An algorithm for determining the composition of a musical chord using Fourier transformations

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1 Rationale

As someone who is interested in signals, music and mathematics it was obvious for me that I wanted my mathematics internal assessment to be related to music. Although I have known of and used the Fourier transformations before, I have never used Fourier transformations to find individual notes in sounds. I decided, therefore, to try and identify chords using Fourier transformations, and try to find which are the composing notes of a chord.

2 Introduction

Even now, in the 21st century, an algorithm for identifying tones in music does not exist[1]. Research has been done which does allow simple chord progressions to be identified, but complex progressions remain a mystery.

The likes of Hausner, Kurnia and, of course, Wang, have developed algorithms which present themselves as being promising. Although currently, simple, with the exception of Wang's, these algorithms are powerful enough to follow chord progressions. However, with advancements in the field, it is likely that these algorithms would improve to a point where they could follow highly complex, highly noisy melodies, for example, in music like jazz and rock.

The research goal of this paper is to create a simple algorithm with decent performance which allows for recognizing chords and the individual tones that make them up. This simple algorithm could prove useful as a piece of a bigger algorithm for identifying chord progressions in songs.

2.1 Musical background

A chord is a musical unit in which three or more tones[2] are played simultaneously (Benward, and Saker). The most simple chords are *triads*, as they are composed of only three tones. Although they are modeled by three composing sine waves (for example, the C major chord is $f_{C_4maj}(t) = sin(41.63t) + sin(52.46t) + sin(62.39t)$), it is very important to note that actual musical instruments do not produce sine waves, as there

[1] Trained models are able to identify tones, however, there doesn't exist a single untrained algorithm which can achieve this.
[2] Air vibrations at a consistent pitch

are physical events that prevent this - reverb[3], distortion[4], harmonic series[5], etc.

Another major issue stands - the timbre of a musical instrument. *Timbre* does not have a clear-cut definition as it is quite abstract, but can be thought of as the color of the sound a musical instrument produces, ie. it is the property (or, rather *properties*) that allow us to discern musical instruments from one another.

All of these effects hinder the possibility of easily and effectively fitting a function using the least squares method. To find the individual frequencies composing a chord, it is necessary to convert the chord signal from the time domain to the frequency domain. For this, the *Fourier transformation* is used.

2.2 Fourier transformation

The Fourier transformation is based on the Fourier series.

2.2.1 Fourier series

The Fourier series is a way of representing any periodic waveform as a series of sine and cosine waves.

Given a periodic signal such as the one given in figure 1, it is possible to express it as a sum of cosine waves:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(2\pi n t)$$

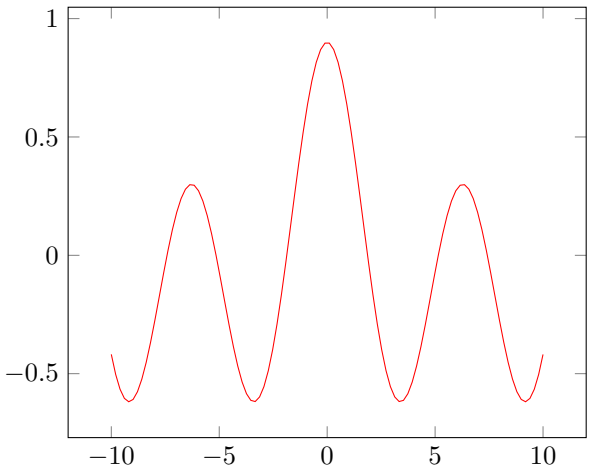

Figure 1: An arbitrary even function

[3]Reverberation is the effect of reflecting sound waves. When sound waves reflect off of surfaces many of them build up and then decay as they become absorbed (Valente et al.). It is what allows us to discern from big halls to small studios when hearing audio.

[4]Distortion is most easily modeled as clipping. If a signal function is defined as $f(t) = A(a)sin(\omega t)$ then clipping will, for example, occur when:

$$A(a) = \begin{cases} -4 & \text{when } a < -4 \\ a & \text{when } -4 < a < 4 \\ 4 & \text{when } a > 4 \end{cases}$$

This is an example of hard clipping, and is the simplest example, but other, more complicated examples exist.

