

# Chapter 1

## Known Results on Finite Generation

*{ch:known}*

Throughout this chapter  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  will be an RGD system of type  $(W, S)$  with the following assumptions:

$W$  has rank 3,  $S = \{s, t, u\}$ ,  $a = m(s, t), b = m(s, u), c = m(t, u)$  and  $3 \leq a \leq b \leq c$  (A)  
 $[U_\alpha, U_\beta] = 1$  when  $\alpha, \beta$  are nested

Let  $\Sigma$  be the Coxeter complex of  $W$  with fundamental chamber  $C$ , and  $\Phi_+$  be the positive roots of  $\Sigma$ . We will also let  $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$  be the subgroup of  $G$  generated by the positive root groups. We will also note that properties of RGD systems tell us that  $a, b, c \in \{2, 3, 4, 6, 8\}$  and thus by (A) we know that  $a, b, c \in \{3, 4, 6, 8\}$ .

For any vertex  $v$  of  $\Sigma$ , there will be some walls of  $\Sigma$  which pass through  $v$ , and for each of these walls we have a unique *positive* root. We will call these the **positive roots at  $v$**  and denote them by  $\Phi_+^v$ . Recall that  $\text{st}(v)$  is defined as all the chambers containing  $v$  as a vertex. If there are  $n$  positive roots at  $v$  then  $|\text{st}(v)| = 2n$ . Furthermore, it is possible to label the positive roots at  $v$  as  $\alpha_1, \dots, \alpha_n$  in such a way that  $\alpha_i \cap \alpha_j \subset \alpha_k$  for any  $1 \leq i \leq k \leq j \leq n$ . This ordering is unique up to a reversal of the form  $\alpha_i \mapsto \alpha_{n+1-i}$ . This possible reversal will not matter in most cases and if it does then a choice of  $\alpha_1$  will be specified. It does however allow us to unambiguously define  $\alpha_1$  and  $\alpha_n$  as the **simple** roots at  $v$ . They are the unique positive roots at  $v$  whose intersection is contained in all other positive roots at  $v$ .

Now we can define  $U_v$  to be the subgroup of  $G$  generated by all of the root groups of the positive roots at  $v$ . That is

$$U_v = \langle U_\alpha | \alpha \text{ is a positive root at } v \rangle = \langle U_\alpha | \alpha \in \Phi_+^v \rangle$$

If  $\alpha_1, \alpha_2, \dots, \alpha_n$  is a standard ordering of the positive roots at  $v$  then we can simplify notation by letting  $U_i = U_{\alpha_i}$  for all  $\alpha_i$  through  $v$ . Since  $v$  is a simplex of  $\Sigma$  of co-dimension 2, we know from the theory of RGD systems that  $U_v$  will also have the structure of a spherical, rank 2 RGD system as well. Let  $U'_v = \langle U_1, U_n \rangle$  be the subgroup of  $U_v$  generated by the simple root groups, where  $|\text{st}(v)| = 2n$ . Then it is known that  $U_v = U'_v = \langle U_1, U_n \rangle$  with the exception of a few cases which we will explicitly state in the following Lemma.

{lem:index}

**Lemma 1.** *Let  $v$  be a vertex of  $\Sigma$  with  $|\text{st}(v)| = 2n$  and let  $U'_v = \langle U_1, U_n \rangle$  where  $U_1, U_n$  are the root groups of the simple roots at  $v$ . Then the group  $U_v$  has the structure of a spherical, rank 2 RGD system and  $U_v = U'_v$  unless  $U_v$  is isomorphic to one of the following groups:*

$$C_2(2) \quad G_2(2) \quad G_2(3) \quad {}^2F_4(2)$$

*In fact, we also know the index  $[U_v : U'_v]$  in each of these cases which is summarized in the following table.*

$U_v$	$[U_v : U'_v]$
$C_2(2)$	2
$G_2(2)$	4
$G_2(3)$	3
${}^2F_4(2)$	2

*and  $[U_v : U'_v] = 1$  in all other cases.*

We can see from the previous lemma that even when  $U'_v \neq U_v$ , it is still a fairly large subgroup and in some cases it will even be normal. This will allow us to construct helpful homomorphisms later, but before we do so we will explicitly state the desired result.

{lem:normal}

**Lemma 2.** *Suppose  $v$  is a vertex of  $\Sigma$  with  $|\text{st}(v)| = 2n$  such that  $[U_v : U'_v] \geq 2$ . If  $U_v$  is isomorphic to  $C_2(2), G_2(3)$ , or  ${}^2F_4(2)$  then  $U'_v$  is a normal subgroup of  $U_v$ . If  $U_v \cong G_2(2)$  then  $U'_v$  is not a normal subgroup of  $U_v$ , but there is a standard labeling of the positive roots through  $v$  so that  $U''_v = \langle U_1, U_5, U_6 \rangle$  is a normal subgroup of  $U_v$  with  $[U_v : U''_v] = 2$ .*

*Proof.* If  $U_v \cong C_2(2)$  or  ${}^2F_4(2)$  then  $U'_v$  is a subgroup of index 2 and thus it is normal. If  $U_v \cong G_2(3)$  then  $U_v$  is a 3-group and thus 3 is the smallest prime dividing  $|U_v|$  and we know that  $U'_v$  is normal in this case as well.

