Chapter 9

Question 0.1. How many strings of length n with digits 0, 1, or 2 are there if 102 is not allowed in the string?

ends in

Case 0: Case 1: Case 2:

Answer 0.2. If $S(n)$ is the number of such strings of length n, then

$S(n-1)$ $S(n-1)$ $S(n-1)$

$$S(n) = 3S(n-1) - S(n-3)$$

$-S(n-3)$

with $S(1) = 3$, $S(2) = 9$, and $S(3) = 26$.

sum together / had to avoid ...10 2

The above is what is called a **recurrence equation.**

a.k.a. recurrence relation difference equation

Definition 0.3. A **recurrence equation** is an equation using the symbols

$$n, s_n, s_{n+1}, s_{n+2}, \ldots, s_{n+k}$$

for some k, with an integer function $S : \mathbb{Z} \to \mathbb{R}$ a solution if for all $n \in \mathbb{Z}$,

$$s_n = S(n), s_{n+1} = S(n+1), \ldots, s_{n+k} = S(n+k)$$

satisfies the equation.

A **linear recurrence equation** is a recurrence equation of the form

$c_k s_n + c_{k-1} s_{n-1} + \cdots + c_1 s_{n-k+1} + c_0 s_{n-k} = g(n)$

$$\boxed{c_k s_{n+k} + c_{k-1} s_{n+k-1} + \cdots + c_1 s_{n+1} + c_0 s_n = g(n),}$$

where $c_0, c_1, c_2, \ldots, c_k$ are constants. A linear recurrence equation is **homogeneous** if $g(n) = 0$, and **nonhomogeneous** otherwise.

Simple Equation:

$s_n = s_{n-1} + 1$

$s_0 = 1$ ← initial condition

$(\underset{s_0}{1}, \underset{s_1}{2}, \underset{s_2}{3}, \underset{s_3}{4}, 5, 6, \ldots)$

$n=1$

$s_1 = s_0 + 1$ holds, because $2 = 1+1$

$n=2$

$s_2 = s_1 + 1$ holds because $3 = 2+1$

$n=3$

$\vdots$

$n=4$

Note: linear means we can't have $S(\sim) \cdot S(\sim)$

(so for example $S(n-3) \cdot S(n-1)$ is not allowed)

and we can't have (function of n) $\cdot S(\sim)$

(so for example $-3n^3 \cdot S(n)$ is not allowed)

Recurrence Equ. | Differential Equation

There are a lot of parallels between differential equations and recurrence relations.

$n \in \mathbb{N} = \{0,1,2,3,\dots\}$ indep. variable | $x \in \mathbb{R}$

$S(n) : \mathbb{N} \to \mathbb{R}$ dep. variable | $f(x) : \mathbb{R} \to \mathbb{R}$

$S(n-1), S(n-2), \dots, S(n-k)$ $\xleftrightarrow[\text{variable}]{\text{operator on dep.}}$ $f'(x), f''(x), \dots, f^{(k)}(x)$

linear homogeneous equations

$c_0 S(n) + c_1 S(n-1) + c_2 S(n-2) + \dots + c_{k-1} S(n-k+1) + c_k S(n-k) = 0$ $\xleftrightarrow[\text{(can't have } S(n)\cdot S(n-1) \text{ or } n\cdot S(n))]{}$ $c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \dots + c_{k-1} f^{(k-1)}(x) + c_k f^{(k)}(x) = 0$

← homog. because RHS is 0

homog. example

$S(n) - 3S(n-1) + S(n-3) = 0$

linear nonhomogeneous equation (when we replace 0 on RHS by some function $g(n)$)

example of non-homog. lin. eq'n

$S(n) - 3S(n-1) + S(n-3) = n^2$

Idea: Replace g_n with r^n, can then divide out by r^n, which leaves a polynomial we can solve

Exercise 0.4 (Exercise 9.3 from textbook). Find the general solution of the following recurrence relation:

$$g_{n+2} = 3g_{n+1} - 2g_n.$$

linear equation: the space of solutions is linear.

- So if a_n is a solution, and b_n is also a solution, then so is $a_n + b_n$
- if a_n is a solution, and c is some number, then $c \cdot a_n$ is also a solution.

