Chapter 1

Known Results on Finite Generation

{ch:known}

Throughout this chapter $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) with the following assumptions:

$$\begin{aligned}
&W \text{ has rank } 3,\ S = \{s, t, u\},\ a = m(s,t), b = m(s,u), c = m(t,u)\\
&3 \le a \le b \le c \\
&[U_\alpha, U_\beta] = 1 \text{ when } \alpha, \beta \text{ are nested}
\end{aligned} \tag{1.1}$$

Let Σ be the Coxeter complex of W with fundamental chamber C, and Φ_+ be the positive roots of Σ. We will also let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ be the subgroup of G generated by the positive root groups. We will also note that properties of RGD systems tell us that $a, b, c \in \{2, 3, 4, 6, 8\}$ and thus by (A) we know that $a, b, c \in \{3, 4, 6, 8\}$.

For any vertex v of Σ, there will be some walls of Σ which pass through v, and for each of these walls we have a unique *positive* root. We will call these the **positive roots at** v and denote them by Φ_+^v. Recall that $\mathrm{st}(v)$ is defined as all the chambers containing v as a vertex. If there are n positive roots at v then $|\mathrm{st}(v)| = 2n$. Furthermore, it is possible to label the positive roots at v as $\alpha_1, \ldots, \alpha_n$ in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \le i \le k \le j \le n$. This ordering is unique up to a reversal of the form $\alpha_i \mapsto \alpha_{n+1-i}$. This possible reversal will not matter in most cases and if it does then a choice of α_1 will be specified. It does however allow us to unambigiously define α_1 and α_n as the **simple** roots at v. They are the unique positive roots at v whose intersection is contained in all other positive roots at v.

Now we can define U_v to be the subgroup of G generated by all of the root groups of the positive roots at v. That is

$$U_v = \langle U_\alpha | \alpha \text{ is a positive root at } v\rangle = \langle U_\alpha | \alpha \in \Phi_+^v\rangle$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a standard ordering of the positive roots at v then we can simplify notation by letting $U_i = U_{\alpha_i}$ for all α_i through v. Since v is a simples of Σ of co-dimension 2, we know from the theory of RGD systems that U_v will also have the structure of a spherical, rank 2 RGD system as well. Let $U_v' = \langle U_1, U_n\rangle$ be the subgroup of U_v generated by the simple root groups, where $|\mathrm{st}(v)| = 2n$. Then it is known that $U_v = U_v' = \langle U_1, U_n\rangle$ with the exception of a few cases which we will explictly state in the following Lemma.

{lem:index}

Lemma 1. *Let v be a vertex of Σ with $|\mathrm{st}(v)| = 2n$ and let $U_v' = \langle U_1, U_n \rangle$ where U_1, U_n are the root groups of the simple roots at v. Then the group U_v is has the structure of a spherical, rank 2 RGD system and $U_v = U_v'$ unless U_v is isomorphic to one of the following groups:*

$$C_2(2) \qquad G_2(2) \qquad G_2(3) \qquad {}^2F_4(2)$$

In fact, we also know the index $[U_v : U_v']$ in each of these cases which is summarized in the following table.

U_v	$[U_v : U_v']$
$C_2(2)$	2
$G_2(2)$	4
$G_2(3)$	3
${}^2F_4(2)$	2

We can see from the previous lemma that even when $U_v' \neq U_v$, it is still a fairly large subgroup and in some cases it will even be normal. This will allow us to construct helpful homomorphisms later, but before we do so we will explicitly state the desired result.

{lem:normal}

Lemma 2. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$. If U_v is isomorphic to $C_2(2), G_2(3)$, or ${}^2F_4(2)$ then U_v' is a normal subgroup of U_v. If $U_v \cong G_2(2)$ then U_v' is not a normal subgroup of U_v, but there is a standard labeling of the positive roots through v so that $U_v'' = \langle U_1, U_5, U_6 \rangle$ is a normal subgroup of U_v with $[U_v : U_v''] = 2$.*

Proof. If $U_v \cong C_2(2)$ or ${}^2F_4(2)$ then U_v' is a subgroup of index 2 and thus it is normal. If $U_v \cong G_2(3)$ then U_v is a 3-group and thus 3 is the smallest prime dividing $|U_v|$ and we know that U_v' is normal in this case as well.

Now suppose $U_v \cong G_2(2)$. Need to add this proof later □

Using Lemma 2 and elementary group theory, we get the following result.

{cor:phiv}

Corollary 1. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$. Then there is a cyclic group H and a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$ where U_1 and U_n are the simple root groups at v.*

Proof. If $[U_v : U_v'] \geq 2$ then U_v must be isomorphic to one of $C_2(2), G_2(2), G_2(3), {}^2F_4(2)$. If $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$ then we can apply Lemma 2 to let $H = U_v/U_v'$ and ϕ_v be the quotient map which certainly will be surjective and send U_1 and U_n to $\{1\}$ by the definition of U_v'. The group H is cyclic because it has prime order.

If $U_v \cong G_2(2)$ then we know that $U_v' \subset U_v'' = \langle U_1, U_5, U_6 \rangle$ for an appropreate standard labeling, and we again apply Lemma 2 to set $H = U_v/U_v''$ and ϕ_v as the quotient map. The group H is again cyclic because it has prime order. □

The following corollary will show that we do not have very much wiggle room when defining ϕ_v, and thus if we can write any function which "looks like" ϕ_v then they must be esentially the same.

{cor:uniquephiv}

Corollary 2. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$ and let ϕ_v be defined as in the previous corollary. Then $\ker \phi_v$ is the unique, proper, normal subgroup of U_v which contains U_1 and U_n.*

Proof. If $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$ then U_v' is normal, it is generated by U_1 and U_n,, and it has prime index so there cannot be another proper subgroup containing U_v'. By the construction of ϕ_v, we also know that $\ker \phi_v = U_v'$ so that $\ker \phi_v$ is the unique proper, normal subgroup of U_v containing U_1 and U_n.

If $U_v \cong G_2(2)$ then $\ker \phi_v = U_v'' = \langle U_1, U_5, U_6 \rangle$ under a standard labeling. If N is any normal subgroup containing U_1 and U_n then we can apply the commutator relations in $G_2(2)$ to get

add proof later □

So far we have only considered each vertex v and U_v separately. But in the Coxeter complex Σ, we have not only a collection of vertices, but an action of the group W on the vertices which behaves nicely with properties like the type of a vertex. We will show that the W action also interacts nicely with U_v and ϕ_v in a similar way.

{lem:responder}

Lemma 3. *Suppose v is a vertex of Σ of type s, $|\mathrm{st}(v)| = 2n$, and $[U_v : U_v'] \geq 2$. Also suppose that w is an element of W such that $w\gamma$ is a positive root at wv for every positive root γ at v. Then there are standard labelings $\alpha_1, \ldots, \alpha_n$ and $\alpha_1', \ldots, \alpha_n'$ of the positive roots through v and wv respectively such that $\alpha_i' = w\alpha_i$ for all i. In particular, w sends roots at v which are simple to roots at v' which are also simple. Furthermore, if v' is any vertex of Σ of type s then there is a $w \in W$ such that $wv = v'$ and $w\gamma$ is a positive root at v' for any positive γ at v.*

Proof. Recall a standard labeling is on of the form $\alpha_1, \ldots, \alpha_n$ where $\alpha_i \cap \alpha_j \subset \alpha_k$ for all $1 \leq i \leq k \leq j \leq n$. If w sends all of the positive roots at v to the positive roots at wv then w induces a bijection on the positive roots at v and wv. Now we can define a labeling of the positive roots at wv by $\alpha_i' = w\alpha_i$ for all i. It only remains to check that this is a standard labeling. If $1 \leq i \leq k \leq j \leq n$ then $\alpha_i \cap \alpha_j \subset \alpha_k$ and thus $\alpha_i' \cap \alpha_j' = w\alpha_i \cap w\alpha_j \subset w\alpha_k = \alpha_k'$ so this is a standard labeling as desired.

Now it suffices to show that such a w exists for any vertex v' in Σ. Since the W action on Σ is transitive on vertices of the same type, it will suffice to show the result when v is a vertex of the fundamental chamber C. Let $D = \mathrm{Proj}_{v'}(C)$ so that $d(D, C)$ is minimal among all chambers of $\mathrm{st}(v')$. Then we know that no walls through v' can separate D and C, because crossing one of these walls would produce a chamber in $\mathrm{st}(v)$ which is closer to C. Therefore, a root at v' is positive if and only if it contains D.

Now choose the unique $w \in W$ such that $D = wC$. We claim that w satisfies the desired properties. First of all, v is a vertex of C of type s and thus wv is a vertex of $wC = D$ of type s. But we know that v' is a vertex of D of type s by definition and thus $wv = v'$ as desired. Now suppose γ is any positive root at v. Then $C \in \gamma$ and thus $D = wC \in w\gamma$ and thus $C \in w\gamma$ so $w\gamma$ is positive at $wv = v'$. Now this w sends positive roots at v to positive roots at v' as desired.

$\square$

Before moving on it is worth clarifying that the type s of the vertex v in the previous lemma can by any type, not just the literal type s in the definition of W.

The previous result can also be used to show that the W action on Σ also behaves nicely with respect to the group U_v and the homomorphisms ϕ_v when they exit.

