ISYE 8803 HDDA

Homework 1

May 31st, 2021

1.

$When\ taking\ the\ 2^{nd}\ derivative\ \frac{d^2}{dx}y(x)\ =\ 0\ for\ the\ outside\ intervalfor\ x\ outside\ (\xi_1, \xi_k)$
$we\ get\ the\ following\ result:$

$Taking\ the\ first\ derivative:$

$$\beta_1 + \beta_2 x + \sum_{k=1}^{K} \theta_k (x - \xi_k)_+^2 = 0$$

$$2\sum_{k=1}^{K} \theta_K (x - \xi_K) = 0$$

$Taking\ the\ second\ derivative:$

$$2\sum_{k=1}^{K} \theta_K (x - \xi_K) = 0$$

$$\sum_{k=1}^{K} \theta_K = 0$$

$We\ can\ rearrange\ this\ so\ that:$

$$\theta_K = -\sum_{k=1}^{K} \theta_k$$

$$We\ know\ that \sum_{k=1}^{K} \theta_k (x - \xi_k)_+^2 = \sum_{k=1}^{K-1} \theta_k (x - \xi_k)_+^2 + \theta_K (x - \xi_K)_+^2$$

$Then\ when\ I\ plug\ this\ back\ into\ the\ original\ equation\ I\ get\ the\ following:$

$$\beta_1 + \beta_2 x + \sum_{k=1}^{K-1} \theta_k (x - \xi_k)_+^2 - \theta_k (x - \xi_K)_+^2 = 0$$

$Then\ we\ can\ factor\ out\ the\ summation\ to\ get\ the\ following:$

$$\beta_1 + \beta_2 x + \sum_{k=1}^{K-1} \theta_k \left[(x - \xi_k)_+^2 - (x - \xi_K)_+^2 \right]$$

Therefore, the set of basis functions are:

$$\left\{ 1,\ x,\ \left\{(x-\xi_k)_+^2 - (x-\xi_K)_+^2\right\} \forall k = 1,\ldots,K-1 \right\}$$

2.
When deriving the values of uniform quadratic B – spline basis functions at the various values $B_{0,2}(x)$, $B_{1,2}(x)$, $B_{2,2}(x)$ we can use the matrix representation to visualize the combination of these formulas for the first and higher order terms:

$$\left[\begin{array}{ccccc} B_{0,0}(x) & \longrightarrow & B_{0,1}(x) & \longrightarrow & B_{0,2}(x) \\ & \nearrow & & \nearrow & \\ B_{1,0}(x) & \longrightarrow & B_{1,1}(x) & \longrightarrow & B_{1,2}(x) \\ & \nearrow & & \nearrow & \\ B_{2,0}(x) & \longrightarrow & B_{2,1}(x) & \longrightarrow & B_{2,2}(x) \\ & \nearrow & & \nearrow & \\ B_{3,0}(x) & \longrightarrow & B_{3,1}(x) & & \end{array}\right]$$

Therefore, based on our formulas for first and higher order terms, we can derive $B_{0,2}(x)$ by the following:

$$B_{0,2}(x) \ = \ \frac{x-0}{2-0}\, B_{0,1}(x) + \frac{3-x}{3-1}\, B_{1,1}(x)$$

Then since it's recursive, we know we can also replace the values of $B_{0,1}(x)$ and $B_{1,1}(x)$ with the child attributes used to calculate them by applying the same formula. Since the child terms are first order terms, they will only receive a value of 0 or 1, and therefore makes the formula much easier to solve.

Once we do that we get the following result:

$$= \frac{x}{2}[x\ B_{0,0}(x) + (2-x)\ B_{1,0}(x)] + \frac{3-x}{2}[(x-1)\ B_{1,0}(x) + (3-x)\ B_{2,0}(x)]$$

Now when we factor out $\frac{1}{2}$, and expand the terms we get the following:

$$= \ \frac{1}{2}\left[x^2\ B_{0,0}(x) + x(2-x)\ B_{1,0}(x) + (3-x)(x-1)\ B_{1,0}(x) + (3-x)^2\ B_{2,0}(x)\right]$$

