Revision Notes

Set:
- ✓ Set basics and defining sets
- ✓ Set operations and their formal definitions
- ✓ Set identities, set dual and set closure
- ✓ Power set
- ✓ Partition

Counting Principles
- ✓ Multiplication Principle
- ✓ Addition Principle

Relation:
- ✓ Cartesian product and binary relation
- ✓ Properties of binary relation
- ✓ Equivalence relation and equivalence classes
  - ✓ Equivalence classes form a partition

Functions:
- ✓ Formal definition and its implication
- ✓ Types of functions
- ✓ Cardinality
  - ✓ Finite and infinite sets
  - Countable vs Uncountable sets
  - Equinumerous

Proof Techniques
- ✓ Direct proof
- ✓ Proof by contrapositive
- ✓ Proof by contradiction
- ✓ Proof by construction
- ✓ Proof by induction:
  - ✓ Weak induction
  - ✓ Strong induction
- ✓ Pigeonhole Principle
- Diagonalization principle

Sets:
- a collection of well-defined objects
- no ordering
- no duplicates

Define a set **{ ….. }**:

1. Enumerate all elements in the set

   A = {3,7,9,14}

   - Not applicable for infinite set

2. Define the property or characteristics of the elements in the set with respect to the typical number sets N, Z, R, Q and C.

   A = { x | P(x) } where *P* is the unary predicate

Symbol	Symbol Name	Meaning / definition	Example
N	natural numbers / whole numbers set (with zero)	N = {0,1,2,3,4,...}	$0 \in$ N
Z	integer numbers set	Z = {...-3,-2,-1,0,1,2,3,...}	$-6 \in$ Z
Q	rational numbers set	Q = $\{x \mid x=a/b,\ a,b\in\mathbb{N}\}$	$2/6 \in$ Q
R	real numbers set	R = $\{x \mid -\infty < x < \infty\}$	$6.343434 \in$ R
C	complex numbers set	C = $\{z \mid z=a+bi,\ -\infty<a<\infty,\ -\infty<b<\infty\}$	$6+2i \in$ C

3. Recursively define the elements in the set

   i) $0 \in A$

   ii) if $x \in A$, $x+2 \in A$

   - What is set A?
   - How to define Z?

Set Basics:

- A special set Ø = { }, called empty or null set, which does not have any element
  - Note that ∅ is different from {∅}.
    - ∅ is a set with NO element
    - {∅} is a set with 1 element, which is the empty set.

Symbol	Symbol Name	Meaning / definition	Example
Ø or {}	empty set	Ø = { }	
U	universal set	set of all possible values	
$a \in A$	element of	set membership	A={3,9,14}, 3 ∈ A
$a \notin A$	not element of	no set membership	A={3,9,14}, 1 ∉ A
A ⊆ B	subset	every element in set A is also an element in set B	{9,14,28} ⊆ {9,14,28}
A ⊂ B	proper subset / strict subset	every element in set A is also an element in set B, but not every element in set B is also an element in set A	{9,14} ⊂ {9,14,28}
A ⊄ B	not subset	left set not a subset of right set	{9,66} ⊄ {9,14,28}
A ⊇ B	superset	set A contains all elements in set B	{9,14,28} ⊇ {9,14,28}
A ⊃ B	proper superset / strict superset	set A contains at least one more element than set B	{9,14,28} ⊃ {9,14}
A ⊅ B	not superset	set A is not a superset of set B	{9,14,28} ⊅ {9,66}
A ∩ B	intersection	set of elements that belong to set A and set B	A ∩ B = {9,14}
A ∪ B	union	set of elements that belong to set A or set B	A ∪ B = {3,7,9,14,28}
A = B	equality	both sets have the same members	A={3,9,14}, B={3,9,14}, A=B
A’ or A^c	complement	set of all elements that do not belong to set A	
A - B	relative complement	set of all elements that belong to A and not to B	A={3,9,14}, B={1,2,3}, A-B={9,14}
A Δ B or A ⊖ B	symmetric difference	set of all elements that belong to A or B but not to their intersection	A={3,9,14}, B={1,2,3}, A Δ B={1,2,9,14}

Formal Definitions:

- A ⊆ B iff …
  - Ø ⊆ Ø ?
  - Ø ⊆ A for any set A ?
- A ⊂ B iff …
- A = B iff …

Operations on Set:

- A ∪ B = …
- A ∩ B = …
- A – B = …

Two set are **equal** iff A ⊆ B and B ⊆ A
Two sets are **disjoint** if A ∩ B = Ø

Set Identities & *Set Dual* properties:

A ∪ B = B ∪ A
A ∪ (B ∪ C) = (A ∪ B) ∪ C
A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)
A ∪ A = A
A ∪ Ø = A
A ∪ **U** = **U**

A ∩ B = B ∩ A
A ∩ (B ∩ C) = (A ∩ B) ∩ C
A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)
A ∩ A = A
A ∩ **U** = A
A ∩ Ø = Ø

More Identities:

A – (B ∪ C) = (A – B) ∩ (A – C)
A – (B ∩ C) = (A – B) ∪ (A – C)

Set Closure

A set S is "**closed under operation X**" if the result of performing operation X on elements from set S is also in set S.

- Take the necessary number of arguments from set S, perform the operation X, check to see if the result is still in set S.

With respect to operations +, -, *, / (floating division), DIV (integer division):

- N is closed under:
  - +, *
- Z is closed under:
  - +, -, *
- R is closed under:
  - +, -, *
- N^+ is closed under:
  - +, *

Practice:

- R – {0} is closed under what operations?

- Let F = set of Fibonacci numbers
  - Is F closed under addition?

- Question: Would a finite set of integers closed under +?

More Set Operations and Symbols:

Symbol	Symbol Name	Meaning / definition	Example
2^A *or* $\mathcal{P}(A)$	Power set	Set of all subsets of A	
(a,b)	ordered pair	collection of 2 elements	
A×B	Cartesian product	set of all ordered pairs from A and B	
|A|	Cardinality	the number of elements of set A	A={3,9,14}, |A|=3

Power Set, P(A): Set of sets

P(A) = 2^A = set of all possible subsets

= { x | x ⊆ A }

A power set of any set A must have at least two subsets:

- ∅ ⊆ A
- A ⊆ A

Example:

A = { }

P(A) = { Ø
}

B = {1}

P(B) = { Ø,
{1},
}

C = {1,2}

P(C) = { Ø,
{1}, {2},
{1,2}
}

D = {1,2,3}

P(D) = { Ø,
{1}, {2}, {3},

{1,2}, {1,3}, {2,3}
{1,2,3}
}

E = {1,2,3,4}
P(E) = { Ø,
{1}, {2}, {3}, {4}
{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}
{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4},
{1,2,3,4},
}

F = {1,2,3,4,5}
P(F) = ?

G = {1,2,3,4,5,6}
P(G) = ?

|A| = cardinality of a set
for a finite set A, |A| = number of elements in set A

|A| = 0, |P(A)| = 1
|A| = 1, |P(A)| = 1+1 = 2
|A| = 2, |P(A)| = 1+2+1 = 4
|A| = 3, |P(A)| = 1+3+3+1 = 8
|A| = 4, |P(A)| = 1+4+6+4+1 = 16

Theorem: for any set A,
$|P(A)| = 2^{|A|}$

- Hint: using mathematic induction and note how do you derive P(B) from P(A), P(C) from P(B), P(D) from P(C), and P(E) from P(D) as the size of the set increases.

- Hint: An element is either in or not in a subset of A

Other related topics:

- Pascal Triangle
- Binomial Theorem
- Prove that for any n>r>0,
$$C(n,r) = C(n-1,r-1) + C(n-1,r)$$
- Prove that $\sum_{i=0}^{n} C(n,i) = 2^n$

Partition

- A collection of **nonempty disjoint** subsets of *S* whose **union equals *S***.

Partition π of a set A is a set of subsets of A such that

1. $\forall x \in \pi, x \neq \varnothing$
   - Each element in π is non-empty
2. $\forall x, y \in \pi, x \cap y = \varnothing$
   - Distinct elements in π are disjoint
3. $\bigcup_{x \in \pi} x = A$
   - Union of all elements in π gets back the original set

Example:
A = {1,2,3,4}
P(A) = { Ø,
{1}, {2}, {3}, {4}
{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}
{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4},
{1,2,3,4},
}

π = { Ø, {1,2},{3,4}} ?
π = {{1,2}, {2,3}, {4}} ?
π = {{1}, {3}, {4}} ?
π = {{1}, {2,3},{4}} ?
π = {{1}, {2}, {3}, {4}} ?
π = {{1,2,3,4}} ?

