MIMESIS AS RANDOM GRAPH COLORING

Aidan Rocke
aidanrocke@gmail.com

April 29, 2019

ABSTRACT

Inspired by the thought-provoking masterpiece by René Girard, Le Bouc Emissaire, a simple and tractable model for mimetic behaviour occurred to me. When we change our beliefs, we do so not because of their intrinsic value. Our desire to switch from belief A to belief B is proportional to the number of adherents of belief B that we know.

1 Mimesis as a decentralised process

In this article, I propose that when we change our beliefs, we do so not because of their intrinsic value. Our desire to switch from belief A to belief B is proportional to the number of adherents of belief B that we know. Technically, I modelled the problem of two conflicting beliefs that propagate through a network with N nodes in a decentralised manner. These beliefs are in some sense competing for adherents.

Using vertex notation, two individuals v_i and v_j with identical beliefs are connected with probability q, and $1-q$ otherwise. v_i changes its belief with a probability proportional to the number of nodes connected to v_i that have opposing views.

Two key motivating questions are:

1. Under what circumstances does a belief get completely wiped out?
2. Under what circumstances does a belief completely dominate(i.e. wipe out) all other beliefs?

In the scenario where there are only two possible beliefs these two questions are equivalent and I show that on average it's sufficient that $q > 1-q$ and that initially, one belief has a greater number of adherents than the other.

Nodes carrying the first belief were assigned to the set of red vertices, R, and nodes carrying the second belief were assigned to the set of blue vertices, B. After further reflection, I chose $+1$ and -1 as labels. The reason being that a change of belief using this representation would be equivalent to multiplication by -1. As a result, the N vertices could be represented by an N-dimensional vector:

$$\vec{v} \in \{-1, 1\}^N \tag{1}$$

where $N = |v_i \in R| + |v_j \in B|$.

2 A random graph model

Using this representation, between each pair of vertices we may define a virtual weight matrix W:

$$w_{ij} = v_i \cdot v_j \tag{2}$$

where $w_{ij} = +1$ implies identical beliefs and we have $w_{ij} = -1$ otherwise.

Now, we note that W may be conveniently decomposed as follows:

$$W = W^+ + W^- \tag{3}$$

where W^- denotes potential connections between nodes of different color and W^+ denotes potential connections between nodes of identical colors.

In order to simulate variations in connectivity we may assume that nodes of the same color are connected with probability $\frac{1}{2} < q < 1$ and nodes of different color are connected with probability $1 - q$. Given W we may therefore construct the adjacency matrix A by sampling random matrices:

$$M_1, M_2 \sim \mathcal{U}([0,1])^{N \times N} \tag{4}$$

$$M^+ = 1_{[0,q)} \circ M_1 \tag{5}$$

$$M^- = 1_{(1-q,1]} \circ M_2 \tag{6}$$

where $1_{[0,q)}$ denotes the characteristic function over the set $[0, q)$ and then we compute the Hadamard products:

$$A^+ = M^+ \cdot W^+ \tag{7}$$

$$A^- = M^- \cdot W^- \tag{8}$$

so the adjacency matrix is given by $A = A^+ + A^-$.

3 Stochastic dynamics for computer simulation

Now, in order to simulate stochastic dynamics we simply use majority vote:

$$p(v_i^{n+1} = v_i^n) = \frac{\bar{N}_i}{N_i} \tag{9}$$

$$p(v_i^{n+1} = -1 \cdot v_i^n) = 1 - \frac{\bar{N}_i}{N_i} \tag{10}$$

$$\bar{N}_i = |A(i,-) > 0| - 1 \tag{11}$$

$$N_i = \bar{N}_i + |A(i,-) < 0| \tag{12}$$

where $|A(i,-) > 0| - 1$ denotes the number of connections between v_i and nodes sharing the same belief without counting a connection to itself.

4 Analysis

If we denote the number of red vertices at instant n by α_n and the number of blue vertices by β_n we may observe that the expected number of neighbors is given by:

$$\langle N(v_i \in R)\rangle = q \cdot (\alpha_n - 1) + (1-q) \cdot \beta_n \tag{13}$$

$$\langle N(v_i \in B)\rangle = q \cdot (\beta_n - 1) + (1-q) \cdot \alpha_n \tag{14}$$

Using the above equations we may define the expected value:

$$\langle \alpha_{n+1}\rangle = \alpha_n \left(\frac{q \cdot (\alpha_n - 1)}{q \cdot (\alpha_n - 1) + (1-q) \cdot \beta_n}\right) + \beta_n \left(\frac{(1-q) \cdot \alpha_n}{q \cdot (\beta_n - 1) + (1-q) \cdot \alpha_n}\right) \tag{15}$$

and we may deduce that $\langle \beta_{n+1}\rangle = N - \langle \alpha_{n+1}\rangle$.

4.1 $\alpha_n > \beta_n$ implies that $\lim\limits_{n \to \infty} \langle \alpha_n \rangle = N$

Assuming that $q > 1 - q$, a simple calculation shows that:

$$\langle \alpha_{n+1}\rangle - \alpha_n \geq 0 \iff \alpha_n \geq \beta_n \tag{16}$$

and since:

$$\langle \alpha_{n+1}\rangle - \alpha_n = 0 \iff \alpha_n = \beta_n \tag{17}$$

we may deduce that:

$$\lim_{n \to \infty} \langle \alpha_n \rangle = N \tag{18}$$

4.2 Analysis of $\Delta\alpha$

Using the fact that $\beta_n = N - \alpha_n$ we may derive the following continuous-space variant of $\Delta\alpha_n = \langle \alpha_{n+1}\rangle - \alpha_n$:

$$\Delta\alpha(\alpha, \gamma) = \frac{\alpha \cdot (N - \alpha)}{\gamma \cdot (\alpha - 1) + (N - \alpha)} - \frac{\alpha \cdot (N - \alpha)}{\gamma \cdot (N - \alpha - 1) + \alpha} \tag{19}$$

where $\gamma = \frac{q}{1-q}$.

References

[1] Réka Albert Albert-László Barabási. Statistical mechanics of complex networks. 2002.

[2] René Girard. Le Bouc émissaire. 1986.

[3] P. Erdös and A. Rényi. On the evolution of random graphs. 1960.

[4] G. Grimmett and C. McDiarmid. On colouring random graphs. 1975.