1. Linear Regression

1.1. Univariate Linear Regression

1.1.1. Motivation

The problem that we are trying to solve with most machine learning algorithms is that we are basically trying to predict some new information based on some old information that we already know. So what we want is some mathematical model that given some previous information can tell us with a high degree of probability where any new information might lie. There are many ways to do this , but here we are going to discuss only the most basic form of a model which is a line. Or more specifically if you remember from high school classes this would be the line of best fit. Basically if we have a 2-D scatter plot , we just find the line of best fit for this scatter plot and we're done.

One of the cliché basic machine learning problems is trying to predict housing prices. If the example isn't broken , I'm not going to fix it for sake of novelty , so I'm just going to stick to the same example. Let us say that we have a set of housing prices (y) , and a set of house sizes (x). What we want is a model that based on any new housing price x_{new} would tell us what the price should be based on all of our historical prices. All we do is make a scatter plot of out existing data points and find the line of best fit. Pretty simple.

This is basically what univariate linear regression is and hopefully we now have a good understanding of why we are doing what we are doing. So now lets get to the how.

1.1.2. Hypothesis Equation

The first thing to do is recall the equation of a line from our elementary school days. It was of the form $y = mx + b$, but we want to be fancy machine learning programmers now , so we want to use things like functions and Greek symbols. [i] So our y-intercept b becomes θ_0 , our slope m becomes θ_1 and y becomes $h_\theta(x)$. The x stays as x.

From now on, I will never refer to the highschool version of the equation of a line again, since it has a y in it, and this is likely to cause confusion when talking about our y values from the (x, y) data points. So to reiterate the y from this point forward has nothing to do with the equation of a line, that was only used to help you understand the new notation by comparing it to something familiar.

The function is $h(x)$ is called a ***Hypothesis function***. Given a input value x , the hypothesis function denotes the corresponding ***predicted value*** $\hat{y} = h(x)$. The $\hat{y}$ is often more used in statistics and math while $h(x)$ is more used in machine learning , people's usage of one or the other tends to imply basically where they first learned this linear regression concept , either in a statisics class or in a computer science class. These $\hat{y} = h(x)$ values are also sometimes referred to as ***target variables*** or ***output variables***.

To maintain complete clarity and refer back to our housing price example , y are the housing prices we already know. The $\hat{y}$ is the house price that we guess based on our linear regression line and a new house square footage x.

This gives us our equation for the hypothesis function :

$$h_\theta(x) = \theta_0 + \theta_1 \cdot x \tag{1}$$

or more formally :

$$h : x^{(i)} \to y^{(i)}, \quad x^{(i)} \in X = \{x_1, \ldots, x_m\}$$

such that $h_\theta(x)$ is a good predictor for the corresponding value of y for a training example of (x, y).

Here we only have one variable x which is namely the size of our houses from the earlier example. This variable x is commonly known as our ***input variable*** or a ***feature***. This is a model based on only one feature or variable , which is why this technique is called **uni**variate linear regression. Later we can use not only the

[i]This is only meant facetiously. I thought I should put in this note in case someone reads this and gets put off or something about the sarcasm.

house size , but also the number of bedrooms that house has , the number of bathrooms , whether or not the house has a yard , and so on as multiple variables to use in our model. But for now lets just stick with the one that we have.

We usually split our data into two sets the **training set** , and the **test set**. The training set is what we use to build our line of best fit , and the test set as the name implies is used to test how well our model is working. Each data point is basically a (x, y) pair of values.

A summary of all the notation for this section is given below :

1.1.3. Cost Function

OK , now we know the 'form' of the function which is $h_\theta(x) = \theta_0 + \theta_1 x$. We now have to learn this function. But what does 'learning' a function mean ? To do understand this look at the graph below , each line here denotes a possible h value :

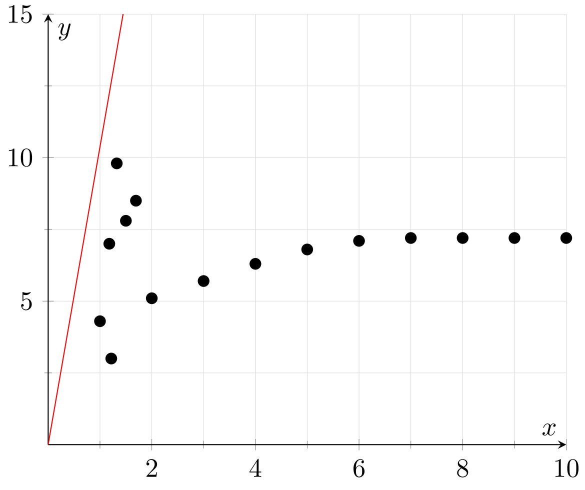


Figure 1

Learning here means figuring out which one of the lines drawn (and many many more not drawn) should we use as our hypothesis function ? Or rather given all the lines , which one of them is the BEST FIT line to our data points.

In our equation we already have a fixed value x , so we need to figure out what is the best y-intercept θ_0 , and slope θ_1. Therefore our learning task now becomes , out of all possible real values , what are the values of θ_0 and θ_1 , such that the difference between our guessed value $h(x)$ and our known value y is as small as we can make it , i.e. , we want the value of $h_\theta(x) - y$ to be as small as possible , and ideally 0.

Keep in mind , that here for every change in our theta values , the distance $h_\theta(x) - y$ will change for all of our $y^{(i)}$'s in our training set and we want a $h(x)$ so that the distance is as small as it can be for all of the $y^{(i)}$ not just 0 for one and some large number for all the others.

These θ_i values are called ***parameters***, and what we need to do is guess good values for these parameters. We need something that tells us whether the values we choose are 'good' or not. This something is called an **evaluation metric**. Evaluation Metrics are used to explain the performance of a model. They compare the difference between the real value and the predicted value of a model. There are a lot of different equations that we can use as our evaluation metric. Each has its own pro's and con's. Some of the most commonly used ones are :

Mean Absolute Error

$$\mathbf{MAE} = \frac{1}{n} \cdot \sum_{j=1}^{n} \{|y_j - \hat{y_j}|\}$$

Mean Squared Error

Mean squared error is popular because the focus of the error is geared towards larger errors. This is due to the squared term exponentially increasing larger errors in comparison to smaller ones.

$$\mathbf{MSE} = \frac{1}{n} \cdot \sum_{i=1}^{n} \left\{ (y_i - \hat{y_i})^2 \right\}$$

Root Mean Squared Error

Root Mean squared error is also very popular since it is interpretable in the same units as the response vector. Which makes it easy to relate its information.

