Finite Generation in RGD Systems with Exceptional Rank-2 Residues

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Abstract

Let $(G, (U_\alpha), T)$ be an RGD system. The most prominent examples are Kac-Moody groups, which are infinite dimensional analogs of semisimple Lie groups. These groups have an associated twin building Δ on which the group G will act strongly transitively. We say that the building Δ satisfies condition (co) of the collection of chambers opposite any chamber is gallery connected. It is known that if Δ satisfies (co), the the subgroup U_+ of G is generated by some finite set of fundamental root groups, and thus is finitely generated if these root groups are finitely generated.

We will help close the gap in the literature relying on condition (co) by proving when RGD systems associated to rank-3 buildings, are and are not finitely generated when the buildings do not satisfy condition (co). Most of the time, the group U_+ will not be finitely generated, and we will give sufficient conditions to guarantee the infinite generation of U_+. We will then modify this aproach to see that another group not covered by the conditions is also not finitely generated. Our main strategy will be to produce a large family of surjective homomorphisms from U_+ which send relatively few U_α to non-identity elements, implying that some of these U_α must be in any generating set.

Finally, we will show that there are two cases where U_+ remains finitely despite Δ not satisfying (co). We will use an aproach which relies on defining a distance between root groups, and showing that most root groups can be expressed in terms of those closer to the fundamental chamber. This approach can also give another proof of the finite generation of U_+ with condition (co).

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Contents

Chapter 0

Introduction

The classification of finite simple groups is one of the foundational problems in group theory, and central to this problem is the study of finite Lie groups. Jacques Tits introduced and developed the theory of buildings in the 1950's and 1960's as a way to study and classify these groups. In the 1980's, these ideas were extended by Tits and Mark Ronan to study Kac-Moody groups, with the introduction of twin buildings. Kac-Moody groups can be described by some group functor $\mathcal{G}$ and a field k, which give the Kac-Moody group $\mathcal{G}(k)$. These groups act as an infinite dimensional analogy of semisimple Lie groups and share many nice properties.

All of the motivating examples in the previous paragraph can be generalized with the notion of RGD systems and Moufang twin buildings. RGD systems are equipped with a Coxeter group W, which is the Weyl group, and a collection $\{U_\alpha\}$ of subgroups associated to every root of W. If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system, then there are several natural subgroups of interest. We can define the subgroup U_+ of G to be the subgroup generated by the positive root groups U_α for all the positive roots of W. A consequence of the RGD axioms is that U_+ admits a nice presentation in terms of the geometry of W. When the Weyl group W is infinite then this presentation will be defined in terms of infinitely many generators, but this does not preclude the possibility that U_+ is still finitely generated. Our main goal in the following will be to show when U_+ is in fact finitely generated and when it is not.

For a twin building Δ, we will say that Δ satisfies condition (co) if the chambers opposite C_-, form a gallery connected subset of the positive half of the twin building. Results in [1] and [2] show that U_+ is finitely generated, and even gives an explicit generating set when that building associated to U_+ satisfies condition (co). Furthermore, [3] shows that a twin building will satisfy condition (co) when all of its rank 2 residues satisfy condition (co). Finally, [1] proves that every rank 2 residue will satisfy condition (co), unless it is the Moufang polygon associated to one of the groups $C_2(2), G_2(2), G_2(3)$ or ${}^2F_4(2)$. Much of the general theory regarding RGD systems, and thus Kac-Moody groups, relies on the (co) condition, and makes assumptions like $|U_\alpha| \geq 4$ for all α to insist it will be satisfied. In this paper we will consider the cases where the rank 2 residues of Δ do not satisfy (co), and thus we cannot rely on (co) to be satisfied in Δ.

Our main approach will be to use the properties of these exceptional rank 2 residues to show that U_+ is not finitely generated. A consequence of the failure of condition (co) is that for each vertex v where (co) fails, the subgroup $U_v \leq U_+$ generated by roots at v will have a proper normal subgroup of small index. We can use these normal subgroups to define homomorphisms which are surjective and where the kernel is well understood. Since we have a presentation of U_+, we will attempt to extend these homomorphisms to all of U_+ in a way that does not help with surjectivity, namely if all other root groups U_α are sent to the identity. If this extension is well defined, and remains surjective, then we should be able to say roughly that any generating set of U_+ will need to contain a root group U_α for some positive root α which goes through v. If we have enough vertices where we can do this, we will be able to show that U_+ is not finitely generated.

In Chapters 1 and 2, we will introduce the necessary theory of Coxeter groups, Coxeter complexes, and buildings that we will use in the remainder of the paper. In particular, results about roots, projections and links will be used extensively in the proof of Theorem 6 and its preceding lemmas. One result of particular interest will be that concerning M-Operations. We will see that we can write elements of a Coxeter group W in a mostly canonical way, which will allow us to determine when any word in a Coxeter group is trivial, or contained in special subgroups. We will use this in the proof of Lemma 15 and 16, when we translate geometric properties of Coxeter complexes, into group theoretic properties of W.

In chapter 3, we will discuss the theory of RGD systems and BN-Pairs, which explain the connections between buildings and the groups of interest. It will show how we can construct buildings from RGD systems and give properties of the action. These connections will allow us to use the geometry from Chapters 1 and 2 to study the group theory of G. We will also see some common examples of RGD systems with Lie groups and Kac-Moody groups.

Chapter 4 will explore the properties of the exceptional rank 2 residues, and will construct the main tools we use when proving the main results. Specifically we give presentations for the exceptional groups U_v and use them to define the homomorphisms ϕ_v to be extended to all of U_+. These homomorphisms will interact nicely with the geometry and group theory of Weyl group W. We will also state the triangle condition, which is a result about the geometry of certain Coxeter complexes showing that triangles formed by the walls of a Coxeter complex, must be single chambers. This will be used when we define our extension maps as we will need to prove properties about the intersections of roots.

In the last two chapters we will record the main results about finite generation in RGD systems. Chapter 5 will introduce condition (A) on certain RGD systems, and discuss how (A) is satisfied by canonical examples like Kac-Moody groups. Lemma 14 will give us a way to construct extension maps for a large family of vertices. Then we can give a necessary condition for the group U_+ to not be finitely generated as described in Theorem 6.

Chapter 6 starts by examining the rank 3 cases not covered by Chapter 5 and explaining why the same approach will not work. There are three remaining cases which will be enumerated in this chapter. The first one will not be finitely generated, and the arguments there will be adaptations of those in Chapter 5 with slight modification. In the rest of the Chapter we will use the presentation of U_+ and the presentations given in Chapter 4 two prove that the remaining cases will yield a U_+ which is in fact finitely generated. We will end by discussing future work including minimal generation in the finitely generated cases, as well as possibilities to extend these results to RGD systems not satisfying (A), especially to higher rank cases.

Before starting, we will try to motivate our curiosity in the group U_+. Finite generation is a fundamental problem in group theory, and so determining answers to questions about finite generation are always interesting from a group theoretic point of view. This question is also specifically of interest because the general theory leaves such a specific gap, and it would be nice to complete the characterization of when the group U_+ is finitely generated. The group U_+ is equipped with a nice presentation that makes it easy to work with, but it also allows us to extend our results found here to other special subgroups of G.

If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system then the subgroup $B_+ = TU_+$ is the stabilizer of the fundamental chamber C_+ in one half of the twin building. This subgroup is known as the Borel subgroup, and can be seen in classical examples like $\mathrm{SL}_n(k)$ as the upper triangular matrices. If T is finite, as in the case for Kac-Moody groups over finite fields, then $[B_+ : U_+] < \infty$, and thus B_+ is finitely generated if and only if U_+ is finitely generated. Knowing stabilizers for certain group actions is an important tool for understanding other finiteness properties of B_+ and G.

We can also define parabolic subgroups P_J for every standard subgroup W_J of the Weyl group W. These subgroups will correspond to stabilizers of lower dimensional simplices in Δ. These subgroups have a very nice decomposition, but without going into too many details here it will suffice to say that each parabolic subgroup P_J has a finite set $\{g_i\} \subset G$ such that $P_J = \langle Bg_iB \rangle$. In particular, if B_+ is finitely generated then so is P_J for every proper parabolic subgroup P_J of G.

Chapter 1

Coxeter Groups and Coxeter Complexes

The RGD systems and Kac-Moody groups discussed in the introduction have a large amount of geometric structure which we can use to study them. The foundations of this geometry come from the idea of reflection groups studied by H.S.M. Coxeter in the 1930's. These reflection groups, and their associate geometry, have particularly nice properties, and they provide the foundation for the study of RGD systems.

1.1 Coxeter Groups

Definition 1. A Coxeter system is a pair (W, S) such that S is a finite set, and W is a group with a presentation

$$W = \langle s \in S | (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

subject to the conditions that $m(s,t) \in \mathbb{N} \cup \{\infty\}$, $m(s,s) = 1$, and $m(s,t) = m(t,s) \geq 2$ if $s \neq t$. If $m(s,t) = \infty$ then we simply discard the relation $(st)^{m(s,t)} = 1$.

Through slight abuse of terminology we will refer to W as a Coxeter group, but we will always have a specific generating set S for the Coxeter system in mind. Coxeter groups have many nice properties, and far too many to discuss here, but we will mention a few which will be of use later. The first of which is the length function. If (W, S) is a Coxeter system then we can define a function $\ell : W \to \mathbb{N}$ by $\ell(w)$ is the minimum number n such that w can be written as $w = s_1 s_2 \cdots s_n$ with $s_i \in S$ for all i. This length function is standard in group theory, and can be defined on any group with any generating set. However, in Coxeter groups this length function takes on a much richer structure which we will describe in more detail.

1.1.1 M-Operations

We say that $(s_1, s_2, \ldots, s_n)$ is a decomposition of w if $w = s_1 s_2 \cdots s_n$, and that it is a reduced decomposition if $n = \ell(w)$. Certainly decompositions, and even reduced decompositions need not be unique, and we will see some ways that we can generate new decompositions. By definition, $m(s,s) = 1$ so $s^2 = 1$ for all $s \in S$. Thus if we ever have an element of s repeated twice in a row in a decomposition, we can simply delete the copies to get another decomposition with smaller length. If $s \neq t \in S$ then $(st)^{m(s,t)} = 1$ and thus we can say

$$\underbrace{sts\cdots t(s)}_{m(s,t)} = \underbrace{tst\cdots s(t)}_{m(s,t)}$$

This again means if we have any alternating string of s and t of the right length, then we can replace it with the swapped alternating string, and get another decomposition of the same length. These operations on decompositions are automatic from the presentation, and we will repeat Theorem 2.33 from [4], which shows these operations are all we need.

Theorem 1. *If (W, S) is a Coxeter system and $(s_1, s_2, \ldots, s_n)$ is a decomposition of w, then we can obtain a new decomposition of W by deleting a sub-string of the form (s, s), or replacing a sub-string of length $m(s,t)$ of the form $(s, t, \ldots, s(, t))$ with a sub-string of the form $(t, s, \ldots, t(, s))$. We will call these two operations* M-Operations *of type 1 and 2 respectively. Furthermore, any decomposition of w can be transformed into a reduced decomposition by repeated application of M-Operations of type 1 and 2, and any two reduced decompositions of w can be transformed into one another by applications of M-Operations of type 2.*

There are many consequences of Theorem 1 but one of the most notable is this, we have a simple algorithm to obtain a reduced decomposition of any w, and we can always check if a decomposition is reduced. In either case we repeatedly apply any possible M-Operations, and applying those of type 1 if possible or noting if none are possible in the case of a decomposition which is already reduced. It also gives us some facts about the length function. For example, if we can write $w = s_{i_1} \cdots s_{i_k}$ then $\ell(w)$ and k are either both even, or both odd, as application of type 1 operations will always reduce the length of a decomposition by 2.

1.1.2 Standard Subgroups and Standard Cosets

Coxeter groups also have a nice subgroup structure will give rise to the rich geometry we will use later. If (W, S) is a Coxeter system then by definition W is generated by S. For any $J \subset S$ we can form a subgroup $W_J = \langle s | s \in J \rangle \leq W$. For example, $W_S = W$ and $W_\emptyset = \{1\}$. We will also define a standard coset to be any coset of the form wW_J for any $w \in W$ and $J \subset S$.

As before, there is nothing special about these definitions, as similar definitions hold for any group, but what is special is the structure on standard subgroups. Proposition 2.13 from [4] tells us that the map which sends $J \to W_J$ is a bijection from subsets of S to standard subgroups. Furthermore, if H is a standard subgroup, then its J can be recovered as $H \cap S$. We can also check that (W_J, J) is also a Coxeter system.

We can use Theorem 1 to derive some basic consequences about standard subgroups. For example, we can show that $W_J \cap W'_J = W_{J\cap J'}$. One inclusion is clear, and if we take $w \in W_J \cap W'_J$ we can write two reduced decompositions of w, one of which only uses letters from J and the other only uses letters from J'. These reduced decompositions can be transformed into one another by M-Operations of type 2, but M-Operations cannot introduce new letters into a reduced decomposition, only change the order. Thus every letter in the initial decompositions must be in J and J'.

One situation which will be very useful later is when the group W is finite. We say that a Coxeter Group or Coxeter System is *spherical* if W is finite. If (W, S) is spherical then we can prove several facts. By Corollary 2.19 in [4], W has a unique element of maximal length, which is usually denoted w_0. It has the property that $\ell(ww_0) = \ell(w_0) - \ell(w)$ for every $w \in W$. One consequence of this fact is that for any $w \in W$ a reduced decomposition of w can be extended to a reduced decomposition of w_0. This element of maximal length will be of some interest in the geometry of W as well. In a similar fashion, we say that J is a spherical subset of S if W_J is spherical. We also say that W is 2-spherical if every subset of S of size 2 is a spherical subset. This is equivalent to saying that $m(s, t) < \infty$ for every $s, t \in S$.

Let Δ be the set of all standard subgroups of W, with a partial order given by reverse inclusion, so that $W_J \leq W_{J'}$ if and only if $J' \subset J$. Using the fact from the previous paragraph, one can check that Δ is isomorphic as a poset to the subsets of S under reverse inclusion. This fact is the basis for our definition of the Coxeter Complex.

1.2 Coxeter Complex

Definition 2. If (W, S) is a Coxeter system, let Σ be the collection of standard cosets of W, ordered by reverse inclusion. Then Σ is a simplicial complex called the Coxeter Complex of W.

Before proceeding we should clarify what is meant by the following definition. A simplicial complex should be a collection of subsets $\mathcal{S} \subset \mathcal{P}(V)$ of some set V such that $B \in \mathcal{S}$ whenever $A \in \mathcal{S}$ and $B \subset A$. For any simplicial complex we can form a poset given by $\mathcal{S}$ ordered by inclusion. As described in Section A.1.1 of [4], posets satisfying certain properties can also be considered as simplicial complexes, where the vertices are the elements which are greater than exactly one other element, and the other faces are determined by which vertices they contain. Theorem 3.5 in [4] proves that the poset Σ satisfies the needed properties, and thus can be viewed as a simplicial complex.

In the standard terminology of simplicial complexes, we will refer to each standard coset as a simplex, and A and B are simplices with $A \leq B$ then we say A is a face of B. One can check that the dimension of any simplex wW_J will be $|S| - |J| - 1$ because the ordering is by reverse inclusion. For this reason, it is sometimes more useful to refer to the rank of a simplex which is the number of vertices, and is also one more than the dimension, so that the rank of $wW_J = |S| - |J|$. We can also draw several conclusions from this fact. First of all, every maximal simplex of Σ has the same dimension, $|S| - 1$, and they will correspond exactly to the elements of W by $w \mapsto wW_\emptyset$. We can also see that the standard subgroup $W = W_S$ is a simplex of dimension -1 and of rank 0 which is a face of every other simplex.

Let Δ be a simplicial complex with vertices $V(\Delta)$, and let I be any set. We say that $\tau : V(\Delta) \to I$ is a type function if the vertices of each maximal simplex map bijectively onto I. If τ is a type function then by definition, each maximal simplex has the same rank and dimension. For any simplex A, we can also extend the type function $\tau : \Delta \to \mathcal{P}(I)$ where $\tau(A) = \{\tau(v) | v \text{ is a face of } A\}$.

By Theorem 3.5 in [4],the Coxeter complex is also equipped with a type function $\tau : \Sigma \to \mathcal{P}(S)$ by $\tau(wW_J) = S \setminus J$. However, for convenience, we will more often refer to the *cotype* of a simplex which is $S \setminus \tau(wW_J) = J$. For example, maximal dimensional simplices will have cotype $\emptyset$, and co-dimension 1 simplices will have cotype $\{s\}$ for some $s \in S$. This convention is also convenient as simplices of cotype J will have rank $|J|$ and dimension $|J| - 1$.

We will call the maximal simplices of Σ *chambers* and the co-dimension 1 simplices of Σ will be called *panels*. A panel will have cotype $\{s\}$ for some $s \in S$, or just cotype s for short. If we take a look at a panel of cotype s, we see that it is a standard subgroup of the form $wW_{\{s\}} = w\{1, s\} = \{w, ws\}$. Thus each panel will be contained in exactly two chambers, corresponding to w and ws, and we will say that the chambers w and ws are s-adjacent. We say that two chambers are adjacent if they are s-adjacent for some $s \in S$. We will also note that there is an obvious chamber which can be distinguished, namely the chamber $W_\emptyset = \{1\}$. We will call this the *fundamental chamber* of Σ and denote it as C.

A *gallery* in Σ is a sequence of chambers $D_0, D_2, \ldots, D_n$ such that D_i and D_{i+1} are adjacent for every i. We will say that a subset $\mathcal{D}$ of chambers of Σ is gallery connected if for all chambers $D, E \in \mathcal{D}$, there is a gallery $D_0, \ldots, D_n$ in $\mathcal{D}$ such that $D_0 = D$ and $D_n = E$. Since every chamber E can be written as wC for some $w \in W$, we can make a gallery $D_0, \ldots, D_n$ from 1 to w where $D_i = s_1 \ldots s_i$ for any decomposition $(s_1, \ldots, s_n)$ of w. This leads to the following proposition

Proposition 1. *If Σ is a Coxeter complex then it is gallery connected.*

It turns out that Σ is sufficiently nice that the geometry of the lower dimension simplices can be recovered from the chambers of Σ and from the s-adjacency relations. Thus we will rarely need to make arguments using simplices other than chambers or panels. This also means when considering subset of Σ, we will instead use the chambers of Σ, which we will denote $\mathcal{C}(\Sigma)$. In fact, for any sub-complex Δ', we will refer to the set of chambers of Δ' as $\mathcal{C}(\Delta)$. In the next chapter we will see how the chambers and notion of s-adjacency will be "enough" when discussing Coxeter complexes.

If D and E are chambers then a minimal gallery between D and E is a gallery of minimal length, that is, any other gallery between D and E is at least as long. Then we can turn $\mathcal{C}(\Sigma)$ into a metric space where $d(D, E)$ is the length of a minimal gallery between D and E. It is not so surprising that there is a direct link between galleries in Σ and decompositions in W. In fact, we have the following facts which can be found in [4]. If $D = w$ and $E = w'$ are chambers of Σ, then $d(D,E) = \ell(w^{-1}w')$. Furthermore, if $(s_{i_1},\ldots,s_{i_n})$ is any decomposition of $w^{-1}w'$ then there is a gallery $D_0,\ldots,D_n$ from D to E where D_j is s_{i_j} adjacent to D_{j+1} for all j. In this case the minimal galleries will correspond to reduced decompositions.

1.2.1 Links and Stars

We saw before that if $J \subset S$ then (W_J, J) is also a Coxeter system. This structure will also carry over into the Coxeter complexes. Before giving the details, we need to define a few more terms. In any simplicial complex, we say that two simplices A and B are joinable if they are contained in a common maximal simplex. In term of the Coxeter complex Σ, two simplices $A = wW_J$ and $B = w'W_{J'}$ are joinable if they share a common element w. We can now make two more definitions which we will use extensively through the rest of the paper.

Definition 3. If A is a simplex of Σ, then the star of A, $\mathrm{st}(A)$, is all of the simplices of Σ which are joinable to A. In terms of chambers $\mathcal{C}(\mathrm{st}(A)) = \{w \in W | w \in A = w'W_J\}$. We can also define the link of A, $\mathrm{lk}(A)$, as the set of all simplices of Σ which are joinable to A, but do not contain A.

Previously we saw for an $J \subset S$, that not only could we form the standard subgroup W_J, but that (W_J, J) was also a Coxeter system in it's own right. Proposition 3.16 in [4] translates this concept in to Coxeter complexes as well.

Proposition 2. *If A is a simplex of Σ of cotype J, then $\mathrm{lk}(A)$ is isomorphic as a simplicial complex to the Coxeter complex Σ_J of (W_J, J).*

We can define $\Sigma_{\geq A}$ to be the set of simplices in Σ which contain A. There is a bijection from $\mathrm{lk}(A)$ to $\Sigma_{\geq A}$ given by $B \mapsto B \cup A$ which is also an isomorphism as posets. Using this fact we can check that the chambers of $\mathrm{st}(A)$ will be in 1-1 correspondence with the maximal simplices of $\mathrm{lk}(A)$ which are also the chambers of Σ_J. For a simplex A, the star and link of A will give more or less the same combinatorial information, and thus which one we use will be somewhat a matter of convenience.

Stars and links have other nice properties which we will take advantage of later. First of all $\mathcal{C}(\mathrm{st}(A))$ is gallery connected, and the galleries in $\mathrm{st}(A)$ correspond exactly to galleries in Σ_J. Furthermore, suppose that $D_0,\ldots,D_n$ is a minimal gallery between two chambers in $\mathrm{st}(A)$ where A has cotype J. Then we know that D_i and D_{i+1} are s_i adjacent for some $s_i \in S$. But in fact, $s_i \subset J$ for every i. In fact, the types of these adjacencies is exactly the same as those in the minimal gallery of Σ_J.

We say that a Coxeter complex Σ is spherical or 2-spherical if W is spherical or 2-spherical. If Σ is spherical then we will define C^{op} to be the chamber of Σ corresponding to w_0. Then C^{op} is the unique chamber of Σ at maximal distance from C, and it has the property that every chamber of Σ is part of a minimal gallery from C to C^{op}.

Now suppose that Σ is a 2-spherical Coxeter complex, and let A be a simplex of Σ of co-dimension 2. Then A is a simplex of cotype $J = \{s,t\}$ for some $s,t \in S$. By definition of 2-spherical, this means W_J is spherical and thus there are finitely many chambers in $\mathrm{st}(A)$.

Every chamber in $\mathrm{st}(A)$ also has a unique chamber at maximal distance away in $\mathrm{st}(A)$ which we will call opposite in $\mathrm{st}(A)$. If we examine the structure of W_J we can even see that it is the dihedral group of order $2m(s,t)$, and the simplicial complex Σ_J will be a $2m(s,t)$-gon with edges as chambers and vertices as panels. Translating to Σ this means that $\mathrm{st}(A)$ consists of $2m(s,t)$ chambers arranged in a circular pattern around A, and opposite chambers in $\mathrm{st}(A)$ will be at distance $m(s,t)$ away from each other.

1.2.2 Projections

Another useful tool for studying the geometry of Σ is the concept of projections. Proposition 3.105 from [4] yields the following theorem.

