Chapter 1

Known Results on Finite Generation

We know that if $\mathcal{U}$ is the unipotent subgroup of any excpetional rank 2 Chevalley group then $\langle \mathcal{U}_\alpha, \mathcal{U}_\beta \rangle \subsetneq \mathcal{U}$ where α, β are the simple roots of the root system. However, we do have the following results about generation.

Lemma 1. *Let $\alpha, \beta, \beta+\alpha, \beta+2\alpha$ be the positive roots of a root system of type C_2 and $\mathcal{U}$ the unipotent subgroup of $C_2(\mathbb{F}_2)$. Then $\mathcal{U} = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+\alpha} \rangle = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+2\alpha} \rangle$.*

Lemma 2. *Let $\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha$ be the positive roots of a root system of type G_2 and $\mathcal{U}$ the unipotent subgroup of $G_2(\mathbb{F}_3)$. Then $\mathcal{U} = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+\alpha} \rangle = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+3\alpha} \rangle$.*

Lemma 3. *Let $\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha$ be the positive roots of a root system of type G_2 and $\mathcal{U}$ the unipotent subgroup of $G_2(\mathbb{F}_2)$. Then $\mathcal{U} = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+\alpha} \rangle$ but $\langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_{\beta+3\alpha} \rangle \subsetneq \mathcal{U}$.*

Lemma 4. *If $\mathcal{G} = \mathcal{G}_2(\mathbb{F}_2)$ then there is a unique surjective homomorphism $\phi : \mathcal{U} \to K$ where K is the cyclic group of order 2, such that $\phi(\mathcal{U}_\alpha) = \phi(\mathcal{U}_\beta) = 1$.*

Chapter 2

Conditions for Infinite Generation

2.1 Existience of $\tilde{\phi}_v$

The general heuristic in this section will be as follows. For each vertex v with an exceptional root group $\mathcal{U}_v$ then we know there is a surjective homomorphism $\phi_v : \mathcal{U}_v \to K$ where K is the cyclic group of order $|k|$. We will try to extend the map ϕ_v to a map $\tilde{\phi}_v : \mathcal{U} \to K$ in such a way that $\mathcal{U}_\gamma$ is sent to the identity for all γ which do not pass through v. This will show that any generating set for $\mathcal{U}$ must contain at least one root through v. If we can do this for enough vertices v then we can show that $\mathcal{U}$ is not finitely generated. We will proceed in showing a large set of v for which this is possible.

Our first lemma is the primary key to showing the existence of $\tilde{\phi}_v$.

Lemma 5. *Suppose v is a vertex of Σ. Also suppose that any non-simple, positive root γ at v is simple at all other vertices on $\partial\gamma$. Then $\tilde{\phi}_v$ exists and is well defined.*

Proof. Since we have a presentation, it suffices to define $\tilde{\phi}_v$ on the generators, and check that the relations are satisfied by the image. We can define $\tilde{\phi}_v$ as follows:

$$\tilde{\phi}_v(x_\gamma(u)) = \begin{cases} \phi_v(x_\gamma(u)) & \text{if } v \text{ lies on } \partial\gamma \\ 1 & \text{otherwise} \end{cases}$$

Now we just need to check the relations in $\mathcal{U}$. There are three types of relations in the presentation for $\mathcal{U}$. There are relations within the same root group so that $\mathcal{U}_\alpha \cong (k, +)$ for all positive roots α. There are also relations between root groups whose walls intersect, and those whose walls don't intersect.

Let R_α be a relation for $\mathcal{U}_\alpha$ where R_α is considered as a word with letters in $\mathcal{U}_\alpha$. If v lies on $\partial\alpha$ then $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$ since ϕ_v is a well defined homomorphism. Otherwise, every element of $\mathcal{U}_\alpha$ is sent to 1 and thus $\tilde{\phi}_v(R_\alpha) = 1$ as well so that R_α is mapped to the identity as desired.

Now suppose that α and β are any two positive roots. If $\partial\alpha \cap \partial\beta = \emptyset$ then properties of Kac-Moody groups tell us that $[\mathcal{U}_\alpha, \mathcal{U}_\beta] = 1$. Since the codomain of $\tilde{\phi}_v$ is an abelian group, then any relation of the form $[x, y] = 1$ will be satisfied by the image.

Now suppose that $\partial\alpha$ and $\partial\beta$ meet at a point y and consider any relation of the form $[x_\alpha(u), x_\beta(t)] = w$ where w is a word in $\mathcal{U}_{(\alpha,\beta)} \subset \mathcal{U}_y$. Again, note that the image of the left side of this equation will always be the identity as the codomain is still abelian. If $y = v$ then $\mathcal{U}_y = \mathcal{U}_v$ and thus $\tilde{\phi_v}(w) = \phi_v(w) = 1$ because ϕ_v is well defined.

Now suppose that $y \neq v$. Then we can label the positive roots passing through y as $\gamma_1, \cdots, \gamma_n$ in such a way that $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$ whenever $i < j$. In this case we can can say without loss of generality that $\alpha = \gamma_l$ and $\beta = \gamma_m$ with $l < m$. There can be at most one root whose wall passes through y and v, which we will call γ_k if it exists. If γ_k does not exist, or $k \leq l$ or $k \geq m$ then the root γ_k is not contained in (α, β) and thus $\tilde{\phi_v}(\mathcal{U}_\delta) = 1$ for all $\delta \in (\alpha, \beta)$. This means $\tilde{\phi_v}(w) = 1$ and the relation is satisfied.

Now we suppose that γ_k exists and $l < k < m$. Then γ_k is not simple at y and thus γ_k must be simple at v by assumption. This means $\tilde{\phi_v}(\mathcal{U}_{\gamma_k}) = \phi_v(\mathcal{U}_{\gamma_k}) = 1$ by the construction of ϕ_v. Since $\tilde{\phi_v}(\mathcal{U}_{\gamma_i}) = 1$ for all $i \neq k$ by definition, this means that $\tilde{\phi_v}(w) = 1$ showing the relation is satisfied and giving the desired result. $\square$

Now Lemma 5 gives a sufficient condition for the existence of $\tilde{\phi_v}$ which is fairly easy to check. In fact, we will use this condition to choose appropriate vertices to constuct a large class of $\tilde{\phi_v}$.

Suppose that $W = \langle s, t, u\rangle$ so that $m(s,t) = a, m(s,u) = b$, and $m(t,u) = c$. Assume that $a \leq b \leq c$ with $a, b, c \in \{3, 4, 6\}$ and furthermore assume that $c \geq 4$ if $k = \mathbb{F}_2$ and $c = 6$ if $k = \mathbb{F}_3$. Let C be the fundamental chamber of Σ and let x be the vertex of C of type s. Then by our assumptions, $\mathcal{U}_x$ is not generated by its simple roots and ϕ_x exists.

We can label the roots through x as $\alpha_1, \ldots, \alpha_n$ so that α_1 and α_n are the simple roots at x. Also note that $n = c$. The ordering on these roots is chosen so that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. This is equivalent to the condition that $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ for any $i < j$.

