Chapter 8

1 Section 8.2

Example 1.1. Recall the Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Note that $(1+x)^n$ encodes all the different combinations where you choose from n objects. If you want to determine how many ways there are to choose k objects from a set of n objects, you look at the coefficient of x^k.

We thus say that $(1+x)^n$ is the **generating function** for the sequence of choosing k objects from a set of n objects.

Definition 1.2. Let $S = \{a_0, a_1, a_2, \dots,\}$ be a sequence of numbers. We say that a power series $F(x)$ is a **generating function** of S if the for every $k = 0, 1, 2, \dots$ we have that a_k is the coefficient of x^k in $F(x)$. That is,

$S = (a_0, a_1, a_2, a_3, \dots)$

$$F(x) = \sum_{k=0}^{\infty} a_k x^k.$$

g.f. for $S = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots$

$$(1+x)^5 = \binom{5}{0}x^0 + \binom{5}{1}x^1 + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 + 0\cdot x^6 + 0\cdot x^7 + \dots$$

this is the generating function for the # of ways to choose k objects from a set of 5

how many ways are there to choose $\boxed{2}$ objects from a set of 5?

I go to the term whose degree is $\boxed{2}$

Look at its coefficients $\binom{5}{2}$, which is the answer.

Why do we like generating functions? They preserve information under multiplication.

$f(x)$ is the generating function for the # of ways to do <u>this</u>

$g(x)$ is the g.f. for the # of ways to do <u>that</u>

$\Rightarrow f(x) \cdot g(x)$ the g.f. for the # of ways to do this ~~&~~ that
and

Exercise 1.3. You are packing for a trip. You can pick from 4 red shirts, 4 blue shirts, and 4 green shirts. Write the generating function for the number of ways to pack k shirts? (Assume that shirts of the same color are identical.)

Idea: Split ~~into~~ task of choosing k shirts of any color into the simpler task of choosing ~~#~~ shirts of just one color.

g.f. for # of ways to pack red shirts $R(x) = 1x^0 + 1x^1 + 1\cdot x^2 + 1\cdot x^3 + 1\cdot x^4 + 0x^5 + 0x^6 + \ldots$

$= (1+x+x^2+x^3+x^4)$

g.f.'s for picking just blue shirts $B(x)$ or just green shirts $G(x)$ will be the same

$= B(x)$

$= G(x)$

so our answer will be

$B(x)\cdot G(x)\cdot R(x)$

$= \boxed{(1+x+x^2+x^3+x^4)^3}$

Why multiply: In above answer, look at # of ways to choose 3 shirts (regardless of color)? Many combinations of colors:

1 red, 2 blue, 0 green $(x \cdot x^2 \cdot 1$

0 red, 1 blue, 2 green $(1 \cdot x \cdot x^2)$

3 red, 0 blue, 0 green $(x^3 \cdot 1 \cdot 1)$

1 red, 1 blue, 1 green $(x \cdot x \cdot x)$

⋮

Exercise 1.4. How does the above generating function change if you now can pick from up to 7 red shirts, and you have to pack at least 1 blue shirt?

Same strategy as in 1.3: Compute $R(x)$, $G(x)$, $B(x)$ individually, then multiply all three together.

$G(x) = 1+x+x^2+x^3+x^4$

$R(x) = 1+x+x^2+x^3+x^4+x^5+x^6+x^7$

$B(x) = 0\cdot x^0 + 1x^1 + 1x^2 + 1x^3 + 1x^4$

$= x+x^2+x^3+x^4$

so answer

$$\boxed{G(x)R(x)B(x) = (1+x+x^2+x^3+x^4)(1+x^2+\ldots+x^7)(x+x^2+x^3+x^4)}$$

$(1+x+x^2+x^3+\dots)(\binom{5}{0}x+\binom{5}{1}x^2+\binom{5}{2}x^3\dots)(1^2+2^2x+3^2x^2+4^2x^3\dots)$

What about extracting the coefficients from a generating function? Especially a generating function written in closed form like this:

$$\frac{1+x}{(1-x^2)}?$$

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ geometric series

$\frac{1}{1-\bigcirc} = \sum_{k=0}^{\infty} \bigcirc^k$

closed form
Advantage: easy to manipulate

series form
Advantage: can look up coefficients of x^k's.