We can then use this formula to derive the values of each basis function across each impacted time interval, each interval is the result of plugging in the corresponding values

$into\ the\ top\ equation.\ For\ our\ first\ row\ below,\ only\ B_{0,0}(x)\ =\ 1\ in\ the\ interval\ [0,1],$ $and\ the\ rest\ of\ the\ equations\ cancels\ out\ because\ B_{1,0}(x)\ and\ B_{2,0}(x) = 0.\ The\ same\ logic$ $can\ then\ be\ applied\ to\ all\ other\ intervals.\ Since\ we\ have\ a\ uniform\ quadratic\ function$ $with\ order\ M\ =\ 3,\ we\ know\ that\ each\ basis\ function\ spands\ across\ at\ most\ 3\ intervals,$ $so\ therefore\ all\ other\ values\ of\ the\ basis\ function\ are\ 0\ when\ not\ in\ the\ impacted$ $intervals.$

$From\ this\ we\ get:$

$$B_{0,2}(x)\ =\ \left\{\begin{matrix} \frac{1}{2}x^2 & 0 \le x < 1 \\ \frac{1}{2}[-2(x-1)^2 + 2(x-1) + 2 & 1 \le x < 2 \\ \frac{1}{2}[(x-2)^2 - 2(x-2) + 1 & 2 \le x < 3 \\ 0 & otherwise \end{matrix}\right\}$$

$For\ the\ other\ basis\ functions,\ we\ can\ just\ imagine\ that\ we're\ shifting\ our\ time\ interval$ $in\ the\ matrix\ by\ 1,\ and\ therefore\ we\ get\ the\ following\ results:$

$$B_{1,2}(x)\ =\ \left\{\begin{matrix} \frac{1}{2}(x-1)^2 & 1 \le x < 2 \\ \frac{1}{2}[-2(x-2)^2 + 2(x-2) + 2 & 2 \le x < 3 \\ \frac{1}{2}[(x-3)^2 - 2(x-3) + 1 & 3 \le x < 4 \\ 0 & otherwise \end{matrix}\right\}$$

$$B_{2,2}(x)\ =\ \left\{\begin{matrix} \frac{1}{2}(x-2)^2 & 1 \le x < 2 \\ \frac{1}{2}[-2(x-3)^2 + 2(x-3) + 2 & 2 \le x < 3 \\ \frac{1}{2}[(x-4)^2 - 2(x-4) + 1 & 3 \le x < 4 \\ 0 & otherwise \end{matrix}\right\}$$

3.

a)