{cor:respectphiv}

Corollary 3. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ and $[U_v : U'_v] \geq 2$ and v' is any other vertex of Σ of the same type. Then there is an isomorphism between U_v and $U_{v'}$ which sends U'_v to $U'_{v'}$. Consequently, $[U_v : U'_v] = [U_{v'} : U'_{v'}]$, ϕ_v exists if and only if $\phi_{v'}$ exists, and if ϕ_v exists then this isomorphism sends $\ker \phi_v$ to $\ker \phi_{v'}$. If w is any element of W such that $wv = v'$ and $w\gamma$ is positive for all positive γ at v, then this isomorphism can be defined by the property that U_γ is sent to $U_{w\gamma}$ for every γ at v.*

Proof. Let w be any element of W with $wv = v'$ which sends positive roots at v to positive roots at v'. Such a w is guaranteed to exist by Lemma 3. By the theory of RGD systems there is an element $\tilde{w} \in G$ such that $\tilde{w}U_\alpha(\tilde{w})^{-1} = U_{w\alpha}$ for all $\alpha \in \Phi$. Let $f_w : G \to G$ be the isomorphism of conjugation by $\tilde{w}$. Since $w\gamma$ is positive at v' for every positive root γ at v we know that $f_w(U_\gamma) = U_{w\gamma} \subset U_{v'}$ and thus f_w restricts to a homomorphism $\bar{f}_w : U_v \to U_{v'}$ which is necessesarily injective. But w also give a bijection on positive roots at v and v', and $U_{v'}$ is generated by positive root groups at v' so $\bar{f}_w$ is surjective and thus an isomorphism. Now it remains to check it statisfies the rest of the properties.

Since w preserves standard labelings at v and v' we know that it also preserves simple roots. Thus $\bar{f}_w(U_{\alpha_1}) = U_{\alpha'_1}$ for a standard labeling, and similarly for U_{α_n} and $U_{\alpha'_n}$. Since $U'_v = \langle U_{\alpha_1}, U_{\alpha_n} \rangle$ and $U'_{v'} = \langle U_{\alpha'_1}, U_{\alpha'_n} \rangle$ we can also see that $\bar{f}_w$ sends U'_v to $U'_{v'}$. Since $\bar{f}_w$ is an isomorphism it also preserves index so $[U_v : U'_v] = [U_{v'} : U'_{v'}]$.

For any vertex v, the map ϕ_v exists if and only if $[U_v : U'_v] \geq 2$ and thus ϕ_v will exist exactly when $\phi_{v'}$ exists. By Corollary 2 we know that $\ker \phi_v$ is a proper normal subgroup of U_v contaning U'_v and thus $\bar{f}_w(\ker \phi_v)$ will be a proper, normal subgroup of $U_{v'}$ containing $U'_{v'}$. By Corollary 2 again this means $\bar{f}_w(\ker \phi_v) = \ker \phi_{v'}$ which completes the result.

$\square$

{thm:knownfg}

The general theory gives us the following result

Theorem 1. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S) of any rank. If $U_v = U'_v$ for every vertex v of Σ then U_+ is finitely generated.*

Remark: In fact, we can make an even stronger statement. Let α_s be the positive root defined by the wall which separates C and sC and similarly define α_t and α_u. If $U'_v = U_v$ for all $v \in \Sigma$ then U is generated by $U_{\alpha_s}, U_{\alpha_t}$, and U_{α_u}.

Chapter 2

Conditions for Infinite Generation

{ch:general}

2.1 Extension of ϕ_v

Throughout this chapter $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) with the following assumptions:

$$\begin{aligned}
&W \text{ has rank } 3,\ S = \{s,t,u\},\ a = m(s,t), b = m(s,u), c = m(t,u)\\
&3 \le a \le b \le c,\ 4 \le c\\
&[U_\alpha, U_\beta] = 1 \text{ when } \alpha, \beta \text{ are nested}
\end{aligned} \tag{A}$$

Let Σ be the Coxeter complex of W with fundamental chamber C, and Φ_+ be the positive roots of Σ. We will also let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ be the subgroup of G generated by the positive root groups. We will also note that properties of RGD systems tell us that $a, b, c \in \{2, 3, 4, 6, 8\}$ and thus by (A) we know that $a, b, c \in \{3, 4, 6, 8\}$. We assume that $c \ge 4$ because otherwise Lemma 1 and Theorem 1 tell us that U_+ is finitely generated and there is nothing to show.

We can also recall some terminology from the last chapter. We will say that α is a positive root at v if α is positive and the wall $\partial\alpha$ passes through v and we will denote the positive roots at v as Φ_+^v. Then we can define $U_v = \langle U_\alpha | \alpha \in \Phi_+^v\rangle$. We can also label the roots of Φ_+^v as $\alpha_1, \dots, \alpha_n$, where $2n = |\mathrm{st}(v)|$ in Σ, in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for $1 \le i \le k \le j \le n$. With this labeling we will call α_1, α_n the simple roots at v and we will note that they do not depend on the labeling. We will use this labeling many times throughout the section and we will refer to it as the standard labeling. This definition is a slight abuse as this labeling scheme is not unique, however, the only other possible labeling is given by flipping the order and sending $\alpha_i \mapsto \alpha_{n+1-i}$. In practice, this ambiguity will not matter and so most of the time we can simply refer to the standard labeling without any further detail.

We say that two distinct positive roots α, β are a *pre-nilpotent* pair if $\alpha\cap\beta$ and $(-\alpha)\cap(-\beta)$ both contain a chamber. There is a very nice characterization of pre-nilpotent roots which we will use in the remainder of the chapter. Two roots α, β form a pre-nilpotent pair if and only if one of the following holds:

$$\text{(i) } \partial\alpha \cap \partial\beta \neq \emptyset \qquad \text{(ii) } \alpha, \beta \text{ are nested}$$

where we say α, β are nested if $\alpha \subset \beta$ or vice versa. By definition, $\partial\alpha \cap \partial\beta = \emptyset$ if α, β are nested so only one of the previous conditions can be satisfied.

We will also briefly recall the definitions of open and closed intervals of roots. If α, β are two pre-nilpotent, positive roots then we define the closed interval

$$[\alpha, \beta] = \{\gamma \in \Phi_+ | \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset -\gamma\}$$

and the open interval $(\alpha, \beta) = [\alpha, \beta] \setminus \{\alpha, \beta\}$. In a similar manner as before, we will define $U_{(\alpha,\beta)} = \langle U_\gamma | \gamma \in (\alpha, \beta)\rangle$.

One feature of the standard labeling is that it allows us to describe some of these intervals in a very natural way. If v is some vertex of Σ and $\alpha_1, \dots, \alpha_n$ are the positive roots through v with the standard labeling, then $[\alpha_i, \alpha_j] = \{\alpha_k | i \leq k \leq j\}$ whenever $i \leq j$. Similarly we get $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ whenever $i < j$.

By definition, U_+ is generated by the U_α for all positive roots α. However we can say a little bit more about U_+. Each U_α will have its own set of relations $\mathcal{R}_\alpha$. The theory of RGD systems tells us that we have a presentation of U_+ of the following form

$$U_+ = \langle U_\alpha, \alpha \in \Phi_+ | \mathcal{R}_\alpha, \alpha \in \Phi_+, [u, u'] = v, u \in U_\alpha, u' \in U_\beta, \{\alpha, \beta\} \text{ a pre-nilpotent pair}\rangle$$

where v is a word in $U_{(\alpha,\beta)}$ which depends on u, u'. Furthermore, by condition (A) we know that $[u, u'] = 1$ if α and β are nested. Therefore, the only non-trivial commutator relations will occur when $\partial\alpha \cap \partial\beta \neq \emptyset$.

Let $U'_v = \langle U_1, U_n\rangle$ for any vertex $v \in \Sigma$, where U_1 and U_n are the simple roots at v. By Theorem 1 we know that U is finitely generated if $U'_v = U_v$ for all $v \in \Sigma$. What we will show in the rest of the chapter is that if $U'_v \neq U_v$ for some $v \in \Sigma$, then most of the time U will not be finitely generated. Our general strategy will be as follows. If v is some vertex of Σ such that $U'_v \neq U_v$ then Corollary 1 shows the existence of a surjective group homomorphism $\phi_v : U_v \to H$ where H is a cyclic group of the appropriate order. If we can extend this map to all of U_+ in a certain way then we will be able to show certain root groups must be in any generating set of U_+. If we can do this for enough v then we will be able to show that U_+ is not finitely generated.

Our first lemma will define our notion of extending ϕ_v, and give a sufficient condition for this extension to exist.

{lem:existence}

Lemma 4. *Suppose that v is a vertex of Σ such that $U'_v = \langle U_1, U_n\rangle \neq U_v$, where U_1, U_n are the simple roots at v. Then there is a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$, where H is a cyclic group. Also suppose that for any positive root γ with $v \in \partial\gamma$ which is not simple at v, that γ is simple at y for all $y \in \partial\gamma$ with $y \neq v$. Then the map $\tilde{\phi}_v : \cup_{\gamma\in\Phi_+} U_\gamma \to H$ defined by*

$$\tilde{\phi}_v(u) = \begin{cases} \phi_v(u) & \text{if } u \in U_\gamma \text{ and } v \text{ lies on } \partial\gamma \\ 1 & \text{otherwise} \end{cases}$$

Extends uniquely to a well defined group homormoprhism $\tilde{\phi}_v : U_+ \to H$.

Proof. Since $U_v' \neq U_v$ we know that the map ϕ_v exists by Corollary 1. We have a presentation for U_+ and we have defined $\tilde{\phi}_v$ on the generators of U_+, so in order to check that it is well defined we will need to verify that the relations of U_+ are satisfied in the image.

There are three types of relations in the presentation for U_+. There are relations within the same root group U_α for all positive roots α. There are also relations between root groups of pre-nilpotent pairs where either the walls intersect or the roots are nested.

Let R_α be a relation for U_α where R_α is considered as a word with letters in U_α. If v lies on $\partial\alpha$ then $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$ since ϕ_v is a well defined homomorphism. Otherwise, every element of U_α is sent to 1 and thus $\tilde{\phi}_v(R_\alpha) = 1$ as well so that R_α is mapped to the identity as desired.

