ISYE 8803 HDDA

Homework 1

May 31st, 2021

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When deriving the values of uniform quadratic B – spline basis functions at the various values $B_{0,2}(x)$, $B_{1,2}(x)$, $B_{2,2}(x)$ *we can use the matrix representation to visualize the combination of these formulas for the first and higher order terms:*

$$\begin{bmatrix} B_{0,0}(x) & \longrightarrow & B_{0,1}(x) & \longrightarrow & B_{0,2}(x) \\ & \nearrow & & \nearrow & \\ B_{1,0}(x) & \longrightarrow & B_{1,1}(x) & \longrightarrow & B_{1,2}(x) \\ & \nearrow & & \nearrow & \\ B_{2,0}(x) & \longrightarrow & B_{2,1}(x) & \longrightarrow & B_{2,2}(x) \\ & \nearrow & & \nearrow & \\ B_{3,0}(x) & \longrightarrow & B_{3,1}(x) & & \end{bmatrix}$$

Therefore, based on our formulas for first and higher order terms, we can derive $B_{0,2}(x)$ *by the following:*

$$B_{0,2}(x) \;=\; \frac{x-0}{2-0}\, B_{0,1}(x) + \frac{3-x}{3-1}\, B_{1,1}(x)$$

Then since it's recursive, we know we can also replace the values of $B_{0,1}(x)$ *and* $B_{1,1}(x)$ *with the child attributes used to calculate them by applying the same formula. Since the child terms are first order terms, they will only receive a value of 0 or 1, and therefore makes the formula much easier to solve.*

Once we do that we get the following result:

$$= \frac{x}{2}[x\, B_{0,0}(x) + (2-x)\, B_{1,0}(x)] + \frac{3-x}{2}[(x-1)\, B_{1,0}(x) + (3-x)\, B_{2,0}(x)]$$

Now when we factor out $\frac{1}{2}$, *and expand the terms we get the following:*

$$= \frac{1}{2}\left[x^2\, B_{0,0}(x) + x(2-x)\, B_{1,0}(x) + (3-x)(x-1)\, B_{1,0}(x) + (3-x)^2\, B_{2,0}(x)\right]$$

We can then use this formula to derive the values of each basis function across each impacted time interval, each interval is the result of plugging in the corresponding values *into the top equation. For our first row below, only* $B_{0,0}(x) = 1$ *in the interval* $[0,1]$, *and the rest of the equations cancels out because* $B_{1,0}(x)$ *and* $B_{2,0}(x) = 0$. *The same logic can then be applied to all other intervals. Since we have a uniform quadratic function with order* $M = 3$, *we know that each basis function spands across at most* 3 *intervals, so therefore all other values of the basis function are* 0 *when not in the impacted intervals.*

From this we get:

$$B_{0,2}(x) = \left\{ \begin{array}{cc} \frac{1}{2}x^2 & 0 \le x < 1 \\ \frac{1}{2}[-2(x-1)^2 + 2(x-1) + 2 & 1 \le x < 2 \\ \frac{1}{2}[(x-2)^2 - 2(x-2) + 1 & 2 \le x < 3 \\ 0 & otherwise \end{array} \right\}$$

For the other basis functions, we can just imagine that we're shifting our time interval in the matrix by 1, *and therefore we get the following results:*

$$B_{1,2}(x) = \left\{ \begin{array}{cc} \frac{1}{2}(x-1)^2 & 1 \le x < 2 \\ \frac{1}{2}[-2(x-2)^2 + 2(x-2) + 2 & 2 \le x < 3 \\ \frac{1}{2}[(x-3)^2 - 2(x-3) + 1 & 3 \le x < 4 \\ 0 & otherwise \end{array} \right\}$$

$$B_{2,2}(x) = \left\{ \begin{array}{cc} \frac{1}{2}(x-2)^2 & 1 \le x < 2 \\ \frac{1}{2}[-2(x-3)^2 + 2(x-3) + 2 & 2 \le x < 3 \\ \frac{1}{2}[(x-4)^2 - 2(x-4) + 1 & 3 \le x < 4 \\ 0 & otherwise \end{array} \right\}$$