Number of knots with lowest MSE of 21.05 = 7 knots

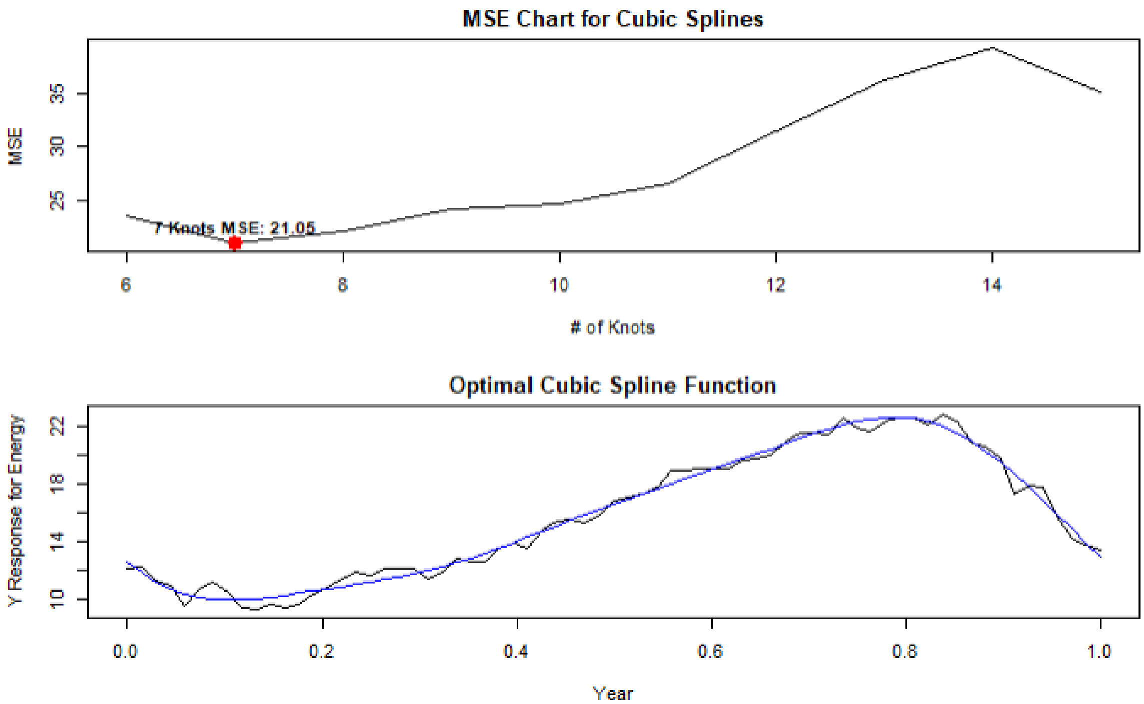

Figure 1: Cubic Spline MSE & Optimal Function Plots

b)

Number of knots with lowest MSE of 20.11 = 11 knots

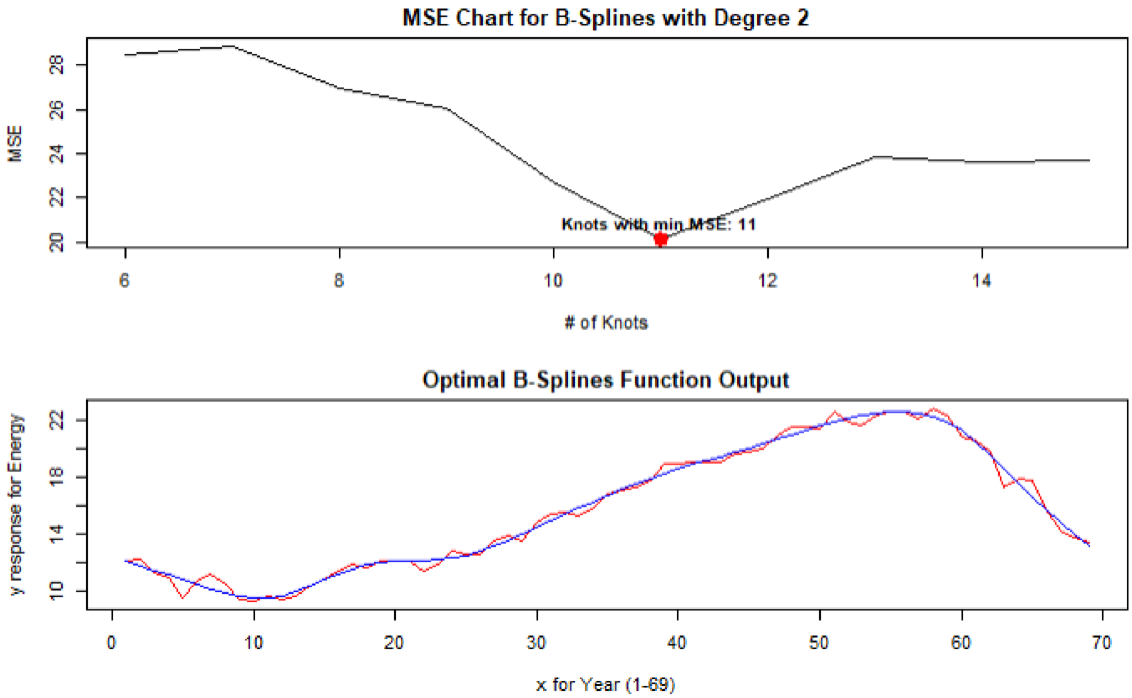

Figure 2: B-Splines MSE & Optimal Function Plots

c)