Now suppose that α and β are any two positive roots. If α, β nested, then (A) tells us that $[U_\alpha, U_\beta] = 1$. Since the codomain of $\tilde{\phi}_v$ is an abelian group, then any relation of the form $[x, y] = 1$ will be satisfied by the image.

Now suppose that $\partial\alpha$ and $\partial\beta$ meet at a point y and consider any relation of the form $[u_\alpha, u_\beta] = w$ where $u_\alpha \in U_\alpha$, $u_\beta \in U_\beta$, and w is a word in $U_{(\alpha,\beta)} \subset U_y$. Again, note that the image of the left side of this equation will always be the identity as the codomain is still abelian. If $y = v$ then $U_y = U_v$ and thus $\tilde{\phi}_v(w) = \phi_v(w) = 1$ because ϕ_v is well defined.

Now suppose that $y \neq v$. Then we can label the positive roots passing through y as $\gamma_1, \cdots, \gamma_n$ in such a way that $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$ whenever $i < j$. In this case we can can say without loss of generality that $\alpha = \gamma_l$ and $\beta = \gamma_m$ with $l < m$. There can be at most one root whose wall passes through y and v, which we will call γ_k if it exists. If γ_k does not exist, or $k \leq l$ or $k \geq m$ then the root γ_k is not contained in (α, β) and thus $\tilde{\phi}_v(U_\delta) = 1$ for all $\delta \in (\alpha, \beta)$. This means $\tilde{\phi}_v(w) = 1$ and the relation is satisfied.

Now we suppose that γ_k exists and $l < k < m$. Then γ_k is not simple at y and thus γ_k must be simple at v by assumption. This means $\tilde{\phi}_v(U_{\gamma_k}) = \phi_v(U_{\gamma_k}) = 1$ by the construction of ϕ_v. Since $\tilde{\phi}_v(U_{\gamma_i}) = 1$ for all $i \neq k$ by definition, this means that $\tilde{\phi}_v(w) = 1$ showing the relation is satisfied and giving the desired result. □

Now Lemma 4 gives a sufficient condition for the existence of $\tilde{\phi}_v$ which is fairly easy to check. This will be the main tool we use in the remainder of the section.

Recall our assumptions in (A) that (W, S) is a rank 3 Coxeter system with $S = \{s, t, u\}$. We also assumed that $a = m(s, t)$, $b = m(s, u)$, and $c = m(t, u)$ with $3 \leq a \leq b \leq c$. Let x be the vertex of C of type s and assume that $[U_x : U_x'] \geq 2$. Our first step in the main proof will be to show that $\tilde{\phi}_x$ exists. We will do this by applying Lemma 4 and to do this we need to prove the following result about roots through x.

{lem:xpos}

Lemma 5. *Let x be the vertex of C of type s. If γ is any positive root at x, and y is any other vertex on $\partial\gamma$, then γ is simple at y.*

Proof. Suppose that γ is not simple at y. Then we can label the positive roots at y as $\delta_1, \ldots, \delta_m$ in such a way that $\delta_i \cap \delta_j \subset \delta_k$ for $1 \leq i \leq k \leq j$. In this case we have δ_1, δ_m

are simple at y and $\gamma = \delta_r$ for some $1 < r < m$. But x is a vertex of C and $C \in \delta_1 \cap \delta_m$ and thus $x \in \delta_1 \cap \delta_m$ as well. We know that x lies on $\partial\delta_r$ by assumption and thus x is an element of $\partial\delta_r \cap \delta_1 \cap \delta_m$. But this is impossible as we can observe from the geometry of Σ that $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$ for all $1 < i < m$. Thus γ is simple at y as desired. □

Despite some of the technical details the previous result should be intuitively clear. The walls through y will divide Σ into $2m$ regions, and the region which contains C will be bounded by the two simple roots. Since x lies on $\partial\gamma$, it is impossible for any other roots through y to be any "closer" to C and thus γ must be simple at y as we proved.

{cor:phix}

Corollary 4. *Let x be the vertex of C of type s, and assume that $[U_x : U'_x] \geq 2$. Then the map $\tilde{\phi}_x$ as defined in Lemma 4 is well defined.*

Proof. Let γ be any non-simple, positive root through x and let y be another vertex on $\partial\gamma$. Then by the previous lemma, γ is simple at y and thus $\tilde{\phi}_x$ exists by Lemma 4. □

The remainder of the section will be used to show that we can use $\tilde{\phi}_x$ and the W action on Σ to construct a large family of vertices for which $\tilde{\phi}_v$ exists.

We can label the roots through x as $\alpha_1, \ldots, \alpha_n$ so that α_1 and α_n are the simple roots at x. Also note that $n = c$. The ordering on these roots is chosen so that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. This is equivalent to the condition that $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ for any $i < j$.

We can describe any root in terms of a pair of adjacent chambers. We can also identify $\mathcal{C}(\Sigma)$ with W where the chamber wC is associated to w. If we use this identification then we can describe the roots as follows

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}
\end{aligned}$$

In a similar way we can define two more roots

$$\begin{aligned}
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Now we can define $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. These roots are chosen and $\mathcal{D}$ is defined in such a way to give the following lemma:

{lem:containD}

Lemma 6. *Let x be the vertex of C of type s and assume $W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^a = (su)^b = (tu)^c = 1\rangle$. Let $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where $\alpha_1, \alpha_n, \beta, \beta'$ are roots of Σ defined by*

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

{fig:defineD}

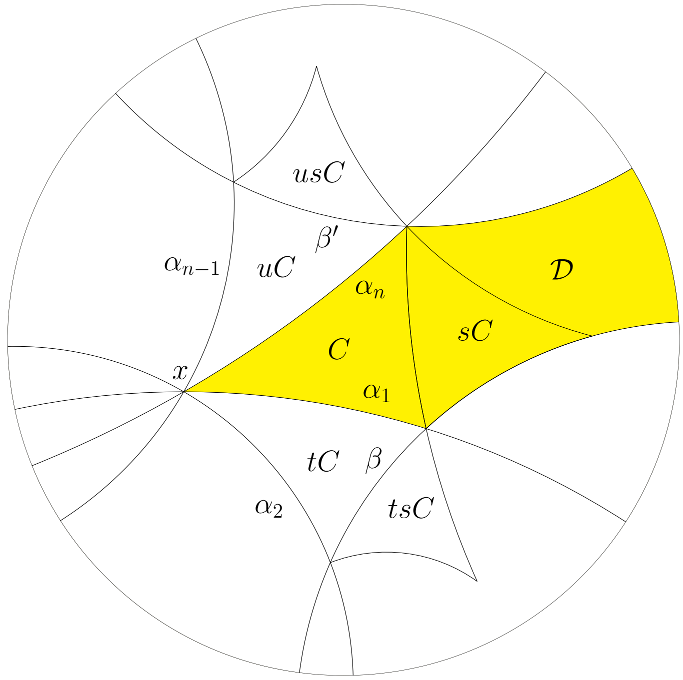

Figure 2.1: The Roots $\alpha_1, \alpha_n, \beta, \beta'$ with the region $\mathcal{D}$ in yellow.

If γ is a positive root at x which is not simple at x, and δ is any other positive root such that $\partial\gamma \cap \partial\delta \neq \emptyset$, then $\mathcal{D} \subset \gamma \cap \delta$.

Proof. By assumption, γ is a positive root through x so $\gamma = \alpha_i$ for some i. Furthermore, we assumed that γ was not simple which means $2 \leq i \leq n-1$. Since α_1 and α_n are simple at x we can see that $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$. Thus it will suffice to prove that $\mathcal{D} \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$ and thus $\mathcal{D} \subset \delta$ so that $\mathcal{D} \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_n$. Thus the point y will lie in exactly one of α_1 or α_n.

First suppose that $y \in \alpha_n$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilites would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D} \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_n$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_n$. Since neither y or y' lies on $\partial\alpha_n$ this means that $y' \in \alpha_n$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_n$. If $E = D'$ then $D' \in \alpha_n$ since x, y, y' all lie in α_n. But this is a contradiction as α_n cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D} \subset \beta = \delta$ and $\mathcal{D} \subset \gamma \cap \delta$ as desired.

If we assume insteach that $y \in \alpha_1$ so that $y \notin \alpha_n$ then identical arguments show that $\delta = \beta'$ and we can again conclude that $\mathcal{D} \subset \gamma \cap \delta$ as desired.

$\square$

We are now ready to construct a large family of vertices $\{v\}$ for which $\tilde{\phi_v}$ will exist. The idea is as follows. If we take any chamber in $\mathcal{D}$ and treat it as a new "C" then $\tilde{\phi_x}$ would exist for this "C." So what we do is apply elements of W which map the chambers of $\mathcal{D}$ to C, and use these choices of w to get new vertices v. We can use Lemma 3 to show that this W action will play nicely with the map ϕ_v.

{lem:Dexists}

Lemma 7. *Let x be the vertex of C of type s, and assume $U'_x \neq U_x$. If v is a vertex in $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ of type s then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. Let $D = \mathrm{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 3.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 4 at the vertex wx. Now suppose that γ is a non-simple, positive root through wx and y is another vertex on $\partial\gamma$. We must show that γ is simple at y. Since γ is positive through wx we know that $C, E \in \gamma$. If we apply w^{-1} then we get the following facts. We know that $w^{-1}\gamma$ is a root such that $\partial(w^{-1}\gamma)$ passes through $w^{-1}wx = x$. We also know that $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is also a positive root. Since w^{-1} sends positive roots at wx to positive roots at x we can apply Lemma 3 when necessary.

The first claim is that $w^{-1}\gamma$ is not simple at x. Suppose that δ is any positive root at wx. Then $E \subset \delta$ and so applying w^{-1} we get that $w^{-1}E = C \subset w^{-1}\delta$. By Lemma 3 this means that w^{-1} sends simple roots at wx to simple roots at x. Since γ is not simple at wx this means that $w^{-1}\gamma$ is not simple at x.