$\Rightarrow g_{n+2} - 3g_{n+1} + 2g_n = 0$

try $g_n = r^n$, assuming $r \neq 0$

$\Rightarrow g_{n+1} = r^{n+1}$

$\Rightarrow g_{n+2} = r^{n+2}$

$\Rightarrow$ substitute in,

$$r^{n+2} - 3r^{n+1} + 2r^n = 0$$

divide both sides by r^n

$\Rightarrow r^2 - 3r + 2 = 0$

$\Rightarrow (r-2)(r-1) = 0$

$\Rightarrow r=2, r=1 \Rightarrow$ get $\underbrace{2^n, 1^n}$ as my "atoms"

gen. solution:

$$c_1 2^n + c_2 \cdot (1)^n = \boxed{c_1 2^n + c_2 \cdot 1^n}$$

so, 2^n, 1^n sol'ns means $2^n + 1^n$ is also a solution

- $c \cdot 2^n$ is a sol'n
- $d \cdot 1^n$ is a sol'n

think of 2^n and 1^n as the "basis" (from linear algebra) of the solution space, and the general solution is how we describe all possible solutions in this space.

So, it looks like solving a homogeneous linear recurrence equation is a matter of turning the equation

$$c_k s_{n+k} + c_{k-1} s_{n+k-1} + \cdots + c_1 s_{n+1} + c_0 s_n = 0,$$

into a polynomial

$$P(r) = c_k r^k + c_{k-1} r^{k-1} + \cdots + c_1 r + c_0,$$

finding $P(r)$'s roots

$$r_1, r_2, \ldots, r_k,$$

and thus the general solution is

$$d_1 r_1^n + d_2 r_2^n + \cdots + d_{k-1} r_{k-1}^n + d_k r_k^n.$$

root -2 with multiplicity 3

Question 0.5. What if $P(r)$ has roots with multiplicity? E.g. $P(r) = (r-2)\underbrace{(r+3)^2}(r+2)^3$?

↑ root (-3) with multiplicity 2

If you have multiplicity k, ~~root~~ root r_0
you have to ~~add in~~ have
k different atoms / basis elements:

~~e.g.~~ $c_0 r_0^n + c_1 n r_0^n + c_2 n^2 r_0^n + \ldots + c_{k-1} n^{k-1} r_0^n$

with r_0^n term; distinguish using powers of n!

So for Ex: answer will be
$c_0 \cdot 2^n +$
$c_1(-3)^n + c_2 n(-3)^n +$
$c_3(-2)^n + c_4 n(-2)^n$
$+ c_5 n^2(-2)^n$

Question 0.6. What about nonhomogeneous linear equations?

(1) pretend it's homogeneous, i.e. the RHS is just 0. Find the general solution to this homogeneous version

(2) find a particular solution; find a solution that looks like the RHS

different

these two components added together is the general sol'n for the nonhomog. eq'n

Question 0.7. We covered general solutions, when do specific solutions come into play?

Specific solution, derive from general solution using initial conditions to solve for the unknown constants

Ex on back →

Diagram of Solutions:

gen. sol'n to hom. + part. sol'n

gen. sol'n to non. hom.

plug in initial conditions

↓

specific solution

Continue Ex. C.4

Find the specific solution to

$$g_{n+2} = 3g_{n+1} - 2g_n$$

subject to $g_0 = 1$, $g_1 = 2$

~~First, fi~~

First, find general sol'n. We did this in Ex C.4, found the gen. sol'n to be

$$g_n = c_1 2^n + c_2$$

Second, plug in initial conditions:

$g_0 = 1$, when $n=0$, we have

$$1 = g_0 = c_1 \cdot 2^0 + c_2 = c_1 + c_2$$

$$\Rightarrow 1 = c_1 + c_2$$

$g_1 = 2$, when $n=1$, we have

$$2 = g_1 = c_1 \cdot 2^1 + c_2$$

$$\Rightarrow 2 = 2c_1 + c_2$$

solve system of equations for c_1 and c_2

$$\begin{array}{r} 2 = 2c_1 + c_2 \\ -(1 = c_1 + c_2) \\ \hline 1 = c_1 \end{array}$$

$2 =$

so $2 = 2(1) + c_2 \Rightarrow c_2 = 0$

So specific sol'n is

$$g_n = 1 \cdot 2^n + 0 = \boxed{2^n}$$

particular solution should look like this

Exercise 0.8 (Exercise 9.9a from textbook). Find the general solution to the nonhomogeneous recurrence equation

$$(A-5)(A+2)f_n = 3^n.$$

(1) First, pretend it's homogeneous,

$$(A-5)(A+2)f_n = 0$$

roots are $A=5$, $A=-2$

$\Rightarrow (A^2-3A-10)f_n = 0$

$\Rightarrow A^2 f_n - 3Af_n - 10f_n = 0$

$\Rightarrow f_{n+2} - 3f_{n+1} - 10f_n = 0$

Note: If we substitute in $r^n = f_n$, we end up back where we started

(and thus $r^{n+1} = f_{n+1}$, $r^{n+2} = f_{n+2}$)

