Chapter 1

Exceptional Cases

In the previous chapter we were able to show that U_+ is not finitely generated for a large family of Coxeter groups W with labels $a \leq b \leq c$. These results were based on assuming $b \geq 4$ which allowed us to show that $\mathcal{D}$ was infinite and proceed from there. In fact, we didn't even describe all of the chambers in $\mathcal{D}$, just an infinite family. However, the same approach will not work in the remaining cases because of the following lemma.

Lemma 1. *If W is a Coxeter group with labels $a \leq b \leq c$ as before, then $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ as defined in the previous chapter is infinite if and only if $b \geq 4$.*

Proof. We know by Lemma **??** that $\mathcal{D}$ is infinite if $b \geq 4$. Thus it remains to show that $\mathcal{D}$ is finite if $b = 3$. If $b = 3$ then $a = 3$ also, and by definition of a, b, c this means $m(s,t) = m(s,u) = 3$. We will also recall the definition of $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D,C) < d(D,tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D,C) < d(D,uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D,tC) < d(D,tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D,uC) < d(D,usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Let $w \in W$ and suppose $\ell(w) \geq 2$. Then we can write $w = s_1 s_2 w'$ where $\ell(w') = \ell(w) - 2$. If $s_1 = t$ then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows $w \notin \alpha_1$ and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_1 = u$.

Now we assume $s_1 = s$ and so we can also assume $s_2 = t, u$. First let $s_2 = t$ so that $w = stw'$. If $w \notin \alpha_1$ then $w \notin \mathcal{D}$ and so we will suppose $w \in \alpha_1$. Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \leq \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_2 = u$.

We have shown that if $\ell(w) \geq 2$ then $w \notin \mathcal{D}$ and thus $\mathcal{D}$ must be finite as desired. In fact, if $a = b = 3$ then we can check relatively easily that $\mathcal{D} = \{C, sC\}$ which proves the desired result. $\square$

The previous lemma shows that generating results for the remaining cases is not just a matter of being slightly more clever when we look for vertices, but changing the strategy as a whole. In fact, in some cases our proof strategy needs to switch entirely since U_+ will be finitely generated in some cases as we will see. First, we will show which of the remaining cases are not finitely generated.

All of the remaining rank 3 cases have the property that $m(s,u) = m(s,t) = 3$. If x is the vertex of C of type s then x is the only possible vertex of type C with the property that $[U_x : U'_x] \geq 2$. With two edge labels of 3 it is impossible for $U_x \cong {}^2F_4(2)$ and so the only remaining possibilites are $U_x \cong C_2(2), G_2(2)$, and $G_2(3)$. We will enumerate through each of these cases individually.

1.1 Case: $U_x \cong G_2(2)$

We saw in the previous chapter that a vertex contained in $\mathcal{D}$ was a sufficient condition to construct a corresponding map $\tilde{\phi_v}$. However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct $\tilde{\phi_v}$ for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statment similar to Lemma **??**.

For the remainder of the section we will assume that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) where $S = \{s, t, u\}$ and

$$W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^6 = 1\rangle$$

Furthermore, let x be the vertex of C of type s and assume that $U_x \cong G_2(2)$. Recall that this means $[U_x : U_x'] = 4$ and $[U_v : U_v'] = 4$ for all vertices v of type s by Lemma **??** and Lemma **??**.

Recalling from the previous chapter, we know that there is a presentation of U_+ generated by U_α for all $\alpha \in \Phi_+$. Again, there are several types of relations we need to consider. There are relations among the U_α and there are relations between U_α and U_β when $\{\alpha, \beta\}$ is a prenilpotent pair. By (**??**) we know that $[U_\alpha, U_\beta] = \{1\}$ if α and β are nested. We also know that when $\partial\alpha \cap \partial\beta \neq \emptyset$ that $[u, u'] = w$ for some word $w \in U_{(\alpha,\beta)}$ where $u \in U_\alpha$ and $u' \in U_\beta$.

Now recall from Chapter **??** that there is a surjective homormorphism $\phi_x : U_x \to H$ where H is a cyclic group. We can also choosea standard labeling $\alpha_1, \dots, \alpha_6$ of the positive roots through x in such a way that $\ker \phi_x = U_x'' = \langle U_1, U_5, U_6\rangle$. Similarly to the last chapter, if v is any vertex of type s, our goal is to construct an extension of the form $\tilde{\phi_v}$ in such a way that

$$\tilde{\phi_v}(U_\alpha) = \begin{cases} \phi_v(U_\alpha) & v \in \partial\alpha \\ 1 & \text{otherwise} \end{cases}$$

If we can do this for enough vertices v then we will be able to show that U_+ is not finitely generated in the same way as the previous chapter. Our first step is to prove an analagous result to Lemma **??** in the current context.

