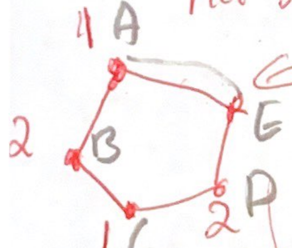
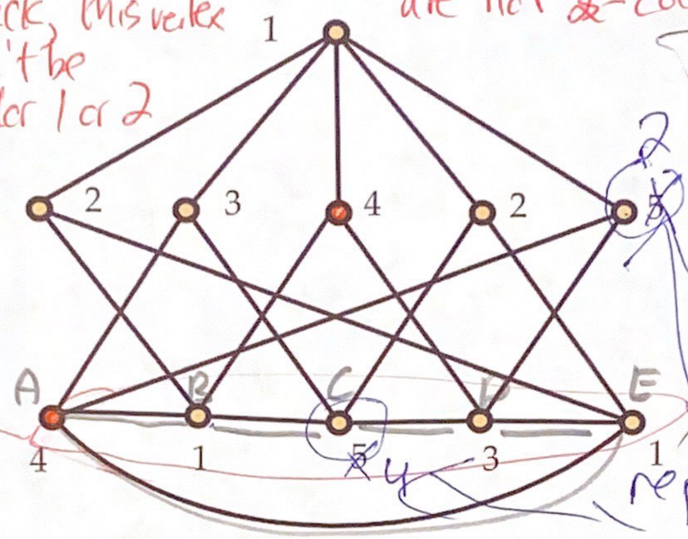
not 2-colorable because bottom 5 vertices (which form a cycle of length 5) are not 2-colorable



stuck, this vertex can't be color 1 or 2



this graph is 5-colorable $\therefore \chi(G) \leq 5$

replace with color 2

replace with ~~colors 2 or~~ color 4

now have a proper 4-coloring $\therefore \chi(G) \leq 4$

Figure 1

3 Section 5.4: Graph Coloring

Definition 3.1. Let G be a graph.

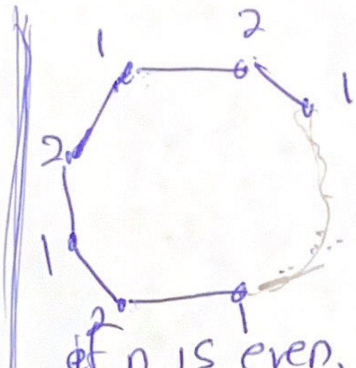
- A **coloring** of G is a function from the vertex set of G to a set of elements $\mathcal{C}$. (Usually we set $\mathcal{C}$ to be the set of integers from 1 to n, so the coloring looks like
  $\phi : V(G) \to \{1, 2, \ldots, n\}$.)
- A **proper coloring** of G is a coloring of G where any two adjacent vertices have to have different colors. That is, if $x, y \in V(G)$, then x adjacent to y implies $\phi(x) \neq \phi(y)$. We say that G is k**-colorable** if it has a proper coloring $\phi : V(G) \to \mathcal{C}$ where $\mathcal{C}$ has size k.
- The **chromatic number** $\chi(G)$ of G is the smallest integer k for which G is k-colorable.

Exercise 3.2. The graph in Figure 1 is 5-colorable. Is it 4-colorable? How about 3-colorable?

Yes, let's go to the two vertices with color 5, and replace with one of the four other colors.

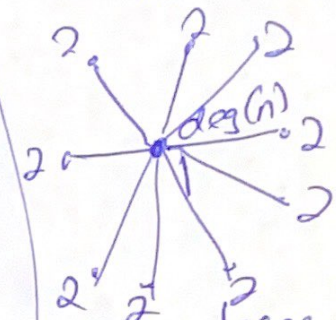
Remark 3.3. The number of vertices in G does not impose a constraint on how high the chromatic number $\chi(G)$ must be. Neither does the degree of a vertex.

In n vertices, no edges that has chromatic number is 1



if n is even, we can have our graph be a cycle of length n, chromatic number 2

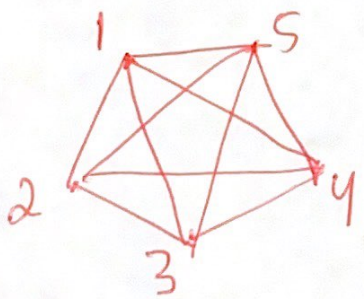
$n+1$ vertices degree n, n vertex

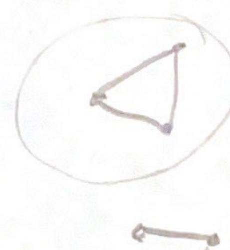


n vertices surrounding center vertex chromatic number 2

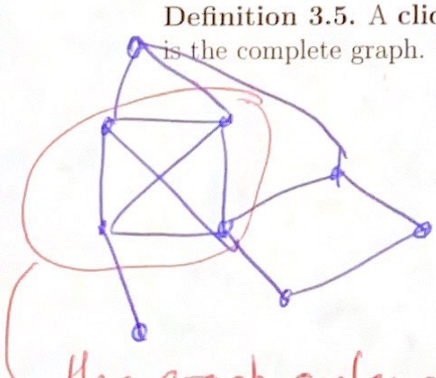
Remark 3.4. The chromatic number can get arbitrarily high. That is, for any positive integer n we can find a graph G whose chromatic number is n.

K_n, complete graph on n vertices, has chromatic number n





Definition 3.5. A **clique** of a graph G is a subset of the vertices whose induced subgraph is the complete graph. The **clique number** $\omega(G)$ is the size of the largest clique in G.

