Chapter 1

Conditions for Infinite Generation

1.1 Existience of $\tilde{\phi}_v$

The general heuristic in this section will be as follows. For each vertex v with an exceptional root group $\mathcal{U}_v$ then we know there is a surjective homomorphism $\phi_v : \mathcal{U}_v \to K$ where K is the cyclic group of order $|k|$. We will try to extend the map ϕ_v to a map $\tilde{\phi}_v : \mathcal{U} \to K$ in such a way that $\mathcal{U}_\gamma$ is sent to the identity for all γ which do not pass through v. This will show that any generating set for $\mathcal{U}$ must contain at least one root through v. If we can do this for enough vertices v then we can show that $\mathcal{U}$ is not finitely generated. We will proceed in showing a large set of v for which this is possible.

Our first lemma is the primary key to showing the existence of $\tilde{\phi}_v$.

Lemma 1. *Suppose v is a vertex of Σ. Also suppose that any non-simple, positive root γ at v is simple at all other vertices on $\partial\gamma$. Then $\tilde{\phi}_v$ exists and is well defined.*

Proof. Since we have a presentation, it suffices to define $\tilde{\phi}_v$ on the generators, and check that the relations are satisfied by the image. We can define $\tilde{\phi}_v$ as follows:

$$\tilde{\phi}_v(x_\gamma(u)) = \begin{cases} \phi_v(x_\gamma(u)) & \text{if } v \text{ lies on } \partial\gamma \\ 1 & \text{otherwise} \end{cases}$$

Now we just need to check the relations in $\mathcal{U}$. There are three types of relations in the presentation for $\mathcal{U}$. There are relations within the same root group so that $\mathcal{U}_\alpha \cong (k, +)$ for all positive roots α. There are also relations between root groups whose walls intersect, and those whose walls don't intersect.

Let R_α be a relation for $\mathcal{U}_\alpha$ where R_α is considered as a word with letters in $\mathcal{U}_\alpha$. If v lies on $\partial\alpha$ then $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$ since ϕ_v is a well defined homomorphism. Otherwise, every element of $\mathcal{U}_\alpha$ is sent to 1 and thus $\tilde{\phi}_v(R_\alpha) = 1$ as well so that R_α is mapped to the identity as desired.

Now suppose that α and β are any two positive roots. If $\partial\alpha \cap \partial\beta = \emptyset$ then properties of Kac-Moody groups tell us that $[\mathcal{U}_\alpha, \mathcal{U}_\beta] = 1$. Since the codomain of $\tilde{\phi}_v$ is an abelian group, then any relation of the form $[x, y] = 1$ will be satisfied by the image.

Now suppose that $\partial\alpha$ and $\partial\beta$ meet at a point y and consider any relation of the form $[x_\alpha(u), x_\beta(t)] = w$ where w is a word in $\mathcal{U}_{(\alpha,\beta)} \subset \mathcal{U}_y$. Again, note that the image of the left side of this equation will always be the identity as the codomain is still abelian. If $y = v$ then $\mathcal{U}_y = \mathcal{U}_v$ and thus $\tilde{\phi_v}(w) = \phi_v(w) = 1$ because ϕ_v is well defined.

Now suppose that $y \neq v$. Then we can label the positive roots passing through y as $\gamma_1, \cdots, \gamma_n$ in such a way that $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$ whenever $i < j$. In this case we can can say without loss of generality that $\alpha = \gamma_l$ and $\beta = \gamma_m$ with $l < m$. There can be at most one root whose wall passes through y and v, which we will call γ_k if it exists. If γ_k does not exist, or $k \leq l$ or $k \geq m$ then the root γ_k is not contained in (α, β) and thus $\tilde{\phi_v}(\mathcal{U}_\delta) = 1$ for all $\delta \in (\alpha, \beta)$. This means $\tilde{\phi_v}(w) = 1$ and the relation is satisfied.

