1 Section 11.1/11.2 Logic and Truth Tables

We are going to spend a few days talking about logic. Logic is a way of formalizing our ideas about truth, falsehood, and logical arguments. We are going to start out with a lot of definitions.

A **statement** is a declarative sentence which is either true or false. Examples and non-examples:

- This course is the best
- I enjoy teaching this course
- It takes 3 hours to get home
- It is green (this is not a declarative sentence)
- Is this boring?

The **truth value** of a statement is either true or false We can combine logical statements as we do in english, the 4 main connectors we will use will be **or**, **and**, **not**, and **if...then...**. These phrases are called connectives. A **compound** statment is a statement is a statment formed by other statements with connectives. A **simple** statement is one that is not compound. Examples:

- The wall is brown (simple)
- You will get an A in the class or you will get a B (compound)
- If you study, then you will pass (compound)
- I will not answer emails after midnight (could be either, but helpful to think about not)
- I have office hours on Friday and I have office hours on Thursday.

Now we want to work a little bit more symbolically. We will denote simple statements by lowercase letters like p, q, r. Then we also have symbols for our connections

1. **Conjunction**: the symbol $\wedge$ represents and
2. **Disjunction**: the symbol $\vee$ represents or
3. **Negation**: the symbol $\sim$ represents not
4. **Implication**: the symbol $\rightarrow$ represents if...then...

Examples: Suppose p, q, r are simple statements. Write the following statements in english

1. $p \wedge q$
2. $\sim q \vee p$
3. $r \rightarrow p \vee q$
4. $p \wedge \sim q \rightarrow r$

This gives the following definition: a **Statement Form** is an expression formed from simple statements and connectives accoring to the following rules:

1. a simple statement is a statement form
2. if P is a statement form then $\sim P$ is a statement form
3. if P, Q are statement forms then $P \vee Q$, $P \wedge Q$ and $P \rightarrow Q$ are statement forms

In other words a statment form is a symbolic representation of a compound statement. We will usually denote these by capital letters.

How do we know when statement forms are true or false. We look at something called their **truth table**. It is a table summarizing the truth value of a statement form for every possiblilty of inputs.

Let's look at truth tables for the basic connectors:

p	$\sim p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Let's talk for a little bit about implicicaiton. The second two rows may seem strange, but let's put this in the context of a real world example. Consider the following statement: If you study, then you will do well on the next quiz. What if you don't study but still do well? Does this mean I was wrong? Does it mean the statement was false? No, in that case the statement has no bearing on outcome. So if p is false then $p \to q$ will always be true regardless of q. p is called the hypothesis and q is called the conclusion.

We have two more definitions. A **tautology** is a statement form which is always true. A **contradiction** is a statement form which is always false. For each of the following statement forms, decide if it is a tautology, contradiction, or neither by constructing the truth table.

- $p \vee \sim p$
- $p \wedge \sim p$
- $(p \vee q) \to (p \wedge q)$
- $(p \wedge q) \to (p \vee q)$
- $(p \wedge q) \wedge (\sim p \to q)$

Truth tables also give us the idea of **logical equivalence**. Two statement forms are logically equivalent if they have the same truth tables.Example: Show that $\sim (P \vee Q)$ is logically eqivialent to $\sim P \wedge \sim Q$. Also prove that $\sim (P \wedge Q)$ is logically eqivalent to $\sim P \vee \sim Q$. These are called demorgan's laws.

There are two more very common operations which can be build out of basic operations.

Find the truth table for $(p \vee q) \wedge \sim (p \wedge q)$. This operation is called exclusive or or xor and it is denoted $p \oplus q$. It is true when exactly one of p or q is true.

Another one will be very important when we discuss proofs and logical arguments. Find the truth table for $(p \to q) \wedge (q \to p)$. This is called a biconditional and it is more ofter refered to as "if and only if". It is written as $\leftrightarrow$ and it is true when p and q have the same value.

Consider the statement $p \to q$. Then there are 3 very closely related statements. The **converse** is $q \to p$. The **inverse** is $\sim p \to \sim q$. The **contrapositive** is the statement $\sim q \to \sim p$.

Show that a statement and its contrapositive are logically equivalent. Show that a statement's inverse and converse are logically equivalent. Show that a statement and it's converse are not logically equivalent.