$$\mathbf{RMSE} = \sqrt[2]{\frac{1}{n} \cdot \sum_{i=1}^{n} \left\{ (y_i - \hat{y_i})^2 \right\}}$$

Relative Absolute Error

The Relative Absolute Error , also known as the Residual Sum of Square , takes the total absolute value and normalizes it by dividing by the absolute sum of the simple predictor.

$$\mathbf{RAE} = \frac{\sum_{j=1}^{n} \{|y_j - \hat{y_j}|\}}{\sum_{j=1}^{n} \{|y_j - \hat{y}|\}}$$

Relative Squared Error

$$\mathbf{RSE} = \frac{\sum_{j=1}^{n} \left\{ (y_j - \hat{y_j})^2 \right\}}{\sum_{j=1}^{n} \left\{ (y_j - \hat{y})^2 \right\}}$$

$$R^2 = 1 - RSE$$

For the sake of this writeup I will be using the **Mean Squared Error**. The reason is that based on previous evidence of hundreds of times doing this with other cost functions , the square error function is a reasonable choice that works well for most regression problems.

The equation is given again below :

$$MSE = \frac{1}{n} \sum_{i=0}^{n} \{(\hat{y_i} - y_i)^2\}$$

This is the 'generic' or statistics version of the equation , which means that I have not used the machine learning variable forms in it. I wanted to show it in this form first. Because after this point we'll probably be staring at the other version for a long while.

The computer science version of the same mean squared error equation is commonly called the ***cost function*** and is represented using the character J. The function is defined as follows :

$$J(\theta_0, \theta_1) = \frac{1}{2} \cdot \frac{1}{m} \cdot \sum_{i=0}^{m} \{(h_\theta(x)^{(i)} - y^{(i)})^2\} \tag{2}$$

Keep in mind here are that the superscripts (i) are not powers they are indexes into our training set , so dont get confused.

Oh , also note here that we multiply by the additional term $\frac{1}{2}$. It doesnt really change much about the equation it just scales the values down , but it helps a lot in cleaning up the gradient descent calculations that we are just about to do.

Speaking of gradient descent , what in the world is that ? Keep in mind that so far , our function $J(\theta_0, \theta_1)$ only represents how much error there is in our guesses for θ_0 and θ_1. But our job doesn't end here , we actually have to minimize this error value and keep making our model better. Therefore our task now becomes to minimize the cost function J :

$\min_{\theta_0,\theta_1} J(\theta_0, \theta_1)$

and the algorithm we are going to use to minimize this function is gradient descent.

1.1.4. Gradient Descent

Gradient Descent is an algorithm that solves our minimization problem. It is sometimes also called the hill climbing algorithm, but that is just a naming convention depending on what your source of learning is. I'm just gonna stick with gradient descent since going downhill I feel is preferable to going uphill. This algorithms main job is reduce the value of any given function to a minimum value based on some given parameters. Gradient descent works not only for linear regression but for all sorts of minimization problems in machine learning. Minimizing the cost function for linear regression is only one small application of gradient descent.

We are dealing with univariate linear regression right now, but the same algorithm would work in the multivariate case , i.e. , over an arbitrary number of parameters $\theta_0, \theta_1, \ldots, \theta_n$. This is useful to know for the future , but for right now we will only be minimizing with respect to our two parameters θ_0 and θ_1 , i.e. , we are going to be running gradient descent on our linear regression cost function $J(\theta_0, \theta_1)$.

Intuition

Try to imagine yourself standing on top of a grassy hill. You now want to get back downhill. What is the easiest and fastest way to do that ? You can just look around 360° , and take a step in which ever direction seems to have the biggest slope downhill. This seems like best most intuitive way to ensure that at every moment you are constantly heading downhill.

One caveat to mention here is that you are also extremely short sighted so you can only see enough for one step forward , so you cannot go up for a while for a bigger descent later.

This is basically what gradient descent algorithm does. More formally the algorithm is defined as follows :

```
repeat until convergence {
    $\theta_j := \theta_j - \alpha \cdot \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$    (for j=0 and j=1)
}
```

Convergence means that after running the algorithm another time , the value of $theta_j$ will not change. In practice however we basically wait until there is no **significant** change in our value for θ_j. This is because it might not be worth waiting around for another day or a week just to get another 0.0001% increase in accuracy for your model. As the machine learning engineer it would be up to you to decide how accurate you want it to be and how long you are willing to wait for it get that accurate.

Also note that in the equation above := indicates assignment and not equality.

Following is the caculation of the partial derivative term from the gradient descent algorithm for linear regression.

$$
\begin{aligned}
&\frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \\
&= \frac{\partial}{\partial \theta_j} \left(\frac{1}{2} \cdot \frac{1}{m} \cdot \sum_{i=0}^{m} \left\{ \left(h_\theta(x)^{(i)} - y^{(i)} \right)^2 \right\} \right) \\
&= \frac{\partial}{\partial \theta_j} \left(\frac{1}{2} \cdot \frac{1}{m} \cdot \sum_{i=0}^{m} \left\{ \left(\left(\theta_0 + \theta_1 \cdot x^{(i)} \right) - y^{(i)} \right)^2 \right\} \right)
\end{aligned}
$$

This is straighforward because we only have to do the differential for two values of θ_j , namely θ_0 and θ_1 :

$$
\begin{aligned}
&\frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1) \\
&= \frac{\partial}{\partial \theta_0} \left(\frac{1}{2} \cdot \frac{1}{m} \cdot \sum_{i=0}^{m} \left\{ \left(\left(\theta_0 + \theta_1 \cdot x^{(i)} \right) - y^{(i)} \right)^2 \right\} \right) \\
&= \quad \left(\frac{1}{m} \cdot \sum_{i=0}^{m} \left\{ \left(h_\theta(x)^{(i)} - y^{(i)} \right)^2 \right\} \right)
\end{aligned}
$$

$$\begin{aligned}
&\frac{\partial}{\partial\theta_1} J(\theta_0,\theta_1)\\
&= \frac{\partial}{\partial\theta_1}\left(\frac{1}{2}\cdot\frac{1}{m}\cdot\sum_{i=0}^{m}\left\{\left(\left(\theta_0+\theta_1\cdot x^{(i)}\right)-y^{(i)}\right)^2\right\}\right)\\
&= \quad\left(\frac{1}{m}\cdot\sum_{i=0}^{m}\left\{\left(h_\theta(x)^{(i)}-y^{(i)}\right)^2\right\}\right)\cdot(x)^{(i)}
\end{aligned}$$

Now that we have a way of getting new and improved θ_0 and θ_1 values , we have to actually somehow put these new values into our old equation. This means that we need an update rule.

