4 Combinations

Definition 4.1. Given a set X, a **combination** is an unordered collection of objects in that set X. In other words, a combination is a permutation, except we forget about the order of the objects.

Example 4.2. Returning one last time to $X = \{a, b, c\}$, we have that all the possible combinations are as follows:

in this example, $n=3$

- size 0 combinations: $\{\}$ ⇐ $k=0$; $\binom{3}{0} = \frac{3!}{0!\,3!} = 1$
- size 1 combinations: $\{a\}, \{b\}, \{c\}$ ⇐ $k=1$; $\binom{3}{1} = \frac{3!}{2!\,1!} = 3$ same
- size 2 combinations: $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and the ⇐ $k=2$; $\binom{3}{2}$
- size 3 combinations: $\{a, b, c\}$ ⇐ $k=3$; $\binom{3}{3} = \frac{3!}{3!\,0!} = 1$

Note that for combinations, since order doesn't matter, $\{a, b\}$ is the same as $\{b, a\}$; $\{a, b, c\}$ is the same $\{b, c, a\}$; etc.

Formula 4.3. If X is a set of size n, then the number of size k combinations we can make from X is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{P(k,k)}$$

Example 4.4. You are packing for a vacation. Available for packing are nine shirts, five pairs of pants, and seven pairs of socks.

(a) How many ways are there to pack three shirts, two pairs of pants, and four pairs of socks?

(b) You also have the option of taking a hat. You may choose from one of three hats, or choose not to take a hat. How many ways are there to pack the same number of shirts, pants, and socks as in part (a), now with the hat option added in?

a) choose 3 shirts from 9 available AND choose 2 pants from 5 AND choose 4 socks from 7

$$\boxed{\binom{9}{3} \times \binom{5}{2} \times \binom{7}{4}}$$

(b)

same as above, but also with AND 4 options, choose 1 ⇐ 4 because in addition to the three hats we have the option of no hat

$$\boxed{\binom{9}{3}\binom{5}{2}\binom{7}{4} \times \binom{4}{1}}$$

	Replacement	Order Matters
String	YES	YES
Permutation	NO	YES
Combination	NO	NO

Table 1: Comparing how strings, permutations, and combinations differ in regards to whether replacement is allowed and whether order matters.

Example 4.5. How many ways are there to re-arrange the letters in the word "banana"?

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}$ think of as a combination

our set is the six different places for the arrangement:

$X = \{$1st place, 2nd place, ..., 5th place, 6th place$\}$

so can think of outcome as

choose 3 places from 6 available for letter "a" AND choose 2 places from remaining 3 for letter "n" AND choose last remaining place for "b"

$$\boxed{\binom{6}{3}\binom{3}{2}\binom{1}{1}}$$

Remark 4.6. Strings, permutations, and combinations differ only in regard to two aspects: (i) whether you can choose with replacement and (ii) whether order matters. See Table 1.

5 Counting Poker Hands

The standard set of French-suited playing cards consists of 52 cards. There are four different **suits**:

Clubs, Diamonds, Hearts, Spades,

for simplicity, we always assume the Ace is "high"

and thirteen different **"ranks"**:

can be here in the ordering

A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. X

but not here

How many different types of hands are there?

6 Combinatorial Proofs

Normally, when we want to show that an equation is true, we use algebra to show that the left hand side equals the right hand side. In a **combinatorial proof**, we eschew algebra and instead show that the two sides of the equation count the same thing.

Example 6.1 (Example 2.17 from the book). Show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Answer 6.2. Claim: Both sides of the equation count the number of strings of length n whose entries are 0 or 1.

One Way to Count

for each string

choose 0 or 1 for 1st place AND choose 0 or 1 for 2nd place AND...

$\Downarrow$

$\underbrace{2 \cdot 2 \cdot \ldots 2}_{n \text{ total}}$

$\boxed{2^n}$

no 1's

strings with one 1

strings with two 1's

⋮

strings with n 1's

set of strings of length n with entries 0 and 1

Another Way to Count

divide into cases:

of {0,1} strings of length n with exactly k 1's: like Example 4.5, choose k places for the ones. $\binom{n}{k}$

then sum up from 0 to n:

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$

both count the same thing, so they must be equal

Example 6.3 (Example 2.19 from the book). ~~Show that~~

$$\binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n = 3^n.$$

Answer 6.4. Claim: Both sides of the equation count the number of strings of length n whose entries are 0, 1, or 2.

set of strings of length n, entries {0,1,2}

One Way to Count

n places, 3 choices for each place:

$\underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{n \text{ total}}$

$\Downarrow$

$\boxed{3^n}$

strings with n 2's

strings with (n-1) 2's

strings with 2 2's

strings with 1 2

strings with 0 2's

Another Way to Count

divide into cases

case: # of strings of length n and entries {0,1,2} where precisely (n-k) of those places ~~are~~ 2's

choose which k places will have 0's or 1's (rest of the places will have 2's) $\cdot \underbrace{2 \cdot 2 \cdot \ \ \cdot 2}$ for each of those k places, choose 0 or 1

gives us $\binom{n}{k}2^k$, then sum up

Example 6.5. Show that

$$\boxed{\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}} = \boxed{\binom{m+n}{r}}.$$

can be rewritten as

$$\binom{m}{0}\binom{n}{r-0}+\binom{m}{1}\binom{n}{r-1}+\binom{m}{2}\binom{n}{r-2}+\ldots+\binom{m}{r-1}\binom{n}{r-(r-1)}+\binom{m}{r}\binom{n}{r-r}$$

(right hand side)

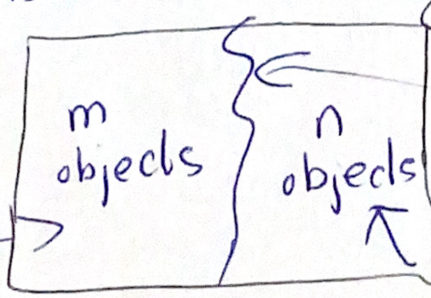
the RHS shows the number of ways to choose r things from a set of m+n objects

left hand side

we will show that the LHS counts the ~~same~~ number of ways to choose r things from a set of m+n objects as well

here's the set of m+n objects, call it X

split X into two disjoint sets, one having m objects and the other n objects



call the set of m objects Y

call the set of n objects

whenever we choose r objects from X we choose a certain # of objects from Y ~~and a~~ (possibly zero.) Let's condition based on the # of objects taken from Y:

(take 0 objects from Y AND thus take r-0 objects from Z) OR (take one object from Y AND thus take r-1 objects from Z) OR . . .

. . . OR (take r objects from Y AND thus take r-r = 0 objects from Z)

this is the LHS

$$\boxed{\binom{m}{0}\binom{n}{r-0}+\binom{m}{1}\binom{n}{r-1}+\ldots+\binom{m}{r}\binom{n}{r-r}}$$