Chapter 2, Sections 2.4-2.6

1 Section 2.4: The Ubiquitous Nature of Binomial Coefficients

1.1 Counting Integer Solutions to Equations

Example 1.1 (Example 2.21 from the book)**.** The office assistant is distributing supplies. In how many ways can he distribute 18 identical folders among four office employees: Audrey, Bart, Cecilia and Darren, with the additional restriction that each will receive at least one folder?

Remark 1.2. This problem is equivalently asking: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

if each of x_1, x_2, x_3, x_4 are integers and $x_1, x_2, x_3, x_4 \geq 1$? x_1 is the number of folders Audrey receives, x_2 is the number of folders Bart receives, etc.

Remark 1.3. The building blocks we are working with here:

- identical objects that we are distributing (the folders in the example above)
- distinct bins in which we place the objects (Audrey, Bart, Cecilia, and Darren)

So $x_1 = 2$, $x_2 = 7$, $x_3 = 4$, and $x_4 = 5$ is different from $x_1 = 7$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$.

Formula Intuition:

line up M objects (in this case 18)

have n-1 (in this case 3) dividers that when placed between objects will divide them into n sections

*_*_*_*_*_*_*_*_*_*_*_*_*_*_*_*_*_*

17 spaces between 18 objects choose 3 of them to place a single divider in

$\binom{M-1}{n-1}$ which in this case $(M=18, n=4)$ is $\binom{17}{3}$

formula on next page ⟹

objects to distribute

Formula 1.4. Given an equation

n variables to distribute objects to

$$\underbrace{x_1 + x_2 + \cdots + x_{n-1} + x_n}_{} = \boxed{M},$$

the number of integer solutions to the equation where $x_1, x_2, \ldots, x_{n-1}, x_n$ are all ≥ 1 is

$$\binom{M-1}{n-1}$$

spaces between objects that we may select for a divider

dividers to place

What about cases when one of the $x_i \geq k$, for some integer $k > 1$? We "take away" a unit from M, reducing it to $M-1$, and "give" it to that x_i, reducing its constraint to $x_i \geq k-1$. We continue these changes until all the constraints are at ≥ 1.

Example 1.5. How many integer solutions are there to $x_1 + x_2 + x_3 = 16$ where $x_1 \geq 2$, $x_2 \geq 4$, and $x_3 \geq 1$?

Idea:

We take this problem given to us:

$$\boxed{\begin{array}{c} x_1+x_2+x_3=16 \\ x_1 \geq 2,\ x_2 \geq 4,\ x_3 \geq 1 \end{array}} \Longrightarrow \boxed{\begin{array}{c} x_1+x_2+x_3 = \{\ \}? \\ x_1, x_2, x_3 \geq 1 \end{array}}$$

and transform it into this problem here

which has the same answer, but because the constraints are now all $x_i \geq 1$, we can use Formula 1.4.

Stage One:

$$\boxed{\begin{array}{c} x_1+x_2+x_3=16 \\ x_1 \geq 2 \\ x_2 \geq 4 \\ x_3 \geq 1 \end{array}}$$

$\Longrightarrow$ take one object, give to x_1

Stage Two:

$$\boxed{\begin{array}{c} x_1+x_2+x_3=15 \\ x_1 \geq 1 \\ x_2 \geq 4 \\ x_3 \geq 1 \end{array}}$$

$\Longrightarrow$ take 3 objects, give to x_2

Stage Three

$$\boxed{\begin{array}{c} x_1+x_2+x_3=\boxed{12} \\ x_1 \geq 1 \\ \boxed{x_2 \geq 1} \\ x_3 \geq 1 \end{array}}$$

can now use Formula 1.4:

$$\boxed{\binom{11}{2}}$$

What about when $x_i \geq 0$? In that case we "borrow" a unit from that x_i which we have to give back later. Thus the constraint for that x_i rises to ≥ 1, and M increases to $M+1$.

Example 1.6. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$, where $x_1, x_2, x_3, x_4 \geq 0$?