Now we suppose that γ_k exists and $l < k < m$. Then γ_k is not simple at y and thus γ_k must be simple at v by assumption. This means $\tilde{\phi_v}(\mathcal{U}_{\gamma_k}) = \phi_v(\mathcal{U}_{\gamma_k}) = 1$ by the construction of ϕ_v. Since $\tilde{\phi_v}(\mathcal{U}_{\gamma_i}) = 1$ for all $i \neq k$ by definition, this means that $\tilde{\phi_v}(w) = 1$ showing the relation is satisfied and giving the desired result. $\square$

Now Lemma 1 gives a sufficient condition for the existence of $\tilde{\phi_v}$ which is fairly easy to check. In fact, we will use this condition to choose appropriate vertices to constuct a large class of $\tilde{\phi_v}$.

Suppose that $W = \langle s, t, u \rangle$ so that $m(s,t) = a, m(s,u) = b$, and $m(t,u) = c$. Assume that $a \leq b \leq c$ with $a, b, c \in \{3, 4, 6\}$ and furthermore assume that $c \geq 4$ if $k = \mathbb{F}_2$ and $c = 6$ if $k = \mathbb{F}_3$. Let C be the fundamental chamber of Σ and let x be the vertex of C of type s. Then by our assumptions, $\mathcal{U}_x$ is not generated by its simple roots and ϕ_x exists.

We can label the roots through x as $\alpha_1, \ldots, \alpha_n$ so that α_1 and α_n are the simple roots at x. Also note that $n = c$. The ordering on these roots is chosen so that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. This is equivalent to the condition that $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ for any $i < j$.

We can describe any root in terms of a pair of adjacent chambers. We can also identify $\mathcal{C}(\Sigma)$ with W where the chamber wC is associated to w. If we use this identification then we can describe the roots as follows

$$\alpha_1 = \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\}$$
$$\alpha_n = \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}$$

In a similar way we can define two more roots

$$\beta = \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}$$
$$\beta' = \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}$$

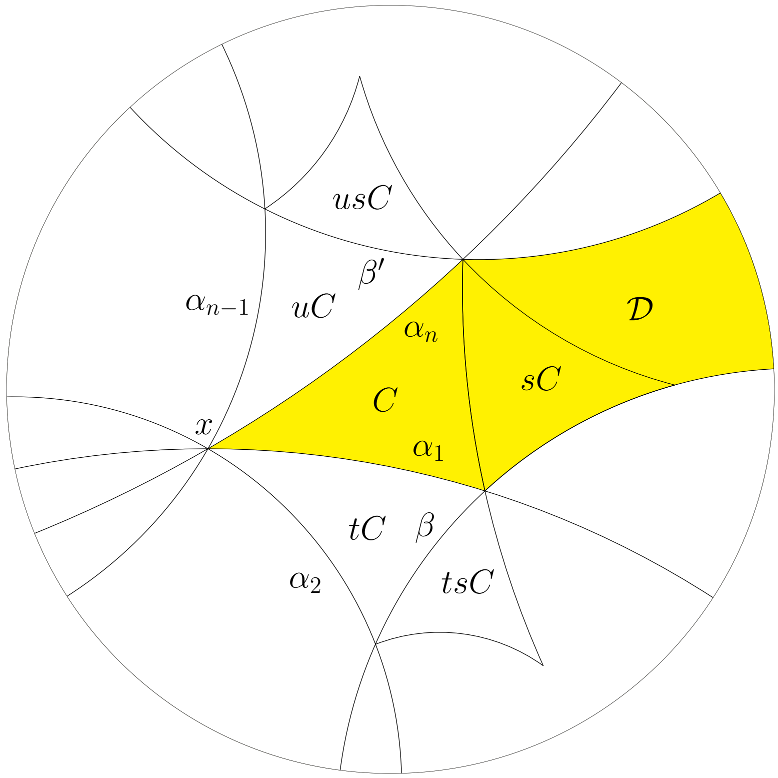

Figure 1.1: The Roots $\alpha_1, \alpha_n, \beta, \beta'$ with the region $\mathcal{D}$ in yellow.