[5]The harmonic series is the result of multiple parts of a vibrating body vibrating themselves (Benward, and Saker). A tone, therefore, is not composed of only one sine wave, but multiple sine waves of varying amplitudes. Usually, the longest harmonic is the strongest.

Note that n is the frequency of the wave in Hz. This derivation will employ Hz, however, no difference does it pose to use a radial frequency either.

It is required to calculate the amplitudes of the individual cosine waveforms a_n to get the constituting waveforms.

First, the left-hand and the right-hand side are multiplied by $\cos(2\pi mt)$.

$$f(t)\cos(2\pi mt) = \sum_{n=0}^{\infty} a_n \cos(2\pi nt)\cos(2\pi mt)$$

Then, integration is done on both sides of the equation and the trigonometric product of cosines identity is used:

$$\begin{aligned}\int_T f(t)\cos(2\pi mt)dt &= \int_T \sum_{n=0}^{\infty} a_n \cos(2\pi nt)\cos(2\pi mt)dt \\ &= \sum_{n=0}^{\infty} a_n \int_T \cos(2\pi nt)\cos(2\pi mt)dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_T \Big(\cos\big((m+n)2\pi t\big) + \cos\big((m-n)2\pi t\big)\Big)dt\end{aligned} \tag{1}$$

Another premise is that $m \geq 0$ (m is arbitrary and impossible for frequency to be negative). Because of this premise it follows that $\cos\big((m+n)2\pi t\big)$ has $m+n$ oscillations in a period. Since this is true, when this function is integrated, the result is 0[6].

Equation 1 is then simplified to:

$$\begin{aligned}\int_T f(t)\cos(2\pi mt)dt &= \frac{1}{2}\sum_{n=0}^{\infty} a_n \Big(\int_T \cos((m+n)2\pi t)dt + \int_T \cos((m-n)2\pi t)dt\Big) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_T \cos((m-n)2\pi t)dt\end{aligned} \tag{2}$$

Since the cosine function is even, in one period (ie. the interval of the integration - from $-\frac{T}{2}$ to $\frac{T}{2}$) the following is true:

$$\cos((m-n)2\pi t) = \cos((n-m)2\pi t)$$

When $m = n$:

$$\cos((m-n)2\pi t) = \cos(0) = 1$$

Due to the orthogonality of sines and cosines[7] it follows that:

$$\int_T \cos((m-n)2\pi t) = \begin{cases}\int_T \cos((m-n)2\pi t)dt = 0 & \text{when } m \neq n \\ \int_T 1 \cdot dt = T & \text{when } m = n\end{cases}$$

The summation in equation 2 is then reduced to the case when $m = n$, because otherwise the sum is 0:

$$\int_T = f(t)\cos(m2\pi t)dt = \frac{1}{2}a_m T$$

[6]The proof for this is trivial, however, it is published by a third party online and is available to the reader at the following link: `http://planetmath.org/integraloveraperiodinterval`.

[7]The orthogonality of sines and cosines is outside of the scope of this paper, but in a nutshell, it is the property of sines and cosines which states that:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{(n+m)\pi}{T}t\right) + \cos\left(\frac{(n-m)\pi}{T}t\right)dt = 0 \text{ when } m \neq n$$

This paper considers it as a premise.

