Scrap Paper - Test 3 (Module 5 and 6)

1.

$$\int x^3 e^x \, dx$$

integration by parts:

$u = x^3$ (poly is easy to differentiate)

$dv = e^x dx$ (exp easy to integrate)

$du = 3x^2 dx$

$v = \int e^x dx = e^x$

Using formula

$$uv - \int v\,du = x^3 e^x - \int e^x \cdot 3x^2$$

or:

$$= x^3 e^x - 3\int x^2 e^x.$$

2.

$$\frac{-x-1}{12x^2-25x+12} \rightarrow \frac{f(x)}{4x-3} + \frac{g(x)}{3x-4}$$

Find $f(x)$ & $g(x)$

Factor bottom:

$12x^2 - 25x + 12$

$\rightarrow 12\times1 \quad 3\times4 \quad 4\times3$

$\overline{4x-3}(3x-4)$

$12x^2$ ~~$-48x$~~ $+12$

$12x^2$

$-16x$

$-9x$

$$12x^2 - 25x + 12 = (4x-3)(3x-4)$$

$$\left(\frac{-x-1}{(4x-3)(3x-4)}\right) = \left(\frac{A}{4x-3} + \frac{B}{3x-4}\right) \times (4x-3)(3x-4)$$

$$\frac{x}{(4x-3)(3x-4)}$$

$$-x-1 = B(4x-3) + A(3x-4)$$

group like terms

$$= (3A + 4B)x + (-4A - 3B)$$

Equate coefficients:

x: $3A + \cancel{B}\ 4B = -1$

Constant $\rightarrow$ $-4A - 3B = -1$

System of equation solve:

1. $3A + 3B = -1 \quad (4)$
2. $-4A - 3B = -1 \quad (3)$

$($

$(12A - 12A) + (16B - 9B) =$

$-4 - 3 \Rightarrow 7B = -7 \rightarrow B = -1$

sub $B = -1$ back

$3A + 4(-1) = -1 \rightarrow 3A - 4 = -1 \rightarrow 3A = 3$

$A = 1$

$A = 1$, $B = -1$ or $f(x) = 1$ & $g(x) = -1$

3. $\displaystyle\int_{-\infty}^{5} \frac{1}{x^2+1}\,dx$

standard integral

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$$

$$\int_{-\infty}^{5} \frac{1}{x^2+1}\,dx = \lim_{c \to -\infty} \int_{c}^{5} \frac{1}{x^2+1}\,dx$$

since lower limit is $-\infty$

$$= \int_c^5 \frac{1}{x^2+1}\,dx = \tan^{-1}(5) - \tan^{-1}(c)$$

$$= \lim_{c \to -\infty} \left(\tan^{-1}(5) - \tan^{-1}(c) \right)$$

= remain same some numerical value.

$\lim_{c \to -\infty} \tan^{-1}(c) = -\frac{\pi}{2}$

Horizontal asymptote.

$$= \tan^{-1}(5) - \left(-\frac{\pi}{2}\right)$$

$$= \tan^{-1}(5) + \frac{\pi}{2} \quad \text{or} \quad \arctan(5) + \frac{\pi}{2}$$

4. <u>width of each subinterval (w)</u>

$-9 \to 6 \quad n=4$

$$w = \frac{6-(-9)}{4} = \frac{15}{4} = 3.3/4 \text{ or } 3.75$$

<u>Endpoints:</u>

$x_i = a + i \cdot h$

		$\sqrt{5(x)+45}$
$x_0 = -9 + 0.375 = -9$		0
$x_1 = \;\;\; \cdot +1 \;\;\; = -5.25$		4.3301
$x_2 = \;\;\; \cdot +2 \;\;\; = -1.5$		6.1237
$x_3 = \;\;\; \cdot +3 \;\;\; = 2.25$		7.5
$x_4 = \;\;\; \cdot +4 \;\;\; = 6$		8.66
$\;\;\;\;\;\;\; +5$		

Trapezoid Rule

$$T_4 = \frac{3.75}{2}\left[f(-9) + 2\cdot f(-5.25) + 2f(-1.5) + 2f(2.25) + f(6)\right]$$

$$= 1.875(44.568)$$

$$= 83.565$$

need exact value of integral to find absolute error btw trapezoid method

$$\int_{-9}^{6} \sqrt{5x+45}\,dx.$$

sub.