Lemma 2. *Let v be a vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$. Assume $\gamma_1, \dots, \gamma_6$ is a standard ordering of the positive roots through v such that $U_{\gamma_5} \subset \ker \phi_v$. If $\gamma_2, \gamma_3,$ and γ_4 are simple at all other vertices they meet, then $\tilde{\phi_v}$ as defined in Lemma **??** exists.*

Proof. To check $\tilde{\phi_v}$ is well defined is a matter of checking the relations are satisfied by the images under $\tilde{\phi_v}$. Since $\tilde{\phi_v}$ has a cyclic group as its codomain, we can see immediately that the first two types of relations will be satisfied regardless of α and β. Now to check the third type.

Suppose α and β are any two positive roots with $y = \partial\alpha \cap \partial\beta$. Then there is a relation in U_+ of the form $[u, u'] = w$ where $u \in U_\alpha, u' \in U_\beta$, and $w \in U_{(\alpha,\beta)}$. Since $[u_\alpha, u_\beta]$ must be mapped to the identity then we just need to check that w is also mapped to the identity. If $y = v$ then u_α, u_β, w all lie in U_v and $\tilde{\phi_v}(w) = \phi_v(w)$ which must be the identity because ϕ_v is a well defined homomorphism.

Now suppose $y \neq v$. Let $\delta_1, \dots, \delta_n$ be the positive roots through y, with a standard labeling, and assume that $\alpha = \delta_i$ and $\beta = \delta_j$ with $i < j$. There is at most one positive root whose wall can pass through both v and y, call it δ_k if it exists. If δ_k does not exist, then no positive roots through y pass through v and so $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all m. Thus $\tilde{\phi_v}(w) = 1$ as desired.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{1, 5, 6\}$. Then we know $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all $m \neq k$ and $\tilde{\phi_v}(U_{\delta_k}) = \tilde{\phi_v}(U_{\gamma_r}) = \phi_v(U_{\gamma_r}) = \{1\}$ by the construction of ϕ_v. Thus $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all m and so $\tilde{\phi_v}(w) = \{1\}$ as well.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{2, 3, 4\}$. Then by assumption, δ_k is simple at y and thus $k = 1, n$. Thus $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all $2 \leq m \leq n - 1$. But w is a word in $U_{(\alpha,\beta)} \subset U_{(\delta_2,\delta_{n-1})}$ and thus $\tilde{\phi_v}(w) = 1$ again, which gives the result. $\square$

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma **??**, and so we have a hope of constructing more $\tilde{\phi}_v$ then the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Let x be the vertex of C of type s as in the previous chapter and let $\alpha_1, \ldots, \alpha_6$ be the positive roots through x, labeled as usual. Recall from the previous chapter that

$$\begin{aligned} \alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\ \alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\ \beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \end{aligned}$$

Also assume without loss of generality that $\phi_x(U_{\alpha_5}) = \{1\}$. Now let $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$. We can now prove a lemma similar to Lemma **??**.

picture of $\mathcal{D}'$

Lemma 3. *Let x be the vertex of C of type s so that $|\mathrm{st}(x)| = 12$. Let $\alpha_1, \ldots, \alpha_6$ be the positive roots at x with the standard ordering. Also assume that $\phi_x(U_{\gamma_5}) = 1$. Suppose $\gamma = \alpha_i$ for $i \in \{2, 3, 4\}$. If δ is any positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$ then $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta \subset \gamma \cap \delta$ where*

$$\beta = \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}$$

as in the previous chapter.

Proof. Since γ is a positive root at x, and α_1, α_6 are the simple roots at x, we know that $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$ and thus it will suffice to show that $\mathcal{D}' \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_6 \subset \alpha_j = \delta$ and thus $\mathcal{D}' \subset \delta$ so that $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_6 = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_6$. Thus the point y will lie in exactly one of α_1 or α_6.