Then a_m is:

$$a_m = \frac{2}{T}\int_T f(t)\cos(m2\pi t)dt$$

And, we can replace m with n:

$$a_n = \frac{2}{T}\int_T f(t)\cos(m2\pi t)dt \tag{3}$$

The case of $m = 0$ gives the y axis offset a_0. Substituting with $m = 0$ in equation 1, we get:

$$\int_T f(t)\cos(2\pi mt)dt = \sum_{n=0}^{\infty} a_n \int_T \cos(2\pi nt)\cos(2\pi mt)dt$$

$$\int_T f(t)\cos(0)dt = \sum_{n=0}^{\infty} a_n \int_T \cos(2\pi nt)\cos(0)dt$$

$$\int_T f(t)dt = \sum_{n=0}^{\infty} a_n \int_T \cos(2\pi nt)dt$$

$$\int_T f(t)dt = Ta_0$$

This gives the final result for the y axis offset:

$$a_0 = \frac{1}{T}\int_T f(t)dt \tag{4}$$

Odd functions are represented with sine waves:

$$f_o(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi nt)$$

The coefficient b_n is given by:

$$b_n = \frac{2}{T}\int_T f_o(t)\sin(2\pi nt)dt$$

The derivation is almost the same as for even functions and is therefore omitted.

Since it is supposed any arbitrary function $f(t)$ is a sum of odd and even functions:

$$f_o(t) = \frac{1}{2}(f(t) - f(-t))$$

$$f_e(t) = \frac{1}{2}(f(t) + f(-t))$$

$$f(t) = f_o(t) + f_e(t)$$

The Fourier series' components can be applied:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

where

$$a_0 = \frac{1}{T}\int_T f(t)dt$$

$$a_n = \frac{2}{T}\int_T f(t)\cos(2\pi nt)dt, n \neq 0$$

$$b_n = \frac{2}{T}\int_T f(t)\sin(2\pi nt)dt$$

Euler's formulae allow for expressing the Fourier series in complex form:

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)) \\ &= a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{i2\pi nt} + e^{-i2\pi nt}}{2} + b_n \frac{e^{i2\pi nt} - e^{-i2\pi nt}}{2i}\right) \\ &= a_0 + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{2\pi nt} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-2\pi nt} \end{aligned}$$

Next, c_n is defined as:

$$c_n = \frac{a_n - ib_n}{2}$$

This gives the complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi t}$$

This is referred to as the *synthesis* equation.

c_n is expressed in the following:

$$\begin{aligned} c_n &= \frac{1}{2} \cdot \left(\frac{2}{T}\int_T f(t)\cos(2\pi nt)dt - i\frac{2}{T}\int_T f(t)\sin(2\pi nt)dt\right) \\ &= \frac{1}{T}\int_T f(t)\cos(2\pi nt) - if(t)\sin(2\pi nt)dt \\ &= \frac{1}{T}\int_T f(t)\left(\frac{e^{i2\pi nt} + e^{-i2\pi nt}}{2} - i\frac{e^{i2\pi nt} - e^{-i2\pi nt}}{2i}\right) \\ &= \frac{1}{T}\int_T f(t)\frac{2e^{i2\pi nt}}{2}dt \\ c_n &= \frac{1}{T}\int_T f(t)e^{-i2\pi nt}dt \end{aligned}$$

This is the *analysis* equation.

2.3 The Fourier transformation

The Fourier transformation is the Fourier series but brought into a continuous form:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\nu t}dt$$

The Fourier transformation is the limit of the Fourier series as T approaches infinity:

$$\hat{f}(t) = \lim_{T\to\infty} c_n = \lim_{T\to\infty}\int_{-T/2}^{T/2} f(t)e^{-i2\pi nt}dt$$

The factor $\frac{1}{T}$ is ignored, due to its limit being 0:

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

which would make the whole function 0. It is not relevant in this discussion, because here we define:

$$f(t) \propto \frac{k}{T}$$

where k is varied with the period T.

This assumption cannot be made when using the inverse Fourier transformation, but the inverse Fourier transformation is outside of the scope of this paper.

The limit is then evaluated to:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\nu t} dt$$

This equation gives a continuous relationship between frequency and amplitude, which allows for identifying individual fundamental waveforms inside complex waveforms. Its applications are almost limitless - from analogue signal, to digital signal processing, image processing, seismic analysis and even medicine.

3 The chord identification algorithm

Given an input signal such as the one given in figure 2, the algorithm operates in 4 distinct stages.