$u = 5x + 45$

$du = 5\,dx$

$dx = \frac{1}{5}\,du$

$x = -9$

$u = 5(-9) + 45 = 0$

$x = 6,\ u = 5(6) + 45 = 75$

$$\therefore \int_0^{75} \sqrt{u} \cdot \frac{1}{5}\,du = \frac{1}{5}\int_0^{75} u^{1/2}\,du$$

$$= \frac{1}{5}\left[\frac{u^{3/2}}{3/2}\right]_0^{75} = \frac{2}{15}\left[u^{3/2}\right]_0^{75}$$

$$= \frac{2}{15}\left(75^{3/2} - 0^{3/2}\right) = \frac{2}{15}$$

simplify:

$$-\frac{2}{15}\left(75\sqrt{75}\right) = \frac{2}{15}\left(75 \cdot 5\sqrt{3}\right)$$

$$= \frac{2}{15}\left(375\sqrt{3}\right) = 50\sqrt{3}$$

$$= 86.6025$$

absolute error

$$EV - AV = 86.6025 - 83.5648$$

$$= 3.0377$$

Relative Error:

$$\frac{EV - AV}{EV} = \frac{3.0377}{86.6025} = 3.51\%$$

5. $\int_2^8 \frac{9}{x-4}\,dx$

since $\frac{9}{4-4} = \frac{9}{0}$ undefined

discontinuity @ $x=4$, that is

within $[2,8]$

split $\int$ @ discontinuity

$$\int_2^8 \frac{9}{x-4}\,dx = \int_2^4 \frac{9}{x-4}\,dx + \int_4^8 \frac{9}{x-4}\,dx$$

check limit

Left:

$$\lim_{c \to 4^-} \int_2^c \frac{9}{x-4}\,dx.$$

By Rule:

antiderivative:

$$\int \frac{9}{x-4}\,dx = 9\ln|x-4| + C$$

$$\lim_{c \to 4^-} \left(9\ln|x-4|\right)\Big|_2^c = \left[9\ln|c-4| - 9\ln|2-4|\right]$$

$$= 9\ln|c-4| - 9\ln(2)$$

$$c \to 4^- \Rightarrow |c-4| \to 0^+$$

$$\Rightarrow \ln|c-4| \to -\infty$$

$\Rightarrow$ DIVERGES on one side $\Rightarrow$ integral diverges

DNE

6

$$\int_2^7 (-x^4 + 18x^3 - 84x^2 - 3x - 1)\,dx$$

Trap Rule

$y' = -4x^3 + 54x^2 - 168x - 3$

$y'' = -12x^2 + 108x - 168$

max on interval

cp of y''

$y'' = -12x^2 + 108x - 168$

$y''' = -24x + 108 = 0$

$x = \frac{-108}{-24}$

4.5

evaluate endpoints & cp

$f''(2) = 0$

$f''(7) = 0$

$f''(4.5) = 75$

$\therefore$ max of $y'' = 75$

Back to

trapRule (plug n)

$$E_T \le \frac{(7-2)^3}{12n^2} \cdot .75 = \frac{125}{12n^2} \cdot .75 =$$

$$\frac{93.75}{12n^2} \le 0.1$$

which was

$$n^2 \ge \frac{93.75}{1.2} = 78.125$$

$$n \ge \sqrt{78.125} \approx 8.84 = 9$$

$n = 9$

Simpson Rule

use y''''

$y'' = -12x^2 + 108x - 168$

$y''' = -24x + 108$

$y'''' = -24$

$\max |y''''(x)| = \cancel{25}\ 24.$

use Formula:

$|E_S| \leq \frac{(7-2)^5}{180n^4} \cdot 24 = \frac{3125}{180n^4} \cdot 24 = \frac{75k}{180n^4} =$