First suppose that $y \in \alpha_6$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilites would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D}' \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_6$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_6$. Since neither y or y' lies on $\partial\alpha_6$ this means that $y' \in \alpha_6$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_6$. If $E = D'$ then $D' \in \alpha_6$ since x, y, y' all lie in α_6. But this is a contradiction as α_6 cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D}' \subset \beta = \delta$ and $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

If we assume insteach that $y \in \alpha_1$ so that $y \notin \alpha_6$ then we have the same two possibilites. If $\partial\alpha_6 \cap \partial\delta = \emptyset$ then by similar arguments we get $\mathcal{D}' \subset \alpha_6 \subset \delta$ and thus $\mathcal{D}' \subset \gamma \cap \delta$ as desired. If $\partial\alpha_6 \cap \partial\delta = y'$ then the vertices x, y, y' form a chamber with y' on α_6. Again, by similar arguments as before, this would imply that $\gamma = \alpha_5$ or α_1, both of which are impossible.

Therefore, regardless of case we have $\mathcal{D}' \subset \gamma \cap \delta$ as desired. I still don't like this proof

□

We now have a condition for $\tilde{\phi}_v$ to exist which we can check and so it remains to find potential candidates to use at v. We know by Lemma **??** that ϕ_v will exist for all vertices v of type s. We also know from Lemma **??** that there is a compatibility of standard orderings, which we can use the check the hypothesis in Lemma **??**. We now prove the analogue of Lemma **??**.

Lemma 4. *Let x be the vertex of C of type s and suppose that v is any vertex in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ of type s. Then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. Let $D = \mathrm{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary **??**.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 2 at the vertex wx. Let $\gamma_1, \ldots, \gamma_6$ be a standard labeling of the positive roots through wx such that $U_{\gamma_5} \subset \ker \phi_{wx}$. We need to check that if $y \neq wx$ is on $\partial\gamma_i$ for $i \in \{2, 3, 4\}$ then γ_i is simple at y. First we will show that w^{-1} sends positive roots at wx to positive roots at x. Suppose γ is any positive root at wx. Then we know that $E \in \gamma$ and thus $C = w^{-1}E \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is positive, and thus w^{-1} sends positive roots at wx to positive roots at x.

If we apply Lemma **??** then we know that $w^{-1}\gamma_1 = \alpha_1, \ldots, w^{-1}\gamma = \alpha_6$ is a standard labeling of the of the positive roots at x. If we apply this isomorphism given by Corollary **??** then we know that $U_{w^{-1}\gamma_5} = U_{\alpha_5} \subset \ker \phi_x$ since $U_{\gamma_5} \subset \ker \phi_{wx}$.

Now we fix $i \in \{2, 3, 4\}$ and we need to check γ_i is simple at all vertices $y \neq v$ on $\partial\gamma_i$. If we apply w^{-1} we get that $w^{-1}y \neq x$ is a vertex on $\partial\alpha_i$. Thus by Lemma **??** we know that α_i is simple at $w^{-1}y$. Now suppose that δ is any positive root at $w^{-1}y$. Recall that $D \in \mathcal{D}'$ and we can apply Lemma 3 to see that $D \in \mathcal{D}' \subset \delta$. If we apply w we get $C = wD \in w\delta$ where $w\delta$ is a positive root through $w(w^{-1}y) = y$. Thus w sends positive roots at $w^{-1}y$ to positive roots at y. We can apply Lemma **??** again to say that w sends the simple roots at $w^{-1}y$ to the simple roots at y. Since α_i is simple at $w^{-1}y$ we know that $w\alpha_i = \gamma_i$ is simple at y as desired. We now for all positive roots γ_i for $i \in \{2, 3, 4\}$ at wx that γ_i is simple at all other vertices, and thus we can apply Lemma 2 to say that $\tilde{\phi}_{wx}$ exists as desired.

□

As in the previous chapter, we now have a potentially large class of vertices for which $\tilde{\phi}_v$ exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. In fact, we can even use the same vertices as in the previous chapter. Let $w_k = (tus)^k$ for all $k \geq 0$ and let $v_k = w_k x$. Recall in our current setup that $m(t, u) = 6$ and $m(s, u) = m(s, t) = 3$.