Theorem 2. *If A is a simplex of Σ, and D is a chamber of Σ, then there is a chamber $E \in \mathrm{st}(A)$ such that $d(D,E) \leq d(D,E')$ for all $E' \in \mathrm{st}(A)$. Additionally, the chamber E is unique and we define the projection of D on to A, or $\mathrm{Proj}_A(D)$ to be the chamber E. The projection E is also characterized by the property that $d(D,E') = d(D,E) + d(E,E')$ for all $E' \in \mathrm{st}(A)$.*

The property $d(D,E') = d(D,E) + d(E,E')$ is known as the gate property because it means for any $E' \in \mathrm{st}(A)$, there is a minimal gallery from D to E' which passes through E. Projections also allow us to define a notion of convexity in a Coxeter complex.

Definition 4. We say that a sub-complex Δ of Σ is convex, if $\mathrm{Proj}_A(D) \in \Delta$ whenever A is a simplex of Δ and D is a chamber of Δ.

Convexity also has another interpretation, which can be taken as the definition if desired. A chamber sub-complex Δ of Σ is convex if for any chambers D, E of Δ, any minimal gallery from D to E in Σ is contained in Δ. This means that we can look for minimal galleries in a convex chamber sub-complex of Σ, and still be sure that it will be minimal in all of Σ. One of the most common uses for this is to apply the result to the convex chamber sub-complex $\mathrm{st}(A)$ for some simplex A. In particular, if A is a simplex of cotype $\{s,t\}$ then we can find minimal galleries from D to E by examining the Dihedral group $D_{2m(s,t)}$ which is not too challenging.

1.2.3 Roots

Intuitively we should think of Coxeter groups as reflection groups in some space, which is how they were originally considered. If we think about a reflection in Euclidean space, then there should be a few properties that are satisfied. A reflection should divide our space into two halves, which are interchanged by the reflection. Furthermore, there should be some set of fixed points of the reflection which would ideally have co-dimension 1. While it is natural to view some Coxeter groups as reflections in Euclidean space, in this section we will show how the notion of roots will generalize these ideas to any Coxeter group.

Definition 5. For any adjacent chambers D, D', let $\alpha_{D,D'}$ be the sub-complex of Σ defined by $\mathcal{C}(\alpha_{D,D'}) = \{E \in \Sigma | d(E, D) < d(E, D')\}$. Then $\alpha_{D,D'}$ is called a root, and the collection of all $\alpha_{D,D'}$ for adjacent chambers D and D' are called the roots of Σ.

We will denote the set of all roots of Σ by Φ. If D and D' are adjacent then for any gallery from E to D, there is a gallery from E to D' of length 1 more. As a consequence of Theorem 1, it is impossible for $d(E, D) = d(E, D')$ for any chamber E, and thus the condition $d(E, D) < d(E, D')$ is equivalent to $d(E, D) \leq d(E, D')$. If D and D' are adjacent chambers then both $\alpha_{D,D'}$ and $\alpha_{D',D}$ will be roots, and we will have $\mathcal{C}(\alpha_{D,D'}) \cap \mathcal{C}(\alpha_{D',D}) = \emptyset$ and $\mathcal{C}(\alpha_{D,D'}) \cup \mathcal{C}(\alpha_{D',D}) = \mathcal{C}(\Sigma)$.

The roots $\alpha_{D,D'}$ and $\alpha_{D',D}$ are very closely related, and roughly correspond to the two half spaces defined by a reflection. To differentiate between these roots, we say a root is *positive* if it contains the fundamental chamber C. This choice is of course arbitrary, but the chamber C is a convenient choice. Similarly, we say a root is negative if it does not contain C, and we say that $\alpha_{D,D'}$ and $\alpha_{D',D}$ are opposite roots. We will also denote this with the notation $\alpha_{D',D} = -\alpha_{D,D'}$.

If roots are roughly analogous to the half spaces defined by a reflection, then we should also have some notion of the reflection line. If α is a root of Σ then we define the *wall* of α, denoted by $\partial\alpha$ or $\mathcal{H}_\alpha$, to be $\alpha \cap (-\alpha)$. Then certainly $\partial\alpha$ will contain no chambers, but will not be non-empty, as the panel contained in D and D' will be in $\partial\alpha$ if $\alpha = \alpha_{D,D'}$. Since the maximal simplices of $\partial\alpha$ will be co-dimension 1 in Σ, this is consistent with our intuition that reflections should have a co-dimension 1 set of fixed points.

There are several facts about roots and walls which we will use later. By Proposition 3.94 from [4], every root is gallery connected, and is also a convex chamber sub-complex of Σ. Furthermore, if $\mathcal{D}$ is a collection of chambers of Σ, then $\mathcal{D}$ is convex if and only if $\mathcal{D}$ is the intersection of $\mathcal{C}(\alpha)$ for some collection of roots $\{\alpha\}$.

What is even more interesting is the interaction between roots and links. Suppose A is a simplex of Σ of cotype J. Then we can recall that $\operatorname{lk}(A) \cong \Sigma_J$ where Σ_J is is a Coxeter complex for (W_J, J). Then there is a natural correspondence between roots in Σ to roots in $\operatorname{lk}(A)$ as described in Proposition 3.79 of [4]. The map $\alpha \to \alpha \cap \operatorname{lk}(A)$ is a bijection between the roots of Σ such that $A \in \alpha$, and the roods of $\operatorname{lk}(A)$ viewed as a Coxeter complex in its own right. Furthermore, this map is also a bijection between walls as well. These results further reiterate the fact that when working in $\operatorname{lk}(A)$, we can essentially forget about the rest of Σ and consider only the Coxeter complex for (W_J, J). This will be especially useful when discussing links of co-dimension 2 simplices.

There is also a connection between roots and the distance function d. By Proposition 3.78 in [4], for any two chambers D, E, the distance $d(D, E)$ is equal to the number of roots α such that $D \in \mathcal{C}(\alpha)$ and $E \notin \mathcal{C}(\alpha)$. Equivalently, $d(D, E)$ is equal to the number of walls which separate D and E.

While discussing roots we should also discuss intervals of roots which will become important later on when discussing RGD systems. We say that two roots α and β are pre-nilpotent if both $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ contain a chamber. In this case we define $[\alpha, \beta] = \{\gamma \in \Phi | \alpha \cap \beta \subset \gamma$ and $(-\alpha) \cap (-\beta) \subset -\gamma\}$ and $(\alpha, \beta) = [\alpha, \beta] \setminus \alpha, \beta$. While these definitions seem arbitrary, they will be very important in later chapters, and they also have very nice interpretations in the context of links. Suppose α and β form a pre-nilpotent pair. Then Lemma 8.42 of [4] says that either $\partial\alpha$ and $\partial\beta$ will meet, or α and β are nested, meaning $\alpha \subset \beta$ or $\beta \subset \alpha$.

If α and β are nested then we can think of $\partial\alpha$ and $\partial\beta$ as parallel hyperplanes in some space. Then the closed interval $[\alpha, \beta]$ is the set of roots whose walls lie between $\partial\alpha$ and $\partial\beta$. If $\partial\alpha$ and $\partial\beta$ meet then we can choose a maximal simplex A of $\partial\alpha \cap \partial\beta$. As described in Example 8.44 of [4], the simplex A will have have co-dimension 2, and $\text{lk}(A)$ will be a rank 2 buildings, which is a $2n$-gon. In this case, intervals of roots can be described by enumerating the roots around a $2n$-gon in clockwise or counterclockwise order, and then lifting those roots to Σ. It is also worth noting that the condition on pre-nilpotence is natural in the context of spherical Coxeter complexes, where two roots α and β are pre-nilpotent as long as $\beta \neq -\alpha$.

Thus far we have discussed many properties and attributes of Σ, but we have not really described how the group theory of W interacts with Σ besides in the notion of galleries. In the next section we will see that we can say much more about the interaction of the group W and the Coxeter complex Σ.

1.3 W-Action

Proposition 3. *There is a well defined action of W on Σ by $w'(wW_J) = w'wW_J$. Then each $w \in W$ induces an isomorphism of Σ which also preserves (co)type of each simplex.*

As Σ is built directly from W, it is unsurprising that this W action plays very nicely with the geometry of Σ, and we will briefly collect the more relevant facts. The W-action sends galleries to galleries, minimal galleries to minimal galleries, and thus $d(D, E) = d(wD, wE)$ for all $D, E \in \mathcal{C}(\Sigma)$ and $w \in W$.

Because of how natural our definition of the W action is, we can also check relatively easily that W interacts nicely with all of the concepts we have defined so far. If A is a simplex and D is a chamber then we have $\text{Proj}_{wA}(wD) = w\text{Proj}_A(D)$ for all $w \in W$. If α is a root of Σ then $w\alpha$ is also a root with wall $w\partial\alpha$. Furthermore, if $\partial\alpha$ is a wall which separates D and D' then $w\partial\alpha$ will separate wD and wD'. This also means $w\alpha_{D,D'} = \alpha_{wD,wD'}$.

It will also be useful to provide some properties of this action. The first result is immediate from the definition of the action.

Theorem 3. *The action of W is transitive on simplices of Σ of cotype J. If $A = wW_J$ is a simplex in Σ of cotype J, then $\text{stab}_W(A) = wW_Jw^{-1}$. In particular, W acts simply transitively on the chambers of Σ.*

The previous theorem also shows, among other things, that W will send links to links of the same type. There is also nice interplay between this w action and the roots of Σ. In light of Theorem 3 we can identify the chambers of Σ with the set of all wC for $w \in W$ where C is the fundamental chamber. Note, this identification also agrees with the identification $wW_\emptyset \mapsto w$. We can check that the chambers wC and wsC are s-adjacent. We can define a root $\alpha_s = \{D \in \mathcal{C}(\Sigma) | d(D, C) < d(D, sC)\} = \{w \in W | \ell(w) < \ell(sw)\}$. The root α_s is positive, and the wall $\partial\alpha_s$ is also the set of all simplices fixed by s.

For any $w \in W$ and $s \in S$, $w\alpha_s$ will also be a root which contains w but not ws. Since any adjacent chambers D and D' must be s adjacent for some $s \in S$, we can also say that $D = wC$ and $D' = wsC$, and thus $\alpha_{D,D'} = w\alpha_s$. This shows every root is a W translate of α_s for some $s \in S$, and similarly every wall is a W translate of some $\partial\alpha_s$.

Chapter 2

Buildings

In Chapter 1 we saw that for a Coxeter system (W, S), we can define a simplicial complex Σ which will encapsulate the group theoretic structure of W in its geometry. The theory of finite reflection groups has also long been part of the study of classical groups, as the associated Weyl group will always be a spherical Coxeter group. The theory of buildings, as developed largely by Tits and Bruhat in the 50's and 60's, looks to extend the information given by the Weyl group W in a natural way. We do not however have the time to discuss the full history of the building axioms, so we begin with a definition which we will study for the rest of the chapter.

Definition 6. A *building* is a simplicial complex Δ which can be expressed as a union of sub-complexes Σ, called Apartments, such that

(B0) Every apartment Σ is a Coxeter complex

(B1) For any two simplices $A, B \in \Delta$, there is an apartment containing A and B.

(B2) For any two apartments Σ, Σ', there is an isomorphism from Σ to Σ' which fixes $\Sigma \cap \Sigma'$ pointwise.

We are using much of the same notation and terminology as [4] but we have changed (B2). It is possible to replace (B2) by an apparently weaker axiom which is actually equivalent. However, we choose here to just list the stronger result as an axiom as it will be easier to work with. When looking at the axioms, (B1) and (B2) seem to be fairly natural connectedness and coherence relations, but (B0) seems to be somewhat arbitrary. Besides the appearance of Coxeter groups in classical groups, it is unclear why we should require sub-complexes to be Coxeter complexes. At the end of the chapter we will see that Coxeter complexes are in fact the natural (and only) choice for these apartments.

As buildings are defined as unions of Coxeter complex, it should come as no surprise that many of the properties of Chapter 1 will still hold, possibly with some slight modification. In fact, a Coxeter complex Σ is an example of a building with a single apartment, so nearly every result about buildings in general will also hold for Coxeter complexes.

First of all, we will note that every maximal simplex of Δ will have the same dimension as any two maximal simplices will lie in some apartment Σ, and apartments, which are Coxeter complexes, have the property that every maximal simplex has the same dimension. As with any simplicial complex, we will say the dimension of Δ is the dimension of a maximal simplex. We will call these maximal simplices Chambers, and we will call co-dimension 1 simplices panels, using the same terminology as in the Coxeter complex.

As with Coxeter complexes, we will say that two chambers are adjacent if they share a panel as a common face. One key difference between buildings and Coxeter complexes is that in a Coxeter complex, exactly two chambers will be adjacent on every panel, where in a building, there can be any number of chambers sharing the same panel, possibly infinitely many. We say a building is *thick* if each panel is a face of at least 3 chambers. We say a building is *thin* if each panel is the face of exactly 2 chambers. As alluded to before, thin buildings are exactly Coxeter complexes. The definition of a thick building may again seem arbitrary, but we will see throughout the rest of the paper that many strong results related to buildings rely on thickness. It also turns out the most buildings arising naturally from the study of interesting groups, will be thick, and the number of chambers containing a given panel will have a nice interpretation as well.

As in the previous chapter, a sequence of chambers $D_0, \ldots, D_n$ is called a gallery if D_i and D_{i+1} are adjacent for all i. A building Δ will be gallery connected as any two chambers will be contained in an apartment, and apartments are gallery connected. We can use galleries to define a metric on the set of chambers of Δ, where $d(D, E)$ is the length of a minimal gallery from D to E. Even though we know that any two chambers can be connected through an apartment, there is no guarantee a priori that such a gallery would be minimal, or that a minimal gallery can even be contained in a single apartment. However, Corollary 4.34 from [4] gives us the following lemma.

Lemma 1. *Suppose Δ is a building with chambers D and E. If Σ is an apartment of Δ which contains D and E, then any minimal gallery connecting D and E in Σ will also be minimal in Δ.*

A consequence of the previous lemma is that when trying to determine the distance between any two chambers of Δ, it is enough consider any apartment between the two chambers. When working with Coxeter complexes, we had a stronger notion of adjacency coming from the type function on Σ. We were able to say that two chambers D and E were s-adjacent if they shared a panel of cotype s. It turns out that we can construct a type function for buildings as well. By Proposition 4.6 in [4] we get the following theorem.

Theorem 4. *If Δ is a building of rank n, and S is a set of size n, then there is a type function τ which takes values in S. Furthermore, it is possible to choose isomorphisms satisfying (B2) which are type preserving.*

We will not include the proof of this theorem, but the idea is as follows. If we fix a chamber C then each apartment containing C will have a type function with values in S. By some permutation of S, we can choose all of these type functions so that they agree on C. Then we glue these type functions together to get a type function on the union of apartments containing C, where compatibility is ensured by (B2). But (B1) ensures that the union of apartments containing C is all of Δ so we have a well defined type function. This also allows us to define the types and cotypes of simplices, and refer to s-adjacent chambers as we did with Coxeter complexes. It also means if we have any gallery $D_0, \ldots, D_k$, they we can define the type of this gallery to be a tuple $(s_1, \ldots, s_k)$ such that D_{i-1} and D_i are s_i-adjacent for all i.

Using S-adjacency, and gallery types, we can introduce the notion of residues on Δ. Assume Δ has a type function taking values in S, $J \subset S$, and D is a chamber of Δ. Then we can define the J-residue of Δ containing D, denoted $\mathcal{R}_J(D)$, to be the chamber sub-complex of Δ where the chambers are those which can be connected to D through galleries consisting of only J-adjacencies. More precisely, a chamber E is in $\mathcal{R}_J(D)$ if and only if there is a gallery $D = D_0, \ldots, D_k = E$ of type $(s_1, \ldots, s_k)$ where $s_i \in J$ for all i. There are two ideas which should be discussed before developing more of the general theory started in Coxeter complexes.

If Δ is a building, then each apartment Σ is a Coxeter complex for some Coxeter group W. Since every apartment is isomorphic, and Coxeter complexes are isomorphic if and only if their associated Coxeter groups are isomorphic, we can assign to each building Δ a well defined (up to isomorphism) Coxeter group W. In this case we say that W is the Weyl group of Δ, and Δ is a building of type W. It is also worth mentioning that W can be recovered purely from the combinatorial information of Δ. If τ is a type function on Δ taking values in S, then we can define a Coxeter group W generated by S with $m(s,t) = \operatorname{diam}(\mathcal{R}_J(D))$ where D is any chamber of Δ and $J = \{s,t\}$. A consequence of Corollary 4.36 of [4] is that each apartment Σ will be isomorphic to the Coxeter complex of this W. It also allows for the definition of a Coxeter system (W,S) where S is the range of the type function on Δ.

Finally, we will discuss the multiple ways in which we can treat buildings, a topic that we have mostly glossed over thus far. Our definition of a buildings involved with simplicial complexes, where we view lower dimensional simplices as being contained in chambers. In the previous chapter we often just considered the chambers of Σ, and we even alluded to the fact that this was "enough". We are now ready to formalize what we mean. If D and E are chambers in the same J-residue, then D and E will share a face of cotype J and vice versa. This gives a natural correspondence between residues and stars of simplices.

For example, let's consider Panels. Simplicially, a panel is a simplex of co-dimension 1 and cotype $\{s\}$ for some $s \in S$. If D is a chamber containing a panel P of cotype s, then $\mathcal{R}_s(D)$ will be all of the chambers which are s-adjacent to D. This also corresponds exactly to the chambers in $\operatorname{st}(P)$, and thus we can associate to each rank 1 residue, a panel P. The connection works for simplices of all ranks, and Lemma 5.88 of [4] gives the following result.

Theorem 5. *Suppose Δ is a building. Let $\mathcal{C}$ be the set of residues of Δ, ordered by reverse inclusion. Then $\mathcal{C}$ can be viewed as a simplicial complex, and it is canonically isomorphic to Δ. The isomorphism is given by $\mathcal{R}_J(D) \mapsto \cap_{E \in \mathcal{R}_J(D)} E$.*

The previous theorem allows us to ignore lower dimensional simplices all together, and instead focus on chambers and J-residues. In practice, we will not devote completely to one approach or the other, but use whichever is more convenient at the time. The biggest difference between the two approaches is language. For example, in the simplicial viewpoint we think of a panel as a co-dimension 1 simplex, and chambers contain a panel, where in the residue viewpoint, we think of a panel as a J residue where $|J| = 1$, and we say that a panel contains a chamber. This mixing of terminology will not be confusing in context however, as it will be clear what approach is being used at any given time.

2.1 Links, Projections, and Roots

Throughout this section, assume that Δ is a building with a type function taking values in S. In this section we will examine some of the ideas introduced in the previous chapter.

Suppose that $J \subset S$ and A is a simplex of Δ of cotype J. As with Coxeter complexes, we define $\mathrm{st}(A)$ to be the set of all simplices which are joinable to A, and $\mathrm{lk}(A)$ is the set of all simplices in $\mathrm{st}(A)$ which are disjoint from A. As alluded to before, the set of all chambers in $\mathrm{st}(A)$ will form a J-residue of Δ. Proposition 4.9 in [4] gives us the following analog of Proposition 2, If A is a simplex of cotype J, then $\mathrm{lk}(A)$ is a building of type (W_J, J), and the type function on $\mathrm{lk}(A)$ is the restriction of the type function on Δ. The same proposition also says if $\mathcal{A}$ is a set of apartments for Δ, then $\{\mathrm{lk}(A) \cap \Sigma | A \in \Sigma\}_{\Sigma \in \mathcal{A}}$ is a set of apartments for $\mathrm{lk}(A)$.

Now suppose that A is a simplex of cotype J and D is any chamber of Δ. By Proposition 4.95 in [4], there is a unique chamber $E \in \mathrm{st}(A)$ such that $d(D, E) \leq d(D, E')$ for all chambers $E' \in \mathrm{st}(A)$. In this case we call E the projection of D onto A and denote it $\mathrm{Proj}_A(D)$. If we use the chamber complex point of view then we say $\mathrm{Proj}_R(D)$ where R is the J-residue corresponding to A. The projection still possesses the gate property so that $d(D, E') = d(D, E) + d(E, E')$ for all chambers $E' \in \mathrm{st}(A)$. Proposition 4.95 also says that projections can be computed in any apartment containing A and D. More precisely, $\mathrm{Proj}^{\Sigma}_A(D) = \mathrm{Proj}^{\Delta}_A(D)$ where the projections are taken in an apartment Σ and Δ respectively.

A subset of $\mathcal{M}$ of Δ is called convex if for every simplex A and chamber D of M, we have that $\mathrm{Proj}_A(D) \in M$. The condition that A is contained in $\mathcal{M}$ is replaced by the assumption that the residue R meets $\mathcal{M}$ in the chamber complex viewpoint. Similar to the case for Coxeter complexes, the condition that $\mathcal{M}$ is convex is equivalent to ensuring that for any chambers D, E of $\mathcal{M}$, any minimal gallery connecting D and E will be completely contained in $\mathcal{M}$, by Proposition 4.115 in [4]. As projections can be computed in any apartment, some of our previous comments show that apartments and residues are both convex chamber sub-complexes of Δ.

Recall that in a Coxeter complex, for every pair of adjacent chambers D, D', we define the root $\alpha_{D,D'}$ to be the chambers which are closer to D than to D'. In a building Δ, a subset α is called a root if it is a root of some apartment, and thus every apartment Σ containing α.

We have mentioned it before but it is worth reiterating, most of the properties and definitions for buildings can be defined in terms of apartments. For this reason, you will see throughout the remainder our our work that we will rarely reference the building Δ at all, but will instead choose appropriate apartments and work there. This will be especially be the case once we introduce group actions in the next chapter, which allow us to view every apartment as some translate of a fixed choice of apartment Σ.

2.2 Spherical Buildings

We say a building Δ is spherical if the Weyl group W is spherical, or equivalently if each apartment Σ is a spherical Coxeter complex. As stated before, this means that W has a unique element of maximal length and the diameter of any apartment will also be this length. We say that two chambers C and D are opposite if $d(C, D)$ is maximal, and we write $C\mathrm{op}D$. By Theorem 4.70 of [4], we know that for each pair of opposite chambers, there is a unique apartment containing the pair, and that apartment is the minimal convex chamber complex containing the pair.

Results about spherical buildings will be especially useful when consider 2-spherical buildings, where $m(s, t) < \infty$ for all $s, t \in S$. In this case, every co-dimension 2 link will be a spherical building and we can use facts about opposition to study local properties of Δ. In particular every rank 2 residue will be a generalized m-gon, which is defined as a rank 2 building.

2.3 Thick Buildings

Earlier in the chapter we said that a building was thin if every panel is a face of exactly two chambers, and it is thick if each panel is the face of at least 3 chambers. The definition of thin seems natural as Coxeter complexes are thin buildings, and in fact the only thin buildings. In this section we will seek to justify the thick condition. In the next chapter we will see how we can associate to any vector space V, a canonical building $\Delta(V)$. This discussion is better saved for the following, but we will introduce one fact here. In $\Delta(V)$, each panel is the face of exactly $|k| + 1$ chambers, where k is the base field for V. Since $|k| + 1 \geq 3$ for any field k, this building will be a thick building. The buildings associated to other classical groups will also be thick, as will nearly all buildings which arise naturally.