1.1.5. Update Rule

One thing to keep in mind when implementing gradient descent , is the correct order of the parameter update operation. We want to update our parameters all at the same time , because if we dont do that then we are running gradient descent on an effectively different paramter set than the one we want to optimize over.

Following is the **INCORRECT** way of implementing the update rule :

$$\begin{aligned}
temp0 &:= \theta_0 - \alpha\cdot\frac{\partial}{\partial\theta_0}J(\theta_0,\theta_1)\\
\theta_0 &:= temp0\\
temp1 &:= \theta_1 - \alpha\cdot\frac{\partial}{\partial\theta_1}J(\theta_0,\theta_1)\\
\theta_1 &:= temp1
\end{aligned}$$

Following is the **CORRECT** way to implement a simultaneous update.

$$\begin{aligned}
temp0 &:= \theta_0 - \alpha\cdot\frac{\partial}{\partial\theta_0}J(\theta_0,\theta_1)\\
temp1 &:= \theta_1 - \alpha\cdot\frac{\partial}{\partial\theta_1}J(\theta_0,\theta_1)\\
\theta_0 &:= temp0\\
\theta_1 &:= temp1
\end{aligned}$$

Basically do all your calculations before you do all your assignments.

This update rule marks the end of our basic univariate linear regression algorithm.

1.2. Multivariate Linear Regression

1.2.1. Motivation

The motivation at this point is to generalize our existing linear regression algorithm. This means that we want to get out of two dimensions where we are only using the equation of a line and two parameters θ_0 and θ_1 , and build a version of linear regression that can use an arbitrary number of parameters $\theta_0, \theta_1, \ldots, \theta_n$. This means that we are going to have a whole bunch of equations , and a whole bunch of derivatives we are going to have to calculate when doing gradient descent. Since we have to solve a large number of equations simultaneously , the best way would be to use Matrices because that is kind of their raison d'être.

But before we do any calculations , we need to transform our existing equations into their new vector / matrix forms and deal with some more notation.

1.2.2. Hypothesis Equation

Independant / Input Variable / Feature : x

Number of features x : n , $n \in \mathbb{N}$

Output Variable / Dependant Variable / Target : y

Number of training examples y : m , $m \in \mathbb{N}$

the j^{th} feature : x_j for $j \in \{1, \dots, n\}$, where features are columns in a table

the i^{th} training set / vector : $x^{(i)}$ for $i \in \{1, \dots, m\}$, $x^{(i)} \in \mathbb{R}^n$, where indexes are rows in a table

value of feature j in i^{th} training example : $x_j^{(i)}$, in an $m \times n$ data table it is the value of the i^{th} row , j^{th} column

i^{th} training example : $(x_j^{(i)}, y^{(i)})$ Learning Rate : α So continuing with our housing prices example from univariate linear regression. Let us now assume that we have a dataset of housing prices , for four houses x_0, x_1, x_2, x_3. We can represent these in a 1-dimensional vector as follows :

$$X = \text{Housing Prices} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We want to do linear regression , i.e. , we want to figure out house prices in the future given data we already have.

In addition to the prices of the house , we have two features that we know affect the house price. Lets assume they are square footage , and number of bedrooms. These can be represented by the θ_0 , and θ_1 respectively. So far everything look similar to what we have seen before. But this time , the only difference from what we did in univariate regression earlier is that instead of just having one x , we have multiple x_0, x_1, x_2, x_3 values represented in a vector X. Which means that we have multiple hypothesis equations :

$$\begin{aligned} h_\theta(x_0) &= \theta_0 + \theta_1 x_0 \\ h_\theta(x_1) &= \theta_0 + \theta_1 x_1 \\ h_\theta(x_2) &= \theta_0 + \theta_1 x_2 \\ h_\theta(x_3) &= \theta_0 + \theta_1 x_3 \end{aligned}$$

What we want to do , is to represent all the equations above in one succinct form. To do this we need 1 vector for each one of our features θ , and 1 for all our input variables X. The challenge right now is that we have to split up $\theta_1 x$, into an independent X vector and θ_1 vector , so that we can represent this multiplication in terms of matrices.

So first step is to transform our 1-D Housing prices vector (X) to a 2-D matrix :

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}$$

Second step , represent , our parameter values as a vector

$$\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}$$

Now we do matrix multiplication which easily and succinctly gives us our predicted $h_\theta(x)$ values for all four house prices.

$$\begin{aligned} Prediction &= Data & \times Parameters \\ \begin{bmatrix} h_\theta(x_0) \\ h_\theta(x_1) \\ h_\theta(x_2) \\ h_\theta(x_3) \end{bmatrix} &= \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} & \times \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \\ &= \begin{bmatrix} (\theta_0 \cdot 1) + (\theta_1 \cdot x_0) \\ (\theta_0 \cdot 1) + (\theta_1 \cdot x_1) \\ (\theta_0 \cdot 1) + (\theta_1 \cdot x_2) \\ (\theta_0 \cdot 1) + (\theta_1 \cdot x_3) \end{bmatrix} \end{aligned}$$

Not only is this form cleaner to look and more efficient to solve. It provides a huge advantage in that we can optimize our parameters , and minimize the cost function equation $J(\theta_0, \dots, \theta_n)$ very efficiently without having to go through an iterative process using gradient descent like we were doing earlier in univariate linear regression.

Which means that the more input variables and features that we have , the more this matrix multiplication method becomes useful and at a certain point in fact necessary. Not to mention the matrix notation tends to take up less space and look cleaner so a bunch of people prefer it. It can make you want to tear your hair out if you haven't completely understood what is going on though. kek. Which is the exactly why I am sitting here going line by line and typing it all out. Mainly for me to understand it but on the one in a billion chance that anyone else actually reads this , then also for the sake your hairline.

Now that we understand how it works for a finite number of houses $x_0, \dots, x_3$, and parameters θ_0 , θ_1 , our job is to now transform this further into an equation for an arbitrary number of input vectors (houses) $x_0, x_1, \dots, x_n$, and an arbitrary number of parameters $\theta_0, \theta_1, \dots, \theta_n$.

