Higher Category Theory

mark

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1 Background category theory

Definition 1.1 Given $F : \mathcal{C} \to \mathcal{D}$ and $x : \mathcal{D}$ we can define the slice category over F as the pullback:

$$\begin{tikzcd} \mathcal{C}_{/F} \arrow[r] \arrow[d] & \mathcal{D}_{/x} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

Explicitly objects are given as pairs $c : \mathcal{C}$ and morphisms $Fc \to x$.

Example 1.2 Over the Yoneda embedding $よ : \mathcal{C} \to \hat{\mathcal{C}}$ and $F : \hat{\mathcal{C}}$ we have an element of $\mathcal{C}_{/F}$ is a pair of $c : \mathcal{C}$ and a morphism $よ_c \to F$, that is an element $F(c)$.

Lemma 1.3 *Given $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ the natural map $\mathrm{colim}_{c \in \mathcal{C}_{/F}} よ_c \to F$ is an equivalence*

Proof. The natural map is given by choosing, for each $c : \mathcal{C}_{/F}$, the specified map $よ_c \to F$.
We have equivalences, for each presheaf G

$$\hom_{\hat{\mathcal{C}}}(\operatorname*{colim}_{c : \mathcal{C}_{/F}} よ_c, G) \cong \lim_{c \in \mathcal{C}_{/F}} \hom(よ_c, G) \cong \lim_{c \in \mathcal{C}_{/F}} G(c) \simeq \hom(F, G)$$

giving the natural equivalence by Yoneda. The last equivalence is quick to check by hand. □

Definition 1.4 (Kan extension (over inclusions)) Given $i : \mathcal{C}_0 \subseteq \mathcal{C}$, the inclusion of a subcategory, and $\mathcal{D}$ a (co)complete category, the restriction functor $i^* : \hom(\mathcal{C}, \mathcal{D}) \to \hom(\mathcal{C}_0, \mathcal{D})$ has left and right adjoints, given by

$$i_!(F)(x) = \operatorname*{colim}_{c \in \mathcal{C}_0/x} F(c)$$

$$i_*(F)(x) = \lim_{c \in \mathcal{C}_0} F(c)$$

1.1 Simplicial sets

We define the category Δ with objects given by $[n] = \{0, \ldots, n\}$ and morphisms given by monotone maps. We define the category sSet to be presheaves on Δ. We denote the yoneda embedding of n to be Δ^n. This gives "expected" geometric results, such as $\hom_{\mathrm{sSet}}(\Delta^n, X) = X_n$, the n-simplices of X are given by morphisms of n-simplices.

Lemma 1.5 *Given a simplicial set X, we have*

$$X \cong \operatorname*{colim}_{[n]\in\Delta/X} \Delta^n$$

Proof. This is exactly the fact that presheaves are colimits of representables. $\square$

Definition 1.6 For a simplicial set X we define

$$\pi_0^\Delta(X) = X_0/\sim$$

where x y iff there exists $f : X_1$ such that $d_0(f) = x$ and $d_1(f) = y$.

Exercise 1 Find examples of simplicial sets where $\sim$ is not symmetric, or not transitive.

Solution For the first example we consider Δ^1. This has non-degenerate 0-simplices given by $\Delta^1([0]) = [0] \to [1]$ by the constant maps at 0 and 1. And it has non-denerate 1-simplex given by the identity map $(0,1) : [1] \to [1]$. This has $d_0(0,1) = 1$ and $d_1(0,1) = 0$ so that $1 \sim 0$. But there is no 1-simplex giving $0 \sim 1$ so the relation isn't symmetric.
For a non-transitive example consider the spine $I^2 \subseteq \Delta^2$. This has non-degenerate 1-simplices given by $(0,1)$ and $(1,2)$. Hence we have $0 \sim 1$ and $1 \sim 2$ but not $0 \sim 2$.

Exercise 2 Let X : sSet. Every simplex $x : X_n$ is uniquely determined by a surjection $\alpha : [n] \to [m$ and $y : X_m$ with y non-degenerate and $\alpha^*(y) = x$.