Now suppose  $U_v \cong G_2(2)$ . Need to add this proof later □

{cor:phiv}

Using Lemma 2 and elementary group theory, we get the following result.

**Corollary 1.** *Suppose  $v$  is a vertex of  $\Sigma$  with  $|\text{st}(v)| = 2n$  such that  $[U_v : U'_v] \geq 2$ . Then there is a cyclic group  $H$  and a surjective group homomorphism  $\phi_v : U_v \rightarrow H$  with the property that  $\phi_v(U_1) = \phi_v(U_n) = \{1\}$  where  $U_1$  and  $U_n$  are the simple root groups at  $v$ .*

*Proof.* If  $[U_v : U'_v] \geq 2$  then  $U_v$  must be isomorphic to one of  $C_2(2), G_2(2), G_2(3), {}^2F_4(2)$ . If  $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$  then we can apply Lemma 2 to let  $H = U_v/U'_v$  and  $\phi_v$  be the quotient map which certainly will be surjective and send  $U_1$  and  $U_n$  to  $\{1\}$  by the definition of  $U'_v$ . The group  $H$  is cyclic because it has prime order.

If  $U_v \cong G_2(2)$  then we know that  $U'_v \subset U''_v = \langle U_1, U_5, U_6 \rangle$  for an appropriate standard labeling, and we again apply Lemma 2 to set  $H = U_v/U''_v$  and  $\phi_v$  as the quotient map. The group  $H$  is again cyclic because it has prime order. □

The following corollary will show that we do not have very much wiggle room when defining  $\phi_v$ , and thus if we can write any function which “looks like”  $\phi_v$  then they must be essentially the same.

*cor:uniquephiv}*

**Corollary 2.** *Suppose  $v$  is a vertex of  $\Sigma$  with  $|\text{st}(v)| = 2n$  such that  $[U_v : U'_v] \geq 2$  and let  $\phi_v$  be defined as in the previous corollary. Then  $\ker \phi_v$  is the unique, proper, normal subgroup of  $U_v$  which contains  $U_1$  and  $U_n$ .*

*Proof.* If  $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$  then  $U'_v$  is normal, it is generated by  $U_1$  and  $U_n$ , and it has prime index so there cannot be another proper subgroup containing  $U'_v$ . By the construction of  $\phi_v$ , we also know that  $\ker \phi_v = U'_v$  so that  $\ker \phi_v$  is the unique proper, normal subgroup of  $U_v$  containing  $U_1$  and  $U_n$ .

If  $U_v \cong G_2(2)$  then  $\ker \phi_v = U''_v = \langle U_1, U_5, U_6 \rangle$  under a standard labeling. If  $N$  is any normal subgroup containing  $U_1$  and  $U_n$  then we can apply the commutator relations in  $G_2(2)$  to get

add proof later □

So far we have only considered each vertex  $v$  and  $U_v$  separately. But in the Coxeter complex  $\Sigma$ , we have not only a collection of vertices, but an action of the group  $W$  on the vertices which behaves nicely with properties like the type of a vertex. We will show that the  $W$  action also interacts nicely with  $U_v$  and  $\phi_v$  in a similar way.

*{lem:resporder}*

**Lemma 3.** *Suppose  $v$  is a vertex of  $\Sigma$  of type  $s$ ,  $|\text{st}(v)| = 2n$ , and  $[U_v : U'_v] \geq 2$ . Suppose that  $v'$  is any other vertex of  $\Sigma$  of type  $s$ . Then there is an element of  $w \in W$  such that  $v' = wv$  and there is an isomorphism between  $U_v$  and  $U_{v'} = U_{wv}$ . Furthermore, if  $\alpha_1, \dots, \alpha_n$  is a standard ordering of the positive roots through  $v$ , then we can choose  $w$  so that  $w\gamma$  is a positive root at  $wv$  for every positive root  $\gamma$  at  $v$  and  $\alpha'_1 = w\alpha_1, \dots, \alpha'_n = w\alpha_n$  is a standard ordering of the positive roots at  $wv$ .*

*Proof.* Since the  $W$  action on  $\Sigma$  is transitive on vertices of the same type, it will suffice to show the result when  $v$  is a vertex of the fundamental chamber  $C$ . Let  $D = \text{Proj}_{v'}(C)$  so that  $d(D, C)$  is minimal among all chambers of  $\text{st}(v')$ . Then we know that no walls through  $v'$  can separate  $D$  and  $C$ , because crossing one of these walls would produce a chamber in  $\text{st}(v)$  which is closer to  $C$ . Therefore, a root at  $v'$  is positive if and only if it contains  $D$ .