Multivariate Linear Regression : General Form

The equation is written generally in the form :

$$\hat{y} = \theta_0 + \theta_1 \cdot x_1 + \theta_2 \cdot x_2 + \ldots + \theta_n \cdot x_n$$

The dot product of two vectors is the sum of the products of elements with regards to position. The first element of the first vector is multiplied by the first element of the second vector and so on.

in vector form , we can express the equation as a dot product of two vectors :

$$\Theta_{(n+1\times 1)} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \in \mathbb{R}^{n+1} \quad , \ \Theta^T_{(1\times n+1)} = \begin{bmatrix} \theta_0, \theta_1, \theta_2, \cdots, \theta_n \end{bmatrix} \quad , \ \vec{x}_{(n+1\times 1)} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$\begin{aligned} h_\theta(\vec{x}) &= \Theta^T_{(1\times n+1)} \cdot \vec{x}_{(n+1\times 1)} \\ &= \begin{bmatrix} \theta_0, \theta_1, \theta_2, \cdots, \theta_n \end{bmatrix}_{(1\times n+1)} \cdot \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{(n+1\times 1)} \\ &= \begin{bmatrix} x_0\theta_0 + x_1\theta_1 + x_2\theta_2 + \cdots + x_n\theta_n \end{bmatrix}_{(1\times 1)} \end{aligned}$$

In the case that we have multiple features , each feature will contain multiple training examples. Which means each one is own vector. As an example $x^{(1)}$ will contain all the data for the 1¡sup¿st¡/sup¿ training example , and similarly $x^{(i)}$ will contain the input data for the i¡sup¿th¡/sup¿ training example. A training example with respect to the house price prediction problem , would be a full set of data like house size , rooms , bathrooms , etc ... , whereas a data point would be the specific house size or number of rooms for a certain house. So we will have m houses , and n number of things per house to measure and compare.

$$\vec{x}^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} , \dots , \ \vec{x}^{(i)} = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix} , \dots , \ \vec{x}^{(m)} = \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ \vdots \\ x_n^{(m)} \end{bmatrix}$$

Just like before we define an $\vec{x}^{(0)}$, and add it to our final matrix X so that the dimensions of the input matrix will match the dimensions of the parameter vector Θ. This allows us to solve all the equations through a simple inner product.

$$\vec{x}^{(0)} = \begin{bmatrix} x_1^{(0)} = 1 \\ x_2^{(0)} = 1 \\ \vdots \\ x_n^{(0)} = 1 \end{bmatrix}$$

$$X_{m \times n+1} = \begin{bmatrix} \vec{x}^{(0)} & \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(m)} \end{bmatrix}$$

$$= \begin{bmatrix} x_1^{(0)} & x_1^{(1)} & x_1^{2)} & \cdots & x_0^{(m)} \\ x_2^{(0)} & x_2^{(1)} & x_2^{(2)} & \cdots & x_1^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{(0)} & x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(m)} \end{bmatrix}$$

$$h_\theta(X) = \Theta^T_{(1 \times n+1)} \cdot X_{(n+1 \times m)}$$

$$= \begin{bmatrix} h_\theta(x_0) \\ h_\theta(x_1) \\ h_\theta(x_2) \\ \vdots \\ h_\theta(x_m) \end{bmatrix}_{(1 \times m)} = \begin{bmatrix} \theta_0, \theta_1, \theta_2, \cdots, \theta_n \end{bmatrix}_{(1 \times n+1)} \cdot \begin{bmatrix} x_1^{(0)} & x_1^{(1)} & x_1^{(2)} & \cdots & x_0^{(m)} \\ x_2^{(0)} & x_2^{(1)} & x_2^{(2)} & \cdots & x_1^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n^{(0)} & x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(m)} \end{bmatrix}_{(n+1 \times m)}$$

$$= \begin{bmatrix} h_\theta(x_0) \\ h_\theta(x_1) \\ h_\theta(x_2) \\ \vdots \\ h_\theta(x_j) \\ \vdots \\ h_\theta(x_m) \end{bmatrix}_{(1 \times m)} = \begin{bmatrix} x_0^{(1)}\theta_0 + x_1^{(1)}\theta_1 + x_2^{(1)}\theta_2 + \cdots + x_n^{(1)}\theta_n \\ x_0^{(2)}\theta_0 + x_1^{(2)}\theta_1 + x_2^{(2)}\theta_2 + \cdots + x_n^{(2)}\theta_n \\ \vdots \\ x_j^{(i)}\theta_0 + x_j^{(i)}\theta_1 + x_j^{(i)}\theta_2 + \cdots + x_j^{(i)}\theta_n \\ \vdots \\ x_0^{(m)}\theta_0 + x_1^{(m)}\theta_1 + x_2^{(m)}\theta_2 + \cdots + x_n^{(m)}\theta_n \end{bmatrix}_{(1 \times m)}$$

$$0 \le i \le m \ , \ 0 \le j \le n \ , \ m, n \in \mathbb{N}$$

value of feature j in i^{th} training example : $x_j^{(i)}$, in an $n \times m$ data table it is the value of the j^{th} row , i^{th} column

$\hat{y} = \Theta^T \bullet X$ is the equation of a line , when we are dealing with more than one input variable then we will be dealing with the equation of a plane , and when we have even more then the equation of a hyper-plane.

1.2.3. Cost Function

1.2.4. Gradient Descent

1.2.5. Update Rule

2. Logistic Regression

2.1. Motivation

Logistic Regression is a classification algorithm. The objective of logistic regression is to categorize unseen datapoints into one of multiple categories. First we will deal with the simple case for explanation , where we only have to categorize an input data point into one of two categories. Namely it has to be either true / false , 1/0 , spam / not spam , malignant / benign etc . . . So we have :

$$y \in \{0, 1\}$$

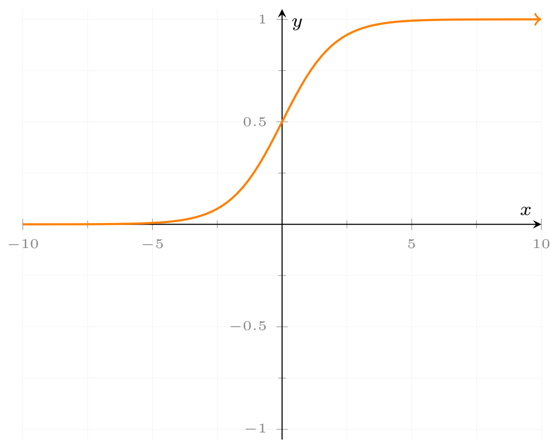
Where 0 is also known as the negative class (e.g. benign , not spam , false . . .) , and 1 is the positive class (e.g. malignant , spam , true . . .). Later we will deal with more than two classes so y will look like :

$$y \in \{0, 1, 2, 3, \ldots, n\}$$

The linear regression hypothesis is a good starting point but our hypothesis $h_\theta(x)$ can be any continuous value.

