Finite Presentability of Groups Acting on Locally Finite Twin Buildings

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Coxeter systems

Definition

A *Coxeter system* is a pair (W, S) consisting of a group W and generating set S such that W admits a presentation $W = \langle S | (st)^{m(s,t)} = 1$ for all $s, t \in S\rangle$, where $m(s,t) \in \mathbb{N} \cup \{\infty\}$, $m(s,t) = m(t,s)$, and $m(s,t) = 1$ if and only if $s = t$ for all $s, t \in S$. In fact, $m(s,t)$ is the order of st, and $m(s,t) = \infty$ means there are no relations.

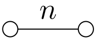
Some examples:

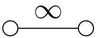
- The dihedral group $D_{2n} = \langle s, t | s^2 = t^2 = (st)^n = 1\rangle$
- The infinite dihedral group $D_\infty = \langle s, t | s^2 = t^2 = 1\rangle$
- The symmetric group

$$S_n = \langle s_1, \ldots, s_{n-1} | s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \text{ if } |i - j| > 1\rangle$$

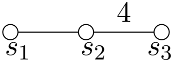
The Coxeter diagram

We can associate a Coxeter diagram to a Coxeter group by assigning a node for each generator and putting an edge between two vertices i and j if $m_{ij} \geq 3$ and labeling that edge with m_{ij} if $m_{ij} > 3$.

D_{2n}: 

D_∞: 

$W = \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_1 s_3)^2 = 1 \rangle$:

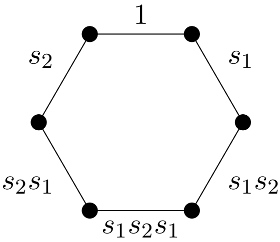


Coxeter complexes

- A *Coxeter complex* of type (W, S) is a simplicial complex associated to a Coxeter system (W, S). Its dimension is $|S| - 1$.
- The *chambers* (maximal dimension simplices) are in 1-1 correspondence with elements of W.
- The codimension-1 simplices are called *panels* and can be labeled with elements of S. If two chambers share an s-panel for some $s \in S$, they are called s*-adjacent*.
- If W is finite, the Coxeter complex is homeomorphic to a $(|S| - 1)$-sphere. In this case, we say (W, S) is a *spherical* Coxeter system.

Examples

$\tilde{A}_2$

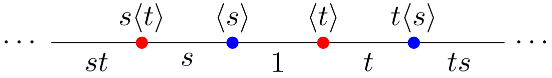


A_2

Buildings and Twin Buildings

- A building of type (W, S) is a simplicial complex built out of Coxeter complexes of type (W, S) satisfying certain axioms.

  Example: A tree without endpoints is a building of type $(D_\infty, \{s, t\})$.



- A twin building of type (W, S) is a pair of buildings of the same type with an opposition relation between them. These generalize spherical buildings.

- The theory of twin buildings was developed by Tits and Ronan to study Kac-Moody groups, which naturally act on twin buildings.

Kac-Moody groups

- These can be thought of as infinite-dimensional analogues of semisimple Lie groups.
- Any Kac-Moody group has an associated Weyl group W and hence Coxeter system (W, S).
- Example: $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ is an affine Kac-Moody group of type $\tilde{A}_{n-1}$ over $\mathbb{F}_q$.
- The key structure we gain from a strongly transitive action on a twin building is a *twin BN-pair*.

- Kac-Moody groups over infinite fields are always infinitely generated, and Kac-Moody groups over finite fields are always finitely generated.

Question

When are Kac-Moody groups over finite fields finitely presented?

- Abramenko and Mühlherr showed that $\mathcal{G}(\mathbb{F}_q)$ is finitely presented in the 2-spherical case (all finite labels in Coxeter diagram).
- Stuhler showed that $\mathrm{SL}_2(\mathbb{F}_q[t, t^{-1}])$ is not finitely presented in 1980 using different methods.

Conjecture

Conjecture

Let G be a group acting strongly transitively on a locally finite twin building. If the Coxeter diagram for G has an ∞ label, then G is not finitely presented.

Tools

Definition

A group is said to be FP_n if there is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules $\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to \mathbb{Z}$ such that $P_0, \ldots, P_n$ are finitely generated.

FP_1 is equivalent to finite generation and finite presentation implies FP_2.

Theorem (Gandini, 2012)

If a group acts cellularly on a product of two trees with finite stabilizers of unbounded order, then G is not FP_2. In particular, G is not finitely presented.

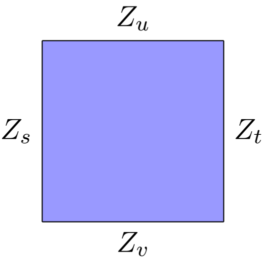
Tools

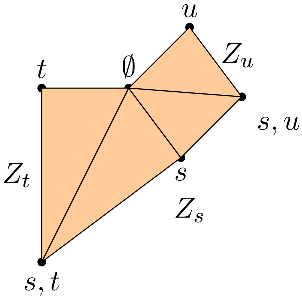
Definition (Z-realization of a building)

Let Z be any topological space with a family of nonempty closed subsets Z_s for each $s \in S$ and let Δ be a building. Then we define $Z(\Delta)$ as a quotient of $Z \times \mathcal{C}$ where we glue copies of Z together by their s-panels if the associated chambers are s-adjacent in the building.

Idea: Z is the model for a closed chamber and Z_s is the s-panel. We want $Z(\Delta)$ to be a tree, so that we can apply Gandini's theorem.

