1

A

$Salt_0 = 70kg$

$V_0 = 2000L$ (water)

$$\text{Concentration} = \frac{\text{how much solute we have (starting)}}{\text{Volume of given solution}}$$

$$= \frac{7\cancel{0}\,kg}{200\cancel{0}\,L}$$

$$= \frac{7}{2000}\ kg/L$$

$$= 0.035\ \frac{kg}{L}$$

B

after t_i, Find equation representing amount of salt in tank?

rate of change of salt in tank $= \dfrac{ds}{dt} = \Delta S$

Convert the flow from minutes to hours

$$9\,\frac{L}{mins} \times 60\ mins/hour = 540\ \frac{L}{hour}.$$

$\Delta S =$ salt in $-$ salt out

salt in = 0.0175 kg/L × flow rate (540 L/hour)

= 9.45 kg/hour

salt out

salt in tank @ t_1 = $\frac{s(t)}{V_0}$ = $\frac{s(t)}{2000L}$

×

Drain rate / flow rate = 540 L/hr

∴ salt out = $\frac{s(t)}{2000L}$ · 540 L/hour = $\frac{540\, s(t)}{2000}$ =

0.27 s(t) kg/hour.

diff. equation

$\frac{ds}{dt} = 9.45 - 0.27\,s(t)$

1st order ⇒ $\frac{ds}{dt} + 0.27\,s(t) = 9.45$

multiply both sides by integrating factor: $e^{0.27t}$

let Q(t) = salt in tank @ t_i (mins)

given = inflow = 0.0175 kg/L × 9 L/min = 0.1575 kg/min

Outflow = $\frac{Q(t)}{2000}$ × 9 = $\frac{9\,Q(t)}{2000}$

differential equation set up

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow}$$

$$= 0.1575 - \frac{9Q}{2000}$$

$$\frac{dQ}{dt} = 0.1575 - \frac{9}{2000}Q$$ $\longrightarrow$ linear 1st order condition solvable.

$$\frac{dQ}{dt} + \frac{9}{2000}Q = 0.1575$$

need integrating factor $\Longrightarrow \mu(t) = e^{\int 9/2000 \, dt} = e^{9t/2000}$

$$\mu(t) \cdot \left(\frac{dQ}{dt} + \frac{9}{2000}Q = 0.1575 \right)$$

$$= e^{9t/2000}\frac{dQ}{dt} + e^{9t/2000} \cdot \frac{9}{2000}Q = e^{9t/2000} \cdot 0.1575$$

$$\int \frac{d}{dt}\left(Qe^{9t/2000}\right) = \int 0.1575e^{9t/2000}$$

$$Q(t)e^{9t/2000} = \int 0.1575e^{9t/2000}\,dt$$

$$\int 0.1575e^{9t/2000}\,dt = \frac{0.1575 \cdot 2000}{9} \cdot e^{9t/2000} + C$$

$$= 35e^{9t/2000} + C$$

$$Q(t)e^{\frac{9t}{2000}} = 35e^{9t/2000} + C \text{ — } Q(t) = 35 + Ce^{-9t/2000}$$

since $Q(0) = 70$

$$70 = 35 + C$$

$$C = 35$$

plug back.

$$Q(t) = 35 + 35e^{-9t/2000} \quad t \text{ in minutes}$$

need t in hours

$$t_{\text{minutes}} = 60t_{\text{hours}}$$

$$Q(t) = 35 + 35e^{\frac{-9(60t)}{2000}}$$

$$= 35 + 35e^{-540t/2000}$$

C. after 4.5 hours.

plug & calculate

$$Q(4.5) = 35 + 35e^{-\frac{540(4.5)}{2000}} = 35 + 35e^{-1.215}$$

$$= 35 + 35(0.2968)$$

$$= 45.38 \text{ kg}$$

(D) $\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} 35 + 35e^{-\frac{540t}{2000}}$

$\lim_{t \to \infty} \Rightarrow e^{\frac{-540t}{2000}}$

we know that

$e^{x} \Uparrow$ fast as $x \longrightarrow +\infty$

$e^{-x} \downarrow$ fast as $x \longrightarrow -\infty$

$e^{-540t/2000} = e^{-0.27t}$

as $t \to \infty$, $-0.27t$ exponents tends towards $-\infty$

the recipical of that number $\left(\frac{1}{e^{0.27}}\right) \to \frac{1}{\infty}$ $=0$