Lemma 5. *Let $w_k = (tus)^k$ for all $k \geq 0$ and let x be the vertex of C of type s. Then the vertices $(w_k)^{-1}x$ are all distinct, and they all lie in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ as defined previously.*

Proof. Many of the proofs will be identical to those in the proof of Lemma **??** and so work will not be repeated when unnecessary. Also note that $w_k^{-1} = (sut)^k$ for all k. We can check that $\ell((w_k)^{-1}) = 3k$ and

$\ell(t(w_k)^{-1}) = 3k+1$ by identical arguments as before. We can also check that

$$\begin{aligned} u(w_k)^{-1} &= u(sutsut\cdots) \\ &= (usu)(tsutsu\cdots) \\ &= (sus)(tsutsu\cdots) \\ &= (su)(sts)(utsuts\cdots) \\ &= (su)(tst)(utsuts\cdots) \\ &= (su)(ts)(tut)(sutsut\cdots) \end{aligned}$$

We have exhausted all possible M-Operations in $u(w_k)^{-1}$ and none of them led to a reduction in length so we can conclude that $\ell(u(w_k)^{-1}) = 3k+1$ also so that $(w_k)^{-1} \in \alpha_1 \cap \alpha_6$.

Now we do the same analysis for $st(w_k)^{-1}$ to see

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut\cdots) = (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) = (ts)(tut)(sutsut) \end{aligned}$$

and since no reductions can be performed we also get $\ell(st(w_k)^{-1}) = 3k+2$ so that $(w_k)^{-1} \in \beta$ as well. Thus each $(w_k)^{-1}x$ lies in $\mathcal{D}'$ as desired. We also know that $(w_m)^{-1}x \neq (w_n)^{-1}x$ if $m > n$ by the same argument as in Lemma **??**. $\square$

The last major step is to show that the $w_k x$ cannot somehow lie on only finitely many walls. The analysis here will be slightly more complicated, but ultimately similar to that done in the previous chapter.

Lemma 6. *Let x be the vertex of C of type s and let $w_k = (tus)^k$ for all $k \geq 0$. Any wall of Σ can contain only finitely many $w_k x$.*

Proof. By arguments identical to those in Lemma **??**, $w_m x$ and $w_n x$ will lie on the same wall if and only if x and $w_{n-m}x$ lie on the same wall. If we assume $m > n$ then it will suffice to show that a wally containing x can contain $(w_k)^{-1}x$ for only finitely many $k > 0$. Using the argument of Lemma **??** again we know that x and $(w_k)^{-1}x$ will lie on the same way if and only if $w_k t w_k^{-1}$ or $w_k u w_k^{-1}$ lies in $\mathrm{stab}_W(x) = \langle u, t\rangle$. If we recall that $m(s,t) = m(s,u) = 3$ and $m(t,u) = 6$ we can check these two conjugates we see

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \\ &= (\cdots tus)(tut)(s)(tut)(sut\cdots) \end{aligned}$$

and then we see also

$$\begin{aligned} w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(s)(tsuts\cdots) \\ &= (\cdots stu)(ststs)(uts\cdots) \\ &= (\cdots stu)(t)(uts\cdots) \\ &= (\cdots stustu)(t)(utsuts\cdots) \\ &= (\cdots stus)(tutut)(suts\cdots) \end{aligned}$$

In the first case, no reduction is possible and thus there will always be an s in any reduced word for $w_k t w_k^{-1}$ and thus $w_k t w_k^{-1} \notin \langle u, t\rangle$. In the second case, We are able to do two reductions in length but then are unable to continue. If we check the relations applied, we will se that the relations cannot continue if $k \geq 3$. For completion we will also note that $w_1 u w_1^{-1} = tst \notin \langle u, t\rangle$ but $w_2 u w_2^{-1} = tutut \in \langle u, t\rangle$. Regardless, we know that $w_m x$ and $w_n x$ cannot lie on the same wall if $|m-n| \geq 3$ so that any wall can contain only finitely many $w_k x$ as edesired.

$\square$

Now we are ready to prove the main result of the section, which is nearly identical to the proof of Theorem **??**.

Theorem 1. *Let $(G, (U_\alpha)_{\alpha \in Phi}, T)$ be an RGD system of type (W, S) with assumptions as in (**??**). Suppose that $a = m(s,t) = b = m(s,t) = 3$ and $U_x \cong G_2(2)$ where x is the vertex of C of type S. Then $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m\rangle$. Let $w_k = (tus)^k)$ for all $k \geq 0$. Now only finitely many of the vertices $w_k x$ lie on the same wall and thus we can choose k so that $v = w_k x$ does not lie on $\partial\beta_i$ for any i. By Lemma 5 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. $\square$

1.2 Finite Generation in the Exceptional Cases

Now there are two cases left to consider, and no ammount of modification to our previous strategies will work since we will see that these remaining cases are finitely generated.