Now choose  $w \in W$  such that  $D = wC$ . We claim that  $w$  satisfies the desired properties. First of all,  $v$  is a vertex of  $C$  of type  $s$  and thus  $wv$  is a vertex of  $wC = D$  of type  $s$ . But we know that  $v'$  is a vertex of  $D$  of type  $s$  by definition and thus  $wv = v'$  as desired. Now suppose  $\gamma$  is any positive root at  $x$ . Then  $C \in \gamma$  and thus  $D = wC \in w\gamma$  and thus  $w\gamma$  is positive at  $wv = v'$ . Therefore, we know that  $w$  induces a bijection between the positive roots at  $v$  and the positive roots at  $v'$ . Suppose  $\alpha_1, \dots, \alpha_n$  is a standard ordering of the positive roots at  $v$ . Then by definition we have  $\alpha_i \cap \alpha_j \subset \alpha_k$  for all  $1 \leq i \leq k \leq j \leq n$  where  $2n = |\text{st}(v)|$ . If we apply the action of  $w$  we get  $w\alpha_i \cap w\alpha_j \subset w\alpha_k$  for all  $1 \leq i \leq k \leq j \leq n$  as well. But since the action of  $w$  is a bijection on the positive roots, we know that each  $w\alpha_i$  is also positive at  $wv$  and thus  $\alpha'_1 = w\alpha_1, \dots, \alpha'_n = w\alpha_n$  is a standard ordering of the roots through  $wv = v'$  as desired.

The last thing we must do is show there is a bijection between  $U_v$  and  $U_{v'}$ . The theory of RGD systems tells us that there is a subgroup  $N \leq G$  with the property that for any  $w \in W$ , there is some  $\tilde{w} \in N$  such that  $\tilde{w}U_\alpha\tilde{w}^{-1} = U_{w\alpha}$  for all  $\alpha \in \Phi$ . Choose such an  $\tilde{w}$  for the  $w$  defined above and let  $f_w : G \rightarrow G$  be the isomorphism of conjugation by  $\tilde{w}$ . Now suppose that  $\alpha$  is any positive root through  $v$ . Then  $f_w(U_\alpha) = U_{w\alpha}$  and  $w\alpha$  is a positive root through  $wv = v'$ . Thus the map  $f_w$  restricts to a group homomorphism  $\bar{f}_w : U_v \rightarrow U_{v'}$  which is necessarily injective.

Now suppose that  $\alpha'$  is any positive root at  $v'$ . The action of  $w$  induces a bijection of the positive roots at  $v$  to the positive roots at  $v'$  so there is some positive root  $\alpha$  at  $v$  such that  $w\alpha = \alpha'$ . But this means  $\bar{f}_w(U_\alpha) = U_{\alpha'}$ . Since  $U_{v'}$  is generated by the positive root groups at  $v'$ , this means  $\bar{f}_w$  is surjective and we get an isomorphism between  $U_v$  and  $U_{v'}$  as desired.  $\square$

Before moving on it is worth clarifying that the type  $s$  of the vertex  $v$  in the previous lemma can be any type, not just the literal type  $s$  in the definition of  $W$ .

The previous result can also be used to show that the  $W$  action on  $\Sigma$  also behaves nicely with respect to the homomorphisms  $\phi_v$  when they exit.

**Corollary 3.** *Suppose  $v$  is a vertex of  $\Sigma$  with  $|\text{st}(v)| = 2n$  and  $[U_v : U'_v] \geq 2$ . If  $v'$  is any other vertex of  $\Sigma$  of the same type then there is an isomorphism between  $U_v$  and  $U_{v'}$  which sends  $U'_v$  to  $U'_{v'}$  and  $\ker \phi_v$  to  $\ker \phi_{v'}$ .*

*Proof.* Let  $\bar{f}_w$  be the isomorphism defined by Lemma ?? and let  $\alpha_1, \dots, \alpha_n$  be a standard ordering of the positive roots through  $v$ . Then by Lemma ?? again we know that there is a standard ordering  $\alpha'_1, \dots, \alpha'_n$  of the positive roots through  $v'$  such that  $\bar{f}_w(U_{\alpha_i}) = U_{\alpha'_i}$  for all  $1 \leq i \leq n$ . Since  $U'_v$  and  $U'_{v'}$  are generated by  $\{U_{\alpha_1}, U_{\alpha_n}\}$  and  $\{U_{\alpha'_1}, U_{\alpha'_n}\}$  respectively, we know that  $\bar{f}_w$  must induce an isomorphism between  $U'_v$  and  $U'_{v'}$  as desired.