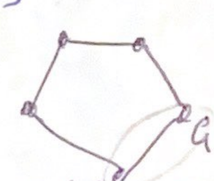


lower bound for $\chi(G)$

$\omega(G) \leq \chi(G)$

can sometimes have

$\omega(G) < \chi(G)$



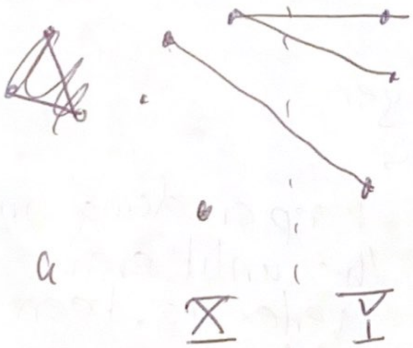
$\chi(G) = 3$

$\omega(G) = 2$

this graph contains K_4 as a subgraph a.k.a. it has a clique of size 4

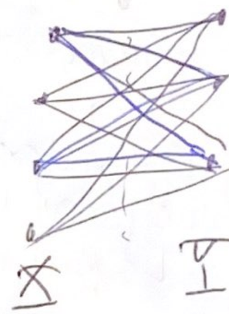
Definition 3.6.

- A graph is **bipartite** if its vertex set $V(G)$ can be partitioned into two subsets X and Y, such that every edge of G has one endpoint in X and the other endpoint in Y.
- A **complete bipartite graph** is a bipartite graph that includes all possible edges between vertices in X and vertices in Y. We use the notation $K_{|X|,|Y|}$.



· forests and trees are bipartite graphs
· not every bipartite graph is a forest or a tree

$K_{4,3}$



Remark 3.7. Being bipartite is the same thing as being 2-colorable.

· give all vertices in X the color 1 and all the vertices in Y the color 2

bipartite ~~is 2~~ ⇒ 2-colorable

· 2-colorable ⇒ bipartite

set X to be the set of all vertices with color 1

set Y to be the set of all vertices with color 2

Theorem 3.8. *A graph is bipartite if and only if it does not contain an odd cycle.*

because trees and forests don't have any cycles at all

bipartite "⇒" no odd cycle

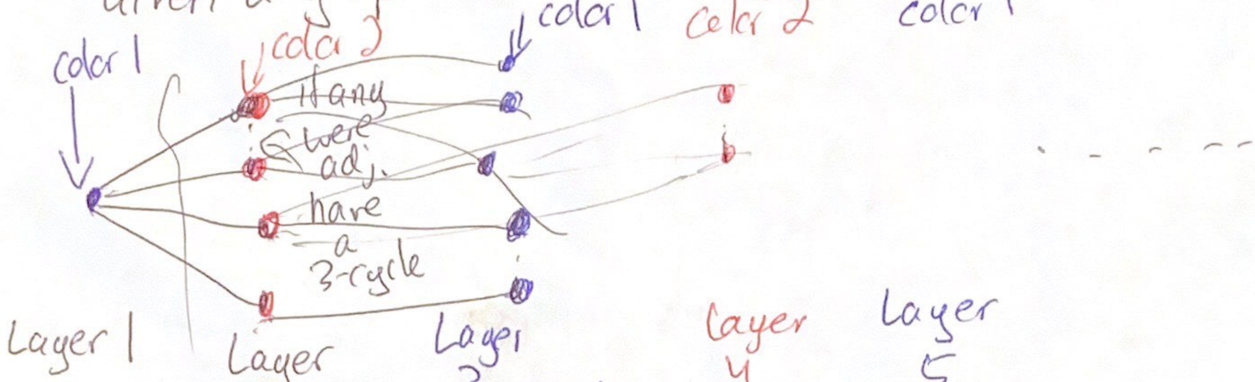
show that if there is an odd cycle then it can't be bipartite, Use Remark 3.7, we saw previously that an odd cycle is not 2-colorable, so by Remark 3.7, it also can't be bipartite.

no odd cycle "⇒" bipartite

'no odd cycles' => bipartite

Use the fact that 2-colorable and bipartite:

Given a graph with no odd cycles, give it a proper 2-coloring



Layer 2 (any vertex adjacent to a Layer 1 vertex)

Layer 3 (any vertex that hasn't been colored in yet, and is adjacent to a Layer 2 vertex

Keep on doing this until ~~every~~ vertex in comp. has been colored in; repeat on other components

$$
\begin{array}{c|c}
 & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \boxed{0} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

an adjacency matrix, way of storing info about a graph in matrix form

chemical 5 can't be stored with chemical 9

chemical 7 and chemical 9 can be stored together

Figure 2

Exercise 3.9 (Exercise 5.15 from textbook). A pharmaceutical manufacturer is building a new warehouse to store its supply of 10 chemicals it uses in production. However, some of the chemicals cannot be stored in the same room due to undesirable reactions that will occur. The matrix in Figure 2 has a 1 in position (i, j) if and only if chemical i and chemical j cannot be stored in the same room. Develop an appropriate graph theoretic model and determine the smallest number of rooms into which they can divide their warehouse so that they can safely store all chemicals in the warehouse.