For any positive root γ, we say that a chamber D borders γ if a panel of D lies on $\partial\gamma$. This allows us to define

$$d(\gamma, C) = \min_{D \text{ borders } \gamma} \{d(D, C)\}$$

It is worth noting that if $d(\gamma, C) = k$ then there is a chamber D which borders γ and $d(\gamma, C) = d(D, C)$. Furthermore, the chamber D must lie in γ since, otherwise, the chamber adjacent to D across $\partial\gamma$ would be closer to C.

We can now define $U_n = \langle U_\gamma | \gamma \in \Phi^+, d(\gamma, C) \leq n\rangle$ which is a subgroup of U_+ for all n. We also have a few facts which are immediate from the definition of U_n. We can see that $U_1 \subset U_2 \subset U_3 \subset \cdots$ and $U_+ = \cup_n U_n$ as any positive root will be some finite distance from C.

Slightly less obvious is the fact that U_n is finitely generated for all n. If $d(\gamma, C) \leq n$ then there must be a chamber D which borders γ with $d(D, C) \leq n$. There are only finitely many such chambers, and each of these chambers borders at most 3 roots, so U_n is finitely generated.

The idea of the remaining proofs will be to use the following lemma

Lemma 7. *For any positive root γ we define $d(\gamma, C) = \min\{d(D, C) | D$ has a panel on $\partial\gamma\}$. Let $U_n = \langle U_\gamma | d(\gamma, C) \leq n\rangle$ for all $n \geq 0$ where $d(\gamma, C)$. If there is some N such that $U_n \subset U_{n-1}$ for $n > N$ then U_+ is finitely generated.*

Proof. If $U_n = U_{n-1}$ for all $n > N$ then inductively we know that $U_n = U_N$ for all $n > N$. Thus

$$U_+ = \cup_{n=N}^{\infty} U_n = \cup_{n=N}^{\infty} U_N = U_N$$

which is fintely generated as desired. $\square$

By the results of Chapter **??** and the previous section, we know that the only cases remaining to consider are when W has a Coxeter diagram defined by edge labels 334, 336, or 338. The 338 case is impossible. And we have already covered the 336 case when Σ has a vertex x with $U_x \cong G_2(2)$. Thus we only need to consider when Σ has a vertex x with $U_x \cong C_2(2)$ or $G_2(3)$.

1.2.1 Case: $U_x \cong C_2(2)$

Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S) where $S = \{s, t, u\}$ and $m(s, t) = m(s, u) = 3$ and $m(t, u) = 6$. Let x be the vertex of the fundamental chamber C of type s and assume $U_x \cong C_2(2)$. We will show that U_+ is finitely generated.

Theorem 2. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ of type (W, S) as above. If x is the vertex of C of type s and $U_x \cong C_2(2)$ then $U_n \subset U_{n-1}$ for all $n > 2$ where $U_n = \langle U_\gamma | d(\gamma, C) \leq n\rangle$.*

Proof. Let γ be any positive root with $d(\gamma, C) = n > 2$. Then choose a chamber D_1 which borders γ such that $d(D_1, C) = d(\gamma, C)$. Now there is another chamber D_2 such that D_1 and D_2 are adjacent and $d(D_2, C) = d(D_1, C) - 1$. Then D_1 and D_2 will share exactly one vertex which lies on $\partial\gamma$, call it v. Recall that $\mathrm{st}(v)$ is the set of chambers of Σ for which v is a vertex. Then we have $|\mathrm{st}(v)| = 6$ or 8.

First suppose $|\mathrm{st}(v)| = 6$. In Σ, we can see that $\mathrm{st}(v)$ consists of the 6 chambers "surrounding" v which each have a vertex on v. Since we have already defined D_1 and D_2 we may label the other 4 chambers in $\mathrm{st}(v)$ as $D_3, \ldots, D_6$ by going in a circular order around v. Equivalently this means that D_i is ajacent to D_{i+1} for $1 \leq i \leq 5$ and D_6 is also adjacent to D_1. We also know that each positive root will contain exactly 3 of these chambers, and those three chambers will be D_i, D_{i+1}, and D_{i+2} for some i, where addition is done modulo 6.

