Internal Assessment
HL Mathematics Analysis & Approaches

Title:

Modelling Simple Harmonic Damping of a Swing

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Table of Contents

I. Introduction

Rationale

As an aspiring engineer, I have always been curious about how the world around us works, and especially how mathematics and the sciences are applied to the real world. This has become a driving factor in my education as I am constantly linking newly learnt knowledge to different aspects of life. For example, recently, when walking through my local park I came across a set of swings and I was motivated to learn more about the mathematics hidden within them. The perfect opportunity to do so arose with the Maths Internal Assessment.

Initially, I was wondering why a swing eventually comes to rest once the swinger stops pushing forward. I quickly realised that the swing set essentially acts as an enlarged pendulum, thus the swing must share properties to the pendulums which we studied in physics class. This connection led me to determine the reason the swing eventually stopped oscillating was due to air resistance acting against the motion of the swing and also because of friction between the swing's rope and the structure supporting it. Consequently, the kinetic energy (the energy possessed by an object due to its motion) of the motion of the swing and the potential energy (the energy possessed by an object due to its position) stored within the oscillatory system are transferred into thermal energy which is dissipated into the surroundings.

The next question which came to mind was how I could mathematically model this trend and if it was possible to predict how the swing would act in certain scenarios. After further online research, I discovered that this trend in motion is described as damped simple harmonic motion.

Aim & Approach

The aim of this investigation is to effectively model the damped simple harmonic motion of a swing to predict how the swing will act under certain conditions. This will be achieved by deriving and solving a second order differential equation from basic force formulas and evaluating the final function's accuracy using primary data collected from a practical investigation. The mathematics required in solving differential equations is new content which further motivates me to learn and explore the topic independently.

Firstly, to achieve this aim, I will apply my knowledge of mathematics and physics to derive a function to model the motion of the swing. My next step in the exploration of the swing is to theoretically model its motion using realistic conditions which I can imitate in real life. To determine the validity of the model I will conduct a practical investigation which replicates the conditions set in my theoretical model. Subsequently, I will plot the results collected from this investigation on a graph. Finally, to achieve my goal of effectively modelling the motion of the swing, I will directly compare the theoretical and practical graphs. This will allow me to evaluate the accuracy of the derived function, and if necessary improve the function to better fit the real life outcome. Overall, the exploration of the model function will enable me to discover how varying certain variables affects the system of the swing's movement in real life.

II. Function Derivations

The damped simple harmonic motion that swings experience will be modelled using pendulums which experience the same periodic oscillations. Simple harmonic motion, as defined by Britannica, is the "repetitive movement [of an oscillator] back and forth through an equilibrium position, so that the maximum displacement on one side of this position is equal to the maximum displacement on the other side"[1]. The adjective damped refers to the decrease of the maximum displacement over time as a result of air resistance and friction.

Initially, pendulums rest at an equilibrium position. When displaced, they begin to oscillate in the opposite direction of the displacement so that their acceleration is proportional to their displacement and angular frequency which acts as a constant. The simple harmonic motion can be described with the use of the formula[2] $a = -\omega^2 x$ where a is acceleration, ω is angular frequency, and x is the displacement of the system from the equilibrium position. The motion of the displaced pendulum can be seen in *figure 1* where t represents the time of an oscillation which lasts for one period, T.

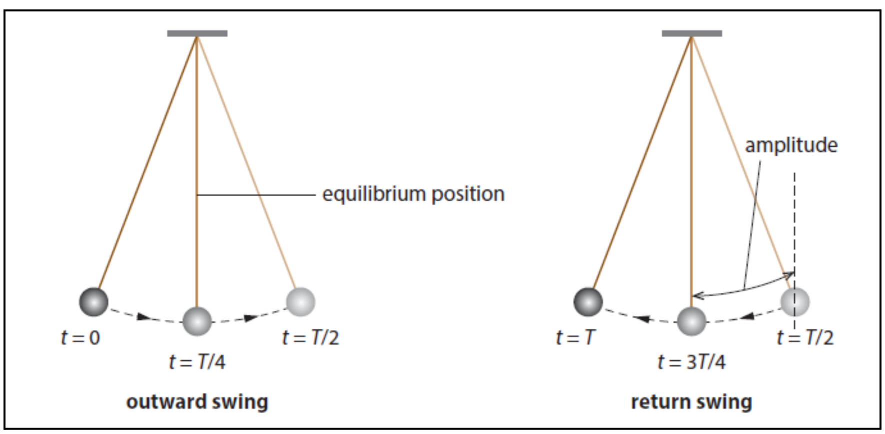

figure 1: Motion of a displaced pendulum with regards to time[3]

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[1] Britannica (2022, May 30). *simple harmonic motion*. Retrieved from: https://www.britannica.com/science/simple-harmonic-motion. Accessed: 10.09.22
[2] University of Birmingham (2022). *Why is simple harmonic motion so important?*. Retrieved from: https://www.birmingham.ac.uk/teachers/study-resources/stem/physics/harmonic-motion.aspx. Accessed: 17.09.22
[3] Tsokos, K, A, (2014). *Physics for the IB Diploma*. Cambridge, UK: Cambridge University Press. Edition six

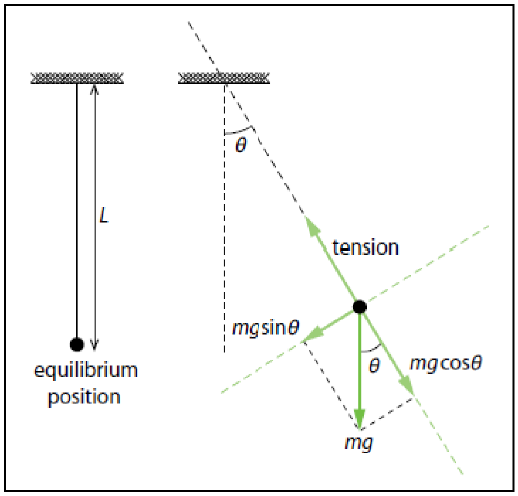

figure 2: A diagram showing the forces acting on a displaced pendulum[4]