By Corollary 2,  $\ker \phi_v$  is the unique proper, normal subgroup of  $U_v$  containing  $U'_v$ . If we apply the isomorphism  $\bar{f}_w$  once again we get that  $\bar{f}_w(\ker \phi_v)$  is a proper, normal subgroup of  $U_{v'}$  containing  $\bar{f}_w(U'_v) = U'_{v'}$  and thus  $\bar{f}_w(\ker \phi_v) = \ker \phi_{v'}$  by Corollary 2.  $\square$

The general theory gives us the following result

**Theorem 1.** *Let  $\mathcal{G}$  be a Kac-Moody group over  $k$  with rank 3 Weyl group  $W$  as before. For any vertex  $v$  of  $\Sigma$ , let  $U'_v = \langle U_1, U_n \rangle$  where  $U_1, U_n$  are the simple roots at  $v$ . If  $U'_v = U_v$  for all  $v \in \Sigma$  then  $U$  is finitely generated.*

Remark: In fact, we can make an even stronger statement. Let  $\alpha_s$  be the positive root defined by the wall which separates  $C$  and  $sC$  and similarly define  $\alpha_t$  and  $\alpha_u$ . If  $U'_v = U_v$  for all  $v \in \Sigma$  then  $U$  is generated by  $U_{\alpha_s}, U_{\alpha_t}$ , and  $U_{\alpha_u}$ .

# Chapter 2

## Conditions for Infinite Generation

### 2.1 Extension of $\phi_v$

Throughout this chapter  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  will be an RGD system of type  $(W, S)$  with the following assumptions:

$$\begin{aligned} &W \text{ has rank 3, } S = \{s, t, u\}, a = m(s, t), b = m(s, u), c = m(t, u) \text{ and } 3 \leq a \leq b \leq c \\ &[U_\alpha, U_\beta] = 1 \text{ when } \alpha, \beta \text{ are nested} \end{aligned} \quad (\text{A})$$

Let  $\Sigma$  be the Coxeter complex of  $W$  with fundamental chamber  $C$ , and  $\Phi_+$  be the positive roots of  $\Sigma$ . We will also let  $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$  be the subgroup of  $G$  generated by the positive root groups. We will also note that properties of RGD systems tell us that  $a, b, c \in \{2, 3, 4, 6, 8\}$  and thus by (A) we know that  $a, b, c \in \{3, 4, 6, 8\}$ .

We can also recall some terminology from the last chapter. We will say that  $\alpha$  is a positive root at  $v$  if  $\alpha$  is positive and the wall  $\partial\alpha$  passes through  $v$  and we will denote the positive roots at  $v$  as  $\Phi_+^v$ . Then we can define  $U_v = \langle U_\alpha | \alpha \in \Phi_+^v \rangle$ . We can also label the roots of  $\Phi_+^v$  as  $\alpha_1, \dots, \alpha_n$ , where  $2n = |\text{st}(v)|$  in  $\Sigma$ , in such a way that  $\alpha_i \cap \alpha_j \subset \alpha_k$  for  $1 \leq i \leq k \leq j \leq n$ . With this labeling we will call  $\alpha_1, \alpha_n$  the simple roots at  $v$  and we will note that they do not depend on the labeling. We will use this labeling many times throughout the section and we will refer to it as the standard labeling. This definition is a slight abuse as this labeling scheme is not unique, however, the only other possible labeling is given by flipping the order and sending  $\alpha_i \mapsto \alpha_{n+1-i}$ . In practice, this ambiguity will not matter and so most of the time we can simply refer to the standard labeling without any further detail.

We say that two distinct positive roots  $\alpha, \beta$  are a *pre-nilpotent* pair if  $\alpha \cap \beta$  and  $(-\alpha) \cap (-\beta)$  both contain a chamber. There is a very nice characterization of pre-nilpotent roots which we will use in the remainder of the chapter. Two roots  $\alpha, \beta$  form a pre-nilpotent pair if and only if one of the following holds:

$$(i) \partial\alpha \cap \partial\beta \neq \emptyset \quad (ii) \alpha, \beta \text{ are nested}$$

where we say  $\alpha, \beta$  are nested if  $\alpha \subset \beta$  or vice versa. By definition,  $\partial\alpha \cap \partial\beta = \emptyset$  if  $\alpha, \beta$  are nested so only one of the previous conditions can be satisfied.

We will also briefly recall the definitions of open and closed intervals of roots. If  $\alpha, \beta$  are two pre-nilpotent, positive roots then we define the closed interval

$$[\alpha, \beta] = \{\gamma \in \Phi_+ | \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset -\gamma\}$$

and the open interval  $(\alpha, \beta) = [\alpha, \beta] \setminus \{\alpha, \beta\}$ . In a similar manner as before, we will define  $U_{(\alpha, \beta)} = \langle U_\gamma | \gamma \in (\alpha, \beta) \rangle$ .

One feature of the standard labeling is that it allows us to describe some of these intervals in a very natural way. If  $v$  is some vertex of  $\Sigma$  and  $\alpha_1, \dots, \alpha_n$  are the positive roots through  $v$  with the standard labeling, then  $[\alpha_i, \alpha_j] = \{\alpha_k | i \leq k \leq j\}$  whenever  $i \leq j$ . Similarly we get  $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$  whenever  $i < j$ .