Theorem: Equivalence Classes form a Partition.

Practice: Given B = { Ø, {Ø}, {{Ø,Ø}}, (Ø), {(Ø,Ø)}} }
Write out all partitions with exactly 1, 2, 3, 4 and 5 elements.

Cartesian Product

A X B = { (a,b) : a $\in$ A and b $\in$ B }

|A X B| = |A|·|B| (Proof ?)

Binary Relation R:

R $\subseteq$ A X B $\Rightarrow$ $R_i \in$ P(A X B)

$|R_i|$ = |P(A X B)| = $2^{|AXB|} = 2^{|A|.|B|}$

Properties of Relation:
R $\subseteq$ AXA
xRy iff (x,y) $\in$ R
R(x,y) iff (x,y) $\in$ R

1. Reflexive:
   $\forall$x$\in$A, (x,x) $\in$ R
2. Symmetric:
   $\forall$x,y$\in$A, if (x,y) $\in$ R, then (y,x) $\in$ R
3. Transitive:
   $\forall$x,y,z$\in$A, if (x,y) $\in$ R and (y,z) $\in$ R, then (x,z) $\in$ R
4. Anti-symmetric:
   $\forall$x,y$\in$A, if (x,y) $\in$ R and (y,x) $\in$ R, then x = y

Question:
- Not Symmetric == Anti-Symmetric?
- Can a relation be Symmetric and Anti-Symmetric at the same time?

Give example of relations that exhibits 16 different combinations of properties:

- Some cases might not be possible, but which one(s)?

R	S	T	A	Example
T	T	T	T	“=”
T	T	T	F	
T	T	F	T	
T	T	F	F	
T	F	T	T	“≤”
T	F	T	F	
T	F	F	T	
T	F	F	F	
F	T	T	T	
F	T	T	F	
F	T	F	T	
F	T	F	F	
F	F	T	T	“<”
F	F	T	F	
F	F	F	T	
F	F	F	F	

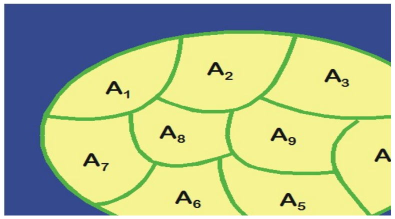
A relation that is Reflexive, Symmetric and Transitive is known as **Equivalence Relation**.

Examples:
Which one of the following relations is equivalence relation?

R1: A = {all human}, xRy iff x is the father of y
R2: A = {all human}, xRy iff x is the brother of y
R3: A = {all males}, xRy iff x is the brother of y
R4: A = {all human}, xRy iff x and y have the same parents
R5: A = {Morehouse students}, xRy iff x and y take CSC311}
R6: A = N, xRy iff x mod 5 = y mod 5
R7: A = N^+, (a/b)R(c/d) iff ad = bc

Equivalence relation R on set A divides set A into a number of **Equivalence Classes** $[a]_R$:

$[a]_R = \{ b \in A: (a, b) \in R \}$



R4:

- $[\text{Billingslea, Marcus}]_{R4}$ = ?
- $[\text{Boadi, Noel}]_{R4}$ = ?
- $[\text{Bradford, Emmanuel}]_{R4}$ = ?
- $[\text{Brown, Mason}]_{R4}$ = ?
- $[\text{Egwuekwe, Aren}]_{R4}$ = ?
- $[\text{Fuller Reed, Jakylan}]_{R4}$ = ?
- $[\text{Gray, Amari}]_{R4}$ = ?
- $[\text{Gray, Christian}]_{R4}$ = ?
- $[\text{Hulett, Kobe Ryan}]_{R4}$ = ?
- $[\text{Hutchins, Julian}]_{R4}$ = ?
- $[\text{Mccoy, Keyshawn}]_{R4}$ = ?
- $[\text{Nurse, Marlon}]_{R4}$ = ?
- $[\text{Okeke, Chimezie}]_{R4}$ = ?
- $[\text{Oliga, John}]_{R4}$ = ?
- $[\text{Ross, Mark}]_{R4}$ = ?
- $[\text{Smith, Jonah}]_{R4}$ = ?
- $[\text{Stokes, Kaleb}]_{R4}$ = ?
- $[\text{Swain, Mark-Anthony}]_{R4}$ = ?
- $[\text{Von Lotten, Zachary}]_{R4}$ = ?
- $[\text{Wilson, Paul}]_{R4}$ = ?
- $[\text{Young, Masai}]_{R4}$ = ?