Once the swing, or in this case pendulum, is displaced from its natural point of equilibrium, there are a number of forces acting on the mass. Since the pendulum is a simple harmonic oscillator, it is clear that a restoring force will be acting on the mass. The restoring force is a force acting against the displacement in order to return the system to a state of equilibrium. For a simple pendulum, the restoring force is a result of the force of gravity and is directly proportional to the displacement.[5] The restoring force is negative as it acts opposite to the direction of the displacement. A formula for restoring force can be derived as visualised in *figure 2*.

$$F_{restoring} = -\ mgsin(\theta)$$

Where m is the mass, g is the gravitational acceleration constant that is equal to approximately[6] 9.81 ms^{-2} on Earth, and θ is the angle of displacement from the equilibrium position.

The swing which is being modelled is present in the open air where the force of drag due to air resistance acting against the motion of the mass is present. The force of drag is directly

[4] Tsokos, K, A, (2014). *Physics for the IB Diploma*. Cambridge, UK: Cambridge University Press. Edition six
[5] StudySmarter (2022). *Restoring Force*. Retrieved from: https://www.studysmarter.co.uk/explanations/physics/oscillations/restoring-force/. Accessed: 17.09.22
[6] International Baccalaureate Organisation (2016). *Physics data booklet*. Retrieved from: https://ibphysics.org/wp-content/uploads/2016/01/ib-physics-data-booklet-2016.pdf. Accessed: 19.11.22

proportional to the instantaneous velocity of the mass.[7] Additionally, a constant of proportionality is required (c) which accounts for the amount of air resistance present in the system. The force of drag is negative as it is acting in the opposite direction to the displacement of the pendulum. To adapt the formula in a format viable for its use in a differential equation, velocity can be represented as the first derivative of the displacement of the pendulum (see to *Velocity as the first derivative of displacement* in the *Appendix*).

$$F_{drag} =- cv =- cl\theta'$$

Where c is the constant of proportionality accounting for air resistance, v is the velocity of the mass, l is the length of the pendulum, and θ' is the first derivative of displacement.

As the pendulum oscillates, its velocity changes, thus Newton's Second Law of Motion is applicable. Newton's Second Law states that "the time rate of change of the momentum of a body is equal in both magnitude and direction to the force imposed on it."[8] To adapt the formula in a format viable for its use in a differential equation, acceleration can be represented as the second derivative of the displacement of the pendulum (see to *Acceleration as the second derivative of displacement* in the *Appendix*).

$$F_{net} = ma = ml\theta''$$

Where m is the mass, a is the acceleration of the mass, l is the length of the pendulum, and θ'' is the second derivative of displacement.

The following mathematical method is an adaptation of Dr. Trefor Bazett's work as seen on his youtube videos on mechanical vibrations[9] and Dibyajyoti Das as seen in his youtube video

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[7] Lumen Learning (2022). *Drag Forces*. Retrieved from: https://courses.lumenlearning.com/suny-physics/chapter/5-2-drag-forces/. Accessed: 17.09.22

[8] Britannica (2022, August 16). *Newton's laws of motion*. Retrieved from: https://www.britannica.com/science/Newtons-laws-of-motion. Accessed: 17.09.22

[9] Bazett, T (2021, April 11 & 14). *Mechanical Vibrations.* Retrieved from: https://www.youtube.com/c/DrTreforBazett. Accessed: 15.09.22

on damped harmonic oscillations[10]. These three fundamental formulas can be combined to ultimately form a second order ordinary differential equation as shown below:

$$F_{net} = F_{restoring} + F_{drag}$$

$$ml\theta'' = (-mgsin(\theta)) + (-cl\theta')$$

$$ml\theta'' =- mgsin(\theta) - cl\theta'$$

$$ml\theta'' + cl\theta' + mgsin(\theta) = 0$$

$$\theta'' + \frac{c\theta'}{m} + \frac{g}{l}sin(\theta) = 0$$

To simplify the equation, let $M = \frac{c}{2m}$ and because the angular frequency of a pendulum $\omega = \sqrt{\frac{g}{l}}$ as derived in the *Appendix*, let $\omega^2 = \frac{g}{l}$. The equation now becomes:

$$\theta'' + 2M\theta' + \omega^2 sin(\theta) = 0$$

This is a nonlinear differential equation. According to Dr. William F. Trench, a professor at Trinity University department of mathematics, there are very few methods of solving nonlinear differential equations exactly and none of which are generalised.[11] However, this equation can be linearised using small angle approximation which applies for angles equal to or less than 0.26 radians at an accuracy of 1.14%[12]. Therefore it is assumed that $sin(\theta) \approx \theta$. Applying small angle approximation, the equation becomes:

$$\theta'' + 2M\theta' + \omega^2\theta = 0$$

This equation can also be represented using Leibniz Notation for ease in later steps:

$$\frac{d^2\theta}{dt^2} + 2M\frac{d\theta}{dt} + \omega^2\theta = 0$$

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[10] Das, D (2021, April 11). *Damped Harmonic Oscillations*. Retrieved from: https://www.youtube.com/c/FortheLoveofPhysics. Accessed: 15.09.22
[11] Trench, W, F (2000, March 28). *ELEMENTARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEMS*. San Antonio, Texas, USA: Trinity University.
[12] Farvard, A (2022, February 28). *Small-Angle Approximation: Formulas, Proofs & Applications*. Retrieved from: https://study.com/learn/lesson/small-angle-approximation-formulas-theorems-examples.html. Accessed: 19.09.22