$\frac{1250}{3n^4}$

goal $= \frac{1250}{3n^4} \leq 0.1 \rightarrow n^4 \geq \frac{1250}{0.3} = 4166.7 = \cancel{8.6}$

$\therefore n \geq \sqrt[4]{4166.67} = 8.042$

simpson require even number so

8.04 rounds up to 10

$n = 10$

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power reduction formulas to solve:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

rewrite $\int \sin^2(x)\cos^4(x)\,dx$

$$\cos^4(x) = \left(\frac{1+\cos(2x)}{2}\right)^2 = \frac{1+2\cos(2x)+\cos^2(2x)}{4}$$

$$\cos^4(x) = \frac{1+2\cos(2x)+\frac{1+\cos(4x)}{2}}{4} = \frac{2+4\cos(2x)+1+\cos(4x)}{8}$$

$$= \frac{3+4\cos(2x)+\cos(4x)}{8}$$

sub back into expression & integrate

$$\int \sin^2(x)\cos^4(x)\,dx = \int\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{3+4\cos(2x)+\cos(4x)}{8}\right)dx$$

$$= \frac{1}{16}\int \left[(1-\cos(x))(3+4(\cos(2x)) + \cos(4x)\right]dx$$

$$= \frac{1}{16}\int \left(3+4\cos(2x)+\cos(4x) - 3\cos(2x) - 4\cos^2(2x) - \cos(2x)\cdot\cos(4x)\right)dx$$

$\int$

$$= \frac{1}{16}\int \left(3+\cos(2x)+\cos(4x)-4\cos^2(2x)-\cos(2x)\cos(4x)\right)dx$$

apply power red Formula

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

product to sum Formula: $\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

$$\therefore \cos(2x)\cos(4x) =$$

$$\frac{1}{2}\left[\cos(2x-4x)+\cos(2x+4x)\right]$$

$$= \frac{1}{2}\left[\cos(-2x)+\cos(6x)\right] = -\frac{1}{2}\left(\cos(2x)+\cos(6x)\right)$$

Back to integral

$$\frac{1}{16}\int\left(3+\cos(2x)+\cos(4x)-4\left(\frac{1+\cos(4x)}{2}-\frac{1}{2}\left(\cos(2x)+\cos(6x)\right)\right)\right)dx$$

$$\frac{1}{16}\int\left(3+\cos(2x)+\cos(4x)-2-2\cos(4x)-\frac{1}{2}\cos(2x)-\frac{1}{2}\left(\cos(6x)\right)\right)dx$$

$$\frac{1}{16}\int\left(1+\frac{1}{2}\cos(2x)-\cos(4x)-\frac{1}{2}\cos(6x)\right)dx$$

By term

$$\frac{1}{16}\left[\int 1\,dx+\frac{1}{2}\int\cos 2x\,dx-\int\cos(4x)\,dx-\frac{1}{2}\int\cos(6x)\,dx\right.$$

$$\frac{1}{16}\left(x+\frac{1}{4}\sin(2x)-\frac{1}{4}\sin(4x)-\frac{1}{12}\sin(6x)\right)+C$$

$$\frac{x}{16}+\frac{1}{64}\sin(2x)-\frac{1}{64}\sin(4x)-\frac{1}{192}\sin(6x)+C$$

$$\frac{x}{16}+\frac{1}{64}\sin(2x)-\frac{1}{64}\sin(4x)-\frac{1}{192}\sin(6x)+C$$

(12) integration by parts

$\int u\,dv = uv - \int v\,du$

for $\int 3x\sin(x)\,dx$

$u = 3x$

$du = 3dx$

$dv = \sin(x)\,dx$

$v = \int \sin(x)\,dx = -\cos(x)$

integrate by parts

$\int 3x\sin(x)\,dx = 3x(-\cos(x)) - \int(-\cos(x)(3x))$

$= -3x\cos(x) - \int -3\cos(x)\,dx$

$= -3x\cos(x) + 3\int \cos(x)\,dx$

$\int \cos(x)\,dx = \sin(x) + c$

$= -3x\cos(x) + 3\cdot(\sin(x) + c_i)$

$= -3x\cos(x) + 3\sin x + 3c_i$ replace w/ C

$= -3x\cos(x) + 3\sin x + C$

constant · constant

$A \cdot c_i = C$ $\Longleftarrow$ constant $\times$ constant $= C$

(11) midpoint is more accurate.
why? middle represents the average height of the line in section: better estimate to find estimate area under curve.