So $w^{-1}\gamma$ is a non-simple positive root at x, and since y lies on $\partial\gamma$ we also know that $w^{-1}y$ lies on $w^{-1}(\partial\gamma)$. If we apply Lemma 5 we can see that $w^{-1}\gamma$ must be simple at $w^{-1}y$.

{fig:mappicture}

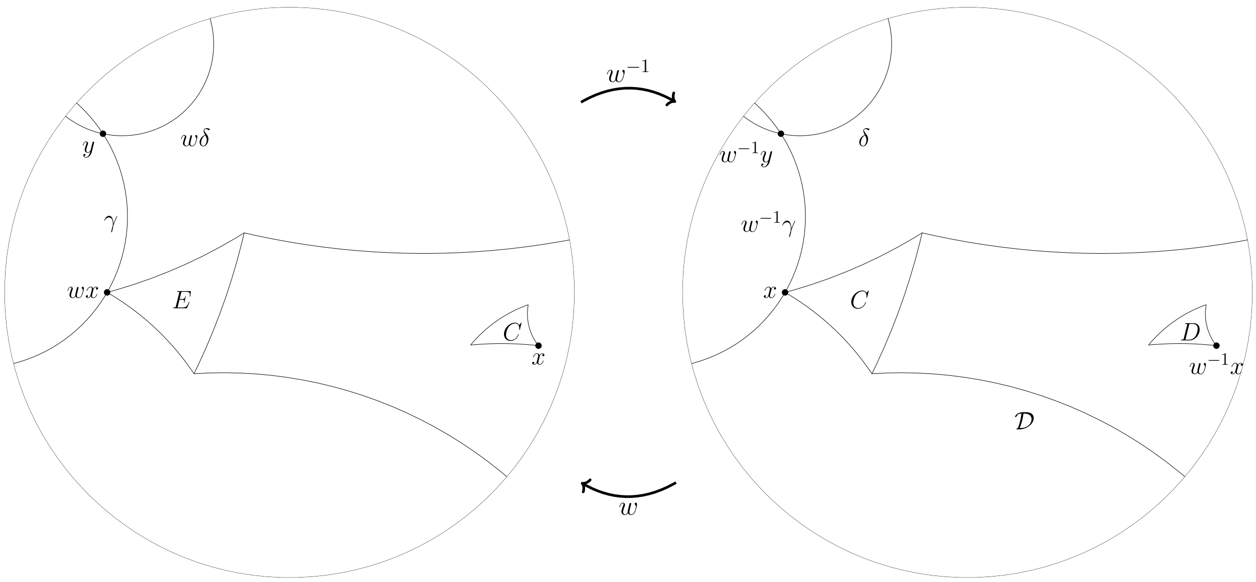

Figure 2.2: The effect of w and w^{-1} on the chambers and roots.

Recall that $D \in \mathcal{D}$ by assumption. Now suppose that δ is any positive root at $w^{-1}y$. Then by Lemma 6 we know that $D \in \mathcal{D} \subset \delta$. If we apply w then we get $C = wD \in w\delta$ and $w\delta$ is a root through y. Thus $w\delta$ is a positive root through y and therefore w sends positive roots through $w^{-1}y$ to positive roots through y. Again we can apply Lemma 3 to say that w must also send simple roots through $w^{-1}y$ to simple roots through y. But $w^{-1}\gamma$ was a simple root through $w^{-1}y$ and thus γ is simple at y as desired.

We now have a vertex wx where $[U_{wx} : U'_{wx}] = [U_x : U'_x] \geq 2$ and the positive roots at wx which are not simple at wx are simple everywhere else. Thus we can apply Lemma 4 to say that $\tilde{\phi}_{wx}$ exists as desired. □

Now we have shown that vertices of $\mathcal{D}$ in some way correspond to $\tilde{\phi}_v$. If our goal is to find infinitely many such v then there is still some work to be done. For instance, we do not yet know if the region $\mathcal{D}$ contains infinitely many chambers, or even if it does, if all the vertices of D lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

2.2 When $\mathcal{D}$ is infinite

Our first task will be two show that the region $\mathcal{D}$ contains infinitely many unique vertices. Intuitively, this will happen if the walls for β and β' do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that W is defined by the edge labels $a = m(s,t), b = m(s,u), c = m(t,u)$ with $a \leq b \leq c$. For the remainder of the section we will also add the assumption that $b \geq 4$. This assumption will allow us to show that the region $\mathcal{D}$ contains infinitely many vertices.

{lem:infmany}

Lemma 8. *Let W as before with diagram labels $3 \leq a \leq b \leq c$, and $b \geq 4$. Also let $w_k = (tus)^k$ for all $k \geq 0$. Then the vertices $(w_k)^{-1}x$ are all distinct from one another, and they all lie in $\mathcal{D}$.*

Proof. Note that $(w_k)^{-1} = (sut)^k$ for all k. First we will show that $(w_k)^{-1}x \in \mathcal{D}$ for all k. Since x is a vertes of C we know that $(w_k)^{-1}x$ is a vertex of $(w_k)^{-1}C$ and thus it will suffice to show $(w_k)^{-1}C$ is contained in $\mathcal{D}$ for all k. Since the roots $\alpha_1, \alpha_n, \beta, \beta'$ can be identified with their corresponding subsets of W, we can use the length function to check containment in these roots.

Now we recall the two M operations on words in a Coxeter group are as follows:

1. Delete a subword ss for some $s \in S$
2. Replace a subword of the form $stst \cdots st(s)$ by a subword of the form $tsts \cdots ts(t)$ where each of these strings has length $m(s,t)$.

Also recall that any word in a Coxeter group can be reduced to its minium length by repeated application of these operations, and any two reduced words can be converted each other by application of operations of type 2. Therefore, in order to check that the lenght relations are satisfied, it will be enough to show that we can never perform an M operation of type 1 as this is the only way to reduce length.

It is immediate from the definition that $\ell((w_k)^{-1}) = 3k$ for all k. We can also see that $\ell(t(w_k)^{-1}) = 3k+1$ and thus $(w_k)^{-1} \in \alpha_1$ for all k. Similarly, $u(w_k)^{-1} = u(sutsut\cdots)$, and no reduction operations can be done as we assumed $m(s,u) \geq 4$. Thus $\ell(u(w_k)^{-1}) = 3k+1$ which means $(w_k)^{-1} \in \alpha_n$ as well.

Now consider the element $st(w_k)^{-1}$. If we write this element out in terms of the generators and apply the only possible Coxeter relations we get

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut\cdots) \\ &= (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) \\ &= (ts)(tut)(sutsut\cdots) \end{aligned}$$

and none of these can be reduced as $m(t,u) \geq 4$. Note that the commutation relation $sts = tst$ may not be possible if $m(s,t) \geq 4$, but it is the only relation possible in $st(w_k)^{-1}$ and even if it does exists then it does not allow $st(w_k)^{-1}$ to be reduced in length. We previously showed $\ell(t(w_k)^{-1}) = 3k+1$ and now we see $\ell(st(w_k)^{-1}) = 3k+2$ and so $(w_k)^{-1} \in \beta$.

Now we can consider $su(w_k)^{-1}$ in a similar manner. Writing $su(w_k)^{-1}$ out as a word in the generators and applying Coxeter relations gives us

$$\begin{aligned} su(w_k)^{-1} &= su(sutsut\cdots) \\ &= (susu)(tsutsu\cdots) \\ &= (usus)(tsutsu\cdots) \\ &= (usu)(sts)(utsuts\cdots) \\ &= (usu)(tst)(utsuts\cdots) \end{aligned}$$

Note once again that not all of these relations may be possible if $m(s,u)=6$ or $m(s,t)\geq 4$. However, these are the only possible relations, and since $su(w_k)^{-1}$ cannot be reduced under these assumptions, it cannot be reduced at all. Thus $\ell(su(w_k)^{-1})=3k+2$ which means $su(w_k)^{-1}\in\beta'$ as well.

Now it only remains to show that $v_m\neq v_n$ for $m\neq n$. Suppose $(w_m)^{-1}x=(w_n)^{-1}x$ for $m>n$. Then we would have $x=w_m(w_n)^{-1}x=w_{m-n}$. Thus it will suffice to show $w_kx\neq x$ for any $k\geq 1$. But we know that $\mathrm{stab}_W(x)=\langle u,t\rangle$ which does not contain w_k for any $k\geq 1$ and thus $(w_k)^{-1}x\neq x$ so that $(w_m)^{-1}x\neq(w_n)^{-1}x$ as desired.

□

We now know that each of the $(w_k)^{-1}x$ is distinct and each of them lies in $\mathcal{D}$. By Lemma 8 we know that $\tilde{\phi}_{w_kx}$ exists for each $k\geq 0$. Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the k at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices w_nx distinct, but also no two lie on the same wall.

{lem:samewall}

Lemma 9. *Let $w_k=(tus)^k$ for all $k\geq 0$ and x the vertex of C of type s. If W as in the rest of this section then w_mx and w_nx do not lie on the same wall of Σ if $m>n\geq 0$.*

Proof. Suppose w_mx and w_nx do lie on the same wall with $m>n$. Then we also know that $w_nw_m^{-1}x=w_{n-m}x$ and x will lie on the same wall. Since $m>n$ we can let $k=m-n$ and thus it will suffice to show that $(w_k)^{-1}x$ and x do not lie on the same wall for any $k\geq 1$.

