Chapter 6

1 Section 6.1: Basic Notation and Terminology

Definition 1.1. A **binary relation** on a set X is a set of ordered pairs (x, y) of elements in X.

$X = \{a, b, c, d, e\}$

$B = \{(a,b), (c,d), (d,c), (d,e), (e,a)\}$

order matters, these are different.

Definition 1.2. A **partial order** on a set X is a binary relation that is:

- reflexive: for every $x \in X$, (x, x) is part of the binary relation
- antisymmetric: if $x \neq y$, then the binary relation can have either (x, y) or (y, x) (or possibly neither), but not both.
- transitive: if (x, y) and (y, z) are in the binary relation, then so is (x, z).

A set with a partial order is called a **partially ordered set** (often abbreviated to **poset**).

Example 1.3. $\mathbb{Z}$ with the binary relation B defined by $(a, b) \in B$ if and only if $a \leq b$ is a partial order.

Example 1.4. $\mathbb{Z}^2$ with the binary relation B defined by

$$((a, b), (c, d)) \in B \text{ if and only if } a \leq c \text{ and } b \leq d$$

is a partial order.

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Example 1.3: $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$B = \{(x,y) \mid \text{if } x \leq y\}$

(so < wouldn't work because $x \leq x$)

- reflexive: any integer, $x \leq x$, so $(x,x) \in B$
- anti-symmetric: if $(x,y), (y,x) \in B \Rightarrow x \leq y$ and $y \leq x \Rightarrow x = y$
- transitive: if $(x,y), (y,z) \in B \Rightarrow x \leq y$ and $y \leq z \Rightarrow x \leq z \Rightarrow (x,z) \in B$

Example 1.4:

pt in $\mathbb{Z}^2$

$X = \mathbb{Z}^2 = \{(x,y) \mid x,y \in \mathbb{Z}\}$

$R = \{\underbrace{((a,b),\ (c,d))}_{\text{ordered pair in binary relation}} \mid a \leq c \text{ and } b \leq d\}$

Note: use '$\leq$' for partial orders in general, not just for the case when it's 'greater than or equal'.

so can rewrite as $(a,b) \leq (c,d)$

Specific Example of what I mean:

$(2,3) \leq (6,4)$ because $2 \leq 6$ and $3 \leq 4$

Note of Caution:

$(-1,5)$ and $(0,2)$: $-1 \leq 0$ but $5 \not\leq 2$

so for this partial order, those two elements are incomparable (neither $(-1,5) \leq (0,2)$ nor $(0,2) \leq (-1,5)$)

In general, cases like the above are true for most posets.

- reflexive: take arbitrary (a,b), we have that $a \leq a$ and $b \leq b$, so $(a,b) \leq (a,b)$. ✓
- anti-symmetric: so suppose $(a,b) \leq (c,d)$ and $(c,d) \leq (a,b)$. $\Rightarrow a \leq c$ and $c \leq a \Rightarrow a = c$. Similarly, $b \leq d$ and $d \leq b \Rightarrow b = d$. So $(a,b) = (c,d)$. ✓
- transitive: ~~(a,b)~~ Assume $(a,b) \leq (c,d)$ and $(c,d) \leq (e,f)$. We have $a \leq c$ and $c \leq e \Rightarrow a \leq e$. Similarly, $b \leq d$ and $d \leq f \Rightarrow b \leq f$. Thus $(a,b) \leq (e,f)$. ✓

Definition 1.5.

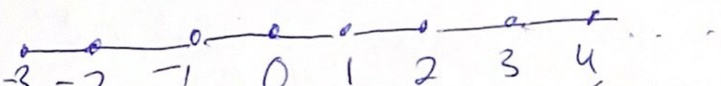
- Take two points x and y in a poset, with $x \neq y$. We say that x **is covered by** y if (x, y) is in the partial order, and there's no $z \neq x, y$ such that both (x, z) in the partial order and (z, y) in the partial order.

|| z that "gets between" x and y

- A **cover graph** of a poset is a graph where the vertices correspond to elements in the poset, and we have an edge between two vertices x and y if and only if either x is covered by y or y is covered by x.
- A **Hasse diagram** is a cover graph where the vertices are placed so that if $x \leq y$, then y is "above" x in the graph.