Another surprising result is the relation between thickness and the building axioms. Theorem 4.131 of [4] says that if Δ is any thick chamber complex, with a collection of thin chamber sub-complexes $\{\Sigma\}$ which satisfy axioms (B1) and (B2), that Δ will be a building, and each Σ will be a Coxeter complex. Not only does this show the strength of the thick condition, but it also shows that Coxeter complexes are in some sense the “correct” type of chamber complexes to consider.

Chapter 3

BN-Pairs and RGD Systems

In the first chapter we saw the interplay between the group theory of W and the geometry of Σ. In the previous chapter we developed the geometry of buildings, and we will now explore the group theoretic consequences of groups acting on a building.

Some arbitrary group action on a building Δ will not be enough to say much, so we need to restrict our attention to stronger group actions. Throughout this chapter we will assume that we have a group G acting on a building Δ and the action is both simplicial and type preserving. We will also assume that $\mathcal{A}$ is a system of apartments for Δ such that $g\Sigma \in \mathcal{A}$ for each $g \in G$ and $\Sigma \in \mathcal{A}$. We will not discuss the details of apartment systems, but it should be noted that this is always possible using the complete system of apartments as described in Theorem 4.54 of [4].

3.1 BN-Pairs

We say that the group G acts *strongly transitively* if G acts transitively on pairs (Σ, C) where Σ is an apartment of Δ, and C is a chamber of Δ contained in Σ. Equivalently, G acts strongly transitively if G acts transitively on chambers, and for any chamber C, the stabilizer of C acts transitively on apartments containing C, or if G acts transitively on apartments, and for any apartment Σ, the stabilizer of Σ acts transitively on the chambers of Σ. Assume for the rest of the chapter that any group action on a building is strongly transitive.

Choose a chamber C and an apartment Σ containing Σ which we will fix and call the fundamental chamber and fundamental apartment respectively. We can define several subgroups of G which will be the basis of most of the section. Define subgroups

$$B = \{g \in G | gC = C\}N \qquad\qquad = \{g \in G | g\Sigma = \Sigma\}$$

We can make a few remarks which can all be found in section 6.1.1 of [4]. First of all, there is a natural, type preserving action of N on Σ, and since the type preserving automorphisms of Σ are exactly those induced by W, we get a homomorphism $\phi : N \to W$ which is surjective by strong transitivity. Let $T = \ker \phi$ be the elements of N which fix Σ pointwise. Then $N/T \cong$

W, and furthermore, N/T has a canonical choice of generators by taking representatives in N which send the fundamental chamber C to adjacent chambers. Since the action of G is type preserving, we can check that any element of G which stabilizes Σ and C will fix Σ pointwise, and thus $T = B \cap N$. Finally, G is generated by B and N, and we can even show $G = BNB$.

Similar to the case with Coxeter groups and Coxeter complexes, we would like to be able to move between the group theory of G and the geometry of Δ. Before we can do this we need to make a few more remarks. As previously stated, $G = BNB$. We also know that there is a surjection of N onto W with kernel $T = B \cap N$. This means that for any $w \in W$, there is some lift $\tilde{w} \in N$ which is sent to w, and any other lift will differ by an element of $T \subset B$. This allows us to unambiguously write expressions of the form BwB which is understood to mean the double coset $B\tilde{w}B$ for any lift of w. Furthermore, the map $W \to B\backslash G/B$ by $w \mapsto BwB$ is a bijection.

Proposition 6.27 in [4] also says that we have a nice description of stabilizers of lower dimensional simplices. If A is a face of C of cotype J, then the stabilizer of A is $P_J = \cup_{w \in W_J} BwB$. In particular, P_J is a subgroup of G for all $J \subset S$, and we will refer to them as standard parabolic subgroups, while cosets of the form gP_J will be called standard parabolic cosets. Corollary 6.29 of [4] says, similar to Coxeter complexes, the building Δ can be recovered as the poset of standard parabolic cosets of G, ordered by reverse inclusion. While the result is similar to that for Coxeter complexes, it is worth nothing that this result goes in the opposite direction, with Coxeter complexes we defined a simplicial complex from the coset data, while here we already had the simplicial complex data, and simply recovered it from the group theory. Before moving on we will explore the conditions necessary to actually construct a building from group theoretic data.

Definition 7. A pair of subgroups B and N of a group G is a *BN-Pair* if $G = \langle B, N\rangle$, $T = B \cap N$ is normal in N, and the quotient $W = N/T$ admits a set of generators S such that $sBw \subset BswB \cup BwB$ and $sBs^{-1} \not\subset B$ for all $s \in S$ and $w \in W$. In this case we also say that the tuple (G, B, N, S) is a *Tits System*.

Despite the suggestive notation, we do not assume that the elements of S have order 2, or even that W is a Coxeter group. These are results which follow from the axioms, as well as others which can be found in Theorem 6.56 of [4]. If (G, B, N, S) is a Tits system, then (W, S) is a Coxeter system, and there is a thick building $\Delta(G, B)$ on which G acts strongly transitively with B the stabilizer of the fundamental chamber, and N the stabilizer of the fundamental apartment. Furthermore, if G acts strongly transitively on Δ, then B and N as defined before form a BN-pair, and the building Δ is canonically isomorphic to $\Delta(G, B)$.

Before moving on it is worth giving at least one example of a BN-Pair. Let $G = \mathrm{GL}_n(k)$ where $n \geq 2$ and k is a field. Then one can let B be the set of upper triangular matrices, and N the set of permutation matrices, which are matrices with exactly one non-zero element in each row and column. The elements S will be the permutation matrices which swap the i and $i + 1$ position. The rest of the axioms can be checked with linear algebra, but it is also of interest what building this group acts on. The complete construction can be found in section 4.3 of [4], but there is a way to associate a building to any vector space V. For any vector space V, we can define a building $\Delta(V)$ where the chambers are complete flags in V, and the apartments roughly correspond to unordered bases of V. This is consistent as the upper triangular matrices B are exactly those that stabilize the standard flag, and the permutation matrices are those that preserve the standard unordered basis.

3.2 Moufang Buildings and RGD Systems

We saw in the previous section that groups acting strongly transitively on a building have a great deal of group theoretic structure. In this section we will explore additional restrictions which can be placed on these group actions to be able to draw even more consequences, and then apply the results to common examples including Kac-Moody groups.

For now, let Δ be a thick spherical building. We will later cover how to generalize the results in the non-spherical case. In the previous section we discussed properties of certain group actions on a building, but we gave no indication on how these actions arise. There is however one group which always acts nicely on a building, namely $\mathrm{Aut}_0(\Delta)$, the group of type preserving automorphisms of Δ. For any root α of Δ we can then define the root group U_α to be the subgroup of $\mathrm{Aut}_0(\Delta)$ which fixes α pointwise, and fixes $\mathrm{st}(P)$ pointwise for any panel of Δ in $\alpha \setminus \partial\alpha$.

Recall that α is a root of Δ if it is a chamber sub-complex of Δ which is a root in any apartment which contains it. Define $\mathcal{A}(\alpha)$ to be the set of apartments of Δ which contain α. If P is a panel contained in $\partial\alpha$ we can also define $\mathcal{C}(P,\alpha)$ to be the set of chambers in P which do not lie in α. Remark 4.118 in [4] says that for any chamber D in $\mathcal{C}(P,\alpha)$ there is a unique apartment of Δ containing D and α, and thus there is a canonical bijection from $\mathcal{C}(P,\alpha)$ to $\mathcal{A}(\alpha)$. We can also see from the definition that U_α will act on both $\mathcal{C}(P,\alpha)$ and $\mathcal{A}(\alpha)$. Lemma 7.25 in [4] says that these actions are equivalent under the canonical bijection, and that these actions are also free as long as the Coxeter diagram of W has no isolated nodes.

We say that a building is *Moufang* if the action of U_α on $\mathcal{A}(\alpha)$ on is transitive, and it is *strictly Moufang* if the action is simply transitive. By the previous remarks, a Moufang building is guaranteed to be strictly Moufang as long as the Coxeter diagram has no isolated nodes.

We would like some way to relate these root groups to strongly transitive actions from the previous section, and we do get this. Let Σ be the fundamental apartment of Σ and Φ its set of roots. Assume that Δ is Moufang and let $G = \langle U_\alpha | \alpha \in \Phi\rangle$. Then Proposition 7.28 from [4] says that the group G will act strongly transitively on Δ with respect to the apartment system $\mathcal{A} = \{g\Sigma | g \in G\}$. It can also be checked that $gU_\alpha g^{-1} = U_g\alpha$ for any $g \in \mathrm{Aut}(\Delta)$. If β is any root of Δ then there is some $g \in G$ such that $g\beta \subset \Sigma$ since G, acts strongly transitively, and thus we have $gU_\beta g^{-1} = U_{g\beta} \leq G$. This means $G = \langle U_\alpha | \alpha \text{ is a root of } \Delta\rangle$ and G does not depend on the choice of Σ as described in Remark 7.29 of [4].

In Chapter 2 we discussed links and the relationship between apartments of Δ and apartments of $\mathrm{lk}(A)$. Recall that for any simplex A of Δ of cotype J that $\mathrm{lk}(A)$ is a building of type (W_J, J). Furthermore, if Σ is an apartment of Δ which contains A then $\Sigma \cap \mathrm{lk}(A)$ is an apartment of $\mathrm{lk}(A)$. Given the connection between roots and apartments, it is not surprising that the roots of $\mathrm{lk}(A)$ are given by $\mathrm{lk}(A) \cap \alpha$ for all roots α of Δ with $A \in \partial\alpha$.

Suppose that A is a simplex of Δ and let $\Delta' = \mathrm{lk}(A)$. Suppose that α is a root of Δ with $A \in \partial\alpha$ and let α' be the corresponding root of Δ'. If P' is a panel on $\partial\alpha'$ then $P = P' \cup A$ is a panel on $\partial\alpha$. As described in section 7.3.2 of [4], we get a homomorphism $\rho : U_\alpha \to U_{\alpha'}$ given by the restriction of the action of U_α on Δ'. There is also a bijection between $\mathcal{C}(P', \alpha')$ and $\mathcal{C}(P, \alpha)$ given by $C' \mapsto C' \cup A$. The consequence, if Δ is Moufang, then $\Delta' = \mathrm{lk}(A)$ is also Moufang, and if Δ is strictly Moufang and the Coxeter diagram for Δ' has no isolated nodes then Δ' is also strictly Moufang and ρ is an isomorphism.

As with BN-Pairs, we would like to develop the theory that allows us to construct a Moufang building from group theoretic data which we will do in the next section.

3.2.1 RGD Systems

Throughout the section (W, S) will be a spherical Coxeter system and Φ will be the set of roots of the Coxeter complex Σ of type (W, S). Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be a triple consisting of a group G, a family of subgroups U_α for each root of Φ, and another subgroup T. Let $\Phi_\pm$ denote the set of positive (negative) roots, and $U_\pm = \langle U_\alpha | \alpha \in \Phi_\pm\rangle$. We also know that for every $s \in S$ there is a root $\alpha_s = \{D \in \Sigma | d(C, D) < d(sC, D)\}$, and we will let $U_s = U_{\alpha_s}$.

We say that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) if the following conditions hold:

RGD0 $U_\alpha \neq \{1\}$ for all $\alpha \in \Phi$

RGD1 $[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)} = \langle U_\gamma | \gamma \in (\alpha, \beta)\rangle$ whenever $\alpha \neq \pm\beta$

RGD2 For every $s \in S$, there is a function $m : U_s^* \to G$ such that $m(u) \in U_{-s}uU_{-s}$ and $m(u)U_\alpha m(u)^{-1} = U_{s\alpha}$ for all $\alpha \in \Phi$. Furthermore, $m(u)^{-1}m(t) \in T$ for all $u, t \in U_s^*$.

RGD3 $U_{-s} \not\leq U_+$ for all $s \in S$.

RGD4 $G = T\langle U_\alpha | \alpha \in \Phi\rangle$

RGD5 $T \leq \cap_{\alpha\in\Phi} N_G(U_\alpha)$

Based on our setup, it should be of no surprise that our first example of RGD systems comes from strictly Moufang buildings. Suppose Δ is a spherical, strictly Moufang building of type (W, S). Fix an apartment Σ and let U_α be the root group of α for each root of Φ. Let $G = \langle U_\alpha | \alpha \in \Phi\rangle$ and $T = \mathrm{Fix}_G(\Sigma)$. Then $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S).

The goal of this section is similar to that for BN-Pairs, which is to show that RGD systems are more or less equivalent to Moufang buildings in the right circumstances. Suppose that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) where W is spherical. Let $B_+ = TU_+$ and $N = \langle T, \{m(u) | u \in U_s^*, s \in S\}\rangle$. Theorem 7.115 in [4] says that B_+ and N form a BN-pair, and (G, B, N, S) is a Tits system. Furthermore, $B_+ \cap N = T$ and the Weyl group

$N/(B_+ \cap N) = N/T$ is isomorphic to W. In particular, there is a building Δ of type (W, S) on which G will act strongly transitively with respect to some apartment system.

Theorem 7.166 in [4] then also goes on to say that Δ is Moufang. Moreover, if the Coxeter diagram of W has no isolated nodes then Δ is strictly Moufang. We can also choose an apartment Σ_0 of δ and identify the roots Φ_0 of Σ_0 with Φ by $\alpha_0 \mapsto \alpha$ such that for every $\alpha \in \Phi$, the subgroup U_α is exactly the root group U_{α_0}. This theorem says that RGD systems and strictly Moufang buildings encode essentially the same information.

3.2.2 Moufang Polygons

Recall that the rank of a Building/Coxeter complex/Coxeter system is the size of S. Then a *Moufang Polygon* is a spherical, Moufang building of rank 2. If Δ is a Moufang polygon then the Weyl group W has presentation $\langle s, t | s^2 = t^2 = (st)^{m(s,t)} = 1\rangle$ and the spherical condition is equivalent to saying that $m(s, t) < \infty$. As long as $m(s, t) > 2$ we also know that the Coxeter diagram of W is connected and so Δ will also be strictly Moufang.

The Moufang condition is very restrictive and it turns out that rank 2, thick Moufang buildings only exist when $m(s, t) = 3, 4, 6, 8$ as shown in something.

Moufang Polygons are of interest because if we assume that Δ is a 2-spherical Moufang building with a Coxeter diagram that has no labels of 2, then every link of a co-dimension 2 simplex will be a Moufang polygon. The structure of these co-dimension 2 links will be very useful for understanding the group theory and geometry of Δ.

probably some more information should go here but I don't know a good reference right now

3.2.3 Non-Spherical RGD Systems and Twin Buildings

To finish the chapter, we need to add some details to complete the discussion of Moufang buildings. Thus far, the reader may have noticed that we have only discussed the topic of Moufang buildings and RGD systems which are spherical. They may have also noticed that when defining RGD systems, we got a BN-pair by defining $B_+ = T\langle U_\alpha | \alpha \in \Phi_+\rangle$. It seems that we made an arbitrary choice to use B_+ instead of the completely symmetric choice of B_- which is defined similarly. In this section, we will address both of these issues at the same time.

Suppose that (W, S) is an arbitrary Coxeter system with roots Φ. Then we say that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system if it satisfies all of RGD axioms outlined before with the replacement of (RGD2) by the condition that $[U_\alpha, U_\beta] \le U_{(\alpha,\beta)}$ whenever α and β form a pre-nilpotent pair. Recall that if W is spherical then α and β form a pre-nilpotent pair if and only if $\alpha \ne \pm\beta$. Thus this new (RGD2) axiom is equivalent to the old.

Under these assumptions we can form a Tits system (G, B_+, N, S) as before, and we can also form another Tits system (G, B_-, N, S) which gives us two buildings Δ_+ and Δ_-. We can also define an opposition relation between Δ_+ and Δ_- so that the fundamental chambers G/B_+ and G/B_- are opposite. The triple $(\Delta_+, \Delta_-, \mathrm{op})$ is called a *twin building*. A full treatment of Twin buildings can be found in sections 5.8, 6.3, and 8.3 of [4] which we will not repeat here, but we will cover some of the highlights. First of all, twin buildings are a generalization of spherical buildings. If $(\Delta_+, \Delta_-, \mathrm{op})$ is a twin building with spherical Weyl group then there is a canonical isomorphism between Δ_+ and Δ_- such that the opposition relation corresponds exactly with opposition in the spherical sense.

Let Δ be the twin building $(\Delta_+, \Delta_-, \mathrm{op})$. Twin apartments of Δ consist of a pair of apartments $(\Sigma_+, \Sigma-)$ so that every chamber in each half is opposite to exactly 1 chamber of the other half. Twin roots of Δ consist of pairs of roots in twin apartments with appropriate opposition relations. The twin roots of of Δ are in exact correspondence with the roots of either half of the twin building. When we start with an RGD system, we also know that the group G will act strongly transitively on both halves of the twin building. The groups U_α will act just as they did in the spherical case, except they are now acing on twin roots instead of standard roots.

We can define links in twin buildings as we did before, but for the most part we will only be using links in a single half of the twin building, so the theory is identical. There is of course much more theory about twin buildings than what was covered here, but we will mostly use information only in the spherical case and will cite any other results later as they are used.

In the next chapter we will develop more of the group theoretic consequences of the RGD axioms, and we will so how we can use them to answer questions about finite generation.

3.2.4 Kac-Moody Groups

We will take this section to motivate some of the topics in this chapter with classical examples. Perhaps the most well known example is the group $G = \mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$ where k is a field. Then G has an RGD system of type S_n where S_n is the symmetric group on n letters with the standard generating set $(i\ i+1)$. The roots of W correspond to those in a classical root system of type A_{n-1}, and so we get a root group $U_\alpha = U_{ij}$ for each pair $1 \leq i \neq j \leq n$. Each root group U_{ij} consists of matrices with 1's on the diagonal, and all 0's except for possibly in the ij position. The associated building is the same one as described earlier in the chapter, which is Moufang. The BN-pair associated to the RGD system is the same as that described earlier as well.

There are also many more examples. Classical groups such as simplectic, unitary, and orthogonal groups all have RGD systems of spherical type, but we also would like to describe some examples which are not spherical. Let $G = \mathrm{SL}_n(k[t, t^{-1}])$. A more complete treatment of this group can be found in [5] but we will show some of the details here. The Weyl group of G is the affine Coxeter group of type $\tilde{A}_{n-1}$. The roots of W correspond to ij pairs with $i \neq j$ as well as an exponent ℓ of t. Then we have a root group $U_{ij\ell}$ consisting of matrices with 1's on the diagonal, and some multiple of $t^{-\ell}$ in the ij position. The subgroup B_+ is the subgroup $\mathrm{SL}_n(k[t])$ and similarly for B_-.

Perhaps the most motivating example to justify RGD systems is that of Kac-Moody groups as treated in [6]. Kac-Moody groups are a natural extensions of Chevalley groups and have similar presentations by "Steinberg relations". We start with a triple $(\Lambda, (\alpha_i)_{i\in I}, (h_i)_{i\in I})$

where Λ is a free $\mathbb{Z}$ modules with $\alpha_i \in \Lambda$ and h_i in the dual so that $(\langle \alpha_j, h_i \rangle)$ is a generalized Cartan matrix. For any field k we get an RGD system $(\mathcal{G}_D(k), (\mathcal{U}_\alpha(k))_{\alpha \in \Phi}, \mathcal{T}(k))$ where the root system Φ is that for the Weyl group associated to the generalized Cartan matrix. It is also worth noting that each root group $\mathcal{U}_\alpha(k)$ is isomorphic to the additive group of k, and $\mathcal{T}(k)$ is a torus. These Kac-Moody groups also have some additional properties which we will use later.

Chapter 4

Known Results on Finite Generation

Throughout this chapter $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ will be an RGD system of type (W, S). As discussed in the previous chapter, this means that there is a twin building $\Delta = (\Delta_+, \Delta_-, \mathrm{op})$ on which G will act strongly transitively. Our main goal in the rest of the paper is to prove some results about finiteness properties of G and its subgroups. Our main tools will be the geometry of the fundamental apartment Σ, and links of co-dimension 2 simplices in Δ. In the next section we will develop the theory about roots which we will use in the main results. Since the roots of a twin apartment are in exact correspondence with the roots of a single half, we will consider Σ as a standard apartment or Coxeter complex, and not a twin apartment.

4.1 Local Roots and Root Groups

Assume that they Weyl group W of G has rank 3 and that W is 2-spherical. Then the fundamental apartment Σ will be a Coxeter complex which is 2 dimensional, and thus co-dimension 2 simplices of Σ will be points, or vertices. We will also assume that $m(s,t) \geq 3$ for all $s, t \in S$ so that Δ is strictly Moufang, and every link of a vertex will also by strictly Moufang.

For any vertex v of Σ, there will be some walls of Σ which pass through v, and for each of these walls we have a unique *positive* root. We will call these the **positive roots at** v and denote them by Φ_+^v. Recall that $\mathrm{st}(v)$ is the set of all simplices with v as a face, but we will view it as a chamber complex and only consider the chambers with v as a face. If there are n positive roots at v then $|\mathrm{st}(v)| = 2n$.

Furthermore, it is possible to label the positive roots at v as $\alpha_1, \ldots, \alpha_n$ in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. We will call this a *standard labeling* or *standard ordering* of the positive roots at v. This ordering is unique up to a reversal of the form $\alpha_i \mapsto \alpha_{n+1-i}$. In most cases this reversal will not matter, and when it does we will specify a choice of α_1. While this definition may seem strange, it is worth noting that a standard ordering will give an ordering which increases as we move clockwise or counterclockwise, depending on our choice of α_1. The standard labeling also has a nice interpretation for open intervals. If $\alpha_1, \ldots, \alpha_n$ is a standard labeling of the roots through v, and $i < j$ then

$(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$.

If v is a vertex of Δ then the Moufang property, and the assumption $m(s,t) \geq 3$ will imply that $\mathrm{lk}(v)$ is also a Moufang polygon Δ'. We also know that the root groups U_α will be isomorphic to the root groups of Δ'. We define the subgroup $U_v = \langle U_\alpha | \alpha \in \Phi_+^v \rangle$, and note that it is the corresponding U'_+ for the building Δ'. We can also define a subgroup $U'_v = \langle U_1, U_n \rangle \leq U_v$ which is the subgroup of U_v generated by the root groups of simple roots at v. It turns out that "most" of the time, $U'_v = U_v$, which has deep consequences for U_+.

Recall that a spherical building has a notion of opposition when two chambers are as far apart as possible. We will say that a spherical building Δ satisfies condition (co) if for any chamber C, the set $C^{\mathrm{op}} = \{D \in \Delta | C \mathrm{op} D\}$ is gallery connected. Lemma 3 in [1] tells us that for any vertex v of Σ, the index $[U_v : U'_v]$ is equal to the number of connected components, as chamber complexes, of the spherical building $\mathrm{lk}(v)$. In particular, $U_v = U'_v$ if and only if $\mathrm{lk}(v)$ satisfies condition (co). Citing the main result of [1] again we have the following Lemma

Lemma 2. *If v is a vertex of Σ, then $\mathrm{lk}(v)$ satisfies condition (co) unless $\mathrm{lk}(v)$ is the spherical building associated to one of the following finite Chevalley groups*

$$C_2(2) \qquad G_2(2) \qquad G_2(3) \qquad {}^2F_4(2)$$

Moreover, the index $[U_v : U'_v]$ is summarized for all of the exceptional cases in the following table.