We can describe any root in terms of a pair of adjacent chambers. We can also identify $\mathcal{C}(\Sigma)$ with W where the chamber wC is associated to w. If we use this identification then we can describe the roots as follows

$$\begin{aligned}\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\}\\ \alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}\end{aligned}$$

In a similar way we can define two more roots

$$\begin{aligned}\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}\\ \beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}\end{aligned}$$

Now we can define $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. These roots are chosen and $\mathcal{D}$ is defined in such a way to give the following lemma:

Lemma 6. *Let $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. If γ is a non-simple, positive root with x on $\partial\gamma$, and δ is any other positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$, then $\mathcal{D} \subset \gamma \cap \delta$.*

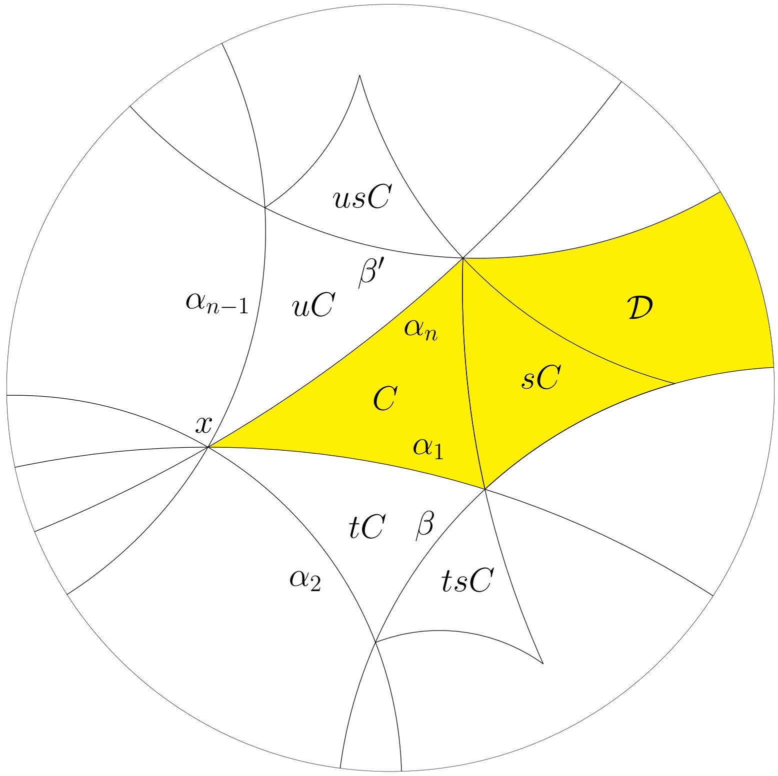

Figure 2.1: The Roots $\alpha_1, \alpha_n, \beta, \beta'$ with the region $\mathcal{D}$ in yellow.

Proof. rewrite this lemma using the better structure of Lemma 14
By assumption, γ is a positive root through x so $\gamma = \alpha_i$ for some i. Furthermore, we assumed that γ was not simple which means $2 \leq i \leq n-1$. Since α_1 and α_n are simple at x we can see that $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$. Thus it will suffice to prove that $\mathcal{D} \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$ and thus $\mathcal{D} \subset \delta$ as desired.

Now suppose $y \neq x$ and also assume that x and y are not adjacent. Suppose that $\partial\delta$ meets $\partial\alpha_1$ at a point y'. Then the points x, y, y' form a triangle, whose sides lie on the walls $\partial\gamma$, $\partial\delta$, and $\partial\alpha_1$. The triangle condition then implies that xyy' must be a chamber of Σ, which is a contradiction since x and y are not adjacent. Thus $\partial\delta$ does not meet $\partial\alpha_1$ and a similar argument shows that $\partial\delta$ does not meet α_n.

From the geometry of the Coxeter complex, we can observe that for any $1 < i < n$ we have $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = x$. Since $y \neq x$ this means that y does not lie in $\alpha_1 \cap \alpha_n$. We can assume without loss of generality that y does not lie in α_1. We know that α_1 and δ are two positive roots whose walls do not meet, and thus there are exactly three possibilites. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$ or $-\delta \subset \alpha_1$. The later two cases are impossible as both would imply that $y \in \alpha_1$ which contradicts our assumption. Thus we have $\alpha_1 \subset \delta$ which gives $\mathcal{D} \subset \alpha_1 \subset \delta$ as desired.

Now we suppose again that $y \neq x$ but x and y are adjacent. Then once again there are

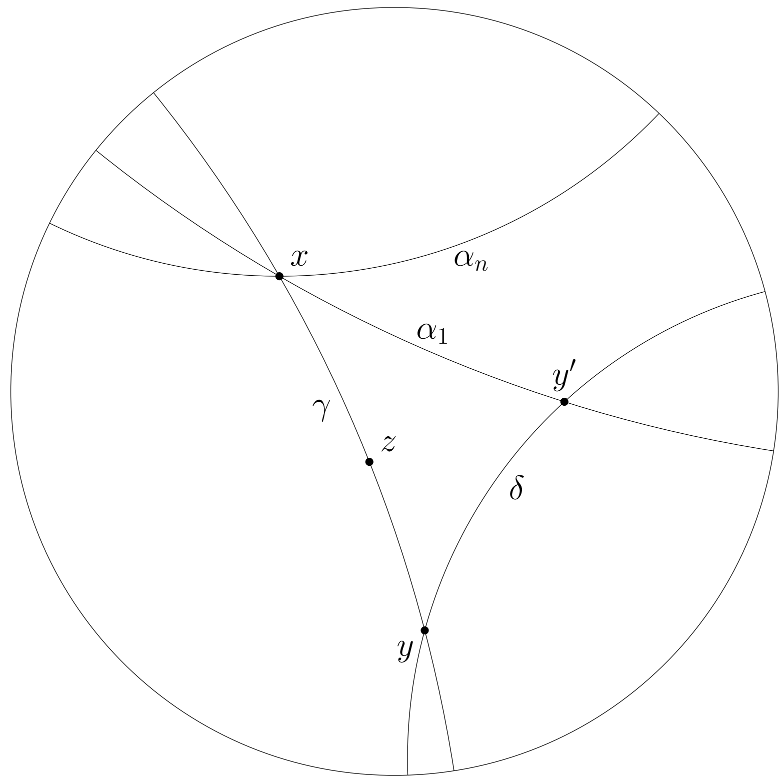

Figure 2.2: The point y' cannot exist as it would form a triangle which is not a chamber.

two possibilities. If $\partial\delta$ does not meet $\partial\alpha_1$ or $\partial\alpha_n$ then the identical argument from the last paragraph shows that $\mathcal{D} \subset \delta$.

So now we suppose that $\partial\delta$ does meet $\partial\alpha_1$ or $\partial\alpha_n$. If $\partial\delta$ meets $\partial\alpha_1$ at a point y', then the vertices xyy' will form a triangle which must be a chamber call it C'. This chamber contains, x, and has 2 sides on $\partial\alpha_1$ and $\partial\gamma$ respectively. Since γ is not simple, an observation of the chambers around x in Figure 2.1 shows that γ must be α_2 C' must be tC in which case $\delta = \beta$ by definition. Thus we get $\mathcal{D} \subset \beta = \gamma$ which proves the result.

If $\partial\delta$ meets α_n then an identical argument shows that $\gamma = \beta'$ which also proves the result.

□

The next two lemmas will allow us to construct new $\tilde{\phi_v}$ for vertices v based on the region $\mathcal{D}$.

Lemma 7. *Suppose γ is any positive root with $x \in \partial\gamma$, and y is another vertex on $\partial\gamma$. Then γ is simple at y.*

Proof. Suppose that γ is not simple at y. Then we can label the positive roots at y as $\delta_1, \ldots, \delta_m$ in such a way that $\delta_i \cap \delta_j \subset \delta_k$ for $1 \leq i \leq k \leq j$. In this case we have δ_1, δ_m are simple at y and $\gamma = \delta_r$ for some $1 < r < m$. But x is a vertex of C and $C \in \delta_1 \cap \delta_m$ and thus $x \in \delta_1 \cap \delta_m$ as well. We know that x lies on $\partial\delta_r$ by assumption and thus x is an element of $\partial\delta_r \cap \delta_1 \cap \delta_m$. But this is impossible as we can observe from the geometry of Σ that $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$ for all $1 < i < m$. Thus γ is simple at y as desired. □

Despite some of the technical details the previous result should be intuitively clear. The walls through y will divide Σ into $2m$ regions, and the region which contains C will be bounded by the two simple roots. Since x lies on $\partial\gamma$, it is impossible for any other roots through y to be any "closer" to C and thus γ must be simple at y as we proved.

Corollary 1. *$\tilde{\phi}_x$ exists.*

Proof. Let γ be any non-simple, positive root through x and let y be another vertex on $\partial\gamma$. Then by the previous lemma, γ is simple at y and thus $\tilde{\phi}_x$ exists by Lemma 5. □

We are now ready to construct a large family of vertices $\{v\}$ for which $\tilde{\phi}_v$ will exist. The idea is as follows. If we take any chamber in $\mathcal{D}$ and treat it as a new "C" then $\tilde{\phi}_x$ would exist for this "C." So what we do is apply elements of W which map the chambers of $\mathcal{D}$ to C, and use these choices of w to get new vertices v.

