ISYE 6669 Deterministic Optimization

Homework 2

January 30th, 2021

1.

Expand the following summations:
(For example, the answer to part (a) is $x_1 + x_2 + x_3$.)

(a) $\sum_{i=1}^{3} x_i$

(b) $\sum_{t=1}^{3} \frac{x^{2t}}{t!}$

(c) $\sum_{i=1}^{3} \sum_{j=1}^{i} x^{i+j}$

(d) $\sum_{i=1}^{2} \sum_{j=2}^{4} (x_{ij} - y_i)$

(e) $\sum_{k=-1}^{1} (2k+1) x_{k-1}^2$

(f) $\sum_{n=2}^{4} \sum_{m=n}^{n+2} x_n y_m$

Note that by definition $t! = 1 \cdot 2 \cdots (t-1) \cdot t$ for integer $t \geq 1$.

a) $x_1 + x_2 + x_3$

b) $x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

c) $x^2 + x^3 + 2x^4 + x^5 + x^6$

d) $(x_{12} - y_1) + (x_{13} - y_1) + (x_{14} - y_1) + (x_{22} - y_2) + (x_{23} - y_2) + (x_{24} - y_2)$
$= x_{12} + x_{13} + x_{14} + x_{22} + x_{23} + x_{24} - 3y_1 - 3y_2$

e) $\left(-x_{-2}^2\right) + \left(x_{-1}^2\right) + \left(3x_0^2\right)$

f) $x_2y_2 + x_2y_3 + x_2y_4 + x_3y_3 + x_3y_4 + x_3y_5 + x_4y_4 + x_4y_5 + x_4y_6$

2. Consider the following two vectors: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and a matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$.

(a) Let n be the dimension of $\boldsymbol{x}$ and $\boldsymbol{y}$. What is the value of n?

(b) Compute $2\boldsymbol{x} - \boldsymbol{y}$.

(c) Compute the inner product $\boldsymbol{x}^\top \boldsymbol{y}$.

(d) Compute the Euclidean norm $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. Also called the ℓ_2-norm.

(e) Compute the ℓ_1-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \sum_{i=1}^{n} |x_i - y_i|$.

(f) Compute the ℓ_∞-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$.

(g) Compute $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}$.

a) n is 3 for both vectors x and y

b)

$$2x-y=\begin{bmatrix}2\\4\\6\end{bmatrix}-\begin{bmatrix}3\\2\\1\end{bmatrix}$$
$$=\begin{bmatrix}-1\\2\\5\end{bmatrix}$$

c) 10

d) $=\sqrt{(1-3)^2+(2-2)^2+(3-1)^2}$
$= 2\sqrt{2}$

e) = $|-2|+|0|+|2|$
= 4

f) = max(|-2|,|0|,|2|) = 2

g)

x^TAy =

= [1,2,3]$\begin{bmatrix}1&-2&3\\-2&3&-1\\3&-1&-2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}$

= [1,2,3]$\begin{bmatrix}2\\-1\\9\end{bmatrix}$

= 2-2+27

3. x

a)

$X=\left\{\left(x_1,x_2\right)\in R^2|\ x_1^2+x_2^2\geqslant 1\right\}$

From the results below we can see that everything where y>0 would be considered our epigraph, because we can show that for every $y \geqq\ f(x)$. However, if we were draw any point going through the circle we can see that these points would not be contained within the set since our epigraph is contained in the shaded red region.

Recall the definion, a set is said to be convex if for any pair of points in the set, all the points between them, are members of the set as well. Therefore, X is not convex because we have several points where we can draw a line segment going through the circle, and none of the points in the interior of our circle are in the set.

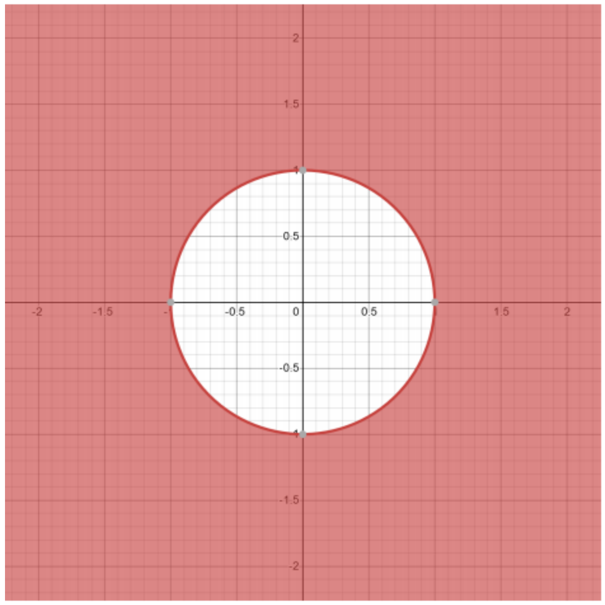

Figure 1: Geometric set of X

b)

For this graph, we can see our set is the roots of our equation $-\sqrt{2}\ and\ \sqrt{2}$.