$(\mathbb{Z}, \leq)$

non-Hasse diagram cover graph of $(\mathbb{Z}, \leq)$



edge between 0 and 1 because 1 covers 0.

2 is not covered by 4 because even though $2 \leq 4$, we have another element 3, that gets between them ($2 \leq 3$ and $3 \leq 4$)

Hasse diagram of $(\mathbb{Z}, \leq)$

$X = \{a, b, c, d, e, f, g\}$

$P = \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (g,g),$

$(a, d),$ → a is covered by d

$(b, d),$ → b is covered by d

$(c, e),$ → c is covered by e

$(f, a), (f, c), (f, d), (f, e),$ → f is covered by a, c

$(g, a), \underbrace{(g, d)}\}$ → g is covered by a

↑ not a covering relation, a gets in the way

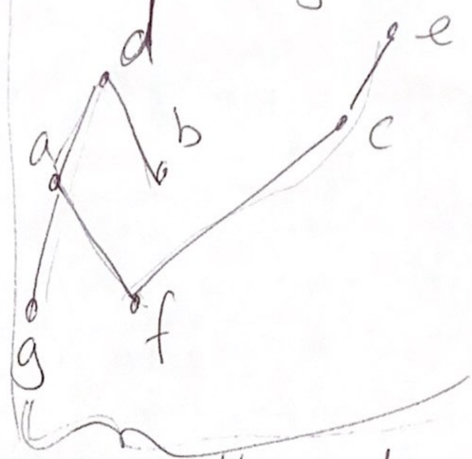
back ⇒

covering relations:

a covered by d
b covered by d
c covered by e
f covered by a, c
g covered by a

cover graph

Hasse diagram



In a Hasse diagram, two elements are comparable if and only if there is a path from one to the other that only goes up or only goes down

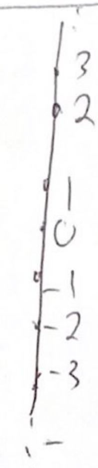
Definition 1.6.

- Two distinct points x, y in a poset are **comparable** if either (x, y) or (y, x) is in the partial order. Otherwise we say that x and y are **incomparable**.
- A partial order is a **total order** if any two distinct points are comparable.

(or can also it a linear order)

~~Idea:~~

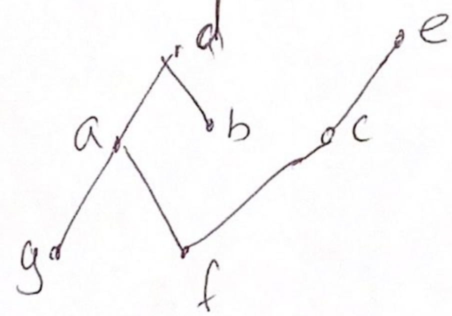
Example: $(\mathbb{Z}, \leq)$ is a total/linear order.



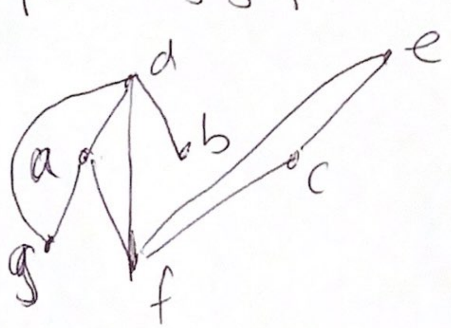
← Hasse diagram is a line when it's a linear order

