Chapter 7

1 Section 7.2: The Inclusion-Exclusion Formula

Exercise 1.1. A survey of 100 people found that 57 could speak French, 45 could speak German, and 30 could speak both French and German. How many people could speak French or German?

Fred - speaks French, not German

Jill - speaks German, not French

Jane - speaks both

# of people who speak French + # of people who speak German − $\boxed{\text{\# of people who speak both}}$

undo double counting

$$\underbrace{57}_{\text{count Fred, count Jane}} + \underbrace{45}_{\text{count Jill, count Jane}} - \underbrace{30}_{\text{count Jane}} = \boxed{72}$$

Exercise 1.2. A survey of 200 people asked whether they like to ski, surf, or hike. 75 said they like to ski, 32 said they like to surf, and 103 said they liked to hike. Furthermore, 25 said they liked to both ski and surf, 40 said they liked to ski and hike, and 5 said they liked to surf and hike. There were 2 people who liked to do all three recreational activities. How many people didn't like doing any of the three activities?

200 − (# who ~~don't~~ like doing at least one of the three)

$$200 - (\underbrace{75}_{\text{count Bill, Bob, Mary}} + \underbrace{32}_{\text{count Bob, Mary}} + \underbrace{103}_{\text{count Mary}} - (\underbrace{25}_{\text{Bob, Mary}} + \underbrace{40}_{\text{Mary}} + \underbrace{5}_{\text{Mary}} - (\underbrace{2}_{\text{Mary}})))$$

Bill - likes to ski

Bob - likes to ski and surf

Mary - likes to do all three

$$\underbrace{200}_{\text{total people}} - \underbrace{(75+32+103)}_{\text{like to do one}} + \underbrace{(25+40+5)}_{\text{like to do two}} - \underbrace{(2)}_{\text{like to do all three}}$$

$$= 200 - 210 + 70 - 2 = \boxed{58}$$

in Exercise 2 $n=3$
P_1 = likes to ski
P_2 = likes to surf
P_3 = likes to hike

Theorem 1.3. *Suppose we have a set X, and elements of the set X can have properties $P_1, P_2, \ldots, P_n$. An element of X can have many of these P_i's, or none.*

- *Define X_i to be the subset of X whose elements satisfy property P_i.*
- *Define X_{ij} to be the subset of X whose elements satisfy properties P_i and P_j.*
- *Define X_{ijk} to be the subset of X whose elements satisfy properties P_i, P_j, and P_k.*

  $\vdots$

- *Define $X_{12\ldots n}$ to be the subset of X whose elements satisfy all properties P_1, P_2, $\ldots$, P_n.*

Then the formula for the number of elements of X that don't satisfy any single property P_i is:

$$|X| - \underbrace{\sum_{i=1}^{n} |X_i|}_{\text{for each property } P_i\text{, the \# of things that satisfy } P_i} + \underbrace{\sum_{\{i,j\}\in\{1,2,\ldots,n\}} |X_{ij}|}_{\text{for each pairing } P_i \text{ and } P_j \text{ count \# of things that satisfy both } P_i \text{ and } P_j} - \underbrace{\sum_{\{i,j,k\}\in\{1,2,\ldots,n\}} |X_{ijk}|}_{\text{for each combination of three } P_i\text{'s count \# of things that satisfy all three}} + \cdots + (-1)^n |X_{123\ldots n}|$$

count # of things that satisfy all n properties

Corollary 1.4. *The number of elements with at least 1, 2, $\ldots$, k many properties is:*

take this formula,

satisfy at least one property: $\sum_{i=1}^{n} |X_i| - \sum |X_{ij}| + \sum |X_{ijk}| - \ldots + (-1)^{n+1} |X_{123\cdots n}|$

satisfy at least two properties: $\sum |X_{ij}| - \sum |X_{ijk}| + \sum |X_{ijkl}| - \ldots + (-1)^{n} |X_{123\cdots n}|$

satisfy at least k many properties: $\sum |X_{i_1 i_2 \cdots i_k}| - \sum |X_{i_1 i_2 \cdots i_k i_{k+1}}| + \ldots + (-1)^{n-k+1} |X_{123\cdots n}|$

Exercise 1.5 (Exercise 7.3 from textbook). How many positive integers less than or equal to 100 are divisible by neither 2 nor 5?