R6:

$[0]_{R6} = \{ 0,5,10,15,\ldots\}$
$[1]_{R6} = \{ 1,6,11,16,\ldots\}$
$[2]_{R6} = \{ 2,7,12,17,\ldots\}$
$[3]_{R6} = \{ 3,8,13,18,\ldots\}$
$[4]_{R6} = \{ 4,9,14,19,\ldots\}$

R7:

$[1/2]_{R7}$ = ?
$[3/10]_{R7}$ = ?
$[4/13]_{R7}$ = ?

Questions for discussion:

R5:

Is R5 an equivalence relation?
How to make R5 into an equivalence relation, R5E?
If so, what are the equivalence classes R5E?

R5E:

[Billingslea, Marcus]$_{R5E}$ = ?
[Boadi, Noel]$_{R5E}$ = ?
[Bradford, Emmanuel]$_{R5E}$ = ?
[Brown, Mason]$_{R5E}$ = ?
[Egwuekwe, Aren]$_{R5E}$ = ?
[Fuller Reed, Jakylan]$_{R5E}$ = ?
[Gray, Amari]$_{R5E}$ = ?
[Gray, Christian]$_{R5E}$ = ?
[Hulett, Kobe Ryan]$_{R5E}$ = ?
[Hutchins, Julian]$_{R5E}$ = ?
[Mccoy, Keyshawn]$_{R5E}$ = ?
[Nurse, Marlon]$_{R5E}$ = ?
[Okeke, Chimezie]$_{R5E}$ = ?
[Oliga, John]$_{R5E}$ = ?
[Ross, Mark]$_{R5E}$ = ?
[Smith, Jonah]$_{R5E}$ = ?
[Stokes, Kaleb]$_{R5E}$ = ?
[Swain, Mark-Anthony]$_{R5E}$ = ?
[Von Lotten, Zachary]$_{R5E}$ = ?
[Wilson, Paul]$_{R5E}$ = ?
[Young, Masai]$_{R5E}$ = ?

- How many equivalence classes are there?

- Number of distinct equivalence classes finite or infinite?
  - Give an equivalence relation for each case

Theorem: **Equivalence Classes form a Partition**

Proof:

1. Each equivalence class is non-empty.
   - $\forall a \in S$, $[a]_R \neq \emptyset$
     ???

2. Distinct equivalence classes are disjoint
   - $\forall a,b \in S$, $[a]_R \cap [b]_R = \emptyset$ unless $[a]_R = [b]_R$
     ???

3. Union of all equivalence classes = S
   - $\bigcup_{a \in S} [a]_R = S$
     ???

Closure on Relations:

A binary relation R^* on set S is the **closure** of a relation R on S with respect to property P if:

1. R^* has the property P *(though R doesn't have property P)*
2. $R \subseteq R^*$
3. R^* is a subset of any other relation on S that includes R and has the property P *(i.e. R* is the smallest relation that has P)*

Given a binary relation R, find the reflexive, symmetric and transitive closure.

1. Reflexive closure?
2. Symmetric closure?
3. Transitive closure?

- Note that finding the reflexive and symmetric closure is a one-pass operation.
  - Is finding the transitive closure also a one-pass operation?
    - No, the new added ordered pairs may introduce new transitive relationship with the old ordered pairs
  - Do we need to find the anti-symmetric closure?

Practice: Find the reflexive, symmetric and transitive closure:
$S = \{1, 2, 3\}$
$R = \{(1,1), (1,2), (1,3), (3,1), (2,3)\}$.