The smoothing splines with an MSE = 24.29586 with an optimal lambda = 0.4994995

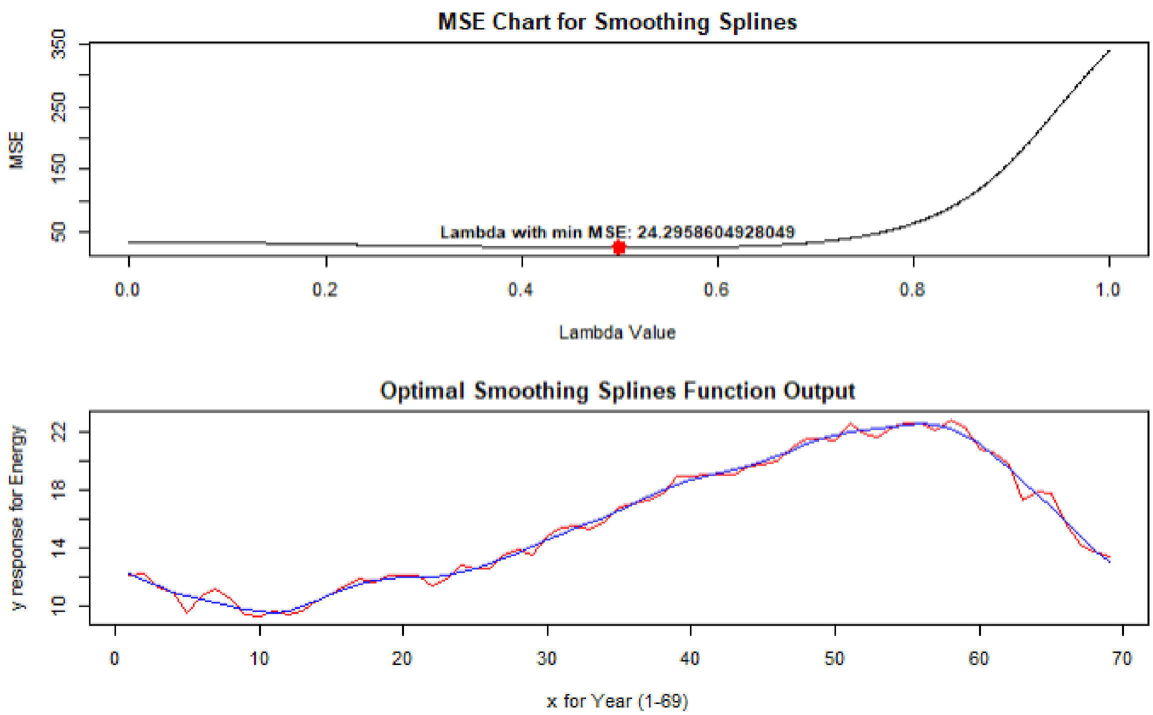

Figure 3: Optimal MSE & Lambda For Smoothing Splines

d)