By definition,  $U_+$  is generated by the  $U_\alpha$  for all positive roots  $\alpha$ . However we can say a little bit more about  $U_+$ . Each  $U_\alpha$  will have its own set of relations  $\mathcal{R}_\alpha$ . The theory of RGD systems tells us that we have a presentation of  $U_+$  of the following form

$$U_+ = \langle U_\alpha, \alpha \in \Phi_+ | \mathcal{R}_\alpha, \alpha \in \Phi_+, [u, u'] = v, u \in U_\alpha, u' \in U_\beta, \{\alpha, \beta\} \text{ a pre-nilpotent pair} \rangle$$

where  $v$  is a word in  $U_{(\alpha, \beta)}$  which depends on  $u, u'$ . Furthermore, by condition (A) we know that  $[u, u'] = 1$  if  $\alpha$  and  $\beta$  are nested. Therefore, the only non-trivial commutator relations will occur when  $\partial\alpha \cap \partial\beta \neq \emptyset$ .

Let  $U'_v = \langle U_1, U_n \rangle$  for any vertex  $v \in \Sigma$ , where  $U_1$  and  $U_n$  are the simple roots at  $v$ . By Theorem 1 we know that  $U$  is finitely generated if  $U'_v = U_v$  for all  $v \in \Sigma$ . What we will show in the rest of the chapter is that if  $U'_v \neq U_v$  for some  $v \in \Sigma$ , then most of the time  $U$  will not be finitely generated. Our general strategy will be as follows. If  $v$  is some vertex of  $\Sigma$  such that  $U'_v \neq U_v$  then Corollary ?? shows the existence of a surjective group homomorphism  $\phi_v : U_v \rightarrow H$  where  $H$  is a cyclic group of the appropriate order. If we can extend this map to all of  $U_+$  in a certain way then we will be able to show certain root groups must be in any generating set of  $U_+$ . If we can do this for enough  $v$  then we will be able to show that  $U_+$  is not finitely generated.

Our first lemma will define our notion of extending  $\phi_v$ , and give a sufficient condition for this extension to exist.

**Lemma 4.** *Suppose that  $v$  is a vertex of  $\Sigma$  such that  $U'_v = \langle U_1, U_n \rangle \neq U_v$ , where  $U_1, U_n$  are the simple roots at  $v$ . Then there is a surjective group homomorphism  $\phi_v : U_v \rightarrow H$  with the property that  $\phi_v(U_1) = \phi_v(U_n) = \{1\}$ , where  $H$  is a cyclic group. Also suppose that for any positive root  $\gamma$  with  $v \in \partial\gamma$  which is not simple at  $v$ , that  $\gamma$  is simple at  $y$  for all  $y \in \partial\gamma$  with  $y \neq v$ . Then the map  $\tilde{\phi}_v : \cup_{\gamma \in \Phi_+} U_\gamma \rightarrow H$  defined by*

$$\tilde{\phi}_v(u) = \begin{cases} \phi_v(u) & \text{if } u \in U_\gamma \text{ and } v \text{ lies on } \partial\gamma \\ 1 & \text{otherwise} \end{cases}$$

{existence} *Extends uniquely to a well defined group homomorphism  $\tilde{\phi}_v : U_+ \rightarrow H$ .*

*Proof.* We know that the map  $\phi_v$  exists by Corollary ?. We have a presentation for  $U_+$  and we have defined  $\tilde{\phi}_v$  on the generators of  $U_+$ , so in order to check that it is well defined we will need to verify that the relations of  $U_+$  are satisfied in the image.

There are three types of relations in the presentation for  $U_+$ . There are relations within the same root group so that  $U_\alpha$  for all positive roots  $\alpha$ . There are also relations between root groups of pre-nilpotent pairs where either the walls intersect or the roots are nested.

Let  $R_\alpha$  be a relation for  $U_\alpha$  where  $R_\alpha$  is considered as a word with letters in  $U_\alpha$ . If  $v$  lies on  $\partial\alpha$  then  $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$  since  $\phi_v$  is a well defined homomorphism. Otherwise, every element of  $U_\alpha$  is sent to 1 and thus  $\tilde{\phi}_v(R_\alpha) = 1$  as well so that  $R_\alpha$  is mapped to the identity as desired.

Now suppose that  $\alpha$  and  $\beta$  are any two positive roots. If  $\alpha, \beta$  nested, then (A) tells us that  $[U_\alpha, U_\beta] = 1$ . Since the codomain of  $\tilde{\phi}_v$  is an abelian group, then any relation of the form  $[x, y] = 1$  will be satisfied by the image.

Now suppose that  $\partial\alpha$  and  $\partial\beta$  meet at a point  $y$  and consider any relation of the form  $[u_\alpha, u_\beta] = w$  where  $u_\alpha \in U_\alpha$ ,  $u_\beta \in U_\beta$ , and  $w$  is a word in  $U_{(\alpha, \beta)} \subset U_y$ . Again, note that the image of the left side of this equation will always be the identity as the codomain is still abelian. If  $y = v$  then  $U_y = U_v$  and thus  $\tilde{\phi}_v(w) = \phi_v(w) = 1$  because  $\phi_v$  is well defined.

