Chapter 2, Sections 2.1-2.3

1 Addition and Multiplication Rules

For the following two definitions, suppose we have m disjoint sets, and that there are r_1 different objects in the first set, r_2 different objects in the second set, ..., r_m different objects in the mth set. By a **set**, we mean that every object in it is unique, there are no duplicates. By a **disjoint set**, we mean that sets don't share objects. Every object belongs in no more than one set.

$\{a, b, c\}$ ← this is a set

$\{a, b, b\}$ ← this is not a set

same element repeats itself

$\{a, b, c\}$ $\{d, e, f\}$ ← these sets are disjoint

$\{a, b, c\}$ $\{c, e, f\}$

c is in both of these sets, so the two sets are not disjoint

Definition 1.1 (The Addition Rule). Suppose we have m sets. There are r_1 different objects in the first set, r_2 different objects in the second set, ..., and r_m different objects in the mth sets, the the number of ways to select a single object from one of the m sets is $r_1 + r_2 + \cdots + r_m$.

Definition 1.2 (The Multiplication Rule). Same m sets as above. The number of ways to select m objects, with one object from each of the m sets, is $r_1 \times r_2 \times ... \times r_m$.

Associate 'OR' with + add.

choose from A OR choose from B

⇓

\# of choices in A + # of choices in B

Associate 'AND' with × mult.

choose from A AND choose from B

⇓

\# of choices in A × # of choices in B

Example 1.3. There are five math books, six history books, and four science books.

(a) How many ways are there to select a single book?

(b) How many ways are there to select one book from each topic?

(c) How many ways are there to select one book from only two topics?

(a)

choose a math book OR choose a history book OR choose a science book

↓

\# of math books + # of history books + # of science books

$= \boxed{5+6+4}$

(b) choose math book AND choose history book AND choose science book

↓

\# of math × # of hist × # of science $= \boxed{5 \cdot 6 \cdot 4}$

(c) (choose math book AND choose history) OR (choose math AND science) OR (choose history AND science)

↓

$(5 \times 6) + (5 \times 4) + (6 \times 4)$

2 Strings

Definition 2.1. Let X be a set of objects. A **string** is an ordering of objects taken from X, with the possibility that an object can be chosen more than once (**with replacement**). The **length** of the string is the number of objects in the ordering.

Example 2.2. Suppose $X = \{a, b, c\}$. One possible string is bca. This string has length 3. The string ab has length 2. We can also create a string using the letters multiple times: $ababaaacb$ is a valid string. The important thing to note is that order matters. For example, ab and ba are different strings even though they both use the same number of each type of letter.

Formula 2.3. Given a set X containing m objects, the number of strings of length n is

$$S(m, n) = m^n.$$

Idea: $\underbrace{\_\ \_\ \_\ \ldots\ \_}_{}$

n spots, need to fill with objects from X

m choices for first spot
m choices for second spot
etc.
m choices for each spot, $\underbrace{m \times m \times \ldots \times m}_{n \text{ times}}$

Example 2.4. Five houses are standing in a row on a street. Each house can be painted one of three colors: red, blue, or green.

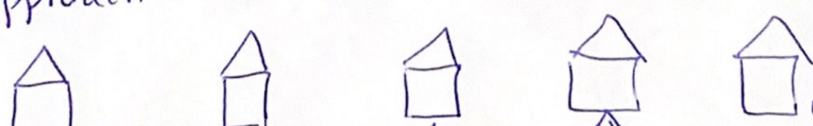
(a) How many different ways are there to paint the houses?

(b) How many different ways are there to paint the houses if the houses on the ends of the row can't be painted green, and the house in the middle must be painted blue?

(c) How many different ways are there to paint the houses if two adjacent houses can't be painted the same color?

(a) This is a string. $X = \{R, G, B\}$ (R stands for red, G for green, B for blue), and so RGRBB represents the first and third houses being red, the second house being green, and the fourth and fifth houses blue. Thus this part of the problem is equivalently asking how many strings of length 5 there are. Formula 2.3 says the answer is $\boxed{3^5}$

(b) Let's approach each house and see what the choices are:



end house, so 2 choices (red or blue)

3 choices

1 choice (must be blue)

3 choices

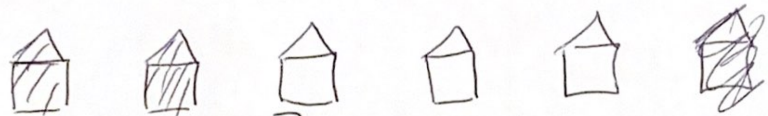
end house, so 2 choices (red or blue)

so answer is $2 \cdot 3 \cdot 1 \cdot 3 \cdot 2 = \boxed{2^2 \cdot 3^2}$ ← (use Multiplication Rule; we paint first house AND second house AND third...)

(c) Work from left to right, and apply the rule of "adjacent houses can't be same color" when it comes up.

paint 1st house; 3 choices

paint 2nd house; 2 choices (can't have color of 1st house



same ~~rate~~ situation for remaining houses as for 2nd house; 2 choices; can't have the color of the house to its left

so answer is

$$3\cdot2\cdot2\cdot2\cdot2=\boxed{3\cdot2^4}$$

3 Permutations

Definition 3.1. A **permutation** is a string with the constraint that it is built **without replacement**. That is, an object can't be chosen for the permutation more than once.

Example 3.2. Back to the example where $X = \{a, b, c\}$. bca and abc are valid permutations, because no letter is used more than once. So is ab (note that for a permutation you don't have to use all the objects in the set). aab and bb are not permutations since a letter is used more than once. Note that order still matters for permutations: ab and ba are different.

Formula 3.3. Given a set X containing m objects, the number of permutations of length n is

$$P(m, n) = \frac{m!}{(m-n)!}$$

Idea: can't use choice for 1st spot here; can't use choices for first 2 spots here; can't use choices for first (n-1) spots here

m choices for 1st spot, (m-1) choices for 2nd spot, (m-2) choices for 3rd spot, ... $(m-(n-1)) = m-n+1$ choices for nth spot

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{permutation of length } n}$

Example 3.4. Let $X = \{J, A, C, K, E, T, S\}$.

(a) How many permutations of length four are there? How many permutations of length six?

(b) How many permutations of length six are there if the first letter in the permutation is a vowel? How many permutations of length six are there if the first and fourth letters must be a vowel?

(a) Use the formula. ~~X~~ has 7 objects, so $m=7$.

- For permutations of length 4, $n=4$, so the answer is $\frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7\cdot6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \boxed{7\cdot6\cdot5\cdot4}$

- For permutations of length 6, $n=6$, so the answer is $\frac{7!}{(7-6)!} = \frac{7!}{1!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = \boxed{7!}$

(b) 2 choices, 6 choices left, 5 choices left, 4, 3, 2

can only use 'A' or 'E' here

so answer is $2\cdot6\cdot5\cdot4\cdot3\cdot2 = \boxed{2\cdot6!}$

case when the first letter must be a vowel resolve red first, since these are the most restrictive

2 choices for the first	5 choices left	4 choices left	1 choice for 4th spot, since only one vowel left	3 choices left	2 choices left

case when 1st and 4th spots must be vowels

so answer is

$$2 \cdot 5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 = \boxed{2 \cdot 5!}$$