Lambda value of 0.03 produces a MSE of 15.23 for Gaussian Kernel Smoother

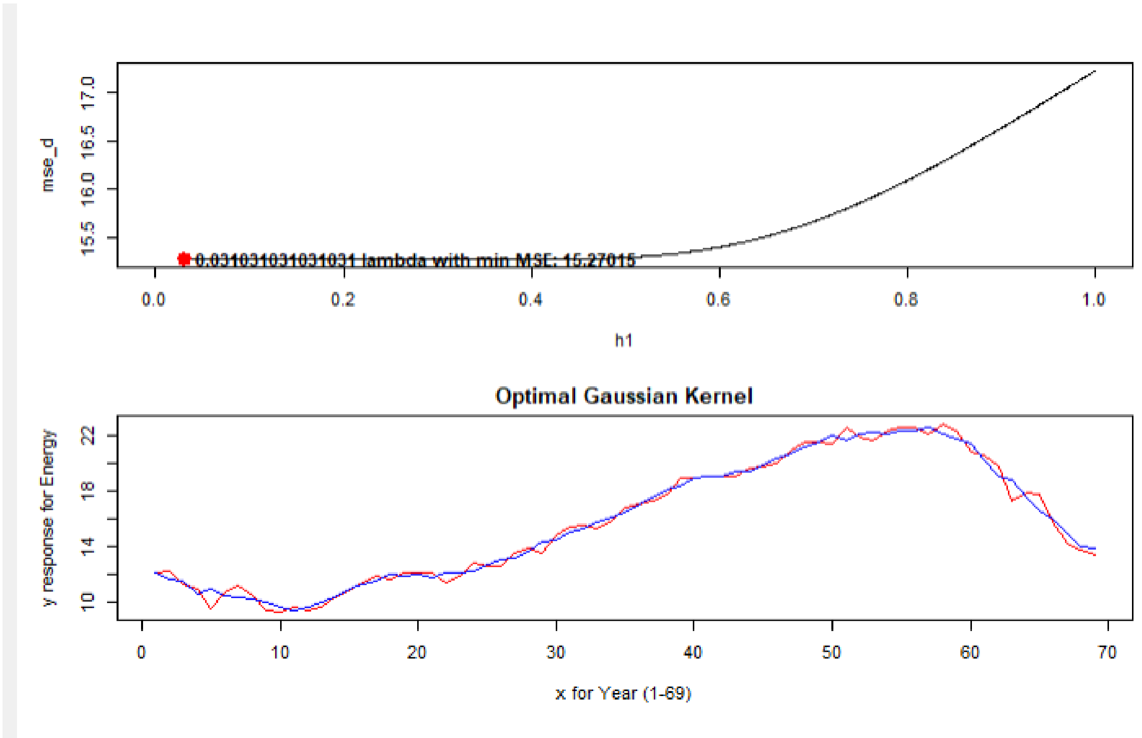

Figure 4: Optimal MSE & Lambda For Gaussian Kernel

4.

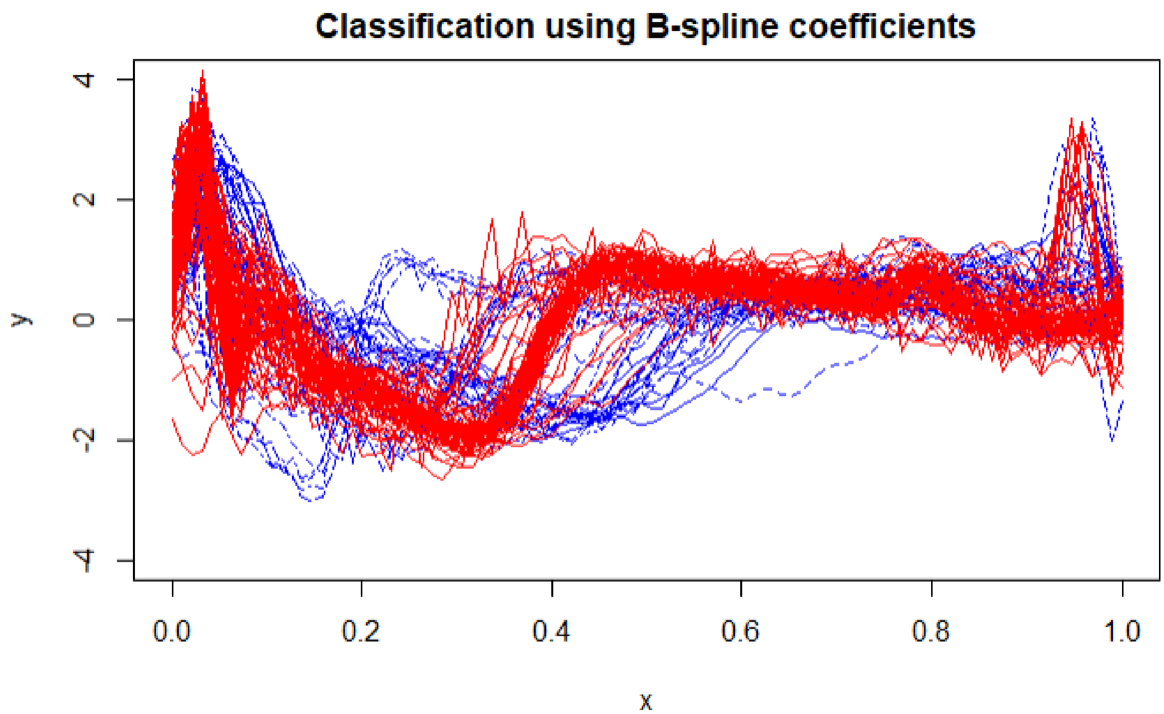

Figure 5: ECG Hearbeat Classification using B-spline

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          Reference
Prediction  0  1
         0 30  6
         1  6 58

