Chapter 1

Known Results on Finite Generation

Throughout this section, $\mathcal{G}$ will be a Kac-Moody group with rank 3 Weyl group W over a field k. We will also assume that W is defined by the coxeter diagram with edge labels $a, b, c \in \{3, 4, 6\}$ with $a \leq b \leq c$ and $c \geq 4$. This last condition ensures that W is hyperbolic. Let Σ be the Coxeter complex of W. Let Φ^+ be the positive roots of Σ, and for any $\alpha \in \Phi^+$ we will let $\mathcal{U}_\alpha$ be the root group associated to α.

For any vertex v of σ, there will be some walls of Σ which pass through v, and for each of these walls we have a unique *positive* root. We will call these the **positive roots at v** and denote them by Φ_v. Recall that $\mathrm{st}(v)$ is defined as all the chambers containing v as a vertex. If there are n positive roots at v then $|\mathrm{st}(v)| = 2n$. Furthermore, it is possible to label the positive roots at v as $\alpha_1, \ldots, \alpha_n$ in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. This ordering is unique upto a reversal of the form $\alpha_i \mapsto \alpha_{n+1-i}$. This possible reversal will not matter in most cases and if it does then a choice of α_1 will be specified. It does however allow us to unambigiously define α_1 and α_n as the **simple** roots at v. They are the unique positive roots at v whose intersection is contained in all other positive roots at v.

Now we can define $\mathcal{U}_v$ to be the subgroup of $\mathcal{G}$ generated by all of the root groups of the positive roots at v. That is

$$\mathcal{U}_v = \langle \mathcal{U}_\alpha | \alpha \text{ is a positive root at } v \rangle = \langle \mathcal{U}_\alpha | \alpha \in \Phi_v \rangle$$

Most of the time the group $\mathcal{U}_v$ is generated by $\mathcal{U}_1, \mathcal{U}_n$ which are the simple root groups at v. However, there are some exceptions to this. Let $\mathcal{U}'_v = \langle \mathcal{U}_1, \mathcal{U}_n \rangle$ where $2n = |\mathrm{st}(v)|$. Then we have the following results about the $\mathcal{U}_v$ which comes from the known theory about rank 2 Moufang Polygons.

Lemma 1. *Let v be a vertex of Σ with $|\mathrm{st}(v)| = 2n$. Let $\mathcal{U}'_v = \langle \mathcal{U}_1, \mathcal{U}_n \rangle$ where $\mathcal{U}_1, \mathcal{U}_n$ are the root groups of the simple roots at v. Then we can describe $[\mathcal{U}_v : \mathcal{U}'_v]$ with the following table*

n	$\lvert k\rvert$	$[\mathcal{U}_v : \mathcal{U}'_v]$
4	2	2
6	2	4
6	3	3

and $[\mathcal{U}_v : \mathcal{U}'_v] = 1$ *in all other cases. In other words,* $\mathcal{U}' = \mathcal{U}$ *with the exception of the 3 cases above.*

We can in fact say a little more than that when $|k| = 2$ and $n = 6$.

Lemma 2. *Suppose that* $\mathcal{U}$ *is defined over* $k = \mathbb{F}_2$ *and* v *is a vertex of* Σ *with* $|\mathrm{st}(v)| = 2n = 12$*. Then it is possible to label the positive roots at* v *as* $\mathcal{U}_1, \ldots, \mathcal{U}_6$ *in such a way that* $\mathcal{U}''_v = \langle \mathcal{U}_1, \mathcal{U}_5, \mathcal{U}_6 \rangle$ *has index 2 in* $\mathcal{U}_v$*.*

These two lemmas together give the following corollary.