Since the construction of these $\tilde{\phi}_v$ depends on properties of simple roots, we want to know the simplicity behaves nicely with the action of W to this end we have the following lemma.

Lemma 8. *Suppose v is a vertex with simple roots γ, γ' at v. If w is an element of w such that $w\delta$ is a positive root for all positive δ at v, then $w\gamma$ and $w\gamma'$ are the simple roots at wv.*

Proof. Let δ be a positive root at wv. Since w induces an isomorphism of simplical complexes, and it sends positive roots at v to positive roots at wv, it must also send negative roots at v to negative roots at wv. So $w^{-1}\delta$ is a root at v, and $w(w^{-1}\delta) = \delta$ is positive, so $w^{-1}\delta$ is also positive. Thus by definition of simple, we have $\gamma \cap \gamma' \subset w^{-1}\delta$. But we can now apply w to get $w\gamma \cap w\gamma' \subset \delta$. Since the choice of δ was arbitrary we must have $w\gamma$ and $w\gamma'$ are simple as desired. □

We can now use the previous lemma to actually construct $\tilde{\phi}_v$ for a certain collection of vertices v.

Lemma 9. *If $v = w^{-1}x \in \mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ then $\tilde{\phi}_v = \tilde{\phi_{wx}}$ exists.*

Proof. Let $D = \mathrm{Proj}_{w^{-1}x}(C)$ and let $D = (w')^{-1}C$. By the definition of projections, $w^{-1}x$ is a vertex of D of type s, but $(w')^{-1}x$ is also a vertex of D of type s, and thus $(w')^{-1}x = w^{-1}x$. Now without loss of generality we may assume that $w' = w$. Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, if γ is any positive root through wx then $E \in \gamma$ by the properties of projections.

Now suppose that γ is a non-simple, positive root through wx and y is another vertex on $\partial\gamma$. We must show that γ is simple at y. Since γ is positive through wx we know that $C, E \in \gamma$.

If we apply w^{-1} then we get the following facts. We know that $w^{-1}\gamma$ is a root such that $\partial(w^{-1}\gamma)$ passes through $w^{-1}wx = x$. We also know that $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is also a positive root.

The first claim is that $w^{-1}\gamma$ is not simple at x. Suppose that δ is any positive root at wx. Then $E \subset \delta$ and so applying w^{-1} we get that $w^{-1}E = C \subset w^{-1}\delta$. Thus w^{-1} sends positive roots at wx to positive roots at x. By Lemma 8 this means that w^{-1} sends simple roots at wx to simple roots at x. Since γ is not simple at wx this means that $w^{-1}\gamma$ is not simple at x.

So $w^{-1}\gamma$ is a non-simple positive root at x, and since y lies on $\partial\gamma$ we also know that $w^{-1}y$ lies on $w^{-1}(\partial\gamma)$. If we apply Lemma 7 we can see that $w^{-1}\gamma$ must be simple at $w^{-1}y$.

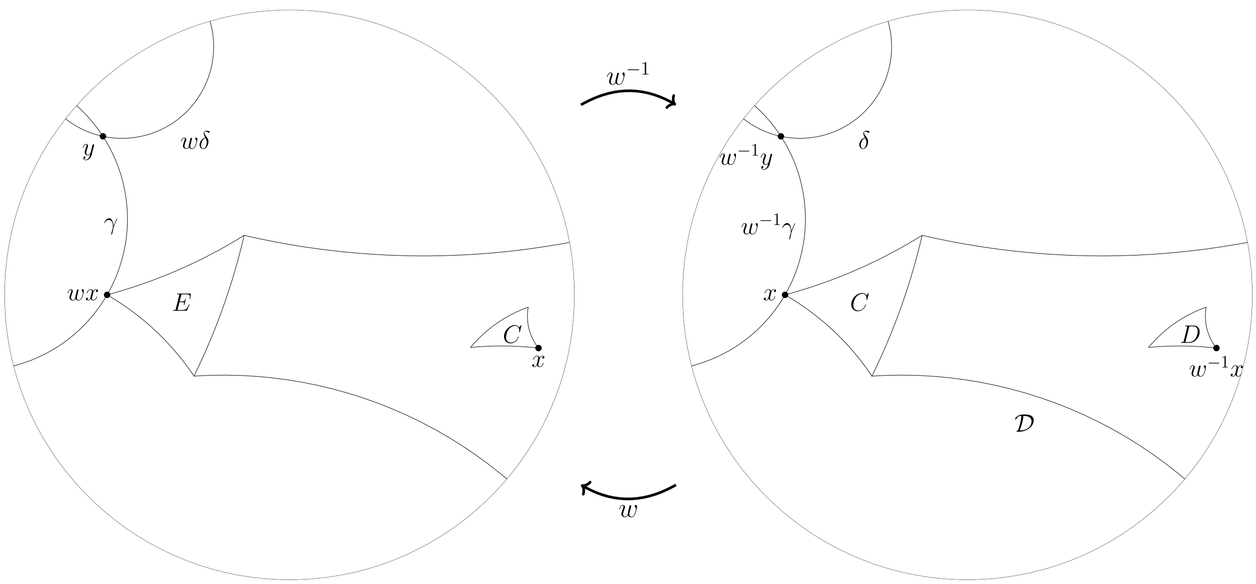

Figure 2.3: The effect of w and w^{-1} on the chambers and roots.

Now suppose that δ is any positive root at $w^{-1}y$. Then by Lemma 6 we know that $D \in \mathcal{D} \subset \delta$. If we apply w then we get $C = wD \in w\delta$ and $w\delta$ is a root through y. Thus $w\delta$ is a positive root through y and therefore w sends positive roots through $w^{-1}y$ to positive roots through y. Again we can apply Lemma 8 to say that w must also send simple roots through $w^{-1}y$ to simple roots through y. But $w^{-1}\gamma$ was a simple root through $w^{-1}y$ and thus γ is simple at y as desired. $\square$

Now we have shows that vertices of $\mathcal{D}$ in some way correspond to $\tilde{\phi_v}$. If our goal is to find infinitely many such v then there is still some work to be done. For instance, we do not yet know if the region $\mathcal{D}$ contains infinitely many chambers, or even if it does, if all the vertices of D lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

2.2 When $\mathcal{D}$ is infinite

Our first task will be two show that the region $\mathcal{D}$ contains infinitely many unique vertices. Intuitively, this will happen if the walls for β and β' do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that W is the Weyl group of $\mathcal{G}$ and W is defined by the edge labels $a = m(s,t), b = m(s,u), c = m(t,u)$ with $a \leq b \leq c$. For the remainder of the section we will also add the assumption that $b \geq 4$. Let $w_k = (sut)^k$ and $v_k = w_k x$. We will see in the following that with this additional assumption on b that the vertices v_k will be sufficiently many to prove $\mathcal{U}$ is not finitely generated.

Lemma 10. *The vertices v_k are distinct for all k and they all lie in $\mathcal{D}$.*

Proof. First we will show that $v_k \in \mathcal{D}$ for all k. Since x is a vertes of C we know that v_k is a vertex of $w_k C$ and thus it will suffice to show $w_k C$ is contained in $\mathcal{D}$ for all k. Since the roots $\alpha_1, \alpha_n, \beta, \beta'$ can be identified with their corresponding subsets of W, we can use the length function to check containment in these roots.

Recall that words in a Coxeter group can only be reduced by removal of consecutive repeated letters, or application of the Coxeter relations. It is immediate from the definition that $\ell(w_k) = 3k$ for all k. We can also see that $\ell(tw_k) = 3k+1$ and thus $w_k \in \alpha_1$ for all k. Similarly, $uw_k = u(sutsut\cdots)$, and no reduction operations can be done as we assumed $m(s,u) \geq 4$. Thus $\ell(uw_k) = 3k+1$ which means $w_k \in \alpha_n$ as well.