However we only want it predict binary values 0 or 1. We need to find a way to restrict it to this range , i.e. we want $0 \leq h_\theta(x) \leq 1$, where $h_\theta(x) = \theta_0 + \theta_1 x$ or in vector notation : $h_\theta(X) = \Theta^T X$. To achieve this goal we will be using the logistic function (also called the sigmoid). Given by the equation :

$$g(x) = \frac{1}{1+e^{(-x)}}$$



This is a function that will always output a value between 1 and 0 , i.e. , $0 \leq h_\theta(x) \leq 1$. So we just take the output of our hypothesis function , and pipe it through to the sigmoid in order to map it down to the range we want. Which looks like :

$$\begin{aligned} h_\theta(x) &= g(\theta_0 + \theta_1 x) = \frac{1}{1+e^{-(\theta_0+\theta_1 x)}} \\ h_\theta(X) &= g(\Theta^T X) \quad = \frac{1}{1+e^{-(\Theta^T X)}} \end{aligned}$$

However $h_\theta(x) \in [0, 1]$ or rather we are getting continuous values in the interval between 0 and 1 and not really discrete values of 0 and 1 , which is what the end goal of our binary classifier should be. Before we move on and learn how to get discrete values , we should note that the values on this continuous interval can also be considered useful. The hypothesis values on this continuous interval represent the probability of x being in the positive class. So the higher our $h_\theta(x)$ value , the higher the probability that x is 1 (or true , or spam , or malignant etc ...)

$$h_\theta(x) = P(y = 1|x; \theta)$$

$$P(y = 1|x; \theta) + P(y = 0|x; \theta) = 1$$

These equations also show that now , our hypothesis h_θ is giving us a probability value between 1 and 0 of how likely it is that our hypothesis is true. The probability that the hypothesis is true is 0 at x = 0 and steadily increases as $x \to 1$. The function $h_\theta(x)$ is also asymptotic at 1 and 0.

Anyway , what we actually want is an equation that classifies things into groups of true or not true , and not an equation that predicts probabilities of whether or not something is true. That is easy to do from what we already have. All we have to do is come up with some **decision boundary** . This means that if $x \geq$ some threshold value then we predict true or 1 , and if $x >$ some threshold value then we predict false or 0. The equals can go on either side of these two threshold values it doesn't really matter and is up to you to set your own threshold values. One easy to pick threshold value is just 0.5 (which is also the y intercept) , i.e., as long as $h_\theta(x) \geq 0.5$ we will predict 1 (or true , or spam ...) and 0 otherwise.

2.2. Hypothesis Equation

2.3. Cost Function

Training set :

$$\left\{\left(x^{(1)},y^{(1)}\right),\left(x^{(2)},y^{(2)}\right),\left(x^{(3)},y^{(3)}\right),\dots\left(x^{(m)},y^{(m)}\right)\right\}$$

where we have m samples / examples for each data point x :

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}_{1\times(n+1)} \in \mathbb{R}^{n+1}\ ,\ x_0 = 1\ ,\ y \in 0,1$$

There is no concept of residuals in logistic regression as compared to linear regression. As a reminder a residual is the difference between the prdicted value and the actual value. This doesnt make any sense as a concept when we are trying to make discrete predictions.

This means that we cannot use least squares for figuring our how much error there was in our prediction like we did in linear regression.

The new thing that we use is called Maximum Likelihood Estimation. The goal of maximum likelihood is to find the optimal way to fit a distribution to the data. Basically , whenever we have a data set we can fit a distribution to it.

So lets assume that the data you have can be fit according to a normal distribution. But the quesiton is where and how do we center the normal distribution. This is where maximum likelihood comes in. We basically just start at 0 and go till our maximum value centering our distribution at each value. The maximum likelihood equation tells us , how likely is it that this is the correct center given the data that we have. We just do this for every value possible in the range that we have for our values. Once we have tried (used the likelihood function on) all of the locations possible for our center , we just pick the center that gives the maximum output from the likelihood function. Since in this example we are dealing with the normal distribution ,

[ii]

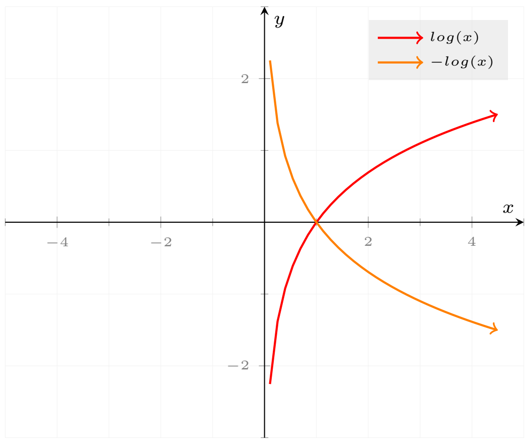
Which means that we have a new cost function $J(\theta)$:

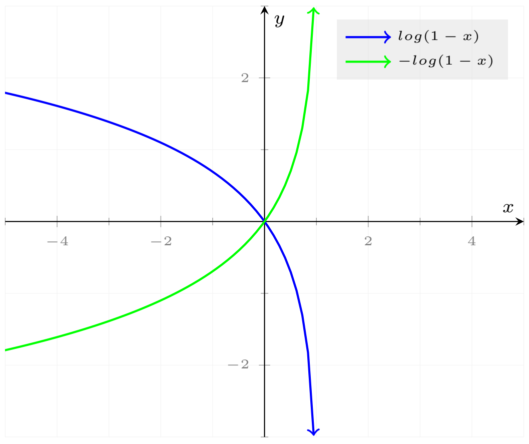
$$\begin{aligned} J(\theta) &= \frac{1}{m}\sum_{i=1}^{m}\left\{\mathrm{Cost}(h_\theta(x^{(i)}),y^{(i)})\right\} \\ \mathrm{Cost}(h_\theta(x),y) &= \begin{cases} -\log(h_\theta(x)) & \text{if y = 1} \\ -\log(1-h_\theta(x)) & \text{if y = 0} \end{cases} \end{aligned}$$