Now we can define $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. These roots are chosen and $\mathcal{D}$ is defined in such a way to give the following lemma:

Lemma 2. *Let $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. If γ is a non-simple, positive root with x on $\partial\gamma$, and δ is any other positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$, then $\mathcal{D} \subset \gamma \cap \delta$.*

Proof. rewrite this lemma using the better structure of Lemma **??**

By assumption, γ is a positive root through x so $\gamma = \alpha_i$ for some i. Furthermore, we assumed that γ was not simple which means $2 \leq i \leq n-1$. Since α_1 and α_n are simple at x we can see that $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$. Thus it will suffice to prove that $\mathcal{D} \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$ and thus $\mathcal{D} \subset \delta$ as desired.

Now suppose $y \neq x$ and also assume that x and y are not adjacent. Suppose that $\partial\delta$ meets $\partial\alpha_1$ at a point y'. Then the points x, y, y' form a triangle, whose sides lie on the walls $\partial\gamma$, $\partial\delta$, and $\partial\alpha_1$. The triangle condition then implies that xyy' must be a chamber of Σ, which is a contradiction since x and y are not adjacent. Thus $\partial\delta$ does not meet $\partial\alpha_1$ and a similar argument shows that $\partial\delta$ does not meet α_n.

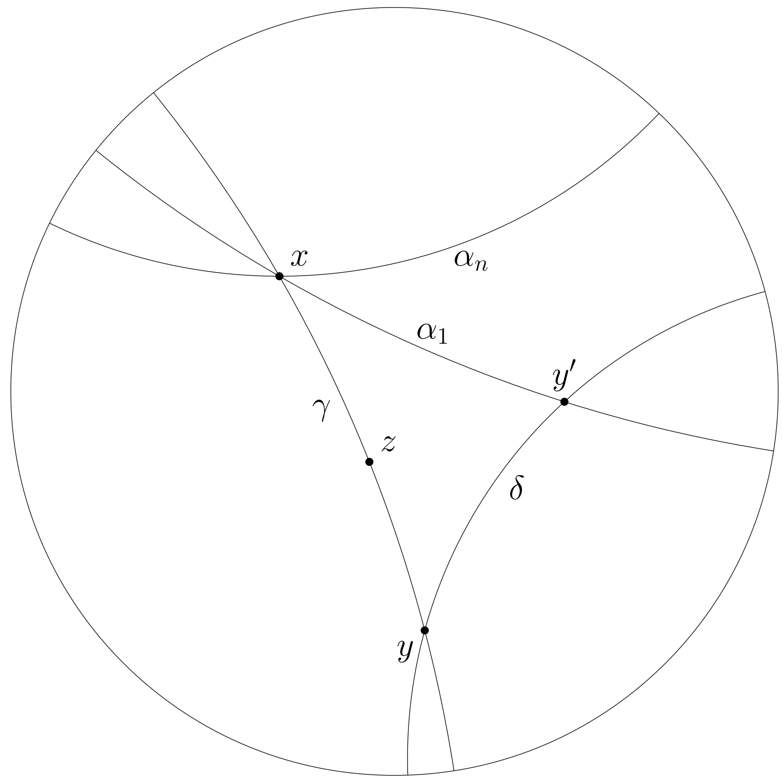

Figure 1.2: The point y' cannot exist as it would form a triangle which is not a chamber.

From the geometry of the Coxeter complex, we can observe that for any $1 < i < n$ we have $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = x$. Since $y \neq x$ this means that y does not lie in $\alpha_1 \cap \alpha_n$. We can assume without loss of generality that y does not lie in α_1. We know that α_1 and δ are two positive roots whose walls do not meet, and thus there are exactly three possibilites. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$ or $-\delta \subset \alpha_1$. The later two cases are impossible as both would imply that $y \in \alpha_1$ which contradicts our assumption. Thus we have $\alpha_1 \subset \delta$ which gives $\mathcal{D} \subset \alpha_1 \subset \delta$ as desired.

Now we suppose again that $y \neq x$ but x and y are adjacent. Then once again there are two possibilities. If $\partial\delta$ does not meet $\partial\alpha_1$ or $\partial\alpha_n$ then the identical argument from the last paragraph shows that $\mathcal{D} \subset \delta$.