Stage One:

$x_1+x_2+x_3+x_4=20$
$x_1 \geq 0$
$x_2 \geq 0$
$x_3 \geq 0$
$x_4 \geq 0$

$\Rightarrow$ "take" an object from x_1, add it to pile of 20 to distribute; "owe x_1 an object"

Stage Two:

$x_1+x_2+x_3+x_4=21$
$x_1 \geq 1$
$x_2 \geq 0$
$x_3 \geq 0$
$x_4 \geq 0$

$\Rightarrow$ "take" an object from x_2, x_3, x_4 each

Stage Three

$x_1+x_2+x_3+x_4=24$
$x_1 \geq 1$
$x_2 \geq 1$
$x_3 \geq 1$
$x_4 \geq 1$

Can now use Formula 1.4

$\boxed{\binom{23}{3}}$

Formula 1.7. Given an equation

$$x_1 + x_2 + \cdots + x_{n-1} + x_n = M,$$

the number of integer solutions to the equation where $x_1, x_2, \ldots, x_{n-1}, x_n$ are all ≥ 0 is

$$\binom{M+n-1}{n-1}$$

just have to know either this or Formula 1.4

What if there is a constraint for an x_i that is an upper bound? Count the number of solutions where x_i goes past that upper bound and subtract them from the total.

Example 1.8. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 36$ if $x_1, x_2, x_3, x_4 \geq 0$ and $x_4 \leq 10$?

Strategy: answer = total # of sol'ns without upper bound
$x_1+x_2+x_3+x_4=36$
$x_1, x_2, x_3, x_4 \geq 0$

− # of sol'ns that violate upper bound that is ~~x4~~ x_4 is not less than or equal to 10.
$x_1+x_2+x_3+x_4=36$
$x_1, x_2, x_3 \geq 0;\ x_4 \geq 11$

give 11 to x_4

becomes
$x_1+x_2+x_3+x_4=\boxed{25}$
$x_1, x_2, x_3, x_4 \geq 0$

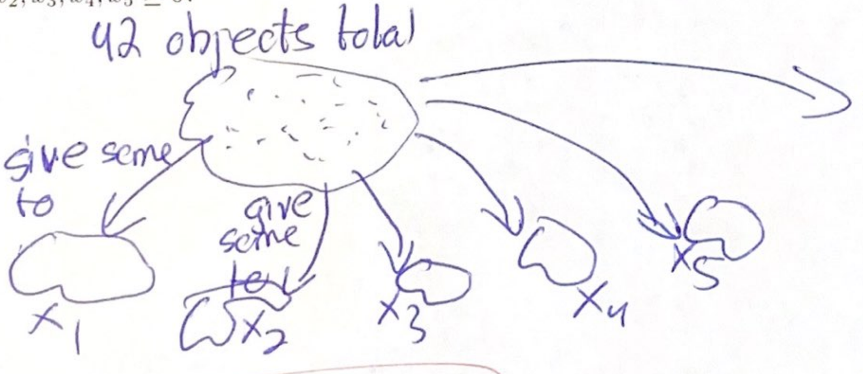
$\binom{39}{3}$ by Formula 1.7

$\binom{28}{3}$ by Formula 1.7

so answer is $\boxed{\binom{39}{3} - \binom{28}{3}}$

What if instead of an equation we have an inequality? Then we can change it to an equation by adding an extra variable, y, that will count as $M - x_1 - x_2 - \cdots - x_n$.

Example 1.9. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 \leq 42$, where $x_1, x_2, x_3, x_4, x_5 \geq 0$?



may have some leftover; assign the leftover ones to a new variable x_6

so $x_1+x_2+x_3+x_4+x_5 \leq 42$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

becomes

$x_1+x_2+x_3+x_4+x_5+x_6 = 42$
$x_1, x_2, x_3, x_4, x_5, \boxed{x_6 \geq 0}$

≥ 0 will always be the constraint of the new variable that we introduce

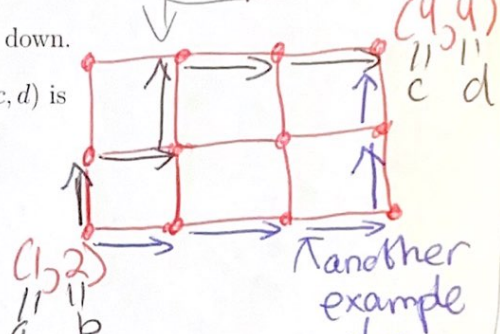
$= \boxed{\binom{47}{5}}$ by Formula 1.7

1.2 Counting Lattice Paths

Definition 1.10. Let (a, b) and (c, d) be two points in the coordinate plane $\mathbb{R}^2$, where a, b, c, d are all integers, and we have $a \leq c$ and $b \leq d$. A **lattice path** is a path from (a, b) to (c, d) that is composed of line segments whose endpoints are either $\{(x, y), (x+1, y)\}$ or $\{(x, y), (x, y+1)\}$. In other words