$\Rightarrow r^{n+2} - 3r^{n+1} - 10r^n = 0$

$\Rightarrow r^2 - 3r - 10 = 0$

$\Rightarrow (r-5)(r+2) = 0$

roots are $r=5$, $r=-2$

same roots for poly. involving r and poly. involving A

gen. sol'n for hom. version is

$$c_1 5^n + c_2(-2)^n$$

what is A? Advancement Operator

Think of A as like the differential:

$\frac{d}{dx}(f) = f'$, the derivative of f

So A acts on ~~th~~ the sequence f_n like this

$A(f_n) = f_{n+1}$

$f_n = (f_0, f_1, f_2, f_3, f_4, \ldots)$

$Af_n = (f_1, f_2, f_3, f_4, f_5, \ldots)$

$A^2 f_n = A(A(f_n)) = A(f_{n+1}) = f_{n+2}$

$\Rightarrow$

Ex 0.8) $(A-5)(A+2)f_n = 3^n$

2) ~~Try~~ RHS is 3^n,

so try something like $\boxed{d\cdot 3^n = f_n} \to f_{n+1} = d\cdot 3^{n+1}$

Try this out:

verify that this works; find the value of d that makes this work

$$(A-5)(A+2)(d\cdot 3^n) = 3^n$$

$$\Rightarrow (A-5)(A\cdot d3^n + 2d3^n) = 3^n$$

$$\Rightarrow (A-5)(d\cdot 3^{n+1} + 2d3^n) = 3^n$$

$$\Rightarrow Ad3^{n+1} + A(2d3^n) - 5d3^{n+1} - 10d3^n = 3^n$$

$$\Rightarrow d3^{n+2} + 2d3^{n+1} - 5d3^{n+1} - 10d3^n = 3^n$$

divide out both sides by 3^n

$$\Rightarrow d\cdot 3^2 + 2d\cdot 3 - 5d\cdot 3 - 10d = 1$$

$$\Rightarrow 9d + 6d - 15d - 10d = 1$$

$$\Rightarrow -10d = 1$$

$\Rightarrow d = \frac{-1}{10}$, so particular sol'n is this

$$\boxed{-\frac{1}{10}3^n}$$

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so general solution (to the nonhomogeneous eq'n) will just be our answers from (1) and (2) added together

$$\boxed{c_1 5^n + c_2(-2)^n - \frac{1}{10}3^n}$$

Solving nonhom. lin. eq'ns

(1) Pretend it's hom., and get the gen. sol'n to that

(2) Find a part. sol'n → gen sol'n to the nonhom. eq'n

Remark 0.9. For nonhomogeneous linear recurrence equations, if the right hand side contains...

- ...an exponential component c^n for some constant c, start by trying $d * c^n$ (where d is some unknown constant). If that doesn't work, try $d * nc^n$, then $d * n^2c^n$, then $d * n^3c^n$, and so forth.
- ...a power component n^k, then use a polynomial of degree k. E.g., if the right hand side is n^3, then you should try $d_3n^3 + d_2n^2 + d_1n + d_0$.

Exercise 0.10 (Exercise 9.9e from textbook). Find the general solution for the nonhomogeneous linear recurrence equation

$$(A-2)(A-40)f = 3n^2 + 9^n$$

$3n^2 \to b_2n^2 + b_1n + b_0$

$9^n \to$ start by trying $d\cdot 9^n$ → if it doesn't work $dn9^n$ → if it doesn't work $dn^2 9^n$ → if it doesn't work $dn^3 9^n$ ⋮

(I) First find gen. sol'n of hom. version:

$(A-2)(A-40)f_n = 0$

$c_1 2^n + c_2 (40)^n$

Final Answer:

$$\boxed{c_1 2^n + c_2(40)^n + 13n^2 + \frac{80}{3}n + \frac{3278}{117} - \frac{1}{217}\cdot 9^n}$$

(II) Second, find a particular solution:

RHS is $3n^2 + 9^n$

$\boxed{b_2n^2 + b_1n + b_0 + d\cdot 9^n}$ ← try this

$(A-2)(A-40)(b_2n^2 + b_1n + b_0 + d9^n) = 3n^2 + 9^n$

$(A-2)(b_2(n+1)^2 + b_1(n+1) + b_0 + d9^{n+1}$
$- 40b_2n^2 - 40b_1n - 40b_0 - 40d9^n) = 3n^2 + 9^n$

⋮ after some algebra

$39b_2n^2 - 80b_2n + 39b_1n + b_2 - 40b_1 + 39b_0 - 39 - 7\cdot 31d9^n = 3n^2 + 9^n$

n^2 term: $39b_2 = 3$

n term: $-80b_2 + 39b_1 = 0$

const: $b_2 - 40b_1 + 39b_0 - 39 = 0$

9^n term: $-7\cdot 31d = 1$

solve $b_2 = 13$, $b_1 = \frac{80}{3}$, $b_0 = \frac{3278}{117}$, $d = -\frac{1}{217}$

Extension of Exercise 0.11)

What if we have initial conditions?

$$f_0 = 0$$

$$f_1 = \frac{145}{140}$$

$$f_2 = \frac{-90}{140}$$

Take the general solution to the nonhom. lin. equation

$$f_n = c_1(-2)^n + c_2 5^n + c_3 + \frac{1}{140} n 5^n$$

and substitute

$$\boxed{f_n = -\frac{1}{3}(-2)^n + \frac{1}{3} + \frac{1}{140} n 5^n}$$

$n=0$

$$f_0 = 0 = c_1(-2)^0 + c_2 5^0 + c_3 + \frac{1}{140}\cdot 0 \cdot 5^0 = c_1 + c_2 + c_3$$

$$f_1 = \frac{145}{140} = c_1(-2)^1 + c_2 5^1 + c_3 + \frac{1}{140}\cdot 5 \Rightarrow \frac{145}{140} - \frac{5}{140} = -2c_1 + 5c_2 + c_3$$

$$f_2 = \frac{-90}{140} = c_1(-2)^2 + c_2\cdot 5^2 + c_3 + \frac{1}{140}\cdot 2\cdot 5^2 \Rightarrow \frac{-90}{140} - \frac{50}{140} = 4c_1 + 25c_2 + c_3$$

system

$$\begin{cases} 0 = c_1 + c_2 + c_3 \\ 1 = -2c_1 + 5c_2 + c_3 \\ -1 = 4c_1 + 25c_2 + c_3 \end{cases}$$

$$\boxed{c_3 = \frac{1}{3}}$$

$$c_3 = -(c_1 + c_2)$$

$$\boxed{c_2 = 0}$$

$$1 = -2c_1 + 5c_2 - c_1 - c_2$$

$$\Rightarrow 1 = -3c_1 + 4c_2 \Rightarrow \frac{1+3c_1}{4} = c_2$$

$$\cancel{f_n = -\frac{1}{3}(-2)^n}$$

$$-1 = 4c_1 + 25\left(\frac{1}{4} + \frac{3}{4}c_1\right) - \left(c_1 + \frac{1}{4} + \frac{3}{4}c_1\right)$$

$$-1 = \left(4c_1 + \frac{75}{4}c_1 - c_1 - \frac{3}{4}c_1\right) + \frac{25}{4} - \frac{1}{4}$$

$$\Rightarrow -\frac{4}{4} + \frac{1}{4} - \frac{25}{4} = c_1\left(\frac{16}{4} + \frac{75}{4} - \frac{4}{4} - \frac{3}{4}\right)$$

$$\Rightarrow -\frac{28}{4} = c_1\left(\frac{84}{4}\right) \Rightarrow -7 = 21c_1 \Rightarrow \boxed{c_1 = \frac{-1}{3}}$$

Exercise 0.11 (Exercise 9.9f from textbook). Find the general solution for the nonhomogeneous linear recurrence equation

$$(A+2)(A-5)(A-1)f_n = 5^n.$$

(1) Pretend it's hom., and solve that version

$$\underbrace{(A+2)(A-5)(A-1)f_n = 0}$$

$\hookrightarrow \underline{c_1(-2)^n + c_2 5^n + c_3(1)^n}$

(2) Look for particular sol'n:

RHS is 5^n

so let's try $d \cdot 5^n = f_n$

$\cancel{(A+2)(A-5)(A-1)f_n = 5^n}$

$\Rightarrow \cancel{(A+2)(A-5)(f_{n+1} - f_n) = 5^n}$

$(A+2)(A-5)(A-1)d\cdot 5^n = 5^n$

$\Rightarrow (A+2)(A-5)(d5^{n+1} - d5^n) = 5^n$

$\Rightarrow (A+2)(d5^{n+2} - d5^{n+1} - 5d5^{n+1} + 5d5^n) = 5^n$

$\Rightarrow (A+2)(\cancel{d5^{n+2}} \cancel{- d5^{n+1}} - \cancel{d5^{n+2}} + \cancel{d5^{n+1}}) = 5^n$

$\Rightarrow (A+2)\cdot 0 = 5^n$

$\Rightarrow 0 = 5^n$???

doesn't work

$\Rightarrow$

Ex 0.11 cont'd)

so now try $d \cdot n \cdot 5^n$

$$(A+2)(A-5)(A-1)\,dn\cdot 5^n = 5^n$$

$$\Rightarrow (A+2)(A-5)\left(d(n+1)5^{n+1} - dn5^n\right) = 5^n$$

$$\Rightarrow (A+2)(A-5)\left[dn5^{n+1} + d5^{n+1} - dn5^n\right] = 5^n$$

$$\Rightarrow (A+2)\left[d(n+1)5^{n+2} + d5^{n+2} - d(n+1)5^{n+1} - 5dn5^{n+1} - 5d5^{n+1} + 5dn5^n\right] = 5^n$$

$$\Rightarrow (A+2)\left[\cancel{dn5^{n+2}} + d5^{n+2} \cancel{+ d5^{n+2}} \cancel{- dn5^{n+1}} - d5^{n+1} \cancel{5d} \cancel{- dn5^{n+2}} \cancel{- d5^{n+2}} \cancel{+ dn5^{n+1}}\right] = 5^n$$

$$\Rightarrow (A+2)\left(d5^{n+1}(5-1)\right) = 5^n$$

$$\Rightarrow (A+2)(4d5^{n+1}) = 5^n$$

$$\Rightarrow 4d5^{n+2} + 8d5^{n+1} = 5^n$$

divide out by 5^n

$$\Rightarrow 4d\cdot 5^2 + 8d\cdot 5 = 1$$

$$\Rightarrow 100d + 40d = 1$$

$$\Rightarrow d = \frac{1}{140}$$

and so part. sol'n is

$$\frac{1}{140} n 5^n$$

so our gen. sol'n

$$\boxed{f_n = c_1(-2)^n + c_2 5^n + c_3 + \frac{1}{140} n 5^n}$$

Idea: Need to solve for
$\{r_n\} = (r_0, r_1, r_2, r_3, \dots)$

take the generating function for $\{r_n\} = r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \dots$
$= \sum_{k=0}^{\infty} r_k x^k$

call it $g(x)$ $\quad$ $g(x)$

1 Section 9.6: Using generating functions to solve recurrences

Idea:

- Start with the generating function

$$g(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n + \dots .$$

This represents the generating function for the solution $S(n)$ of the recurrence equation. By figuring out the closed form of this generating function we solve the recurrence equation.

(*)
- Make the following substitutions into the recurrence equation; replace r_n with $g(x)$, replace r_{n-1} with $xg(x)$, replace r_{n-2} with $x^2 g(x)$, ..., replace r_{n-k} with $x^k g(x)$, etc.
- Now simplify the expression you created from the substitution. If r_{n-k} was the smallest index, you should be left with a polynomial of degree $< k$. All that's left is to solve for $g(x)$.

Exercise 1.1 (Exercise 9.15 from textbook). Use generating functions to solve the recurrence equation

$$\boxed{r_n = r_{n-1} + 6r_{n-2},} \; r_0 = 1, \; r_1 = 3.$$

① create LHS of functional equation

② "align" the LHS series term by term

③ eval. alignment to get RHS

move everybody onto the left side
$\Rightarrow r_n - r_{n-1} - 6r_{n-2} = 0$

set up functional equation:
LHS will be the LHS of the recurrence eq'n after we do (*) to it