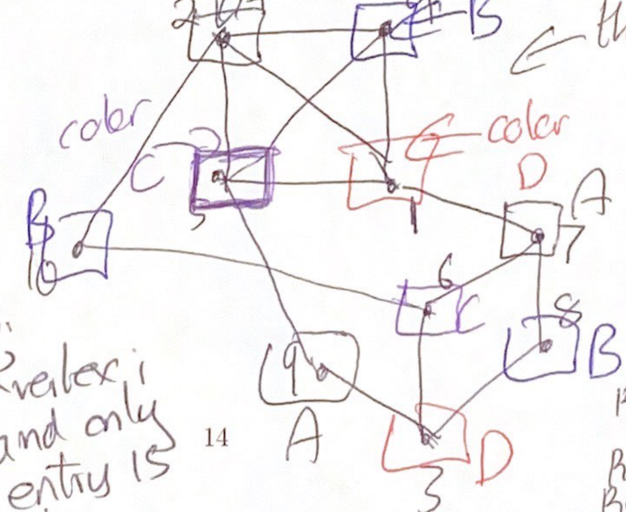
chemicals $\Rightarrow$ vertices

2 chemicals can't be stored together $\Rightarrow$ ~~edge~~ there is an edge between those vertices

rooms $\Rightarrow$ colors

find chromatic number k and a proper k-coloring

color A
color B
color C
color D



this is the graph of the above adjacency matrix

tells us how to construct the graph
put down 10 vertices;
put an edge between vertex i and vertex j if and only (i, j)th entry is 1

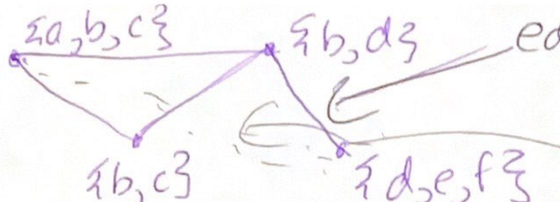
Rooms to store chemicals
Room A: 2, 7, 9
Room B: 4, 8, 10
Room C: 5, 6
Room D: 1, 3

$\{a, b, c\}$

$\{b, d\}$

$\{b, c\}$

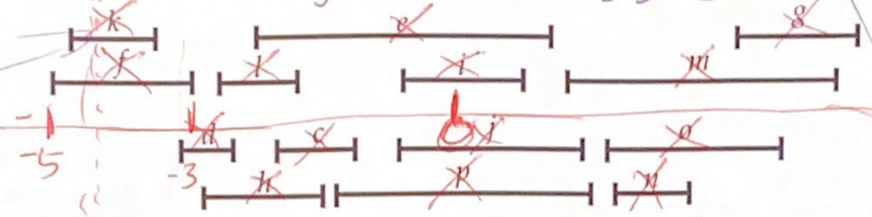
$\{d, e, f\}$



edge, because $\{b,d\} \cap \{d,e,f\} = \{d\} \neq \emptyset$

intersect the two sets

nonempty



$f = [-5, -3]$

$k = [-4.75, -3.5]$

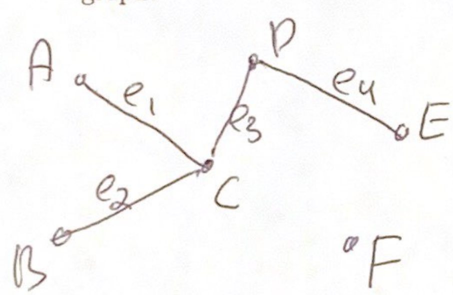
$\{a,b,c\} \cap \{d,e,f\} = \emptyset$

Figure 3

Definition 3.10. If we have a familiy of sets $\{S_\alpha\}_{\alpha \in I}$, the **intersection graph** is the graph where each vertex corresponds to a set S_α, and two vertices are adjacent if and only if their intersection is nonempty.

Remark 3.11. For any graph G, one can find a family of sets for which G is the intersection graph.

every graph is an intersection graph



sets will be sets of edges of graph

set of edges corresponding to vertex X will be all edges that touch X:

A: $\{e_1\}$ E: $\{e_4\}$

B: $\{e_2\}$ F: $\{\} = \emptyset$

C: $\{e_1, e_2, e_3\}$

D: $\{e_3, e_4\}$

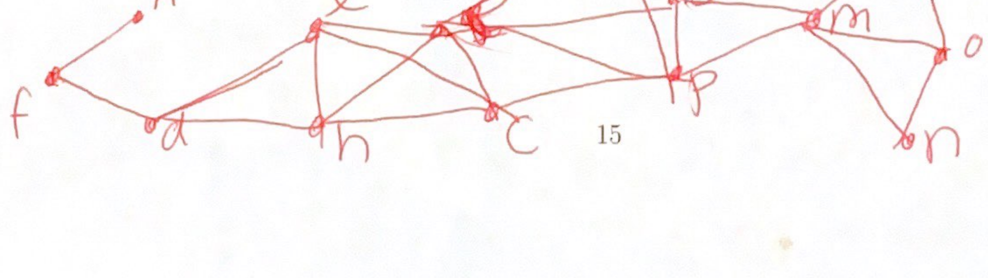
$\{\} \cap S = \{\}$

Definition 3.12. An **interval graph** is an intersection graph where the sets in question are closed intervals of the real line $\mathbb{R}$.

Exercise 3.13 (Exercise 5.25 from textbook)**.** What is the interval graph of the intervals in Figure 3?

see red markings above; intervals of Figure 3 are all on the ~~inter~~ real line, vertical spacing is just to make them easier to distinguish.

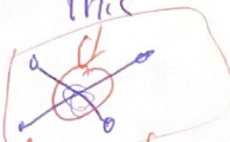
- two intervals have nonempty intersection $\iff$ there is a vertical line running through both of them



4 Section 5.4: Planar Graphs

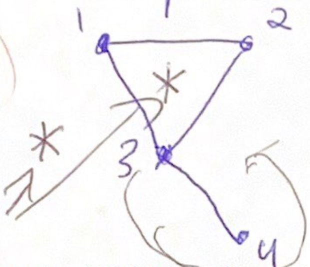
Definition 4.1. A graph is **planar** if it can be drawn in the plane $\mathbb{R}^2$ without any edges crossing. More precisely, the only time two edges meet each other is at a vertex.

draw it so that so this



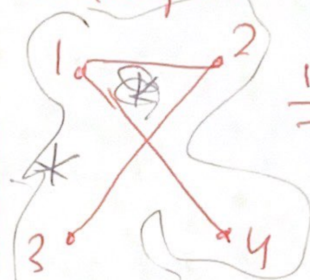
does not occur

This graph is planar:

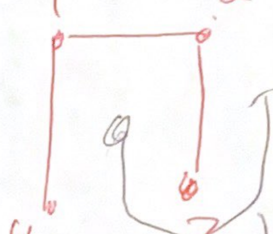


two faces inside the triangle and outside the triangle

This graph is also planar:



because we can redraw it as follows:



now there are no edge crossings

Formula 4.2 (Euler's Formula). In a planar graph G, the equation

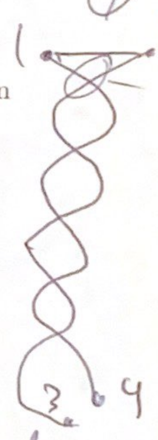
$$\boxed{n - m + f = 2}$$

holds only when you draw it without edge crossings

is always satisfied, where

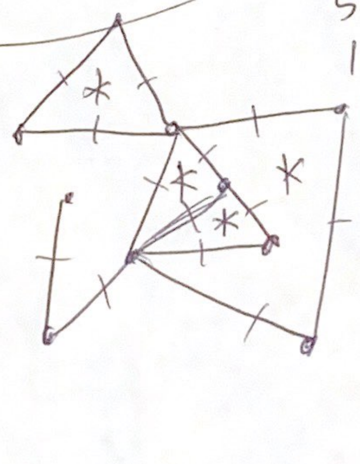
- n is the number of vertices of G
- m is the number of edges of G
- f is the number of faces of G

$\{\} \cap \{a, b, c\} = \{\}$



Definition 4.3. A face of a planar graph G is a region of $\mathbb{R}$ that G divides $\mathbb{R}^2$ into when G is drawn without any edge crossings.

same reason:
-why Euler's formula only holds for a planar drawing (no edge crossings)
-and why we only count the faces of a graph when we have no edge crossings



5 faces
10 vertices
13 edges
$10 - 13 + 5 = 2$

if we allow edge crossings, cancel as many faces as we want.

If for graph with n vertices and m edges:
$m > 3n-6 \Rightarrow$ graph is not planar
$m \le 3n-6 \Rightarrow$ can't say anything/could be either

meaning

satisfying this inequality is necessary, but not sufficient for planarity

Theorem 4.4. *If G is a planar graph with n vertices and m edges, then* $\boxed{m \le 3n-6.}$

derive from Euler's formula

Idea: to get 2 faces we need at least 3 edges, so we have this ratio:

K4 is planar

$$\frac{m = \#\text{ of edges}}{f = \#\text{ of faces}} \ge \frac{3}{2}$$

, plug into Euler's formula

$\Rightarrow 2m \ge 3f$

$n - m + f = 2$

$\Rightarrow n - m + \frac{2}{3}m \ge 2$

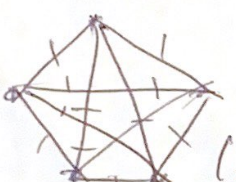
$n - \frac{1}{3}m \ge 2$

$\Rightarrow 3n - m \ge 6$

$\Rightarrow 3n - 6 \ge m$

Exercise 4.5. Is K_5, the complete graph on 5 vertices, planar?

can now apply to a non-planar drawing



$n = 5$

$m = 10 \Rightarrow$ plug into $m \le 3n-6$

$\binom{5}{2} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2 \cdot 1} = 5 \cdot 2 = 10$

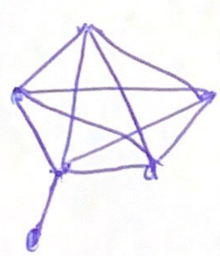
$m = 10$ $\quad 3n-6 = 15-6 = 9$

$m > 3n-6$

so K_5 is not planar

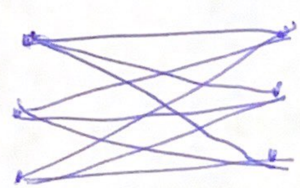
what if we appended an extra vertex to this graph

Remark 4.6. Even though $K_{3,3}$ doesn't fail the above theorem's requirement, it is also not planar.



nonplanar (contains a nonplanar graph, K_5, as a subgraph

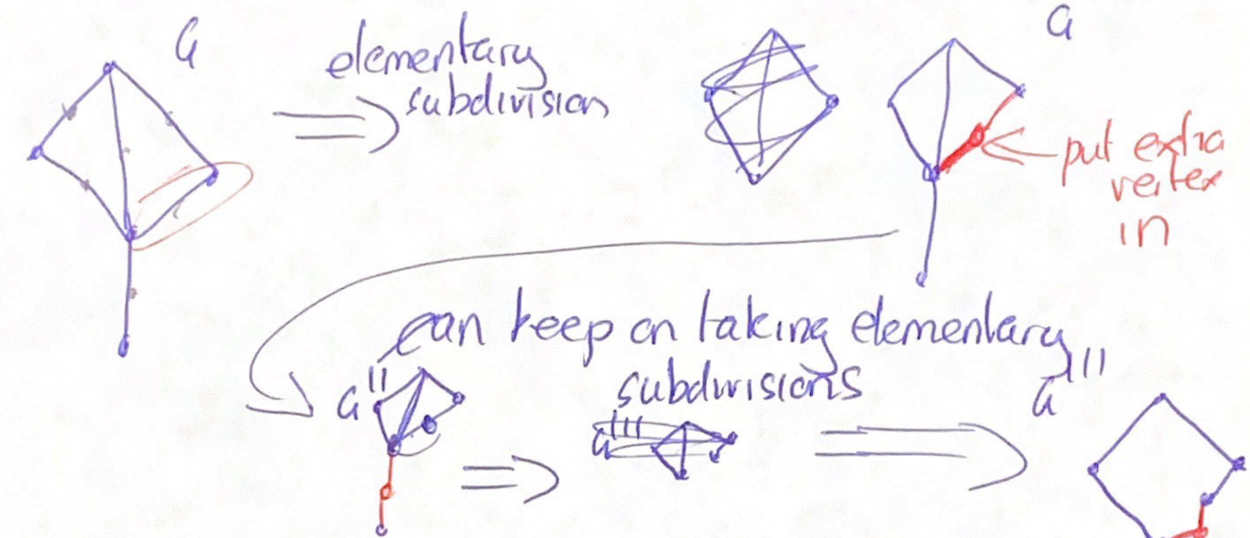
$n = 6$
$m = 11$
$11 \le 3 \cdot 6 - 6 = 12$