(3) Solve the separable diff. equation

$$\frac{dy}{dx} = -3y$$

initial condition $\Rightarrow$ $y(0) = -8$

~~seperate x & y~~

$$\frac{dy}{dx} = -3y$$

$$dx\left(\frac{1}{y}\frac{dy}{dx} = -3\right)$$

$$\frac{1}{y}\,dy = -3\,dx$$

integrate

$$\int \frac{1}{y}\,dy = \int -3\,dx$$

$$\ln|y| + c_x = -3x + c_B \longrightarrow \quad c_x + c_B = c$$

$$\ln|y| = -3x + c$$

solve for y

exponentiate both sides.

$$e^{\cancel{\ln|y|}} = e^{-3x+c}$$

$$y = e^{-3x} \cdot e^{c}$$

let $A = \pm e^c$

$$y = A \cdot e^{-3x}$$

since $y(0) = -8$

$$y(0) = A \cdot e^{-3(0)}$$

$$= A \cdot 1$$

$$y(0) = A$$

$$-8 = A$$

$$\therefore y = -8 \cdot e^{-3x}$$

(4)

$$\frac{dy}{dt} = 5y + ty + \sec(t)$$

<u>Combine terms</u>

$$\frac{dy}{dt} = y(5+t) + \sec(t)$$

<u>Rewrite in linear form</u>

$$\frac{dy}{dt} + P(t)y = Q(t)$$

$$\frac{dy}{dt} - (5+t)y = \sec(t)$$

5 general solution

$y' = 2\cos(4t)$ → note 1st order diff. eq. can be solved using integration.

$$\frac{dy}{dt} = 2\cos(4t)$$

$$\int \frac{dy}{dt}\,dt = \int 2\cos(4t)\,dt$$

$$\to 2 \cdot \int \cos(4t)\,dt$$

substitution

$u = 4t$

$du = 4dt$

$dt = \frac{du}{4}$

$$\int \cos(4t)\,dt = \frac{1}{4}\int \cos(u)\,du =$$

$$\frac{1}{4}\sin(u) = \frac{1}{4}\sin(4t)$$

$$= 2 \cdot \frac{1}{4}\sin(4t) = \frac{1}{2}\sin(4t)$$

$$\therefore y(t) = \frac{1}{2}\sin(4t) + C$$

7

$y = \frac{1}{5}x + b$

note parallel lines have same slope

$\therefore \; y_2 = \frac{1}{5}x + \theta$ & $y_1 = \frac{1}{5}x + 4$

b) How long will it take for the population to increase to 4800 (half of the carrying capacity)?

given point $(-1, 0)$

$y = \frac{1}{5}x + \theta$

$0 = \frac{1}{5}(-1) + \theta$

$0 = \frac{1}{-5} + \theta$

$\theta = \frac{1}{5}$

$\therefore \; y_2 = \frac{1}{5}x + \frac{1}{5}$

6

general solution for logistic equation

$$P(t) = \frac{k}{1 + Ae^{-kt}}$$

$k = 9600$

$A =$ constant from initial condition 18.2

$k =$ constant solving for

↳ when $P(0) = 500$

$$\therefore P(0) = 500 = \frac{9600}{1 + Ae^{0}} = \frac{9600}{1 + A}$$

$$1 + A = \frac{9600}{500} = 19.2.$$

$$1 + A = 19.2$$
$$A = 18.2$$

solve k

pop doubles at $P(1)$

$$\therefore P(1) = 1000 = \frac{9600}{1 + 18.2e^{-k}} \quad 1 + 18.2e^{-k}$$

$$\frac{1000(1 + 18.2e^{-k})}{1000} = \frac{9600}{1000}$$

$$1 + 18.2e^{-k} = 9.6$$

$$\frac{\cancel{18.2}e^{-k}}{\cancel{18.2}} = \frac{8.6}{18.2}$$

$$\ln\left(e^{-k} = \frac{8.6}{18.2}\right)$$

$$-1 \quad -k = -1\ln\left(\frac{8.6}{18.2}\right)$$

$$k = -\ln\left(\frac{8.6}{18.2}\right)$$

simplify $k = -\ln\left(\frac{\cancel{42}\ 43}{91}\right)$

$$k = \ln\left(91/43\right)$$

$B =$ known $K = 9600$

$A = 18.2$ or $\frac{91}{5}$

$k = \ln\left(\frac{91}{43}\right)$

sub into logistic formula

$$P(t) = \frac{9600}{1 + \frac{91}{5}e^{-kt}}.$$

$$e^{-kt} = e^{-t\ln\left(\frac{91}{43}\right)} = \left(\frac{91}{43}\right)^{-t} = \left(\frac{43}{91}\right)^{t}$$

$$\therefore\ P(t) = \frac{9600}{1 + \frac{91}{5}\left(\frac{43}{91}\right)^{t}}$$