① $g(x) - xg(x) - 6x^2 g(x) = \underbrace{r_0 + (r_1 - r_0)x}$

②
$$\begin{array}{lllllll}
g(x) = r_0 & + r_1 x & + r_2 x^2 & + r_3 x^3 & + r_4 x^4 + \dots & + r_k x^k + \dots \\
-xg(x) = & - r_0 x & - r_1 x^2 & - r_2 x^3 & - r_3 x^4 - \dots & - r_{k-1} x^k - \dots \\
-6x^2 g(x) = & & - 6r_0 x^2 & - 6r_1 x^3 & - 6r_2 x^4 - \dots & - 6r_{k-2} x^k - \dots \\
\hline
\end{array}$$

③ $= r_0 + (r_1 - r_0)x + (r_2 - r_1 - 6r_0)x^2 + 0x^3 + 0x^4 + \dots + 0x^k$
$+ 0x^2$

$r_0 - r_1 - 6r_2 = 0$, $r_0 = 1, r_1 = 3$

functional equation

$g(x) - xg(x) - 6x^2 g(x) = r_0 + (r_1 - r_0)x$

plug in

$g(x) - xg(x) - 6x^2 g(x) = 1 + 2x$ now solve for $g(x)$

$g(x)(1 - x - 6x^2) = 1 + 2x$

$\Rightarrow g(x) = \dfrac{1+2x}{(1 - x - 6x^2)}$

$\Rightarrow g(x) = \dfrac{\cancel{1+2x}}{(-3x+1)\cancel{(2x+1)}} = \dfrac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0} \boxed{3^n} x^n$

we were looking for $(r_0, r_1, r_2, \dots, r_n, \dots)$

$\sum_{n=0}^{\infty} \boxed{r_n} x^n = g(x) = \sum_{n=0}^{\infty} \boxed{3^n} x^n$

$\boxed{r_n = 3^n}$

Ex: $a_n = 2a_{n-1} + 2^n$, $\boxed{a_0 = 1}$

Assume sequence
$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$
solves the above rec. eq'n

Take $A(x)$ the generating function of $a_0, a_1, a_2, \ldots$

$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

① Get LHS of functional equation by convert a_n's to $A(x)$'s

$a_n = 2a_{n-1} + 2^n$ ← keep part not involving a_n on the RHS

$\Rightarrow a_n - 2a_{n-1} = 2^n$ — replace a_n with $A(x)$, a_{n-1} with $xA(x)$, a_{n-2} with $x^2A(x)$, a_{n-3} with $x^3A(x)$, ...

from so, it will be $A(x) - 2xA(x)$

② LHS, create alignment

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$
$$-2xA(x) = \quad -2a_0 x - 2a_1 x^2 - \ldots - 2a_{n-1} x^n - \ldots$$

③ sum

use initial condition

$$A(x) - 2xA(x) = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \ldots + (a_n - 2a_{n-1})x^n + \ldots$$
$$= 1 + 2^1 x + (2^2)x^2 + \ldots + 2^n x^n + \ldots$$

$\Rightarrow$ functional equation
$$A(x) - 2xA(x) = \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}$$

solve for $A(x)$ $\Rightarrow$ $A(x) - 2xA(x) = \frac{1}{1-2x}$

$\Longrightarrow$

$$A(x) - 2xA(x) = \frac{1}{1-2x}$$

$$\Rightarrow A(x)(1-2x) = \frac{1}{1-2x}$$

Convert to power series form

$$\Rightarrow A(x) = \underbrace{\frac{1}{(1-2x)^2}}$$

$$\int \frac{1}{(1-2x)^2}\,dx \qquad u = 1-2x$$

$$du = -2\,dx$$

$$= \int \frac{1}{u^2}\frac{du}{-2} \qquad \frac{du}{-2} = dx$$

$$= \frac{1}{-2}\int u^{-2}\,du$$

$$= \frac{1}{2}\left[u^{-1}\right] = \frac{1}{2}\frac{1}{(1-2x)}$$

$$A(x) = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{1-2x}\right)$$

$$A(x) = \frac{1}{2}\frac{d}{dx}\sum_{n=0}^{\infty}(2x)^n$$

~~$A(x) = \frac{1}{2}$~~

$$A(x) = \frac{1}{2}\frac{d}{dx}\left(1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \dots + 2^n x^n + \dots\right)$$

$$= \frac{1}{2}\left(1\cdot 2^0 + 2\cdot 2^2 x + 3\cdot 2^3 x^2 + 4\cdot 2^4 x^3 + \dots + \underbrace{n2^n x^{n-1}}_{\text{rewrite as } (n+1)2^{n+1}x^n} + \dots\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)2^{n+1}x^n$$