This is a linear second order ordinary differential equation which can be solved by assuming that the solution of $\theta = Ae^{kt}$ is true. Where e^{kt} is the continuous exponential model in which k represents the continuous growth rate ($k > 0$), and t represents time.

$$\frac{d^2}{dt^2}Ae^{kt} + 2M\frac{d}{dt}Ae^{kt} + \omega^2 Ae^{kt} = 0$$

$$Ak^2e^{kt} + 2MAke^{kt} + \omega^2 Ae^{kt} = 0$$

$$Ae^{kt}(k^2 + 2Mk + \omega^2) = 0$$

$$k^2 + 2Mk + \omega^2 = 0$$

The quadratic can be solved using the quadratic formula:

$$k = \frac{-2M \pm \sqrt{(2M)^2 - 4(1)(\omega^2)}}{2(1)}$$

$$k = \frac{-2M \pm \sqrt{4M^2 - 4\omega^2}}{2}$$

$$k = \frac{-2M \pm \sqrt{4(M^2 - \omega^2)}}{2}$$

$$k = \frac{-2M \pm 2\sqrt{M^2 - \omega^2}}{2}$$

$$k = - M \pm \sqrt{M^2 - \omega^2}$$

Therefore the general solution of the second order ordinary differential equation can be demonstrated as:

$$\theta(t) = A_1e^{(-M+\sqrt{M^2-\omega^2})t} + A_2e^{(-M-\sqrt{M^2-\omega^2})t}$$

$$\theta(t) = A_1e^{-Mt+\sqrt{M^2-\omega^2}t} + A_2e^{-Mt-\sqrt{M^2-\omega^2}t}$$

$$\theta(t) = e^{-Mt}(A_1e^{\sqrt{M^2-\omega^2}t} + A_2e^{-\sqrt{M^2-\omega^2}t})$$

The general solution has 4 different possible cases depending on the values of M and ω, hence the square root within the solution. Possible cases:

I. If $M = 0$
II. If $M^2 < \omega^2$
III. If $M^2 = \omega^2$
IV. If $M^2 > \omega^2$

For each case a different function modelling the motion of the swing can be derived. For this reason, it is important to analyse each case to find the case that correctly models the motion of the swing in the investigation.

Case I where $M = 0$ can be expressed as $\frac{c}{2m} = 0$ and simplified to $c = 0$ where c is the constant of proportionality for drag. Therefore, in this case there is no drag present within the system causing the total energy within the system to remain constant, and thus the system is undamped. It should be noted that case I is very difficult to practically model as it is the case where a force due to drag does not exist. Case I can however be practically modelled by performing the investigation in a vacuum or by using software to simulate frictionless conditions.

Case II where $M^2 < \omega^2$ can also be expressed as $(\frac{c}{2m})^2 < \frac{g}{l}$ and simplified to $c^2 < \frac{4gm^2}{l}$ where c is the constant of proportionality for drag. In this case the drag force is present, although it is not strong enough to bring the mass to rest without it oscillating at least once through its equilibrium position. This means that during each oscillation energy will be dissipated and the total energy of the system will decrease. When this occurs, the system is underdamped.[13] Case II is the most likely case which will model the movement of the practical pendulum the closest. This is due to the fact that the force of drag due to air resistance will likely

[13]OpenStax (2022). *College Physics*. Retrieved from: https://pressbooks.uiowa.edu/clonedbook/chapter/damped-harmonic-motion/. Accessed: 18.09.22

be small as the area of the mass is small and thus there is only a small area that the air resistance can act upon.

Case III where $M^2 = \omega^2$ can be expressed as $(\frac{c}{2m})^2 = \frac{g}{l}$ and simplified to $c^2 = \frac{4gm^2}{l}$ where c is the constant of proportionality for drag. In this case, c has the exact value such that the drag force brings the pendulum to rest as soon as possible. When the mass is displaced it travels towards the equilibrium and comes to rest exactly at the equilibrium position.[14] This means that all of the kinetic energy of the mass is dissipated by the time the mass reaches equilibrium. This, of course, is very difficult to emulate, however, when it occurs, the system is critically damped. In the situation of the swing in this investigation, case III is impossible to occur. This is because case III requires the force due to drag and the angular frequency of the pendulum to be equal. The force due to gravity is far too small for this case to occur.

The final case, case IV, where $M^2 > \omega^2$ can be expressed as $(\frac{c}{2m})^2 > \frac{g}{l}$ and simplified to $c^2 > \frac{4gm^2}{l}$ where c is the constant of proportionality for drag. This case is similar to critical damping which occurred in case III in the sense that when the mass is displaced it travels towards equilibrium and comes to rest at the equilibrium position. However, as indicated with the higher value of c, the system experiences a larger drag force which causes the mass to move more slowly towards equilibrium.[15] This often occurs when the surrounding medium of the mass has a high viscosity such as in water, oil, or honey. When this occurs, the system is overdamped. Similar to case III, it is impossible for case IV to occur in the situation of the swing in this investigation as the drag force is not greater than the angular frequency.