U_v	$[U_v : U'_v]$
$C_2(2)$	2
$G_2(2)$	4
$G_2(3)$	3
${}^2F_4(2)$	2

As mentioned in the previous chapter, twin buildings are a generalization of spherical buildings and we define condition (co) in the same manner for twin buildings, where opposition is now the twin building opposition in the two halves of the twin building. Theorem 1.5 in [3] says that a twin building will satisfy property (co) if all of it's rank 2 residues satisfy (co) when viewed as spherical buildings. As a result, it is enough to check that none of the links of co-dimension 2 vertices of Δ are one of the 4 types described above.

For any twin building Δ of type (W, S), and choice of fundamental apartment and fundamental chamber Σ and C we have a canonical set of fundamental roots $\{\alpha_s\}_{s \in S}$ where α_s is the root which contains C and not sC. Then we have the subgroup $U' = \langle U_{\alpha_s} | s \in S \}$ and Lemma 3 of [1] says that $U' = U_+$ if and only if Δ satisfies (co). Note that if Δ has rank 2 then U' is identical to that described above and we get the same result as before. This also gives the following Corollary

Corollary 1. *Let $(G, (U_\alpha), T)$ is an RGD system of type (W, S). If W is 2-spherical, and the associated building Δ does not have any rank 2 residues associated to $C_2(2), G_2(2), G_2(3)$, or ${}^2F_4(2)$ then U_+ is finitely generated.*

Much of the theory of twin buildings relies on condition (co), and thus uses the assumption that no rank 2 residues are among the 4 exceptional types listed above. For example, if Δ

satisfies (co) then U_+ is finitely generated. Our goal for the remainder will be to fill in some of this theory to include cases where Δ does not satisfy (co). Before we can do this, we will need to collect some more results about the 4 exceptional rank 2 buildings listed above.

The groups $C_2(2), G_2(2)$, and $G_2(3)$ are all finite Chevalley groups and so they have well known presentations. The group ${}^2F_4(2)$ is a twisted Chevalley group, but we will not have as much cause to work with this group specifically so we will not work with it as much. A full construction of Chevalley groups can be found in [7], among other places, but we will record the specific presentations found in Corollary 5.2.3.

Lemma 3. *If v is a vertex of Σ such that* $\mathrm{lk}(v)$ *is the building associated to $C_2(2)$, then there is a standard labeling $\alpha_1, \dots, \alpha_4$ of the positive roots at v so that U_v has the following presentation:*

$$U_{\alpha_i} = U_i = \{1, u_i\} \textit{ for all } i$$
$$U_+ = \langle u_i | 1 \leq i \leq 4,\ u_i^2 = 1,\ [u_1, u_4] = u_2 u_3,\ [u_i, u_j] = 1 \textit{ if } |i-j| < 3 \rangle$$

To get the presentations in the $G_2(2)$ and $G_2(3)$ cases, we can derive both presentations at the same time from the results in [7]. If $\mathrm{lk}(v)$ is the building associated to $G_2(k)$, then for each positive root α through v, the group U_α is isomorphic to the additive group of the finite field of order k. This means we can write $U_\alpha = \{x_\alpha(t) | t \in \mathbb{F}_k\}$ and $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$. As a consequence, $x_\alpha(0) = 1$ for all roots α. Then there is a standard labeling $\alpha_1, \dots, \alpha_6$ of the positive roots through v, with $U_i = U_{\alpha_i}$ such that U_v is generated by $U_i = \{x_i(t) | t \in \mathbb{F}_k\}$ subject to the relations

$$\begin{aligned}
[x_1(u), x_6(t)] &= x_5(\pm tu) x_3(\pm tu^2) x_2(\pm tu^3) x_4(\pm 2t^2u^3) \\
[x_1(u), x_5(t)] &= x_3(\pm 2tu) x_2(\pm 3tu^2) x_4(\pm 3t^2u) \\
[x_2(u), x_6(t)] &= x_4(\pm tu) \\
[x_1(u), x_3(t)] &= x_2(\pm 3tu) \\
[x_3(u), x_5(t)] &= x_4(\pm 3tu) \\
[x_i(u), x_j(t)] &= 1 \quad \text{otherwise}
\end{aligned}$$

There is some ambiguity in this presentation from the signs in the relations, but this will not be a problem as when $k = 2$ the signs are irrelevant and when $k = 3$ then sign change replaces a generator of U_i by its inverse. This presentation applies to any Chevalley group of type G_2, so next we will apply it to the two specific groups in question.

Lemma 4. *Suppose v is a vertex of Σ so that* $\mathrm{lk}(v)$ *is the building associated to $G_2(2)$. Then there is a standard labeling of the positive roots through v with $U_i = U_{\alpha_i}$ such that*

$U_v = \langle U_i | 1 \leq i \leq 6 \rangle$ and a presentation is given by the following relations

$$\begin{aligned}
U_i &= \{1, u_i\} \\
[u_1, u_6] &= u_5 u_3 u_2 \\
[u_1, u_5] &= u_2 u_4 \\
[u_2, u_6] &= u_4 \\
[u_1, u_3] &= u_2 \\
[u_3, u_5] &= u_4 \\
[u_i, u_j] &= 1 \quad \textit{otherwise}
\end{aligned}$$

Lemma 5. *Suppose v is a vertex of Σ so that $\mathrm{lk}(v)$ is the building associated to $G_2(3)$. Then there is a standard labeling of the positive roots through v with $U_i = U_{\alpha_i}$ such that $U_v = \langle U_i | 1 \leq i \leq 6 \rangle$ and a presentation is given by the following relations*

$$\begin{aligned}
U_i &= \{1, x_i(1), x_i(-1)\} \\
[x_1(u), x_6(t)] &= x_5(c_1 tu) x_3(c_2 tu^2) x_2(c_3 tu) x_4(c_4 t^2 u) \\
[x_1(u), x_5(t)] &= x_3(c_5 tu) \\
[x_2(u), x_6(t)] &= x_4(c_6 tu) \\
[x_i(u), x_j(t)] &= 1 \quad \textit{otherwise}
\end{aligned}$$

where each $c_i \in \{\pm 1\}$.

So far, we know that for the 4 exceptional cases listed above we have $U_v' \neq U_v$ and we know the index. The next few results will be to collect properties about U_v' and U_v which we will use later when proving results about finite generation.

Lemma 6. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$. If $\mathrm{lk}(v)$ is the building associated to $C_2(2), G_2(3)$, or ${}^2F_4(2)$ then U_v' is a normal subgroup of U_v. If $\mathrm{lk}(v)$ is the building associated to $G_2(2)$ then U_v' is not normal, but there is a standard labeling of the positive roots through v such that $U_v'' = \langle U_1, U_2, U_6 \rangle$ is proper and normal.*

Proof. First suppose $\mathrm{lk}(v)$ is the building associated to either $C_2(2)$ or ${}^2F_4(2)$. Then U_v' is an index 2 subgroup of U_v and thus it is normal.

Now suppose $\mathrm{lk}(v)$ is the building associated to $G_2(3)$. To show that U_v' is normal it will suffice to show that $x_i(t) U_j x_i(-t) \in U_v'$ for $1 \leq i \leq 6$ and $j \in \{1, 6\}$. Since most of the commutators are trivial, we can use the presentation in Lemma 5 to calculate

$$\begin{aligned}
[[x_1(u), x_6(t)], x_6(t')] &= [x_5(c_1 tu) x_3(c_2 tu^2) x_2(c_3 tu) x_4(c_4 t^2 u), x_6(t')] \\
&= [x_2(c_3 tu), x_6(t')] \\
&= x_4(c_3 c_6 t t' u)
\end{aligned}$$

which means $U_4 \subset U_v'$. A similar computation shows that $[x_1(t'), [x_1(u), x_6(t)]]$ is a non-trivial element of U_3 so that $U_3 \subset U_v'$ as well.

The presentation in Lemma 5 tells us that U_i and U_j commute when $|i-j| \leq 3$. Therefore we have $x_i(t)U_1x_i(-t) = U_1 \subset U_v'$ for all $i \geq 3$, and similarly we have $x_i(t)U_6x_i(-t) = U_6$ for all $i \leq 4$. It is also clear that $x_1(t)U_6x_1(-t) \subset U_v'$ and $x_6(t)U_1x_6(-t) \subset U_v'$. There are only two conjugates left to check and we can see that

$$x_5(u)x_1(t)x_5(-u) = [x_5(u), x_1(t)]x_1(t)x_5(u)x_5(-u) \subset U_3U_1 \subset U_v'$$

and similarly we have

$$x_2(u)x_6(t)x_2(-u) = [x_2(u), x_6(t)]x_6(t)x_2(u)x_2(-u) \subset U_4U_6 \subset U_v'$$

which shows U_v' is normal as desired.

Now suppose that $\mathrm{lk}(v)$ is the building associated to $G_2(2)$ and choose the standard labeling of the positive roots through v so that we have the presentation given in Lemma 4. Let f be a map from $\{u_i\}_{1\leq i\leq 6}$ to the group with two elements $\{\pm 1\}$ such that $f(u_i) = 1$ if $i \in \{1,2,4,6\}$ and $f(u_i) = -1$ if $i \in \{3,5\}$. If we check the presentation given in Lemma 4 we can see that f will extend to a well defined group homomorphism $f : U_v \to \{\pm 1\}$ which is surjective. Thus $\ker f$ has index 2 in U_v and it contains $U_v'' = \langle U_1, U_2, U_6\rangle$ by definition.

The group U_v is generated by the groups U_i and thus is generated by the elements u_i for $1 \leq i \leq 6$. Since $u_2, u_6 \in U_v''$ we also know that $[u_2, u_6] = u_4 \in U_v''$ as well. By the presentation in Lemma 4 this means that $[u_i, u_3], [u_j, u_5] \in U_v''$ for all i, j. Using the fact that $xy = yx[y,x]^{-1}$ we can commute u_3 and u_5 past any element of U_v'', picking up only other elements of U_v''. Since u_3 and u_5 commute we can say that $U_v = U_5U_3U_v''$. The cosets of U_v'' are $U_v'', u_5U_v'', u_3U_v'', u_5u_3U_v''$, but since $u_5u_3 = [u_1, u_6]u_2 \in U_v''$ we get $U_v'' = u_5u_3U_v''$ and $u_5U_v'' = u_3U_v''$ so that $[U_v : U_v''] \leq 2$. Since $U_v'' \subset \ker f$ and $[U_v : \ker f] = 2$, we must have $U_v'' = \ker f$ so that U_v'' is proper an normal. Now it remains to show that U_v' is not normal.

If U_v' was normal then there would be a surjective map $g : U_v \to U_v/U_v' = H$ where $|H| = 4$ and therefore is abelian. This means that $g(u_2) = g([u_1, u_3]) = [g(u_1), g(u_3)] = 1$ and thus $u_2 \in \ker g$. But $u_1, u_6 \in \ker g$ by definition and thus $U_v'' \subset \ker f$. This is a contradiction as $\ker f = U_v'$ by definition, and U_v'' strictly contains U_v', and thus U_v' is not normal.

□

Using Lemma 6 and elementary group theory, we get the following result.

Corollary 2. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$. Then there is a non-trivial cyclic group H and a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$ where U_1 and U_n are the simple root groups at v.*

Proof. If $[U_v : U_v'] \geq 2$ then $\mathrm{lk}(v)$ must be isomorphic to the building associated to one of $C_2(2), G_2(2), G_2(3), {}^2F_4(2)$. If the associated group is one of $C_2(2), G_2(3), {}^2F_4(2)$ then we can apply Lemma 6 to let $H = U_v/U_v'$ and ϕ_v be the quotient map which certainly will be surjective and send U_1 and U_n to $\{1\}$ by the definition of U_v'. The group H is cyclic because it has prime order.

If $\mathrm{lk}(v)$ is isomorphic to the building associated to $G_2(2)$ then we know that $U'_v \subset U''_v = \langle U_1, U_2, U_6 \rangle$ for an appropriate standard labeling, and we again apply Lemma 6 to set $H = U_v/U''_v$ and ϕ_v as the quotient map. The group H has order equal to $[U_v : U''_v]$ which must be 2 as $U''_v \neq U_v$ and $U''_v \neq U'_v$, and thus H is cyclic as desired. □

The following corollary will show that we do not have very much wiggle room when defining ϕ_v, and $\ker \phi_v$ is uniquely determined by the fact that ϕ_v sends U_1 and U_n to the identity.

Corollary 3. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U'_v] \geq 2$ and let ϕ_v be defined as in the previous corollary. Then $\ker \phi_v$ is the unique, proper, normal subgroup of U_v which contains U_1 and U_n.*

Proof. First suppose that $\mathrm{lk}(v)$ is the building associated to $C_2(2), G_2(4)$, or ${}^2F_4(2)$. Then by the construction in Corollary 2 we know that $\ker \phi_v = U'_v$ and U'_v contains U_1, U_n by definition. In all of these cases the index $[U_v : U'_v]$ is prime and thus U'_v is the unique proper, normal subgroup of U_v which contains U_1, U_n.

Now suppose that $\mathrm{lk}(v)$ is the building associated to $G_2(2)$. The proof of Corollary 2 shows that any normal subgroup of U_v which contains U_1 and U_6 must also contain U_2 and thus U''_v. Since $\ker \phi_v = U''_v$ and $[U_v : U''_v] = 2$, $\ker \phi_v$ is once again the unique proper normal subgroup of U_v containing U_1 and U_6. □

Despite the fact that U_v is not generated by U_1 and U_n, it will be helpful to show which root groups will generate U_v. This will be necessary later when we prove that U_+ is finitely generated in certain cases.

Lemma 7. *Suppose v is a vertex of Σ such that $\mathrm{lk}(v)$ is the Moufang polygon associated to $C_2(2)$ or $G_2(3)$. If $\alpha_1, \ldots, \alpha_n$ is a standard ordering of the positive roots through v which gives the presentation as in Lemma 3 and 5, then $U_v = \langle U_1, U_2, U_n \rangle = \langle U_1, U_{n-1}, U_n \rangle$.*

Proof. Let $H = \langle U_1, U_2, U_n \rangle$ and let $K = \langle U_1, U_{n-1}, U_n \rangle$. In both cases we have $U'_v \leq H, K \leq U_v$, and since $[U_v : U'_v]$ is prime, we get $H = U'_v$ or U_v and similarly for K.

Now suppose $\mathrm{lk}(v)$ is associated to $C_2(2)$. Using the presentation we know that $u_1, u_2, u_4 \in H$ and thus $u_2[u_1, u_4] = u_3 \in H$. Since U_v is generated by $\{u_1, u_2, u_3, u_4\}$ we get $H = U_v$. Similarly, $u_2 = [u_1, u_4]u_3 \in K$ so $K = U_v$ as well.

Now suppose $\mathrm{lk}(v)$ is associated to $G_2(3)$. Since the presentation is more complicated in this case we can use a slightly different argument. By Corollary 2, there is a surjective homomorphism $\phi_v : U_v \to C$ where C is a non-trivial cyclic group such that $U'_v \subset \ker \phi_v$. Since C is cyclic we get $\phi_v(x_4(c_6)) = \phi_v([x_2(1), x_6(2)]) = [\phi_v(x_2(1)), \phi_v(x_6(1))] = 1$ and thus $U_4 \subset \ker \phi_v$. A similar argument shows that $U_3 \subset \ker \phi_v$.

If $H = U'_v$ then get $U_2 \subset \ker \phi_v$ as well. This means that $x_5(c_1) = [x_1(1), x_6(1)]x_4(-c_4)x_2(-c_3)x_3(-c_2) \in \ker \phi_v$ and thus $U_5 \subset \ker \phi_v$. Since $\ker \phi_v$ contains U_i for all $1 \leq i \leq 6$, it must be the trivial map which is a contradiction, as it is a surjection onto a non-trivial group. Thus $H \neq U'_v$ and $H = U_v$ as desired. A similar argument shows that $K = U_v$ and thus $U_v = \langle U_1, U_2, U_n \rangle = \langle U_1, U_{n-1}, U_n \rangle$ as desired. □

So far we have only considered each vertex v and U_v separately. But in the Coxeter complex Σ, we have not only a collection of vertices, but an action of the group W on the vertices which behaves nicely with properties like the type of a vertex. We will show that the W action also interacts nicely with U_v and ϕ_v in a similar way.

Lemma 8. *Suppose v is a vertex of Σ of type s, $|\mathrm{st}(v)| = 2n$, and $[U_v : U_v'] \geq 2$. Also suppose that w is an element of W such that $w\gamma$ is a positive root at wv for every positive root γ at v. Then there are standard labelings $\alpha_1, \ldots, \alpha_n$ and $\alpha_1', \ldots, \alpha_n'$ of the positive roots through v and wv respectively such that $\alpha_i' = w\alpha_i$ for all i. In particular, w sends roots at v which are simple to roots at v' which are also simple. Furthermore, if v' is any vertex of Σ of type s then there is a $w \in W$ such that $wv = v'$ and $w\gamma$ is a positive root at v' for any positive γ at v.*

Proof. Recall a standard labeling is on of the form $\alpha_1, \ldots, \alpha_n$ where $\alpha_i \cap \alpha_j \subset \alpha_k$ for all $1 \leq i \leq k \leq j \leq n$. If w sends all of the positive roots at v to the positive roots at wv then w induces a bijection on the positive roots at v and wv. Now we can define a labeling of the positive roots at wv by $\alpha_i' = w\alpha_i$ for all i. It only remains to check that this is a standard labeling. If $1 \leq i \leq k \leq j \leq n$ then $\alpha_i \cap \alpha_j \subset \alpha_k$ and thus $\alpha_i' \cap \alpha_j' = w\alpha_i \cap w\alpha_j \subset w\alpha_k = \alpha_k'$ so this is a standard labeling as desired.

Now it suffices to show that such a w exists for any vertex v' in Σ. Since the W action on Σ is transitive on vertices of the same type, it will suffice to show the result when v is a vertex of the fundamental chamber C. Let $D = \mathrm{Proj}_{v'}(C)$ so that $d(D, C)$ is minimal among all chambers of $\mathrm{st}(v')$. Then we know that no walls through v' can separate D and C, because crossing one of these walls would produce a chamber in $\mathrm{st}(v)$ which is closer to C. Therefore, a root at v' is positive if and only if it contains D.

Now choose the unique $w \in W$ such that $D = wC$. We claim that w satisfies the desired properties. First of all, v is a vertex of C of type s and thus wv is a vertex of $wC = D$ of type s. But we know that v' is a vertex of D of type s by definition and thus $wv = v'$ as desired. Now suppose γ is any positive root at v. Then $C \in \gamma$ and thus $D = wC \in w\gamma$ and thus $C \in w\gamma$ so $w\gamma$ is positive at $wv = v'$. Now this w sends positive roots at v to positive roots at v' as desired.

□

Before moving on it is worth clarifying that the type s of the vertex v in the previous lemma can by any type, not just the literal type s in the definition of W.

The previous result can also be used to show that the W action on Σ also behaves nicely with respect to the group U_v and the homomorphisms ϕ_v when they exit.

Corollary 4. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ and $[U_v : U_v'] \geq 2$ and v' is any other vertex of Σ of the same type. Then there is an isomorphism between U_v and $U_{v'}$ which sends U_v' to $U_{v'}'$. Consequently, $[U_v : U_v'] = [U_{v'} : U_{v'}']$, ϕ_v exists if and only if $\phi_{v'}$ exists, and if ϕ_v exists then this isomorphism sends $\ker \phi_v$ to $\ker \phi_{v'}$. If w is any element of W such that $wv = v'$ and $w\gamma$ is positive for all positive γ at v, then this isomorphism can be defined by the property that U_γ is sent to $U_{w\gamma}$ for every γ at v.*

Proof. Let w be any element of W with $wv = v'$ which sends positive roots at v to positive roots at v'. Such a w is guaranteed to exist by Lemma 8. By Proposition 8.54 in [4] and the there is an element $\tilde{w} \in G$ such that $\tilde{w}U_\alpha(\tilde{w})^{-1} = U_{w\alpha}$ for all $\alpha \in \Phi$. Let $f_w : G \to G$ be the isomorphism of conjugation by $\tilde{w}$. Since $w\gamma$ is positive at v' for every positive root γ at v we know that $f_w(U_\gamma) = U_{w\gamma} \subset U_{v'}$ and thus f_w restricts to a homomorphism $\bar{f}_w : U_v \to U_{v'}$ which is necessarily injective. But w also give a bijection on positive roots at v and v', and $U_{v'}$ is generated by positive root groups at v' so $\bar{f}_w$ is surjective and thus an isomorphism. Now it remains to check it satisfies the rest of the properties.

Since w preserves standard labelings at v and v' we know that it also preserves simple roots. Thus $\bar{f}_w(U_{\alpha_1}) = U_{\alpha_1'}$ for a standard labeling, and similarly for U_{α_n} and $U_{\alpha_n'}$. Since $U_v' = \langle U_{\alpha_1}, U_{\alpha_n}\rangle$ and $U_{v'}' = \langle U_{\alpha_1'}, U_{\alpha_n'}\rangle$ we can also see that $\bar{f}_w$ sends U_v' to $U_{v'}'$. Since $\bar{f}_w$ is an isomorphism it also preserves index so $[U_v : U_v'] = [U_{v'} : U_{v'}']$.

For any vertex v, the map ϕ_v exists if and only if $[U_v : U_v'] \geq 2$ and thus ϕ_v will exist exactly when $\phi_{v'}$ exists. By Corollary 3 we know that $\ker\phi_v$ is a proper normal subgroup of U_v containing U_v' and thus $\bar{f}_w(\ker\phi_v)$ will be a proper, normal subgroup of $U_{v'}$ containing $U_{v'}'$. By Corollary 3 again this means $\bar{f}_w(\ker\phi_v) = \ker\phi_{v'}$ which completes the result. □

The main idea of our results will be to extend the map ϕ_v in a certain way to a map on all of U_+, and the main difficulty in the proof will be to show that this extension is well defined. Perhaps the easiest way to prove this is to use a presentation of U_+ and the universal property which says if we define a map on generators, which sends all relations to the identity, then the map defines a homomorphism. The group U_+ does admit a nice presentation as shown in Theorem 8.84 of [4], which we will repeat here in the following lemma

Lemma 9. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be and RGD system. For each $\alpha \in \Phi$ choose a set $S_\alpha \subset U_\alpha$, and a set of words R_α with letters in S_α so that $\langle S_\alpha | R_\alpha\rangle$ is a presentation of U_α. Then for each pre-nilpotent pair $\{\alpha, \beta\}$ and any $u_\alpha \in S_\alpha$ and $u_\beta \in S_\beta$, we can write $[u_\alpha, u_\beta] = v$ where v is a word in $\cup_{\gamma\in(\alpha,\beta)} S_\gamma$. Furthermore, one obtains a presentation of U_+ by combining the relations R_α for all α as well as the commutator relations $[u_\alpha, u_\beta] = v$ where α, β range over all pre-nilpotent pairs, and u_α, u_β range over all of S_α and S_β.*

There is another result which we will use extensively in the following chapters, but which is slightly different from what we have done so far. We state the following fact about the geometry of certain Coxeter complexes.