So now we suppose that $\partial\delta$ does meet $\partial\alpha_1$ or $\partial\alpha_n$. If $\partial\delta$ meets $\partial\alpha_1$ at a point y', then the vertices xyy' will form a triangle which must be a chamber call it C'. This chamber contains, x, and has 2 sides on $\partial\alpha_1$ and $\partial\gamma$ respectively. Since γ is not simple, an observation of the chambers around x in Figure 1.1 shows that γ must be α_2 C' must be tC in which case $\delta = \beta$ by definition. Thus we get $\mathcal{D} \subset \beta = \gamma$ which proves the result.

If $\partial\delta$ meets α_n then an identical argument shows that $\gamma = \beta'$ which also proves the result.

□

The next two lemmas will allow us to construct new $\tilde{\phi}_v$ for vertices v based on the region $\mathcal{D}$.

Lemma 3. *Suppose γ is any positive root with $x \in \partial\gamma$, and y is another vertex on $\partial\gamma$. Then γ is simple at y.*

Proof. Suppose that γ is not simple at y. Then we can label the positive roots at y as $\delta_1, \ldots, \delta_m$ in such a way that $\delta_i \cap \delta_j \subset \delta_k$ for $1 \leq i \leq k \leq j$. In this case we have δ_1, δ_m are simple at y and $\gamma = \delta_r$ for some $1 < r < m$. But x is a vertex of C and $C \in \delta_1 \cap \delta_m$ and thus $x \in \delta_1 \cap \delta_m$ as well. We know that x lies on $\partial\delta_r$ by assumption and thus x is an element of $\partial\delta_r \cap \delta_1 \cap \delta_m$. But this is impossible as we can observe from the geometry of Σ that $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$ for all $1 < i < m$. Thus γ is simple at y as desired. □

Despite some of the technical details the previous result should be intuitively clear. The walls through y will divide Σ into $2m$ regions, and the region which contains C will be bounded by the two simple roots. Since x lies on $\partial\gamma$, it is impossible for any other roots through y to be any "closer" to C and thus γ must be simple at y as we proved.

Corollary 1. *$\tilde{\phi}_x$ exists.*

Proof. Let γ be any non-simple, positive root through x and let y be another vertex on $\partial\gamma$. Then by the previous lemma, γ is simple at y and thus $\tilde{\phi}_x$ exists by Lemma 1. □

We are now ready to construct a large family of vertices $\{v\}$ for which $\tilde{\phi}_v$ will exist. The idea is as follows. If we take any chamber in $\mathcal{D}$ and treat it as a new "C" then $\tilde{\phi}_x$ would exist for this "C." So what we do is apply elements of W which map the chambers of $\mathcal{D}$ to C, and use these choices of w to get new vertices v.

Since the construction of these $\tilde{\phi}_v$ depends on properties of simple roots, we want to know the simplicity behaves nicely with the action of W to this end we have the following lemma.

Lemma 4. *Suppose v is a vertex with simple roots γ, γ' at v. If w is an element of w such that $w\delta$ is a positive root for all positive δ at v, then $w\gamma$ and $w\gamma'$ are the simple roots at wv.*

Proof. Let δ be a positive root at wv. Since w induces an isomorphism of simplical complexes, and it sends positive roots at v to positive roots at wv, it must also send negative roots at v to negative roots at wv. So $w^{-1}\delta$ is a root at v, and $w(w^{-1}\delta) = \delta$ is positive, so $w^{-1}\delta$ is also positive. Thus by definition of simple, we have $\gamma \cap \gamma' \subset w^{-1}\delta$. But we can now apply w to get $w\gamma \cap w\gamma' \subset \delta$. Since the choice of δ was arbitrary we must have $w\gamma$ and $w\gamma'$ are simple as desired. □

We can now use the previous lemma to actually construct $\tilde{\phi}_v$ for a certain collection of vertices v.