Now consider the element stw_k. If we write this element out in terms of the generators and apply the only possible Coxeter relations we get

$$\begin{aligned} stw_k &= st(sutsut\cdots) \\ &= (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) \\ &= (ts)(tut)(sutsut\cdots) \end{aligned}$$

and none of these can be reduced as $m(t,u) \geq 4$. Note that the commutation relation $sts = tst$ may not be possible if $m(s,t) \geq 4$, but it is the only relation possible in stw_k and even if it does exists then it does not allow stw_k to be reduced in length. We previously showed $\ell(tw_k) = 3k+1$ and now we see $\ell(stw_k) = 3k+2$ and so $w_k \in \beta$.

Now we can consider suw_k in a similar manner. Writing suw_k out as a word in the generations and applying Coxeter relations gives us

$$\begin{aligned} suw_k &= su(sutsut\cdots) \\ &= (susu)(tsutsu\cdots) \\ &= (usus)(tsutsu\cdots) \\ &= (usu)(sts)(utsuts\cdots) \\ &= (usu)(tst)(utsuts\cdots) \end{aligned}$$

Note once again that not all of these relations may be possible if $m(s,u)=6$ or $m(s,t)\geq 4$. However, these are the only possible relations, and since suw_k cannot be reduced under these assumptions, it cannot be reduced at all. Thus $\ell(suw_k)=3k+2$ which means $suw_k \in \beta'$ as well.

Now it only remains to show that each v_k is unique. Suppose $v_m = v_n$ for $m>n$. Then we would have $w_m x = w_n x$ and thus $w_n^{-1} w_m x = w_{m-n} x = x$. Thus it will suffice to show $w_k x \neq x$ for any $k>1$. But we know that $\mathrm{stab}(x)=\langle u,t\rangle$ which does not contain w_k for any $k\geq 1$ and thus $w_k x \neq x$ as desired.

□

We now know that each of the v_k is distinct and each of them lies in $\mathcal{D}$. By Lemma 9 we know that $\tilde{\phi}'_v$ exists for each $v' = w_k^{-1}x$. Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the v' at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices $w_n^{-1}x$ distinct, but also no two lie on the same wall.

Lemma 11. *If $m\neq n$ then $w_m^{-1}x$ and $w_n^{-1}x$ do not lie on the same wall.*

Proof. Suppose $w_m^{-1}x$ and $w_n^{-1}x$ do lie on the same wall with $m>n$. Then we also know that $w_m w_n^{-1} x = w_{m-n} x$ and x will lie on the same wall and thus it will suffice to show that $w_k x$ and x do not lie on the same wall for any $k\geq 1$.

We know from Lemma 10 that $w_k \in \mathcal{D}$. Thus if $w_k x$ and x lie on the same wall, it must be a wall through x, and since $w_k x \in \mathcal{D}$ this wall must be either H_u or H_t, the fixed points of u and t respectively. Thus we either have $uw_k x = w_k x$ or $tw_k x = w_k x$ which implies that either $w_k^{-1} u w_k$ or $w_k^{-1} t w_k$ is contained in $\mathrm{stab}(s)=\langle u,t\rangle$. However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced. The only possible relations we have are

$$\begin{aligned} w_n^{-1} t w_n &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \end{aligned}$$

or

$$\begin{aligned} w_n^{-1} u w_n &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(sus)(tsuts\cdots) \\ &= (\cdots stu)(sts)u(sts)(uts\cdots) \\ &= (\cdots stu)(tst)u(tst)(uts\cdots) \end{aligned}$$

In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in $\langle u,t\rangle$. Thus the $w_n^{-1}x$ all lie on distinct walls as desired. □

We now have all the ingredients and are ready to prove the main theorem.

Theorem 1. *Let $\mathcal{G}$ be a Kac-Moody group over $\mathbb{F}_2$ or $\mathbb{F}_3$ with unipotent subgroup $\mathcal{U}$. If let W be the Weyl group of $\mathcal{G}$ with coxeter diagram defined by the edge labels $a \leq b \leq c$ with $a, b, c \in \{3, 4, 6\}$. If $b \geq 4$ and $c = 6$ over $\mathbb{F}_2$ then $\mathcal{U}$ is not finitely generated.*

Proof. Suppose that $\mathcal{U}$ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $\mathcal{U} = \langle \mathcal{U}_{\beta_i} | 1 \leq i \leq m \rangle$. Now no two of the vertices $w_k^{-1}x$ lie on the same wall and thus we can choose k so that $v = w_k^{-1}$ does not lie on $\partial\beta_i$ for any i. By Lemma 10 we know that $\tilde{\phi_v}$ exists, and by definition it is a surjective map from $\mathcal{U} \to C$. However, we can also see by definition that $\tilde{\phi_v}(\mathcal{U}_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi_v}$ sends all of the generators of $\mathcal{U}$ to the identity and thus it must be the trivial map which is a contradiction. Thus $\mathcal{U}$ is not finitely generated as desired. $\square$

Chapter 3

Exceptional Cases

In the previous chapter we were able to show that $\mathcal{U}$ is not finitely generated for a large family of Weyl groups W with labels $a \leq b \leq c$. These results were based on assuming $b \geq 4$ which allowed us to show that $\mathcal{D}$ was infinite and proceed from there. In fact, we didn't even describe all of the chambers in $\mathcal{D}$, just an infinite family. However, the same approach will not work in the remaining cases because of the following lemma.

Lemma 12. *If W is the Weyl group of $\mathcal{G}$ with labels $a \leq b \leq c$ as before, then $\mathcal{D}$ is infinite if and only if $b \geq 4$.*

Proof. We know by Lemma 10 that $\mathcal{D}$ is infinite if $b \geq 4$. Thus it remains to show that $\mathcal{D}$ is finite if $b = 3$. If $b = 3$ then $a = 3$ also, and by definition of a, b, c this means $m(s,t) = m(s,u) = 3$. We will also recall the definition of $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Let $w \in W$ and suppose $\ell(w) \geq 2$. Then we can write $w = s_1 s_2 w'$ where $\ell(w') = \ell(w) - 2$. If $s_1 = t$ then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows $w \notin \alpha_1$ and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_1 = u$.

Now we assume $s_1 = s$ and so we can also assume $s_2 = t, u$. First let $s_2 = t$ so that $w = stw'$. If $w \notin \alpha_1$ then $w \notin \mathcal{D}$ and so wewill suppose $w \in \alpha_1$. Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \leq \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_2 = u$.

We have shown that if $\ell(w) \geq 2$ then $w \notin \mathcal{D}$ and thus $\mathcal{D}$ must be finite as desired. In fact, if $a = b = 3$ then we can check relatively easily that $\mathcal{D} = \{C, sC\}$ which proves the desired result. $\square$

The previous lemma shows that generating results for the remaining cases is not just a matter of being slightly more clever when we look for vertices, but changing the strategy as a whole. In fact, in some cases our proof strategy needs to switch entirely since $\mathcal{U}$ will be finitely generated in some cases as we will see. First, we will show which of the remaining cases are not finitely generated.

3.1 Case: 336 over $\mathbb{F}_2$

We saw in the previous chapter that a vertex contained in $\mathcal{D}$ was a sufficient condition to construct a corresponding map $\tilde{\phi}_v$. However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct $\tilde{\phi}_v$ for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statment similar to Lemma 5.

