ISYE 6420 Bayesian Statistics

Homework 3

October 16th, 2021

1.

Our Jeffrey's prior is the following:

$$\pi(\theta) = \frac{1}{\sqrt{\theta}}$$

And our observations: $x_1 = 1,\ x_2 = 2, x_3 = 0, x_4 = 1$

Since we're dealing with Poisson we know our likelihood function is in the following form:

$$L(\theta|x) = \frac{\theta^{x_i} e^{-\theta}}{x_i!}$$

When inputting these values we get the following for the likelihood function:

$$= e^{-4\theta}\left(\frac{\theta^1}{1} * \frac{\theta^2}{2} * \frac{\theta^0}{1} * \frac{\theta^1}{1}\right)$$

$$= \theta^4 e^{-4\theta}$$

$$L(\theta|x) * \pi(\theta) = \theta^{\frac{7}{2}} e^{-4\theta}$$

Now that we have the result we can use this and plug it into the bayes estimator formula:

$$\delta_B(x) = \frac{\int \theta^{\frac{9}{2}} e^{-4\theta} d\theta}{\int \theta^{\frac{7}{2}} e^{-4\theta} d\theta}$$

Now since we know we can introduce the gamma function and have that integrate to 1 in order to simplify:

$$\alpha = \frac{11}{2}$$

$$\beta = 4$$

$$= \frac{\frac{\Gamma\left(\frac{11}{2}\right)}{4^{\frac{11}{2}}}\int \frac{4^{\frac{11}{2}})}{\Gamma(\frac{11}{2}}\, \theta^{\frac{11}{2}-1} e^{-4\theta} d\theta}{\frac{\Gamma\left(\frac{9}{2}\right)}{4^{\frac{9}{2}}}\int \frac{4^{\frac{9}{2}})}{\Gamma(\frac{9}{2}}\, \theta^{\frac{9}{2}-1} e^{-4\theta} d\theta}$$

$$= \frac{\Gamma\left(\frac{11}{2}\right) * 4^{\frac{11}{2}}}{\Gamma\left(\frac{9}{2}\right) * 4^{\frac{9}{2}}}$$

$$= 9/2 * 1/4$$

$$= \frac{9}{8}$$

Which is very close to 1!

b)

Credible set can be derived using the available function in R:
Where the shape value is our alpha = 11/2 - 1 = 4.5 and our rate value which represents Beta = 4.

```
```{r cars}
qgamma(c(0.025,0.975),4.5,4)
```

[1] 0.3375487 2.3778460
```

c)

Same can be done for the HDP credible set and we get a similar result:

```
## Including Plots

You can also embed plots, for example:

```{r pressure, echo=FALSE}
library(HDInterval)
int <- hdi(qgamma,shape=4.5,rate=4)
print(int)

```
```

```
    lower     upper
0.2378233 2.1740332
attr(,"credMass")
[1] 0.95
```

d)

From the derivation we know that the MAP esimtator for poisson is the following:

$$\arg\max_{\lambda} \; l(\lambda)$$

$$\frac{\partial}{\partial \lambda} l(\lambda) = 0$$

$$\frac{\partial}{\partial \lambda} \left\{ \sum_{i=1}^{n} (-\lambda) + \sum_{i=1}^{n} x_i \ln \lambda - \sum_{i=1}^{n} \ln x_i! \right\} = 0$$

$$-n + \frac{1}{\lambda} \cdot \sum_{i=1}^{n} x_i = 0$$

$$\therefore \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Therefore, it equals = 1/4 (4) = 1