Lemma 5. *If $v = w^{-1}x \in \mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ then $\tilde{\phi}_v = \tilde{\phi_{wx}}$ exists.*

Proof. Let $D = \mathrm{Proj}_{w^{-1}x}(C)$ and let $D = (w')^{-1}C$. By the definition of projections, $w^{-1}x$ is a vertex of D of type s, but $(w')^{-1}x$ is also a vertex of D of type s, and thus $(w')^{-1}x = w^{-1}x$. Now without loss of generality we may assume that $w' = w$. Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus

$\text{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \text{Proj}_{wx} wD = \text{Proj}_{wx} C \qquad C = wD = \text{Proj}_{w(w^{-1}x)} wC = \text{Proj}_x E$$

In particular, if γ is any positive root through wx then $E \in \gamma$ by the properties of projections.

Now suppose that γ is a non-simple, positive root through wx and y is another vertex on $\partial\gamma$. We must show that γ is simple at y. Since γ is positive through wx we know that $C, E \in \gamma$. If we apply w^{-1} then we get the following facts. We know that $w^{-1}\gamma$ is a root such that $\partial(w^{-1}\gamma)$ passes through $w^{-1}wx = x$. We also know that $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is also a positive root.

The first claim is that $w^{-1}\gamma$ is not simple at x. Suppose that δ is any positive root at wx. Then $E \subset \delta$ and so applying w^{-1} we get that $w^{-1}E = C \subset w^{-1}\delta$. Thus w^{-1} sends positive roots at wx to positive roots at x. By Lemma 4 this means that w^{-1} sends simple roots at wx to simple roots at x. Since γ is not simple at wx this means that $w^{-1}\gamma$ is not simple at x.

So $w^{-1}\gamma$ is a non-simple positive root at x, and since y lies on $\partial\gamma$ we also know that $w^{-1}y$ lies on $w^{-1}(\partial\gamma)$. If we apply Lemma 3 we can see that $w^{-1}\gamma$ must be simple at $w^{-1}y$.

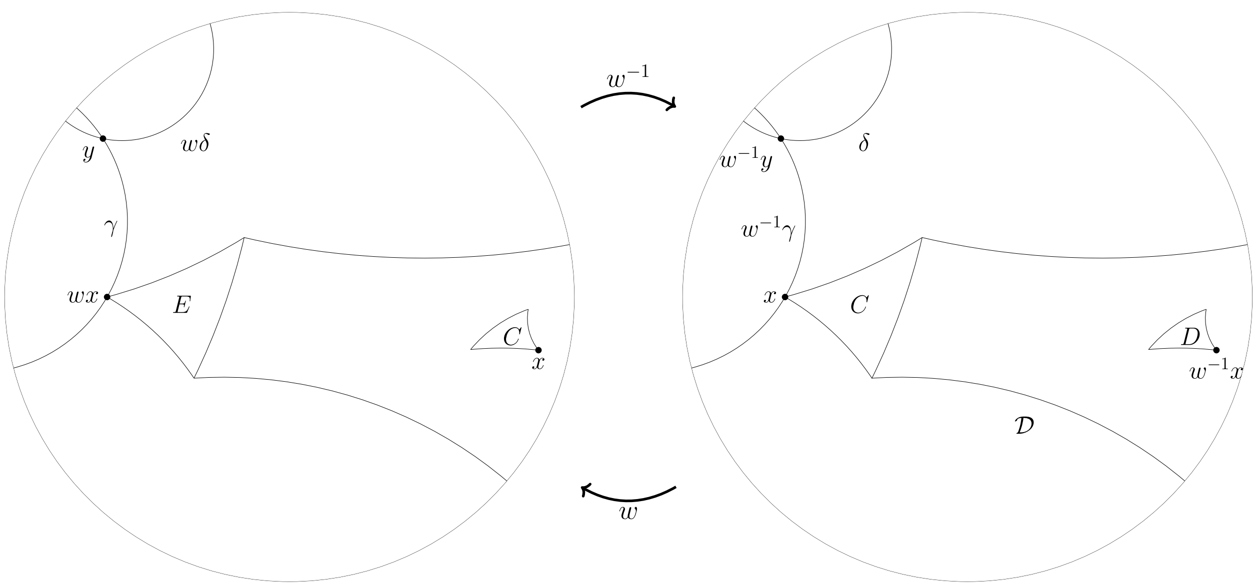

Figure 1.3: The effect of w and w^{-1} on the chambers and roots.