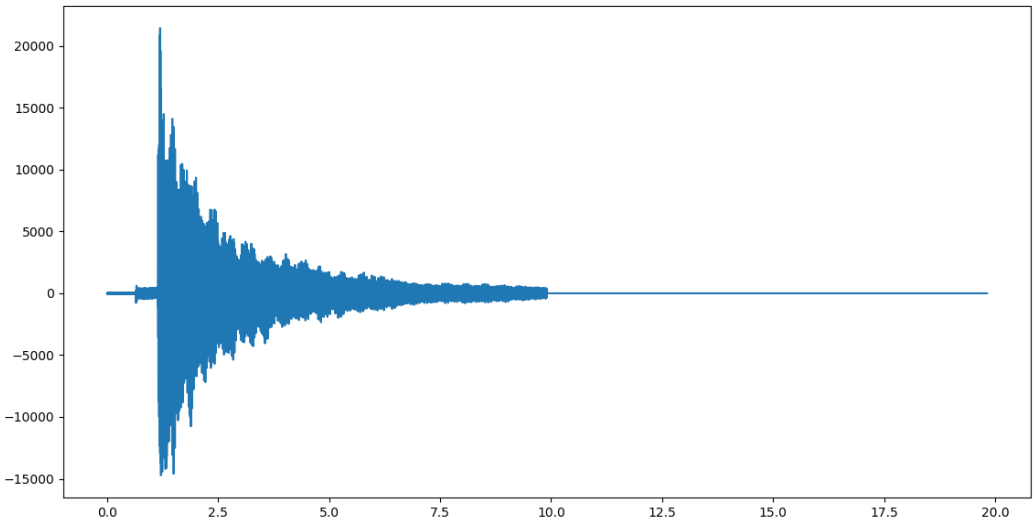

Figure 2: An example of an input signal - in this case a C_{maj} chord

The computer is instructed to load the amplitude samples of the signal in a column vector A.

$$A = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ \vdots \end{pmatrix} \text{ where, } a, b, c, d \text{ and } e \text{ are sample values}$$

The *Microsoft Waveform* datatype[8]does not give information on the x axis, as these values are inferred to be a linear space[9] from 0 to the norm[10] of the y vector. To get precise time information, ie. to get the number of seconds for every point in the y vector, it is necessary to use the sampling frequency of the *.wav* file. This sampling frequency, F_s, gives the number of samples taken in a single second, ie. the number of y values per single second. Finally, to get the values of the x axis, it is necessary to multiply the aforementioned linear space by the reciprocal of the sampling frequency. The domain of the *.wav* file is stored in the t column vector:

$$t = \frac{1}{F_s} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{pmatrix}, \text{ where } \|t\| = \|A\|$$

With these values, it is possible to proceed in the algorithm.

3.1 The Fourier transformation

A fast Fourier transformation[11] is ran on the amplitude vector.

$$F = \text{fft}(t, A)$$

The Fourier transformation gives a symmetric vector of both positive and negative values, so the negative values are ignored, as their absolute values are equal to the positive ones, and they provide no relevant information.

$$F' = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{\frac{\|F\|}{2}} \end{pmatrix}$$

Since the numbers given by the Fourier transformation are complex, to get the amplitudes of the frequencies it is necessary to get absolute values for all of the values in the vector given by the Fourier transformation.

$$F'' = \begin{pmatrix} |F'_1| \\ |F'_2| \\ |F'_3| \\ \vdots \\ |F'_{\|F'\|}| \end{pmatrix}$$

A vector f is created to store the domain of the Fourier transformation, such that it is a linear space from 0 to 1 with the norm being equal to the one returned Fourier transformation, multiplied by the sampling frequency of the original file. This is in turn, creates the frequency domain.

$$f = F_s \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}, \text{ such that } \|f\| = \|F''\|$$

3.2 Fourier transform peak identification

In this section of the algorithm the computer tries to identify which peaks play a significant role in composing a chord.

[8]The commonly used datatype for storing signals in computers

[9]A vector of linearly equidistant values

[10]The number of elements in a vector

[11]The fast Fourier transformation is a computer program implementation of the Fourier transformation which is faster than the standard implementation of the Fourier transformation, but produces equal results. It utilizes functions exclusive to computer science.