Solution Note if x is degenerate, then by definition there exists some surjection α and some y such that $\alpha^*(y) = x$. Similarly if x is non-degenerate then there also exists a surjection, namely $\mathrm{id} : [n] \to [n]$, and this satisfies $\mathrm{id}^*(x) = x$.
We take the minimum $[m]$ such that there exists an α and a y. We claim such a y is non-degenerate. If it were then we would have a surjection $[k] \to [m]$ with $k < m$, composing would give a smaller value. Hence y is non-degenerate.
We now claim this is unique. Suppose we had $\beta : [n] \to [k]$ surjective with $\beta^*(z) = x$. Then by assumption $m \leq k$. By surjectivity we can construct a map $[k] \to [m]$ which composing with β gives α. Since z is not degenerate we deduce $k = m$ and β is the identity.

Exercise 3 Compute the limit and colimit of a simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$.

Solution Since the category Δ^{op} has an initial object, we can compute that the limit of X is precisely X_0. The colimit of X is given by the quotient of $\bigsqcup X_m$ by the relation generated by $x \sim \sigma^*(x)$. Note that all elements are related to an element of X_0 by the first vertex map. And pairs of elements of X_0 are related exactly when there is an $f : X_1$ connecting them. So we have that the colimit of X is precisely $\pi_0^\Delta(X)$.

Definition 1.7 In the category of topological spaces, we define Δ^n_{Top} to be the subset of $\mathbb{R}^{n+1}$ of tuples who sum to 1. This defines a cosimplicial object in Top by

$$[n] \mapsto \Delta^n_{\mathrm{Top}}$$

$$f : [n] \to [m] \mapsto f_*$$

where f_* is the unique affine linear extension of the map sending vertices of Δ^n_{Top} to their image under f. More precisely $f_*(t_0, \dots t_n) = (v_0, \dots, v_m)$ where $v_m = \sum_{j \in f^{-1}(i)} t_j$

Definition 1.8 Given a topological space X, its singular simplicial set is given by

$$S(X) = [n] \mapsto \hom(\Delta^n_{\text{Top}}, X)$$

.
We have a functor in the opposite direction given by Kan extension of the map sending $\Delta^n \mapsto \Delta^n_{\text{Top}}$. Explicitly

$$|X| = \operatorname*{colim}_{\Delta^n \in \Delta_{/X}} \Delta^n_{\text{Top}}$$

We have $|\_|$ is left adjoint to S.

Definition 1.9

1. $\partial\Delta^n$ has m-cells given by non-surjective $[m] \to [n]$.
2. Given $S \subseteq [n]$ the **horn** Λ^n_S has m-cells given $f : [m] \to [n]$ so there is $i \in [n] \backslash S$ s.t. $i \notin \operatorname{im}(f)$.
3. The **spine** I^n has m-cells given by $[m] \to [n]$ with image $\{j\}$ or $\{j, j+1\}$.

Definition 1.10 We can form the subcategory $\Delta_{\leq n} \subseteq \Delta$. The natural adjunction given by kan extending the inclusion gives monads on simplicial sets called the skeleta and coskeleta. Specifically

$$\operatorname{sk}_n(X) = i_! i^* X$$

$$\operatorname{cosk}_n(X) = i_* i^* X$$

We have $\operatorname{cosk}_n(X)_k = \hom(\operatorname{sk}_n(\Delta^k), X)$

Exercise 4 For all $n \geq 0$ we have a pushout

$$\begin{tikzcd} \coprod_{J_n} \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1}(X) \arrow[d] \\ \coprod_{J_n} \Delta^n \arrow[r] & \operatorname{sk}_n(X) \end{tikzcd}$$

Where J_n are the non-degenerate n-simplices.

Solution We will turn this into a Yoneda style gluing argument to make our lives easier. Let Y be a simplicial set. We have to show that

$$\begin{tikzcd} \hom(\coprod_{J_n} \partial\Delta^n, Y) & \hom(\operatorname{sk}_{n-1}(X), Y) \arrow[l] \\ \hom(\coprod_{J_n} \Delta^n, Y) \arrow[u] & \hom(\operatorname{sk}_n(X), Y) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback. By using the adjunction, and the Yoneda lemma this diagram is equivalent to

$$\begin{tikzcd}
\prod_{J_n} \hom(\partial\Delta^n, Y) & \hom(i_{n-1}X, i_{n-1}Y) \arrow[l] \\
(Y_n)^{J_n} \arrow[u] & \hom(i_nX, i_nY) \arrow[l] \arrow[u]
\end{tikzcd}$$

Further we note that a map $\partial\Delta^n \to Y$ is determined by its values on the $n-1$ cells of $\partial\Delta^n$, realising this set as a subset of $\prod_{i=0}^n Y_{n-1}$.
Now we can explicitly describe the maps in this diagram. $\hom(i_nX, i_nY)$ are given by natural transformations, collections of $f_i : X_i \to Y_i$ for $0 \leq i \leq n$.

