Chapter 3

1 Sections 3.1: Introduction and 3.2: The Positive Integers are Well Ordered

Theorem 1.1. *The positive integers $\mathbb{Z}^+ := \{1, 2, 3, \dots\}$ are **well-ordered**, meaning that any subset X of $\mathbb{Z}^+$, even an infinite subset, will have a smallest integer.*

Remark 1.2.

- As a brief example, consider the set of even integers $\{2, 4, 6, \dots\}$. This is a subset of $\mathbb{Z}^+$, and it clearly has a smallest integer, 2.
- Note that the set of all integers, $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ is NOT well-ordered.
- The importance of the well-ordering principle will become apparent further on.

2 Section 3.3: The Meaning of Statements

In this chapter, we are going to be dealing with "sequences of concepts". By a "sequence of concepts", I mean a family of definitions, formulas, or true-false statements, each organized and corresponding to its own positive integer.

They could be formulas, e.g. factorials:

n	1	2	3	4	...
nth factorial	$1! = 1$	$2! = 2\cdot 1$	$3! = 3\cdot 2\cdot 1$	$4! = 4\cdot 3\cdot 2\cdot 1$	

They could be definitions, e.g. prime numbers:

n	1	2	3	4	5	...
nth prime number	2	3	5	7	11	...

They could be counts of objects, e.g. the numbers of (0, 1)-strings:

n	1	2	3	...
# of (0,1)-strings of length n	$3^1=3$	$3^2=9$	$3^3=27$	...

They could be true-false statements, e.g. for any positive integer n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}:$$

n	1	2	3	4	...
	$1=\frac{1(1+1)}{2}$	$1+2=\frac{2(2+1)}{2}$	$1+2+3=\frac{3(3+1)}{2}$	$1+2+3+4=\frac{4(4+1)}{2}$	...

Example 2.1. In this class, we have defined the nth factorial for $n \geq 1$ the following way:

$$n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

However, we could alternatively give it a recursive definition:

Definition 2.2. For $n \geq 1$, the nth factorial is defined to be:

$$n! := \begin{cases} 1, & \text{if } n = 1, \\ n * (n-1)!, & \text{if } n > 1. \end{cases}$$

Note the two important parts of the recursive definition, the **base case** and the "**step up**".

n	1	2	3	4	5	...
nth factorial	1!	2!	3!	4!	5!	

Base Case $2!=2\cdot 1!$ $3!=3\cdot 2!$ $n \longrightarrow n+1$ Inductive Step

3 Sections 3.4: Binomial Coefficients Revisited and 3.5: Solving Combinatorial Problems Recursively

It is also possible to use recursive steps in more complex ways than going from the $(k-1)$th step to the kth step. Here are two such examples:

(i) Using not just the $(k-1)$th case to define the kth case, but the $(k-2)$th, $(k-3)$th, or even smaller cases.

(ii) Including more than one indeterminate; e.g. recursively defining something depending on both n and k

Exercise 3.1 (Example with Case(i)).

(a) How many different ternary strings of length n are there?

(b) How many different ternary strings of length n do not contain 20?

(c) How many different ternary strings of length n do not contain 102?

$S(n) :=$ # of strings with entries 0, 1, 2 that have length n and don't contain 102

(c)

n	1	2	3	4	5	. . .
$S(n)$	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	

Use a recursive definition

Recursive Step: look at $S(n)$; can we define it in terms of $S(n-1)$, $S(n-2)$, . . . ?

$S(n)$

$\underbrace{\_\ \_\ \_\ \cdots\ \_\ \_}_{n \text{ spaces for digits}}$

break into cases, based on what the nth digit is

Case 0:

$\underbrace{\_\ \_\ \_\ \ldots\ 1\ 0}_{n-1 \text{ spaces here}}\ \underline{0}$

n−1 spaces here; populate the first n−1 spaces with a (0,1,2)-string w/o 102 and of length n−1 $\boxed{S(n-1)}$

Case 1:

$\_\ \_\ \_ \ldots \_\ 1$

exactly the same as Case 0 $\boxed{S(n-1)}$

Case 2:

$\_\ \_\ \_ \ldots 2$

back $\Rightarrow$

Case 2:

$\underbrace{1\ 2\ 0 \cdots 1\ 0}\ 2$

what if we have a string of length $(n-1)$ with no 102, but the last two digits are $10\boxed{?}$

the # of strings for case 2 is

$S(n-1) - \boxed{\text{\# of } (n-1) \text{ length strings that end in } 10}$

$\overbrace{\_\ \_\ \_ \cdots}^{n-3 \text{ spaces}}\ \underbrace{\_\ \_\ \_ \cdots 1\ 0}_{n-1}$

$= S(n-3)$

can't have 102 in string whose length < 3

$$\boxed{\begin{aligned} S(1) &= 3 \\ S(2) &= 3^2 = 9 \\ S(3) &= 3 \cdot 3 \cdot 3 - 1 \\ &= 26 \end{aligned}}$$

answer

Case 1: $S(n-1)$

Case 2: $S(n-1)$

Case 3: $S(n-1) - S(n-3)$

Recursive Step $\boxed{S(n) = 3S(n-1) - S(n-3)}$

Now need base cases: get $S(1), S(2), S(3)$, because $S(n-3)$ is in the recursive step

Exercise 3.2 (Example with Case (ii)).