Programming Practice:
Write a program to read in a set S of elements and a set R of ordered pairs, and then generate the reflexive, symmetric and transitive closure of R.

Function (Total Function):

$f : A \rightarrow B \subseteq R$: AXB such that

1. $\forall a \in A, \exists b \in B$ such that $f(a) = b$
2. $\forall a \in A, \forall b,c \in B$, if $f(a)=b$ and $f(a)=c$, then $b=c$

Function without property one is called *Partial Function*.

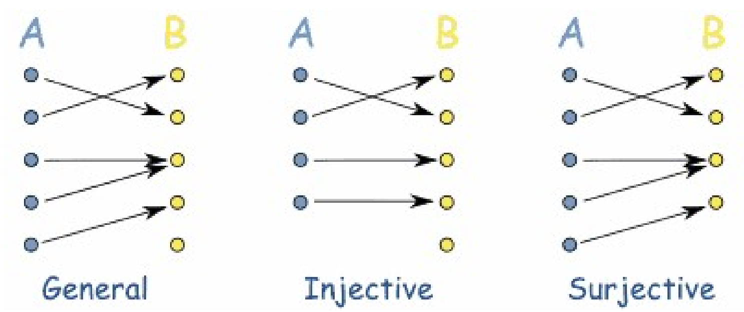
Types of functions:

1. One-to-One (Injective):
   $\forall a,b \in A, c \in B$, if $f(a)=c$ and $f(b)=c$, then $a=b$

2. Onto (Surjective):
   $\forall b \in B, \exists a \in A$ such that $f(a) = b$

3. Bijection:
   One-to-One and Onto
   Inverse function f^{-1} exists

4. None of above



Questions:

- Inverse of a One-to-one is also a function?
- Inverse of a Onto is also a function?

Questions:

Given $|A| = p$, $|B|=q$, how many

1. relations R:AXB can be defined?
2. functions $f:A \rightarrow B$ can be defined?
3. One-to-One function can be defined ($p \leq q$)?
4. Onto function can be defined ($p \geq q$)?
5. Bijection function can be defined ($p = q$)?

Composition of functions and relations:

Given $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$

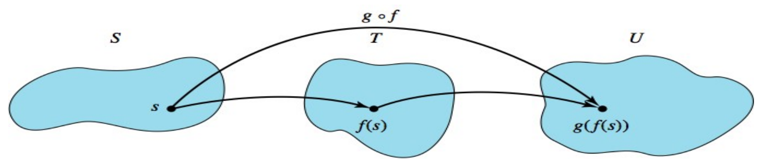
$R_1 \circ R_2 = \{(x,z) : \exists y \in T \text{ such that } (x,y) \in R_1 \text{ and } (y,z) \in R_2\}$

Given $f : S \rightarrow T$ and $g : T \rightarrow U$

The composition function, $g \circ f$: $S \rightarrow U$

$g \circ f\ (x) = g(f(x))$

- The function $g \circ f$ is applied right to left; function f is applied first and then function g.



f	g	$g \circ f$
One-to-One	One-to-One	One-to-One
One-to-One	Onto	??
One-to-One	Bijection	One-to-One
Onto	One-to-One	??
Onto	Onto	Onto
Onto	Bijection	Onto
Bijection	One-to-One	One-to-One
Bijection	Onto	Onto
Bijection	Bijection	Bijection

Finite vs Infinite Sets

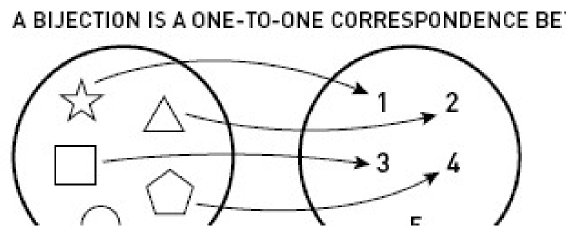
|A| denotes the cardinality (number of elements) of a set

- For a finite set A:
  - |A| = number of elements in set A, which is simply a single integer.
  - e.g. A={1,2,3}, B={a,b,c}, then |A| = |B| = 3
- For infinite sets:
  - How do we count the number of elements in the set?
  - What is the number of the elements?
  - For N, Z and R, which one has more elements?
    - |N| = |Z|?
    - |N| = |R|?
  - Cardinality is not a single number. There are two kinds of infinite set:
    - Countable infinite
    - Uncountable infinite

"Equinumerous":

2 sets A and B are "**equinumerous**" iff $\exists$a bijection f: A $\rightarrow$ B.
$\Rightarrow$ these sets have the same number of elements

- A bijection f is a function such that every element in A is mapped to a distinct element in B, and vice versa.
  - In order for f to be a bijection, |A| = |B|



  - Is there any difference if f: B $\rightarrow$ A instead?