- The line segments all have length 1.
- Each line segment is either horizontal or vertical.
- Each line segment either goes up or right; never left or down.

an example of a lattice path



another example of a lattice path

Formula 1.11. The number of lattice paths from (a, b) to (c, d) is

$$\binom{(c-a)+(d-b)}{(c-a)}.$$

# of times moved to the right (traveled along a horizontal line segment)

# of times moved up (traveled along a vertical line segment)

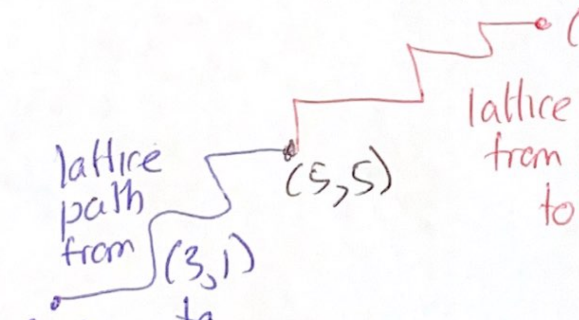
- total of $(c-a)+(d-b)$ segments on which to travel
- put down $(c-a)+(d-b)$ spaces representing segments on which we travel, choose $c-a$ of them to be when travel horizontally

~~Example~~

H H H V V

V H V H H

Example 1.12. How many lattice paths from $(3, 1)$ to $(7, 9)$ are there that pass through $(5, 5)$?



equivalently asking for

# of lattice paths from (3,1) to (5,5) × # of lattice paths from (5,5) to (7,9)

$$\binom{(5-3)+(5-1)}{(5-3)} \qquad \binom{(7-5)+(9-5)}{(7-5)}$$

$$\boxed{\binom{6}{2} \cdot \binom{6}{2}}$$

2 Section 2.5: The Binomial Theorem

We know from the distributive property that $(x+y)^2 = x^2 + 2xy + y^2$. But what about $(x+y)^n$ for some $n > 2$? We have the following:

Theorem 2.1 (The Binomial Theorem).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 2.2. Multiply out $(x+y)^3$, but leave the order of the variables in the terms the same:

$(x+y)(x+y)(x+y)$

$= (xx+xy+yx+yy)(x+y)$

$= (xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy)$

$= \underbrace{xxx}_{\text{3 spots, choose 3 for } x\text{'s: } \binom{3}{3}x^3} + \underbrace{xxy+xyx+yxx}_{\text{3 spots choose 2 for } x\text{'s: } \binom{3}{2}x^2y} + \underbrace{xyy+yxy+yyx}_{\text{3 spots, choose 1 for } x\text{'s: } \binom{3}{1}xy^2} + \underbrace{yyy}_{\text{choose 0 for } x\text{'s } \binom{3}{0}y^3}$

We see that when you multiply out $(x+y)^n$, for each k such that $0 \le k \le n$, we get precisely each different way to order k x's and $(n-k)$ y's in a string. Thus the coefficient for $x^k y^{n-k}$ is $\binom{n}{k}$, and thus why we call these **binomial coefficients**.

Example 2.3. If we were to multiply out $(x^2+y^3)^9$, what would be the coefficient of x^6y^{18}?

substitute so we have no exponents in the binomial we are multiplying out:

$a = x^2$

$b = y^3$

turns int $(a+b)^9$

plug a and b into x^6y^{18}

$(x^2)^3(y^3)^6$

$= \boxed{a^3b^6}$

our answer is the coefficient of this

which is $\boxed{\binom{9}{3}}$

Example 2.4. What is the coefficient of x^6 in $(2x+3)^{10}$?

look at the coefficient of $(2x)^6(3^4)$

Binomial Thm tells us this is $\binom{10}{6}$

so we have that the term is $\binom{10}{6}(2x)^6(3^4)$

isolate x

$\boxed{\binom{10}{6}2^6 \cdot 3^4} \cdot x^6$

answer

3 Section 2.6: Multinomial Coefficients

Definition 3.1. Say that we want to partition a set of n objects into r subsets, with the size of the first subset being k_1, the size of the second subset being k_2, ..., the size of the $(r-1)$th subset being k_{r-1}, and the size of the rth subset being k_r. The number of ways to make this partition is given by the **multinomial coefficient**

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!k_2!\cdots k_r!}.$$

$k_1+k_2+k_3+\ldots+k_r=n$

Remark 3.2.