Lemma 10 (Triangle Condition)**.** *Suppose Σ is a Coxeter complex of type (W, S) where $S = \{s, t, u\}$, $3 \leq m(s, t) \leq m(s, u) \leq m(t, u) < \infty$, and $m(t, u) \geq 4$. Let $\alpha_1, \alpha_2, \alpha_3$ be roots of Σ such that $\partial\alpha_i \cap \partial\alpha_j \neq \emptyset$ for all i, j but $\partial\alpha_1 \cap \partial\alpha_2 \cap \partial\alpha_3 = \emptyset$. If we assume that $\partial\alpha_i \cap \partial\alpha_j \subset \alpha_k$ for $i \neq j \neq k$ then $\alpha_1 \cap \alpha_2 \cap \alpha_3$ is a chamber of Σ.*

The previous lemma essentially says that a "Triangle" formed by 3 walls of Σ under the specified conditions must be a single chamber. One way we will use this lemma is by showing two walls cannot intersect, if the resulting triangle would contain more than one chamber.

In the next two chapters we will prove new results about finite generation in RGD systems when the associated building Δ has exceptional links as described in this chapter.

Chapter 5

Conditions for Infinite Generation

Throughout this chapter $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) and associated twin building Δ with the following assumptions:

$$\begin{aligned}&S = \{s,t,u\},\ a = m(s,t), b = m(s,u), c = m(t,u)\\&3 \le a,b,c\\&U_\alpha \text{ is finitely generated for all } \alpha \in \Phi\\&[U_\alpha, U_\beta] = 1 \text{ when } \alpha,\beta \text{ are nested}\end{aligned} \tag{A}$$

Before moving on we should address each one of these conditions to determine what role it will play. The first condition simply says that the twin building Δ is 2-dimensional. The second condition excludes the possibility of Moufang quadrangles as rank 2 links, and it ensures that every link will be strictly Moufang. As we are interested in questions about finite generation, there is no hope in proving G or U_+ is finitely generated if the root groups themselves are not finitely generated, so we must include this restriction to say anything at all. Finally, the condition that $[U_\alpha, U_\beta] = 1$ if α, β are nested is a strengthening of the commutator relations in G which will allow us to define certain homomorphisms in a nice way.

It is important to note that we are not being too restrictive, and there are still lots of examples of RGD systems which satisfy these properties. For example, any Kac-Moody group of rank 3 over a finite field will satisfy the second two conditions, and the first two conditions are determined by the Weyl group W.

We know by Corollary 1 that U_+ is finitely generated if Δ has no exceptional rank 2 links. In the next two chapters we will determine when the same result will hold if Δ does have exceptional links. The general idea of the proof is as follows. For any vertex v with an exceptional link we have a surjective homomorphism $\phi_v : U_v \to H$ where H is cyclic and thus abelian. We will attempt to extend this homomorphism to all of U_+ in such a way that U_β is sent to the identity if $U_\beta \not\subset U_v$. Since ϕ_v is surjective, the extension will also be surjective, and therefore any generating set of U_+ must contain at least 1 element of U_v. If we can do this for "enough" U_v then we can potentially use this to prove that U_+ cannot be finitely generated. The difficulty lies in checking that the desired extension will be well defined, but we have a presentation of U_+ so it becomes a matter of checking that certain commutator relations are satisfied. Moreover, since the co-domain is abelian, commutators will automatically vanish which simplifies the conditions which need to be checked.

5.1 Extension of ϕ_v

Let Σ be the Coxeter complex of W with fundamental chamber C, and Φ_+ be the positive roots of Σ. We will also let $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$ be the subgroup of G generated by the positive root groups. The Moufang property implies that $a, b, c \in \{2, 3, 4, 6, 8\}$ and thus (A) implies that $a, b, c \in \{3, 4, 6, 8\}$. We will also assume that Δ, and thus Σ, has a vertex v such that $\mathrm{lk}(v)$ is the building associated to one of the 4 exceptional Moufang polygons. Without loss of generality we will say that v has type s, and thus $c = m(t, u) \geq 4$.

Let v be a vertex of Σ such that $\mathrm{lk}(v)$ is the Moufang polygon associated to $C_2(2), G_2(2), G_2(3)$, or ${}^2F_4(2)$. Equivalently, this means $[U_v : U_v'] \geq 2$. As described in the previous chapter we will let $\alpha_1, \ldots, \alpha_n$ be a standard ordering of the positive roots through v and we will define $U_i = U_{\alpha_i}$ for all $1 \leq i \leq n$. By Corollary 2, there is a surjective homomorphism $\phi_v : U_v \to H$ where H is a cyclic group and $\phi_v(U_1) = \phi_v(U_n) = \{1\}$. We would like to extend ϕ_v to a map $\tilde{\phi}_v : U_v \to H$ in a specific way to use later. Our first lemma will define our notion of extending ϕ_v, and give a sufficient condition for this extension to exist.

Lemma 11. *Suppose that v is a vertex of Σ such that $U_v' = \langle U_1, U_n \rangle \neq U_v$, where U_1, U_n are the simple root groups at v. Then there is a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$, where H is a cyclic group. Also suppose that for any positive root γ with $v \in \partial\gamma$ which is not simple at v, that γ is simple at y for all $y \in \partial\gamma$ with $y \neq v$. Then the map $\tilde{\phi}_v : \cup_{\gamma \in \Phi_+} U_\gamma \to H$ defined by*

$$\tilde{\phi}_v(u) = \begin{cases} \phi_v(u) & \text{if } u \in U_\gamma \text{ and } v \text{ lies on } \partial\gamma \\ 1 & \text{otherwise} \end{cases}$$

Extends uniquely to a well defined group homomorphism $\tilde{\phi}_v : U_+ \to H$.

Proof. Since $U_v' \neq U_v$ we know that the map ϕ_v exists by Corollary 2. We have a presentation for U_+ and we have defined $\tilde{\phi}_v$ on the generators of U_+, so in order to check that it is well defined we will need to verify that the relations of U_+ are satisfied in the image.

There are three types of relations in the presentation for U_+. There are relations within the same root group U_α for all positive roots α. There are also relations between root groups of pre-nilpotent pairs where either the walls intersect or the roots are nested.

Let R_α be a relation for U_α where R_α is considered as a word with letters in U_α. If v lies on $\partial\alpha$ then $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$ since ϕ_v is a well defined homomorphism. Otherwise, every element of U_α is sent to 1 and thus $\tilde{\phi}_v(R_\alpha) = 1$ as well so that R_α is mapped to the identity as desired.

Now suppose that α and β are any two positive roots. If α, β nested, then (A) tells us that $[U_\alpha, U_\beta] = 1$. Since the co-domain of $\tilde{\phi_v}$ is an abelian group, then any relation of the form $[x, y] = 1$ will be satisfied by the image.

Now suppose that $\partial\alpha$ and $\partial\beta$ meet at a point y and consider any relation of the form $[u_\alpha, u_\beta] = w$ where $u_\alpha \in U_\alpha$, $u_\beta \in U_\beta$, and w is a word in $U_{(\alpha,\beta)} \subset U_y$. Again, note that the image of the left side of this equation will always be the identity as the co-domain is still abelian. If $y = v$ then $U_y = U_v$ and thus $\tilde{\phi_v}(w) = \phi_v(w) = 1$ because ϕ_v is well defined.

Now suppose that $y \neq v$. Then we can label the positive roots passing through y as $\gamma_1, \cdots, \gamma_r$ in such a way that $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$ whenever $i < j$. In this case we can can say without loss of generality that $\alpha = \gamma_l$ and $\beta = \gamma_m$ with $l < m$. There can be at most one root whose wall passes through y and v, which we will call γ_k if it exists. If γ_k does not exist, or $k \leq l$ or $k \geq m$ then the root γ_k is not contained in (α, β) and thus $\tilde{\phi_v}(U_\delta) = 1$ for all $\delta \in (\alpha, \beta)$. This means $\tilde{\phi_v}(w) = 1$ and the relation is satisfied.

Now we suppose that γ_k exists and $l < k < m$. Then γ_k is not simple at y and thus γ_k must be simple at v by assumption. This means $\tilde{\phi_v}(U_{\gamma_k}) = \phi_v(U_{\gamma_k}) = 1$ by the construction of ϕ_v. Since $\tilde{\phi_v}(U_{\gamma_i}) = 1$ for all $i \neq k$ by definition, this means that $\tilde{\phi_v}(w) = 1$ showing the relation is satisfied and giving the desired result. □

Now Lemma 11 gives a sufficient condition for the existence of $\tilde{\phi_v}$ which is fairly easy to check. This will be the main tool we use in the remainder of the section.

Recall from our assumption (A) that (W, S) is a rank 3 Coxeter system with $S = \{s, t, u\}$, and $m(s, t) = a, m(s, u) = b, m(t, u) = c$. We assumed that $3 \leq a, b, c$. Let C be the fundamental chamber of Σ and let x be the vertex of C of type s, so that $|\mathrm{st}(v)| = 2c$. If we assume that Σ does contain exceptional links then we can say without loss of generality that $[U_x : U'_x] \geq 2$ so that ϕ_x exists. We would like to apply Lemma 11 to show that $\tilde{\phi_x}$ exists, but before we do so we need the following result.

Lemma 12. *Let x be the vertex of C of type s. If γ is any positive root at x, and y is any other vertex on $\partial\gamma$, then γ is simple at y.*

Proof. Suppose that γ is not simple at y. Then we can label the positive roots at y as $\delta_1, \ldots, \delta_m$ in such a way that $\delta_i \cap \delta_j \subset \delta_k$ for $1 \leq i \leq k \leq j$. In this case we have δ_1, δ_m are simple at y and $\gamma = \delta_r$ for some $1 < r < m$. But x is a vertex of C and $C \in \delta_1 \cap \delta_m$ and thus $x \in \delta_1 \cap \delta_m$ as well. We know that x lies on $\partial\delta_r$ by assumption and thus x is an element of $\partial\delta_r \cap \delta_1 \cap \delta_m$. But this is impossible as we can observe from the geometry of Σ that $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$ for all $1 < i < m$. Thus γ is simple at y as desired. □

Despite some of the technical details the previous result should be intuitively clear. The walls through y will divide Σ into $2m$ regions, and the region which contains C will be bounded by the two simple roots. Since x lies on $\partial\gamma$, it is impossible for any other roots through y to be any "closer" to C and thus γ must be simple at y as we proved.

Corollary 5. *Let x be the vertex of C of type s, and assume that $[U_x : U'_x] \geq 2$. Then the map $\tilde{\phi_x}$ as defined in Lemma 11 is well defined.*

Proof. Let γ be any non-simple, positive root through x and let y be another vertex on $\partial\gamma$. Then by the previous lemma, γ is simple at y and thus $\tilde{\phi}_x$ exists by Lemma 11. □

The remainder of the section will be used to show that we can use $\tilde{\phi}_x$ and the W action on Σ to construct a large family of vertices $\{v\}$ for which $\tilde{\phi}_v$ exists.

Let $\alpha_1, \ldots, \alpha_n$ be a standard ordering of the positive roots through x. Recall from chapter 1 that we can describe any root in terms of a pair of adjacent chambers. We can also identify $\mathcal{C}(\Sigma)$ with W where the chamber wC is associated to w. If we use this identification then we can describe the roots as follows

$$\alpha_1 = \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\}$$
$$\alpha_n = \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}$$

In a similar way we can define two more roots

$$\beta = \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}$$
$$\beta' = \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}$$

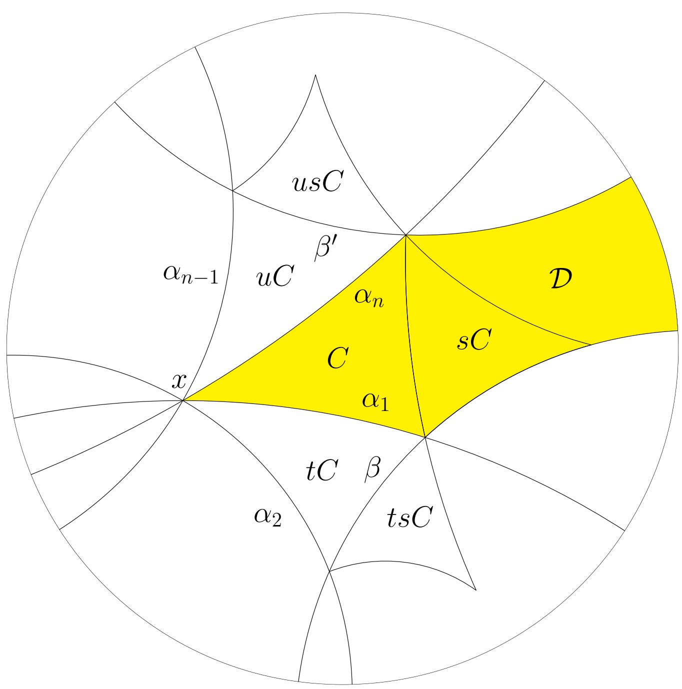


Figure 5.1: The Roots $\alpha_1, \alpha_n, \beta, \beta'$ with the region $\mathcal{D}$ in yellow.

Now we can define $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. These roots are chosen and $\mathcal{D}$ is defined in such a way to give the following lemma:

Lemma 13. *Let x be the vertex of C of type s and assume $W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^a = (su)^b = (tu)^c = 1 \rangle$. Let $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where $\alpha_1, \alpha_n, \beta, \beta'$ are roots of Σ defined by*

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

If γ is a positive root at x which is not simple at x, and δ is any other positive root such that $\partial\gamma \cap \partial\delta \neq \emptyset$, then $\mathcal{D} \subset \gamma \cap \delta$.

Proof. By assumption, γ is a positive root through x so $\gamma = \alpha_i$ for some i. Furthermore, we assumed that γ was not simple which means $2 \leq i \leq n-1$. Since α_1 and α_n are simple at x we can see that $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$. Thus it will suffice to prove that $\mathcal{D} \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$ and thus $\mathcal{D} \subset \delta$ so that $\mathcal{D} \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_n$. Thus the point y will lie in exactly one of α_1 or α_n.

First suppose that $y \in \alpha_n$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilities would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D} \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_n$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_n$. Since neither y or y' lies on $\partial\alpha_n$ this means that $y' \in \alpha_n$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. Let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_n$. If $E = D'$ then $D' \in \alpha_n$ since x, y, y' all lie in α_n. But this is a contradiction as α_n cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D} \subset \beta = \delta$ and $\mathcal{D} \subset \gamma \cap \delta$ as desired. A depiction of this situation can be found in Figure 5.2.

If we assume instead that $y \in \alpha_1$ so that $y \notin \alpha_n$ then identical arguments show that $\delta = \beta'$ and we can again conclude that $\mathcal{D} \subset \gamma \cap \delta$ as desired. □

We are now ready to construct a large family of vertices $\{v\}$ for which $\tilde{\phi}_v$ will exist. The idea is as follows. If we take any chamber in $\mathcal{D}$ and treat it as a new "C" then $\tilde{\phi}_x$ would

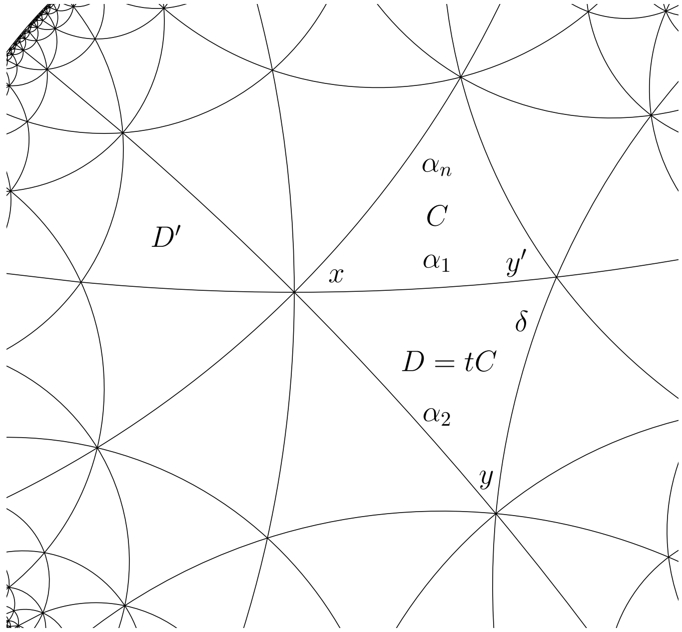


Figure 5.2: $\partial\alpha_1 \cap \partial\delta = y'$

exist for this "C." So what we do is apply elements of W which map the chambers of $\mathcal{D}$ to C, and use these choices of w to get new vertices v. We can use Lemma 8 to show that this W action will play nicely with the map ϕ_v.

Lemma 14. *Let x be the vertex of C of type s, and assume $U'_x \neq U_x$. If v is a vertex in $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ of type s then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. Let $D = \mathrm{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 4.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 11 at the vertex wx. Now suppose that γ is a non-simple, positive root through wx and y is another vertex on $\partial\gamma$. We must show that γ is simple at y. Since γ is positive through wx we know that $C, E \in \gamma$. If we apply w^{-1} then we get the following facts. We know that $w^{-1}\gamma$ is a root such that $\partial(w^{-1}\gamma)$ passes through $w^{-1}wx = x$. We also know that $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is also a positive root. Since w^{-1} sends positive roots at wx to positive roots at x we can apply Lemma 8 when necessary.

The first claim is that $w^{-1}\gamma$ is not simple at x. Suppose that δ is any positive root at wx. Then $E \subset \delta$ and so applying w^{-1} we get that $w^{-1}E = C \subset w^{-1}\delta$. By Lemma 8 this means that w^{-1} sends simple roots at wx to simple roots at x. Since γ is not simple at wx this means that $w^{-1}\gamma$ is not simple at x.

So $w^{-1}\gamma$ is a non-simple positive root at x, and since y lies on $\partial\gamma$ we also know that $w^{-1}y$ lies on $w^{-1}(\partial\gamma)$. If we apply Lemma 12 we can see that $w^{-1}\gamma$ must be simple at $w^{-1}y$.

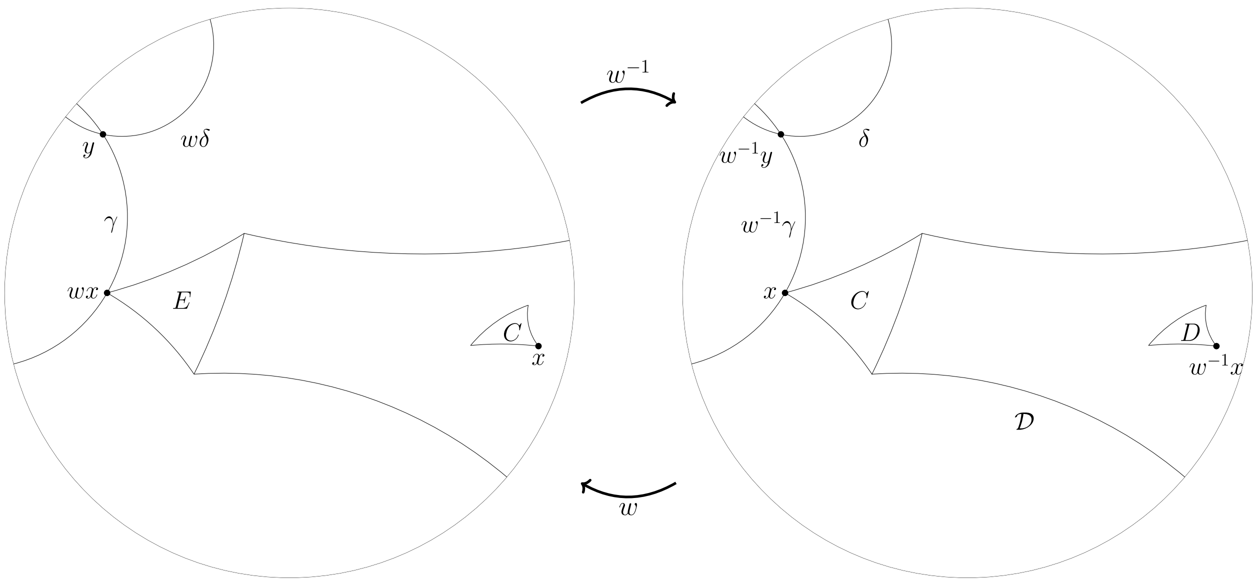


Figure 5.3: The effect of w and w^{-1} on the chambers and roots.

Recall that $D \in \mathcal{D}$ by assumption. Now suppose that δ is any positive root at $w^{-1}y$. Then by Lemma 13 we know that $D \in \mathcal{D} \subset \delta$. If we apply w then we get $C = wD \in w\delta$ and $w\delta$ is a root through y. Thus $w\delta$ is a positive root through y and therefore w sends positive roots through $w^{-1}y$ to positive roots through y. Again we can apply Lemma 8 to say that w must also send simple roots through $w^{-1}y$ to simple roots through y. But $w^{-1}\gamma$ was a simple root through $w^{-1}y$ and thus γ is simple at y as desired.

We now have a vertex wx where $[U_{wx} : U'_{wx}] = [U_x : U'_x] \geq 2$ and the positive roots at wx which are not simple at wx are simple everywhere else. Thus we can apply Lemma 11 to say that $\tilde{\phi}_{wx}$ exists as desired. $\square$

Now we have shown that vertices of $\mathcal{D}$ in some way correspond to $\tilde{\phi}_v$. If our goal is to find infinitely many such v then there is still some work to be done. For instance, we do not yet know if the region $\mathcal{D}$ contains infinitely many chambers, or even if it does, if all the vertices of D lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

5.2 When $\mathcal{D}$ is infinite

Our first task will be two show that the region $\mathcal{D}$ contains infinitely many unique vertices. Intuitively, this will happen if the walls for β and β' do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that W is defined by the edge labels $a = m(s,t), b = m(s,u), c = m(t,u)$ we assumed that the vertices of type s were exceptional which implies $c \geq 4$. We will show that if we also assume that $b \geq 4$ then the region $\mathcal{D}$ will contain infinitely many chambers.

Lemma 15. *Let (W,S) be a rank 3 Coxeter system defined by $a = m(s,t), b = m(s,u), c = m(t,u)$ with $3 \leq a$ and $4 \leq b, c$. If we let $w_k = (tus)^k$ for all $k \geq 0$, then the vertices $(w_k)^{-1}x$ are all distinct from one another and all lie in $\mathcal{D}$.*

Proof. Note that $(w_k)^{-1} = (sut)^k$ for all k. First we will show that $(w_k)^{-1}x \in \mathcal{D}$ for all k. Since x is a vertex of C we know that $(w_k)^{-1}x$ is a vertex of $(w_k)^{-1}C$ and thus it will suffice to show $(w_k)^{-1}C$ is contained in $\mathcal{D}$ for all k. Since the roots $\alpha_1, \alpha_n, \beta, \beta'$ can be identified with their corresponding subsets of W, we can use the length function to check containment in these roots.

Now we recall the two M operations on words in a Coxeter group are as follows:

1. Delete a sub-word ss for some $s \in S$
2. Replace a sub-word of the form $stst \cdots st(s)$ by a sub-word of the form $tsts \cdots ts(t)$ where each of these strings has length $m(s,t)$.

Also recall that any word in a Coxeter group can be reduced to its minimum length by repeated application of these operations, and any two reduced words can be converted each other by application of operations of type 2. Therefore, in order to check that the length relations are satisfied, it will be enough to show that we can never perform an M operation of type 1 as this is the only way to reduce length.

It is immediate from the definition that $\ell((w_k)^{-1}) = 3k$ for all k. We can also see that $\ell(t(w_k)^{-1}) = 3k+1$ and thus $(w_k)^{-1} \in \alpha_1$ for all k. Similarly, $u(w_k)^{-1} = u(sutsut\cdots)$, and no reduction operations can be done as we assumed $m(s,u) \geq 4$. Thus $\ell(u(w_k)^{-1}) = 3k+1$ which means $(w_k)^{-1} \in \alpha_n$ as well.