For the remainder of the section, $\mathcal{G}$ will be the Kac-Moody group over $\mathbb{F}_2$ with Weyl group defined by a the 336 Coxeter diagram, and $\mathcal{U}$ will be its unipotent subgroup. Then for any positive root α of Σ, we know that $\mathcal{U}_\alpha \cong (\mathbb{F}_2, +)$ and thus each $\mathcal{U}_\alpha$ is a cyclic group of order 2. This means we can let u_α be the non-identity element of $\mathcal{U}_\alpha$ for all $\alpha \in \Phi^+$. Then we know that $\mathcal{U}$ is generated by $\{u_\alpha\}$ for all $\alpha \in \Phi^+$ and there are exactly 3 types of relations:

$$\begin{aligned}
u_\alpha^2 &= 1 && \text{For all } \alpha \in \Phi^+ \\
[u_\alpha, u_\beta] &= 1 && \text{if } \partial\alpha \cap \partial\beta = \emptyset \\
[u_\alpha, u_\beta] &= w && \text{where } w \text{ is a word in } \mathcal{U}_{(\alpha,\beta)} \subset \mathcal{U}_y \text{ where } y = \partial\alpha \cap \partial\beta
\end{aligned}$$

Let v be any vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$. Then we showed previouly that there is a map $\phi_v : \mathcal{U}_v \to K$ where K is a cyclic group of order 2. If we label the positive roots through v as $\gamma_1, \ldots, \gamma_6$ with $\gamma_i \cap \gamma_j \subset \gamma_k$ for $1 \le i \le k \le j \le 6$, then we also know that at least one of $\mathcal{U}_{\gamma_2}$ or $\mathcal{U}_{\gamma_5}$ must be sent to the identity by ϕ_v. By reversal of the numbering, we can assume without loss of generality that $\phi(\mathcal{U}_{\gamma_5}) = 1$. As in the previous chapter we want to define an extension of ϕ_v to a map $\tilde{\phi}_v : \mathcal{U} \to K$. We define this extension by

$$\tilde{\phi}_v(u_\alpha) = \begin{cases} \phi_v(u_\alpha) & \text{if } v \text{ lies on } \partial\alpha \\ 1 & \text{otherwise} \end{cases}$$

Since we have defined $\tilde{\phi}_v$ for all generators, to check it is well defined is a matter of checking the relations in our presentation.To this end we have to following lemma.

Lemma 13. *Let v be a vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$, and let $\gamma_1, \ldots, \gamma_6$ be the positive roots through v, labeled as before. Also suppose that $\phi_v(\mathcal{U}_{\gamma_5}) = 1$. If $\gamma_2, \gamma_3,$ and γ_4 are simple at all other vertices they meet, then $\tilde{\phi}_v$ exists.*

Proof. To check $\tilde{\phi}_v$ is well defined is a matter of checking the relations are satisfied by the images under $\tilde{\phi}_v$. Since $\tilde{\phi}_v$ has a cyclic group of order 2 as its codomain, we can see immediately that the first two types of relations will be satisfied regardless of α and β. Now to check the third type.

Suppose α and β are any two positive roots with $y = \partial\alpha \cap \partial\beta$. Since $[u_\alpha, u_\beta]$ must be mapped to the identity then we just need to check that w is also mapped to the identity. If $y = v$ then u_α, u_β, w all lie in $\mathcal{U}_v$ and $\tilde{\phi_v}(w) = \phi_v(w)$ which must be the identity because ϕ_v is a well defined homomorphism.

Now suppose $y \neq v$. Let $\delta_1, \ldots, \delta_n$ be the positive roots through y, labeled as normal, and assume that $\alpha = \delta_i$ and $\beta = \delta_j$ with $i < j$. There is at most one positive root whose wall can pass through both v and y, call it δ_k if it exists. If δ_k does not exist, then no positive roots through y pass through v and so $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all m. Thus $\tilde{\phi_v}(w) = 1$ as desired.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{1, 5, 6\}$. Then we know $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all $m \neq k$ and $\tilde{\phi_v}(u_{\delta_k}) = \tilde{\phi_v}(u_{\gamma_r}) = \phi_v(u_{\gamma_r}) = 1$ by the construction of ϕ_v. Thus $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all m and so $\tilde{\phi_v}(w) = 1$ as well.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{2, 3, 4\}$. Then by assumption, δ_k is simple at y and thus $k = 1, n$. Thus $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all $2 \leq m \leq n-1$. But w is a word in $\mathcal{U}_{(\alpha,\beta)} \subset \mathcal{U}_{(\delta_2,\delta_{n-1})}$ and thus $\tilde{\phi_v}(w) = 1$ again, which gives the result. □

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma 5, and so we have a hope of constructing more $\tilde{\phi_v}$ then the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Let x be the vertex of C of type s as in the previous chapter and let $\alpha_1, \ldots, \alpha_6$ be the positive roots through x, labeled as usual. Also assume without loss of generality that $\phi_x(u_{\alpha_5}) = 1$. Now let $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ where β is defined as in the previous chapter. We can now prove a lemma similar to Lemma 6.

picture of $\mathcal{D}'$

Lemma 14. *Suppose $\gamma = \alpha_i$ for $i \in \{2, 3, 4\}$. If δ is any positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$ then $\mathcal{D}' \subset \gamma \cap \delta$.*

Proof. By assumption, γ is a positive root through x and thus we have $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$. Thus it remains to show that $\mathcal{D}' \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a positive root through x and so $\mathcal{D}' \subset \delta$ as desired. Now suppose $y \neq x$. Then there are two cases to consdier. First suppose that $\partial\delta$ does not meet $\partial\alpha_1$ or $\partial\alpha_6$. Then arguments identical to those made in Lemma 6 show that $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \delta$ as desired.

Now suppose that $\partial\delta$ does meet α_1 or α_6 at a point y' which cannot be x as $\delta \neq \gamma$. Then the vertices x, y, y' form a triangle which must be a chamber, call it C', by the triangle condition. This chamber will have a vertex of x and a vertex on $\partial\alpha_1$ or $\partial\alpha_6$ and thus C' is either C, tC, or uC. But none of the vertices of C or uC lie on $\partial\alpha_i$ for $2 \leq i \leq 4$ and thus C' must be tC. But then $\gamma = \alpha_2$ and $\delta = \beta$ by definition and thus $\mathcal{D}' \subset \beta' = \delta$ as desired. □

The proofs in the previous chapter relied heavily on facts about simple roots, and to aid these proofs we had Lemma 8 which shows the W action on Σ preserves simplicity under certain condidtions. Now that we are dealing more than just simple roots we need to extend this lemma to the current context.

Lemma 15. *Suppose v is a vertex of Σ and $w \in W$ such that $w\gamma$ is a positive root at wv for all positive roots γ at v. If δ is a positive root at v such that $\phi_v(u_\delta) = 1$ then $\phi_{wv}(u_{w\delta}) = 1$ as well.*

Proof. We know from the theory of Moufang twin buildings that there is some $\tilde{w} \in \mathrm{Aut}(\Delta)$ such that $\tilde{w}\mathcal{U}_\alpha\tilde{w}^{-1} = \mathcal{U}_{w\alpha}$ for all roots $\alpha \in \Phi$. Let $\psi_w : \mathcal{G} \to \mathcal{G}$ be the conjugation isomorphism defined by $\tilde{w}$. For any positive root γ at v, we know $w\gamma$ is positive at wv by assumption, and thus $\psi_w(u_\gamma) = u_{w\gamma} \in \mathcal{U}_{w\gamma}$. Thus the map ψ_w restricts to a map from $\mathcal{U}_v$ to $\mathcal{U}_{wv}$ which is necessarily injective. Now suppose γ' is a positive root at wv. There are only finitely many roots at v and wv, and since w sends positive roots to positive roots, it must also send negative roots to negative roots. Thus w^{-1} must also send positive roots at wv to positive roots at v. Thus $w^{-1}\gamma'$ is a positive root at v. Thus $\psi_w(u_{w^{-1}\gamma'}) = \gamma'$ which means $\psi_w : \mathcal{U}_v \to \mathcal{U}_{wv}$ is surjective and thus an isomorphism.

Now consider the map $f = \phi_{wv}\psi_w : \mathcal{U}_v \to K$. We know ψ_w is an isomorphism, and ϕ_{wv} is surjective and thus f is surjective. By Lemma 8 we know that if γ is simple at wv then $w^{-1}\gamma$ is simple at v. and $f(u_{w^{-1}\gamma}) = \phi_{wv}(\gamma) = 1$ by the definition of ϕ_{wv}. By Lemma 4 this means that $f = \phi_v$ and the conclusion follows as $\phi_{wv} = \phi_v\psi_w^{-1}$. ☐

Another way of viewing this lemma is as follows. The local homomorphisms ϕ_v assign the two simple roots at v to the short and long roots of a root system of type G_2, depeding on which other roots are sent to the identity. We cannot tell just from the information of the Coxeter complex which way this assignment will be. However, we have just proved that the W action respects this assignment. We have essentially proved that if α is a long root at v, then under suitable conditions, $w\alpha$ is a long root at wv. We are now prepared to prove a new result corresponding to Lemma 9.