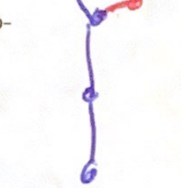
$K_{3,3}$
nonplanar

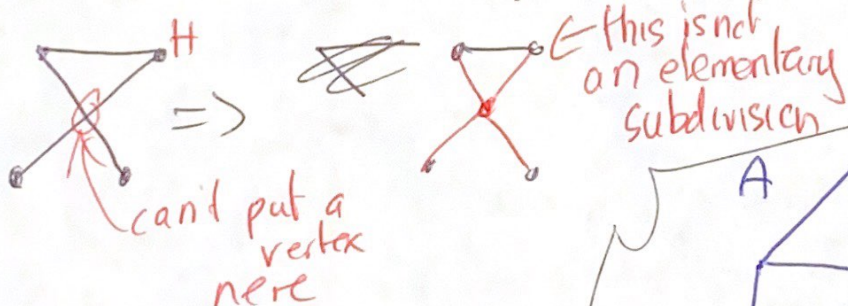
$n = 6$
$m = 9$
$3n - 6 = 12$
$9 \le 12$
$m \le 3n-6$

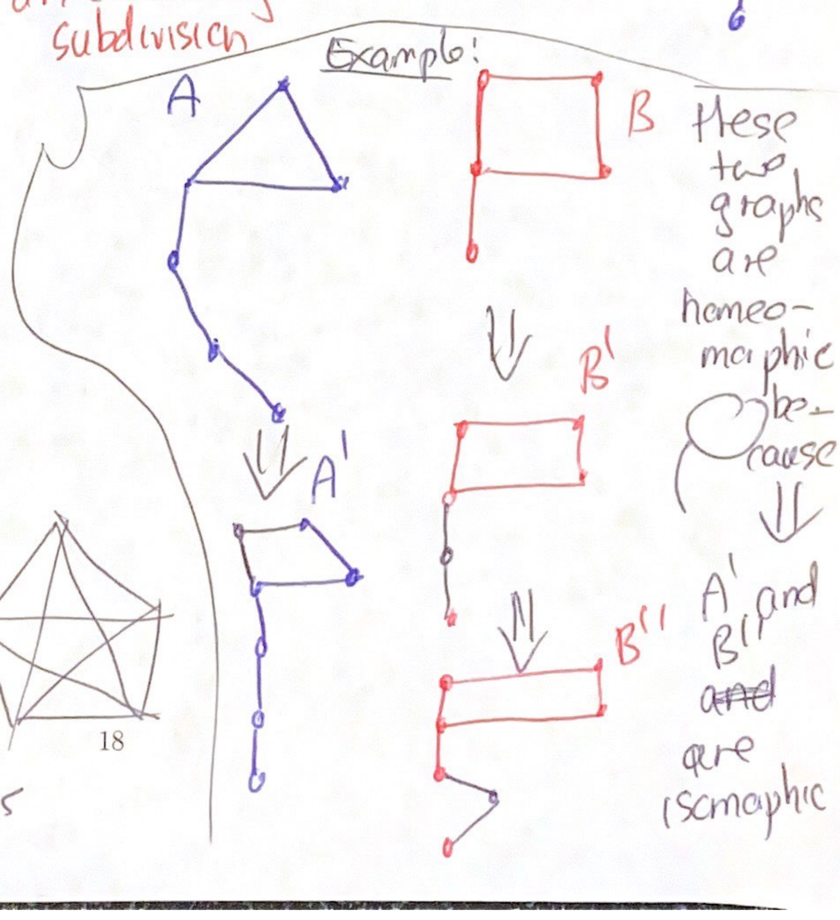
Definition 4.7. If G is a graph, then an **elementary subdivision** G' of G is a graph G' that is formed by "splitting" an edge into two. We take an edge xy (x and y are the endpoints of the edge), and place a vertex z in the middle of the edge. So, now we have two edges, xz and zy.

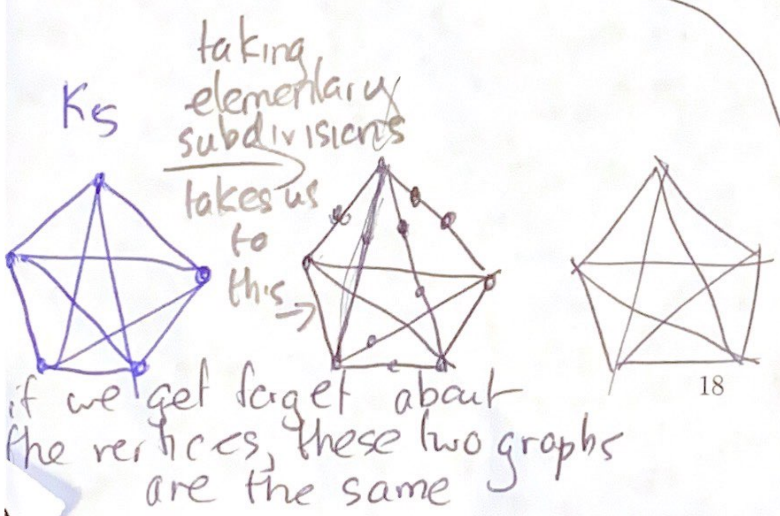


Definition 4.8. Two graphs G and H are **homeomorphic** if there is an elementary subdivision G' of G that is isomorphic to an elementary subdivision H' of H.







