① Estimate Error using partial sum of 20 terms

$$\sum_{k=1}^{\infty} \frac{3}{k^3}$$

$$R_{20} = \sum_{k=1}^{\infty} 3/k^3$$

using integral test remainder estimate

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,d(x)$$

where $f(x) = 3/x^3$ & $n = 20$

$$\int_{20+1}^{\infty} 3/x^3\,dx \le R_{20} \le \int_{20}^{\infty} \frac{3}{x^3}\,dx$$

use upper bound

$$\int_{20}^{\infty} 3/x^3\,dx = \int_{20}^{\infty} x^{-3}\,dx = 3\left[\frac{x^{-2}}{-2}\right]_{20}^{\infty}$$

$$3\left(0 - \left(\frac{1}{-2\cdot 20^2}\right)\right) = 3/800 = 0.00\cancel{7}375$$

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$$\sum_{n=1}^{\infty} \frac{9}{1+n^2}$$

find value of improper integral

$$\int_1^{\infty} \frac{9}{1+x^2}\,dx$$

known Relationship antiderivative

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

$$\therefore \int_1^{\infty} \frac{9}{1+x^2}\,dx = 9\int_1^{\infty} \frac{1}{1+x^2}\,dx = 9\tan^{-1}(x)\Big|_1^{\infty}$$

$$\tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$\int_1^{\infty} \frac{9}{1+x^2} = 9\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \underbrace{\frac{2\pi}{4} - \frac{\pi}{4}}_{\text{comm. denominator}} = \frac{\pi}{4}$$

$$= 9\left(\pi/4\right) = \frac{9\pi}{4}$$

$f(x) = \frac{9}{1+x^2}$

since series is
① positive
② continuous
③ Decreasing on interval $(1, \infty)$

we can use the integral test to evaluate convergence

we know

$$\int_1^\infty \frac{9}{1+x^2}\,dx = \frac{9\pi}{4}$$ which is finite →

If the integral is finite then by definition, it converges & the series also converges.

area under curve is limited → converges

(10) Convergence test using root test

$$\sum_{n=1}^{\infty} \left(\frac{2n}{9n+1}\right)^n$$

Root test. ⟹ reviews the following limit.

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Let $a_n = \left(\dfrac{2n}{9n+1}\right)^n$

Use Root test

$$L = \lim_{n\to\infty} \cancel{\sqrt[n]{}} \left(\frac{2n}{9n+1}\right)^{\cancel{n}}$$

→ Exponent of n & root of n cancel each other out.

$$\therefore\ L = \lim_{n\to\infty} \frac{2n}{9n+1}$$

$f(n)$

now we can simplify rational function & take limit

$$= \frac{2\cancel{n}}{\cancel{n}(9+1/n)}$$ cancel n terms.

$$\therefore\ \frac{2}{9+\frac{1}{n}} \Rightarrow \lim_{n\to\infty} \frac{2}{9+1/n}$$ as $n \to \infty$

$\frac{1}{n}$ will trend to 0

$$\therefore\ L = \lim_{n\to\infty} \frac{2}{9+1/n} = \frac{2}{9+0} = \frac{2}{9}$$

Review for convergence

given $L = 2/9$

Since $L = \frac{2}{9} < 1$

The series converges.

if $L = 1$ → inconclusive another test needed.

if $L < 1$ → Convergence

if $L => 1$ → diverges or if $L = \infty$

(12)

$$\sum_{k=1}^{\infty} \frac{3}{k(k+2)}$$

~~Geo~~ ~~p-series~~

determine convergence & if so, find series sum

partial fractions approach:

$$\cancel{k(k+2)} \cdot \frac{3}{\cancel{k(k+2)}} = \frac{Z}{k} + \frac{\beta}{k+2} \cdot k(k+2)$$

$$3 = Z \cdot \frac{\cancel{k}(k+2)}{(k+2)} + \frac{\beta \cdot k\,\cancel{(k+2)}}{\cancel{k+2}}$$

$$3 = Z(k+2) + \beta k$$

$$3 = Zk + 2Z + \beta k$$

$$3 = (Z+\beta)k + 2Z$$

Match coefficients

$$\beta + Z = 0$$

$$2Z = 3$$

$$Z = \frac{3}{2} \quad \beta = -3/2$$

Partial sums

plug into series

$$\sum_{k=1}^{\infty} \frac{3}{k(k+2)} = \sum_{k=1}^{\infty} \frac{3}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{3}{2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$$