We know from Lemma 8 that $(w_k)^{-1}x\in\mathcal{D}$. Thus if $(w_k^{-1})x$ and x lie on the same wall, it must be a wall through x and thus it must be $\partial\alpha_i$ for some i. We know that $(w_k^{-1})x\in\alpha_1\cap\alpha_n$ since $\mathcal{D}\subset\alpha_1\cap\alpha_n$ by defintion. But we can also recall that $\partial\alpha_j\cap\alpha_1\cap\alpha_n=\{x\}$ for $2\leq j\leq n-1$. Thus we have $i=1$ or $i=n$ so that $(w_k^{-1})x$ either lies on $\partial\alpha_1$ or $\partial\alpha_n$. Therefore, we either have $u(w_k)^{-1}x=(w_k)^{-1}x$ or $t(w_k)^{-1}x=(w_k)^{-1}x$ which implies that either $w_kuw_k^{-1}$ or $w_ktw_k^{-1}$ is contained in $\mathrm{stab}_W(x)=\langle u,t\rangle$. However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced. The only possible relations we have are

$$\begin{aligned}
w_ktw_k^{-1}&=(\cdots tustus)t(sutsut\cdots)\\
&=(\cdots tustu)(sts)(utsut\cdots)\\
&=(\cdots tustu)(tst)(utsut\cdots)\qquad m(t,u)\geq 4
\end{aligned}$$

or

$$\begin{aligned}
w_kuw_k^{-1}&=(\cdots stustus)u(sutsuts\cdots)\\
&=(\cdots stust)(ususu)(tsuts\cdots)\\
&=(\cdots stust)(sus)(tsuts\cdots)\\
&=(\cdots stu)(sts)u(sts)(uts\cdots)\\
&=(\cdots stu)(tst)u(tst)(uts\cdots)
\end{aligned}$$

Similarly as before, even these relations are only possible if $m(s,u)=4$, but even in that case we cannot eliminate every instance of s in $w_k u w_k^{-1}$. In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in $\langle u,t\rangle$. Thus the $w_n x$ all lie on distinct walls as desired. □

We now have all the ingredients and are ready to prove the main theorem.

{thm:notfg}

Theorem 2. *Let $(G,(U_\alpha)_{\alpha\in\Phi},T)$ be an RGD system of type (W,S). Assume W is defined by a Coxeter diagram with edge labels $3\le a\le b\le c$ and also assume that $b\ge 4$. Let $U_+=\langle U_\alpha|\alpha\in\Phi_+\rangle$ and suppose that $[U_x:U_x']\ge 2$ where x is the vertex of C of type s. Then U_+ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1,\dots,\beta_m$ such that $U_+=\langle U_{\beta_i}|1\le i\le m\rangle$. Now no two of the vertices $(tus)^{-k}x$ lie on the same wall and thus we can choose k so that $v=(tus)^{-k}x$ does not lie on $\partial\beta_i$ for any i. By Lemma 8 and Lemma 7 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+\to H$ where H is a cyclic group. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i})=1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. □

A remark worth noting is that the previous proof actually shows something a bit stronger. Since H is abelian, the map $\tilde{\phi}_v$ will factor through the abelianization $(U_+)_{\mathrm{ab}}$. Then the same arguments as before also show that $(U_+)_{\mathrm{ab}}$ cannot be finitely generated either.

Chapter 3

Exceptional Cases

{exceptional}

In the previous chapter we were able to show that U_+ is not finitely generated for a large family of Coxeter groups W with labels $a \le b \le c$. These results were based on assuming $b \ge 4$ which allowed us to show that $\mathcal{D}$ was infinite and proceed from there. In fact, we didn't even describe all of the chambers in $\mathcal{D}$, just an infinite family. However, the same approach will not work in the remaining cases because of the following lemma.

{lem:infD} **Lemma 10.** *If W is a Coxeter group with labels $a \le b \le c$ as before, then $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ as defined in the previous chapter is infinite if and only if $b \ge 4$.*

Proof. We know by Lemma 8 that $\mathcal{D}$ is infinite if $b \ge 4$. Thus it remains to show that $\mathcal{D}$ is finite if $b = 3$. If $b = 3$ then $a = 3$ also, and by definition of a, b, c this means $m(s,t) = m(s,u) = 3$. We will also recall the definition of $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D,C) < d(D,tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D,C) < d(D,uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D,tC) < d(D,tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D,uC) < d(D,usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Let $w \in W$ and suppose $\ell(w) \ge 2$. Then we can write $w = s_1 s_2 w'$ where $\ell(w') = \ell(w) - 2$. If $s_1 = t$ then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows $w \notin \alpha_1$ and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_1 = u$.

Now we assume $s_1 = s$ and so we can also assume $s_2 = t, u$. First let $s_2 = t$ so that $w = stw'$. If $w \notin \alpha_1$ then $w \notin \mathcal{D}$ and so we will suppose $w \in \alpha_1$. Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \le \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_2 = u$.

We have shown that if $\ell(w) \ge 2$ then $w \notin \mathcal{D}$ and thus $\mathcal{D}$ must be finite as desired. In fact, if $a = b = 3$ then we can check relatively easily that $\mathcal{D} = \{C, sC\}$ which proves the desired result. $\square$

The previous lemma shows that generating results for the remaining cases is not just a matter of being slightly more clever when we look for vertices, but changing the strategy as a whole. In fact, in some cases our proof strategy needs to switch entirely since U_+ will be finitely generated in some cases as we will see. First, we will show which of the remaining cases are not finitely generated.

All of the remaining rank 3 cases have the property that $m(s,u) = m(s,t) = 3$. If x is the vertex of C of type s then x is the only possible vertex of type C with the property that $[U_x : U_x'] \geq 2$. With two edge labels of 3 it is impossible for $U_x \cong {}^2F_4(2)$ and so the only remaining possibilites are $U_x \cong C_2(2), G_2(2)$, and $G_2(3)$. We will enumerate through each of these cases individually.

3.1 Case: $U_x \cong G_2(2)$

We saw in the previous chapter that a vertex contained in $\mathcal{D}$ was a sufficient condition to construct a corresponding map $\tilde{\phi}_v$. However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct $\tilde{\phi}_v$ for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statment similar to Lemma 4.

For the remainder of the section we will assume that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) where $S = \{s, t, u\}$ and

$$W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^6 = 1\rangle$$

Furthermore, let x be the vertex of C of type s and assume that $U_x \cong G_2(2)$. Recall that this means $[U_x : U_x'] = 4$ and $[U_v : U_v'] = 4$ for all vertices v of type s by Lemma 1 and Lemma 3.

Recalling from the previous chapter, we know that there is a presentation of U_+ generated by U_α for all $\alpha \in \Phi_+$. Again, there are several types of relations we need to consider. There are relations among the U_α and there are relations between U_α and U_β when $\{\alpha, \beta\}$ is a prenilpotent pair. By (A) we know that $[U_\alpha, U_\beta] = \{1\}$ if α and β are nested. We also know that when $\partial\alpha \cap \partial\beta \neq \emptyset$ that $[u, u'] = w$ for some word $w \in U_{(\alpha,\beta)}$ where $u \in U_\alpha$ and $u' \in U_\beta$.

Now recall from Chapter 1 that there is a surjective homormorphism $\phi_x : U_x \to H$ where H is a cyclic group. We can also choosea standard labeling $\alpha_1, \ldots, \alpha_6$ of the positive roots through x in such a way that $\ker\phi_x = U_x'' = \langle U_1, U_5, U_6\rangle$. Similarly to the last chapter, if v is any vertex of type s, our goal is to construct an extension of the form $\tilde{\phi}_v$ in such a way that

$$\tilde{\phi}_v(U_\alpha) = \begin{cases} \phi_v(U_\alpha) & v \in \partial\alpha \\ 1 & \text{otherwise} \end{cases}$$

If we can do this for enough vertices v then we will be able to show that U_+ is not finitely generated in the same way as the previous chapter. Our first step is to prove an analagous result to Lemma 4 in the current context.

{lem:336f2ex}

Lemma 11. *Let v be a vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$. Assume $\gamma_1, \ldots, \gamma_6$ is a standard ordering of the positive roots through v such that $U_{\gamma_5} \subset \ker \phi_v$. If γ_2, γ_3, and γ_4 are simple at all other vertices they meet, then $\tilde{\phi}_v$ as defined in Lemma 4 exists.*

Proof. To check $\tilde{\phi}_v$ is well defined is a matter of checking the relations are satisfied by the images under $\tilde{\phi}_v$. Since $\tilde{\phi}_v$ has a cyclic group as its codomain, we can see immediately that the first two types of relations will be satisfied regardless of α and β. Now to check the third type.

Suppose α and β are any two positive roots with $y = \partial\alpha \cap \partial\beta$. Then there is a relation in U_+ of the form $[u, u'] = w$ where $u \in U_\alpha, u' \in U_\beta$, and $w \in U_{(\alpha,\beta)}$. Since $[u_\alpha, u_\beta]$ must be mapped to the identity then we just need to check that w is also mapped to the identity. If $y = v$ then u_α, u_β, w all lie in U_v and $\tilde{\phi}_v(w) = \phi_v(w)$ which must be the identity because ϕ_v is a well defined homomorphism.

