CS7643: Deep Learning
Spring 2022
Problem Set 0

Instructor: Zsolt Kira

Due: Wednesday, January 18, 11:59pm AOE

Instructions

1. PS0 will not be graded however you are required to make a submission on gradescope (submission is worth some points).
2. We generally encourage you to collaborate with other students. You may talk to a friend, discuss the questions and potential directions for solving them. However, you need to write your own solutions and code separately, and *not* as a group activity. Please list the students you collaborated with.
   Exception: PS0 is meant to serve as a background preparation test. You must NOT collaborate on PS0.
3. We encourage LaTeX but handwritten solutions are fine for PS0. For LaTeX you can use a generic template and write down the answer to each question or the template of your choice as long as it is clear which answer corresponds to which question.
4. Regardless of how you complete the assignment, collate all the answers into a single pdf and submit to "Problem Set 0" assignment in Gradescope. **When submitting to Gradescope, make sure you select/tag ONLY corresponding pages for each question. This is how other assignment reports and/or problem sets will be collected; consider this an on-boarding example for this concept. Failure to follow this may result in that part of the submission not being graded.**

1 Multiple Choice Questions

1. (1 point) true/false We are machine learners with a slight gambling problem (very different from gamblers with a machine learning problem!). Our friend, Bob, is proposing the following payout on the roll of a dice:

$$\text{payout} = \begin{cases} \$2 & x = 1 \\ -\$1/4 & x \neq 1 \end{cases} \tag{1}$$

where $x \in \{1, 2, 3, 4, 5, 6\}$ is the outcome of the roll, $(+)$ means payout to us and $(-)$ means payout to Bob. Is this a good bet i.e. are we expected to make money?

◯ True ◯ False

2. (1 point) X is a continuous random variable with the probability density function:

$$p(x) = \begin{cases} 4x & 0 \leq x \leq 1/2 \\ -4x + 4 & 1/2 < x \leq 1 \end{cases} \tag{2}$$

Which of the following statements are true about equation for the corresponding cumulative density function (CDF) $C(x)$?
[*Hint:* Recall that CDF is defined as $C(x) = Pr(X \leq x)$.]

◯ $C(x) = 2x^2$ for $0 \leq x \leq 1/2$
◯ $C(x) = -2x^2 + 4x - 1$ for $1/2 < x \leq 1$
◯ All of the above
◯ None of the above

3. (2 point) A random variable x in standard normal distribution has the following probability density

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{3}$$

Evaluate following integral

$$\int_{-\infty}^{\infty} p(x)(ax^2 + bx + c)dx \tag{4}$$

[*Hint:* We are not sadistic (okay, we're a little sadistic, but not for this question). This is not a calculus question.]

◯ a + b + c ◯ c ◯ a + c ◯ b + c

4. (2 points) Consider the following function of $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$:

$$f(\mathbf{x}) = \sigma\left(\log\left(5\left(\max\{x_1, x_2\} \cdot \frac{x_3}{x_4} - (x_5 + x_6)\right)\right) + \frac{1}{2}\right) \tag{5}$$

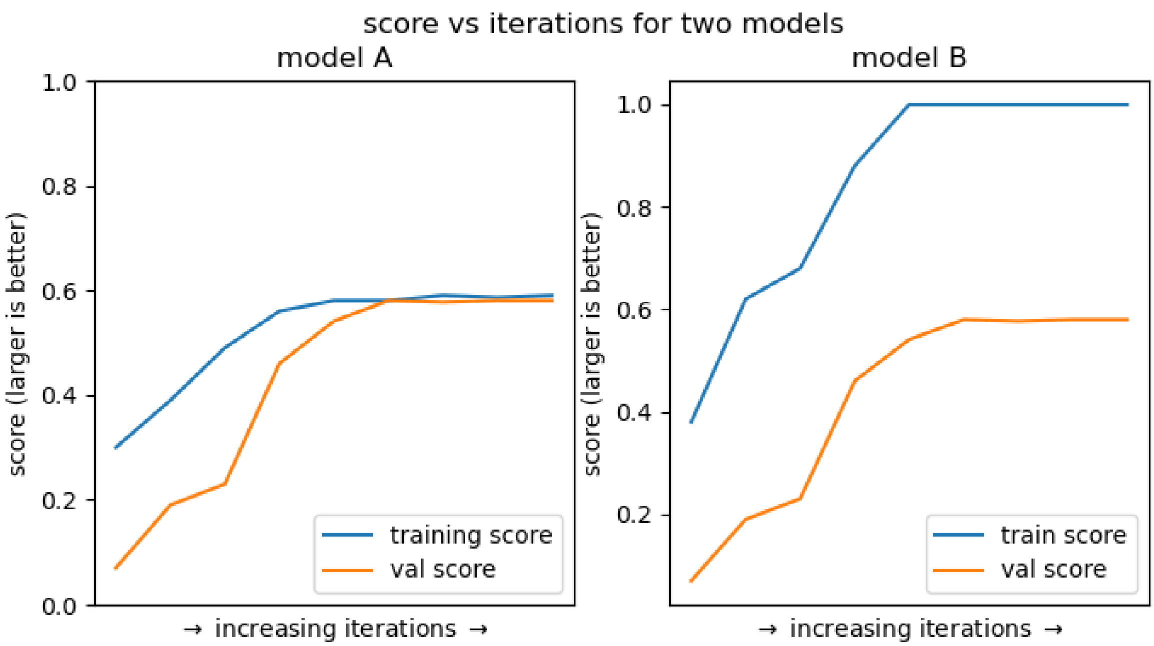
where σ is the sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{6}$$

Compute the gradient $\nabla_{\mathbf{x}} f(\cdot)$ and evaluate it at at $\hat{\mathbf{x}} = (-1, 3, 4, 5, -5, 7)$.