Now suppose that δ is any positive root at $w^{-1}y$. Then by Lemma 2 we know that $D \in \mathcal{D} \subset \delta$. If we apply w then we get $C = wD \in w\delta$ and $w\delta$ is a root through y. Thus $w\delta$ is a positive root through y and therefore w sends positive roots through $w^{-1}y$ to positive roots through y. Again we can apply Lemma 4 to say that w must also send simple roots through $w^{-1}y$ to simple roots through y. But $w^{-1}\gamma$ was a simple root through $w^{-1}y$ and thus γ is simple at y as desired. $\square$

Now we have shows that vertices of $\mathcal{D}$ in some way correspond to $\tilde{\phi}_v$. If our goal is to find infinitely many such v then there is still some work to be done. For instance, we do not yet know if the region $\mathcal{D}$ contains infinitely many chambers, or even if it does, if all the vertices of D lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

1.2 When $\mathcal{D}$ is infinite

Our first task will be two show that the region $\mathcal{D}$ contains infinitely many unique vertices. Intuitively, this will happen if the walls for β and β' do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that W is the Weyl group of $\mathcal{G}$ and W is defined by the edge labels $a = m(s,t), b = m(s,u), c = m(t,u)$ with $a \leq b \leq c$. For the remainder of the section we will also add the assumption that $b \geq 4$. Let $w_k = (sut)^k$ and $v_k = w_k x$. We will see in the following that with this additional assumption on b that the vertices v_k will be sufficiently many to prove $\mathcal{U}$ is not finitely generated.

Lemma 6. *The vertices v_k are distinct for all k and they all lie in $\mathcal{D}$.*

Proof. First we will show that $v_k \in \mathcal{D}$ for all k. Since x is a vertes of C we know that v_k is a vertex of $w_k C$ and thus it will suffice to show $w_k C$ is contained in $\mathcal{D}$ for all k. Since the roots $\alpha_1, \alpha_n, \beta, \beta'$ can be identified with their corresponding subsets of W, we can use the length function to check containment in these roots.

Recall that words in a Coxeter group can only be reduced by removal of consecutive repeated letters, or application of the Coxeter relations. It is immediate from the definition that $\ell(w_k) = 3k$ for all k. We can also see that $\ell(tw_k) = 3k+1$ and thus $w_k \in \alpha_1$ for all k. Similarly, $uw_k = u(sutsut\cdots)$, and no reduction operations can be done as we assumed $m(s,u) \geq 4$. Thus $\ell(uw_k) = 3k+1$ which means $w_k \in \alpha_n$ as well.

Now consider the element stw_k. If we write this element out in terms of the generators and apply the only possible Coxeter relations we get

$$\begin{aligned} stw_k &= st(sutsut\cdots) \\ &= (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) \\ &= (ts)(tut)(sutsut\cdots) \end{aligned}$$

and none of these can be reduced as $m(t,u) \geq 4$. Note that the commutation relation $sts = tst$ may not be possible if $m(s,t) \geq 4$, but it is the only relation possible in stw_k and even if it does exists then it does not allow stw_k to be reduced in length. We previously showed $\ell(tw_k) = 3k+1$ and now we see $\ell(stw_k) = 3k+2$ and so $w_k \in \beta$.

Now we can consider suw_k in a similar manner. Writing suw_k out as a word in the generations and applying Coxeter relations gives us

$$\begin{aligned}
suw_k &= su(sutsut\cdots) \\
&= (susu)(tsutsu\cdots) \\
&= (usus)(tsutsu\cdots) \\
&= (usu)(sts)(utsuts\cdots) \\
&= (usu)(tst)(utsuts\cdots)
\end{aligned}$$

Note once again that not all of these relations may be possible if $m(s,u) = 6$ or $m(s,t) \geq 4$. However, these are the only possible relations, and since suw_k cannot be reduced under these assumptions, it cannot be reduced at all. Thus $\ell(suw_k) = 3k + 2$ which means $suw_k \in \beta'$ as well.