Now suppose that  $y \neq v$ . Then we can label the positive roots passing through  $y$  as  $\gamma_1, \dots, \gamma_n$  in such a way that  $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$  whenever  $i < j$ . In this case we can say without loss of generality that  $\alpha = \gamma_l$  and  $\beta = \gamma_m$  with  $l < m$ . There can be at most one root whose wall passes through  $y$  and  $v$ , which we will call  $\gamma_k$  if it exists. If  $\gamma_k$  does not exist, or  $k \leq l$  or  $k \geq m$  then the root  $\gamma_k$  is not contained in  $(\alpha, \beta)$  and thus  $\tilde{\phi}_v(U_\delta) = 1$  for all  $\delta \in (\alpha, \beta)$ . This means  $\tilde{\phi}_v(w) = 1$  and the relation is satisfied.

Now we suppose that  $\gamma_k$  exists and  $l < k < m$ . Then  $\gamma_k$  is not simple at  $y$  and thus  $\gamma_k$  must be simple at  $v$  by assumption. This means  $\tilde{\phi}_v(U_{\gamma_k}) = \phi_v(U_{\gamma_k}) = 1$  by the construction of  $\phi_v$ . Since  $\tilde{\phi}_v(U_{\gamma_i}) = 1$  for all  $i \neq k$  by definition, this means that  $\tilde{\phi}_v(w) = 1$  showing the relation is satisfied and giving the desired result.  $\square$

Now Lemma 4 gives a sufficient condition for the existence of  $\tilde{\phi}_v$  which is fairly easy to check. This will be the main tool we use in the remainder of the section.

Recall our assumptions in (A) that  $(W, S)$  is a rank 3 Coxeter system with  $S = \{s, t, u\}$ . We also assumed that  $a = m(s, t)$ ,  $b = m(s, u)$ , and  $c = m(t, u)$  with  $3 \leq a \leq b \leq c$ . Let  $x$  be the vertex of  $C$  of type  $s$  and assume that  $[U_x : U'_x] \geq 2$ . By the characterization of such  $U_x$  we know that  $c \geq 4$ . Our first step in the main proof will be to show that  $\tilde{\phi}_x$  exists. We will do this by applying Lemma 4 and to do this we need to prove the following result about roots through  $x$ .

**Lemma 5.** *Let  $x$  be the vertex of  $C$  of type  $s$ . If  $\gamma$  is any positive root at  $x$ , and  $y$  is any other vertex on  $\partial\gamma$ , then  $\gamma$  is simple at  $y$ .*

*Proof.* Suppose that  $\gamma$  is not simple at  $y$ . Then we can label the positive roots at  $y$  as  $\delta_1, \dots, \delta_m$  in such a way that  $\delta_i \cap \delta_j \subset \delta_k$  for  $1 \leq i \leq k \leq j$ . In this case we have  $\delta_1, \delta_m$  are simple at  $y$  and  $\gamma = \delta_r$  for some  $1 < r < m$ . But  $x$  is a vertex of  $C$  and  $C \in \delta_1 \cap \delta_m$  and thus  $x \in \delta_1 \cap \delta_m$  as well. We know that  $x$  lies on  $\partial\delta_r$  by assumption and thus  $x$  is an element of  $\partial\delta_r \cap \delta_1 \cap \delta_m$ . But this is impossible as we can observe from the geometry of  $\Sigma$  that  $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$  for all  $1 < i < m$ . Thus  $\gamma$  is simple at  $y$  as desired.  $\square$



Despite some of the technical details the previous result should be intuitively clear. The walls through  $y$  will divide  $\Sigma$  into  $2m$  regions, and the region which contains  $C$  will be bounded by the two simple roots. Since  $x$  lies on  $\partial\gamma$ , it is impossible for any other roots through  $y$  to be any “closer” to  $C$  and thus  $\gamma$  must be simple at  $y$  as we proved.

**Corollary 4.** *Let  $x$  be the vertex of  $C$  of type  $s$ , and assume that  $[U_x : U'_x] \geq 2$ . Then the map  $\tilde{\phi}_x$  as defined in Lemma 4 is well defined.*

*Proof.* Let  $\gamma$  be any non-simple, positive root through  $x$  and let  $y$  be another vertex on  $\partial\gamma$ . Then by the previous lemma,  $\gamma$  is simple at  $y$  and thus  $\tilde{\phi}_x$  exists by Lemma 4.  $\square$

The remainder of the section will be used to show that we can use  $\tilde{\phi}_x$  and the  $W$  action on  $\Sigma$  to construct a large family of vertices for which  $\tilde{\phi}_v$  exists.

We can label the roots through  $x$  as  $\alpha_1, \dots, \alpha_n$  so that  $\alpha_1$  and  $\alpha_n$  are the simple roots at  $x$ . Also note that  $n = c$ . The ordering on these roots is chosen so that  $\alpha_i \cap \alpha_j \subset \alpha_k$  for any  $1 \leq i \leq k \leq j \leq n$ . This is equivalent to the condition that  $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$  for any  $i < j$ .