○ $\begin{bmatrix} 0 \\ 0.031 \\ 0.026 \\ -0.013 \\ -0.062 \\ -0.062 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0.157 \\ 0.131 \\ -0.065 \\ -0.314 \\ -0.314 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0.358 \\ 0.269 \\ -0.215 \\ -0.846 \\ -0.846 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0.358 \\ 0.269 \\ -0.215 \\ -0.448 \\ -0.448 \end{bmatrix}$

5. (2 points) Suppose your machine learner friend trains two classification models A and B on the same data set. Their training routine produces the following learning curves. You are told one of the models has higher model complexity (by some sufficient definition of model complexity). Generally this would lead us to believe which of the following is True?



○ Model A complexity > Model B complexity

○ Model A complexity < Model B complexity

6. (2 points) Which of the following functions are convex?

○ $\|\mathbf{x}\|_{\frac{1}{2}}$

○ $\min_{i=1}^{k} \mathbf{a}_i^T \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, and a finite set of arbitrary vectors: $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$

- ◯ $\log\left(1 + \exp(\mathbf{w}^T\mathbf{x})\right)$ for $\mathbf{w} \in \mathbb{R}^d$
- ◯ All of the above

7. (2 points) Suppose you want to predict an unknown value $Y \in \mathbb{R}$, but you are only given a sequence of noisy observations $x_1, \ldots, x_n$ of Y with i.i.d. noise ($x_i = Y + \epsilon_i$). If we assume the noise is I.I.D. Gaussian ($\epsilon_i \sim N(0, \sigma^2)$), the maximum likelihood estimate ($\hat{y}$) for Y can be given by:
   - ◯ A: $\hat{y} = \text{argmin}_y \sum_{i=1}^{n} (y - x_i)^2$
   - ◯ B: $\hat{y} = \text{argmin}_y \sum_{i=1}^{n} |y - x_i|$
   - ◯ C: $\hat{y} = \frac{1}{n} \sum_{i=1}^{n} x_i$
   - ◯ Both A & C
   - ◯ Both B & C

2 Proofs

This proof section is optional but highly encouraged.

8. (3 points) Prove that

$$\log_e x \leq x - 1, \qquad \forall x > 0 \tag{7}$$

with equality if and only if $x = 1$.

[*Hint:* Consider differentiation of $\log(x) - (x - 1)$ and think about concavity/convexity and second derivatives.]

9. (6 points) Consider two discrete probability distributions p and q over k outcomes:

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i = 1 \tag{8a}$$

$$p_i > 0, q_i > 0, \quad \forall i \in \{1, \ldots, k\} \tag{8b}$$

The Kullback-Leibler (KL) divergence (also known as the *relative entropy*) between these distributions is given by:

$$KL(p, q) = \sum_{i=1}^{k} p_i \log \left(\frac{p_i}{q_i} \right) \tag{9}$$

It is common to refer to $KL(p, q)$ as a measure of distance (even though it is not a proper metric). Many algorithms in machine learning are based on minimizing KL divergence between two probability distributions. In this question, we will show why this might be a sensible thing to do.

[*Hint:* This question doesn't require you to know anything more than the definition of $KL(p, q)$ and the identity in Q8]

(a) Using the results from Q8, show that $KL(p, q)$ is always non-negative.

(b) When is $KL(p, q) = 0$?

(c) Provide a counterexample to show that the KL divergence is not a symmetric function of its arguments: $KL(p, q) \neq KL(q, p)$

10. (6 points) In this question, we will get familiar with a fairly popular and useful function, called the log-sum-exp function. For $\mathbf{x} \in \mathbb{R}^n$, the log-sum-exp function is defined (quite literally) as:

$$f(\mathbf{x}) = \log\left(\sum_{i=1}^{n} e^{x_i}\right) \tag{10}$$

(a) Prove that $f(\mathbf{x})$ is differentiable everywhere in $\mathbb{R}^n$.

[*Hint:* Multivariable functions are differentiable if the partial derivatives exist and are continuous.]

(b) Prove that $f(\mathbf{x})$ is convex on $\mathbb{R}^n$.
[*Hint:* One approach is to use the second-order condition for convexity.]

(c) Show that $f(\mathbf{x})$ can be viewed as an approximation of the max function, bounded as follows:

$$\max\{x_1, \ldots, x_n\} \leq f(\mathbf{x}) \leq \max\{x_1, \ldots, x_n\} + \log(n) \tag{11}$$