- As for A= {1,2,3}, B={a,b,c}, sets A and B are equinumerous because a bijection f: A $\rightarrow$ B exists:
  f(1)=a, f(2)=b, f(3)=c
  or
  f(1)=b, f(2)=a, f(3)=c
  or
  f(1)=c, f(2)=b, f(3)=a
  or …
  (there could be 6 different definitions of the bijection f)
- Given C = { $x \in N : x = 2y$ for some $y \in N$ }
  D = { $x \in N : x = 2y+1$ for some $y \in N$ }
  - Are C and D equinumerous?
  - If yes, give an explicit bijection f: C $\rightarrow$ D.

Countably vs Uncountably Infinite

Countably infinite (Denumerable) Set:

- Able to select a first element, second element, and so on.
- There exists a counting scheme (bijection function) so that sooner or later I can reach/list out every element in the set.

- Axiom: N is **countably infinite**
  - Why?
  - N is the “smallest/basis” infinite set

Formal definition:
A set S is **countably infinite** iff S is **equinumerous** with N
(i.e $\exists$a bijection f: S $\rightarrow$ N or f: N $\rightarrow$ S).

- Is set of even natural number set C countably infinite?

  ➢ Yes, because $\exists$a bijection f: N $\rightarrow$ C such that
  f(x) = 2x

- Is set of odd natural number set D countably infinite?

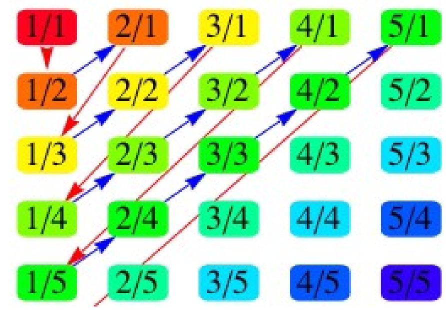
  ➢ Yes, because $\exists$a bijection f: N $\rightarrow$ D such that
  f(x) = 2x + 1

- Is Z countably infinite?

Integers	0	1	−1	2	−2	3	−3	4
Naturals	0	1	2	3	4	5	6	7

  ➢ Yes, because $\exists$a bijection f: N $\rightarrow$ Z such that
  f(x) = ???

- Is Q^+ countably infinite?



➢ Yes, because $\exists$a bijection f: $Q^+ \rightarrow$ N such that

f(1/1) = 0

f(1/2) = 1
f(2/1) = 2

f(1/3) = 3
f(2/2) = 4
f(3/1) = 5

f(1/4) = 6
f(2/3) = 7
f(3/2) = 8
f(4/1) = 9

f(1/5) = 10
….
f(i/j) = ???

➢ Hint: note the pattern and set up a recurrence relation and solve it.

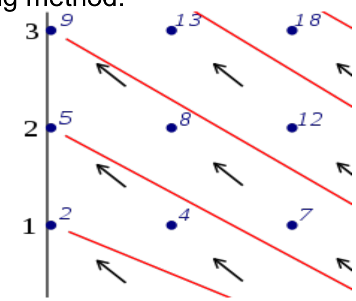
Theorems:

1. **Union of 2 countably infinite sets is countably infinite.**
   - Proof?

2. **Union of countably infinite number of countably infinite sets is countably infinite.**

Prove that N X N is countably infinite (Theorem 2):

Using dovetailing method:



∃a bijection f: N X N → N such that

f(0,0) = 0

f(1,0) = 1
f(0,1) = 2

f(2,0) = 3
f(1,1) = 4
f(0,2) = 5

f(3,0) = 6
f(2,1) = 7
f(1,2) = 8
f(0,3) = 9
…
f(i,j) = ???