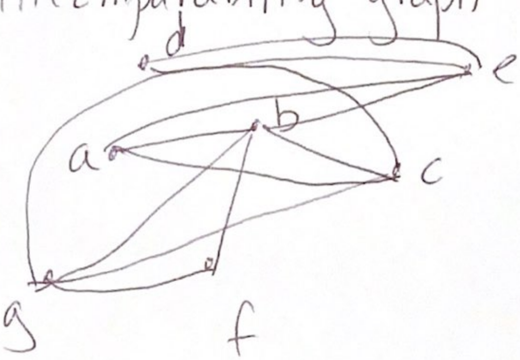
Definition 1.7. A **comparability graph** is a graph where the vertices correspond to the elements in the poset and there is an edge between two vertices if they are comparable. An **incomparability graph** is the same as a comparability graph except the edges are between pairs of incomparable vertices.

Ex from previous page:



comparability graph ← complement incomparability graph





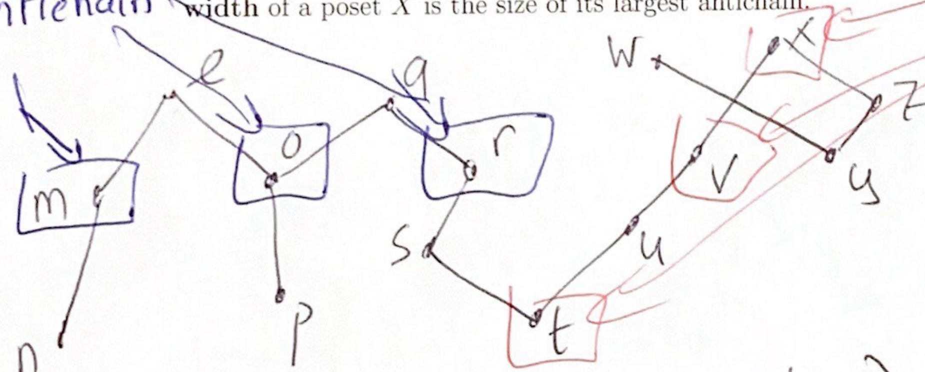
Definition 1.8.

m o r

example of antichain

- If X is a poset whose partial order is P and Y is a subset of X, then we can turn Y into a poset by giving it the partial order P restricted just to the elements of Y. We then say that Y is a **subposet** of X.
- A chain is a subposet where every pair of distinct elements is comparable. The **height** of a poset X is the size of its largest chain.
- An antichain is a subposet where every pair of distinct elements is incomparable. The **width** of a poset X is the size of its largest antichain.

example of a chain



Hasse diagram of a poset X

height of $X = 4$ (chain $\{t, u, v, x\}$)

width of $X = 6$ (antichain $\{m, o, r, v, w, z\}$)

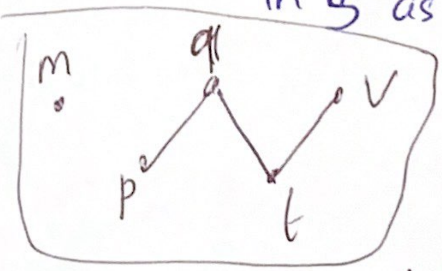
Take subset $S = \{m, p, q, t, v\}$

how to draw Hasse diagram of S? A.k.a. how to determine the ordered pairs in the ~~part~~ inherited partial order of S?

in original poset	in subposet
m: (m,ℓ), (n,m)	~
p: (p,o), (p,ℓ), (p,q)	(p,q) ← keep (p,q) because q is in S as well
q: (p,q), (o,q), (r,q), (s,q), (t,q)	(p,q), (t,q)
t: (t,s), (t,r), (t,q), (t,u), (t,v), (t,x)	(t,q), (t,v)
v: (v,x), (u,v), (t,v)	(t,v)

throw out because ℓ and n are not in S

these are all the (non-reflexive) ordered pairs ~~pairs~~



Hasse diagram of S

Exercise 1.9 (similar to Exercise 6.3 from textbook). Let $X = \{1, 2, 3, 4, 5\}$, and let P be a binary relation on X defined by

$$P = \{(1,1), (2,2), (3,3), (4,4), (1,3), (2,4), (4,5), (5,2), (1,5)\}.$$

No

Is P a partial order? If not, can we turn P into a partial order by adding ordered pairs to it? No

- reflexive: need to add (5,5)
- anti-symmetric: can't have flipped versions of ordered pairs where the elements are different

forbidden: (3,1), (4,2), (5,4), (2,5), (5,1)

good for now..