Things To Remember:

- $\boxed{\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k}$
- $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$
- $\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$
- $\frac{1}{1-x^n} = \sum_{k=0}^{\infty} x^{nk}$
- $\boxed{\frac{1}{(1-x)^2} = \frac{d}{dx}(\frac{1}{1-x}) = \frac{d}{dx}(\sum_{k=0}^{\infty} x^k) = \sum_{k=1}^{\infty} kx^{k-1}}$ ← derivative trick
- $\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}(\frac{1}{1-x}) = \frac{1}{2}\frac{d^2}{dx^2}(\sum_{k=0}^{\infty} x^k) = \sum_{k=2}^{\infty}(\frac{1}{2}k(k-1))x^{k-2}$
- $\frac{1}{(1-x)^4} = \frac{1}{3!}\frac{d^3}{dx^3}(\frac{1}{1-x}) = \frac{1}{3!}\frac{d^3}{dx^3}(\sum_{k=0}^{\infty} x^k) = \sum_{k=3}^{\infty}(\frac{1}{3!}k(k-1)(k-2))x^{k-3}$
- $\frac{1}{(1-x)^n} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(\frac{1}{1-x}) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(\sum_{k=0}^{\infty} x^k) = \sum_{k=n}^{\infty}(\frac{1}{(n-1)!}k(k-1)(k-2)\cdots(k-n+2))x^{k-n+1}$

- Build up generating function as a product of many simpler g.f.'s. → gives us a product of power series
- Want to combine the multiple power series into one big power series,
  - To do that, we convert to closed form, multiply together, then convert back to power series form.

power series form of $\frac{1+x}{(1-x^2)}$

$= (1+x)\frac{1}{(1-x^2)}$ ← geom. series

$= (1+x)\sum_{k=0}^{\infty}(x^2)^k$

$= (1+x)\sum_{k=0}^{\infty} x^{2k}$

$= \sum_{k=0}^{\infty} x^{2k} + \sum_{k=0}^{\infty} x^{2k+1}$

⇒ back

cont'd from front.

$$\sum_{k=0}^{\infty} x^{2k} + \sum_{k=0}^{\infty} x^{2k+1} = \sum_{k=0}^{\infty} x^{2k} + \cancel{x}^{2k+1} = \sum_{k=0}^{\infty} x^{k}$$

$$\sum_{k=0}^{\infty} x^{2k} = (x^0 + x^2 + x^4 + x^6 + \ldots)$$

$$\sum_{k=0}^{\infty} x^{2k+1} = (x^1 + x^3 + x^5 + x^7 + \ldots)$$

What is the coeff. of ~~x~~ x^5 in $\frac{1+x}{(1-x)^2}$?

$$(1+x) \cdot \frac{1}{(1-x)^2}$$

$1-x=u$

$-dx=du$

$$\frac{1}{u^2}(-du)$$

$$= u^{-2}(-du)$$

$$= \cancel{+}\frac{1}{1}u^{-1}(\cancel{-}du)$$

$$= u^{-1} = \frac{1}{u} = \frac{1}{1-x}$$

$$=(1+x)\frac{d}{dx}\left[\frac{1}{1-x}\right]$$

$$=(1+x)\frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right)$$

$$\frac{d}{dx}(1+x+x^2+x^3+\ldots)$$

$$= 1+2x+3x^2+4x^3+\ldots$$

$$= 1x^0+2x^1+3x^2+\ldots$$

$$= \sum_{k=0}^{\infty}(k+1)x^k$$

$$=(1+x)\left(\sum_{k=1}^{\infty} kx^{k-1}\right)$$

$$= \underbrace{\sum_{k=1}^{\infty} kx^{k-1}}_{\text{find coeff. of } x^5 \text{ here}} + \underbrace{\sum_{k=1}^{\infty} kx^{k}}_{\text{find coeff. of } x^5 \text{ here}}$$

find coeff. of x^5 here

x^5 when $k-1=5$

$\Rightarrow k=6$

so coeff. is $\boxed{6}$

find coeff. of x^5 here

~~k=5 gets~~ us x^5 term,

so coeff. is $\boxed{5}$

add together will get our answer

$6+5=\boxed{11}$

$$1+x+x^2+x^3+x^4+\ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$ $\boxed{\text{Bonus Trick \#1}}$

$$x^5+x^6+x^7+x^8+\ldots = \sum_{k=0}^{\infty} x^{k+5} = x^5 \sum_{k=0}^{\infty} x^k = \frac{x^5}{1-x}$$