- The numbers in the bottom part of the multinomial coefficient have to sum up to the number in the top part.
- At first glance, it seems like this formula is only for when you will use up all of the n objects you are given. However, this formula also works for when after choosing your groups, there are still some objects left over. Just "choose" the leftover objects as your last group, and you will still get the correct answer.

The multinomial coefficient is really just a shorthand way of writing binomial coefficients multiplied together.

Example 3.3. There are sixteen people that you need to assign into three teams. One team should have seven people in it, the second team should have five, and the third should have four. How many different ways can you assign the people into teams?

$16 \to 7, 5, 4$

$\to \binom{16}{7,5,4} = \frac{16!}{7!5!4!}$

$7+5+4=16$ bottom sums up to top

choose group of 7 from 16 AND choose group of 5 from remaining 9 AND choose group of 4 from remaining 4

$$\binom{16}{7}\cdot\binom{9}{5}\cdot\binom{4}{4} = \frac{16!}{7!\cancel{9!}}\cdot\frac{\cancel{9!}}{5!\cancel{4!}}\cdot\frac{\cancel{4!}}{4!\cdot 0!} = \frac{16!}{7!5!4!}$$

Multinomial coefficients can also represent cases where not ~~everything~~ all of the n objects are in a group

16 people; teams
divide into ~~groups~~ s
of 7, 5, and 4

16 people → 7 people Team 1, 5 people Team 2, 4 people Team 3

$$\binom{16}{7,5,4}$$

$$\binom{16}{7}\binom{9}{5}\binom{4}{4} \xleftrightarrow{\text{same}} \binom{16}{7}\binom{9}{5}$$

n = 16 people
choose ~~peop~~ a team of 7 ($= r_1$)
then choose a team of 5, ($= r_2$)
and that's it

16 people → 7 people Team 1, 5 people Team 2, 4 leftover, pretend they are their own group

$$\frac{16!}{7!\,\cancel{9!}} \cdot \frac{\cancel{9!}}{5!\,4!} = \frac{16!}{7!\,5!\,4!} = \binom{16}{7,5,4}$$

<u>Moral of the Story:</u> ~~$\binom{n}{r_1, r_2, \ldots, r_k}$~~
Multinomial coefficients can also count cases where not all of the n people are put into groups. (i.e. when $r_1 + r_2 + \ldots + r_k < n$)

remember ~~to~~ include difference in multinomial coefficient

$$\binom{n}{r_1, r_2, \ldots, r_k, r_{k+1}}$$

$r_{k+1} = n - r_1 - r_2 - \ldots - r_k$

Theorem 3.4 (The Multinomial Theorem).

$$\underbrace{(x_1 + x_2 + \cdots + x_r)^n}_{} = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

n multi-nomials

(Note that in addition to $k_1 + k_2 + \cdots + k_r = n$, we also have $k_1, k_2, \ldots, k_r \geq 0$.)

Example 3.5. When $(x + y + z)^{20}$ is multiplied out, what is the coefficient of $x^6y^2z^{12}$?

$\underbrace{(x_1+x_2+\ldots+x_r)}_{\text{1st multinomal}} \cdot \underbrace{(x_1+x_2+\ldots+x_r)}_{\text{2nd multi.}} \cdot \;\ldots\; \cdot \underbrace{(x_1+x_2+\ldots+x_r)}_{\text{nth multinomial}}$

$x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$ $\qquad x + y^2 + 3z$, $v = y^2$

# of ways to choose

k_1 ~~x's~~ multinomials for x_1's

k_2 multinomials from remainder for x_2's

k_3 multi's from rem. for x_3's

⋮

k_r multi's from rem. for x_r's

answer to this is $\boxed{\binom{n}{k_1, k_2, \ldots, k_r}}$

Ex 3.5 $(x+y+z)^{20}$ coefficient of $x^6y^2z^{12}$

Multinomial Theorem $\quad n=20$, $k_1=6$, $k_2=2$, $k_3=12$ $\qquad \boxed{\binom{20}{6, 2, 12}}$

$(x+y^2+3z)^{20}$, what is the coefficient of

choose 6 multi's for x

$\cancel{x^6 y^2 z^{12}}$

choose 1 for y^2

$$\underbrace{x^7 y^2}_{} z^{12}$$

choose 12 multi's for z

change of variable

$v=y^2$

$$x^7 v z^{12}$$

$$\underbrace{\binom{20}{7,1,12}\cdot 3^{12}}_{\text{coefficient of}} \quad x^7 y^2 z^{12}$$

Moral: Techniques we used for addressing extra stuff in Binomial Theorem also work here