$$A(x) = \sum_{n=0}^{\infty}\underbrace{(n+1)2^n}x^n$$

pull the coefficient out of the power series

$$\boxed{a_n = (n+1)2^n}$$

verify answer:

$$a_n = (n+1)2^n \qquad a_n = 2a_{n-1} + 2^n$$

$$a_{n-1} = n\cdot 2^{n-1} \qquad (n+1)2^n = 2(n2^{n-1}) + 2^n$$

$$\Rightarrow n2^n + 2^n = n2^n + 2^n \checkmark$$

$b_{n+3} - 4b_{n+2} + b_{n+1} + 6b_n = 3^n$ Ex 9.17 from textbook

$b_0 = 1$

$b_1 = -5$

$b_2 = 1$

$b_3 = 4$

$\Rightarrow b_n - 4b_{n-1} + b_{n-2} + 6b_{n-3} = 3^{n-3}$

$\hookrightarrow b_3 - 4b_2 + b_1 + 6b_0 = 3^0$ plug in b_0, b_2, b_3, solve for b_1!

$b_1 = -5$

Step I) Get the functional equation:

b_{n+3}: $B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \dots + b_n x^n + \dots$

b_{n+2}: $-4xB(x) = \quad -4b_0 x - 4b_1 x^2 - 4b_2 x^3 - 4b_3 x^4 - \dots - 4b_{n-1} x^n - \dots$

b_{n+1}: $x^2 B(x) = \quad b_0 x^2 + b_1 x^3 + b_2 x^4 + \dots + b_{n-2} x^n + \dots$

b_n: $6x^3 B(x) = \quad 6b_0 x^3 + 6b_1 x^4 + \dots + 6b_{n-3} x^n + \dots$

$$B(x) - 4xB(x) + x^2 B(x) + 6x^3 B(x) = b_0 + (b_1 - 4b_0)x + (b_2 - 4b_1 + b_0)x^2 + 3^0 x^3 + 3^1 x^4 + \dots + 3^{n-3} x^n + \dots$$

$$\Rightarrow B(x) - 4xB(x) + x^2 B(x) + 6x^3 B(x) = 1 - 9x + 22x^2 + \underbrace{\sum_{n=3}^{\infty} 3^{n-3} x^n}$$

$$\sum_{n=0}^{\infty} 3^n x^{n+3} = x^3 \sum_{n=0}^{\infty} 3^n x^n = \frac{x^3}{1-3x}$$

$$\Rightarrow B(x)(1 - 4x + x^2 + 6x^3) = 1 - 9x + 22x^2 + \frac{x^3}{1-3x}$$

take common denominator

$$\Rightarrow B(x)(1 - 4x + x^2 + 6x^3) = \frac{-65x^3 + 49x^2 - 12x + 1}{(1-3x)}$$

$$\Rightarrow B(x) = \frac{-65x^3 + 49x^2 - 12x + 1}{(1 - 4x + x^2 + 6x^3)(1-3x)}$$

Step II)

We have $B(x)$, now to convert it to power series form

$$B(x) = \frac{-65x^3+49x^2-12x+1}{(1-4x+x^2+6x^3)(1-3x)}$$

← partial fractions, first, factor the denominator

Factoring $(1-4x+x^2+6x^3)$

see that $x=-1$ is a root, so factor out $x+1$

$$\begin{array}{r} 6x^2-5x+1 \\ x+1\,\overline{)\,6x^3+x^2-4x+1} \\ -(6x^3+6x^2) \\ \hline -5x^2-4x \\ -(-5x^2-5x) \\ \hline x+1 \end{array}$$

$(x+1)(6x^2-5x+1)$

$=(x+1)(-2x+1)(-3x+1)$

$$= \frac{-65x^3+49x^2-12x+1}{(1+x)(1-2x)(1-3x)^2}$$

now partial fractions

each factor in the denominator gets its own rational function with an unknown constant on top

(two since $(1-3x)$ has mult. 2)

$$= \frac{P}{(1+x)} + \frac{Q}{(1-2x)} + \frac{R}{(1-3x)} + \frac{S}{(1-3x)^2}$$

$$\frac{-65x^3+49x^2-12x+1}{(1+x)(1-2x)(1-3x)^2} = \frac{P}{(1+x)} + \frac{Q}{(1-2x)} + \frac{R}{(1-3x)} + \frac{S}{(1-3x)^2}$$

clear denominators: multiply by $(1+x)(1-2x)(1-3x)^2$

$$\Rightarrow -65x^3+49x^2-12x+1 = P(1-2x)(1-3x)^2 + Q(1+x)(1-3x)^2 + R(1+x)(1-2x)(1-3x)$$
$$+S(1+x)(1-2x) \Longrightarrow$$