$\boxed{\text{Why } \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ ?}}$

$$\cancel{\frac{1}{1-x}} \quad (1-x)(1+x+x^2+x^3+x^4+\ldots)$$

$$= (1+x+x^2+x^3+x^4+\ldots) - (x+x^2+x^3+x^4+\ldots)$$

$$= (1+(x-x)+(x^2-x^2)+(x^3-x^3)+\ldots)$$

$$= 1$$

$$\Rightarrow \frac{1}{1-x} = (1+x+x^2+x^3+x^4+\ldots)$$

$\boxed{\text{Bonus Trick \#2}}$

$$1+x+x^2+x^3+x^4+\ldots+x^{10}$$

$$= (1+x+x^2+x^3+\ldots) - (x^{11}+x^{12}+x^{13}+x^{14}+\ldots)$$

$$= \frac{1}{1-x} - \frac{x^{11}}{1-x} = \frac{(1-x^{11})}{1-x}$$

Exercise 1.5 (Exercise 8.5 from textbook). Find the way to create a bunch of n balloons selected from white, gold, and blue balloons so that the bunch contains at least one white balloon, at least one gold balloon, and at most two blue balloons. How many ways are there to create a bunch of 10 balloons subject to these requirements?

1) Find the generating function for # of bunches of balloons according to these restrictions.

2) Find the # of bunches of 10 balloons subj. to the restrictions.

$$g(x) = \sum_{k=0}^{\infty} b_k x^k$$

b_k is the # of bunches of k balloons

Split into simpler components:

$G(x)$ = g.f. for # of bunches of gold balloons subj. to at least one gold balloon

$$= x^1 + x^2 + x^3 + \ldots = \frac{x}{1-x}$$

$W(x)$ = g.f. for # of bunches of white balloons subj. to at least one white balloon

$$= x + x^2 + x^3 + \ldots = \frac{x}{1-x}$$

$B(x)$ = g.f. for # " " " blue " subj. to ~~at least~~ at most two blue balloons

$$= 1 + x + x^2$$

$$\Rightarrow g(x) = (1+x+x^2)\frac{x^2}{(1-x)^2}$$

back ⟹

$$g(x) = (1+x+x^2)\frac{x^2}{(1-x)^2}$$

convert to power series form, then pick out the term x^{10}

$$= (x^2+x^3+x^4)\frac{1}{(1-x)^2}$$

$$\int \frac{1}{(1-x)^2}\,dx \qquad 1-x = u \qquad -dx = du$$

$$= (x^2+x^3+x^4)\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$\Rightarrow \int \frac{1}{u^2}(-du)$$

$$\Rightarrow -\int u^{-2}\,du$$

$$\Rightarrow (-)\frac{1}{-1}u^{-1} = u^{-1} = \frac{1}{1-x}$$

$$= (x^2+x^3+x^4)\frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right)$$

$$= (x^2+x^3+x^4)\frac{d}{dx}(1+x+x^2+x^3+x^4+\dots)$$

$$= (x^2+x^3+x^4)(1+2x+3x^2+4x^3+\dots)$$

$$= \cancel{(x^2+x^3+x^4)\left(\sum_{k=0}^{\infty}(k+1)x^{k+1}\right.}\ ($$

equivalent to $\sum_{k=1}^{\infty} kx^{k-1}$

$$= (x^2+x^3+x^4)\left(\sum_{k=0}^{\infty}(k+1)x^{k}\right)$$

$$= \underbrace{\sum_{k=0}^{\infty}(k+1)x^{k+2}}_{\text{coeff. of } x^{10}} + \underbrace{\sum_{k=0}^{\infty}(k+1)x^{k+3}}_{\text{coeff. of } x^{10}} + \underbrace{\sum_{k=0}^{\infty}(k+1)x^{k+4}}_{\text{coeff. of } x^{10}}$$

$\leftarrow$ extract coefficients from all three then add together

$k+2=10$	$k+3=10$	$k+4=10$
$k=8$	$k=7$	$k=6$
coeff. $k+1=9$	coeff. $k+1=8$	coeff. $k+1=7$

$$\boxed{9+8+7} = \boxed{24}$$