Now consider the element $st(w_k)^{-1}$. If we write this element out in terms of the generators and apply the only possible Coxeter relations we get

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut\cdots) \\ &= (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) \\ &= (ts)(tut)(sutsut\cdots) \end{aligned}$$

and none of these can be reduced as $m(t,u) \geq 4$. Note that the commutation relation $sts = tst$ may not be possible if $m(s,t) \geq 4$, but it is the only relation possible in $st(w_k)^{-1}$ and even if it does exists then it does not allow $st(w_k)^{-1}$ to be reduced in length. We previously showed $\ell(t(w_k)^{-1}) = 3k+1$ and now we see $\ell(st(w_k)^{-1}) = 3k+2$ and so $(w_k)^{-1} \in \beta$.

Now we can consider $su(w_k)^{-1}$ in a similar manner. Writing $su(w_k)^{-1}$ out as a word in the generators and applying Coxeter relations gives us

$$\begin{aligned} su(w_k)^{-1} &= su(sutsut\cdots) \\ &= (susu)(tsutsu\cdots) \\ &= (usus)(tsutsu\cdots) \\ &= (usu)(sts)(utsuts\cdots) \\ &= (usu)(tst)(utsuts\cdots) \end{aligned}$$

Note once again that not all of these relations may be possible if $m(s,u) = 6$ or $m(s,t) \geq 4$. However, these are the only possible relations, and since $su(w_k)^{-1}$ cannot be reduced under these assumptions, it cannot be reduced at all. Thus $\ell(su(w_k)^{-1}) = 3k+2$ which means $su(w_k)^{-1} \in \beta'$ as well.

Now it only remains to show that $v_m \neq v_n$ for $m \neq n$. Suppose $(w_m)^{-1}x = (w_n)^{-1}x$ for $m > n$. Then we would have $x = w_m(w_n)^{-1}x = w_{m-n}$. Thus it will suffice to show $w_k x \neq x$ for any $k \geq 1$. But we know that $\mathrm{stab}_W(x) = \langle u, t\rangle$ which does not contain w_k for any $k \geq 1$ and thus $(w_k)^{-1}x \neq x$ so that $(w_m)^{-1}x \neq (w_n)^{-1}x$ as desired.

□

We now know that each of the $(w_k)^{-1}x$ is distinct and each of them lies in $\mathcal{D}$. By Lemma 15 we know that $\tilde{\phi}_{w_k x}$ exists for each $k \geq 0$. Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the k at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices $w_n x$ distinct, but also no two lie on the same wall.

Lemma 16. *Let $w_k = (tus)^k$ for all $k \geq 0$ and x the vertex of C of type s. If W as in the rest of this section with $3 \leq a, 4 \leq b, c$ then $w_m x$ and $w_n x$ do not lie on the same wall of Σ if $m > n \geq 0$.*

Proof. Suppose $w_m x$ and $w_n x$ do lie on the same wall with $m > n$. Then we also know that $w_n w_m^{-1} x = w_{n-m} x$ and x will lie on the same wall. Since $m > n$ we can let $k = m - n$ and thus it will suffice to show that $(w_k)^{-1}x$ and x do not lie on the same wall for any $k \geq 1$.

We know from Lemma 15 that $(w_k)^{-1}x \in \mathcal{D}$. Thus if $(w_k^{-1})x$ and x lie on the same wall, it must be a wall through x and thus it must be $\partial\alpha_i$ for some i. We know that $(w_k^{-1})x \in \alpha_1 \cap \alpha_n$ since $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ by definition. But we can also recall that $\partial\alpha_j \cap \alpha_1 \cap \alpha_n = \{x\}$ for $2 \leq j \leq n-1$. Thus we have $i = 1$ or $i = n$ so that $(w_k^{-1})x$ either lies on $\partial\alpha_1$ or $\partial\alpha_n$. Therefore, we either have $u(w_k)^{-1}x = (w_k)^{-1}x$ or $t(w_k)^{-1}x = (w_k)^{-1}x$ which implies that either $w_k u w_k^{-1}$ or $w_k t w_k^{-1}$ is contained in $\mathrm{stab}_W(x) = \langle u, t\rangle$. However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced.

The only possible relations we have are

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \end{aligned}$$

or

$$\begin{aligned} w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(sus)(tsuts\cdots) \\ &= (\cdots stu)(sts)u(sts)(uts\cdots) \\ &= (\cdots stu)(tst)u(tst)(uts\cdots) \end{aligned}$$

since $m(t,u) \geq 4$. Similarly as before, even these relations are only possible if $m(s,u) = 4$, but even in that case we cannot eliminate every instance of s in $w_k u w_k^{-1}$. In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in $\langle u, t\rangle$. Thus the $w_n x$ all lie on distinct walls as desired. □

We now have all the ingredients and are ready to prove the main theorem.

Theorem 6. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S). Assume W is defined by a Coxeter diagram with edge labels a, b, c with $3 \leq a$ and $4 \leq b, c$. Let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ and suppose that $[U_x : U'_x] \geq 2$ where x is the vertex of C of type s. Then U_+ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m\rangle$. Now no two of the vertices $(tus)^{-k}x$ lie on the same wall and thus we can choose k so that $v = (tus)^{-k}x$ does not lie on $\partial\beta_i$ for any i. By Lemma 15 and Lemma 14 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$ where H is a cyclic group. However, we can also see by the definition of $\tilde{\phi}_v$ that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. □

In the previous proof we assumed that $b = m(s,u)$ and $c = m(t,u)$ are larger than 4, but the labels are arbitrary and the previous theorem implies that U_+ is not finitely generated if there is an exceptional vertex, and any other vertex v' has $|st(v')| \geq 8$. The proof of Theorem 6 also implies a stronger statement.

Corollary 6. *If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system as defined in Theorem 6, then $(U_+)_{ab}$ is not finitely generated.*

Proof. Suppose that $(U_+)_{\text{ab}}$ is finitely generated. Then there is a finite set of roots $\beta_1, \ldots, \beta_m$ such that the images $[U_{\beta_1}], \ldots, [U_{\beta_m}]$ of the root groups generate $(U_+)_{\text{ab}}$. As in the previous proof, we choose a vertex v which does not lie on any $\partial\beta_i$ such that $\tilde{\phi}_v$ exists. We know $\tilde{\phi}_v(U_{\beta_i}) = \{1\}$ for all i. But the co-domain of $\tilde{\phi}_v$ is abelian, and thus the map will factor through $(U_+)_{\text{ab}}$ and we get a map $f : (U_+)_{\text{ab}} \to H$ where $f([u)] = \tilde{\phi}_v(u)$. But then f is also surjective, but $f([U_{\beta_i}]) = \{1\}$ for all i which is again a contradiction. $\square$

In this chapter we were able to prove that U_+ will not be finitely generated when we have exceptional links and at least two labels which are more than 4. In the next chapter we will examine what happens in the remaining cases, meaning when two of our edge labels are 3.

Chapter 6

Exceptional Cases

Throughout the chapter, $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) which also satisfies (A). If we fix a fundamental apartment and fundamental chamber, then we will also assume that $[U_x : U'_x] \geq 2$ where x is the vertex of C of type s. In particular, this implies that $c = m(t, u) \geq 4$. In the previous chapter we showed that U_+ was not finitely generated if either a or b was also at least 4. Therefore, in this chapter we will assume that $a = b = 3$.

In the previous chapter, one of the key steps to showing that U_+ was finitely generated was to show that the region $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ was infinite, where $\alpha_1, \alpha_n, \beta, \beta'$ are defined as before. We were able to demonstrate an infinite set of chambers contained in $\mathcal{D}$ from specific elements of the Coxeter group W. These proofs did rely on the assumption that $b \geq 4$, and so we certainly cannot use identical arguments as those that came before. There might be some hope that we can choose the elements of W more carefully to find another infinite family, of chambers, but the following lemma shows this is not possible.

Lemma 17. *Let (W, S) be a rank 3 Coxeter system defined by the labels a, b, c as before. Also assume without loss of generality that $a \leq b \leq c$. Then the region $\mathcal{D}$, defined as before, will contain infinitely many chambers if and only if $b \geq 4$.*

Proof. We know by Lemma 15 that $\mathcal{D}$ is infinite if $b \geq 4$. Thus it remains to show that $\mathcal{D}$ is finite if $b = 3$. If $b = 3$ then $a = 3$ also, and by definition of a, b, c this means $m(s, t) = m(s, u) = 3$. We will also recall the definition of $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Let $w \in W$ and suppose $\ell(w) \geq 2$. Then we can write $w = s_1 s_2 w'$ where $\ell(w') = \ell(w) - 2$. If $s_1 = t$ then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows $w \notin \alpha_1$ and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \alpha_n$ and thus $w \notin \mathcal{D}$ if $s_1 = u$.

Now we assume $s_1 = s$ and so we can also assume $s_2 = t, u$. First let $s_2 = t$ so that $w = stw'$. If $w \notin \alpha_1$ then $w \notin \mathcal{D}$ and so we will suppose $w \in \alpha_1$. Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \leq \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_2 = u$.

We have shown that if $\ell(w) \geq 2$ then $w \notin \mathcal{D}$ and thus $\mathcal{D}$ must be finite as desired. In fact, if $a = b = 3$ then we can check relatively easily that $\mathcal{D} = \{C, sC\}$ which proves the desired result. $\square$

The previous lemma shows that finding chambers of $\mathcal{D}$ is not simply a matter of more clever choices of W. We will therefore have to find a new approach to the remaining cases. Before tackling these cases, we will first enumerate what remains to be shown.

With the assumptions of this chapter we know (W, S) is a Coxeter system with $S = \{s, t, u\}$. We assume that $m(s,t) = m(s,u) = 3$ so that $\mathrm{lk}(v')$ will not be one of the exceptional Moufang polygons if v' has type u or t. We want to assume that the building Δ does have an exceptional link as otherwise there is nothing new two show, so we will assume that there is a vertex v, of type s, with link corresponding to one of the 4 exceptional Moufang polygons. This implies that every vertex of type s will have an exceptional link, and we let x be the vertex of C of type s for some choice of fundamental chamber C and fundamental apartment Σ.

Citing Lemma 2 again we see that $\mathrm{lk}(x)$ must be the Moufang Polygon associated to one of the groups $C_2(2), G_2(2), G_2(3), {}^2F_4(2)$. According to private communication with Bernhard Mühlherr the case where $\mathrm{lk}(x)$ is associated to ${}^2F_4(2)$ is impossible and so we have just the 3 possibilities to consider. We will handle these in separate sections.

6.1 Case: $\mathrm{lk}(x)$ associated to the group $G_2(2)$

Assume that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) as in the setup. Choose a fundamental apartment and chamber Σ and C of the associated building Δ, and assume that $\mathrm{lk}(x)$ is the building associated to $G_2(2)$ where x is the vertex of C of type s. Notably this implies that $m(t,u) = 6$.

We saw in the previous chapter that a vertex contained in $\mathcal{D}$ was a sufficient condition to construct a corresponding map $\tilde{\phi}_v$. However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct $\tilde{\phi}_v$ for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statement similar to Lemma 11.

We still have the presentation of U_+ as in Lemma 9 and so extending ϕ_v is still a matter of checking the commutator relations in U_+. We want to extend the map in the same way by defining $\tilde{\phi}_v$ by

$$\tilde{\phi}_v(u) = \begin{cases} \phi_v(u) & v \in \partial\alpha \text{ and } u \in U_\alpha \\ 1 & \text{otherwise} \end{cases}$$

Using the properties outlined in Chapter 4 we can prove the following Lemma.

Lemma 18. *Let v be a vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$. Assume $\gamma_1, \dots, \gamma_6$ is a standard ordering of the positive roots through v such that $U_{\gamma_2} \subset \ker \phi_v$. If γ_3, γ_4, and γ_5 are simple at all other vertices they meet, then $\tilde{\phi_v}$ as defined in Lemma 11 exists.*

Proof. First we mention that it is always possible to choose a standard ordering of $\gamma_1, \dots, \gamma_6$ such that $U_{\gamma_2} \subset \ker \phi_v$ by Lemma 6 and Corollary 2.

To check $\tilde{\phi_v}$ is well defined is a matter of checking the relations of U_+ are satisfied by the images under $\tilde{\phi_v}$. The proof is similar to that for Lemma 11. In fact, the identical argument shows that relations in U_α and commutator relations with nested roots will again be satisfied. Thus it remains to check commutator relations between roots with intersecting walls.

Suppose α and β are any two positive roots with $y = \partial\alpha \cap \partial\beta$. Then there is a relation in U_+ of the form $[u, u'] = w$ where $u \in U_\alpha, u' \in U_\beta$, and $w \in U_{(\alpha,\beta)}$. Since $[u_\alpha, u_\beta]$ must be mapped to the identity then we just need to check that w is also mapped to the identity. If $y = v$ then u_α, u_β, w all lie in U_v and $\tilde{\phi_v}(w) = \phi_v(w)$ which must be the identity because ϕ_v is a well defined homomorphism.

Now suppose $y \neq v$. Let $\delta_1, \dots, \delta_n$ be the positive roots through y, with a standard labeling, and assume that $\alpha = \delta_i$ and $\beta = \delta_j$ with $i < j$. There is at most one positive root whose wall can pass through both v and y, call it δ_k if it exists. If δ_k does not exist, then no positive roots through y pass through v and so $\tilde{\phi_v}(u_{\delta_m}) = 1$ for all m. Thus $\tilde{\phi_v}(w) = 1$ as desired.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{1, 2, 6\}$. Then we know $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all $m \neq k$ and $\tilde{\phi_v}(U_{\delta_k}) = \tilde{\phi_v}(U_{\gamma_r}) = \phi_v(U_{\gamma_r}) = \{1\}$ by the construction of ϕ_v. Thus $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all m and so $\tilde{\phi_v}(w) = \{1\}$ as well.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{3, 4, 5\}$. Then by assumption, δ_k is simple at y and thus $k = 1, n$. Thus $\tilde{\phi_v}(U_{\delta_m}) = \{1\}$ for all $2 \leq m \leq n - 1$. But w is a word in $U_{(\alpha,\beta)} \subset U_{(\delta_2,\delta_{n-1})}$ and thus $\tilde{\phi_v}(w) = 1$ again, which gives the result. ☐

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma 11, and so we have a hope of constructing more $\tilde{\phi_v}$ than the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Now recall that x is the vertex of C of type s so that $[U_x : U'_x] = 2$. Let $\alpha_1, \dots, \alpha_6$ be a standard labeling of the positive roots through x such that $\phi_x(U_{\alpha_2}) = \{1\}$, which we may do by Lemma 6 and Corollary 2. As in the previous chapter we define roots

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_6 &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Now define $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta'$. We will now prove a result similar to Lemma 13 in the current context.

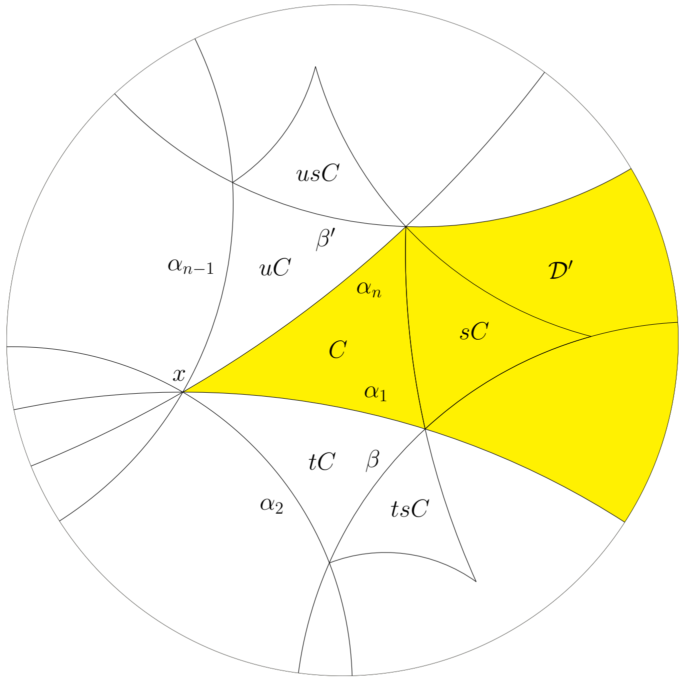


Figure 6.1: The roots $\alpha_1, \alpha_n, \beta'$ with the region $\mathcal{D}'$ in yellow.

Lemma 19. *Let x be the vertex of C of type s so that $|\mathrm{st}(x)| = 12$. Let $\alpha_1, \dots, \alpha_6$ be the positive roots at x with a standard ordering. Also assume that $\phi_x(U_{\gamma_2}) = 1$. Suppose $\gamma = \alpha_i$ for $i \in \{3, 4, 5\}$. If δ is any positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$ then $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta' \subset \gamma \cap \delta$ where*

$$\alpha_1 = \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\}$$
$$\alpha_6 = \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}$$
$$\beta' = \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}$$

as in the previous chapter.

Proof. Since γ is a positive root at x, and α_1, α_6 are the simple roots at x, we know that $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$ and thus it will suffice to show that $\mathcal{D}' \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_6 \subset \alpha_j = \delta$ and thus $\mathcal{D}' \subset \delta$ so that $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_6 = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_6$. Thus the point y will lie in exactly one of α_1 or α_6.

First suppose that $y \in \alpha_1$ so that $y \notin \alpha_6$. If $\partial\alpha_6 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_6 \subset \delta$, $\delta \subset \alpha_6$, or $-\delta \subset \alpha_6$. But the last two possibilities would contradict our assumption that $y \notin \alpha_6$ and thus we get $\alpha_6 \subset \delta$ and thus $\mathcal{D}' \subset \alpha_6 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_6 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_6$. Since y and y' are adjacent this means that either $\gamma = \alpha_5$ or $\gamma = \alpha_1$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_5$. We know that y and y' are adjacent and $y \in \alpha_1$. Since neither y or y' lies on $\partial\alpha_1$ this means that $y' \in \alpha_1$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_6$ and $\partial\alpha_5$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_6 is the only wall separating C and tC which means $D = tC \in \alpha_1$. If $E = D'$ then $D' \in \alpha_1$ since x, y, y' all lie in α_1. But this is a contradiction as α_1 cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D}' \subset \beta = \delta$ and $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

If we assume instead that $y \in \alpha_1$ so that $y \notin \alpha_6$ then we have the same two possibilities. If $\partial\alpha_6 \cap \partial\delta = \emptyset$ then by similar arguments we get $\mathcal{D}' \subset \alpha_6 \subset \delta$ and thus $\mathcal{D}' \subset \gamma \cap \delta$ as desired. If $\partial\alpha_6 \cap \partial\delta = y'$ then the vertices x, y, y' form a chamber with y' on α_6. Again, by similar arguments as before, this would imply that $\gamma = \alpha_2$ or α_6, both of which are impossible.

Therefore, regardless of case we have $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

□

We now have a condition for $\tilde{\phi}_v$ to exist which we can check and so it remains to find potential candidates to use at v. We know by Lemma 8 that ϕ_v will exist for all vertices v of type s. We will us a strategy similar to that of the previous chapter which relies on the definition of $\mathcal{D}'$ to show $\tilde{\phi}_v$ exists for certain v. To this end we now prove the analogue of Lemma 14.

Lemma 20. *Let x be the vertex of C of type s and suppose that v is any vertex in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ of type s. Then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. The proof is nearly identical to that of Lemma 14. Let $D = \mathrm{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 4.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}'$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}'$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}'$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D}' \subset \alpha_1 \cap \alpha_6$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 18 at the vertex wx. Let $\gamma_1, \ldots, \gamma_6$ be a standard labeling of the positive roots through wx such that $U_{\gamma_2} \subset \ker \phi_{wx}$. We need to check that if $y \neq wx$ is on $\partial\gamma_i$ for $i \in \{3, 4, 5\}$ then γ_i is simple at y. First we will show that w^{-1} sends positive roots at wx to positive roots at x. Suppose γ is any positive root at wx. Then we know that $E \in \gamma$ and thus $C = w^{-1}E \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is positive, and thus w^{-1} sends positive roots at wx to positive roots at x.

If we apply Lemma 8 then we know that $w^{-1}\gamma_1 = \alpha_1, \dots, w^{-1}\gamma = \alpha_6$ is a standard labeling of the of the positive roots at x. If we apply this isomorphism given by Corollary 4 then we know that $U_{w^{-1}\gamma_2} = U_{\alpha_2} \subset \ker \phi_x$ since $U_{\gamma_2} \subset \ker \phi_{wx}$.

Now we fix $i \in \{3, 4, 5\}$ and we need to check γ_i is simple at all vertices $y \neq v$ on $\partial\gamma_i$. If we apply w^{-1} we get that $w^{-1}y \neq x$ is a vertex on $\partial\alpha_i$. Thus by Lemma 12 we know that α_i is simple at $w^{-1}y$. Now suppose that δ is any positive root at $w^{-1}y$. Recall that $D \in \mathcal{D}'$ and we can apply Lemma 19 to see that $\mathcal{D}' \subset \delta$ so that $D \in \delta$. If we apply w we get $C = wD \in w\delta$ so that $w\delta$ is a positive root through $w(w^{-1}y) = y$. Thus w sends positive roots at $w^{-1}y$ to positive roots at y. We can apply Lemma 8 again to say that w sends the simple roots at $w^{-1}y$ to the simple roots at y. Since α_i is simple at $w^{-1}y$ we know that $w\alpha_i = \gamma_i$ is simple at y as desired. We now for all positive roots γ_i for $i \in \{3, 4, 5\}$ at wx that γ_i is simple at all other vertices, and thus we can apply Lemma 18 to say that $\tilde{\phi}_{wx}$ exists as desired.

□

As in the previous chapter, we now have a potentially large class of vertices for which $\tilde{\phi}_v$ exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. The approach will be similar as in the previous chapter, and recall in our current setup that $m(s,t) = m(s,u) = 3$ and $m(t,u) = 6$.

Lemma 21. *Let $w_k = (uts)^k$ for all $k \geq 0$ and let x be the vertex of C of type s. Then the vertices $(w_k)^{-1}x$ are all distinct, and they all lie in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ as defined previously.*

Proof. The proof will be similar to that of Lemma 15. First note that $(w_k)^{-1} = (stu)^k$ for all k. Since $(w_k)^{-1}x$ is a vertex of $(w_k)^{-1}C$, it will suffice to show that $(w_k)^{-1}C$ is a chamber of $\mathcal{D}'$. Now we may use the identification of chambers with the elements of W, and work with the length function and M-Operations. There are certainly no M-Operations possible in w_k^{-1} so we have $\ell(w_k^{-1}) = 3k$. There are also no M-Operations possible in $uw_k^{-1} = u(stustu\cdots)$ which means $\ell(uw_k^{-1}) = 3k+1$ so that $w_k^{-1} \in \alpha_6$. Some computation also shows that

$$\begin{aligned}
tw_k^{-1} &= t(stustus\cdots) \\
&= (tst)(ustus\cdots) \\
&= (sts)(ustus\cdots) \\
&= (st)(sus)(tus\cdots) \\
&= (st)(usu)(tus\cdots) \\
&= (st)(us)(utus\cdots)
\end{aligned}$$

which exhausts all of the possible M-Operations in tw_k^{-1}. Since no operations of type 1 were performed, we have $\ell(tw_k^{-1}) = 3k+1$ so that $tw_k^{-1} \in \alpha_1$ as well.