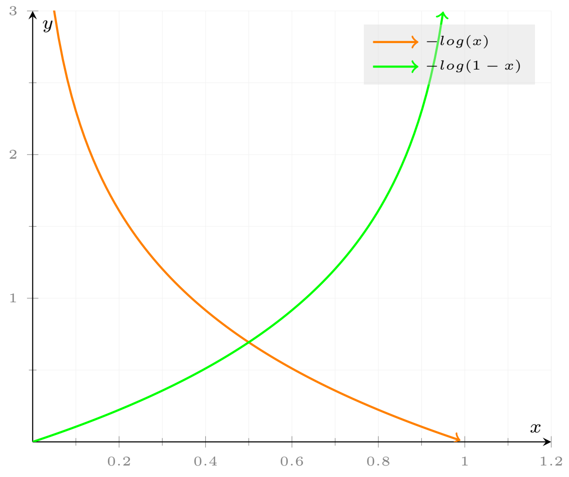
$$\mathrm{Cost}(h_\theta(x),y)\begin{cases} = 0 & \text{if } h_\theta(x) = y \\ \to \infty & \text{if } y = 0 \text{ and } h_\theta(x) \to 1 \\ \to \infty & \text{if } y = 1 \text{ and } h_\theta(x) \to 0 \end{cases}$$

Case y = 1

[ii] Good explainer video : https://www.youtube.com/watch?v=XepXtl9YKwc







Taking only the negative log from the graph and then using the interval only between $0 \le x \le 1$, we get the following graph :

Case y = 0

$$
\begin{aligned}
\text{Cost}(h_\theta(x), y) &= 0 \quad \text{if } h_\theta(x) = y \\
\text{Cost}(h_\theta(x), y) &\to \infty \text{ if } y = 0 \text{ and } h_\theta(x) \to 1 \\
\text{Cost}(h_\theta(x), y) &\to \infty \text{ if } y = 1 \text{ and } h_\theta(x) \to 0
\end{aligned}
$$

$$
\begin{aligned}
J(\theta) &= \frac{1}{m} \sum_{i=1}^{m} \left\{ \text{Cost}(h_\theta(x^{(i)}), y^{(i)}) \right\} \\
\text{Cost}(h_\theta(x), y) &= \begin{cases} -\log(h_\theta(x)) & \text{if y = 1} \\ -\log(1 - h_\theta(x)) & \text{if y = 0} \end{cases} \\
&= -y \log(h_\theta(x)) + (1 - y) \log(1 - h_\theta(x))
\end{aligned}
$$

$$\begin{aligned} J(\theta) &= \frac{1}{m}\sum_{i=1}^{m}\left\{\text{Cost}(h_\theta(x^{(i)}),y^{(i)})\right\} \\ &= \frac{1}{m}\left(\sum_{i=1}^{m}\left\{-y^{(i)}\log(h_\theta(x^{(i)}))+(1-y^{(i)})\log(1-h_\theta(x^{(i)}))\right\}\right) \end{aligned}$$

2.4. Gradient Descent

¡h3¿Logistic Regression : Gradient Descent¡/h3¿

Just for reference cause we will need it later.

$$\nabla J(\Theta)= \left\langle \frac{\partial}{\partial\theta_0}J(\Theta), \frac{\partial}{\partial\theta_1}J(\Theta), \frac{\partial}{\partial\theta_2}J(\Theta)\ldots, \frac{\partial}{\partial\theta_m}J(\Theta)\right\rangle$$

$$\begin{aligned} \sigma(x) &= \tfrac{1}{1+e^{-x}} \\ \tfrac{d}{dx}\sigma(x) &= \tfrac{d}{dx}\left(\tfrac{1}{1+e^{-x}}\right) \\ &= \tfrac{0\cdot(1+e^{-x})-(-e^{-x})\cdot 1}{(1+e^{-x})^2} && \text{Quotient rule} \\ &= \tfrac{e^{-x}}{(1+e^{-x})^2} \\ &= \tfrac{(1+e^{-x})-1}{(1+e^{-x})^2} \\ &= \tfrac{(1+e^{-x})}{(1+e^{-x})^2}-\left(\tfrac{1}{1+e^{-x}}\right)^2 \\ &= \tfrac{1}{1+e^{-x}}-\left(\tfrac{1}{1+e^{-x}}\right)^2 \\ &= \sigma(x)-\sigma(x)^2 \\ \sigma^{'}(x) &= \sigma(x)(1-\sigma(x)) \end{aligned}$$

We also have the following :

$$h_\theta(\Theta^T x) = \sigma(\Theta^T x)= \tfrac{1}{1+e^{-\Theta^T x}}$$

$$\begin{aligned} P(y^{(i)} = 1|x^{(i)};\theta) &= h_\theta(x) &&= \sigma(x) &&= \tfrac{1}{1+e^{-x}} \\ P(y^{(i)} = 0|x^{(i)};\theta) &= 1-h_\theta(x) &&= 1-\sigma(x) &&= 1-\tfrac{1}{1+e^{-x}} \end{aligned}$$

This gives us the overall likelihood of any outcome y given x and theta as :

$$L(\theta)= P(y^{(i)}|x^{(i)};\theta)= h(x^{(i)})^{y^{(i)}}\cdot(1-h(x^{(i)}))^{1-y^{(i)}}$$

overall i's

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{m}\left\{h\left(x^{(i)}\right)^{\left(y^{(i)}\right)}\cdot\left(1-h(x^{(i)})\right)^{\left(1-y^{(i)}\right)}\right\} \\ \log(L(\theta)) &= \mathscr{L}(\theta) \\ \mathscr{L}(\theta) &= \log\left(\prod_{i=1}^{m}\left\{h\left(x^{(i)}\right)^{\left(y^{(i)}\right)}\cdot\left(1-h(x^{(i)})\right)^{\left(1-y^{(i)}\right)}\right\}\right) \\ &= \sum_{i=1}^{m}\left\{\log\left(h\left(x^{(i)}\right)^{\left(y^{(i)}\right)}\cdot\left(1-h(x^{(i)})\right)^{\left(1-y^{(i)}\right)}\right)\right\} \\ &= \sum_{i=1}^{m}\left\{\log\left(h\left(x^{(i)}\right)^{\left(y^{(i)}\right)}\right)+\log\left(\left(1-h(x^{(i)})\right)^{\left(1-y^{(i)}\right)}\right)\right\} \\ &= \sum_{i=1}^{m}\left\{\left(y^{(i)}\right)\cdot\log\left(h\left(x^{(i)}\right)\right)+\left(1-y^{(i)}\right)\cdot\log\left(\left(1-h\left(x^{(i)}\right)\right)\right)\right\} \end{aligned}$$

We want to maximize the likelihood so we take the derivative :