Now suppose $y \neq v$. Let $\delta_1, \ldots, \delta_n$ be the positive roots through y, with a standard labeling, and assume that $\alpha = \delta_i$ and $\beta = \delta_j$ with $i < j$. There is at most one positive root whose wall can pass through both v and y, call it δ_k if it exists. If δ_k does not exist, then no positive roots through y pass through v and so $\tilde{\phi}_v(u_{\delta_m}) = 1$ for all m. Thus $\tilde{\phi}_v(w) = 1$ as desired.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{1, 5, 6\}$. Then we know $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all $m \neq k$ and $\tilde{\phi}_v(U_{\delta_k}) = \tilde{\phi}_v(U_{\gamma_r}) = \phi_v(U_{\gamma_r}) = \{1\}$ by the construction of ϕ_v. Thus $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all m and so $\tilde{\phi}_v(w) = \{1\}$ as well.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{2, 3, 4\}$. Then by assumption, δ_k is simple at y and thus $k = 1, n$. Thus $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all $2 \leq m \leq n-1$. But w is a word in $U_{(\alpha,\beta)} \subset U_{(\delta_2,\delta_{n-1})}$ and thus $\tilde{\phi}_v(w) = 1$ again, which gives the result. $\square$

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma 4, and so we have a hope of constructing more $\tilde{\phi}_v$ then the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Let x be the vertex of C of type s as in the previous chapter and let $\alpha_1, \ldots, \alpha_6$ be the positive roots through x, labeled as usual. Recall from the previous chapter that

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}
\end{aligned}$$

Also assume without loss of generality that $\phi_x(U_{\alpha_5}) = \{1\}$. Now let $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$. We can now prove a lemma similar to Lemma 6.

picture of $\mathcal{D}'$

{lem:336f2D}

Lemma 12. *Let x be the vertex of C of type s so that $|\mathrm{st}(x)| = 12$. Let $\alpha_1, \ldots, \alpha_6$ be the positive roots at x with the standard ordering. Also assume that $\phi_x(U_{\gamma_5}) = 1$. Suppose $\gamma = \alpha_i$*

for $i \in \{2,3,4\}$. *If* δ *is any positive root with* $\partial\gamma \cap \partial\delta \neq \emptyset$ *then* $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta \subset \gamma \cap \delta$ *where*

$$\beta = \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}$$

as in the previous chapter.

Proof. Since γ is a positive root at x, and α_1, α_6 are the simple roots at x, we know that $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$ and thus it will suffice to show that $\mathcal{D}' \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_6 \subset \alpha_j = \delta$ and thus $\mathcal{D}' \subset \delta$ so that $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_6 = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_6$. Thus the point y will lie in exactly one of α_1 or α_6.

First suppose that $y \in \alpha_6$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilites would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D}' \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_6$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_6$. Since neither y or y' lies on $\partial\alpha_6$ this means that $y' \in \alpha_6$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_6$. If $E = D'$ then $D' \in \alpha_6$ since x, y, y' all lie in α_6. But this is a contradiction as α_6 cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D}' \subset \beta = \delta$ and $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

If we assume insteach that $y \in \alpha_1$ so that $y \notin \alpha_6$ then we have the same two possibilites. If $\partial\alpha_6 \cap \partial\delta = \emptyset$ then by similar arguments we get $\mathcal{D}' \subset \alpha_6 \subset \delta$ and thus $\mathcal{D}' \subset \gamma \cap \delta$ as desired. If $\partial\alpha_6 \cap \partial\delta = y'$ then the vertices x, y, y' form a chamber with y' on α_6. Again, by similar arguments as before, this would imply that $\gamma = \alpha_5$ or α_1, both of which are impossible.

Therefore, regardless of case we have $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

□

We now have a condition for $\tilde{\phi}_v$ to exist which we can check and so it remains to find potential candidates to use at v. We know by Lemma 3 that ϕ_v will exist for all vertices v of type s. We will us a strategy similar to that of the previous chapter which relies on the definition of D' to show $\tilde{\phi}_v$ exists for certain v. To this end we now prove the analogue of Lemma 7.

{lem:336f2Dex}

Lemma 13. *Let* x *be the vertex of* C *of type* s *and suppose that* v *is any vertex in* $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ *of type* s. *Then there is a* $w \in W$ *such that* $w^{-1}x = v$ *and* $\tilde{\phi}_{wx}$ *exists.*

Proof. Let $D = \text{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 3.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \text{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\text{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\text{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \text{Proj}_{wx} wD = \text{Proj}_{wx} C \qquad C = wD = \text{Proj}_{w(w^{-1}x)} wC = \text{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 11 at the vertex wx. Let $\gamma_1, \ldots, \gamma_6$ be a standard labeling of the positive roots through wx such that $U_{\gamma_5} \subset \ker \phi_{wx}$. We need to check that if $y \neq wx$ is on $\partial\gamma_i$ for $i \in \{2, 3, 4\}$ then γ_i is simple at y. First we will show that w^{-1} sends positive roots at wx to positive roots at x. Suppose γ is any positive root at wx. Then we know that $E \in \gamma$ and thus $C = w^{-1}E \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is positive, and thus w^{-1} sends positive roots at wx to positive roots at x.

If we apply Lemma 3 then we know that $w^{-1}\gamma_1 = \alpha_1, \ldots, w^{-1}\gamma = \alpha_6$ is a standard labeling of the of the positive roots at x. If we apply this isomorphism given by Corollary 3 then we know that $U_{w^{-1}\gamma_5} = U_{\alpha_5} \subset \ker \phi_x$ since $U_{\gamma_5} \subset \ker \phi_{wx}$.

Now we fix $i \in \{2, 3, 4\}$ and we need to check γ_i is simple at all vertices $y \neq v$ on $\partial\gamma_i$. If we apply w^{-1} we get that $w^{-1}y \neq x$ is a vertex on $\partial\alpha_i$. Thus by Lemma 5 we know that α_i is simple at $w^{-1}y$. Now suppose that δ is any positive root at $w^{-1}y$. Recall that $D \in \mathcal{D}'$ and we can apply Lemma 12 to see that $D \in \mathcal{D}' \subset \delta$. If we apply w we get $C = wD \in w\delta$ where $w\delta$ is a positive root through $w(w^{-1}y) = y$. Thus w sends positive roots at $w^{-1}y$ to positive roots at y. We can apply Lemma 3 again to say that w sends the simple roots at $w^{-1}y$ to the simple roots at y. Since α_i is simple at $w^{-1}y$ we know that $w\alpha_i = \gamma_i$ is simple at y as desired. We now for all positive roots γ_i for $i \in \{2, 3, 4\}$ at wx that γ_i is simple at all other vertices, and thus we can apply Lemma 11 to say that $\tilde{\phi}_{wx}$ exists as desired.

□

As in the previous chapter, we now have a potentially large class of vertices for which $\tilde{\phi}_v$ exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. In fact, we can even use the same vertices as in the previous chapter. Let $w_k = (tus)^k$ for all $k \geq 0$ and let $v_k = w_k x$. Recall in our current setup that $m(t, u) = 6$ and $m(s, u) = m(s, t) = 3$.

{lem:336f2inf}

Lemma 14. *Let $w_k = (tus)^k$ for all $k \geq 0$ and let x be the vertex of C of type s. Then the vertices $(w_k)^{-1}x$ are all distinct, and they all lie in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ as defined previously.*

Proof. Many of the proofs will be identical to those in the proof of Lemma 8 and so work will not be repeated when unnecessary. Also note that $w_k^{-1} = (sut)^k$ for all k. We can check that $\ell((w_k)^{-1}) = 3k$ and $\ell(t(w_k)^{-1}) = 3k+1$ by identical arguments as before. We can also check that

$$\begin{aligned} u(w_k)^{-1} &= u(sutsut\cdots) \\ &= (usu)(tsutsu\cdots) \\ &= (sus)(tsutsu\cdots) \\ &= (su)(sts)(utsuts\cdots) \\ &= (su)(tst)(utsuts\cdots) \\ &= (su)(ts)(tut)(sutsut\cdots) \end{aligned}$$

We have exhausted all possible M-Operations in $u(w_k)^{-1}$ and none of them led to a reduction in length so we can conclude that $\ell(u(w_k)^{-1}) = 3k+1$ also so that $(w_k)^{-1} \in \alpha_1 \cap \alpha_6$.

Now we do the same analysis for $st(w_k)^{-1}$ to see

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut\cdots) = (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) = (ts)(tut)(sutsut) \end{aligned}$$

and since no reductions can be performed we also get $\ell(st(w_k)^{-1}) = 3k+2$ so that $(w_k)^{-1} \in \beta$ as well. Thus each $(w_k)^{-1}x$ lies in $\mathcal{D}'$ as desired. We also know that $(w_m)^{-1}x \neq (w_n)^{-1}x$ if $m > n$ by the same argument as in Lemma 8. □

The last major step is to show that the $w_k x$ cannot somehow lie on only finitely many walls. The analysis here will be slightly more complicated, but ultimately similar to that done in the previous chapter.

{lem:336f2walls}

Lemma 15. *Let x be the vertex of C of type s and let $w_k = (tus)^k$ for all $k \geq 0$. Any wall of Σ can contain only finitely many $w_k x$.*

Proof. By arguments identical to those in Lemma 9, $w_m x$ and $w_n x$ will lie on the same wall if and only if x and $w_{n-m}x$ lie on the same wall. If we assume $m > n$ then it will suffice to show that a wally containing x can contain $(w_k)^{-1}x$ for only finitely many $k > 0$. Using the argument of Lemma 9 again we know that x and $(w_k)^{-1}x$ will lie on the same way if and only if $w_k t w_k^{-1}$ or $w_k u w_k^{-1}$ lies in $\mathrm{stab}_W(x) = \langle u, t\rangle$. If we recall that $m(s,t) = m(s,u) = 3$ and $m(t,u) = 6$ we can check these two conjugates we see

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \\ &= (\cdots tus)(tut)(s)(tut)(sut\cdots) \end{aligned}$$

and then we see also

$$\begin{aligned}
w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\
&= (\cdots stust)(ususu)(tsuts\cdots) \\
&= (\cdots stust)(s)(tsuts\cdots) \\
&= (\cdots stu)(ststs)(uts\cdots) \\
&= (\cdots stu)(t)(uts\cdots) \\
&= (\cdots stustu)(t)(utsuts\cdots) \\
&= (\cdots stus)(tutut)(suts\cdots)
\end{aligned}$$

In the first case, no reduction is possible and thus there will always be an s in any reduced word for $w_k t w_k^{-1}$ and thus $w_k t w_k^{-1} \notin \langle u, t\rangle$. In the second case, We are able to do two reductions in length but then are unable to continue. If we check the relations applied, we will se that the relations cannot continue if $k \geq 3$. For completion we will also note that $w_1 u w_1^{-1} = tst \notin \langle u, t\rangle$ but $w_2 u w_2^{-1} = tutut \in \langle u, t\rangle$. Regardless, we know that $w_m x$ and $w_n x$ cannot lie on the same wall if $|m - n| \geq 3$ so that any wall can contain only finitely many $w_k x$ as edesired.