Lemma 16. *If $v = w^{-1}x \in \mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ then $\tilde{\phi_{wx}}$ exists.*

Proof. The proof will proceed in a manner very similar to the proof of Lemma 9. Let $D = \mathrm{Proj}_{w^{-1}x}(C)$ and let $D = (w')^{-1}C$. By the definition of projections, $w^{-1}x$ is a vertex of D of type s, but $(w')^{-1}x$ is also a vertex of D of type s, and thus $(w')^{-1}x = w^{-1}x$. Now without loss of generality we may assume that $w' = w$. Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx}wD = \mathrm{Proj}_{wx}C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)}wC = \mathrm{Proj}_x E$$

In particular, if γ is any positive root through wx then $E \in \gamma$ by the properties of projections.

Recall that the positive roots through x are $\alpha_1, \ldots, \alpha_6$ and we assumed that $\phi_x(u_{\alpha_5}) = 1$. For any posititive root through x, say α_i, we know that $D \in \alpha_i$ and thus $C = wD \in w\alpha_i$. We also know $w\alpha_i$ will be a root through wx and thus $w\alpha_i$ is a positive root through x. Since w sends positive roots at x to positive roots at wx we can use Lemma 8 and Lemma 15.

Now we can label the positive roots at wx as $\gamma_1, \ldots, \gamma_6$ in such a way that $\gamma_i = w\alpha_i$ for all i. We need to check that this labeling satisfies all of the properties we normally use for labeling the positive roots through a vertex. If $1 \leq i \leq k \leq j \leq 6$ then we know $\alpha_i \cap \alpha_j \subset \alpha_k$ and thus $w\alpha_i \cap w\alpha_j \subset w\alpha_k$ which shows $(\gamma_i, \gamma_j) = \{\gamma_k | i < k < j\}$ as desired. We also know by Lemma 15 that $\phi_{wx}(u_{\gamma_5}) = 1$.

Now we can try to apply Lemma 13 to show $\tilde{\phi_{wx}}$ exists. Consider γ_i for $2 \leq i \leq 4$. Let $y \neq wx$ be any other vertex on $\partial\gamma_i$. If we apply w^{-1} we get that $w^{-1}y \neq x$ is a vertex on α_i and thus α_i is simple at $w^{-1}y$ by Lemma **??**. Now suppose δ is any positive root at $w^{-1}y$. Then $D \in \mathcal{D}' \subset \delta$ by Lemma 14 and so $C, D \in \delta$. But this means that $E, C \in w\delta$ and thus $w\delta$ is a positive root at y. So w sends positive roots at $w^{-1}y$ to positive roots at y, and so by Lemma 8 it must also send simple roots at $w^{-1}y$ to simple roots at y. Since α_i is simple at $w^{-1}y$ then γ_i is simple at y as desired, and $\tilde{\phi_{wx}}$ exists by Lemma 13.

□

As in the previous chapter, we now have a potentially large class of vertices for which $\tilde{\phi}_v$ exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. In fact, we can even use the same vertices as in the previous chapter. Let $w_k = (sut)^k$ for all $k \geq 0$ and let $v_k = w_k x$. Recall in our current setup that $m(t, u) = 6$ and $m(s, u) = m(s, t) = 3$.

Lemma 17. *The vertices v_k are all distinct and all lie in $\mathcal{D}'$.*

Proof. Many of the proofs will be identical to those in the proof of Lemma 10 and so work will not be repeated when unnecessary. We can check that $\ell(w_k) = 3k$ and $\ell(tw_k) = 3k + 1$ by identical arguments as before. We can also check that

$$\begin{aligned} uw_k &= u(sutsut\cdots) \\ &= (usu)(tsutsu\cdots) \\ &= (sus)(tsutsu\cdots) \\ &= (su)(sts)(utsuts\cdots) \\ &= (su)(tst)(utsuts\cdots) \\ &= (su)(ts)(tut)(sutsut\cdots) \end{aligned}$$

We have exhausted all possible Coxeter relations in uw_k and none of them led to a reduction in length so we can conclude that $\ell(uw_k) = 3k + 1$ also so that $w_k \in \alpha_1 \cap \alpha_6$.

Now we do the same analysis for stw_k to see

$$\begin{aligned} stw_k = st(sutsut\cdots) &= (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) = (ts)(tut)(sutsut) \end{aligned}$$

and since no reductions can be performed we also get $\ell(stw_k) = 3k + 2$ so that $w_k \in \beta$ as well. Thus each v_k lies in $\mathcal{D}'$ as desired. Each v_k is unique by an identical argument as in Lemma 10. $\square$

The last major step is to show that the $w_k^{-1}x$ cannot somehow lie on only finitely many walls. The analysis here will be slightly more complicated, but ultimately similar to that done in the previous chapter.

Lemma 18. *Any wall of* Σ *can contain only finitely many* $w_k^{-1}x$.

Proof. By arguments identical to those before, $w_m^{-1}x$ and $w_n^{-1}x$ will lie on the same wall if and only if x and v_k lie on the same wall for some $k \geq 0$, and this will only happen if and only if either $w_k^{-1}uw_k$ or $w_k^{-1}tw_k$ lies in $\langle u, t\rangle$. We will again apply the Coxter relations to show this is impossible for infinitely many k. First we check

$$\begin{aligned} w_k^{-1}tw_k &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \\ &= (\cdots tus)(tut)(s)(tut)(sut\cdots) \end{aligned}$$

and the we see also

$$\begin{aligned} w_k^{-1}uw_k &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(s)(tsuts\cdots) \\ &= (\cdots stu)(ststs)(uts\cdots) \\ &= (\cdots stu)(t)(uts\cdots) \\ &= (\cdots stustu)(t)(utsuts\cdots) \\ &= (\cdots stus)(tutut)(suts\cdots) \end{aligned}$$

Now in the second case we were able to do some reductions so it is possible that $w_k^{-1}uw_k \in \langle s, t\rangle$ for small k, but as long as k is large enough, say $k \geq 3$ then this is no longer a possibility as we showed no further reductions are possible. Thus $w_m^{-1}x$ and $w_n^{-1}x$ can only lie on the same wall if $|n - m| \leq 3$.

$\square$

Now we are ready to prove the main result of the section, which is nearly identical to the proof of Theorem 1.

Theorem 2. *Let* $\mathcal{G}$ *be the Kac-Moody group over* $\mathbb{F}_2$ *with Weyl group defined by the edge labels* $3, 3, 6$*. Then* $\mathcal{U}$ *is not finitely generated.*

Proof. Suppose that $\mathcal{U}$ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $\mathcal{U} = \langle \mathcal{U}_{\beta_i} | 1 \leq i \leq m \rangle$. Now only finitely many of the vertices $w_k^{-1}x$ lie on the same wall and thus we can choose k so that $v = w_k^{-1}x$ does not lie on $\partial\beta_i$ for any i. By Lemma 17 we know that $\tilde{\phi_v}$ exists, and by definition it is a surjective map from $\mathcal{U} \to C$. However, we can also see by definition that $\tilde{\phi_v}(\mathcal{U}_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi_v}$ sends all of the generators of $\mathcal{U}$ to the identity and thus it must be the trivial map which is a contradiction. Thus $\mathcal{U}$ is not finitely generated as desired. □

3.2 Finite Generation in the Exceptional Cases

Now there are two cases left to consider, and no ammount of modification to our previous strategies will work since we will see that these remaining cases are finitely generated.

For any positive root γ, we say that a chamber D borders γ if a panel of D lies on $\partial\gamma$. This allows us to define

$$d(\gamma, C) = \min_{D \text{ borders } \gamma} \{d(D, C)\}$$

It is worth noting that if $d(\gamma, C) = k$ then there is a chamber D which borders γ and $d(\gamma, C) = d(D, C)$. Furthermore, the chamber D must lie in γ since, otherwise, the chamber adjacent to D across $\partial\gamma$ would be closer to C.