We know from Chapter 2 that the binomial coefficient $\binom{n}{k}$, the number of ways to choose k objects from a set of n objects, is given by the formula:

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}.$$

However, we can also define this binomial coefficient recursively:

$$\binom{n}{k} := \binom{n-1}{k} + \binom{n-1}{k-1}$$

What should the base cases be?

$k \rightarrow$

$n \downarrow$

$$\begin{array}{llll} \binom{0}{0} = 1 & & & \\ \boxed{\binom{1}{0}} = 1 & \binom{1}{1} = 1 & & \\ \binom{2}{0} = 1 & \boxed{\binom{2}{1}} & \binom{2}{2} = 1 & \\ \binom{3}{0} = 1 & \binom{3}{1} & \binom{3}{2} & \\ \vdots & \vdots & & \end{array}$$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$\leftarrow$ base cases

$$\binom{n-1}{k-1} \quad \binom{n-1}{k}$$
$$\nwarrow \quad \uparrow$$
$$\binom{n}{k}$$

$\binom{n}{k}$ depends on the formula directly above it, $\binom{n-1}{k}$, and the formula above and to the left of it $\binom{n-1}{k-1}$

4 Section 3.6: Mathematical Induction

We now come to true-false statements. We can try to prove these using **induction**. This method is similar to recursion, we start at a base, prove that's true, and then work our way up from there.

- We prove these statements (assuming they are true) in two steps:

  (i) The **Base Case**: We prove that the statement is true for the case when $n = 1$.

  (ii) The **Inductive Step**: We assume that the statement is true for a particular n. We call this the **inductive hypothesis**. We then prove that the statement is true for $n + 1$.

Essentially, proof by induction sets off a "chain reaction".

- The **Base Case** says the statement is true for $n = 1$.
- The **Inductive Step** says that if it's true for $n = 1$, then it must also be true for $n + 1 = 2$.
- The **Inductive Step** says that if it's true for $n = 2$, then it must also be true for $n + 1 = 3$.
- The **Inductive Step** says that if it's true for $n = 3$, then it must also be true for $n + 1 = 4$.
- and so on, continuing on to infinity through the **Inductive Step.**

Exercise 4.1 (Proposition 3.12 from textbook). Prove that for every $n > 0$, the following holds:

$$\sum_{k=1}^{n} k = \frac{n * (n+1)}{2}.$$

Base Case:

$n=1$

$$\sum_{k=1}^{1} k = 1 \qquad \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

✓

Inductive Step:

Assume $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ is true,

now look at for $(n+1)$:

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

use inductive hypothesis

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+2)(n+1)}{2} \checkmark$$

Exercise 4.2. Use mathematical induction to prove that for all $n \geq 1$, we have that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

n	1	2	3
Stat(n)	1^3+2^3+3		

n	1	2	3 ...
Stat(n)	$1^3+2^3+3^3$ divisible by 9	$2^3+3^3+4^3$ divisible by 9	...

Base Case:

$n=1$:

$1^3+2^3+3^3$

$=1+8+27$

$=36=9\cdot 4$ ✓

Inductive Step:

Inductive Hypothesis

We assume true for n: 9 divides $n^3+(n+1)^3+(n+2)^3$ there exists an integer b such that

$$9b = n^3+(n+1)^3+(n+2)^3$$

Look at $n+1$: $(n+1)^3 + ((n+1)+1)^3 + ((n+1)+2)^3$

$= (n+1)^3 + (n+2)^3 + \boxed{(n+3)^3}$ ⟸ multiply out

$= (n+1)^3+(n+2)^3+(n^3+\binom{3}{1}n^2\cdot 3^1+\binom{3}{2}n^1\cdot 3^2+\binom{3}{3}3^3$

$= (n+1)^3+(n+2)^3+n^3+9n^2+27n+27$

$= (n+1)^3+(n+2)^3+9(n^2+3n+3)$

$= \underbrace{n^3+(n+1)^3+(n+2)^3}_{\text{by Inductive Hypothesis}} + \underbrace{9(n^2+3n+3)}_{\text{factored out } 9}$