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[14] OpenStax (2022). *College Physics*. Retrieved from: https://pressbooks.uiowa.edu/clonedbook/chapter/damped-harmonic-motion/. Accessed: 18.09.22
[15] OpenStax (2022). *College Physics*. Retrieved from: https://pressbooks.uiowa.edu/clonedbook/chapter/damped-harmonic-motion/. Accessed: 18.09.22

Case II

In case II the system is underdamped. The angular frequency of the pendulum will be decreased due to the drag force present, hence it can be assumed that the new, decreased, angular frequency $\omega_1^{\ 2} = \omega^2 - M^2$. By substituting ω_1 into the general solution the following function can be derived:

$$\theta(t) = e^{-Mt}(A_1 e^{\sqrt{M^2-\omega^2}t} + A_2 e^{-\sqrt{M^2-\omega^2}t})$$

$$\theta(t) = e^{-Mt}(A_1 e^{\sqrt{(-1)\omega^2-M^2}t} + A_2 e^{-\sqrt{(-1)\omega^2-M^2}t})$$

$$\theta(t) = e^{-Mt}(A_1 e^{\sqrt{(-1)\omega_1^{\ 2}}t} + A_2 e^{-\sqrt{(-1)\omega_1^{\ 2}}t})$$

$$\theta(t) = e^{-Mt}(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t})$$

To simplify the equation it must be assumed that $A_1 = \frac{1}{2}De^{i\phi}$ and $A_2 = \frac{1}{2}De^{-i\phi}$.

$$\theta(t) = e^{-Mt}(\tfrac{1}{2}De^{i\phi}e^{i\omega_1 t} + \tfrac{1}{2}De^{-i\phi}e^{-i\omega_1 t})$$

$$\theta(t) = e^{-Mt}(\tfrac{1}{2}D)(e^{i\phi}e^{i\omega_1 t} + e^{-i\phi}e^{-i\omega_1 t})$$

$$\theta(t) = e^{-Mt}(\tfrac{1}{2}D)(e^{i\phi+i\omega_1 t} + e^{-i\phi-i\omega_1 t})$$

$$\theta(t) = De^{-Mt}(\tfrac{1}{2})(e^{i(\omega_1 t+\phi)} + e^{-i(\omega_1 t+\phi)})$$

Using an adaptiaton of Euler's Formula (see *Euler's formula adaptation* in the *appendix*) where $cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, the equation can be further simplified to:

$$\theta(t) = De^{-Mt}cos(\omega_1 t + \phi)$$

In this function, $\theta(t)$ represents the instantaneous displacement in radians, D represents the initial amplitude or the maximum displacement of the pendulum in metres, e^{-Mt} expresses exponential decay where M is the rate of decay, ω_1 represents the angular frequency or the number of oscillations per second, and ϕ represents the phase shift or the starting point of the pendulum.

III. Theoretical Analysis

This section will use the function derived in *II. Function Derivations* and set conditions which will be replicated within the *IV. Practical Analysis* section to theoretically predict the movement of the pendulum in order to prove the effectiveness of the function in modelling the swing.

The set conditions are defined as follows:

- The length of the pendulum is equal to 1.00 metre
- The mass of the pendulum is equal to 10.0 grams
- The mass of the pendulum has a radius of 2.00 centimetres
- The mass of the pendulum will be displaced by 0.262 radians (≈15°)
- The acceleration due to gravity which the mass experiences is equal to 9.81 ms^{-1}
- The density of air is equal to 1.23 kgm^{-3}

The following values are required when defining the final theoretical function. They can be calculated from the set conditions.

1. The maximum displacement of the mass is approximately equal to the angle at which the mass is displaced due to small angle approximation.

2. The angular frequency (ω) of the pendulum can be calculated by substituting the respective values into the formula as shown below:

$$\omega = \sqrt{\frac{g}{l}}$$

$$\omega = \sqrt{\frac{9.81}{1}}$$

$$\omega = 3.132091953 \approx 3.13\, rads^{-1}$$

3. The constant for proportionality used within the drag force formula can be determined using the formula for drag force[16] $F = -\frac{1}{2}\rho v^2 cA$ where ρ is the density of air, c is the drag coefficient, and A is the area of the mass:

$$F = -cv \qquad F = -\frac{1}{2}\rho v^2 cA$$

$$c = \frac{1}{2}\rho A$$

$$c = \frac{1}{2}(1.23)(\pi)(0.02)^2$$

$$c \approx 7.73 \cdot 10^{-4}$$

4. The value for M can be calculated as shown below:

$$M = \frac{c}{2m}$$

$$M = \frac{7.73 \cdot 10^{-4}}{2(0.01)}$$

$$M = 0.03865 \approx 3.87 \cdot 10^{-2}$$

5. The angular frequency (ω_1) of the pendulum can be calculated by substituting the respective values into the formula as shown below:

$$\omega_1 = \sqrt{\omega^2 - M^2}$$

$$\omega_1 = \sqrt{3.13^2 - (3.87 \cdot 10^{-2})^2}$$

$$\omega_1 = 3.129760743 \approx 3.13\, rads^{-1}$$

6. The pendulum will always have a starting point at the position of maximum displacement. For this reason the value of $\phi = 0$.

[16] Lumen Learning (2022). *Drag Forces*. Retrieved from: https://courses.lumenlearning.com/suny-physics/chapter/5-2-drag-forces/. Accessed: 17.09.22

The respective values can be substituted into the function defining case II as follows:

$$\theta(t) = De^{-Mt}cos(\omega_1 t + \phi)$$

$$\theta(t) = 0.262e^{-3.87\cdot 10^{-2}t}cos(3.13t + 0)$$

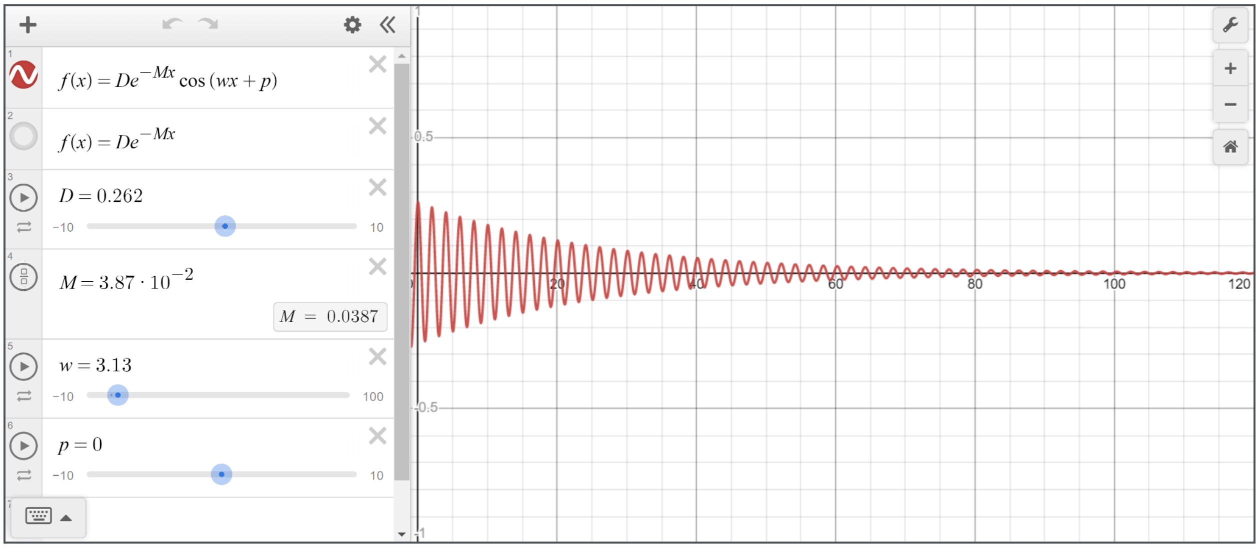

figure 3: A graph to show the theoretical function of case II[17]

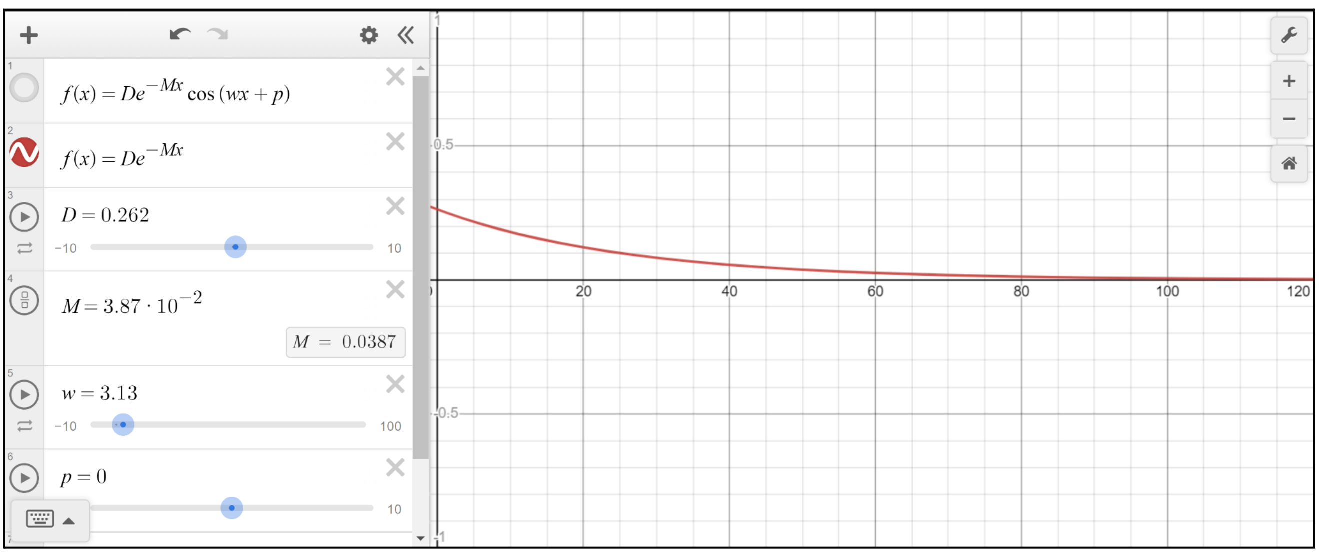

figure 4: A graph to show the the maximum displacement of theoretical function of case II[18]

[17] Desmos (2022). *Desmos - Graphing Calculator*. Retrieved from: https://www.desmos.com/calculator. Accessed: 22.09.22
[18] Desmos (2022). *Desmos - Graphing Calculator*. Retrieved from: https://www.desmos.com/calculator. Accessed: 22.09.22

IV. Practical Analysis

In this section a practical investigation will be completed following the set conditions stated above and the results collected will be plotted. The investigation will be set up as shown in *figure* 5. A video recording of the investigation will be taken and this will later be used to analyse the displacement of the pendulum over 0.5 second intervals until the mass comes to rest. The displacement of the mass of the pendulum at each interval will be recorded. The data collected will be plotted in the graph of displacement against time with a curved line connecting the points. Additionally, a graph of the maximum displacements with a line of best fit will be plotted in order to compare the practical and theoretical values. The collected data can be referenced in the *Appendix* under the title *Data Table: Displacement of the Mass of the Pendulum Over Time*.