Expansion & cancel out

$$\frac{3}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots\right)$$

$$\frac{3}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

as $n \to \infty$, $\frac{1}{n+1} \to 0$

$\frac{1}{n+2} \to 0$

$$\therefore \frac{3}{2}\left(\frac{1}{1}+\frac{1}{2}-0-0\right)$$

$$\frac{3}{2}\left(\frac{3}{2}\right) = \frac{9}{4}$$

since partial sum approaches a finite number $\frac{9}{4}$ the series converges, by defin.

$\sum a_k$ converges if seq. of partial sums approaches a finite Limit as $n \to \infty$

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alternating series test for convergence

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n^2}{\sqrt{n^4+12}}$$

is this series alternating? yes

beginning of series makes signs alternate

$(-1)^{n+1} \Rightarrow -1, 1, -1 \ldots$ etc.

Two conditions:

(A) is $b_n \downarrow$?

(B) is $\lim_{n \to \infty} b_n = 0$

let $b_n = \dfrac{n^2}{\sqrt{n^4+12}}$

(A) $b'(n) \Longrightarrow f(n) = n^2$

$g(n) = (n^4+12)^{1/2}$

quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$f'(n) = 2n$$

$$g'(n) = (n^4+12)^{1/2} = 1/2\,(n^4+12)^{-1/2} \cdot 4n^3 = \frac{2n^3}{(n^4+12)^{1/2}}$$

$$b'(n) = \frac{2n\,(n^4+12)^{1/2} - n^2 \cdot \frac{2n^3}{(n^4+12)^{1/2}}}{(n^4+12)}$$

simplify & factor.

$$\frac{2n\,(n^4+12) - 2n^5}{(n^4+12)^{1/2}\,(n^4+12)} = \frac{2n\,(n^4+12-n^4)}{(n^4+12)^{3/2}} = \frac{24n}{(n^4+12)^{3/2}}$$

for $n > 0$, $b'(n)$ is positive

since $b'(n)$ is positive, first condition of alt. series test does not hold & does not apply to series.

(1f) Root test

$$\sum_{n=1}^{\infty} \left(\frac{2n+1}{5n+4}\right)^n$$

Root test conditions for convergence: $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

$L < 1 \longrightarrow$ converges

$L > 1 \longrightarrow$ diverges

$L = 1, \longrightarrow$ inconclusive

let $a_n = \left(\frac{2n+1}{5n+4}\right)^n$

$$L = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{2n+1}{5n+4}\right)^{\cancel{n}}} = \lim_{n\to\infty} \frac{2n+1 \div 1/n}{5n+4 \div 1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{2\cancel{n}}{\cancel{n}} + \frac{1}{n}}{\frac{5\cancel{n}}{\cancel{n}} + \frac{4}{n}}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{5 + \frac{4}{n}}$$

as $n \to \infty$ $\frac{1}{n} \longrightarrow 0$

$\frac{4}{n} \longrightarrow 0$

$$\therefore \lim_{n \to \infty} \frac{2 + \frac{1}{n}}{5 + \frac{4}{n}} = \frac{2}{5} = L$$

$L = \frac{2}{5} < 1 \longrightarrow$ series converges.