C = use pop equation & solve for t.

$$\frac{4800}{1} = \frac{9600}{1 + \frac{91}{5} e^{-\ln\left(\frac{91}{43}\right)t}}$$

$$\frac{\cancel{4800}\left(1+\frac{91}{5}e^{-\ln\left(\frac{91}{43}\right)t}\right)}{\cancel{4800}} = \frac{\cancel{9600}}{\cancel{4800}}\,2$$

$$1 + \frac{91}{5} e^{-\ln\left(\frac{91}{43}\right)t} = 2$$

$$\frac{\frac{91}{5} e^{-\ln\left(\frac{91}{43}\right)t}}{\frac{91}{5}} = \frac{1}{\frac{91}{5}} \text{ or } 18.2$$

$$\ln\left(= e^{-\ln\left(\frac{91}{43}\right)t} = \frac{1}{18.2} \text{ or } \frac{91}{5}\right)$$

$$-\ln\left(\frac{91}{43}\right)t = \ln\left(\frac{1}{18.2}\right)$$

$$\frac{\cancel{-\ln\left(\frac{91}{43}\right)}t}{\cancel{-\ln\left(\frac{91}{43}\right)}} = \frac{-\ln(18.2) \text{ or } \left(\frac{91}{5}\right)}{-\ln\left(\frac{91}{43}\right)}$$

$$t = \frac{\ln(18.2)}{\ln\left(\frac{91}{43}\right)} \text{ or } \ln\left(\frac{91}{5}\right)$$

8

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{k}\right).$$ logistic growth eq.

we know

$k = 60$

when $y = 28$ / Rate of sales is 8000/month

$\frac{dy}{dt} = 8$

$y = 28$

$\cancel{k} = 60$

$$8 = k \cdot 28\left(1 - \frac{28}{60}\right)$$

simplify & solve for k

$$8 = k \cdot 28\left(1 - \frac{28}{60}\right) \longrightarrow \frac{60-28}{60} = \frac{32}{60} = 8/15.$$

$$8 = k \cdot 28 \cdot \frac{8}{15}$$

$$8 = k \cdot \frac{224}{15}$$

$$k = \frac{8 \cdot 15}{224} = \frac{120}{224} = 15/28$$

$$\therefore \frac{dy}{dt} = \frac{15}{28}y\left(1 - \frac{y}{60}\right)$$

$$= \frac{15}{28}y\left(\frac{60-y}{60}\right) = \frac{15y(60-y)}{28 \cdot 60}$$

$$\frac{dy}{dt} = \frac{y(60-y)}{112}$$

9) $xy' = 3y + x^2$ with $x > 0$

need linear form:

$$y' + P(x)y = Q(x)$$

$$\frac{\cancel{x}y'}{\cancel{x}} = \frac{3y + x^2}{x}$$

$$y' = \frac{3y}{x} + x$$

solve for x

$$x = y' - \frac{3}{x}y$$

$$y' + P(x)y = Q(x)$$

$$P(x) = -\frac{3}{x}, \quad Q(x) = x$$

$$x^{-3}\left(y' - \frac{3}{x}y = x\right) = x^{-3}y' - \frac{3}{x}x^{-3}y = x \cdot x^{-3}$$

$$= x^{-3}y' - 3x^{-4}y = x^{-2}$$

equal to

$\frac{d}{dx}(x^{-3}y)$ $\therefore$

$$\frac{d}{dx}(x^{-3}y) = x^{-2}$$

integrate both sides

$$\int \frac{d}{dx}(x^{-3}y)\,dx = \int x^{-2}\,dx$$

$$= \quad x^{-3}y = -x^{-1} + C$$

solve for y

$$y = x^{-3}(x^{-1} + C) = -x^{2} + Cx^{3}$$

$$y = -x^{2} + Cx^{3}$$

or

$$y = Cx^{3} - x^{2}$$

10

$$\frac{dy}{dt} = -8t$$

integrate both sides

$$y = \int -8t\,dt = -4t^2 + C$$

A. $y(0) = -9$

$$-9 = -4t^2 + C$$
$$= 0 + C$$
$$-9 = C$$
$$\therefore\ y = -4t^2 - 9$$

B. $y(0) = 1$

$$1 = -4t^2 + C$$
$$1 = 0 + C$$
$$1 = C$$
$$y = -4t^2 + 1$$

11

$$\frac{dy}{dx} = yx(x+9)$$

standard form.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} - yx(x+9) = 0$$

$$\therefore\ \frac{dy}{dx} + P(x)y = 0 \qquad = -x(x+9),\ \&\ Q(x) = 0$$

$$\frac{dy}{dx} - x(x+9)y = 0$$

$$\frac{dy}{dx} - (x^2+9x)y = 0$$

13 known values:

$T_0 = 179°$

Ambient T $(T_A) = 65°$

after $t = 13$ minutes, $T = 172°$

unknown:

t when $T = 158$

we can use Newton's law of cooling here

$$\frac{dT}{dt} = k(T - T_A)$$

$$T(t) = T_A + (T_0 - T_A)e^{kt}$$

$$T(t) = 65 + (179-65)e^{kt} = 65 + 114e^{kt}$$

solve for k

at $t = 13$, & $T = 172°$

$$172 = 65 + 114e^{13k}$$

$$107 = 114e^{13k}$$

$$\left(\frac{107}{114} = e^{13k}\right) \ln$$

$$\frac{13k}{13} = \frac{\ln\left(\frac{107}{114}\right)}{13}$$

$$k = -0.00487$$

when $T = 158$

$$T(t) = 65 + 114e^{kt}$$

$$158 = 65 + 114e^{kt}.$$

$$93 = 114e^{kt}.$$

$$\frac{93}{114} = e^{kt} \Longrightarrow \ln\left(\frac{93}{114}\right) = kt$$

$$t = \frac{\ln(93/114)}{k} = \frac{-0.20346}{-0.00487}$$

$$= 41.786$$

$$= 41.80$$

(14) Visual based on ODE, expression & direction field

(15) point $(0.5, 1.5)$

diff. eq $\Rightarrow \frac{dy}{dx} = -xy$

slope of direction field @ point A = dy/dx @ point A.

$\therefore \frac{dy}{dx} = -xy = -(0.5)(1.5)$

$= -0.75$

@ point $(-1, -1.5)$

$\frac{dy}{dx} = -xy = (-1)(-1.5)$

$= 1.5$

(6)

$\frac{dy}{dx}$ $54yx^5$

seperate the differential equation

$\frac{dy}{y} = 54x^5 dx$

integrate both sides:

$\int \frac{dy}{y} = \int 54x^5 dx$

$\ln|y| = 54\int x^5 dx$

$= 54 \cdot \frac{x^{5+1}}{5+1} + C$

$= \frac{\cancel{54}x^6}{\cancel{6}} + C$

$e \left(\ln|y| = 9x^6 + C \right)$

$|y| = e^{9x^6 + c}$

$= e^{9x^6} \cdot e^{c} \longrightarrow$ positive constant.

let $e^c = A$

$y = Ae^{9x^6}$

given y-intercept of $y(x)$ is 2

$y(0) = 2$

$x = 0$

$\therefore \quad y = Ae^{9x^6} \Longrightarrow \quad 2 = Ae^{9 \cdot 0^6}$

$2 = A \cdot \cancel{e^0}\ 1$

$2 = A$

$\therefore \quad y = 2e^{9x^6}$

(17) $\quad \dfrac{dp}{dt} = \dfrac{8}{700} p(7-p)$

$(7-p)$ must be greater than 0 so that $dp/dt > 0$

$\therefore \quad 7 - p > 0$

$7 > p$

$p < 7$

(A) Population should increase when $0 < p < 7$

(B) when $p > 7$ population is decreasing as

$\frac{8}{700} P(-) = -C$

(c) $P(0) = 2$, Find $P(64)$

$$\frac{dP}{dt} = \frac{8}{700} P(7-P)$$
$$= \frac{2}{175} P(7-P) \longrightarrow \text{seperate dif eqn}$$

$$\frac{dP}{P(7-P)} = \frac{2}{175} dt$$

partial fract:

$$\left(\frac{1}{7-P} = \frac{A}{P} + \frac{B}{7-P}\right) P(7-P)$$

$$1 = A(7-P) + BP$$
$$= 7A + (B-A)P$$

look @ coefficients:

$7A = 1 \qquad B - A = 0$

$A = \frac{1}{7} \qquad B = A = 1/2$

∴ equation is now:

$$\left(\frac{\frac{1}{7}}{p} + \frac{\frac{1}{7}}{7-p}\right) dp = \frac{2}{175} dt$$

integrate

$$\frac{1}{7}\int \left(\frac{1}{p} + \frac{1}{7-p}\right) dp = \int \frac{2}{175} dt$$