<u>Implications of Theorems 1 and 2:</u>

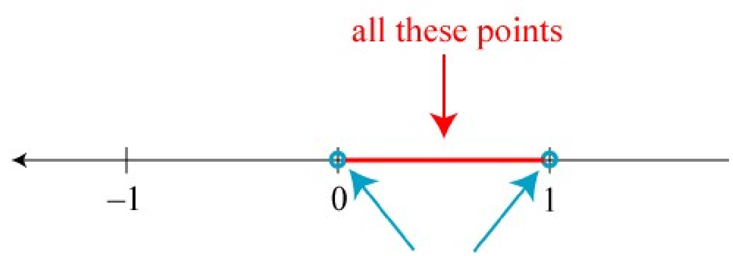
- Countably Infinity + Countably Infinity = Countably Infinity
- Countably Infinity X Countably Infinity = Countably Infinity

- Is there a bigger infinity than N?
- Could there be more infinite than an "infinite" set?

Uncountably Infinite:

- NO counting scheme to list out all the element in the set one by one
- Bijection does not exist!
- Given any hypothetical enumerable scheme, you can always find an element that is not part of the enumeration.
  - In particular with respect to a number set S, if for any two elements a, b in S, there exists an element c from S such that a<c<b.

- Is R countably or uncountably infinite?
  - How to prove bijection does not exist?
  - *Proof by Diagonalization Principle*



- There are uncountably infinite number of decimal values between 0 and 1!!

Proof Techniques:

- Direct proof

- Proof by contrapositive:
  $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$

- Proof by contradiction
  Assume false of Q, arrive contradiction
  $P \wedge \neg Q \to$ False

- Proof by construction:
  $\exists$ "there exists"

- Proof by induction:
  $\forall$" for all " on a total order set (linear ordering)
  - Weak induction:
    - prove base case
    - prove $P(k) \to$ next case: $P(k+1)$
  - Strong induction:
    - prove base case**S**
    - prove $\forall 1 \le i \le k$, $P(i) \to$ next case: $P(k+1)$

- **Pigeonhole Principle**: if $|A| > |B|$, there is no one-to-one function from A to B
  - ➢ There exists at least one element in B such that it is the image of more than one element in A
  - ➢ At least 2 elements in A share the same image in B.

- **Diagonalization Principle**
  - To give you a way to construct another enumeration which is distinct from the existing enumeration.
  - In other words, the bijection cannot exist!

Diagonalization Principle

R is a binary relation on set A
For each $\alpha \in A$,
$R_\alpha = \{ b \in A : (\alpha,b) \in R \}$

Define D: the diagonal set for R
$D = \{ \alpha \in A : (\alpha,\alpha) \notin R \}$

Then **D is distinct from each R_α**

For example,
A = { a,b,c,d,e,f }
R = {(a,a),(a,b),(a,d),(b,e),(c,a),(c,c),(c,d),(c,f),(d,b),(d,f),
(e,c),(e,d),(f,a),(f,d),(f,e),(f,f)}

The order pairs in R can be represented by a table:

	a	b	c	d	e	f
a	x	x		x		
b					x	
c	x		x	x		x
d		x				x
e			x	x		
f	x			x	x	x

R_a = { a,b,d }
R_b = { e }
R_c = { a,c,d,f }
R_d = { b,f }
R_e = { c, d }
R_f = { a,d,e,f }
Diagonal set D = ?

D = { b,d,e } is distinct from each R_a, R_b, R_c, R_d, R_e and R_f

Prove set P(N) is Uncountably Infinite

Proof by contradiction:
Assume P(N) is countably infinite
∃a bijection f: N → P(N).

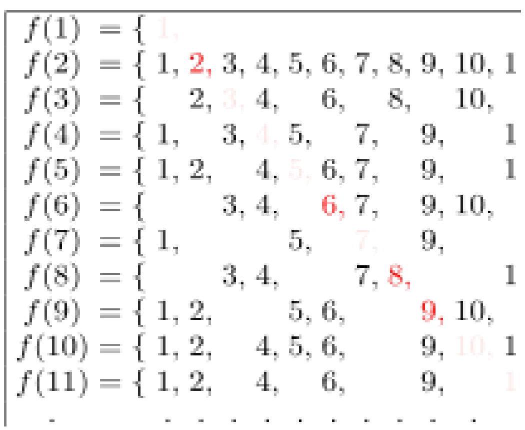
P(N) is a set of sets and can be enumerated as
P(N) = { S_0, S_1, S_2, ….. }

Define f as f(i) = S_i ∀i∈N

Let the list be enumerated as:

f			0 1 2 3 4 5 6 ….
0	→	S_0	?
1	→	S_1	?
2	→	S_2	?
3	→	S_3	?
4	→	S_4	?
….			