& $f(n) = \frac{2n+1}{5n+4}$ & $L = 2/5$

16 Maclaurin Series of

$f(x) = \frac{\sin(x)}{x^9}$ where $x \cancel{=} 0$, $f(0) = 1$

note $f(x)$ is continuous @ $\cancel{x} = 0$

use known maclaurin series for $\sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x^{2n+1}\right)$$

we need to divide series by missing denominator $\cdot x^9$

Rules of exponents $\frac{x^k}{x^n} = x^{k-n}$ or $x^k \cdot \frac{1}{x^n}$

$$\frac{\sin(x)}{x^9} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)} \cdot \frac{1}{x^9}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1-9}}{(2n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n-8}}{(2n+1)}$$

not correct maclaurin series with negative integer powers of x

must shift index to start @ point where exp. is non negative

$2n - 8 \geq 0$
$+8 \quad +8$

$2n \geq 8$

$n \geq 4$

$n=4$, new index $k=n+4$,

put back into series & check

$$\frac{(-1)^n x^{2n-8}}{(2n+1)!} = \frac{(-1)^{k+4} \cdot x^{2(k+4)-8}}{(2(k+4)+1)!} = \frac{(-1)^k x^{2k}}{(2k+9)!}$$

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Check for convergence?

$$\sum_{n=1}^{\infty} \left(\frac{17}{4}\right)^{n-1}$$

geometric series converges when

$|r| < 1$

$r = \frac{17}{4} = 4.25 > 1$

$\therefore$ series diverges

sum DNE

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Find $\lim_{n \to \infty} a_n$ where $\csc(7n)$ to

test for conver

known relationship

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\therefore a_n = \csc(7n) = \frac{1}{\sin(7n)}$$

$$\lim_{n \to \infty} \frac{1}{\sin(7n)}$$ sin(x) is bounded & periodic - between -1 & 1

sin(7n) gets very small ∴ csc(7n) gets large & does not approach any specific value the limit does not exist & the series diverges.

Rules of exponents

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let $a_n = \frac{n^6}{n^3} = n^3$

(A) convergence?

$$\lim_{n \to \infty} n^3 = \infty$$

no diverges to infinity not specific value.

(B) Bound or unbound

$a_n = n^3$

$\uparrow$ w/o bound $\therefore$ unbounded

(C) increasing? (monotonicity)

$a_{n+1} = (n+1)^3 > n^3 = a_n$

it is increasing

(D) since sequence diverges DNE

(4) $1 + \frac{1}{7} + \frac{1}{49 \text{ or } 7^2} + \frac{1}{343 \text{ or } 7^3}$

This is a geometric series in the form

$$\sum_{n=0}^{\infty} \beta \cdot r^n$$

$\beta = 1$

$r = \frac{1}{7}$

$$\text{Sum} = \sum_{n=0}^{\infty} \beta r^n = \frac{\beta}{1-r} = \frac{1}{1-\frac{1}{7}} = \frac{1}{\frac{6}{7}} = \frac{7}{6}$$

is $|r| < 1$?

$\frac{1}{7} < 1 \therefore$ the geometric series converges.

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Taylor series formula:

$f(x) = x^3$ Find taylor series centered @ $x=4$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n$$

coefficients : $\dfrac{f^n(4)}{n!}$

Find $f'(x)$ @ $x=4$ & f'', f''', f''''

$f(x) =$	x^3	64
$f'(x) = x^3$	$3x^2$	48
$f''(x) = 3x^2$	$6x$	24
$f'''(x) = 6x$	6	6
$f^n(x) = 0$ ~~for $n \geq 4$~~		0

coefficients:

$\dfrac{f(4)}{0!} = 64/1$

$\dfrac{f'(4)}{1!} = \dfrac{48}{1}$

$\dfrac{f''(4)}{2!} = \dfrac{24}{2} = 12$

$\dfrac{f'''(4)}{3!} = \dfrac{6}{6}$

$= 1$

(b)

$$\sum_{n=1}^{\infty} a_n, \text{ where } a_n = \frac{n + 2n^3 + 8n^2 + 6n + 9}{n^3}$$

Simplify a_n

$$a_n = \frac{2n^3 + 8n^2 + 7n + 9}{n^3}$$

Divergence Test

$$a_n = 2 + \frac{8}{n} + \frac{7}{n^2} + \frac{9}{n^3}$$

Lim

$$\lim_{n \to \infty} a_n = 2 + \left(\frac{8}{n} \to 0\right) + \left(\frac{7}{n^2} \to 0\right) + \left(\frac{9}{n^3} \to 0\right)$$

$$= 2$$

since $\lim_{n \to \infty} a_n \neq 0$ series diverges for divergence test

Comparison or Limit Comparison Test

Root

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \ ?$$

still tends to 0 except first value

Ratio Test

$$\frac{a_{n+1}}{a_n} = \left| \frac{2 + \frac{8}{n+1}}{2} \right|$$

$= 1$

$L = 1$ (inconclusive.