~~(1,3)~~
(1,3) (1,3)
(2,4) (2,4)
(4,5) (4,5)
(5,2) (5,2)
(1,5) (1,5)
~~(4,5)~~

- transitive: add ~~(4,5)~~ ~~(4~~ (2,5) (5,4) (4,2) (1,2)

see back

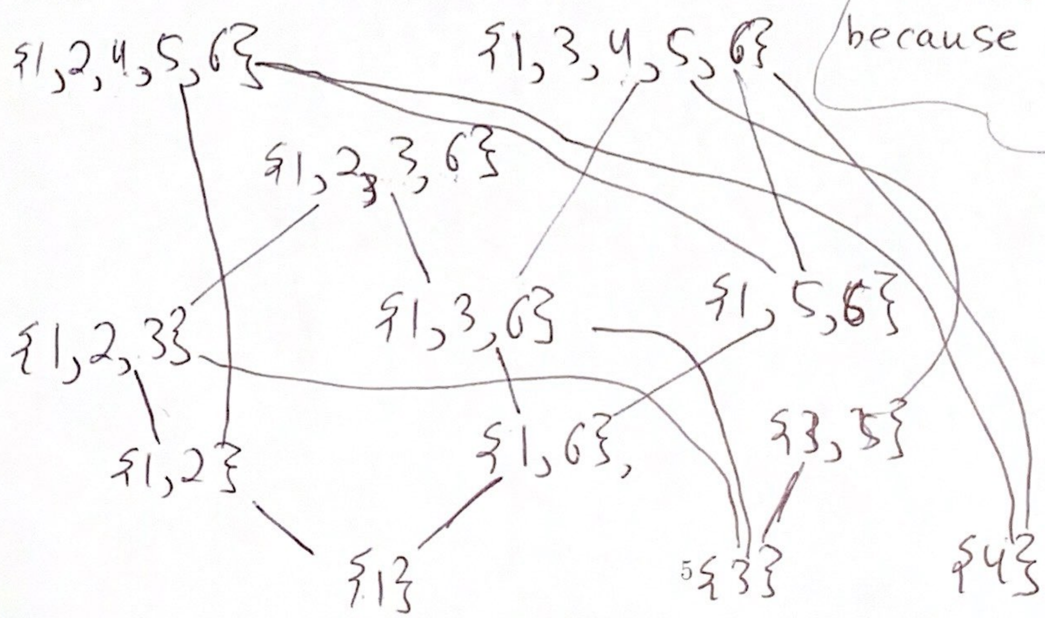
from back we have to include these: (5,4), (2,5), (4,2), (1,2), (1,4) but now we violate anti-symmetric property.

Exercise 1.10 (Exercise 6.5 from textbook). Draw the Hasse diagram of the poset (X, P) where

$$X = \{\{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,6\}, \{1,2,3\}, \{1,3,6\}, \{1,5,6\}, \{1,2\}, \{1,6\}, \{3,5\}, \{1\}, \{3\}, \{4\}\}$$

and P is the partial order given by the subset relationship.