Corollary 1. *Suppose* v *is a vertex of* Σ *with* $|\mathrm{st}(v)| = 2n$ *and* $\mathcal{U}_1, \mathcal{U}_n$ *the simple roots at* v*. Suppose that* $[\mathcal{U}_v : \mathcal{U}'_v] \geq 2$*. Let* H *be the cyclic group of order* $|k|$ *where* k *is the field over which* $\mathcal{U}$ *is defined. Then there is a surjective group homomorphism, call it* $\phi_v : \mathcal{U}_v \to H$ *such that* $\phi_v(\mathcal{U}_1) = \phi_v(\mathcal{U}_n) = \{1\}$*.*

Proof. Let $\mathcal{U}'_v = \langle \mathcal{U}_1, \mathcal{U}_n \rangle$. Since $[\mathcal{U}_v, \mathcal{U}'_v] \neq 1$ we know we must be in one of the three exceptional cases above. If $n = 4$ and $|k| = 2$ then $[\mathcal{U}_v, \mathcal{U}'_v] = 2$ and thus $\mathcal{U}'_v$ is a normal subgroup of $\mathcal{U}_v$ and the quotient has order 2. So we can define $\phi_v : \mathcal{U}_v \to H$ to be the quotient map $\mathcal{U}_v to \mathcal{U}_v / \mathcal{U}'_v$.

If $n = 6$ and $|k| = 3$ then $[\mathcal{U}_v : \mathcal{U}'_v] = 3$. But $\mathcal{U}_v$ is a 3-group and thus $\mathcal{U}'_v$ is normal and the quotient has order 3, so we can construct ϕ_v as before.

Now suppose $n = 6$ and $|k| = 2$. Then by Lemma 2, we can define $\mathcal{U}''_v = \langle \mathcal{U}_1, \mathcal{U}_5, \mathcal{U}_6 \rangle$ so that $[\mathcal{U}_v : \mathcal{U}''_v] = 2$ and thus $\mathcal{U}''_v$ is normal and the quotient has order 2. In this case we can define ϕ_v to be the quotient map $\mathcal{U}_v \to \mathcal{U}_v / \mathcal{U}''_v$. □

The following corollary will show that we do not have very much wiggle room when defining ϕ_v, and thus if we can write any function which "looks like" ϕ_v then they must be esentially the same.

Corollary 2. *Suppose* v *is a vertex of* Σ *with* $|\mathrm{st}(v)| = 2n$ *and* $\mathcal{U}_1, \mathcal{U}_n$ *the simple root groups at* v*. Let* ϕ_v *be defined as in the previous corollary. Then* $\ker \phi_v$ *is the unique, proper, normal subgroup of* $\mathcal{U}_v$ *which contains* $\mathcal{U}_1$ *and* $\mathcal{U}_n$*.*

Proof. If $n = 4$ and $|k| = 2$ or $n = 6$ and $|k| = 3$ then the result is clear as $\mathcal{U}'_v = \langle \mathcal{U}_1, \mathcal{U}_n \rangle = \ker \phi_v$ is normal and has prime index, so there can be no other proper normal subgroups containing it.

If $n = 6$ and $|k| = 2$ then $[\mathcal{U}_v : \mathcal{U}'_v] = 4$ but $\mathcal{U}'_v$ is not a normal subgroup. It can be shown if N is a normal subgroup containing $\mathcal{U}'_v$ then $\mathcal{U}_5 \subset N$ as well, and thus $\mathcal{U}''_v \subset N$. But $[\mathcal{U}_v : \mathcal{U}''_v] = 2$ and thus $\mathcal{U}''_v$ is the only proper normal subgroup containing $\mathcal{U}_1, \mathcal{U}_n$ as desired. □

This isn't really a proof but I will fill in the details later. I was more just reminding myself of the arguments.

The general theory gives us the following result

Theorem 1. *Let $\mathcal{G}$ be a Kac-Moody group over k with rank 3 Weyl group W as before. For any vertex v of Σ, let $\mathcal{U}'_v = \langle \mathcal{U}_1, \mathcal{U}_n \rangle$ where $\mathcal{U}_1, \mathcal{U}_n$ are the simple roots at v. If $\mathcal{U}'_v = \mathcal{U}_v$ for all $v \in \Sigma$ then $\mathcal{U}$ is finitely generated.*

I use this lemma later. This isn't organized yet but I wanted to have it so my reference aren't broken.

Lemma 3. *Let $\alpha, \beta, \beta+\alpha, \beta+2\alpha$ be the positive roots of a root system of type C_2 and $\mathcal{U}$ the unipotent subgroup of $C_2(\mathbb{F}_2)$. Then $\mathcal{U} = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+\alpha} \rangle = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+2\alpha} \rangle$.*