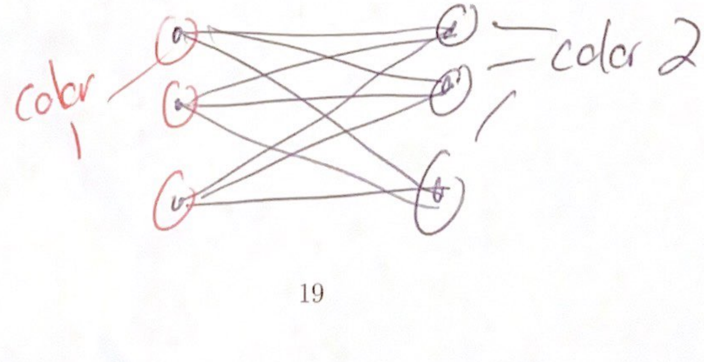


Theorem 4.9 (Kuratowski's Theorem). *A graph is planar if and only if it does not contain a subgraph homeomorphic to either K_5 or $K_{3,3}$.*

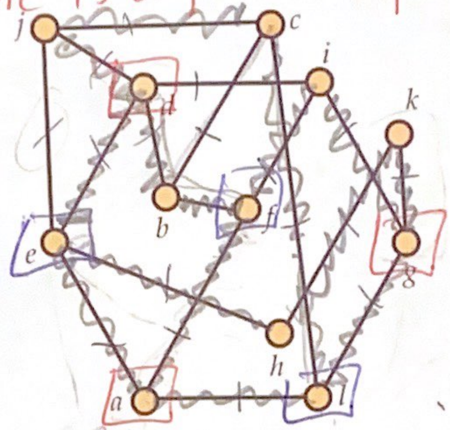
Theorem 4.10 (Four Color Theorem). *If G is a planar graph, then the chromatic number $\chi(G) \leq 4$.*

- Similar to Thm on $m \leq 3n-6$, this gives a necessary, but not sufficient condition for planarity.
  - If G is not 4-colorable, then it is not planar.
  - If G is 4-colorable, can't say anything.

Example: $K_{3,3}$ is not planar, but as a bipartite graph, it is 2-colorable (and thus 4-colorable).



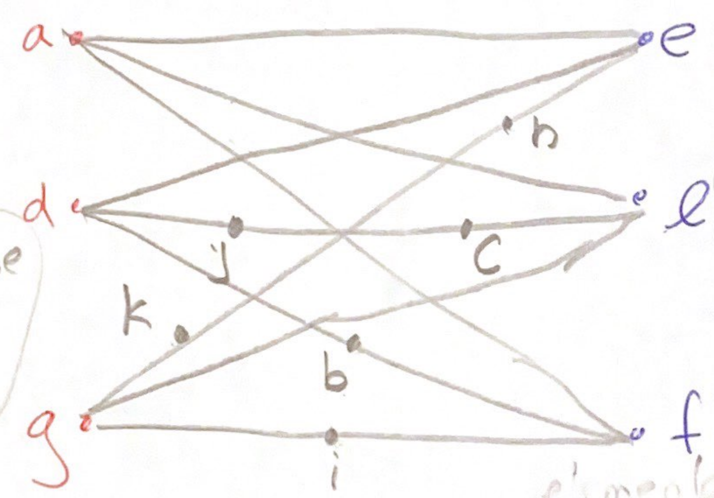
Plan:
1) Try and see if it satisfies $m \leq 3n-6$
2) Try to see if there is a planar representation
↓
3) Try to find subgraph homeomorphic to K_5 or $K_{3,3}$



Idea with Kuratowski's Thm and finding homeo. subgraphs to K_5 and $K_{3,3}$
- look for non-intersecting paths
- e.g. for K_5, choose 5 vertices, then try to find paths between each pair such that no two paths share the same edge or mid-vertex
- same idea for $K_{3,3}$

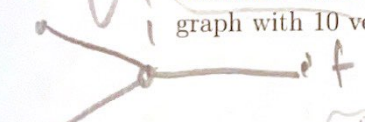
12 vertices $= n$
17 ~~vertices~~ edges $= m$
$17 \leq 30$ ✓

Exercise 4.11 (Exercise 5.29 from textbook). Is the graph shown above planar? If it is, find a drawing without edge crossings. If it is not, give a reason why it is not.