The vector F'' is normalized first:

$$F''' = \frac{1}{\max(F'')} \cdot F''$$

This is useful for the next step, finding values that are bigger than a constant cutoff value c_o and storing them in a vector p:

$$p = \varnothing$$

$$\forall v \in F''' : \left(v > c_o \Rightarrow p = \begin{pmatrix} p \\ v \end{pmatrix} \right)$$

As an example, the normalized Fourier transformation, alongside the cutoff value of $c_o = 0.15$, is plotted in figure 3.

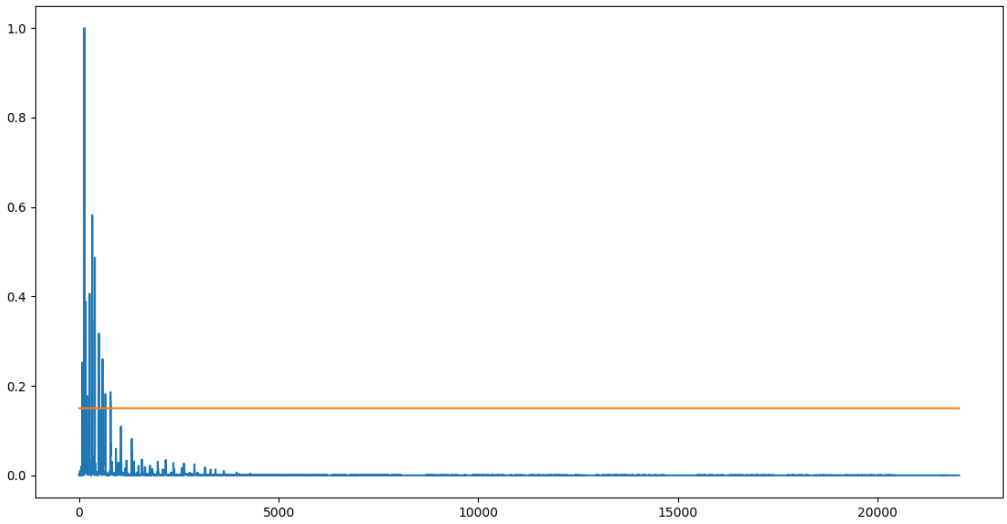

Figure 3: The normalized Fourier transformation of the signal given in figure 2, ie. a C_{maj} chord.

Note that the p vector does not contain the individual frequencies of the notes played, but contains multiple values which resolve to one frequency, due to the fact that the Fourier transformation is a continuous function. To resolve this issue, the average of neighboring values is calculated. Neighboring values w are neighboring to a value v if the absolute difference between them is less than a leeway value l.

$$p' = \varnothing$$

$$\forall v \in p : \left(\left(\left(w \notin p' : |v - w| < l \right) \rightarrow p' = \begin{pmatrix} p' \\ w \end{pmatrix} \right) \wedge \left(\left(w \in p' : |v - w| < l \right) \rightarrow w = (w + v)/2 \right) \right)$$

At this point, the vector p' contains the frequencies of the played chord that are more intense than a given cutoff value.

3.3 Note identification

Notes are identified in this step by finding the closest value in a look-up table of frequencies[12]. As well as identifying the most plausible notes the certainty of a frequency being a single note is calculated according

[12]This table is available at the following link: `https://pages.mtu.edu/~suits/notefreqs.html`.

to the following. In this section, let us denote any value in p' as v. Let us also denote the lookup table as a column vector r. We start from the premise that v lies between two closest values in r:

$$r_n < v < r_{n+1}$$

First, we identify the closest value to v. At this point, we know either r_n or r_{n+1}, but we do not know the other. To calculate the certainty of v being the note r_n or r_{n+1} we need to calculate the unknown value. This is done with the following logical assertion. First denote the index of the known value (r_n or r_{n+1} as k). Next, calculate a value o according to:

$$\Big((|v - r_k| > |v - r_{k+1}|) \Rightarrow o = r_{k+1} \wedge \neg(|v - r_k| > |v - r_{k+1}|) \Rightarrow o = r_{k-1}\Big)$$