Now it only remains to show that each v_k is unique. Suppose $v_m = v_n$ for $m > n$. Then we would have $w_m x = w_n x$ and thus $w_n^{-1} w_m x = w_{m-n} x = x$. Thus it will suffice to show $w_k x \neq x$ for any $k > 1$. But we know that $\operatorname{stab}(x) = \langle u, t\rangle$ which does not contain w_k for any $k \geq 1$ and thus $w_k x \neq x$ as desired.

□

We now know that each of the v_k is distinct and each of them lies in $\mathcal{D}$. By Lemma 5 we know that $\tilde{\phi}'_v$ exists for each $v' = w_k^{-1} x$. Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the v' at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices $w_n^{-1} x$ distinct, but also no two lie on the same wall.

Lemma 7. *If $m \neq n$ then $w_m^{-1} x$ and $w_n^{-1} x$ do not lie on the same wall.*

Proof. Suppose $w_m^{-1} x$ and $w_n^{-1} x$ do lie on the same wall with $m > n$. Then we also know that $w_m w_n^{-1} x = w_{m-n} x$ and x will lie on the same wall and thus it will suffice to show that $w_k x$ and x do not lie on the same wall for any $k \geq 1$.

We know from Lemma 6 that $w_k \in \mathcal{D}$. Thus if $w_k x$ and x lie on the same wall, it must be a wall through x, and since $w_k x \in \mathcal{D}$ this wall must be either H_u or H_t, the fixed points of u and t respectively. Thus we either have $u w_k x = w_k x$ or $t w_k x = w_k x$ which implies that either $w_k^{-1} u w_k$ or $w_k^{-1} t w_k$ is contained in $\operatorname{stab}(s) = \langle u, t\rangle$. However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced. The only possible relations we have are

$$\begin{aligned}
w_n^{-1} t w_n &= (\cdots tustus)t(sutsut\cdots) \\
&= (\cdots tustu)(sts)(utsut\cdots) \\
&= (\cdots tustu)(tst)(utsut\cdots)
\end{aligned}$$

or

$$\begin{aligned} w_n^{-1} u w_n &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(sus)(tsuts\cdots) \\ &= (\cdots stu)(sts)u(sts)(uts\cdots) \\ &= (\cdots stu)(tst)u(tst)(uts\cdots) \end{aligned}$$

In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in $\langle u, t\rangle$. Thus the $w_n^{-1}x$ all lie on distinct walls as desired. $\square$

We now have all the ingredients and are ready to prove the main theorem.

Theorem 1. *Let $\mathcal{G}$ be a Kac-Moody group over $\mathbb{F}_2$ or $\mathbb{F}_3$ with unipotent subgroup $\mathcal{U}$. If let W be the Weyl group of $\mathcal{G}$ with coxeter diagram defined by the edge labels $a \le b \le c$ with $a, b, c \in \{3, 4, 6\}$. If $b \ge 4$ and $c = 6$ over $\mathbb{F}_2$ then $\mathcal{U}$ is not finitely generated.*

Proof. Suppose that $\mathcal{U}$ is finitely generated. Then there is some finite set of roots $\beta_1, \dots, \beta_m$ such that $\mathcal{U} = \langle \mathcal{U}_{\beta_i} | 1 \le i \le m\rangle$. Now no two of the vertices $w_k^{-1}x$ lie on the same wall and thus we can choose k so that $v = w_k^{-1}$ does not lie on $\partial\beta_i$ for any i. By Lemma 6 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $\mathcal{U} \to C$. However, we can also see by definition that $\tilde{\phi}_v(\mathcal{U}_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of $\mathcal{U}$ to the identity and thus it must be the trivial map which is a contradiction. Thus $\mathcal{U}$ is not finitely generated as desired. $\square$