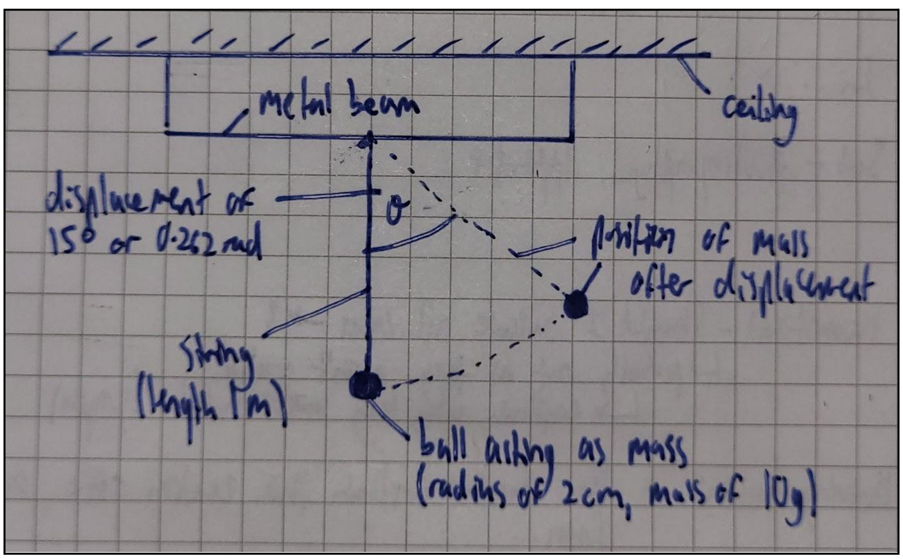

figure 5: Diagram of investigation set-up

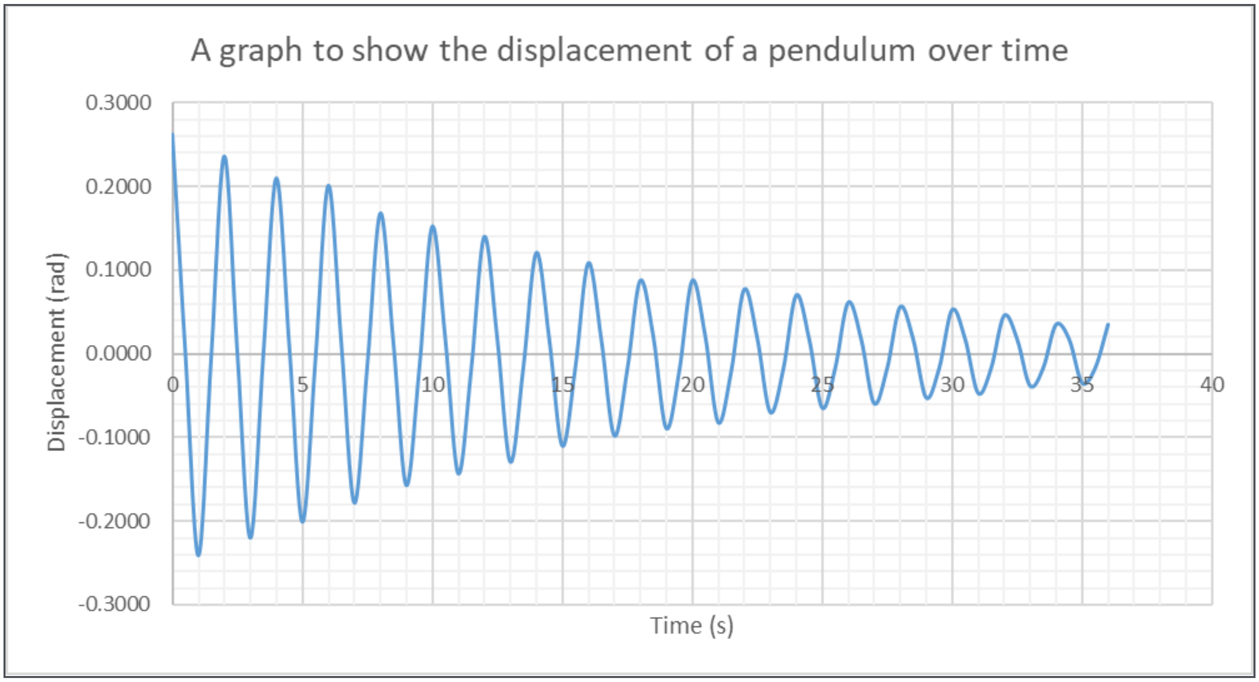

figure 6: A graph to show the displacement of the pendulum over time

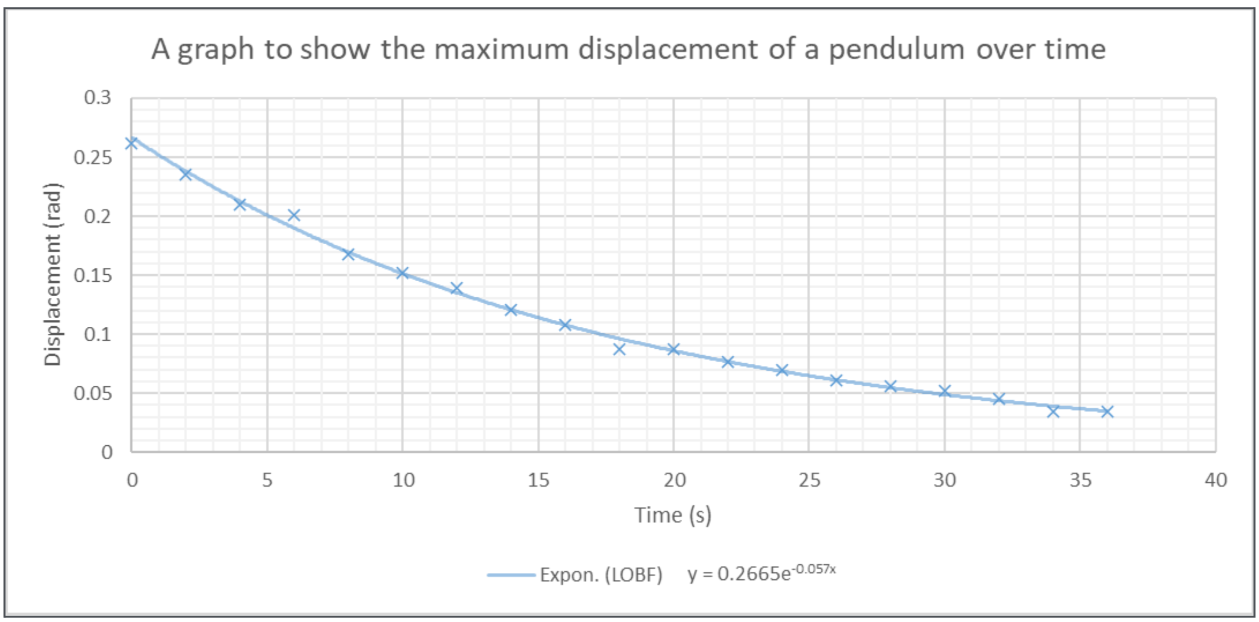

figure 7: A graph to show the maximum displacement of the pendulum over time

V. Theoretical and Practical Comparison

Analysis

After analysing *figure 6*, it is clear that the practical investigation most closely resembles case II. To determine the accuracy of the theoretical function for case II the theoretical and practical functions can be plotted against each other as shown in *figure 7* where *t(x)* represents the theoretical maximum displacement values and *p(x)* represents the practical maximum displacement values.