Finally, calculate the certainty c according to:

$$c = 1 - \frac{|v - r_k|}{|o - r_k|}$$

The full algorithm is available at the following link: `https://github.com/markovejnovic/Chordy`

3.4 Chord identification

Since this algorithm is only designed for chord triads, it was not difficult to design an algorithm which identifies chords from the previously identified notes. A simple look-up table was created and a simple searching algorithm was implemented.

4 Evaluation

To identify how well the algorithm performs tests were created. 10 guitar chord samples were downloaded from the internet (*vide* the works cited). These chord samples were then analyzed using the algorithm. Since it was known which chords were played, it was possible to determine the accuracy of the algorithm.

For a value of $c_o = 0.15$, the algorithm's success rate was 60%. However, when the value of c_o was chosen to be a value greater or lesser than 0.15 different success rates were achieved. A plot of the relationship between c_o and the success rate of the algorithm is given in figure 4.

It is obvious that the maximal success rate is achieved for the minimal value of $c_o = 0.02$. This success rate is equal to 70%. Note, however, that the chords played in the samples might have not been the ones that the authors stated they were. It is possible that the guitars the chords were played on were out of tune, or the chords were played improperly, or the tune of the guitars was not one of $A_4 = 440$, or it is even possible that the guitars used were not precisely manufactured.

The maximal theoretical accuracy is calculated using a linear fit which was calculated using the $RMSE$[13] method. The negative linear fit achieves a maximum value of:

$$(0, 0.70861538)$$

The maximal theoretical accuracy of the algorithm is therefore, using this data,

$$70.9\%$$

at a cutoff frequency of:

$$c_o = 0$$

[13] Root Mean Square Error

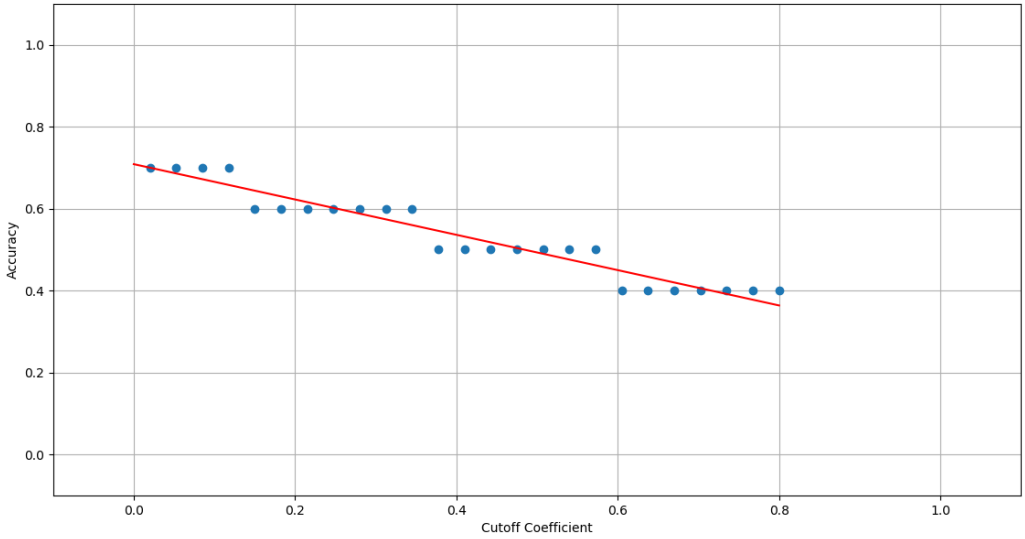

Figure 4: The relationship between c_o and the success rate of the algorithm.

5 Conclusion

The algorithm seems effective in achieving the required result. The evaluation of the algorithm was incomplete, due to a lack of chord samples. For more accurate evaluation it is necessary to use better chord samples.

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