□

Now we are ready to prove the main result of the section, which is nearly identical to the proof of Theorem 2.

{thm:336f2notfg}

Theorem 3. *Let $(G, (U_\alpha)_{\alpha\in Phi}, T)$ be an RGD system of type (W, S) with assumptions as in* (A). *Suppose that $a = m(s, t) = b = m(s, t) = 3$ and $U_x \cong G_2(2)$ where x is the vertex of C of type S. Then $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m\rangle$. Let $w_k = (tus)^k)$ for all $k \geq 0$. Now only finitely many of the vertices $w_k x$ lie on the same wall and thus we can choose k so that $v = w_k x$ does not lie on $\partial\beta_i$ for any i. By Lemma 14 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. □

3.2 Finite Generation in the Exceptional Cases

Now there are two cases left to consider, and no ammount of modification to our previous strategies will work since we will see that these remaining cases are finitely generated.

For any positive root γ, we say that a chamber D borders γ if a panel of D lies on $\partial\gamma$. This allows us to define

$$d(\gamma, C) = \min_{D \text{ borders } \gamma} \{d(D, C)\}$$

It is worth noting that if $d(\gamma, C) = k$ then there is a chamber D which borders γ and $d(\gamma, C) = d(D, C)$. Furthermore, the chamber D must lie in γ since, otherwise, the chamber adjacent to D across $\partial\gamma$ would be closer to C.

We can now define $U_n = \langle U_\gamma | \gamma \in \Phi^+, d(\gamma, C) \leq n \rangle$ which is a subgroup of U_+ for all n. We also have a few facts which are immediate from the definition of U_n. We can see that $U_1 \subset U_2 \subset U_3 \subset \cdots$ and $U_+ = \cup_n U_n$ as any positive root will be some finite distance from C.

Slightly less obvious is the fact that U_n is finitely generated for all n. If $d(\gamma, C) \leq n$ then there must be a chamber D which borders γ with $d(D, C) \leq n$. There are only finitely many such chambers, and each of these chambers borders at most 3 roots, so U_n is finitely generated.

The idea of the remaining proofs will be to use the following lemma

{lem:fgcond} **Lemma 16.** *For any positive root γ we define $d(\gamma, C) = \min\{d(D, C) | D$ has a panel on $\partial\gamma\}$. Let $U_n = \langle U_\gamma | d(\gamma, C) \leq n \rangle$ for all $n \geq 0$ where $d(\gamma, C)$. If there is some N such that $U_n \subset U_{n-1}$ for $n > N$ then U_+ is finitely generated.*

Proof. If $U_n = U_{n-1}$ for all $n > N$ then inductively we know that $U_n = U_N$ for all $n > N$. Thus

$$U_+ = \cup_{n=N}^{\infty} U_n = \cup_{n=N}^{\infty} U_N = U_N$$

which is fintely generated as desired. □

By the results of Chapter 2 and the previous section, we know that the only cases remaining to consider are when W has a Coxeter diagram defined by edge labels 334, 336, or 338. The 338 case is impossible. And we have already covered the 336 case when Σ has a vertex x with $U_x \cong G_2(2)$. Thus we only need to consider when Σ has a vertex x with $U_x \cong C_2(2)$ or $G_2(3)$.

3.2.1 Case: $U_x \cong C_2(2)$

Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S) where $S = \{s, t, u\}$ and $m(s, t) = m(s, u) = 3$ and $m(t, u) = 6$. Let x be the vertex of the fundamental chamber C of type s and assume $U_x \cong C_2(2)$. We will show that U_+ is finitely generated.

{thm:334f2fg} **Theorem 4.** *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ of type (W, S) as above. If x is the vertex of C of type s and $U_x \cong C_2(2)$ then $U_n \subset U_{n-1}$ for all $n > 2$ where $U_n = \langle U_\gamma | d(\gamma, C) \leq n \rangle$.*

Proof. Let γ be any positive root with $d(\gamma, C) = n > 2$. Then choose a chamber D_1 which borders γ such that $d(D_1, C) = d(\gamma, C)$. Now there is another chamber D_2 such that D_1 and D_2 are adjacent and $d(D_2, C) = d(D_1, C) - 1$. Then D_1 and D_2 will share exactly one vertex which lies on $\partial\gamma$, call it v. Recall that $\mathrm{st}(v)$ is the set of chambers of Σ for which v is a vertex. Then we have $|\mathrm{st}(v)| = 6$ or 8.

{fig:334deg6}

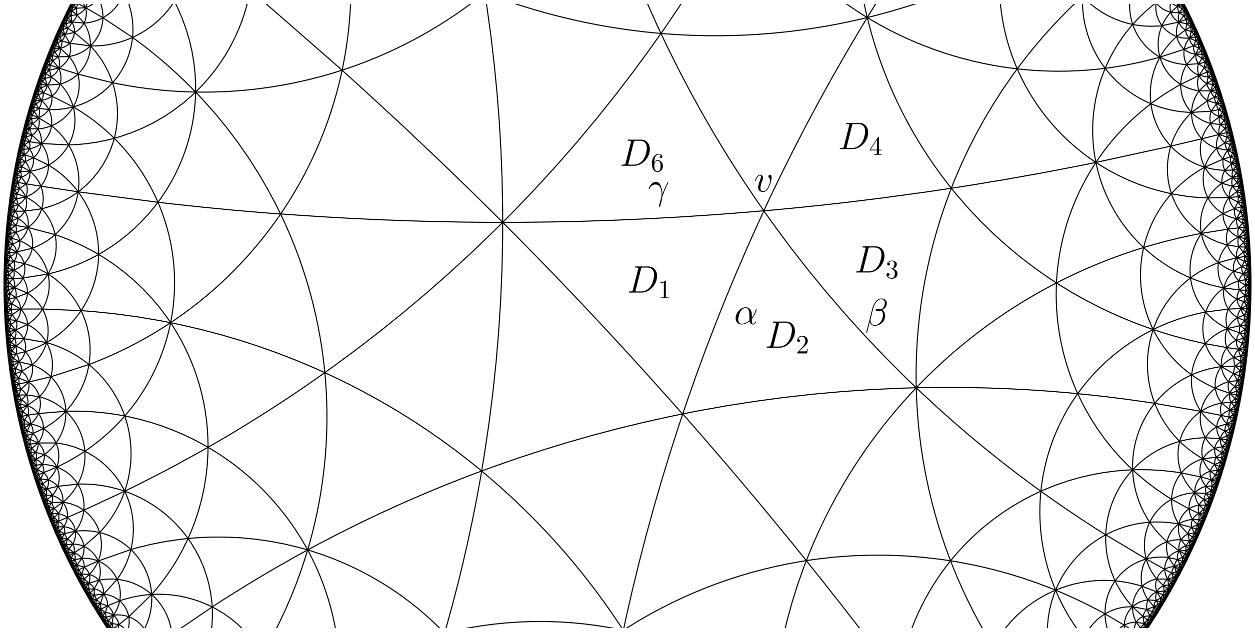

Figure 3.1: Case: $|\mathrm{st}(v)| = 6$

First suppose $|\mathrm{st}(v)| = 6$. In Σ, we can see that $\mathrm{st}(v)$ consists of the 6 chambers "surrounding" v which each have a vertex on v. Since we have already defined D_1 and D_2 we may label the other 4 chambers in $\mathrm{st}(v)$ as $D_3, \ldots, D_6$ by going in a circular order around v. Equivalently this means that D_i is ajacent to D_{i+1} for $1 \leq i \leq 5$ and D_6 is also adjacent to D_1. We also know that each positive root will contain exactly 3 of these chambers, and those three chambers will be D_i, D_{i+1}, and D_{i+2} for some i, where addition is done modulo 6.

By construction, D_2 and D_1 are not adjacent along $\partial\gamma$, but a panel of D_1 lies on $\partial\gamma$, and thus D_1 and D_6 must be adjacent along $\partial\gamma$. Since $D_6 \notin \gamma$, this means that γ must contain D_1, D_2, D_3. Let α and β be the other two positive roots through v. We know that $\partial\gamma$ cannot separate D_2 and D_1 or D_2 and D_3 so we can say again without loss of generality that $\partial\alpha$ separates D_2 and D_1 while $\partial\beta$ separates D_2 and D_3.

Now $D_3 \in \gamma$ but $D_4 \notin \gamma$ which means that D_3 has a panel on $\partial\gamma$. By our choice of D_1 we know that $d(D_3, C) \geq d(D_1, C) > d(D_2, C)$. But D_1 and D_3 are the two chambers adjacent to D_2 in $\mathrm{st}(v)$ and thus D_2 must be the closest chamber to C in $\mathrm{st}(v)$. But this means $D_2 = \mathrm{Proj}_v(C)$ and thus the positive roots at v which border D_2 must be the simple roots at v. These roots are α and β by construction so we know that α and β are simple at v. Since $|\mathrm{st}(v)| = 6$ we know that U_v cannot be an exceptional rank 2 RGD system and thus U_v is generated by the simple root groups through v. Thus $U_x = \langle U_\alpha, U_\beta \rangle$. But α, β border D_2 and $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $d(\alpha, C), d(\beta, C) \leq n - 1$ so that $U_\alpha, U_\beta \subset U_{n-1}$. This means $U_x \subset U_{n-1}$ as well and thus $U_\gamma \subset U_{n-1}$.

