ISYE 8803 HDDA

Homework 5
July 27th, 2021

2.
a)

Yan, et al. [1] formulated the following optimization problem for smooth sparse decomposition.

$$\min_{\theta,\theta_a} \|Y - B\theta - B_a\theta_a\|_2^2 + \lambda\theta^\top\Omega\theta + \gamma\|\theta_a\|_1$$

Here $\Omega = D^\top D$, where D is the difference matrix.

When comparing back to the original formula: $y = \mu + \alpha + e$ we can relate each part to the above equation as follows:

μ : $\lambda\theta^T\Omega\theta$ *acts as L2 norm*
α : $y||\theta_a||_1$ *acts as L1 norm*
e : $||Y - B\theta - B_a\theta_a||$ *uses OLS estimator to generate error term*

We can see from the ITA algorithm that we can create an optimization problem to solve for θ and
θ_a:

$$\mathrm{argmin}_{\theta,\theta_a}\ \lambda\theta' R\theta + \gamma||\theta_a||_1 + ||e||^2, s.t.\ y = B\theta + B_a\theta_a + e$$

- Propose an optimization algorithm based on ITA (Daubechies, et al, 2004)

Iterative Thresholding Algorithm

In the k^{th} iteration, update $\mu^{(k)}$ and $\theta_a^{(k)}$ by
$\mu^{(k)} = H(y - B_a\theta_a^{(k-1)})$, $H = B(B'B + \lambda I)^{-1}B'$ is the projection matrix
$\theta_a^{(k)} = T_{\tau^{(k)}}(\theta_a^{(k-1)} - c^{(k)}B_a'(B_a\theta_a^{(k-1)} + \mu^{(k)} - y))$

- $p = 1$, convex optimization
  - $T(\cdot)$ is the soft-thresholding operator.

The first step in this algorithm to update μ provides a closed-form solution to then use to generate the updated θ_a. Either ITA or accelerated ITA will then solve in a fixed number of interations.

From the following constraint we bound this optimization problem to solve back to the original y vector:

$$y = B\theta + B_a\theta_a + e$$

So from this where p= 1, since the algorithm provides a closed form solution, we know similar to a valid convex optimization problem, we're deriving an optimal solution for $\theta \text{ and } \theta_a$.

Therefore we know we have a valid convex objection function, it's operating on a convex set of values, and therefore we have a valid convex problem.

b)

Provided is the plot showing the output for the smooth, anomaly and error part of the problem:

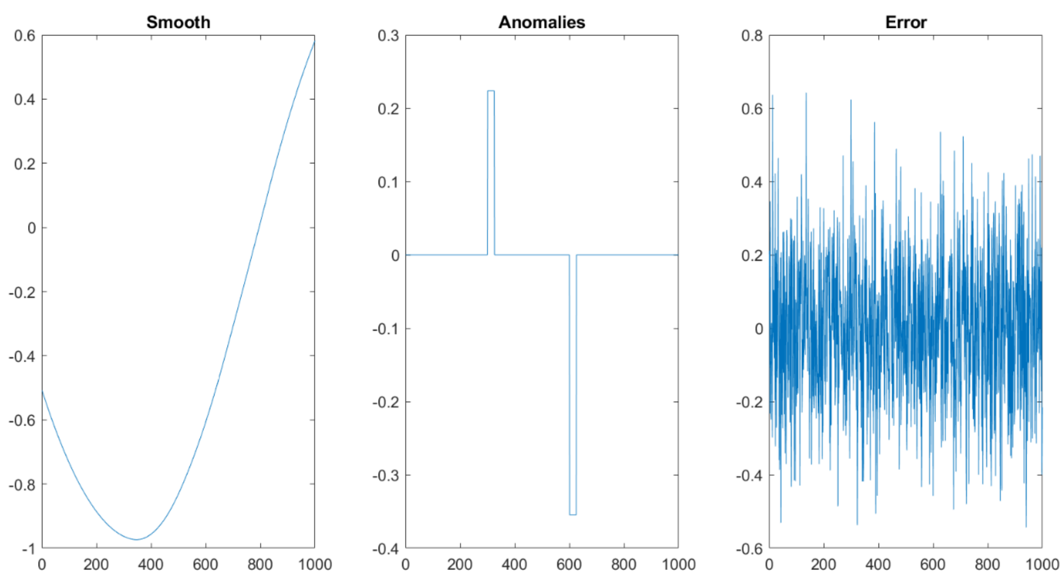

Figure 1: Smooth, Anomaly and Error Output