$\Rightarrow$ after simplifying

$$-65x^3+49x^2-12x+1 = P(-18x^3+21x^2-8x+1)$$
$$+Q(9x^3+3x^2-5x+1)$$
$$+R(6x^3+x^2-4x+1)$$
$$+S(-2x^2-x+1)$$

coefficients of like terms have to be equal

for x^3 term: $-65 = -18P+9Q+6R$
for x^2 term: $49 = 21P+3Q+R-2S$
for x term: $-12 = -8P-5Q-4R-S$
for const. term $\boxed{1 = P+Q+R+S}$

system of eqn's solve for P, Q, R, S

$\downarrow$ $S = 1-P-Q-R$, plug into top three equations, we now have

$$\begin{cases} -65 = -18P+9Q+6R \\ 51 = 23P+5Q+3R \\ -11 = -7P-4Q-3R \end{cases}$$

$\Rightarrow$ add together

$40 = 16P+Q$

$\Rightarrow \boxed{Q = 40-16P}$

plug into top two equations

$-425 = -162P+6R$
$-2(-149 = -57P+3R)$ $\Rightarrow$ multiply bottom by -2, add to the top

$-127 = -48P \Rightarrow \frac{127}{48} = P$

substituting back gives

$\Rightarrow \frac{29}{48} = R$, $Q = -\frac{7}{3}$, $S = \frac{1}{12}$

$$B(x) = \frac{P}{1+x} + \frac{G}{1-2x} + \frac{R}{1-3x} + \frac{S}{(1-3x)^2}$$

$P = \frac{127}{48}$

$G = -\frac{7}{3}$

$R = \frac{29}{48}$

$S = \frac{1}{12}$

completes partial fraction decomposition

$$B(x) = \frac{127}{48}\frac{1}{1+x} - \frac{7}{3}\frac{1}{(1-2x)} + \frac{29}{48}\frac{1}{(1-3x)} + \frac{1}{12}\left(\frac{1}{(1-3x)^2}\right)$$

$$= \frac{127}{48}\sum_{n=0}^{\infty}(-x)^n - \frac{7}{3}\sum_{n=0}^{\infty}(2x)^n + \frac{29}{48}\sum_{n=0}^{\infty}(3x)^n + \frac{1}{12}\sum_{n=0}^{\infty}\frac{1}{3}\cdot 3^{n+1}(n+1)x^n$$

$$= \underbrace{\frac{127}{48}\sum_{n=0}^{\infty}(-1)^n x^n}_{} - \underbrace{\frac{7}{3}\sum_{n=0}^{\infty}2^n x^n}_{} + \underbrace{\frac{29}{48}\sum_{n=0}^{\infty}3^n x^n}_{} + \underbrace{\frac{1}{12}\sum_{n=0}^{\infty}3^n(n+1)x^n}_{}$$

~~then~~ coeff. of x^n

$-\frac{127}{48}$, n odd

$\frac{127}{48}$, n even

coeff. of x^n

$-\frac{7}{3}\cdot 2^n$

coeff. of x^n

$\frac{29}{48}\cdot 3^n$

coeff. of x^n

$\frac{1}{12}3^n(n+1)$

Final Step: Pull out the coefficients of x^n from each series, and add together

Sidebar:

$$\frac{1}{(1-3x)^2} = \frac{d}{dx}\frac{1}{3}\frac{1}{1-3x}$$

$$= \frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}(3x)^n$$

$$= \frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}3^n x^n$$

$$= \frac{1}{3}\sum_{n=1}^{\infty}3^n\cdot n\, x^{n-1}$$

reindex

$$= \frac{1}{3}\sum_{n=0}^{\infty}3^{n+1}(n+1)x^n$$

add together

$$\boxed{b_n = \begin{cases} \frac{1}{12}3^n(n+1) + \frac{29}{48}\cdot 3^n - \frac{7}{3}\cdot 2^n + \frac{127}{48}, & n \text{ even} \\ \frac{1}{12}3^n(n+1) + \frac{29}{48}\cdot 3^n - \frac{7}{3}\cdot 2^n - \frac{127}{48}, & n \text{ odd} \end{cases}}$$