Examples of Z

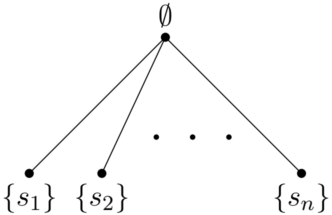




Definition

The *Davis realization* of a building is a specific case of the more general Z-realization where Z is the geometric realization of the flag complex on the spherical subsets of S and Z_s is the geometric realization of the flag complex on the spherical subsets of S containing s.

The Davis realization works if all labels are ∞ but will, in general, have too high dimension to apply Gandini's theorem.



A Z-apartment

Hexagons2.png

Main Result

Theorem (G.)

Suppose G acts strongly transitively on a locally finite twin building and has Coxeter system (W,S) with $S = J \sqcup K$, $|K| \geq 2$ such that $J \cup \{s\}$ is spherical for any $s \in K$ and $m(s,t) = \infty$ for any $s,t \in K$. Then G is not FP_2.

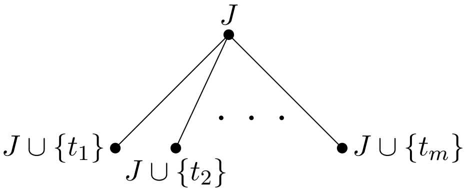
Corollary

Suppose that G has Weyl group W with generating $S = J \cup \{s,t\}$ such that $m(s,t) = \infty$ and both $J \cup \{s\}$ and $J \cup \{t\}$ are spherical. Then G is not FP_2.

Sketch of Proof

Assume G is as in the theorem. Let Δ be one half of the twin building, and suppose $K = \{t_1, \ldots, t_m\}$.

- Define Z to be the geometric realization of the flag complex of spherical subsets of S containing J and Z_s to be the same for spherical subsets containing s:



- Show that $Z(A)$ is a tree for any apartment A of Δ.

Sketch continued

To show that $Z(A)$ is a tree, we move between points in $Z(A)$ using galleries (paths between chambers) in the apartment A. The difficult part is to show that circuits cannot arise.

1. Choose Z wisely so that the copies can only be glued together by panels with no relations between them (i.e. Z_s, Z_t such that $m(s, t) = \infty$)
2. Use a technical lemma about when words in a Coxeter group can be reduced to the trivial word to show that we can't get back to the starting point.

- If $Z(A)$ is a tree, then $Z(\Delta)$ is also a tree.

  One can show this by using retractions (a building has a canonical retraction onto any apartment) or using the fact that if $Z(A)$ is $\mathrm{CAT}(0)$, then $Z(\Delta)$ is $\mathrm{CAT}(0)$.
- Show that the cell stabilizers, which are intersections of parabolic subgroups, are finite and of unbounded order.
- Then G acts on a product of two trees, one for each half of the twin building, which is a contractible 2-D space. Hence G is not FP_2.

Theorem (G.)

Suppose G acts strongly transitively on a locally finite twin building and has Coxeter system (W, S) such that

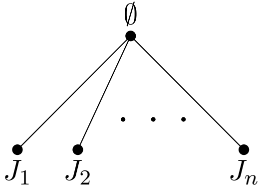
$$S = \coprod_{i=1}^{n} J_i,\ n \geq 2,$$

where all the J_i are spherical subsets of S but $m(s,t) = \infty$ whenever $s \in J_i$ and $t \in J_j$ for $i \neq j$. Then G is not FP_2.

This also takes care of the case when all labels are infinite (in which case it is equivalent to the Davis realization).

Sketch of Proof

Choose Z to be the geometric realization of the flag complex on $\{\emptyset, J_1, \ldots, J_n\}$ and Z_s the same for the subsets containing s:



The strategy is similar to the previous proof, but showing that no circuits exist is easier since

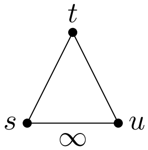
$$W = W_{J_1} * \cdots * W_{J_n}.$$

Rank 3 cases

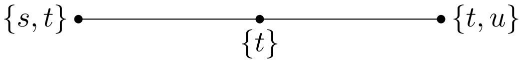
Theorem (G.)

Suppose that G acts strongly transitively on a twin building and has rank 3 Weyl group with at least one ∞ label in the associated Coxeter diagram. Then G is not FP_2 and is therefore not finitely presented.

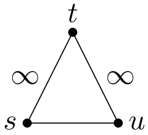
The first case of all ∞ labels is taken care of by the previous result.



- Z is the geometric realization of the flag complex on spherical subsets of S containing t
- $Z_t = Z$
- Z_s, Z_u are geometric realizations of flag complex on spherical subsets containing $\{s,t\}$ and $\{s,u\}$, respectively.

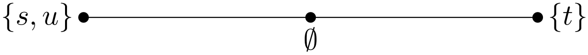


Rank 3, two ∞s



Let $J_1 = \{s, u\}$ and $J_2 = \{t\}$ and use the second result.

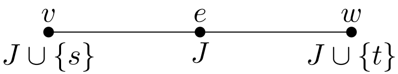
- Z is the geometric realization on the flag complex on $\{\emptyset, J_1, J_2\}$.
- Z_s, Z_t, Z_u are the geometric realizations on the flag complexes on the subsets containing s, t, u, respectively.
- $Z_s = Z_u$



Amalgamated product decomposition

Proposition

If G has Coxeter system (W, S) such that there exist generators $s, t \in S$ such that $m(s, t) = \infty$, then G acts on a tree with a segment as fundamental domain. Furthermore, if we name the edge e with vertices v and w, then $G = G_v *_{G_e} G_w$ is the amalgamated product of the vertex stabilizers over the edge stabilizer.



Thank you!