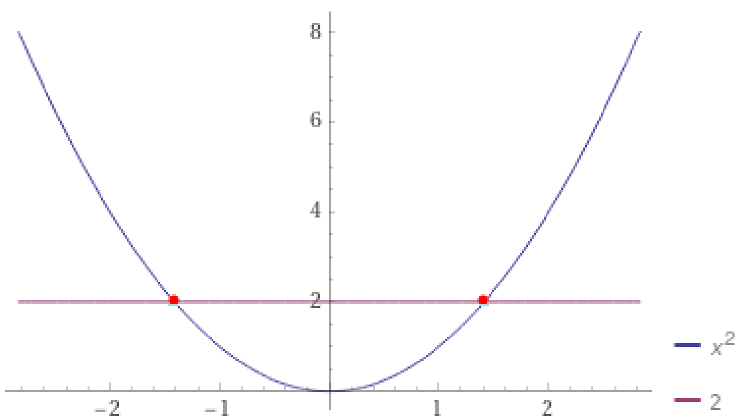

Figure 2: Geometric Representation of X

When we connect a line for both feasible solutions, we notice that any point within the line from $-\sqrt{2}\ and\ \sqrt{2}$ is not within our set, and therefore our problem is not convex.

c)

$$X = \left\{ (x_1, x_2) \in R^2 | \ \frac{x_2}{(x_1 - 1)} \geqslant 1, x_1 \geqslant 1 \right\}$$

We can see that this is a convex set because all line segments are contained without our set.

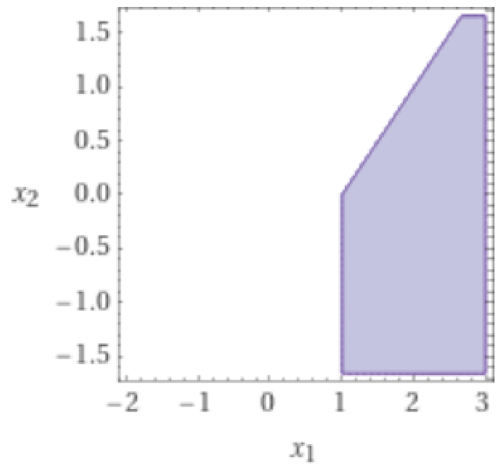

Figure 3: Geometric Representation of X

4.

A convex program is defined as a program where the underlying objective funciton is a convex function and it's set within the domain is a convex set.

a)

$$\min\{x_1^2 + 2x_2^2 \ : \ x_1 \leq 0, \ x_2 \leq 0\}$$

We know that both variables are continious because our objective function is the summation of two quadratic functions

Since sets with a single point are considered as convex, then we know both of our constraints are linear and are convex sets.

We also know that any line segment on our graph for our contour plot is also contained within our set, therefore this is an eligble convex program.

b)

$$\min\{x_1 \cdot x_2 \ : \ x_1^2 + x_2^2 \leq 1\}$$

We know that all variables are continuous. Then by checking if objective function is convex by determining if the hessian matrix is positive semi-definite:

First Partial Derivatives:

$$\frac{d}{d_{x_1}} x_1 \cdot x_2 = x_2$$

$$\frac{d}{d_{x_2}} x_1 \cdot x_2 = x_1$$

Second Partial Derivatives:

$$\frac{d^2}{d_{x_1 x_1}} x_1 \cdot x_2 = 0$$

$$\frac{d^2}{d_{x_2 x_2}} x_1 \cdot x_2 = 0$$

$$\frac{d^2}{d_{x_1 x_2}} x_1 \cdot x_2 = 1$$

$$H_{x_1 x_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

When we try and multiply an arbitrary vectory x = [a,b] to the Hessian matrix we get the following:

$$v^T H v = [a, b] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [b, a] \begin{bmatrix} a \\ b \end{bmatrix} = 2ab$$

So in our case, if a and b were opposing signs, then this would yield a value < 0, therefore we do not have a valid convex program.

c)

$\min\{\sum_{i=1}^{n} \frac{x_i^i}{i!} \ : \ \sum_{i=1}^{n} x_i \geq 5\}.$

Let's try to provide that is a convex by choosing a value of n=3

We also know that the constraints are valid linear constraints, and that we have a valid convex set denoted for our function.

Proof of non-convex objective function by contradiction:

Constraints = $x_1 + x_2 + x_3 \geq 5$

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{6}$$

Taking the first partial derivatives we get the following:

$$\frac{d}{dx_1} = 1$$
$$\frac{d}{dx_2} = x_2$$
$$\frac{d}{dx_3} = \frac{1}{2}x_3^2$$

Taking the second partial derviates we get the following:

$$\frac{d}{dx_1} = 0$$
$$\frac{d}{dx_2} = 1$$
$$\frac{d}{dx_3} = x_3$$

Therefore our Hessian matrix is the following:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

Then when taking an arbitrary vector x = [a,b,c] and taking the following against our Hessian matrix we get the following:

$$\mathrm{x}^T Hx = [a, b, c]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= [a, b, c]\begin{bmatrix} 0 \\ b \\ cx_3 \end{bmatrix}$$

$$= 0 + b^2 + c^2 x_3$$

So long as x_3 is negative, this would yield a result where $c^2 x_3 <$ b^2, and therefore our function is not PSD. Therefore our objective function cannot be considered convex and we do not have a valid convex program.

5.
a)
When we reformulate this we can see that the objective function is:

$$min\{\sum_{i=1}^{n} |y_i - (bx_i + a|)\}$$

This is simply a linear combination of our variables a & b scaled by and we know this is convex.
Therefore, since the objective function is linear and includes only continuous variables, this model is a convex linear program.
b)
Objective function:

$$min\{max\{\sum_{i=1}^{n} |y_i - (bx_i + a|)\text{: x= 1 ... n}\}$$

his is simply a linear combination of our variables a & b scaled by and we know this is convex.
Therefore, since the objective function is linear and includes only continuous variables, this model is a convex linear program.

c)

$$min\{\sum_{i=1}^{n} |y_i - (cx_i^2+bx_i + a)|\}$$

If we realize that and are simply constants while a, b and c are the variables, we see that is just another constant term. Similar to a, we know that this is a convex non-linear program.