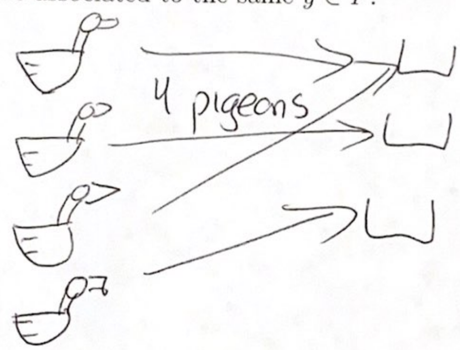
Chapter 4

1 Section 4.1: The Pigeon Hole Principle

The **Pigeon Hole Principle** states that if we have two sets, X and Y, and we are trying to pair up each element $x \in X$ with an element $y \in Y$, then if $|X| > |Y|$ there must be two different x's in X that are associated to the same $y \in Y$.

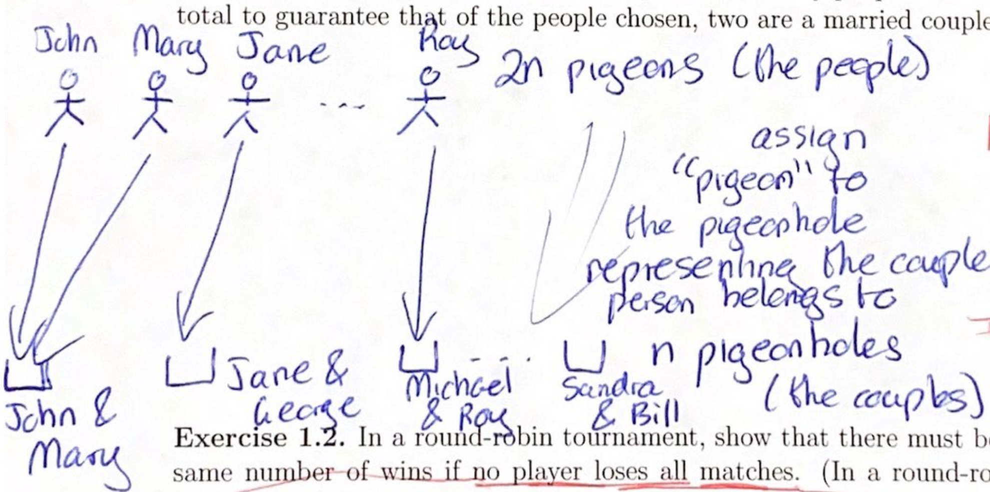
E.g.



3 pigeonholes

If we are putting each pigeon in a pigeonhole, at least one pigeonhole must have more than one pigeon.

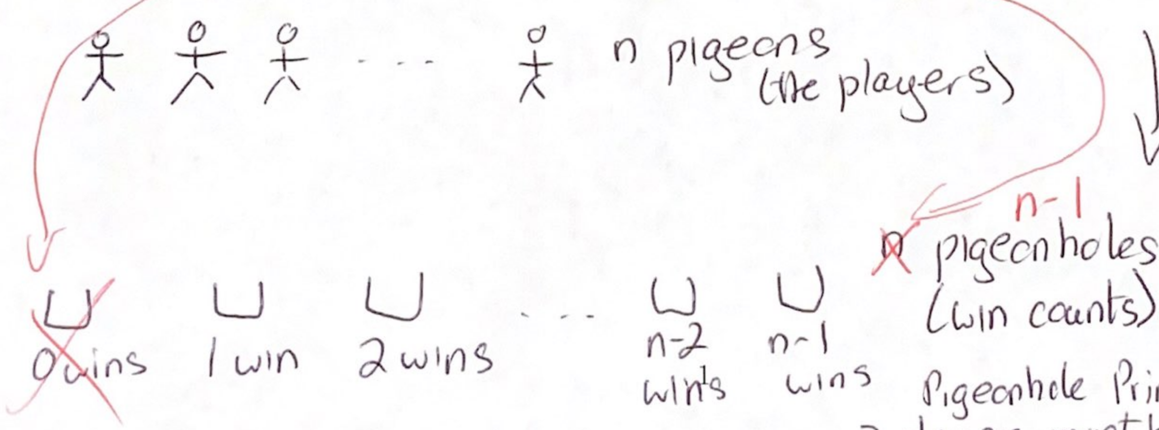
Exercise 1.1. There are n married couples. How many people must we choose from the $2n$ total to guarantee that of the people chosen, two are a married couple?