$\| $

$9\cdot b$

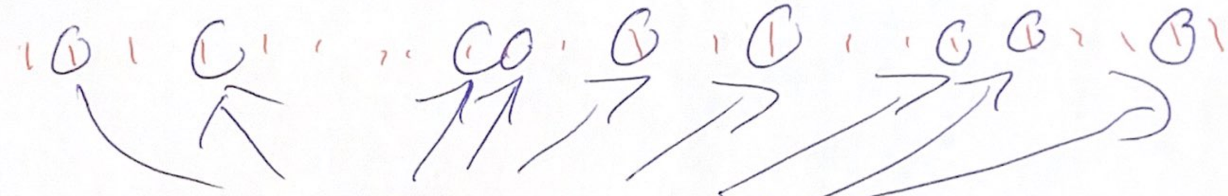
$= \boxed{9\cdot(b+n^2+3n+3)}$

strong induction

n	1	2	3	4	5	6	7	8	...
S(n)	S(1)	S(2)	S(3)	S(4)	S(5)	S(6)	S(7)	S(8)	...

Assume that S(n) DOES NOT hold for all n (we are looking for a contradiction)

Look at the set of integers for which S(n) is not true (DOES NOT hold)



S(n) is ~~not held~~ ~~true~~
does not hold

so by the well-ordering principle, this set of integers n for which S(n) doesn't hold, has a smallest member

1 2 3 4 ... m-1 $\boxed{m}$

so for every ~~thing~~ n less than m, ~~so~~ S(n) must hold

5 Section 3.9: Strong Induction

In Example 3.2, we saw that for recursion, sometimes to define the kth case you need not just the $(k-1)$th case, but some (or maybe even all) of the $(k-2)$th, $(k-3)$th, …, 2nd, and 1st cases.

Similarly, sometimes the inductive step requires more than just assuming the $(k-1)$th case is true. We need to assume that the 1st through $(k-1)$th cases are all true. This is **strong induction**.

Exercise 5.1 (Exercise 3.17 from the textbook)**.** Consider the recursive function given by

$$f(n) := \begin{cases} 2, & \text{if } n = 0, \\ 4, & \text{if } n = 1, \\ 2f(n-1) - f(n-2) + 6, & \text{if } n \geq 2. \end{cases}$$

Prove that $f(n) = 3n^2 - n + 2$ for all $n \geq 0$.

Assume that $f(n) \neq 3n^2 - n + 2$ for all n,
So there is some smallest number in this set, call it m.
So for all $n < m$, it must be that $f(n) = 3n^2 - n + 2$
start at $f(m) = 2\boxed{f(m-1)} - \boxed{f(m-2)} + 6$
try to show that in fact $f(m) = 3m^2 - m + 2$

for $n < m$
$f(n) = 3n^2 - n + 2$

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so $f(m-1) = 3(m-1)^2 - (m-1) + 2$
$f(m-2) = 3(m-2)^2 - (m-2) + 2$

$f(m) = 2(3(m-1)^2 - (m-1) + 2) - (3(m-2)^2 - (m-2) + 2) + 6$
$= 2(3m^2 - 6m + 3 - m + 1 + 2) - (3m^2 - 12m + 12 - m + 2 + 2) + 6$
$\cancel{= 6m^2 - 12m + 2 - 2m + 6 - 3m^2 + 12m - 8 + m + 6}$
$\cancel{= 3m^2 - m + 6}$
$= 6m^2 - 12m + 6 - m + 3 - 3m^2 + 12m - 12 + m - 4 + 6$
$= 3m^2$

$$f(m) = 2f(m-1) - f(m-2) + 6$$

$$f(m-1) = 3(m-1)^2 - (m-1) + 2$$
$$f(m-2) = 3(m-2)^2 - (m-2) + 2$$

$$\begin{aligned}
f(m) &= 2[3(m-1)^2 - (m-1) + 2] - [3(m-2)^2 - (m-2) + 2] + 6 \\
&= 2\{3(m^2 - 2m + 1) - (m-1) + 2\} - \{3(m^2 - 4m + 4) - (m-2) + 2\} + 6 \\
&= 2\{3m^2 - 6m + 3 - m + 1 + 2\} - \{3m^2 - 12m + 12 - m + 2 + 2\} + 6 \\
&= 2\{3m^2 - 7m + 6\} - \{3m^2 - 13m + 16\} + 6 \\
&= 6m^2 - 14m + 12 - 3m^2 + 13m - 16 + 6 \\
&\cancel{= 6} \\
&= \boxed{3m^2 - m + 2}
\end{aligned}$$

Verify Base Cases:

$f(0) = 3 \cdot 0^2 - 0 + 2 = 2$ ✓

$f(1) = 3 \cdot 1^2 - 1 + 2 = 4$ ✓