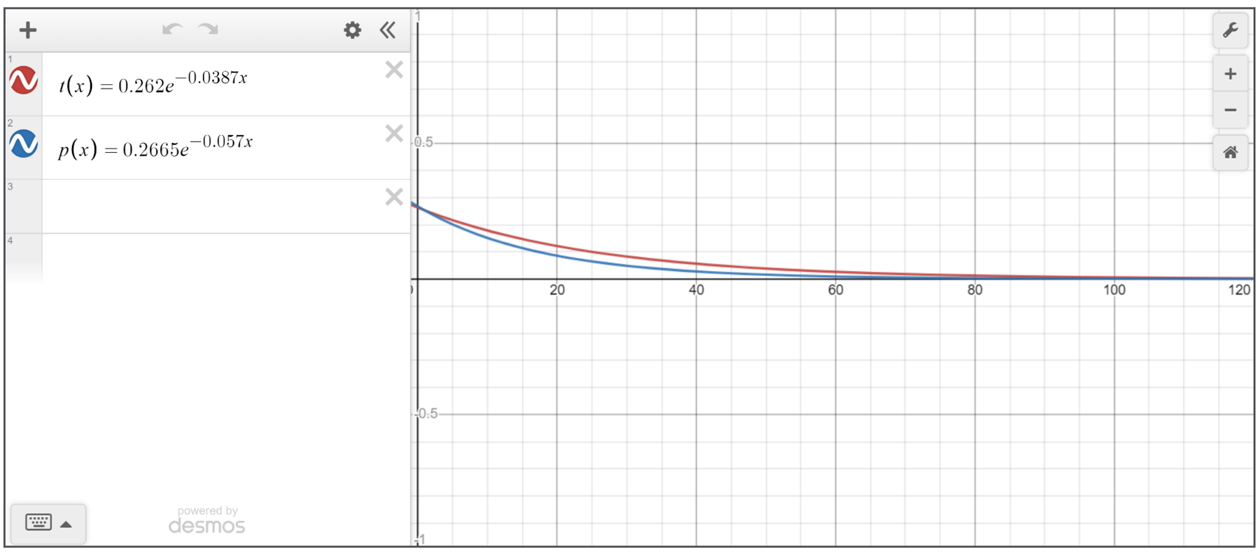

figure 8: A graph to compare the theoretical and practical functions[19]

The functions of *t(x)* and *p(x)* both represent exponential decay functions. The main difference between the two functions is the rate of decay. The rate of decay in *t(x)* (0.0387) is significantly smaller than the rate of decay in *p(x)* (0.057). Accordingly, *p(x)* decays at a faster rate than *t(x)*. This trend can also be visually observed in *figure 8*. Another contrast between the two functions is that the function *p(x)* has a slightly larger coefficient value. The coefficient

[19] Desmos (2022). *Desmos - Graphing Calculator*. Retrieved from: https://www.desmos.com/calculator. Accessed: 22.09.22

value represents the maximum displacement when time is zero, therefore, it should be equal to 15° or 0.262 rad, whereas in *p(x)* it is 0.2665. The reason for this is that *p(x)* is a line of best fit for the maximum displacements of the practical investigation, thus it is merely an estimated value.

Conclusion

The theoretical function did not predict the movement of the practical investigation with 100% accuracy. The difference between the practical and theoretical functions mainly lies with the difference in the value of the coefficient *M* which is essentially a term describing the amount of drag force is acting on the mass due to air resistance. Therefore, the derived function is not liable for this fault, but the environment in which the investigation was performed is. Possible reasons and weaknesses within the investigation which resulted in the higher value of *M* are present in the practical function will be discussed within the *Evaluation*.

In conclusion, the theoretical function modelling case II was the model which matches the data collected during the practical investigation. This outcome was hypothesised as the force due to drag is less than the angular frequency, thus the system is underdamped.

Finally, it can be determined that the function to predict and model the damped simple harmonic motion of a swing is $\theta(t) = De^{-Mt}cos(\omega_1 t + \phi)$.

Evaluation

Overall, the accuracy and precision of the data collected in the practical investigation cannot be effectively evaluated as no repeats of the investigation were conducted.

Nevertheless, the method of the investigation had many strengths regarding the collection of data to minimise the random error. For example, a video recording of the investigation was analysed when gathering data. This eliminated the majority of the human error and allowed for precise measurements of time and displacement to be recorded throughout the investigation. Another strength of the investigation is that pendulums are very reliable and consistent which means that there is not much room for a random error to occur.

With that being said, there are still a number of variables which must be controlled throughout the investigation. For example, the air resistance must be kept constant throughout the investigation. Air resistance was one of the factors which led to a difference between the theoretical and practical functions. This could be because an external force was acting against the mass during the investigation, such as wind from an open window, or because of friction present between the string holding the mass and the structure suspending the mass. Both of these possibilities in errors result in a systematic error and less accurate results. The errors could be avoided by better isolating the pendulum in an environment where no external forces are acting against the mass.