Alternating Series Test

not applicable.

all terms are positive specifically

7) use Ratio test

$$\sum_{n=1}^{\infty} \frac{n^5}{0.9^n} \qquad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

let $a_n = \dfrac{n^5}{0.9^n}$

$$a_{n+1} = \frac{(n+1)^5}{0.9^{n+1}} = \frac{(n+1)^5}{0.9^n \cdot 0.9}$$

calculate $L = \dfrac{(n+1)^5}{0.9^n \cdot 0.9} \cdot \dfrac{0.9^n}{n^5} = \dfrac{(n+1)^5}{n^5} \cdot \dfrac{1}{0.9}$

$$= \left(\frac{n+1}{n}\right)^5 \cdot \frac{1}{0.9} = \left(1 + \frac{1}{n}\right)^5 \cdot \frac{1}{0.9}$$

$f(n) = (1 + 1/n)^5 \cdot 1/0.9$

$$L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^5 \cdot \frac{1}{0.9} = 1^5 \cdot \frac{1}{0.9} = \frac{1}{0.9} = 10/9 = 1.1$$

since $L > 1$, series diverges

8 Limit comparison test

$$\sum_{n=1}^{\infty} \frac{n^4+6}{n^6-6}$$

① Dominant terms

what dictates expression behavior

$n^4+6 = n^4$

$n^6-6 = n^6$

$= \frac{n^4}{n^6}$ $\frac{x^m}{x^n} = x^{m-n}$

$= \frac{1}{n^2}$

$\therefore$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\therefore$ $p=2$

finite ✓

non zero $(n>0$

convergence

(a) (A) $\sum_{n=0}^{\infty} \frac{(x-5)^n}{(6n)!}$, centered @ $x=5$ not $x=0$ X

(B) $\sum_{n=0}^{\infty} \frac{x^n}{(5)^n}$, $x=0$ is the center since it has x^n form with no shifting

(C) $\sum_{n=0}^{\infty} \frac{(x+5)^n}{(6)^n}$, centered @ $x=-5$

(D) $\sum_{n=0}^{\infty} \frac{(x+5)^5}{(6)^n!}$ same, centeded @ $x=-5$

(E) $\frac{x^n}{(5)^n}$ $x=0$ ✓

(F) centered @ $x=-5$

(2) Looking for series centered @ $x=-6$

(A) $\sum \frac{(x-6)^n}{(8)^n}$ no, center is $x=6$

(B) $\sum \frac{x^n}{6^n}$ no, centered @ $x=0$

(C) $\sum \frac{x^n}{(-6)^n}$ no, centered @ $x=0$

(D) $\frac{(x+6)^n}{8^n}$ yes, centered @ $x=-6$

(E) $\frac{(x-6)^n}{(8n)!}$, no, centered @ $x=+6$

15 use alternative series est.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3}{n^{0.6}} \qquad b_n = \frac{3}{n^{0.6}}$$

absolute error < 0.01

e

$$|e| \leq b_{N+1} = 3/(N+1)^{0.6} < 0.01$$

solve

$$\frac{3}{(N+1)^{0.6}} < 0.01$$

$$(N+1)^{0.6} > \frac{3}{0.01}$$

$$(N+1)^{0.6} > 300$$

$$\left((N+1)^{0.6}\right)^{5/3} > 300^{5/3}$$

$$N+1 > 300^{5/3}$$

$$N+1 > (300^{1/3})^5$$

since $6^3 = 216$

$7^3 = 343$

$\therefore \quad 6 < 300^{1/3} < 7$

$300^{1/3}$ is roughly 6.69

Plug back into inequality

$N+1 > (6.69)^5$

$N+1 > 13488.76$

$N > 13487.76$ or 13488