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathscr{L}(\theta) &= \frac{\partial}{\partial\theta}\left(\sum_{i=1}^{m}\left\{\left(y^{(i)}\right)\cdot\log\left(h\left(x^{(i)}\right)\right)+\left(1-y^{(i)}\right)\cdot\log\left(\left(1-h\left(x^{(i)}\right)\right)\right)\right\}\right)\\
&= \frac{\partial}{\partial\theta}\left(\sum_{i=1}^{m}\left\{\left(y^{(i)}\right)\cdot\log\left(\sigma\left(x^{(i)}\right)\right)+\left(1-y^{(i)}\right)\cdot\log\left(\left(1-\sigma\left(x^{(i)}\right)\right)\right)\right\}\right)\\
&= \sum_{i=1}^{m}\left\{\left(y^{(i)}\right)\cdot\frac{\partial}{\partial\theta}\left(\log\left(\sigma\left(x^{(i)}\right)\right)\right)+\left(1-y^{(i)}\right)\cdot\frac{\partial}{\partial\theta}\left(\log\left(\left(1-\sigma\left(x^{(i)}\right)\right)\right)\right\}\right)\\
&= \sum_{i=1}^{m}\left\{\left(y^{(i)}\right)\cdot\left(\frac{1}{\sigma\left(x^{(i)}\right)}\cdot\sigma\prime\left(x^{(i)}\right)\right)+\left(1-y^{(i)}\right)\cdot\left(\frac{1}{1-\sigma\left(x^{(i)}\right)}\cdot-\sigma\prime\left(x^{(i)}\right)\right)\right\}\text{Chain Rule}\\
&=
\end{aligned}$$

Gradient Descent

$$\begin{aligned}
\tfrac{\partial}{\partial\theta_0}J(\Theta) &= \left(\tfrac{1}{m}\cdot\textstyle\sum_{i=1}^{m}\left\{\left(h_\theta(x^{(i)})-y^{(i)}\right)^2\cdot x_0^{(i)}\right\}\right)\\
\tfrac{\partial}{\partial\theta_j}J(\Theta) &= \left(\tfrac{1}{m}\cdot\textstyle\sum_{i=1}^{m}\left\{\left(h_\theta(x^{(i)})-y^{(i)}\right)^2\cdot x_j^{(i)}\right\}\right)+\tfrac{\lambda}{m}\cdot\theta_j
\end{aligned}$$

The second sum, $\frac{\lambda}{2m}\sum_{j=1}^{n}\theta_j^2$ means to explicitly exclude the bias term, θ_0. i.e. the θ vector is indexed from 0 to n (holding n+1 values, θ_0 through θ_n), and this sum explicitly skips θ_0, by running from 1 to n, skipping 0. Thus, when computing the equation, we should continuously update the two following equations:

$$\begin{aligned}
\theta_0 &:= \theta_0-\alpha\cdot\left(\tfrac{1}{m}\cdot\textstyle\sum_{i=1}^{m}\left\{\left(h_\theta(x^{(i)})-y^{(i)}\right)^2\cdot x_0^{(i)}\right\}\right) && \text{for } j=0\\
\theta_j &:= \theta_j-\alpha\cdot\left(\tfrac{1}{m}\cdot\textstyle\sum_{i=1}^{m}\left\{\left(h_\theta(x^{(i)})-y^{(i)}\right)^2\cdot x_j^{(i)}+\left(\tfrac{\lambda}{m}\cdot\theta_j\right)\right\}\right) && \text{for } j=1,2,\dots,n\\
&:= \theta_j\cdot(1-\alpha\cdot\tfrac{\lambda}{m})-\alpha\cdot\left(\tfrac{1}{m}\cdot\textstyle\sum_{i=1}^{m}\left\{\left(h_\theta(x^{(i)})-y^{(i)}\right)^2\cdot x_j^{(i)}\right\}\right)
\end{aligned}$$

2.5. Extra

2.5.1. Advanced Optimization Algorithms

¡h3¿Advanced Optimization Algorithms¡/h3¿

The various advanced optimization algorithms are :

¡ul¿ ¡li¿Gradient Descent¡/li¿ ¡li¿Conjugate gradient¡/li¿ ¡li¿BFGS¡/li¿ ¡li¿L-BFGS¡/li¿ ¡/ul¿

2.5.2. Multiclass Classification

¡h3¿Logistic Regression : Multiclass Classification¡/h3¿

¡p¿

To do multiclass classification , we use a method called one-vs-all (sometimes also called one-vs-rest). In this method we will train a new classfier with a spereate hypothesis function $h_\theta(x)$ for each thing we want to classify. So if we want to classify k different things , we will have k different hypothesis functions. These will be denoted using a superscript $h_\theta^{(k)}(x)$.

$$h_\theta^{(i)}(x)=P(y=i|x;\theta)\ ,\ 1\le i\le k, k\in\mathbb{N}$$

The probability equation is telling us what the probability is that the x value we are looking at belongs in the current class k. ¡br¿ ¡br¿

What this means is that for each class k , when we recieve a new data point we dont have to decide which specific class it belongs into yet. All we have to do is figure out if it belongs in the current class that we are looking at. So does it belong in the current class or one of all of the rest of the classes , doesnt matter which. Then we repeat this process until we figure out which one it belongs to , by iterative comparison of one-vs-all. ¡br¿ ¡br¿

So at the end we just pick the hypothesis that maximizes the probability that a given item x belongs in class k

:

$$\max_i h_\theta^{(i)}(x)$$

¡/p¿

2.5.3. Overfitting

¡h3¿Overfitting¡/h3¿

If we have too many features , the learned hypothesis may fit the training set very well , i.e. ,

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \left\{ \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2 \right\} \approx 0$$

but this function may fail to generalize to new examples.¡br¿ ¡br¿

¡b¿ Underfitting ¡/b¿, or ¡b¿ high bias ¡/b¿, is when the form of our hypothesis function h maps poorly to the trend of the data. It is usually caused by a function that is too simple or uses too few features.¡br¿ ¡br¿

At the other extreme, ¡b¿ overfitting ¡/b¿, or ¡b¿ high variance ¡/b¿, is caused by a hypothesis function that fits the available data but does not generalize well to predict new data. It is usually caused by a complicated function that creates a lot of unnecessary curves and angles unrelated to the data.¡br¿ ¡br¿

There are a couple of things we can do to reduce over fitting :