Another weakness within the method of the investigation was that the displacement of the pendulum was measured at time intervals of 0.5 seconds. This is because the time interval does not necessarily have to line up with the equilibrium position or the maximum or minimum displacements. The reason why they approximately did in this investigation was because of testing conducted prior to the final investigation to determine whether the final investigation would be successful. To improve the method and negate this weakness, the displacement should be recorded at the exact time when the pendulum reaches these points of interest.

Regardless, the function modelling the movement of the swing is reliable and can be used to model the motion of the swing under certain conditions. For example, this could be useful when determining the safety of the swing for children of different weight.

VI. References

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Appendix

Velocity as the first derivative of displacement

The velocity of an object can be defined as the rate of change of the object's position with respect to time.[20] Due to small angle approximation, displacement $x = l\theta$. Therefore velocity can be represented as:

$$v(t) = \frac{dx}{dt} = l\frac{d\theta}{dt} = l\theta'(t)$$

Where v is velocity, t is time, x is displacement, θ is angular displacement, and l is the length of the pendulum.

Acceleration as the second derivative of displacement

The acceleration of an object can be defined as the rate of change of the object's velocity with respect to time.[21] Therefore velocity can be represented as:

$$a(t) = \frac{dv}{dt} = \frac{d\frac{dx}{dt}}{dt} = \frac{dl\frac{d\theta}{dt}}{dt} = l\frac{d^2\theta}{dt} \cdot \frac{1}{dt} = l\frac{d^2\theta}{dt^2} = l\theta''(t)$$

Where a is acceleration, v is velocity, t is time, x is displacement, θ is angular displacement, and l is the length of the pendulum.

Angular frequency of a pendulum

A formula for the angular frequency of a pendulum can be derived from the simple harmonic motion of a pendulum:

$$a = -\omega^2 x$$

$$\omega^2 = \frac{a}{x}$$

[20] Britannica (2022, November 3) *velocity*. Retrieved from: https://www.britannica.com/science/velocity. Accessed: 19.11.22
[21] Britannica (2022, August 29). *acceleration*. Retrieved from: https://www.britannica.com/science/acceleration. Accessed: 19.11.22

The acceleration of the pendulum is equal to the acceleration of the mass due to gravity, and thus the gravitational acceleration constant on Earth.. The displacement of the pendulum is equal to the length of the pendulum due to small angle approximation. Therefore the formula for the angular frequency of a pendulum can be written as:

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Where ω is angular frequency, g is the gravitational acceleration constant that is equal to 9.81 ms^{-2} on Earth, and l is the length of the pendulum.

Euler's formula adaptation

Euler's identity states that[22] $e^{ix} = cos(x) + isin(x)$. This can be rearranged to find an equation for $cos(x)$ and $sin(x)$:

$$cos(x) = e^{ix} - isin(x) \text{ and } isin(x) = e^{ix} - cos(x)$$

These can be substituted accordingly to find an equation for $cos(x)$:

$$cos(x) = e^{ix} - e^{ix} - cos(x)$$

$$2cos(x) = e^{ix} - e^{ix}$$

$$cos(x) = \frac{e^{ix} - e^{ix}}{2}$$

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

[22] Britannica (2022, October 28). *Euler's formula*. Retrieved from: https://www.britannica.com/science/Eulers-formula. Accessed: 19.11.22

Data Table: Displacement of the Mass of the Pendulum Over Time

Time (± 0.01 s)	**Displacement (± 0.1 degrees)**	**Displacement (rad)**
0.00	15.0	0.2618
0.50	0.0	0.0000
1.00	-13.8	-0.2409
1.50	0.0	0.0000
2.00	13.5	0.2356
2.50	0.0	0.0000
3.00	-12.6	-0.2199
3.50	0.0	0.0000
4.00	12.0	0.2094
4.50	0.5	0.0087
5.00	-11.5	-0.2007
5.50	-0.5	-0.0087
6.00	11.5	0.2007
6.50	0.7	0.0122
7.00	-10.2	-0.1780
7.50	-0.8	-0.0140
8.00	9.6	0.1676
8.50	0.8	0.0140
9.00	-9.0	-0.1571
9.50	-1.0	-0.0175
10.00	8.7	0.1518
10.50	1.0	0.0175
11.00	-8.2	-0.1431
11.50	-1.0	-0.0175
12.00	8.0	0.1396

12.50	1.0	0.0175
13.00	-7.4	-0.1292
13.50	-1.0	-0.0175
14.00	6.9	0.1204
14.50	1.0	0.0175
15.00	-6.3	-0.1100
15.50	-1.0	-0.0175
16.00	6.2	0.1082
16.50	1.0	0.0175
17.00	-5.6	-0.0977
17.50	-1.1	-0.0192
18.00	5.0	0.0873
18.50	1.2	0.0209
19.00	-5.1	-0.0890
19.50	-1.2	-0.0209
20.00	5.0	0.0873
20.50	1.2	0.0209
21.00	-4.7	-0.0820
21.50	-1.2	-0.0209
22.00	4.4	0.0768
22.50	1.1	0.0192
23.00	-4.0	-0.0698
23.50	-1.1	-0.0192
24.00	4.0	0.0698
24.50	1.0	0.0175
25.00	-3.7	-0.0646
25.50	-1.0	-0.0175

26.00	3.5	0.0611
26.50	1.0	0.0175
27.00	-3.4	-0.0593
27.50	-1.0	-0.0175
28.00	3.2	0.0559
28.50	1.0	0.0175
29.00	-3.0	-0.0524
29.50	-1.0	-0.0175
30.00	3.0	0.0524
30.50	1.0	0.0175
31.00	-2.7	-0.0471
31.50	-1.0	-0.0175
32.00	2.6	0.0454
32.50	1.0	0.0175
33.00	-2.2	-0.0384
33.50	-1.0	-0.0175
34.00	2.0	0.0349
34.50	1.0	0.0175
35.00	-2.0	-0.0349
35.50	-1.0	-0.0175
36.00	2.0	0.0349