$$\frac{1}{7}\left(\ln|p| - \ln|7-p|\right) = \frac{2}{175}t + C$$

$$\frac{1}{7}\ln\left|\frac{p}{7-p}\right| = \frac{2}{175}t + C$$

$$\ln\left|\frac{p}{7-p}\right| = 7\left(\frac{2}{175}t + C\right)$$

$$= \frac{14}{175}t + C$$

$$\downarrow$$

$$\ln\left|\frac{p}{7-p}\right| = \frac{2}{25}t + k$$

Bring Back initial condition $P(0) = 2$, $t = 0, p = 2$

$$\ln\left|\frac{2}{7-2}\right| = \frac{2}{25}0 + k$$

$$\ln\left|\frac{2}{5}\right) = k.$$

$$k = \ln\left(\frac{2}{5}\right)$$

Return to previous equation

$$e\cdot\left(\ln\left|\frac{P}{7-P}\right| = \frac{2}{25}t + \ln\left(\frac{2}{5}\right)\right)$$

$$\left|\frac{P}{7-P}\right| = e^{\frac{2}{25}t} \cdot \cancel{e^{\ln(2/5)}}$$

$$\left|\frac{P}{7-P}\right| = \frac{2}{5}e^{\frac{2}{25}t}$$

absolute value drops since P will remain between 0 & 7

$$\left(\frac{P}{7-P} = \frac{2}{5}e^{\frac{2}{25}t}\right) 5(7-P)$$

$$5P = 2e^{\frac{2}{25}t}(7-P)$$

$$= 14e^{2/25t} - 2Pe^{2/25t}$$

$$5P + 2Pe^{2/25t} = 14e^{2/25t}$$

$$\frac{P(5+2e^{2/25t})}{\cancel{5+2e^{2/25t}}} = \frac{14e^{2/25t}}{5+2e^{2/25t}}.$$

$$P(t) = \frac{14e^{2/25t}}{5+2e^{2/25t}}$$

sub $P(65) = \frac{14e^{2/25\cdot 65}}{5+2e^{2/25\cdot 65}} = \frac{14e^{128/25}}{5+2e^{128/25}}$

$6.897 \approx 6.90$

(18)

$S_0 = 50\,kg$

$V_0 = 2000L$

Water Flow rate $= 8L/min$

drain $= 4L/min$

(A) given 50kg @ S_0

(B) same format as before -

$\frac{dS}{dt}$ = rate in - rate out $\longrightarrow$ $\frac{s(t)}{v(t)} = \frac{4 \cdot S}{V_0 + \text{drain}}$

0

$$= \frac{4s}{2000+4t}$$

$$\frac{ds}{dt} = 0 - \frac{4s}{2000+4t}$$

$$\frac{ds}{dt} = -\frac{4s}{2000+4t}. \longrightarrow$$

solve diff. eq.

$$\frac{ds}{s} = -\frac{\cancel{4}}{\cancel{4}(500+t)}\,dt$$

$$\text{integrate}\left(\frac{ds}{s} = -\frac{1}{500+t}\,dt\right)$$

$$\int \frac{ds}{s} = \int -\frac{1}{500+t}\,dt$$

$$\ln(s) = -\ln|500+t| + c$$

$$= \ln\left|\frac{1}{500+t}\right| + c$$

$$e^{\left(\ln|s| = \ln\left|\frac{1}{500+t}\right| + c\right)}$$

$$|s| = e^{\ln\frac{1}{500+c}} \cdot e^{c}$$

amount of salt must be non negative so

$A = \pm e^c$ & $A > 0$

$$S(t) = A \cdot \frac{1}{500+t}$$

Find A using initial condition

$t = 0 \Rightarrow S(0) = 50 kg$

$$S(0) = 50 kg = A \cdot \frac{1}{500+0}$$

$$50 = \frac{A}{500}$$

$$A = 25000$$

$\therefore$ amount of salt in tank at t is $S(t) = \frac{25000}{500+t}$

Question

$$S(2.5) = \frac{25000}{500 + 2.5(60)} = \frac{2500\cancel{0}}{65\cancel{0}} = \frac{500}{13}$$

$$= 38.46$$

(c) Theoretically as pure water continuously enters the tank with a fixed amount of salt, the concentration of salt solution would tend to 0.

Mathematically:

$$s(t) = \frac{25000}{500+t} \longrightarrow$$ salt in tank @ time t (min si)

$$v(t) = 2000 + 4t \longrightarrow$$ Volume in tank @ time t

Concentration of salt in tank @ time t.

$$\frac{s(t)}{v(t)} = \frac{\frac{25000}{500+t}}{2000+4t}$$

$$\lim_{t \to \infty} \frac{s(t)}{v(t)} = \frac{25000}{(500+t)(2000+4t)}$$

square

since we have

$$\frac{c}{\text{polynomial}} \rightarrow$$ denominator will increase significant faster than numerator.

so the fraction will tend to 0

$$\lim_{t \to \infty} \frac{s(t)}{v(t)} = 0$$