By construction, D_2 and D_1 are not adjacent along $\partial\gamma$, but a panel of D_1 lies on $\partial\gamma$, and thus D_1 and D_6 must be adjacent along $\partial\gamma$. Since $D_6 \notin \gamma$, this means that γ must contain D_1, D_2, D_3. Let α and β be the other two positive roots through v. We know that $\partial\gamma$ cannot separate D_2 and D_1 or D_2 and D_3 so we can say again without loss of generality that $\partial\alpha$ separates D_2 and D_1 while $\partial\beta$ separates D_2 and D_3.

Now $D_3 \in \gamma$ but $D_4 \notin \gamma$ which means that D_3 has a panel on $\partial\gamma$. By our choice of D_1 we know that $d(D_3, C) \geq d(D_1, C) > d(D_2, C)$. But D_1 and D_3 are the two chambers adjacent to D_2 in $\mathrm{st}(v)$ and thus D_2 must be the closest chamber to C in $\mathrm{st}(v)$. But this means $D_2 = \mathrm{Proj}_v(C)$ and thus the positive roots at v which border D_2 must be the simple roots at v. These roots are α and β by construction so we know that α and β are simple at v. Since $|\mathrm{st}(v)| = 6$ we know that U_v cannot be an exceptional rank 2 RGD system and thus U_v is generated by the simple root groups through v. Thus $U_x = \langle U_\alpha, U_\beta\rangle$. But α, β border D_2 and $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $d(\alpha, C), d(\beta, C) \leq n - 1$ so that $U_\alpha, U_\beta \subset U_{n-1}$. This means $U_x \subset U_{n-1}$ as well and thus $U_\gamma \subset U_{n-1}$.

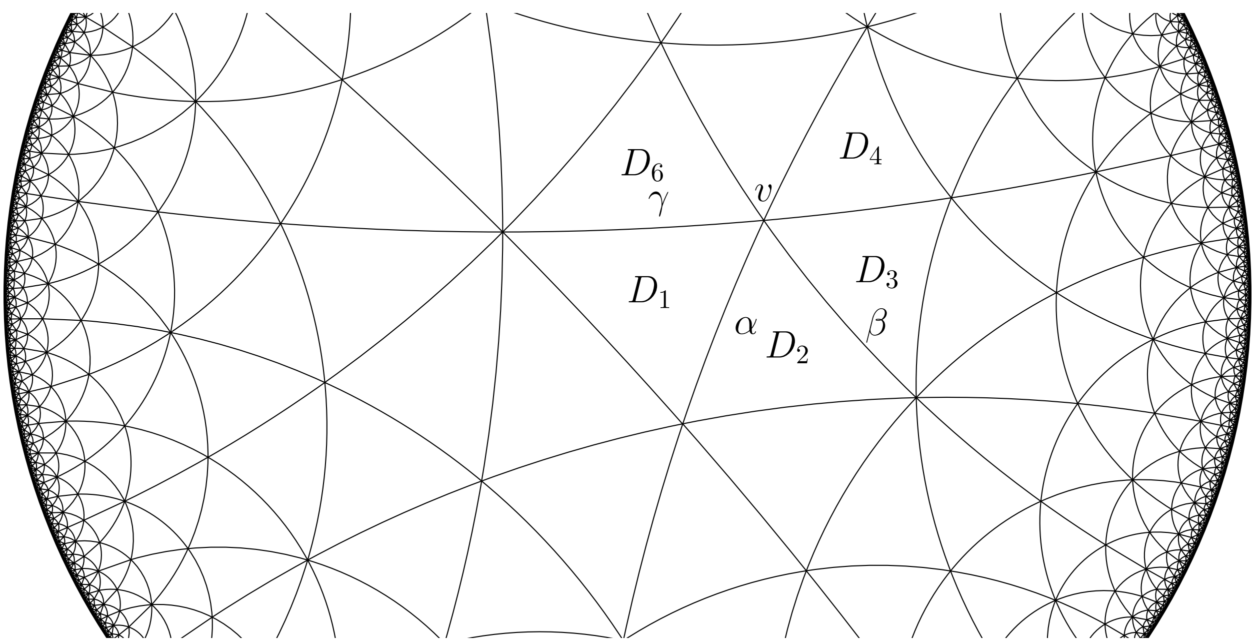

Figure 1.1: Case: $|\mathrm{st}(v)| = 6$

need to tweak this proof and make some edits

Now suppose $|\mathrm{st}(v)| = 8$. Then we will use the same labeling scheme as before except there will be 8 chambers, and each positive root will contain exactly 4 consecutive chambers from $\mathrm{st}(v)$. The same logic as before will still tell us that γ will contain exactly the chambers D_1, D_2, D_3, D_4. Our first claim is that $D_2 = \mathrm{Proj}_v(C)$.