We can now define $\mathcal{U}_n = \langle \mathcal{U}_\gamma | \gamma \in \Phi^+, d(\gamma, C) \leq n$ which is a subgroup of $\mathcal{U}$ for all n. We also have a few facts which are immediate from the definition of $\mathcal{U}_n$. We can see that $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \cdots$ and $\mathcal{U} = \cup_n \mathcal{U}_n$ as any positive root will be some finite distance from $\mathbb{C}$.

Slightly less obvious is the fact that $\mathcal{U}_n$ is finitely generated for all n. If $d(\gamma, C) \leq n$ then ther must be a chamber D which borders γ with $d(D, C) \leq n$. There are only finitely many such chambers, and each of these chambers borders at most 3 roots, so $\mathcal{U}_n$ is finitely generated.

The idea of the remaining proofs will be to use the following lemma

Lemma 19. *If there is some N such that $\mathcal{U}_n \subset \mathcal{U}_{n-1}$ for $n > N$ then $\mathcal{U}$ is finitely generated.*

Proof. If $\mathcal{U}_n = \mathcal{U}_{n-1}$ for all $n > N$ then inductively we know that $\mathcal{U}_n = \mathcal{U}_N$ for all $n > N$. Thus

$$\mathcal{U} = \cup_{n=N}^{\infty} \mathcal{U}_n = \cup_{n=N}^{\infty} \mathcal{U}_N = \mathcal{U}_N$$

which is fintely generated as desired. □

Since the remaining W, k pairs the only exceptional cases in rank 3, it is clear that we will have to use not only the specific commutator relations of the local root groups, but also the geometry in the Coxeter complex specific to these choices of W.

3.2.1 Case: 334 over $\mathbb{F}_2$

Before we start we will note that almost every case must be considered over $\mathbb{F}_2$ and $\mathbb{F}_3$, which ususally have to be done separately as there are difference in the commutator relations. However, a lack of a 6 in the Coxeter diagram of W means that $\mathcal{U}$ is finitely generated by the known theory for this choice of W. Therefore, we will only consider this W over $\mathbb{F}_2$.

Let W be the Coxeter group defined by a 334 diagram and $k = \mathbb{F}_2$. Then we will show $\mathcal{U}$ is finitely generated in this case.

Theorem 3. *If W and k as above then $\mathcal{U}_n \subset \mathcal{U}_{n-1}$ for all $n > 2$.*

Proof. Let γ be any positive root with $d(\gamma, C) = n > 2$. Then choose a chamber D_1 which borders γ such that $d(D_1, C) = d(\gamma, C)$. Now there is another chamber D_2 such that D_1 and D_2 are adjacent and $d(D_2, C) = d(D_1, C) - 1$. Then D_1 and D_2 will share exactly one vertex which lies on $\partial\gamma$, call it v. Recall that $\mathrm{st}(v)$ is the set of chambers of Σ for which v is a vertex. Then we have $|\mathrm{st}(v)| = 6$ or 8.

First suppose $|\mathrm{st}(v)| = 6$. In Σ, we can see that $\mathrm{st}(v)$ consists of the 6 chambers "surrounding" v which each have a vertex on v. Since we have already defined D_1 and D_2 we may label the other 4 chambers in $\mathrm{st}(v)$ as $D_3, \ldots, D_6$ by going in a circular order around v. Equivalently this means that D_i is ajacent to D_{i+1} for $1 \leq i \leq 5$ and D_6 is also adjacent to D_1. We also know that each positive root will contain exactly 3 of these vertices, and those three vertices will be D_i, D_{i+1}, and D_{i+2} for some i, where addition is done modulo 6.

By construction, D_2 and D_1 are not adjacent along $\partial\gamma$, but a panel of D_1 lies on $\partial\gamma$, and thus D_1 and D_6 must be adjacent along $\partial\gamma$. Since $D_6 \notin \gamma$, this means that γ must contain D_1, D_2, D_3. Let α and β be the other two positive roots through v. We know that $\partial\gamma$ cannot separate D_2 and D_1 or D_2 and D_3 so we can say again without loss of generality that $\partial\alpha$ separates D_2 and D_1 while $\partial\beta$ separates D_2 and D_3.

Now $D_3 \in \gamma$ but $D_4 \notin \gamma$ which means that D_3 has a panel on $\partial\gamma$. By our choice of D_1 we know that $d(D_3, C) \geq d(D_1, C) > d(D_2, C)$. We can conclude that $D_2 \in \alpha \cap \beta \cap \gamma$ and thus $D_2 = \mathrm{Proj}_v(C)$. The local isomorphism at v then gives $[\mathcal{U}_\alpha, \mathcal{U}_\beta] = \mathcal{U}_\gamma$. However, we already showed that D_2 borders α and β and $d(D_2, C) = d(D_1, C) - 1 = n - 1$ so that $\mathcal{U}_\alpha, \mathcal{U}_\beta \in \mathcal{U}_{n-1}$ and thus $\mathcal{U}_\gamma \in \mathcal{U}_{n-1}$ as desired.

Now suppose $|\mathrm{st}(v)| = 8$. Then we will use the same labeling scheme as before except there will be 8 chambers, and each positive root will contain exactly 4 consecutive chambers from $\mathrm{st}(v)$. The same logic as before will still tell us that γ will contain exactly the chambers D_1, D_2, D_3, D_4. Our first claim is that $D_2 = \mathrm{Proj}_v(C)$.

We know that $\mathrm{Proj}_v(C)$ must lie in any positive root through v and thus it can only be D_1, D_2, D_3, D_4. We also know it is the chamber A in $\mathrm{st}(v)$ which minimizes $d(A, C)$. Since $d(D_1, C) > d(D_2, C)$ we know that D_1 cannot be the projection. By a similar argument as before we know that D_4 borders γ and thus $d(D_4, C) \geq d(D_1, C)$ by our choice of D_1. Thus D_4 cannot be the projection. Finally, if D_3 were the projection then $d(D_4, C) = d(D_3, C) + 1 < d(D_3, C) + 2 = d(D_1, C)$ which is also a contradiction and thus $D_2 = \mathrm{Proj}_v(C)$.