(10) $\int \sin^2 r \cos^3 r\,dr$

we know
$\cos^2 x = 1 - \sin^2 x$
let $x = r$

rewrite

$$\int \sin^2 r \cdot (\cos^2 r \cdot \cos r)$$

$$\int \sin^2 r \cdot 1 - \sin^2 r\, (\cos r)$$

let $u = \sin r$

$du = \cos r\, dr$.

rewrite as

$$\int u^2 (1 - u^2)\, du = \int u^2 - u^4\, du$$

$$= \int u^2 - \int u^4\, du$$

Power Rule $= \dfrac{u^{2+1}}{2+1} - \dfrac{u^{4+1}}{4+1} + C$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

sub $= \dfrac{\sin^3 r}{3} - \dfrac{\sin^5 r}{5} + C$

9 find $f(x)$ & $g(x)$

$$\frac{12}{x^2 - 9} = \frac{f(x)}{x-3} + \frac{g(x)}{x+3}$$

factor

$x^2 - 9 = (x-3)(x+3)$

$$\frac{12}{(x-3)(x+3)} = \frac{f(x)}{(x-3)} + \frac{g(x)}{x+3}$$

$12 = f(x)(x+3) + g(x)(x-3) \implies$ expression must work for all x other than $x = \pm 3$

$12 = f(x)(x+3) + g(x)(x-3)$

isolate $f(x)$ by setting $x = 3$

$12 = f(3)(3+3) + g(x)(3-3)$

$12 = f(3)(6) + \cancel{g(x)(0)}$

$12 = f(3)(6)$

$\frac{12}{6} = \frac{6f(3)}{6}$

$f(3) = 2$ since $f(x)$ is a constant. as the problem says $f(3) = f(x)$ for all x

$\therefore$ $f(x) = 2$

isolate $g(x)\big|_{x=-3}$

$12 = f\cancel{(-3)(-3+3)} + g(-3)(-3-3)$

$= 0 + g(-3)(-6)$

$\frac{12}{-6} = \frac{\cancel{-6}\,g(-3)}{\cancel{-6}}$

$-2 = g(-3)$

$g(x) = -2$

⑧ Find $f(x)$, $g(x)$ & $h(x)$ for partial fraction decomp. to

$$\frac{-4x^2 - 5x + 1}{x(x+1)^2}$$

$$\frac{f(x)}{x} + \frac{g(x)}{x+1} + \frac{h(x)}{(x+1)^2}$$ common denominator $= x(x+1)^2$

$$\left(\frac{-4x^2 - 5x + 1}{x(x+1)^2} = \frac{f(x)}{x} + \frac{g(x)}{x+1} + \frac{h(x)}{(x+1^2)}\right) x(x+1)^2$$

$-4x^2 - 5x + 1 = f(x)(x+1)^2 + g(x)x(x+1) + h(x)x$

$-4x^2 - 5x + 1 = f(x)(x^2+2x+1) + g(x)(x^2+x) + h(x)x$

isolate $f(x)$

let $x = 0$

$-4(0^2) - 5(0) + 1 = f(0)(0+1)^2 + g(0)(0^2+0) + h(0)(0)$

$1 = f(0) \times 1 + \cancel{g(0)(0) + h(0)(0)}$

$1 = f(0)$

$f(x) = 1$ since problem implys it as constant

isolate $h(x)$

let $x = -1$

$-4(-1)^2 - 5(-1) + 1 = f(-1)\cancel{(-1+1)^2}^{\,0} + g\cancel{(-1)(-1)(-1+1)}^{\,0}$
$+ h(-1)(-1)$