For Example: f(0) is missing in the diagram though.


$f(1) = \{ 1,$
$f(2) = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1$
$f(3) = \{ 2, 3, 4, 6, 8, 10,$
$f(4) = \{ 1, 3, 4, 5, 7, 9, 1$
$f(5) = \{ 1, 2, 4, 5, 6, 7, 9, 1$
$f(6) = \{ 3, 4, 6, 7, 9, 10,$
$f(7) = \{ 1, 5, 7, 9,$
$f(8) = \{ 3, 4, 7, 8, 1$
$f(9) = \{ 1, 2, 5, 6, 9, 10,$
$f(10) = \{ 1, 2, 4, 5, 6, 9, 10, 1$
$f(11) = \{ 1, 2, 4, 6, 9, 1$

Consider the set
T = { n ∈ N : n ∉ S_n }

Obviously, T is a set of natural numbers
∴ T ∈ P(N)
⇒ T = S_k for some k

Consider an element k and set S_k: k ∈ S_k or k ∉ S_k

Case (i): if k ∈ S_k, T = { n ∈ N : n ∉ S_n }
⇒ k ∉ T

Case(ii): if k ∉ S_k, T = { n ∈ N : n ∉ S_n }
⇒ k ∈ T

Both cases end with contradiction that T = S_k for some k
⇒ T ≠ S_k for all k
⇒ T is distinct from the enumeration S_0, S_1, S_2, …..

∴ P(N) is uncountable.

⇒ **The power set of a countably infinite set is uncountable.**

Prove that set of real numbers, R_{01}, between 0 and 1 is uncountable.

Assume the set of real numbers between 0 and 1 is countable

$\Rightarrow R_{01}$ is **equinumerous** with N

$\Rightarrow \exists$a bijection f: N $\rightarrow R_{01}$

$\Rightarrow$ these real numbers can list out one by one

Let the list be enumerated as:

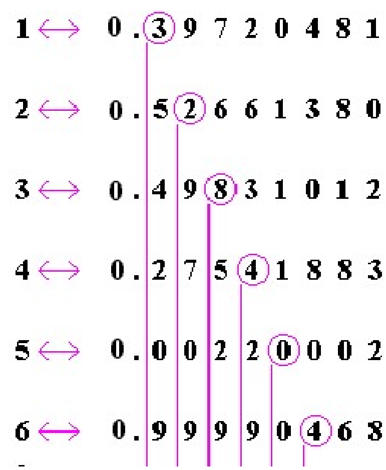
	f	
0	$\rightarrow$	$0.d_{11}d_{12}d_{13}d_{14}\ldots.$
1	$\rightarrow$	$0.d_{21}d_{22}d_{23}d_{24}\ldots.$
2	$\rightarrow$	$0.d_{31}d_{32}d_{33}d_{34}\ldots.$
3	$\rightarrow$	$0.d_{41}d_{42}d_{43}d_{44}\ldots.$
4	$\rightarrow$	….

Look at the diagonal digits, d_{ii}, and construct a new real number using these diagonal digits. For each row, pick a digit that is **different** than the diagonal digit

Construct p = $0.p_1p_2p_3p_4\ldots$
where $p_i = (d_{ii} + 1)$ %10

p is distinct from any real number in the original list.

For example:



- p = 0.439515…. is a real number, but is **different from every other number in the list.**
  - It differs from any number in the list by at least one digit and therefore this number p is **not** in the list.

- However, this list was supposed to have all real numbers
  - Perhaps we have missed p in the first enumeration?
  - What about including p in our first enumeration?

- Therefore, we have a contradiction. Hence our initial assumption must be false
  - ➢ The set of real numbers is not countable(denumerable).