Higher Category Theory

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1 Background category theory

Definition 1.1 Given $F : \mathcal{C} \to \mathcal{D}$ and $x : \mathcal{D}$ we can define the slice category over F as the pullback:

$$\begin{tikzcd} \mathcal{C}_{/F} \arrow[r] \arrow[d] & \mathcal{D}_{/x} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

Explicitly objects are given as pairs $c : \mathcal{C}$ and morphisms $Fc \to x$.

Example 1.2 Over the Yoneda embedding $よ : \mathcal{C} \to \hat{\mathcal{C}}$ and $F : \hat{\mathcal{C}}$ we have an element of $\mathcal{C}_{/F}$ is a pair of $c : \mathcal{C}$ and a morphism $よ_c \to F$, that is an element $F(c)$.

Lemma 1.3 *Given* $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *the natural map* $\mathrm{colim}_{c \in \mathcal{C}_{/F}} よ_c \to F$ *is an equivalence*

Proof. The natural map is given by choosing, for each $c : \mathcal{C}_{/F}$, the specified map $よ_c \to F$.

We have equivalences, for each presheaf G

$$\hom_{\hat{\mathcal{C}}}(\operatorname*{colim}_{c:\mathcal{C}_{/F}} よ_c, G) \cong \lim_{c \in \mathcal{C}_{/F}} \hom(よ_c, G) \cong \lim_{c \in \mathcal{C}_{/F}} G(c) \simeq \hom(F, G)$$

giving the natural equivalence by Yoneda. The last equivalence is quick to check by hand. □

Definition 1.4 (Kan extension (over inclusions)) Given $i : \mathcal{C}_0 \subseteq \mathcal{C}$, the inclusion of a subcategory, and $\mathcal{D}$ a (co)complete category, the restriction functor $i^* : \hom(\mathcal{C}, \mathcal{D}) \to \hom(\mathcal{C}_0, \mathcal{D})$ has left and right adjoints, given by

$$i_!(F)(x) = \operatorname*{colim}_{c \in \mathcal{C}_0/x} F(c)$$

$$i_*(F)(x) = \lim_{c \in \mathcal{C}_0} F(c)$$

1.1 Simplicial sets

We define the category Δ with objects given by $[n] = \{0, \dots, n\}$ and morphisms given by monotone maps. We define the category sSet to be presheaves on Δ. We denote the yoneda embedding of n to be Δ^n. This gives "expected" geometric results, such as $\hom_{\mathrm{sSet}}(\Delta^n, X) = X_n$, the n-simplices of X are given by morphisms of n-simplices.

Lemma 1.5 *Given a simplicial set X, we have*

$$X \cong \operatorname*{colim}_{[n]\in\Delta/X} \Delta^n$$

Proof. This is exactly the fact that presheaves are colimits of representables. □

Definition 1.6 In the category of topological spaces, we define Δ^n_{Top} to be the subset of $\mathbb{R}^{n+1}$ of tuples who sum to 1. This defines a cosimplicial object in Top by

$$[n] \mapsto \Delta^n_{\mathrm{Top}}$$

$$f : [n] \to [m] \mapsto f_*$$

where f_* is the unique affine linear extension of the map sending vertices of Δ^n_{Top} to their image under f. More precisely:

$$f_*(t_0, \dots t_n) = (v_0, \dots, v_m)$$

where

$$v_m = \sum_{j\in f^{-1}(i)} t_j$$

Definition 1.7 Given a topological space X, its singular simplicial set is given by

$$S(X) = [n] \mapsto \hom(\Delta^n_{\mathrm{Top}}, X)$$

.

We have a functor in the opposite direction given by Kan extension of the map sending $\Delta^n \mapsto \Delta^n_{\mathrm{Top}}$. Explicitly

$$|X| = \operatorname*{colim}_{\Delta^n \in \Delta_{/X}} \Delta^n_{\mathrm{Top}}$$

We have $|\_|$ is left adjoint to S.

Definition 1.8

1. $\partial\Delta^n$ has m-cells given by non-surjective $[m] \to [n]$.
2. Given $S \subseteq [n]$ the **horn** Λ^n_S has m-cells given $f : [m] \to [n]$ so there is $i \in [n]\backslash S$ s.t. $i \notin \mathrm{im}(f)$.
3. The **spine** I^n has m-cells given by $[m] \to [n]$ with image $\{j\}$ or $\{j, j+1\}$.

Definition 1.9 We can form the subcategory $\Delta_{\leq n} \subseteq \Delta$. The natural adjunction given by kan extending the inclusion gives monads on simplicial sets called the skeleta and coskeleta. Specifically

$$\mathrm{sk}_n(X) = i_! i^* X$$

$$\mathrm{cosk}_n(X) = i_* i^* X$$

We have $\mathrm{cosk}_n(X)_k = \hom(\mathrm{sk}_n(\Delta^k), X)$

Definition 1.10 Given a category $\mathcal{C}$, its nerve is the simplicial set given by

$$[n] \mapsto \hom([n], \mathcal{C})$$

That is chains of n composable maps.

Definition 1.11 Given a group G its classifying space is the geometric realisation of its nerve as a category with one object.

Definition 1.12 A **Kan complex** is a simplicial set X such that for all Horns Δ^n_j there are solutions to lifting problems

$$\begin{tikzcd} \Delta^n_j \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Example 1.13 The singular simplicial set of a topological space is a Kan complex. This follows quickly by adjunction, noting that a horn is a retract of Δ^n in topological spaces.

Example 1.14 The nerve of a category is a Kan complex if and only if the category is a groupoid.

Definition 1.15 Given $f, g : X \to Y$ maps of simplicial sets, we say they are homotopic if there exist a map $H : \Delta^1 \times X \to Y$ restricting to f, g on boundary. We can define the simplicial homotopy groups as homotopy classes of maps $(\Delta^n, \partial\Delta^n) \to (X, x)$.