¡ol¿ ¡li¿Reduce number of features¡/li¿ ¡ol¿ ¡li¿Manually select which features to keep¡/li¿ ¡li¿Use a model selection algorithm¡/li¿ ¡/ol¿ ¡li¿Use Regularization¡/li¿ ¡ol¿ ¡li¿Keep all the features , but reduce the magnitude / values of the parameters θ_j ¡/li¿ ¡li¿Works well when we have a lot of features , each one of whic contribute a little bit towards predictin y¡/li¿ ¡/ol¿ ¡/ol¿

2.5.4. Regularization

¡h3¿Regularization¡/h3¿

To fix the problem of overfitting one of the things that we prefer is for the hypothesis to be as simple as possible. This means that we want to reduce the effect (as close to zero as we can) of the higher order polynomials as we can. In order to do this we can introduce a regularization term. The purpose of this term is to penalize the cost function (by assigning a higher value) when it assigns higher weight (larger values for θ_j) to higher order polynomials. This means the higher the order of the x value the more the cost function will be rewarded if it DOES NOT include it as a significant contibutor. Keep in mind we are using the same amount of features $x_(j)^{(i)}$, all we are doing is reducing the polynomial order of the features. The cost function is not being penalized for reducing the impact of the features theselves.¡br¿

The regularization term looks like :

$$\frac{1}{2} \cdot \frac{1}{m} \cdot \lambda \cdot \sum_{j=1}^{n} \left\{ \theta_j^2 \right\}$$

λ is called the regularization parameter. Its job is to control the tradeoff between two different goals. The first goal is to actually fit the training data well. This is done by our original cost function $J(\theta)$. The second goal is to keep the parameters small. This makes our Cost function now look like :

$$J(\theta) = \left(\frac{1}{2} \cdot \frac{1}{m} \cdot \left(\sum_{i=1}^{m} \left\{ \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2 \right\} \right) + \left(\frac{1}{2} \cdot \frac{1}{m} \cdot \lambda \cdot \sum_{j=1}^{n} \left\{ \theta_j^2 \right\} \right) \right)$$

Keep in mind though that if we set $lambda$ to be some huge value like

$\lambda = 10^{10}$ then we might not even achieve our primary objective of fitting the data well since we are punishing the cost function too much. This would basically mean that each $\theta_j \approx 0$, $j \geq 1$ and we would effectively only be left with $h_\theta(x) = \theta_0$ which is basically just a straight line through the intercept. This would result in a case of underfitting due to a too large λ value.

¡h3¿Regularized Linear Regression¡/h3¿

$$\min_{\theta} J(\theta) = \tfrac{1}{2} \cdot \tfrac{1}{m} \cdot \left(\left(\textstyle\sum_{i=1}^{m} \left\{ \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2 \right\} \right) + \left(\tfrac{1}{2} \cdot \tfrac{1}{m} \cdot \lambda \cdot \textstyle\sum_{j=1}^{n} \left\{ \theta_j^2 \right\} \right) \right)$$

¡h3¿Logistic Regression¡/h3¿

This is a type of classification algorithm , as opposed to linear or non-linear regression algorithms which all relied on predicting continuous values , a classifcation algirthm will use the independent variables (features) to predict binary values like : Yes / No , True / False , Malignant / Non-Malignant , Pregnant / Not-Pregnant etc... All these examples are binary beacuse that is what I am going to be using in this explanation , but we can use multiple logistic regression with vectors to extend the algorithm for multi-class classification to classify the output into as many fields as we like.¡br¿

Even though the dependant variables are discrete classifications , the independant variables x_j values should all be continous.¡br¿

Logistic Regression measures the probability of a case belonging to a specific class. Logistic Regression can be used to understand the impact of a feature on a dependant variable. ¡br¿

¡ul¿

¡li¿Input / independant Variables : $X \in \mathbb{R}^{m \times n}$¡/li¿ ¡li¿Output / dependant Variables : $y \in \{0, 1\}$¡/li¿ ¡li¿: $\hat{y} = P(y = 1|x)$¡/li¿ ¡li¿: $P(y = 0|x) = 1 - P(y = 1|x)$¡/li¿ ¡/ul¿

We can use the equations we have from linear regression as outr starting poing for logistic regression. We have :

$$\Theta_{(n\times1)} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \ , \ \Theta^T_{(1\times n)} = \begin{bmatrix} \theta_0, \theta_1, \theta_2, \cdots, \theta_n \end{bmatrix} \ , \ X_{(n\times1)} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{aligned} \hat{y} = h_\theta(x) = \Theta^T_{(1\times n)} \cdot X_{(n\times1)} &= \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}_{(1\times n)} \cdot \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{(n\times1)} \\ &= \begin{bmatrix} x_0\theta_0 + x_1\theta_1 + x_2\theta_2 + \cdots + x_n\theta_n \end{bmatrix}_{(1\times1)} \end{aligned}$$

$$\begin{aligned} \Theta^T X &= \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n \\ h_\theta(x) &= \Theta^T X \end{aligned}$$

¡h4¿The Training Process¡/h4¿

¡ol¿ ¡li¿Initialize θ¡/li¿ ¡li¿Calculate $\hat{y} = \sigma(\Theta^T X)$¡/li¿ ¡li¿Compare the output of $\hat{y}$ with actual output of cutomer , y, and record it as error.¡/li¿

$$Error = 1 - \left(\hat{y} = \sigma(\theta^T X) \right)$$

¡li¿Calcualte error for all customers $Cost = J(\theta)$¡/li¿ ¡li¿Change θ to reduce the cost¡/li¿ ¡li¿Go back to step 2¡/li¿ ¡/ol¿

$$\widehat{y} = \sigma(\theta_1 x_1 + \ldots + \theta_n x_n)$$

$$J(\theta) = -\tfrac{1}{m} \cdot \textstyle\sum_{i=1}^{m} \left\{ y^i log(\widehat{y^i}) + (1 - y^i) log(1 - \widehat{y^i}) \right\}$$

$$\tfrac{\partial}{\partial \theta_0} J(\theta_1) = -\tfrac{1}{m} \cdot \textstyle\sum_{i=1}^{m} \left\{ \left(y^i - \widehat{y^i} \right) \cdot x_1^i \right\}$$

Using gradient descent the gradient vector is :

$$\nabla J = \begin{bmatrix} \frac{\partial J(\theta_1)}{\partial \theta_1} \\ \frac{\partial J(\theta_2)}{\partial \theta_2} \\ \vdots \\ \frac{\partial J(\theta_k)}{\partial \theta_k} \end{bmatrix}$$

update equation :

$$\theta_{new} = \theta_{prev} - \eta \nabla J$$

3. References