we want to avoid this, if I chose $d \to i \to f$ and $g \to i \to f$

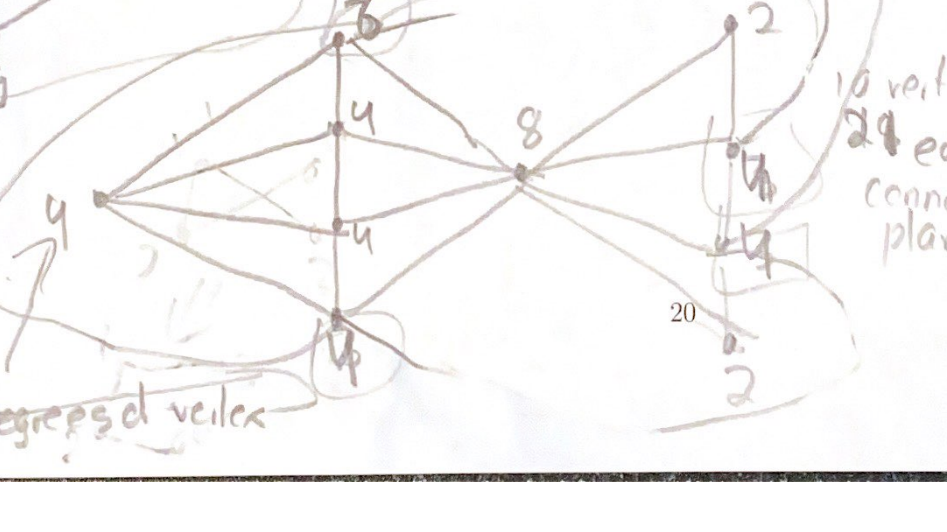
↓



⟸ this is a graph homeomorphic to $K_{3,3}$, so the graph above is nonplanar

Exercise 4.12 (Exercise 5.31 from the textbook). Exhibit a planar drawing of an Eulerian graph with 10 vertices and 21 edges.

- $n=10$
- $m=21$
- planar
- connected
- every vertex has even degree



10 vertices
21 edges
connected
planar

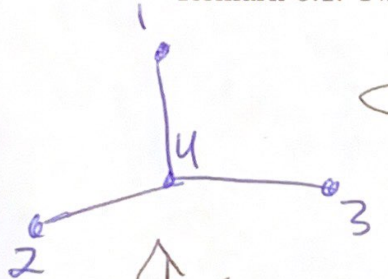
degree of vertex

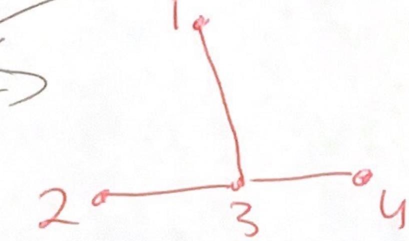
Strategy:
- start with small groups of e's and v's and connect them together
- check degree, # used at each step

5 Section 5.6: Counting Labeled Trees

We are now looking at counting labeled trees with n vertices. That is, how many different types of structures can we create by taking a tree with n vertices, and labeling each vertex with a unique integer from $\{1, 2, \ldots, n\}$?

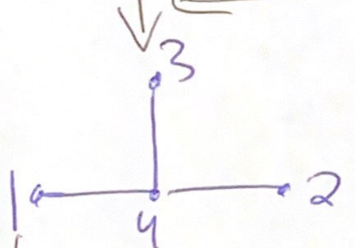
Remark 5.1. Two labeled trees ~~being identical is not~~ the same as two isomorphic trees.



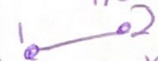


these two trees are isomorphic, but they are not the same labeled tree

these two are the same labeled tree



labeled tree on 2 vertices



Prüfer code $\emptyset$

Theorem 5.2 (Cayley's Formula). *The number of labeled trees on n vertices, where $n \geq 2$, is precisely n^{n-2}.*

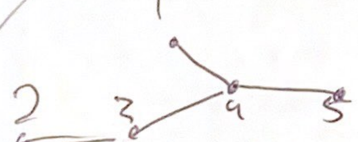
typo $\boxed{n^{n-2}}$

Proof, given by Prüfer codes

labeled trees on n vertices

establish bijection

strings on $\{1, 2, \ldots, n\}$ of length $n-2$



~~build the~~

434

take a labeled tree → Prüfer code of that tree → gives a string of length $n-2$

this also has size n^{n-2}

if there is a bijection between two sets, these sets are the same size

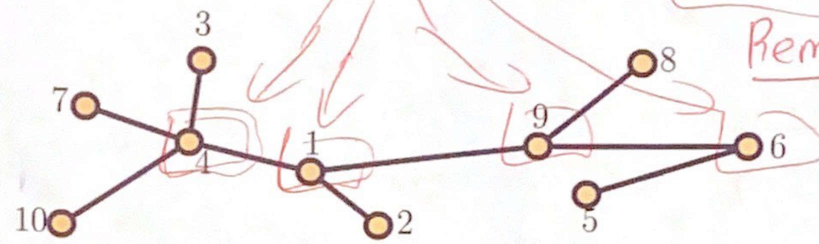
know that n^{n-2} is size of this, so by bijection

1,4,6,9
· only labels in the code
· non-leaves

Prüfer code of this tree

$\boxed{14694914}$

Remark: The digits in the Prüfer code of a tree are precisely the labels of the non-leaves of the tree.



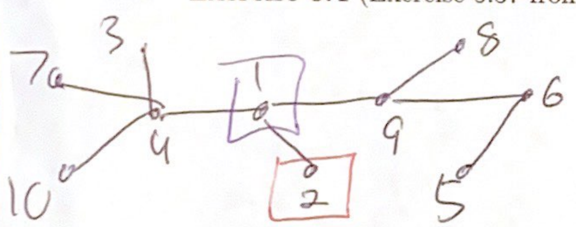
Definition 5.3. Let T be a labeled tree on $n \geq 2$ vertices. The **Prüfer code** $P(T)$ of T is defined recursively as follows:

Base Case: If T is the tree on two vertices, then $P(T)$ is the empty string.