We can describe any root in terms of a pair of adjacent chambers. We can also identify  $\mathcal{C}(\Sigma)$  with  $W$  where the chamber  $wC$  is associated to  $w$ . If we use this identification then we can describe the roots as follows

$$\begin{aligned}\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\ \alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}\end{aligned}$$

In a similar way we can define two more roots

$$\begin{aligned}\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\ \beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}\end{aligned}$$

Now we can define  $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ . These roots are chosen and  $\mathcal{D}$  is defined in such a way to give the following lemma:

{containD}

**Lemma 6.** *Let  $x$  be the vertex of  $C$  of type  $s$  and assume  $W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^a = (su)^b = (tu)^c = 1 \rangle$ . Let  $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$  where  $\alpha_1, \alpha_n, \beta, \beta'$  are roots of  $\Sigma$  defined by*

$$\begin{aligned}\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\ \alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\ \beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\ \beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}\end{aligned}$$

*If  $\gamma$  is a positive root at  $x$  which is not simple at  $x$ , and  $\delta$  is any other positive root such that  $\partial\gamma \cap \partial\delta \neq \emptyset$ , then  $\mathcal{D} \subset \gamma \cap \delta$ .*



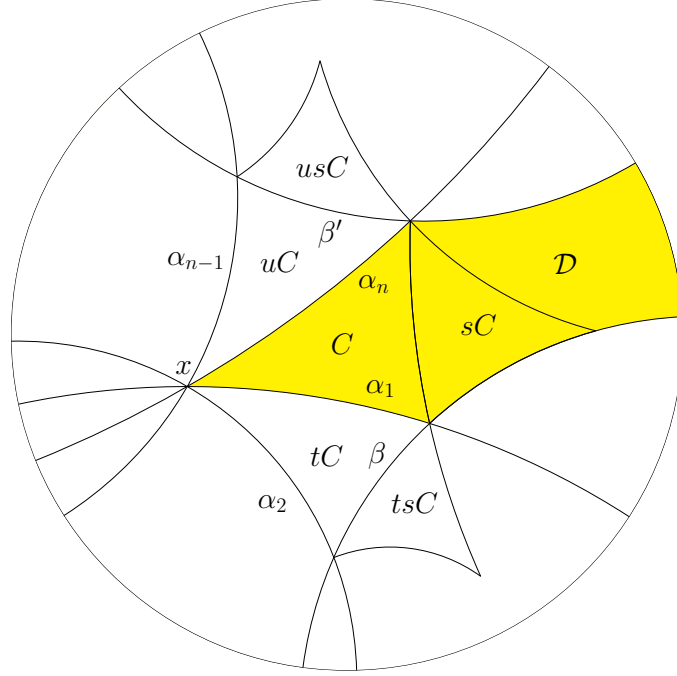
{defineD}

Figure 2.1: The Roots  $\alpha_1, \alpha_n, \beta, \beta'$  with the region  $\mathcal{D}$  in yellow.

*Proof.* By assumption,  $\gamma$  is a positive root through  $x$  so  $\gamma = \alpha_i$  for some  $i$ . Furthermore, we assumed that  $\gamma$  was not simple which means  $2 \leq i \leq n-1$ . Since  $\alpha_1$  and  $\alpha_n$  are simple at  $x$  we can see that  $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$ . Thus it will suffice to prove that  $\mathcal{D} \subset \delta$ .

Let  $y = \partial\gamma \cap \partial\delta$ . If  $y = x$  then  $\delta$  is also a root which passes through  $x$  and so  $\delta = \alpha_j$  for some  $j \neq i$ . Then as before we get  $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$  and thus  $\mathcal{D} \subset \delta$  so that  $\mathcal{D} \subset \gamma \cap \delta$  as desired.

Now suppose that  $\partial\gamma \cap \partial\delta = y \neq x$ . From the local geometry of  $\Sigma$  around  $x$  we can see the following facts. For any  $\alpha_i$  with  $2 \leq i \leq n-1$  we know that  $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = \{x\}$  and  $\partial\alpha_i \subset \alpha_1 \cup \alpha_n$ . Thus the point  $y$  will lie in exactly one of  $\alpha_1$  or  $\alpha_n$ .

First suppose that  $y \in \alpha_n$  so that  $y \notin \alpha_1$ . If  $\partial\alpha_1 \cap \partial\delta = \emptyset$  then there are exactly 3 possibilities. Either  $\alpha_1 \subset \delta$ ,  $\delta \subset \alpha_1$ , or  $-\delta \subset \alpha_1$ . But the last two possibilities would contradict our assumption that  $y \notin \alpha_1$  and thus we get  $\alpha_1 \subset \delta$  and thus  $\mathcal{D} \subset \alpha_1 \subset \gamma \cap \delta$  as desired.