We know that $\mathrm{Proj}_v(C)$ must lie in any positive root through v and thus it can only be D_1, D_2, D_3, D_4. We also know it is the chamber A in $\mathrm{st}(v)$ which minimizes $d(A, C)$. Since $d(D_1, C) > d(D_2, C)$ we know that D_1 cannot be the projection. By a similar argument as before we know that D_4 borders γ and thus $d(D_4, C) \geq d(D_1, C)$ by our choice of D_1. Thus D_4 cannot be the projection. Finally, if D_3 were the projection then $d(D_4, C) = d(D_3, C) + 1 < d(D_3, C) + 2 = d(D_1, C)$ which is also a contradiction and thus $D_2 = \mathrm{Proj}_v(C)$.

Let α be the positive root separating D_1 and D_2, β the positive root separating D_2 and D_3 and δ the positive root separating D_3 and D_4. Recall that γ is the positive root separating D_8 and D_1 as well as D_4 and D_5. We know that D_2 borders α and β with $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $U_\alpha, U_\beta \subset U_{n-1}$. We also know that D_2 lies in all positive roots through v by convexity so $D_2 \in \alpha, \beta, \gamma, \delta$. Since D_2 is bordered by α and β we also know that α and β are the simple roots at v.

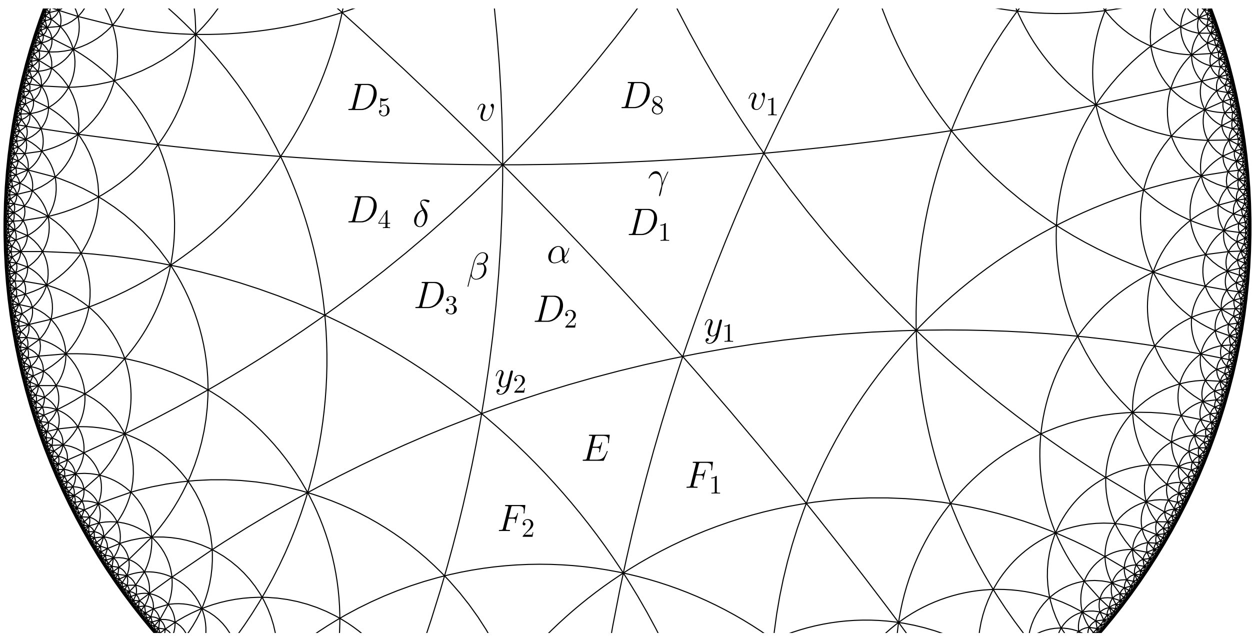

Figure 1.2: Case: $|\mathrm{st}(v) = 8|$

Let E be the third chamber adjacent to D_2. Every chamber must have an adjacent chamber which is closer to C and thus we have $d(E, C) < d(D_2, C)$. We can check that $d(E, C) = d(D_1, C) - 2 \geq 1$ by our choice of γ and thus E is not the fundamental chamber C. We know that D_1 and D_2 share two vertices, and D_2 and E share two vertices, so necessarily we have that D_1, D_2, and E must share at least one, and thus exactly one vertex, call it y_1. By a similar argument, the chambers D_3, D_2, and E will also share a vertex y_2. Let F_1 be the other chamber adjacent to E that has y_1 as a vertex, and let F_2 be the other chamber adjacent to E that has y_2 as a vertex. Note that $|\mathrm{st}(y_1)| = |\mathrm{st}(y_2)| = 6$ since v is the other vertex of D_2. The appropriate labeling can be seen in Figure 1.2.1, and the given diagram is unique up to a mirror image flip, which does not affect any of the following arguments. The labeling of these chambers could have simply been defined by the diagram, but the previous explanation seeks to convince the reader that no choices have been made and this diagram is unique.

Since $d(E, C) < d(D_2, C) < d(D_1, C)$ we know that there is some minimal gallery from D_1 to C which passes through E. If we fix such a minimal gallery we can see that it must pass through either F_1 or F_2.