$\{1\} \leq \{1,3,6\}$ in this poset because $\{1\}$ is a subset of $\{1,3,6\}$



5

searching for all cases where we have

(a,b) (b,c)

ending same as starting

start with	(1,3)	(1,3)
	(2,4)	(2,4)
	(4,5)	(4,5)
	(5,2)	(5,2)
	(1,5)	(1,5)
add based on comparing original five	(2,5)	(2,5)
	(4,2)	(4,2)
	(5,4)	(5,4)
	(1,2)	(1,2)
also add	(1,4)	(1,4)

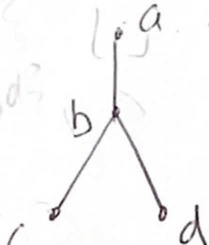
2 Section 6.2: Additional Concepts for Posets

Definition 2.1. Let X and Y be posets. An isomorphism $f : X \to Y$ is a bijection that preserves the partial order. That means, for any two $x, y \in X$, we have

$$x \leq y \text{ if and only if } f(x) \leq f(y)$$

Same Idea as with Graphs: Isomorphic means the two posets are structurally identical

$\underline{X}$ isomorphic $\Longleftrightarrow$ $\underline{Y}$



Hasse diag.

Hasse diag. preserves ordering

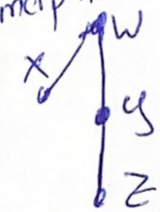
define isomorphism
$a \to p$
$b \to q$
$c \to s$
$d \to r$

e.g. $d \leq a$ and $f(d) = r \leq p = f(a)$

e.g. c and d are not comp. neither are $f(c) = s$ and $f(d) = r$

not isomorphic

$\underline{Z}$



$\underline{X}$ and Z not isomorphic, because $\underline{X}$ has 2 chains of ~~length 3~~ size 3 and Z only has 1 chain of size 3

Definition 2.2. If X is a poset with partial order P, then the **dual** of X, denoted by X^d, is the binary relation given by $\{(x, y) \mid (y, x) \in P\}$.

Remark 2.3.

- The dual of a poset is also a poset.
- The dual of the dual of a poset X is X. That is, $(X^d)^d = X$.
- A chain in X is also a chain in X^d. An antichain in X is also an antichain in X^d. Thus X and X^d will always have the same height and width.

Idea:
- Show that two posets aren't isomorphic by showing they have different number of chains/antichains of a certain size
- Show differing numbers of maximal/minimal elements

$\underline{X}$

$b \leq a$ $\quad$ $d \leq a$
$c \leq b$
$c \leq a$
$d \leq b$

$\Longrightarrow$

$\underline{X}^d$ ← d stands for dual

$a \leq b$ $\quad$ $a \leq d$
$b \leq c$
$a \leq c$
$b \leq d$

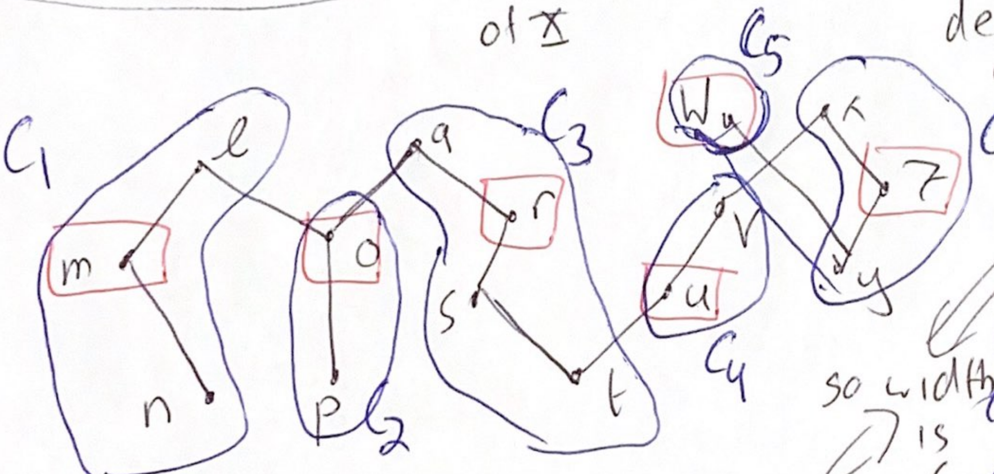
3 Section 6.3: Dilworth's Chain Covering Theorem and its Dual