If we choose n+1 people, by Pigeonhole Principle at least 2 must be in the same ~~coup~~ pigeonhole, thus we have a couple.

Note: Choosing n people doesn't guarantee, since we could have on person from each couple.

Exercise 1.2. In a round-robin tournament, show that there must be two players with the same number of wins if no player loses all matches. (In a round-robin tournament, each player plays all other players exactly once.)



By the Pigeonhole Prin., at least 2 players must be assigned to the same win count.

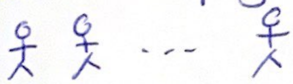
E.g. if I have 31 pigeons and 5 pigeonholes, since $31/5 > 6$, there must be at least one pigeonhole with more than 6 pigeons.

We can also generalize the Pigeonhole Principle to the following: if we have $n * k + 1$ pigeons and n holes, then there must be at least one pigeonhole with more than $n * k$ pigeons.

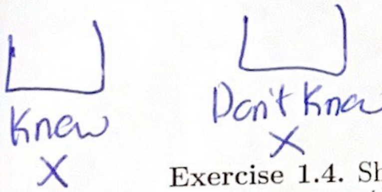
Exercise 1.3. Show that at a party of n people, with $n \geq 6$, there must be either a group of 3 mutual acquaintances or a group of 3 mutual strangers.

Choose one person at the party, call them $\underline{X}$.

Remaining $n-1$ people are the pigeons



assign pigeon to pigeonhole based on whether or not they know $\underline{X}$



2 pigeonholes

$n-1 \geq 5$ pigeons, 2 pigeonholes, so by Pigeonhole Prin. at least one pigeonhole has 3 or more people.

In other words, there are either 3 people who know $\underline{X}$ OR there are 3 people who don't know $\underline{X}$.

back $\Longrightarrow$

Exercise 1.4. Show that if the numbers 1 through 10 are randomly positioned around a circle, then some set of three consecutive numbers must sum to 17 or more.

There are 10 ~~con~~ of these sets (call them "triples"). E.g. on the left the triples are $(8,3,4)$, $(3,4,1)$, $(4,1,10)$, $(1,10,7)$, $(10,7,6)$, $(7,6,9)$, $(6,9,2)$, ~~$(2,5,8)$~~ $(9,2,5)$, $(2,5,8)$, $(5,8,3)$.

Ex:

5	8	3	
2			4
9			1
6		10	
	7		

sums to $22 \geq 17$

Note that every number appears once in exactly one triple. So, if we sum the numbers in each triple, and then add these sums together, we will always get

$1+1+1+2+2+2+3+3+3+\ldots+8+8+8+9+9+9+10+10+10 = 165$

so $\Longrightarrow$ 165 pigeons, distribute to 10 pigeonholes (the triples), at least one pigeonhole must have $> 165/10$ pigeons, a.k.a. ≥ 17.

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1.3 cont'd)

Either 3 people knew X or (at least one of these is true) 3 people don't know X

If this is true.

Call these 3 people who knew X A, B, and C.

If any two of A, B, C know each other (say A and B know each other) then we have 3 people who knew each other (A, B, and X).

If ~~none of A, B, or C kn~~ the above isn't true, and none of A, B, or C knew each other, then we have our 3 mutual strangers (A, B, and C)

If this is true...

almost the same proof, ~~just~~ as what's on the left, just switch "knew" and "don't know"