Recursive Step: If $P(T)$ is a tree on $n > 2$ vertices, find the leaf of T whose label v is smaller than the label of any other leaf of T. Let u be the label of the vertex connected to the leaf with label v. Then

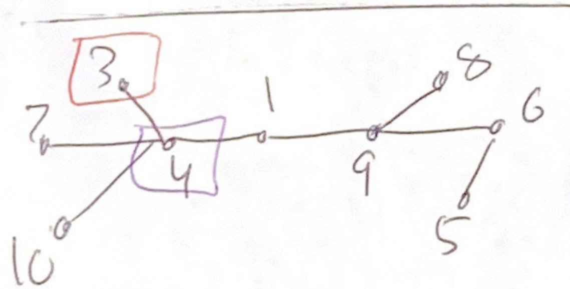
$$P(T) = \{u, P(T \setminus \{v\})\}.$$

Exercise 5.4 (Exercise 5.37 from the textbook). Find the Prüfer code of the above graph:



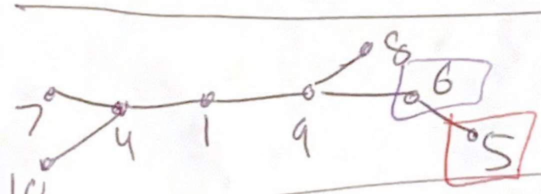
append 1
remove 2

Code



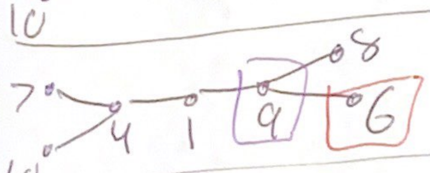
append 4
remove 3

Code
1



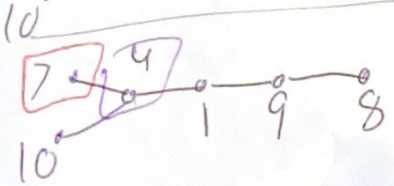
append 6
remove 5

Code
14



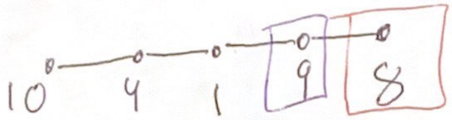
append 9
remove 6

Code
146



append 9
remove 7

Code
1469

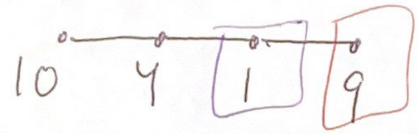


append 9

remove 8

Code

14694

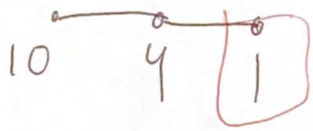


append 1

remove 9

Code

146949

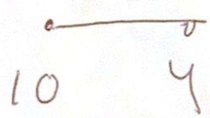


append 4

remove 1

Code

1469491



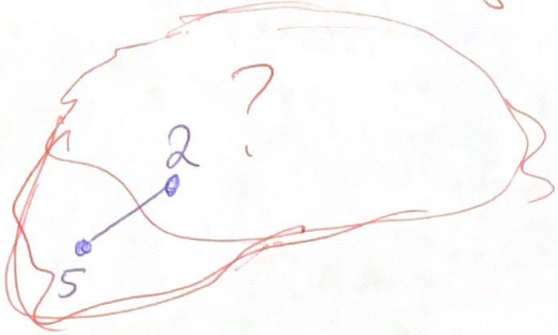
down to two vertices; ~~we finish and say~~ so we are done; this is the Prüfer code

code

$\boxed{14694914}$

Exercise 5.5 (Exercise 5.41 from the textbook). What is the labeled tree whose Prüfer code is 23134?

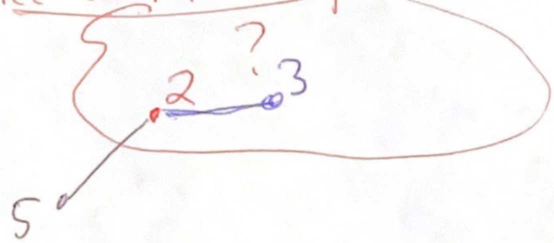
how many vertices? 7, because string has length 5=7-2



Label Set:	Code:	Leaf Set:
{1, 2, 3, 4, 5, 6, 7}	23134	{5, 6, 7}

5 is attached to 2 ⟵ and 5 would be removed first

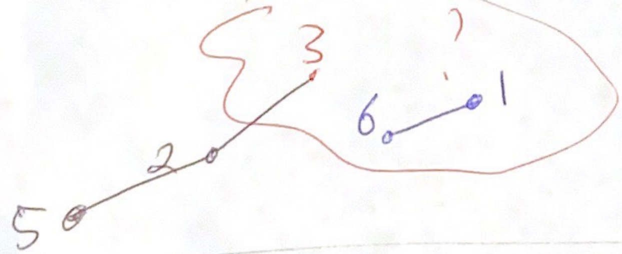
Tree after First Step:



Label Set:	Code:	Leaf Set
{1, 2, 3, 4, 6, 7}	3134	{2, 6, 7}

2 ~~was~~ is attached to 3 ⟵ and now remove 2

Tree after Second Step:



Label Set:	Code:	Leaf Set:
{1, 3, 4, 6, 7}	134	{6, 7}

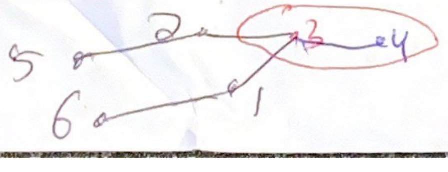
6 is attached to 1 ⟵ and now remove 6

Tree after Third Step:

Label Set:	Code:	Leaf Set
{1, 3, 4, 7}	34	{1, 7}

1 is attached to 3 ⟵ and now remove 1

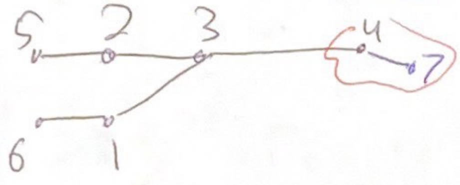
Tree after Fourth Step:



Label Set	Code:	Leaf Set:
{3, 4, 7}	4	{3, 7}

3 is attached to 4 ⟵ and now remove 3

Tree after Fifth Step:



Label Set:	Code	Leaf Set
$\{4, 7\}$		$\{4, 7\}$

down to two vertices, they must be connected, and we're done