$-4 + 5 + 1 = 0 + 0 - 1\,h(-1)$

$\frac{2}{-1} = \frac{-h(-1)}{-1}$

$-2 = h(-1)$

$h(x) = -2$

Isolate $g(x)$

since $f(x)=1$ & $h(x)=-2$ sub back into original expression to find $g(x)$

$$-4x^2-5x+1 = 1(x^2+2x+1)+g(x)(x^2+x)+(-2)x$$

$$-4x^2-5x+1 = x^2+2x+1+g(x)(x^2+x)-2x$$
$$= x^2+1+g(x)(x^2+x)$$

$$-4x^2-5x+1-(x^2+1) = g(x)(x^2+x)$$

$$-4x^2-5x+1-x^2-1 = g(x)(x^2+x)$$

$$\frac{-5x^2-5x}{x^2+x} = \frac{g(x)\cancel{(x^2+x)}}{\cancel{x^2+x}}$$

$$\frac{-5x^2-5x}{x^2+x} = g(x) = \frac{-5x\cancel{(x+1)}}{x\cancel{(x+1)}}$$

$$g(x) = \frac{-5\cancel{x}}{\cancel{x}}$$

$$g(x) = -5$$

⑦ what form for Trig substitution

$$\int \frac{15\sqrt{x^2-100}}{x^4}\,dx \rightarrow \sqrt{x^2-c^2}$$

~~sec~~?

$\sqrt{x^2-c^2}$ | $c=10$, $c^2=100$

$\sqrt{x^2-c^2}$ = integral substitution form case

use $x = c\sec\theta$

$x = 10\sec\theta$

why?

a

$x = 10\sec\theta$

$dx = 10\sec\theta\tan\theta\,d\theta$

put into original integral

$$\int \frac{15(10\tan\theta)}{(10\sec\theta)^4}\cdot 10\sec\theta\tan\theta\,d\theta$$

$$\int \frac{1500\tan^2\theta\sec\theta}{10000\sec^4\theta}\,d\theta$$

$$\int \frac{\cancel{15}\,3\tan^2\theta\,\cancel{\sec\theta}}{\cancel{100}\,20\sec^{\cancel{4}\,3}\theta}\,d\theta$$

$$\int \frac{3\tan^2\theta}{20\sec^3\theta}\,d\theta$$

terms of $\cos\theta$ & $\sin\theta$

$$\int \frac{3}{20}\cdot\frac{\left(\frac{\sin\theta}{\cos\theta}\right)^2}{\left(\frac{1}{\cos\theta}\right)^3}\,d\theta$$

$$\frac{3}{20}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^3\theta}{1}\,d\theta \rightarrow \frac{\cos^3\theta}{\cos^2\theta} = \cos\theta$$

$$\therefore \frac{3}{20}\int(\sin^2\theta\cos\theta)\,d\theta$$

B

use substitution

let $u = \sin\theta$

$du = \frac{d}{d\theta}(\sin\theta)\,d\theta = \cos\theta\,d\theta$

$$\frac{3}{20}\int u^2\,d\theta$$

Power Rule $\rightarrow$ $$\frac{3}{20}\int \frac{u^{2+1}}{2+1} + C_i$$

$$= \frac{\cancel{3}}{20}\cdot\frac{u^3}{\cancel{3}} + C_i$$

$$= \frac{1}{20}u^3 + C_i$$

$$= \frac{1}{20}\sin^3\theta + C_i$$

C in terms of x

remember

$x = 10\sec\theta$

$\sin\theta = \frac{\sqrt{x^2-100}}{x}$

$$\therefore \frac{1}{20}\left(\frac{(x^2-100)^{1/2}}{x^3}\right)^3 + C$$

$$= \frac{1}{20}\cdot\frac{(x^2-100)^{3/2}}{x^3} + C$$

$$= \frac{(x^2-100)^{3/2}}{20x^3} + C$$