First suppose that it passes through F_1. Then $d(F_1, C) = d(D_1, C) - 3$ and so F_1 and D_1 are distance 3 from one another. Since they are both in $\mathrm{st}(y_1)$, this means that D_1 and F_1 are opposite in $\mathrm{st}(y_1)$. Then there is another minimal gallery from D_1 to F_1 which does not pass through D_2 and can also be extended to a minimal gallery from D_1 to C. Let G_1 be the chamber adjacent to D_1 in this new minimal gallery. Then D_1 and G_1 have exactly two vertices in common, on of which is y_1, and the other cannot be v as this would imply $G_1 = D_2$ which contradicts our assumption. Let v_1 be the common vertex which is not y_1. We assumed that v was the unique vertex shared by D_1 and D_2 which lies on $\partial\gamma$. Since y_1 is also shared by D_1 and D_2 this means that y_1 does not lie on $\partial\gamma$. We assumed that D_1 has a panel on $\partial\gamma$ and thus it has two vertices on $\partial\gamma$ which means v_1 must lie on $\partial\gamma$.

Now we have the following situation. We still know that D_1 borders γ with $d(\gamma, C) = d(D_1, C)$ and G_1 is an adjacent chamber such that $d(G_1, C) < d(D_1, C)$. We know that v_1 is a common vertex which lies on $\partial\gamma$ and thus it is the only common vertex which lies on $\partial\gamma$. Finally, v is the uniqe vertex of D_1 with 8 chambers in its star. Thus $|st(v_1)| = 6$. Now we may apply the $|\mathrm{st}(v)| = 6$ case with G_1 as our new choice of D_2 and v_1 the new v. This shows that $U_\gamma \subset U_{n-1}$ as desired.

Now suppose the fixed minimal gallery from before passes through F_2. Then there is also a minimal gallery from D_3 to C which passes through F_2 as well. But then $d(F_2, C) = d(D_3, C) - 3$ which means F_2 and D_3 are opposite in $\mathrm{st}(y_2)$. Since D_3 borders δ, we can use similar arguments as in the previous two paragraphs to show that $U_\delta \subset U_{n-1}$. However, by Lemma **??** we know that $U_v = \langle U_\alpha, U_\beta, U_\delta\rangle$ and thus $U_\gamma \subset U_{n-1}$ as well. Thus for any root γ with $d(\gamma, C) = n \geq 3$ we have $U_\gamma \subset U_{n-1}$ and thus $U_n \subset U_{n-1}$ as desired. $\square$

Corollary 1. *Suppose $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) with $S = \{s, t, u\}$. If $m(s, t) = m(s, u) = 3$ and $U_x \cong C_2(2)$ for the vertex x of C of type s then U_+ is finitely generated.*

1.2.2 Case: 336 over $\mathbb{F}_3$

Now we consider the last exceptional case. In this section we assume $(G, (U_\alpha)_{\alpha\in Phi}, T)$ is an RGD system of type (W, S) with $S = \{s, t, u\}$. Assume that $m(s, t) = m(s, u) = 3$ and $U_x \cong G_2(3)$ where x is the vertex of the fundamental chamber C of type s. We will show that U_+ is finitely generated.