               Accuracy : 0.88
                 95% CI : (0.7998, 0.9364)
    No Information Rate : 0.64
    P-Value [Acc > NIR] : 5.703e-08

                  Kappa : 0.7396

 Mcnemar's Test P-Value : 1

            Sensitivity : 0.8333
            Specificity : 0.9062
         Pos Pred Value : 0.8333
         Neg Pred Value : 0.9063
             Prevalence : 0.3600
         Detection Rate : 0.3000
   Detection Prevalence : 0.3600
      Balanced Accuracy : 0.8698

       'Positive' Class : 0
```

Figure 6: Model Results

We can see with an accuracy of 88% that our model using a b-spline basis matrix is performing well to identify abnormalities in the ECG dataset.

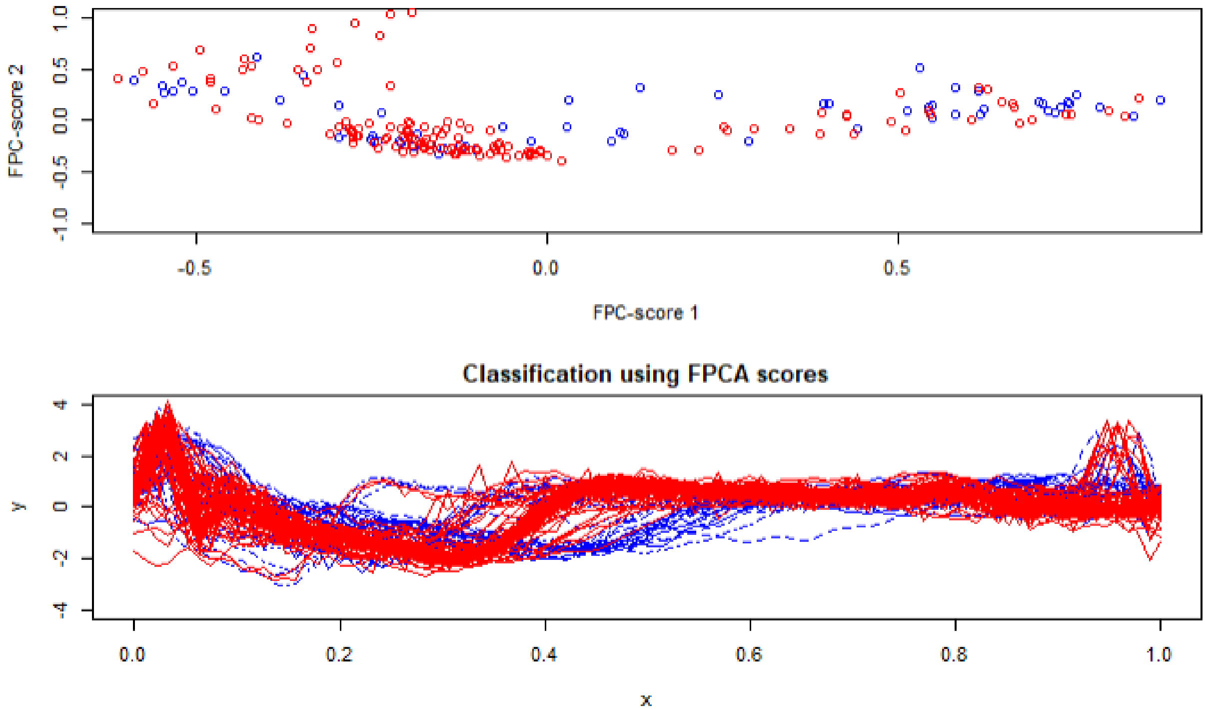

Figure 7: Results of 2 FPCA and Classification using FPCA

We can see good results for the FPCA model with an accuracy of 82% when compared to the
b-splines. This should be expected because smoothing basis matrix is derived from fewer number of features when compared to the b-spline basis matrix.