- The map $\hom(i_nX, i_nY) \to \hom(i_{n-1}X, i_{n-1}Y)$ sends $(f_0, \ldots, f_n)$ to $(f_0, \ldots, f_{n-1})$.
- The map $\hom(i_nX, i_nY) \to (Y_n)^{J_n}$ sends a tuple f to the map $\lambda\sigma.f_n(\sigma)$.
- The map from $(Y_n)^{J_n}$ sends g to the map $\lambda(\sigma, i).d_ig(\sigma)$.
- The top map sends a tuple f to the map $\lambda(\sigma, i).f_{n-1}(d_i\sigma)$.

Thus to check this diagram is a pullback we need to show the following. Suppose we have maps $f : \hom(i_{n-1}X, i_{n-1}Y)$ and $g : (Y_n)^{J_n}$ which satisfy for all $\sigma : J_n$ and i that $d_ig(\sigma) = f_{n-1}(d_i\sigma)$. THen we must construct a unique map $\hom(i_nX, i_nY)$ mapping to f and g. We do this by constructing f_n. If $x : X_n$ then either $x \in J_n$ in which case $f_n(x) := g(x)$, Or x is degenerate and $x = \alpha^*(y)$ for some $y : X_m$ with $m < n$. Then define $f_n(x) = \alpha^*(f_m(y))$. This is well defined, natural and unique.

Definition 1.11 Given a category $\mathcal{C}$, its nerve is the simplicial set given by

$$[n] \mapsto \hom([n], \mathcal{C})$$

That is chains of n composable maps.

Definition 1.12 Given a group G its classifying space is the geometric realisation of its nerve as a category with one object.

Definition 1.13 A **Kan complex** is a simplicial set X such that for all Horns Δ^n_j there are solutions to lifting problems

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Exercise 5 Given a simplicial set X show that for all $n \geq 0$ that $\operatorname{cosk}_n(X)$ is a Kan complex. Further show the natural map $X \to \operatorname{cosk}_n(X)$ induces isomorphisms on π_k for $k < n$ and that $\pi_k(\operatorname{cosk}_n(X)) = 0$ for $k \geq n$.

Solution Suppose we have a map $\Lambda^m_j \to \mathrm{cosk}_n(X)$. By adjunction this is equivalently given by a map $\mathrm{sk}_n(\Lambda^m_j) \to X$. Note that for $m > n+1$ the map $\mathrm{sk_n}(\Lambda^m_j) \to \mathrm{sk_n}(\Delta^m)$ is just the Horn inclusion. Hence we have a lift. If $m \leq n$ then the inclusion becomes an isomorphism, so therefore has a lift. In the case $m = n+1$ we have $\mathrm{sk_n}(\Lambda^{n+1}_j) \to \mathrm{sk_n}(\Delta^{n+1})$ becomes the inclusion $\Lambda^{n+1}_j \to \partial\Delta^{n+1}$. We can fill against this by filling two a the whole simplex, and restricting.
This same argument makes the isomorphism of homotopy groups clear.

Example 1.14 The singular simplicial set of a topological space is a Kan complex. This follows quickly by adjunction, noting that a horn is a retract of Δ^n in topological spaces.

Example 1.15 The nerve of a category is a Kan complex if and only if the category is a groupoid.

Definition 1.16 Given $f, g : X \to Y$ maps of simplicial sets, we say they are homotopic if there exist a map $H : \Delta^1 \times X \to Y$ restricting to f, g on boundary. We can define the simplicial homotopy groups as homotopy classes of maps $(\Delta^n, \partial\Delta^n) \to (X, x)$.

2 ∞-Categories

We will start with a definition which is not of an ∞-category, but a weaker notion. In particular these will be a simplicial set with composition, but the space of composers isn't necessarily contractible.

Definition 2.1 A **composer** is a simplicial set with the extension property against spine inclusions $I^n \to \Delta^n$.

Example 2.2 The singular set of a topological space is a composer. Given paths $p_i : x_i \to x_{i+1}$, the data of a spine map, we define their composition as the composition of the paths. If you aren't convinced this gives an actual n-cell, there is some way to see this either purely topologically, or by inductively filling 2-horns, 3-horns, etc. until you get a full filler of Δ^n, using