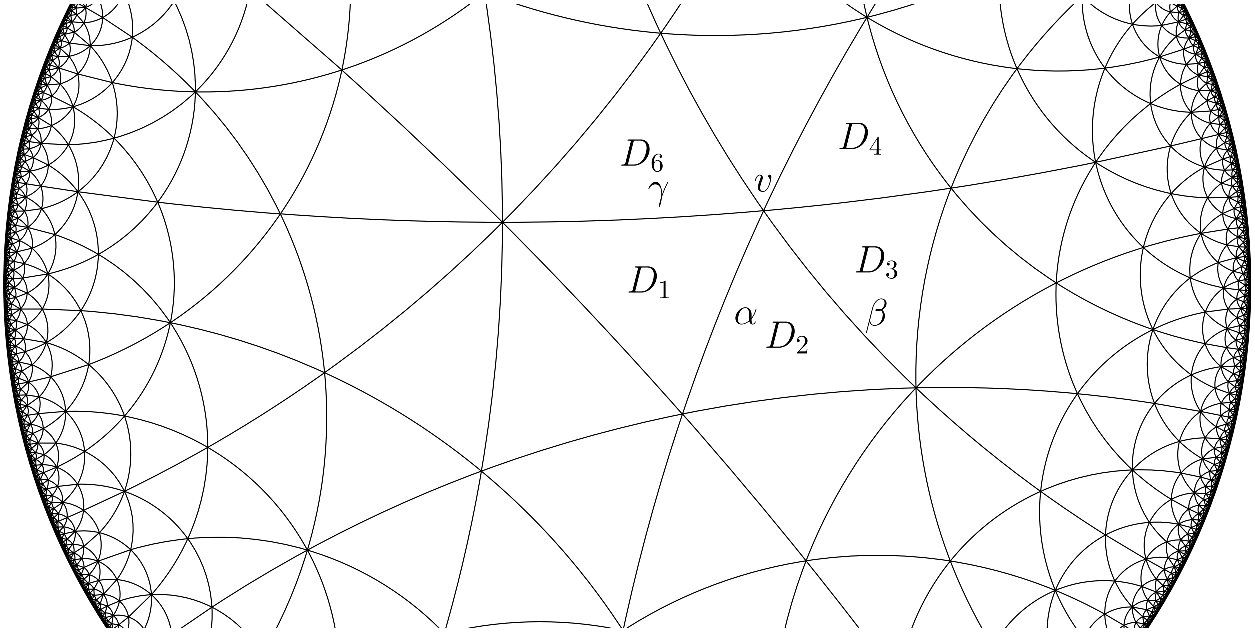

Figure 3.1: Case: $|\mathrm{st}(v)| = 6$

Let α be the positive root separating D_1 and D_2, β the positive root separating D_2 and D_3 and δ the positive root separating D_3 and D_4. Recall that γ is the positive root separating D_8 and D_1 as well as D_4 and D_5. We know that D_2 borders α and β with $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $\mathcal{U}_\alpha, \mathcal{U}_\beta \subset \mathcal{U}_{n-1}$.

Let E be the third chamber adjacent to D_2. Every chamber must have an adjacent chamber which is closer to C and thus we have $d(E, C) < d(D_2, C)$. We can check that $d(E, C) = d(D_1, C) - 2 \geq 1$ by our choice of γ and thus E is not the fundamental chamber C. We know that D_1 and D_2 share two vertices, and D_2 and E share two vertices, so necessarily we have that D_1, D_2, and E must share at least one, and thus exactly one vertex, call it y_1. By a similar argument, the chambers D_3, D_2, and E will also share a vertex y_2. Let F_1 be the other chamber adjacent to E that has y_1 as a vertex, and let F_2 be the other chamber adjacent to E that has y_2 as a vertex. Note that $|\mathrm{st}(y_1)| = |\mathrm{st}(y_2)| = 6$ since v is the other vertex of D_2. The appropriate labeling can be seen in Figure 3.2.1, and the given diagram is unique up to a mirror image flip, which does not affect any of the following arguments. The labeling of these chambers could have simply been defined by the diagram, but the previous explanation seeks to convince the reader that no choices have been made and this diagram is unique.

Since $d(E, C) < d(D_2, C) < d(D_1, C)$ we know that there is some minimal gallery from D_1 to C which passes through E. If we fix such a minimal gallery we can see that it must pass through either F_1 or F_2. First suppose that it passes through F_1. Then $d(F_1, C) = d(D_1, C) - 3$ and so F_1 and D_1 are distance 3 from one another. Since they are both in $\mathrm{st}(y_1)$, this means that D_1 and F_1 are opposite in $\mathrm{st}(y_1)$. Then there is another minimal gallery from D_1 to F_1 which does not pass through D_2 and can also be extended to a minimal gallery from D_1 to C. Let G_1 be the chamber adjacent to D_1 in this new minimal gallery. Then D_1 and G_1 have

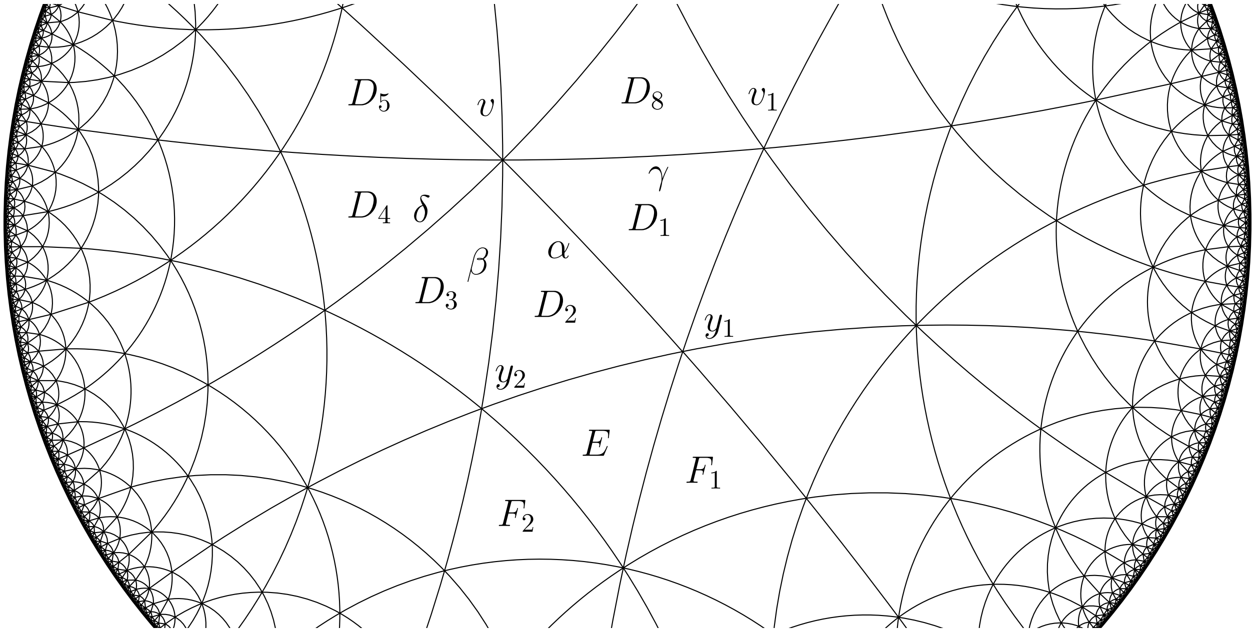

Figure 3.2: Case: $|\mathrm{st}(v) = 8|$

exactly two vertices in common, on of which is y_1, and the other cannot be v as this would imply $G_1 = D_2$ which contradicts our assumption. Let v_1 be the common vertex which is not y_1. We assumed that v was the unique vertex shared by D_1 and D_2 which lies on $\partial\gamma$. Since y_1 is also shared by D_1 and D_2 this means that y_1 does not lie on $\partial\gamma$. We assumed that D_1 has a panel on $\partial\gamma$ and thus it has two vertices on $\partial\gamma$ which means v_1 must lie on $\partial\gamma$.

Now we have the following situation. We still know that D_1 borders γ with $d(\gamma, C) = d(D_1, C)$ and G_1 is an adjacent chamber such that $d(G_1, C) < d(D_1, C)$. We know that v_1 is a common vertex which lies on $\partial\gamma$ and thus it is the only common vertex which lies on $\partial\gamma$. Finally, v is the uniqe vertex of D_1 with 8 chambers in its star. Thus $|st(v_1)| = 6$. Now we may apply the $|\mathrm{st}(v)| = 6$ case with G_1 as our new choice of D_2 and v_1 the new v. This shows that $\mathcal{U}_\gamma \subset \mathcal{U}_{n-1}$ as desired.

Now suppose the fixed minimal gallery from before passes through F_2. Then there is also a minimal gallery from D_3 to C which passes through F_2 as well. But then $d(F_2, C) = d(D_3, C) - 3$ which means F_2 and D_3 are opposite in $\mathrm{st}(y_2)$. Since D_3 borders δ, we can use similar arguments as in the previous two paragraphs to show that $\mathcal{U}_\delta \subset \mathcal{U}_{n-1}$. However, by Lemma **??** we know that $\mathcal{U}_v = \langle \mathcal{U}_\alpha, \mathcal{U}_\beta, \mathcal{U}_\delta \rangle$ and thus $\mathcal{U}_\gamma \subset \mathcal{U}_{n-1}$ as well. Thus for any root γ with $d(\gamma, C) = n \geq 3$ we have $\mathcal{U}_\gamma \subset \mathcal{U}_{n-1}$ and thus $\mathcal{U}_n \subset \mathcal{U}_{n-1}$ as desired. □

Corollary 2. *The group $\mathcal{U}$ with W and k as before is finitely generated.*

3.2.2 Case: 336 over $\mathbb{F}_3$

I haven't typed this stuff up yet either.