Now suppose $|\mathrm{st}(v)| = 8$. Then we will use the same labeling scheme as before except there will be 8 chambers, and each positive root will contain exactly 4 consecutive chambers from $\mathrm{st}(v)$. The same logic as before will still tell us that γ will contain exactly the chambers D_1, D_2, D_3, D_4. Our first claim is that $D_2 = \mathrm{Proj}_v(C)$.

We know that $\mathrm{Proj}_v(C)$ must lie in any positive root through v and thus it can only be D_1, D_2, D_3, D_4. We also know it is the chamber A in $\mathrm{st}(v)$ which minimizes $d(A,C)$. Since $d(D_1,C) > d(D_2,C)$ we know that D_1 cannot be the projection. By a similar argument as before we know that D_4 borders γ and thus $d(D_4,C) \geq d(D_1,C)$ by our choice of D_1. Thus D_4 cannot be the projection. Finally, if D_3 were the projection then $d(D_4,C) = d(D_3,C)+1 < d(D_3,C)+2 = d(D_1,C)$ which is also a contradiction and thus $D_2 = \mathrm{Proj}_v(C)$.

Let α be the positive root separating D_1 and D_2, β the positive root separating D_2 and D_3 and δ the positive root separating D_3 and D_4. Recall that γ is the positive root separating D_8 and D_1 as well as D_4 and D_5. We know that D_2 borders α and β with $d(D_2,C) = d(D_1,C) - 1 = n-1$ and thus $U_\alpha, U_\beta \subset U_{n-1}$. We also know that D_2 lies in all positive roots through v by convexity so $D_2 \in \alpha, \beta, \gamma, \delta$. Since D_2 is bordered by α and β we also know that α and β are the simple roots at v.

{fig:334deg8}

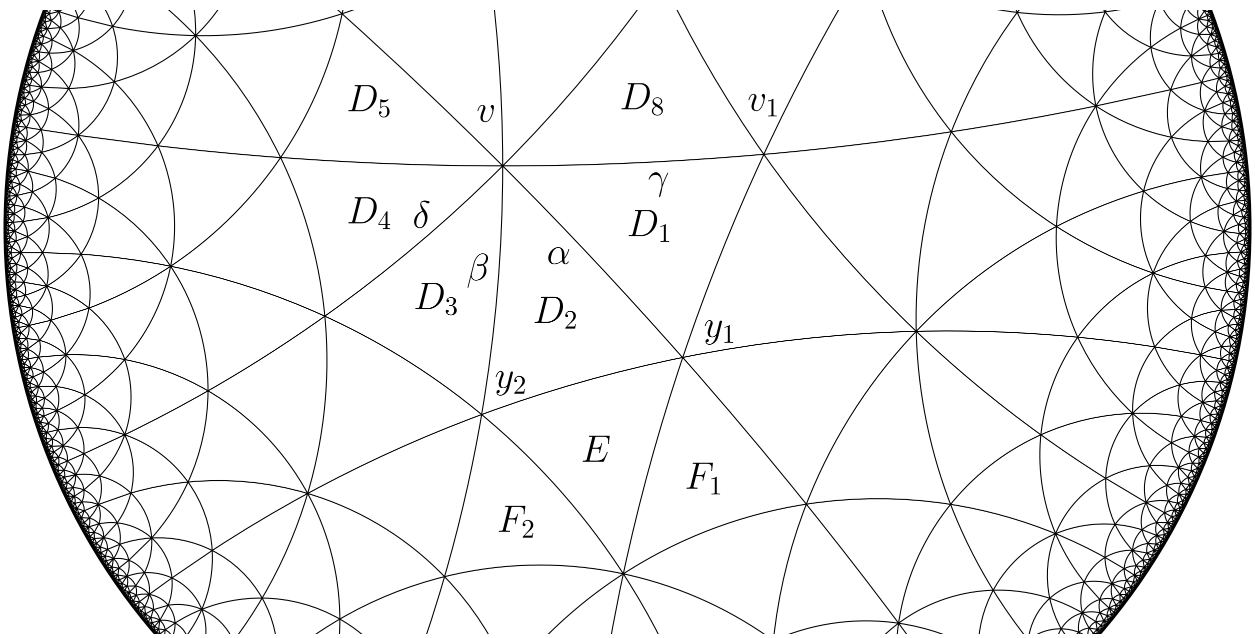

Figure 3.2: Case: $|\mathrm{st}(v) = 8|$

Let E be the third chamber adjacent to D_2. Every chamber must have an adjacent chamber which is closer to C and thus we have $d(E,C) < d(D_2,C)$. We can check that $d(E,C) = d(D_1,C) - 2 \geq 1$ by our choice of γ and thus E is not the fundamental chamber C. We know that D_1 and D_2 share two vertices, and D_2 and E share two vertices, so necessarily we have that D_1, D_2, and E must share at least one, and thus exactly one vertex, call it y_1. By a similar argument, the chambers D_3, D_2, and E will also share a vertex y_2. Let F_1 be the other chamber adjacent to E that has y_1 as a vertex, and let F_2 be the other chamber adjacent to E that has y_2 as a vertex. Note that $|\mathrm{st}(y_1)| = |\mathrm{st}(y_2)| = 6$ since v is the other vertex of D_2. The appropriate labeling can be seen in Figure 3.2.1, and the given diagram is unique up to a mirror image flip, which does not affect any of the following arguments. The labeling of these chambers could have simply been defined by the diagram, but the previous explanation seeks to convince the reader that no choices have been made and this diagram is unique.

Since $d(E, C) < d(D_2, C) < d(D_1, C)$ we know that there is some minimal gallery from D_1 to C which passes through E. If we fix such a minimal gallery we can see that it must pass through either F_1 or F_2. First suppose that it passes through F_1. Then $d(F_1, C) = d(D_1, C) - 3$ and so F_1 and D_1 are distance 3 from one another. Since they are both in $\mathrm{st}(y_1)$, this means that D_1 and F_1 are opposite in $\mathrm{st}(y_1)$. Then there is another minimal gallery from D_1 to F_1 which does not pass through D_2 and can also be extended to a minimal gallery from D_1 to C. Let G_1 be the chamber adjacent to D_1 in this new minimal gallery. Then D_1 and G_1 have exactly two vertices in common, on of which is y_1, and the other cannot be v as this would imply $G_1 = D_2$ which contradicts our assumption. Let v_1 be the common vertex which is not y_1. We assumed that v was the unique vertex shared by D_1 and D_2 which lies on $\partial\gamma$. Since y_1 is also shared by D_1 and D_2 this means that y_1 does not lie on $\partial\gamma$. We assumed that D_1 has a panel on $\partial\gamma$ and thus it has two vertices on $\partial\gamma$ which means v_1 must lie on $\partial\gamma$.

Now we have the following situation. We still know that D_1 borders γ with $d(\gamma, C) = d(D_1, C)$ and G_1 is an adjacent chamber such that $d(G_1, C) < d(D_1, C)$. We know that v_1 is a common vertex which lies on $\partial\gamma$ and thus it is the only common vertex which lies on $\partial\gamma$. Finally, v is the uniqe vertex of D_1 with 8 chambers in its star. Thus $|st(v_1)| = 6$. Now we may apply the $|\mathrm{st}(v)| = 6$ case with G_1 as our new choice of D_2 and v_1 the new v. This shows that $U_\gamma \subset U_{n-1}$ as desired.

Now suppose the fixed minimal gallery from before passes through F_2. The arguments made here will be very similar to those made in the previous paragraphs, as there is an obvious symmetry in the Coxeter complex, but we will explain the arguments again, if a little more briefly. There is also a minimal gallery from D_3 to C which passes through F_2 as well. But then $d(F_2, C) = d(D_3, C) - 3$ which means F_2 and D_3 are opposite in $\mathrm{st}(y_2)$. Then there is another minimal gallery in $\mathrm{st}(y_2)$ from $D_3 \to F_2$ which does not pass through D_2. Let G_2 be the chamber adjacent to D_3 in this new minimal gallery. Then G_2 and D_3 will have two vertices in common and one of them will be y_2. Let v_2 be the other vertex in common. Then v_2 must lie on $\partial\delta$ since y_2 is the only vertex of D_3 which does not lie on $\partial\delta$. We also know that $|\mathrm{st}(v_2)| = 6$ as v is the only vertex of D_3 with $|\mathrm{st}(v)| = 12$.

Since D_3 borders δ we know that $d(\delta, C) \leq n$. If $d(\delta, C) < n$ then $U_\delta \subset U_{n-1}$. If $d(\delta, C) = n$ then we have the following situation: D_3 is a chamber which borders δ and $d(\delta, C) = d(D_3, C)$. Furthermore, G_2 is a chamber adjacent to D_3 with $d(G_2, C) = d(D_3, C) - 1$. The unique, common vertex which lies on $\partial\delta$ is v_2 and it has $|\mathrm{st}(v_2)| = 6$. Thus we can apply that $|st(v) = 6|$ case to see that $U_\delta \subset U_{n-1}$. But $U_v = \langle U_\alpha, U_\beta, U_\delta \rangle$ and thus $U_\gamma \subset U_{n-1}$ as desired.

Now we have shown that $U_\gamma \subset U_{n-1}$ in any case, and since the choice of γ such that $d(\gamma, C) = n > 2$ was arbitrary, we know that $U_n \subset U_{n-1}$ for $n > 2$ as desired. □

{cor:334f2fg}

Corollary 5. *Suppose $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ is an RGD system of type (W, S) with $S = \{s, t, u\}$. If $m(s, t) = m(s, u) = 3$ and $U_x \cong C_2(2)$ for the vertex x of C of type s then U_+ is finitely generated.*

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Now we consider the last exceptional case. In this section we assume $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) with $S = \{s, t, u\}$. Assume that $m(s,t) = m(s,u) = 3$ and $U_x \cong G_2(3)$ where x is the vertex of the fundamental chamber C of type s. We will show that U_+ is finitely generated.