Finally, we check that

$$\begin{aligned} suw_k^{-1} &= su(stustus\cdots) \\ &= (sus)(tustus\cdots) \\ &= (usu)(tustus\cdots) \\ &= (us)(utustus\cdots) \end{aligned}$$

which shows by the same logic that $\ell(suw_k^{-1}) = 3k+2$ and $suw_k^{-1} \in \beta'$. Therefore, $w_k^{-1}x$ is a vertex in $\alpha_1 \cap \alpha_6 \cap \beta'$ for all $k \geq 0$ as desired. □

The previous proof shows that the vertices $w_k^{-1}x$ are all distinct and lie in $\mathcal{D}'$, which means each one will give rise to a $\tilde{\phi}_w x$ for some w. If we check the proof of Lemma 20 then we can verify that w_k will satisfy the properties of the desired w, and thus $\tilde{\phi}_{w_k x}$ will exist for all k. The last major step is to show that these $w_k x$ cannot all lie on finitely many walls.

Lemma 22. *Let x be the vertex of C of type s and let $w_k = (uts)^k$ for all $k \geq 0$. Any wall of Σ can contain only finitely many $w_k x$.*

Proof. The proof is nearly identical to that of Lemma 16. Suppose that $w_m x$ and $w_n x$ lie on the same wall for $m > n \geq 0$. Then we also have $w_{n-m}x$ and x lie on the same wall. The walls passing through x are exactly the walls $\partial\alpha_1, \ldots, \partial\alpha_6$. But $w_{n-m} = w_k^{-1}$ for $k = m-n \geq 1$ so $w_{n-m}x$ lies in $\mathcal{D}' \subset \alpha_1 \cap \alpha_6$. As mentioned before, $\partial\alpha_i \cap \alpha_1 \cap \alpha_6 = \{x\}$ for $2 \leq i \leq 5$ and thus $w_{n-m}x$ will lie on $\partial\alpha_1$ or $\partial\alpha_6$. These walls are the fixed points of the reflections t and u respectively, so we have either $tw_{n-m}x = w_{n-m}x$ or $uw_{n-m}x = w_{n-m}x$. This would mean that one of $w_{n-m}^{-1}tw_{n-m}$ or $w_{n-m}^{-1}uw_{n-m}$ lie in $\mathrm{stab}_W(x) = \langle u, t\rangle$. This can be checked using M-Operations and we see that

$$\begin{aligned} w_{n-m}^{-1}tw_{n-m} &= (\cdots utsutsuts)t(stustustu\cdots) \\ &= (\cdots utsutsut)(sts)(tustustu\cdots) \\ &= (\cdots utsutsut)(tst)(tustustu\cdots) \\ &= (\cdots utsutsu)s(ustustu\cdots) \\ &= (\cdots utsut)(susus)(tustu\cdots) \\ &= (\cdots utsut)(u)(tustu\cdots) \\ &= (\cdots uts)(ututu)(stu\cdots) \end{aligned}$$

and no further M-operations are possible. While we were able to do some reductions in length, we have shown that $w_{n-m}^{-1}tw_{n-m}$ can only be contained in $\langle u, t\rangle$ if $m-n \leq 2$. By a similar computation we can see that

$$\begin{aligned} w_{n-m}^{-1}uw_{n-m} &= (\cdots utsutsuts)u(stustustu\cdots) \\ &= (\cdots utsutsut)(sus)(tustustu\cdots) \\ &= (\cdots utsutsut)(usu)(tustustu\cdots) \\ &= (\cdots utsuts)(utu)s(utu)(stustu\cdots) \end{aligned}$$

and any further M-operations are impossible. In either case we have shown that $w_m x$ and $w_n x$ can only lie on the same wall if $|m-n| \leq 2$ and thus only finitely many $w_k x$ can lie on any wall as desired. □

Now we are ready to prove the main result of the section, which extends the result of Theorem 6 to this new case.

Theorem 7. *Let $(G, (U_\alpha)_{\alpha \in Phi}, T)$ be an RGD system of type (W, S) with assumptions as in (A). Suppose that $a = m(s,t) = b = m(s,t) = 3$. Also suppose that $\mathrm{lk}(x)$ is the Moufang polygon associated to the group $G_2(2)$, where x is the vertex of the fundamental chamber C of type s. Then U_+ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m \rangle$. Let $w_k = (uts)^k$ for all $k \geq 0$. Now only finitely many of the vertices $w_k x$ lie on the same wall and thus we can choose k so that $v = w_k x$ does not lie on $\partial\beta_i$ for any i. By Lemma 21 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. □

In keeping with the theme of copying Chapter 5 nearly verbatim, we also get the following Corollary

Corollary 7. *If $(G, (U_\alpha)_{\alpha \in Phi}, T)$ as in Theorem 7, then $(U_+)_{ab}$ is not finitely generated.*

Proof. The proof is identical to that of Corollary 6. □

There are two more cases to consider and they will be the topic of the next section.

6.2 Case: $\mathrm{lk}(x)$ associated to the group $C_2(2)$ or $G_2(3)$

The two remaining cases to consider are as follows. $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ is an RGD system of type (W, S) satisfying (A). We also assume that x is the vertex of the fundamental chamber C of type s and $\mathrm{lk}(x)$ is the Moufang polygon associated to the group $C_2(2)$ or $G_2(3)$. Finally we assume that $a = m(s,t) = b = m(s,u) = 3$.

In the previous section we were able to modify the strategy of Chapter 5 to see that U_+ was not finitely generated. No amount of modification to that strategy will work in the current context as we will show that in these cases, the group U_+ will be finitely generated. We will do this by defining a filtration of U_+ with nice finiteness properties.

We will say that a chamber D borders a root α if a panel of $\mathcal{D}$ lies on the wall $\partial\alpha$. For any positive root $\alpha \in \Phi$ this allows us to define $d(\alpha, C)$ to be $\min\{d(D, C) | D$ borders α. If $d(\alpha, C) = n$ then we know there is some chamber D such that D borders α and $d(D, C) = n$.

Furthermore, we know that D must be a chamber of α, as otherwise there would be a chamber D' adjacent to D across $\partial\alpha$ with $d(D', C) < d(D, C)$.

We can define subgroups U_k for all $k \geq 1$ where $U_k = \langle U_\gamma | d(\gamma, C) \leq k, \gamma \in \Phi_+ \rangle \leq U_+$. From the definition we have $U_1 \subset U_2 \subset \cdots$ and we also can see that $U_+ = \cup_{k\geq 1} U_k$ since any root of Φ_+ will be some finite distance away from C. Since chambers of Σ correspond to elements of W, for any $k \geq 1$ there are only finitely many chambers at distance k or less away from C. Since each of these chambers borders 3 distinct walls, there are only finitely many positive roots distance k away from C. Since each U_γ is finitely generated by (A), this means that U_k is finitely generated for all $k \geq 1$. The goal for the rest of the section will be to prove that the U_k must eventually stabilize, which would show $U_k = U_+$ and thus U_+ would be finitely generated. First we need some results about the interaction between U_k and U_v.

Lemma 23. *Suppose v is a vertex of Σ with $U_v = U'_v$. If $d(\mathrm{Proj}_v(C), C) = k$ then $U_v \subset U_k$.*

Proof. Let $\alpha_1, \dots, \alpha_n$ be a standard labeling of the positive roots through v and let $E = \mathrm{Proj}_v(C)$. By the properties of projections we know that E is the only chamber in $\mathrm{st}(v)$ which is contained in α_1 and α_n. Suppose that E borders some root α_i for $2 \leq i \leq n-1$. Then we can choose a chamber D which is adjacent to E along $\partial\alpha_i$. Since $\partial\alpha_i$ is the only wall crossed in a gallery from E to D, we must have that $D \in \alpha_1 \cap \alpha_n$ as well. This is a contradiction, and thus E cannot border α_i for $2 \leq i \leq n-1$. But the chambers in $\mathrm{st}(v)$ are arranged in a circular pattern around v, with walls separating each, each chamber of $\mathrm{st}(v)$ must border exactly two of $\alpha_1, \dots, \alpha_n$. Thus E must border α_1 and α_n.

By definition, since E borders α_1, we know that $d(\alpha_1, C) \leq d(E, C) = k$ and similarly for α_n. Thus $U_{\alpha_1}, U_{\alpha_n} \subset U_k$. Since $U'_v = \langle U_{\alpha_1}, U_{\alpha_n} \rangle$ we know that $U'_v \subset U_k$ and thus $U_v \subset U_k$ by assumption as desired. □

When $U'_v \neq U_v$ the situation is slightly more complicated, but we can still prove a similar result.

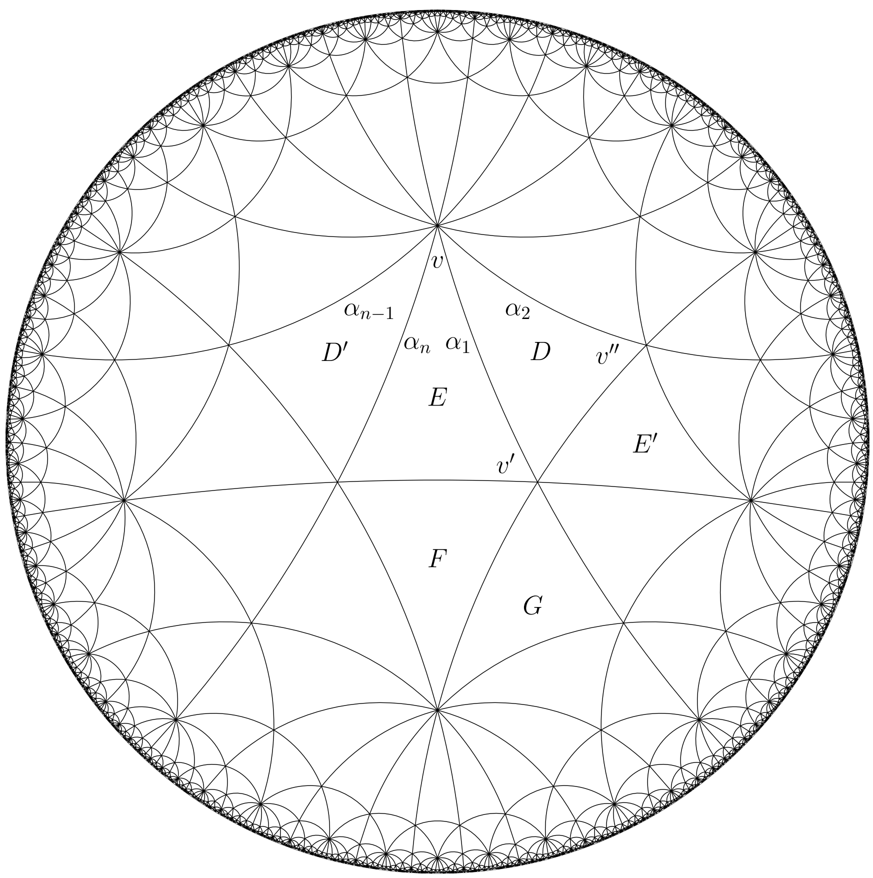
Lemma 24. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S) of rank 3. Furthermore, assume that the Coxeter diagram of W has two labels of 3. Suppose v is a vertex of Σ with $[U_v : U'_v] \geq 2$. If $\alpha_1, \dots, \alpha_n$ is a standard labeling of the positive roots through v then U_{α_1} and U_{α_n} are contained in U_k. Furthermore, if $d(\mathrm{Proj}_v(C), C) = k \geq 2$ then at least one of U_{α_2} and $U_{\alpha_{n-1}}$ is also contained in U_k.*

Proof. The groups U_{α_1} and U_{α_k} are contained in U_k by arguments identical to those in Lemma 23.

Let $E = \mathrm{Proj}_v(C)$. Let D and D' be the chambers in $\mathrm{st}(v)$ which are adjacent to E, and assume that D and E are separated by $\partial\alpha_1$ while D' and E are separated by $\partial\alpha_n$. Since $d(E, C) \geq 2$ by assumption, we know that there is a minimal gallery from E to C containing at least 3 chambers. Choose such a minimal gallery which starts with chambers E, F, G. By definition $d(F, C) = d(E, C) - 1$ and thus F cannot be either D or D' since $d(D, C) = d(D, E) + d(E, C) > d(E, C)$ by the gate property, and similarly for D'. The chambers E and F have two vertices in common, and the chambers F and G have two vertices in common, so E, F, G must have a vertex in common, call it v'. Since $d(F, C) < d(E, C)$ we know that

$F \notin \mathrm{st}(v)$ by the definition of projections, and thus $v' \neq v$. But v' is also a vertex of E so v and v' are two distinct vertices of E. Since $[U_v : U_v'] \geq 2$ we know that $|\mathrm{st}(v)| \geq 8$ and thus $|\mathrm{st}(v')| = 6$ since two of the edge labels for W are 3.

There are exactly 2 chambers in $\mathrm{st}(v')$ which are adjacent to E, and there are exactly 3 vertices in Σ adjacent to E, namely F, D, D'. Thus either D or D' is in $\mathrm{st}(v')$. Assume that $D \in \mathrm{st}(v')$. then we know that $d(D, C) > d(E, C) > d(F, C) > d(G, C)$ and D, E, F, G form a gallery. Therefore, $d(D, C) = d(G, C) + 3$ and since $|\mathrm{st}(v)| = 6$ we know that D and G are opposite in $\mathrm{st}(v')$. This means there is another minimal gallery D, E', F', G in $\mathrm{st}(v')$ from D to G which does not include E or F. This minimal gallery can also be extended to a minimal gallery from D to C by using the original gallery after G.



Since $\partial\alpha_1$ separates D and E, we know that D borders $\partial\alpha_2$. We know that D and E' share two vertices, one of which is v'. The other one cannot be v as the only two chambers which share v and v' are D, E and we assume $E' \neq E$. Thus we can say that D, E' share two vertices, v, v'' and $v'' \neq v$. As before, this means $|\mathrm{st}(v'')| = 6$. Since D borders $\partial\alpha_2$ also know that two vertices of D lie on $\partial\alpha_2$. The vertex v' cannot lie on $\partial\alpha_2$ as we know that $\partial\alpha_1$ contains v and v' and two distinct walls cannot share two vertices. Therefore, v'' lies on $\partial\alpha_2$.

We have that v'' is a vertex of Σ with $|\mathrm{st}(v'')| = 6$ and thus $U_{v''} = U'_{v''}$. We also know that $E' \in \mathrm{st}(v'')$ and $d(E', C) = d(D, C) - 1 = d(E, C) = k$. Thus $d(\mathrm{Proj}_{v''}(C), C) \leq k$. By Lemma 23 this means that $U_{v''} \subset U_k$. But α_2 is a positive root through v'' and thus $U_{\alpha_2} \subset U_k$

as desired.

If $D' \in \mathrm{st}(v')$ from before, then identical arguments show that $U_{\alpha_{n-1}} \subset U_k$ which gives the desired result. $\square$

As we saw in Lemma 7, if $\mathrm{lk}(x)$ is associated to $C_2(2)$ or $G_2(3)$, then the inclusion of either U_2 or U_{n-1} into U_k will also show that all of U_v is contained in U_k as well. Thus in the setup of this chapter, the previous result is an extension of Lemma 23. We are now ready to prove the main result of this section.

Theorem 8. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) satisfying* (A). *Also assume that $S = \{s, t, u\}$ and $m(s, t) = m(s, u) = 3$. If $\mathrm{lk}(x)$ is the Moufang polygon associated to $C_2(2)$ or $G_2(3)$, where x is the vertex of the fundamental chamber of type s, then U_+ is finitely generated.*

Proof. We will show that $U_k \subset U_{k-1}$ for $k \geq 3$ which will show $U_+ = U_2$ and thus U_+ will be finitely generated by our earlier remarks. Let $k \geq 3$ and choose $\gamma \in \Phi_+$ such that $d(\gamma, C) = k$. Then we can find a chamber D of Σ which borders γ such that $d(\gamma, C) = d(D, C) = k$. Let D' be a chamber adjacent D which is closer to C, or in other words, $d(D', C) = d(D, C) - 1$. Since D borders γ we know that D will have two vertices on $\partial\gamma$, and we also know that D and D' will share two vertices, which means one of the common vertices will also lie on $\partial\gamma$. Let v be a vertex shared by D and D' which lies on $\partial\gamma$. By definition, this means γ is a positive root at v and thus $U_\gamma \subset U_v$.

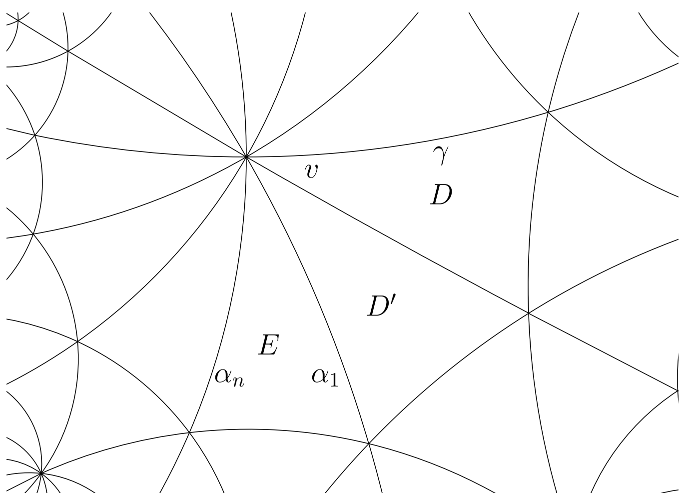


Figure 6.2: An example of the chambers D, D' and E.

Let $E = \mathrm{Proj}_v(C)$. Then E is the chamber in $\mathrm{st}(v)$ which minimizes the distance to C. Since $D' \in \mathrm{st}(v)$ and $d(D', C) < d(D, C)$ we know that $E \neq D$ and $l = d(E, C) < d(D, C) = k$. There are exactly two possibilities for v. If v is a vertex of type t or u then $|\mathrm{st}(v)| = 6$ and $U_v = U_v'$. Then we can apply Lemma 23 to see that $U_v \subset U_l \subset U_{k-1}$, and since $U_\gamma \subset U_v$ we know that $U_\gamma \subset U_{k-1}$ as desired.

Now suppose that v is a vertex of type s. Then $\mathrm{lk}(v) \cong \mathrm{lk}(x)$ is the Moufang polygon for either $C_2(2)$ or $G_2(3)$. If $d(E,C) \geq 2$ then we can apply Lemma 24 to say that $U_{\alpha_1}, U_{\alpha_n}$ and at least one of U_{α_2} and $U_{\alpha_{n-1}}$ is contained in $U_l \subset U_k$. Now we can apply Lemma 7 to see that $U_v \subset U_l \subset U_k$. Since γ is a positive root through v we know that $U_\gamma \subset U_v$ and thus $U_\gamma \subset U_k$ as desired.

If $d(E,C) < 2$ we still have $U_{\alpha_1}, U_{\alpha_n} \subset U_l \subset U_2 \subset U_{k-1}$ by Lemma 24 and the assumption that $k \geq 3$. Let F, F' be the chambers in $\mathrm{st}(v)$ adjacent to E along $\partial\alpha_1$ and $\partial\alpha_n$ respectively. Observation of the local geometry around v shows that F borders α_2 and F' borders α_{n-1}. Since $d(F,C) = d(E,C)+1$ we know that $d(\alpha_2, C) \leq d(F,C) = d(E,C)+1 \leq 2$. This means that $U_{\alpha_2} \subset U_2 \subset U_{k-1}$ since $k \geq 3$. An identical argument shows $U_{\alpha_{n-1}} \subset U_{k-1}$ and thus $U_v \subset U_{k-1}$ by Lemma 7. Since γ is a positive root through v this also shows that $U_\gamma \subset U_{k-1}$ as desired.

We have shown for any $k \geq 3$ and positive root γ with $d(\gamma, C) = k$, that $U_\gamma \subset U_{k-1}$. Since the choice of γ was arbitrary we have shown that $U_k \subset U_{k-1}$, and thus by induction we get $U_k = U_2$ for all $k \geq 2$. By our remarks on U_k this shows that $U_+ = U_2$ and thus U_+ is finitely generated as desired. □

With this theorem, we have completely determined finite generation for U_+ in all RGD systems satisfying (A). In particular, we have determined finite generation for U_+ in all rank 3 Kac-Moody groups where the Weyl group W does not have any 2's in the Coxeter diagram.

Theorem 8 shows that in the current setup, not only is U_+ finitely generated, but it says that $U_+ = U_2$ so that U_+ is generated by the root groups U_α with $d(\alpha, C) \leq 2$. There remains the question of whether this generating set is minimal. The answer here is a resounding no, as we will explain. We will not be able to completely solve the question of finding a minimal generating set but we will be able to make a few remarks. First we will pare down the generators of U_+ as much as possible by using the commutator relations. While there are similarities between the two cases, it will be easiest to consider each one separately. First suppose that $\mathrm{lk}(x)$ is the group associated to $C_2(2)$. Consider the Coxeter complex of W with the labeling of the roots with $d(\alpha, C) \leq 2$ as in Figure 6.3. Of course we do not duplicate roots so every panel does not have a label.

Letting $U_i = U_{\alpha_i}$ for all i, we can apply Lemma 3 from [4], as well as Lemma 7 to write some of these root groups in terms of others. By examining the diagram we get the following relations

$$\begin{aligned}
U_5 &\subset \langle U_1, U_2, U_4 \rangle \\
U_6 &\subset \langle U_2, U_3 \rangle \\
U_7 &\subset \langle U_1, U_3 \rangle \\
U_8 &\subset \langle U_5, U_6 \rangle \\
U_{10} &\subset \langle U_6, U_7, U_9 \rangle \\
U_{11} &\subset \langle U_4, U_7 \rangle
\end{aligned}$$

all of which together show that $U_+ = \langle U_1, U_2, U_3, U_4, U_9 \rangle$.

Now consider the case when $\mathrm{lk}(x)$ is associated to the group $G_2(3)$. Then we have a similar diagram with labels in Figure 6.4. Using the same notation as before and similar analysis we get the following relations

$$\begin{aligned}
U_5, U_8, U_9 &\subset \langle U_1, U_2, U_4 \rangle \\
U_6 &\subset \langle U_2, U_3 \rangle \\
U_7 &\subset \langle U_1, U_3 \rangle \\
U_{10} &\subset \langle U_5, U_6 \rangle \\
U_{12} &\subset \langle U_6, U_7, U_{11} \rangle \\
U_{13} &\subset \langle U_4, U_7 \rangle
\end{aligned}$$

which together show that $U_+ = \langle U_1, U_2, U_3, U_5, U_{11} \rangle$. In both of these cases, every group U_α is cyclic of prime order, and thus U_+ is actually generated by some choice of generator for each U_i, but this is a minor detail and it is more convenient to work with the root groups as a whole.

It is possible to make other choices of relations to get other generating sets as well. For example, replacing U_9 or U_{11} by U_{10} or U_{12} would also produce a generating set, but one which is more or less equivalent up to some relabeling of the diagram. We will not show that this generating set is minimal but we can make at least one remark.