P_1: divisible by 2

P_2: ~~are~~ divisible by 5

set of numbers that don't satisfy P_1 or P_2

$X = \{1, 2, 3, \ldots, 100\}$

$X_1 = \{2, 4, 6, 8, \ldots, 100\}$

$X_2 = \{5, 10, 15, \ldots, 100\}$

$X_{12} = \{10, 20, 30, \ldots, 100\}$

$|X| - \underbrace{\sum_{i=1}^{2} |X_i|} + \cancel{2} |X_{12}|$

$= 100 - (50 + 20) + 10$

$= 100 - 70 + 10 = \boxed{40}$

2 Section 7.3: Enumerating Surjections

If A is a set of m objects, and B is a set of n objects, then the number of functions from A to B is $\boxed{n^m}$.

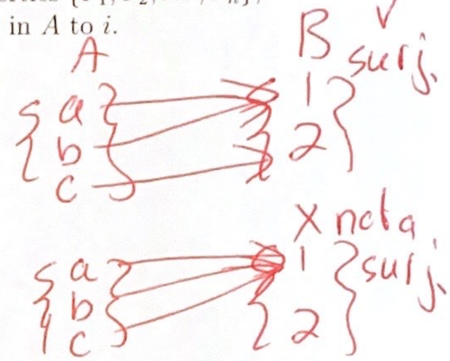
$\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{}$ m choices

But what if we only want to count the number of surjections? That is, we only want to count functions that make sure every element in B receives one element from A.

We use inclusion-exclusion on the set of function from A to B.

Give each of the n objects in B a label 1, 2, ..., n. We then have properties $\{P_1, P_2, \ldots, P_n\}$, where a function having property P_i means f does not map anything in A to i.

$A \; \{1, 2, \ldots, m\}$ $\qquad$ $B \; \{1, 2, \ldots, n\}$



Formula 2.1. The number of surjections from a set A of m objects to a set B of n objects is thus

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n \binom{n}{n}(n-n)^m.$$

- Need to come up with properties $P_1, P_2, \ldots$ such that the # of surjections = # of functions that don't satisfy any property

- Order and label the objects in B $1, 2, \ldots, n$

~~Define: P_i = fun~~

function $f: A \to B$ satisfies P_i if it does not map anybody from A to the ith object in B

$$|X| - \sum_{i=1}^{n} |X_i| + \sum_{i,j} |X_{ij}| - \sum |X_{ijk}| + \cdots + (-1)^n |X_{123\cdots n}|$$

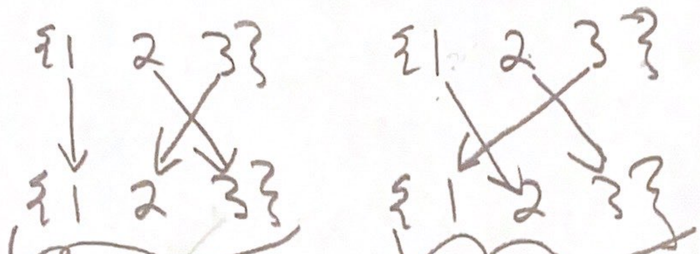
$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \cdots$$

3 Section 7.4: Derangements

Definition 3.1. A **permutation** is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ to itself.

A **derangement** is a permutation where no element i is mapped to itself ($\sigma(i) \neq i$).

set $n=3$



not a derangement because $1 \to 1$

is a derangement

The number of permutations on $\{1, 2, \ldots, n\}$ is $n!$. What about the number of derangements?

Use inclusion-exclusion on the set of permutations. Here, we see a permutation has property P_i if it maps i to itself i. Then the number of derangements is the number of permutations that don't satisfy any of properties $P_1, P_2, \ldots, P_n$.

# of permutations of n objects $\{1, 2, \ldots, n\}$ is $n!$

$$n! = \underbrace{n}_{\text{choices to send 1 to}} \cdot \underbrace{(n-1)}_{\text{choices to send 2 to}} \cdot \underbrace{(n-2)}_{\text{choices to send 3 to}} \cdot \ldots \cdot \underbrace{1}_{\text{choice to send } n \text{ to}}$$

σ permutation σ satisfies P_1 means σ sends 1 to 1

σ satisfies P_2 means σ sends 2 to 2

$\vdots$

σ satisfies P_n means σ sends n to n

$$|X| - \sum |X_i| + \sum |X_{ij}| - \sum |X_{ijk}| + \ldots + (-1)^n \sum |X_{12\cdots n}|$$

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n \cdot 1$$