Theorem 3.1 (Dilworth's Theorem). *If X poset with width w, then the elements of X can be partitioned into w sets, $C_1, C_2, \ldots, C_w$ such that each C_i is a chain.* and I can't do this for anything smaller than w

Theorem 3.2 (Dual of Dilworth's Theorem). *If X poset with height h, then the elements of X can be partitioned into h sets, $A_1, A_2, \ldots, A_h$ such that each C_i is an antichain.*

and I can't do this for any number smaller than h

size of any antichain of X $\leq$ width of X $\leq$ # of chains in a disjoint chain decomposition of X



decomposition into six chains

so width is 6

$C_1 = \{m, l, n\}$

$C_2 = \{o, p\}$

$C_3 = \{q, r, s, t\}$

$C_4 = \{u, v\}$

$C_5 = \{w\}$

$C_6 = \{x, y, z\}$

$A = \{m, o, r, u, w, z\}$ antichain of size 6

A_2 decomp. into four anti chains

A_1

$A_1 = \{n, p, t, y\}$

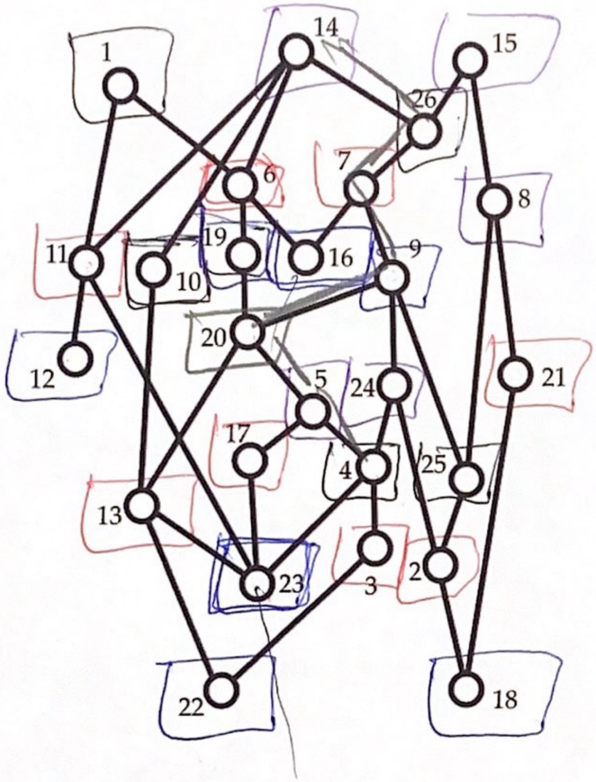
$A_2 = \{m, s, u, z\}$

$A_3 = \{o, r, v\}$

$A_4 = \{l, q, w, x\}$

$C = \{q, r, s, t\}$ is a chain of size 4 $\rightarrow$ so height is 4

Decompose into antichains (and use as few antichains as possible). That is, decompose into h many antichains, where h is the height of the poset.



Idea: Identify minimal elements, they form an antichain, remove, and repeat

Exercise 3.3 (Exercise 6.9 from textbook). For the poset X with the given Hasse diagram, find the height h of X and a decomposition into h antichains.

$A_1 = \{12, 16, 22, 23, 18\}$

$A_2 = \{11, 13, 17, 3, 2, 21\}$

$A_3 = \{10, 4, 25\}$

$A_4 = \{5, 24, 8\}$

$A_5 = \{20\}$

$A_6 = \{19, 9\}$

$A_7 = \{6, 7\}$

$A_8 = \{1, 26\}$

$A_9 = \{14, 15\}$

decomposition into 9 antichains, so height is 9.

14
|
26
|
7
|
9
|
20
|
5 — 4 — 3 — 22

← chain of size 9