Any generating set of U_+ consisting of root groups must contain at least 3 root groups. If $U_+ = \langle U_\alpha, U_\beta \rangle$ then there are 3 possibilities for the pair α, β. If $\partial\alpha$ and $\partial\beta$ meet then $\langle U_\alpha, U_\beta \rangle \subset U_v$ where v is the point of intersection. But U_v is finite so this is impossible. If α and β are nested the condition (A) says that $[U_\alpha, U_\beta] = \{1\}$ so $\langle U_\alpha, U_\beta \rangle = U_\alpha U_\beta$ which is also finite and thus impossible. The last possibility is that the pair α, β is not pre-nilpotent. In this case, as stated in [8], the subgroup $\langle U_\alpha, U_\beta \rangle$ is the free product of U_α and U_β. We can always find a root γ such that $\alpha \subset \gamma$ and thus U_α and U_γ will commute. But the centralizer of U_α in $U_\alpha * U_\beta$ is U_α which is a contradiction. Thus any minimal generating set consisting of root groups must contain at least 3 root groups.

It seems likely based on the commutator relations, and the similarity of these two generating sets that they are in fact minimal generating sets consisting of root groups. There is not much more that can be said at the present time.

6.3 Future Questions

As we have exhausted the question of when groups with RGD systems satisfying (A) the next question is how can these assumptions be weakened to get new results. There are 3 main ways that these ideas could continue. The first idea would be to eliminate the restriction that $[U_\alpha, U_\beta] = 1$ when α and β are nested. This is true for Kac-Moody groups, but is not a consequence of the general RGD axioms, so it is worth considering if we can prove any results without it. Without this assumption, it is still possible to prove groups are finitely generated, but the methods in Chapter 5 would be impossible, as we have no hope of extending ϕ_v without this assumption.

The next possibility would be to allow $m(s,t) = 2$ in the Weyl group W. This path also leads to difficulties as the triangle condition is an important tool which is lost when $m(s,t) = 2$. Also, preliminary explorations seem to indicate that extending ϕ_v is also impossible in the Coxeter complexes which arise in these cases, so once again, a new approach would be needed.

The last area is to allow for W to have higher rank. This seems to have the most promise, and it is even possible that some results about finite generation will follow from the rank 3 case, as there will be subgroups, similar to U_v, for lower dimensional simplices of the building. There are however, two obstacles to face when extending in this manner. This first issue is understanding the geometry of the Coxeter complex Σ. For rank 3 Weyl groups, can draw nice 2 dimensional pictures to get an Idea of what is happening in Σ. A large help was the code seen in the appendix which allowed me to nicely draw Coxeter complexes for rank 3 W. There is some though of writing similar code to possibly 3D print models of the Coxeter complex for rank 4 Weyl groups, which could potentially be helpful.

The other problem, but perhaps the most manageable, is that for higher rank, the Coxeter complex may not live so nicely as a model of hyperbolic space. It does seem that the group theoretic methods described in Chapter 5 will translate however, and it seems promising that something can be said for at least rank 4 cases as well.

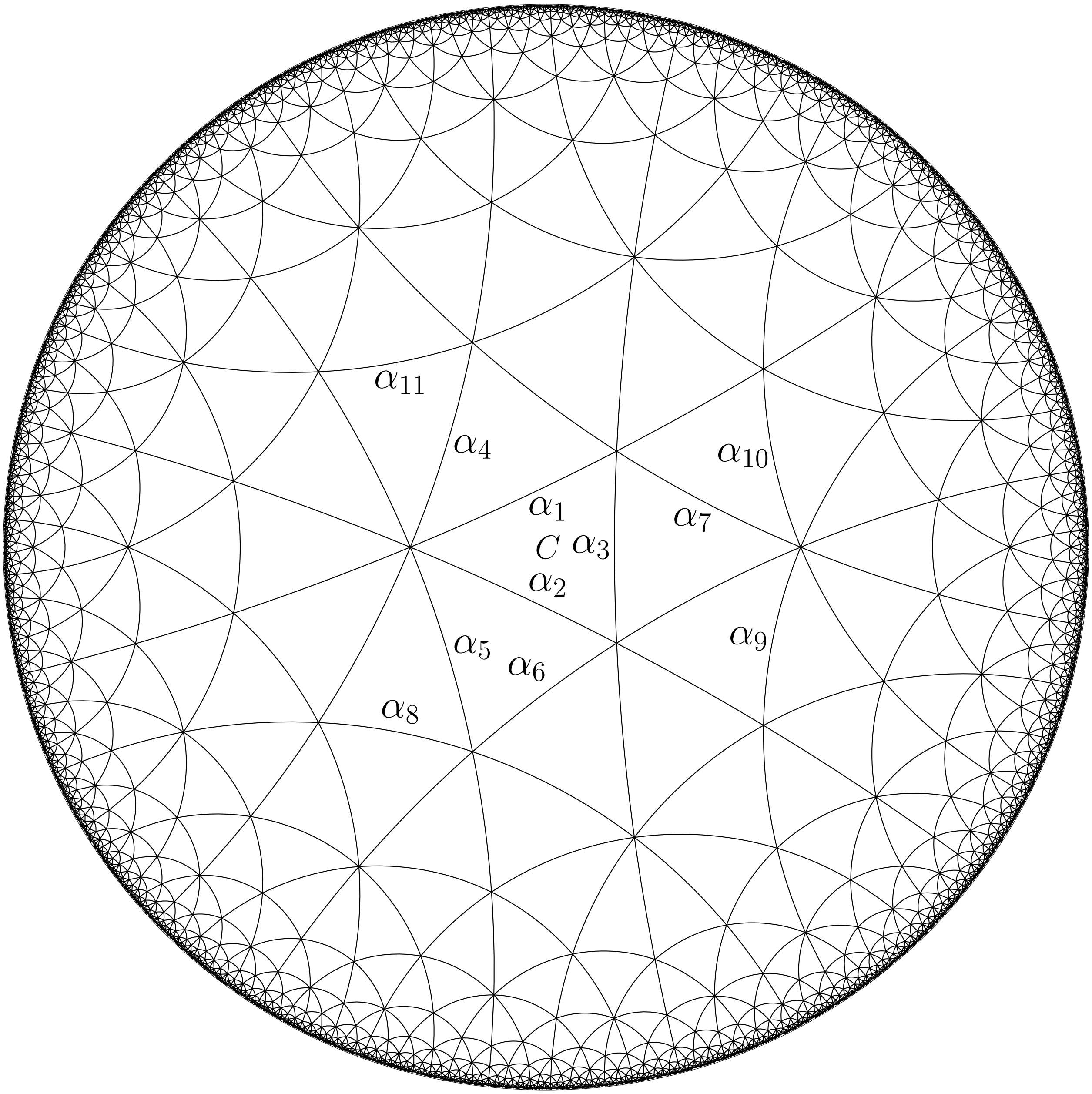


Figure 6.3: Unique roots with $d(\alpha, C) \leq 2$ with $|\mathrm{st}(x)| = 8$

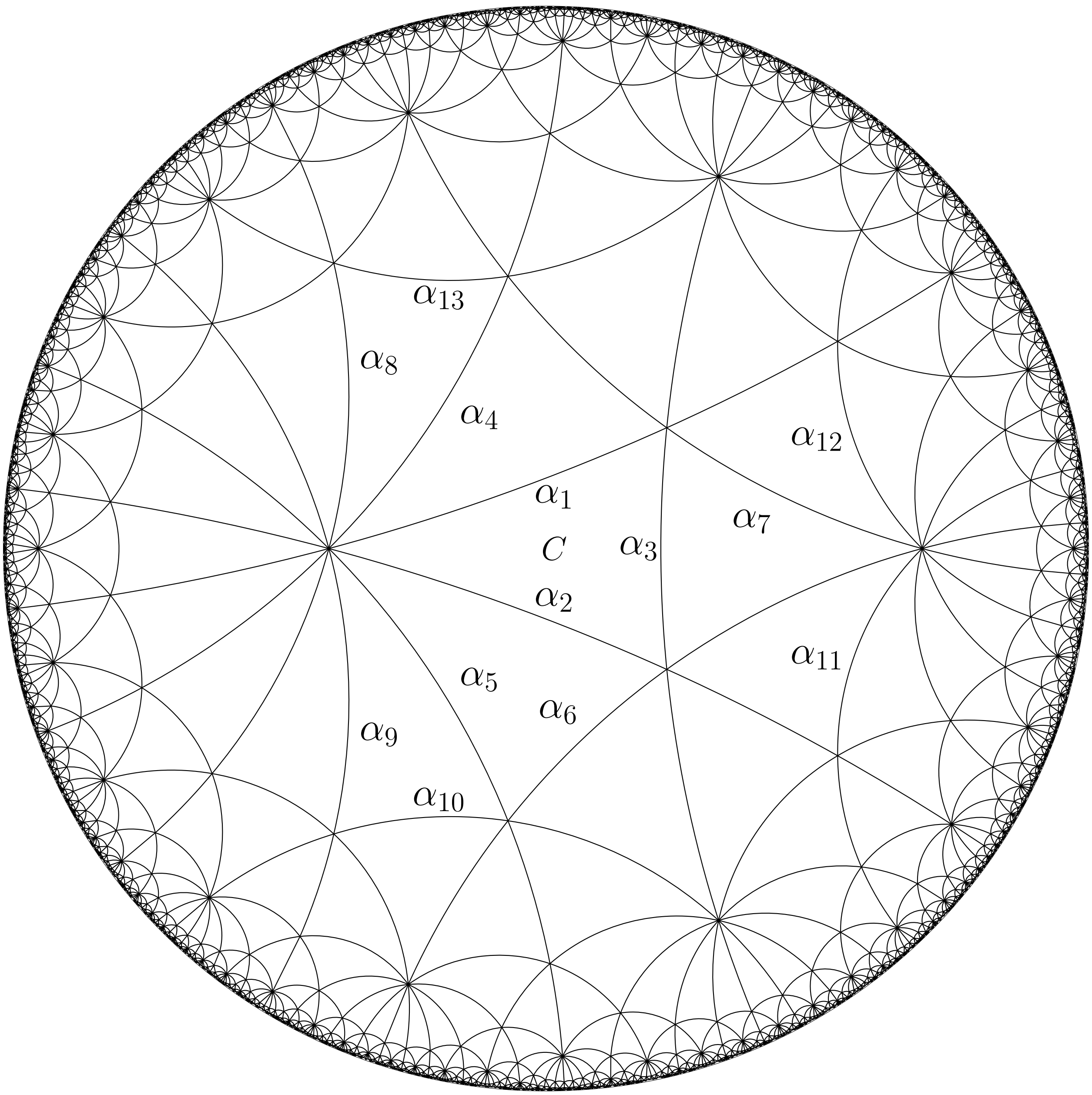


Figure 6.4: Unique roots with $d(\alpha, C) \leq 2$ with $|\mathrm{st}(x)| = 12$

Appendix A

Code for Diagram Generation

Included in Appendix A is the Python code used to generate the various figures. All files should be saved in the same folder, and only hyperbolictilinggenerator.py needs to be run. The code will create (or modify) a document named texcode.tex. This document uses the standalone package, and should be able to be compiled as is, imported into another document, or simply gutted for the tikz code. A word of warning, the code can generate exceptionally large tikz files. There is no check to determine how long the program will run with given inputs, and there is certainly no way of telling how long your $\LaTeX$compiler will take to actually compile the document.

```
import math
import cmath
import copy
from hyptransformations import *


class Vertex:
    def __init__(self,z,t):
        self.pos=z
        self.type=t
        if t not in ["s","t","u"]:
            print("Not a valid vertex type")
            exit(1)

    def reflect(self,p1,p2):
        return Vertex(reflect(p1,p2,self.pos),self.type)

class Panel:
    def __init__(self,v1,v2,tolerance):
        self.start=v1.pos
        self.end=v2.pos
        if v1.type==v2.type:
            print("Two vertices on the same panel cannot share a type")
            exit(1)

        types=["s","t","u"]
        types.remove(v1.type)
        types.remove(v2.type)
        self.cotype=types[0]
        self.tolerance=tolerance


    def liesondiameter(self):
```

```
        return abs(self.start.real*self.end.imag-self.start.imag*self.end.real)<self.tolerance

    def drawingcenter(self):
        if self.liesondiameter():
            return None
        denom=self.start.real*self.end.imag-self.start.imag*self.end.real
        a=(self.start.imag*(abs(self.end)**2)-self.end.imag*(abs(self.start)**2)+self.start.imag-self.end.imag)/denom
        b=(self.end.real*(abs(self.start)**2)-self.start.real*(abs(self.end)**2)+self.end.real-self.start.real)/denom
        return (-a/2)+1j*(-b/2)

    #This function only makes sense when liesondiameter returns false
    def drawingradius(self):
        if self.liesondiameter():
            return None
        return abs(self.drawingcenter()-self.start)


    def gettexcode(self):
        if self.liesondiameter():
            return "\\draw("+str(self.start.real)+","+str(self.start.imag)+") -- ("+str(self.end.real)+","+str(self.end.imag)+");"
        else:
            center=self.drawingcenter()
            return Panel.texcodearc(self.start,self.end,center)

    def getwalltexcode(self):
        if abs(self.start-self.end)<self.tolerance:
            return ""

        if self.liesondiameter():
            step=abs(self.end-self.start)
            startbottom=0
            starttop=3/step
            endbottom=0
            endtop=-3/step
            while abs(starttop-startbottom)>self.tolerance or abs(endbottom-endtop)>self.tolerance:
                startmid=(startbottom+starttop)/2
                endmid=(endbottom+endtop)/2
                startguess=self.start+startmid*(self.end-self.start)
                endguess=self.start+endmid*(self.end-self.start)
                if abs(startguess)>=1:
                    starttop=startmid
                else:
                    startbottom=startmid

                if abs(endguess)>=1:
                    endtop=endmid
                else:
                    endbottom=endmid

            return "\\draw("+str(startguess.real)+","+str(startguess.imag)+") -- ("+str(endguess.real)+","+str(endguess.imag)+");"


        else:
            center=self.drawingcenter()
            r=self.drawingradius()
            c=abs(center)
            x=(r**2-1-c**2)/(-2*c)
            ypos=math.sqrt(1-x**2)
            yneg=-ypos
            start=(x+1j*ypos)*cmath.exp(1j*cmath.phase(center))
            end=(x+1j*yneg)*cmath.exp(1j*cmath.phase(center))
            return Panel.texcodearc(start,end,center)

```

```
    def texcodearc(start,end,center):
        startangle=cmath.phase(start-center)
        endangle=cmath.phase(end-center)
        radius=abs(center-start)

        if abs(endangle-startangle)>cmath.pi:
            if endangle<0:
                endangle+=2*cmath.pi
            else:
                startangle+=2*cmath.pi


        return "\\draw("+str(start.real)+","+str(start.imag)+") arc ("+str(math.degrees(startangle))+":"+str(math.degrees(endangle))+":"+str(radius)+");"



class Chamber:

    def __init__(self,v1,v2,v3,tolerance):
        self.vertices={v1.type:v1,v2.type:v2,v3.type:v3}
        if len(self.vertices)<3:
            print("A chamber must have vertices of 3 different types.")
            exit(1)
        self.tolerance=tolerance

    def center(self):
        ans=0
        for k in self.vertices.keys():
            ans+=self.vertices[k].pos
        return ans/3

    def getpanel(self,cotype):
        types=["s","t","u"]
        types.remove(cotype)
        return Panel(self.vertices[types[0]],self.vertices[types[1]],self.tolerance)

    def translate(self,coxeterword):
        ans=copy.deepcopy(self)
        for c in reversed(coxeterword):
            if c in ans.vertices.keys():
                fixed=list(ans.vertices.keys())
                fixed.remove(c)
                ans=Chamber(ans.vertices[c].reflect(ans.vertices[fixed[0]].pos,ans.vertices[fixed[1]].pos),ans.vertices[fixed[0]],ans.vertices[fixed[1]],ans.tolerance)


            else:
                print("Invalid letter in coxeter word.")
                exit(1)
        return ans
```

Listing A.1: simplex.py

```
import cmath
import numpy

def disktohalfplane(z):
    return (z+1j)/(1+1j*z)

def halfplanetodisk(z):
    return (z-1j)/(1-1j*z)

#all complex numbers, reflects z across the line through p and q
def reflect(z1,z2,point):
    # transform from unit disk to upper half plane
    p=disktohalfplane(z1)
```

```
    q=disktohalfplane(z2)
    z=disktohalfplane(point)


    #check if p and q lie on a vertical line and handle that case on its own
    if p.real==q.real:
        z=complex(p.real-(z.real-p.real),z.imag)
        return halfplanetodisk(z)



    midpoint=(p+q)/2
    #find "center" for hyperbolic line through p and q
    if p.imag==q.imag:
        center=midpoint.real
    else:
        slope=-(p.real-q.real)/(p.imag-q.imag)
        center=-(midpoint.imag/slope)+midpoint.real

    radius=abs(p-center)

    return halfplanetodisk(center+(radius**2)/(numpy.conj(z)-center))
```

Listing A.2: hyptransformations.py

```
class CoxeterGroup:
    def __init__(self,a,b,c):
        self.mst=b
        self.msu=c
        self.mtu=a

    def reduce(self,word):

        if word=="":
            return ""

        for letter in ["s","t","u"]:
            if word.find(letter*2)!=-1:
                return self.reduce(word.replace(letter*2,""))

        stword=("st"*self.mst)[:self.mst]
        tsword=("ts"*self.mst)[:self.mst]
        suword=("su"*self.msu)[:self.msu]
        usword=("us"*self.msu)[:self.msu]
        tuword=("tu"*self.mtu)[:self.mtu]
        utword=("ut"*self.mtu)[:self.mtu]
        substrings=[(stword,tsword),(tsword,stword),(suword,usword),(usword,suword),(tuword,utword),(utword,tuword)]

        finalwords=set()
        newwords=[word]
        while len(newwords)>0:
            newnewwords=[]
            for element in newwords:
                if not element in finalwords:
                    for pair in substrings:
                        instance=1
                        index=CoxeterGroup.findinstancenumber(element,pair[0],instance)
                        while index!=-1:
                            newnewwords.append(element[:index]+pair[1]+element[index+len(pair[0]):])
                            instance+=1
                            index=CoxeterGroup.findinstancenumber(element,pair[0],instance)
                    finalwords.add(element)
            newwords=newnewwords

        for element in finalwords:
```

```
            for letter in ["s","t","u"]:
                if element.find(letter*2)!=-1:
                    return self.reduce(element.replace(letter*2,""))

        return min(finalwords)


    def generatewords(self,maximumlength):
        length=0
        words=[""]
        while length<maximumlength:
            newwords=[]
            for word in words:
                if len(word)==length:
                    for letter in ["s","t","u"]:
                        newword=self.reduce(letter+word)
                        if len(newword)==length+1 and newword not in newwords:
                            newwords.append(newword)
            words=words+newwords
            length+=1

        return words


    def findinstancenumber(word,subword,instance):
        start=0
        numberfound=0
        location=-1

        while numberfound<instance:
            location=word.find(subword,start)
            if location==-1:
                return -1
            else:
                start=location+1
                numberfound+=1

        return location
```

Listing A.3: coxetergroup.py

```
import math
import cmath
import numpy
from coxetergroup import *
from hyptransformations import *
from simplex import *

class CoxeterComplex:
    wiggle=cmath.exp(0.00001j)
    def __init__(self,W,center,angle,tolerance):
        p1=0+0j*CoxeterComplex.wiggle
        p2=math.tanh(CoxeterComplex.sidelengthfromangles(math.pi/W.msu,math.pi/W.mst,math.pi/W.
    mtu)/2)*CoxeterComplex.wiggle
        p3=math.tanh(CoxeterComplex.sidelengthfromangles(math.pi/W.mst,math.pi/W.msu,math.pi/W.
    mtu)/2)*cmath.exp(1j*math.pi/W.mtu)*CoxeterComplex.wiggle

        rotate=CoxeterComplex.conformalmap(0,angle)
        move=CoxeterComplex.conformalmap(-center,0)

        v1=Vertex(move(rotate(p1)),"s")
        v2=Vertex(move(rotate(p2)),"u")
        v3=Vertex(move(rotate(p3)),"t")
        self.fundamentalchamber=Chamber(v1,v2,v3,tolerance)
        self.tolerance=tolerance
        self.W=W

```

```
    def sidelengthfromangles(oppositeangle,adj1,adj2):
        return numpy.arccosh((math.cos(oppositeangle)+math.cos(adj1)*math.cos(adj2))/(math.sin(adj1)*math.sin(adj2)))

    def conformalmap(center,angle):
        def f(z):
            return cmath.exp(1j*angle)*(z-center)/(1-center.conjugate()*z)
        return f

#this is almost done
#just need to add all of the commented labels to the tex code so adding extras isn't so difficult
    def gettexcode(self,resolution,l):
        labellength=l
        ans=""
        comments="\\clip (0,0) circle (1);\n\\draw (0,0) circle (1);\n"
        words=set()
        words.add("")
        length=0

        while True:
            wordstokeeptrying=[]
            for w in words:
                C=self.fundamentalchamber.translate(w)
                distancetoedge=0
                for k in C.vertices.keys():
                    if len(w)%2==0:
                        ans+=C.getpanel(k).gettexcode()+"\n"
                    distancetoedge=max(distancetoedge,1-abs(C.vertices[k].pos))

                if len(w)<=labellength:
                    comments+="%Chamber: "+w[::-1]+"C\n"
                    z=C.center()
                    comments+="\\draw ("+str(z.real)+","+str(z.imag)+") node {$"+w[::-1]+"C$};\n\n"
                    comments+="%Vertices:\n"
                    for k in C.vertices.keys():
                        z=C.vertices[k].pos
                        comments+="% type "+k+": ("+str(z.real)+","+str(z.imag)+")\n"

                    comments+="\n%Walls\n"
                    for k in C.vertices.keys():
                        P=C.getpanel(k)
                        comments+="% "+k+" wall:\n"
                        comments+="%"+P.getwalltexcode()+"\n"
                        comments+="% "+k+" panel:\n"
                        comments+="%"+P.gettexcode()+"\n"
                        if P.drawingcenter()!=None:
                            comments+="%Drawing Center: ("+str(P.drawingcenter().real)+","+str(P.drawingcenter().imag)+")\n"
                        else:
                            comments+="%Drawing Center: None\n"
                        comments+="%Drawing Radius: ("+str(P.drawingradius())+")\n"

                    comments+="\n"+"%"*20+"\n"




                if distancetoedge>resolution:
                    wordstokeeptrying.append(w)

            if len(wordstokeeptrying)==0:
                return comments+"\n"*3+ans


            length+=1
            newwords=set()
            for w in wordstokeeptrying:
```

```
                for v in ["s","t","u"]:
                    possible=self.W.reduce(v+w)
                    if len(possible)==length:
                        newwords.add(possible)
            words=newwords
            biggestside=0

a=3
b=3
c=6

W=CoxeterGroup(c,a,b)
Tolerance=0.01
Sigma=CoxeterComplex(W,-0.4,-math.pi/12,Tolerance)

with open("texcode.tex","w") as f:

    f.write("\\documentclass[crop=true]{standalone}\n\\usepackage[subpreambles=true]{standalone}\n\\usepackage{tikz}\n\n\\begin{document}\n\\begin{tikzpicture}[scale=8.255]\n")
    f.write(Sigma.gettexcode(0.004,2))
    f.write("\n\\end{tikzpicture}\n\\end{document}")
```

Listing A.4: hyperbolictilinggenerator.py

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