Alternatively, assume that  $\partial\alpha_1 \cap \partial\delta = y'$ . Then the points  $x, y, y'$  will form a triangle with sides on walls of  $\Sigma$ . Then by the triangle condition, these three vertices must form a chamber, call it  $E$ . The points  $x, y$  lie on  $\partial\gamma = \partial\alpha_i$  and the points  $x, y'$  lie on  $\partial\alpha_1$ . Since  $y$  and  $y'$  are adjacent this means that either  $\gamma = \alpha_2$  or  $\gamma = \alpha_n$ . The latter is a contradiction of our assumptions and thus  $\gamma = \alpha_2$ . We know that  $y$  and  $y'$  are adjacent and  $y \in \alpha_n$ . Since neither  $y$  or  $y'$  lies on  $\partial\alpha_n$  this means that  $y' \in \alpha_n$  as well.

We know that  $E$  is a chamber in  $\text{st}(x)$  with a side on  $\partial\alpha_1$  and  $\partial\alpha_2$ . let  $D = tC$  and  $D'$  be the chamber opposite  $D$  in  $\text{st}(x)$ . Then either  $E = D$  or  $E = D'$ . By definition,  $\alpha_1$  is the only wall separating  $C$  and  $tC$  which means  $D = tC \in \alpha_n$ . If  $E = D'$  then  $D' \in \alpha_n$  since

$x, y, y'$  all lie in  $\alpha_n$ . But this is a contradiction as  $\alpha_n$  cannot contain two opposite chambers in  $\text{st}(x)$ . Thus  $E = D = tC$  and  $\delta = \beta$  by definition. Thus  $\mathcal{D} \subset \beta = \delta$  and  $\mathcal{D} \subset \gamma \cap \delta$  as desired.

If we assume instead that  $y \in \alpha_1$  so that  $y \notin \alpha_n$  then identical arguments show that  $\delta = \beta'$  and we can again conclude that  $\mathcal{D} \subset \gamma \cap \delta$  as desired. □

We are now ready to construct a large family of vertices  $\{v\}$  for which  $\tilde{\phi}_v$  will exist. The idea is as follows. If we take any chamber in  $\mathcal{D}$  and treat it as a new “ $C$ ” then  $\tilde{\phi}_x$  would exist for this “ $C$ .” So what we do is apply elements of  $W$  which map the chambers of  $\mathcal{D}$  to  $C$ , and use these choices of  $w$  to get new vertices  $v$ .

Since the construction of these  $\tilde{\phi}_v$  depends on properties of simple roots, we want to know the simplicity behaves nicely with the action of  $W$ . To this end we have the following lemma.

**Lemma 7.** *Suppose  $v$  is a vertex of  $\Sigma$  with simple roots  $\gamma, \gamma'$  at  $v$ . If  $w$  is an element of  $w$  such that  $w\delta$  is a positive root for all positive  $\delta$  at  $v$ , then  $w\gamma$  and  $w\gamma'$  are the simple roots at  $wv$ . I don't know if I need this any more, check the lemma from Chapter 1*

{preservesimple}

*Proof.* Let  $\delta$  be a positive root at  $wv$ . Since  $w$  induces an isomorphism of simplicial complexes, and it sends positive roots at  $v$  to positive roots at  $wv$ , it must also send negative roots at  $v$  to negative roots at  $wv$ . So  $w^{-1}\delta$  is a root at  $v$ , and  $w(w^{-1}\delta) = \delta$  is positive, so  $w^{-1}\delta$  is also positive. Thus by definition of simple, we have  $\gamma \cap \gamma' \subset w^{-1}\delta$ . But we can now apply  $w$  to get  $w\gamma \cap w\gamma' \subset \delta$ . Since the choice of  $\delta$  was arbitrary we must have  $w\gamma$  and  $w\gamma'$  are simple as desired. □

We can now use the previous lemma to actually construct  $\tilde{\phi}_v$  for a certain collection of vertices  $v$ .

{Dexists}

**Lemma 8.** *Let  $x$  be the vertex of  $C$  of type  $s$ , and assume  $U'_x \neq U_x$ . If  $w^{-1}x$  is a vertex in  $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$  then  $\tilde{\phi}_{wx}$  exists.*

*Proof.* Let  $D = \text{Proj}_{w^{-1}x}(C)$  and define  $w'$  so that  $D = (w')^{-1}C$ . By the definition of projections,  $w^{-1}x$  is a vertex of  $D$  of type  $s$ , but  $(w')^{-1}x$  is also a vertex of  $D$  of type  $s$ , and thus  $(w')^{-1}x = w^{-1}x$ . Therefore, we can replace  $w$  with  $w'$  which we will still call  $w$  for notational simplicity. Again, the definition of projections means that  $D$  is the closest vertex to  $C$  which has a vertex of  $w^{-1}x$ . Since  $\mathcal{D}$  is convex, and  $w^{-1}x$  and  $C$  both lie in  $\mathcal{D}$ , we also know that  $D = \text{Proj}_{w^{-1}x}(C)$  lies in  $\mathcal{D}$  as well. By a similar argument we know that  $\text{Proj}_x(D)$  must lie in  $\mathcal{D} \subset \alpha_1 \cap \alpha_n$  and thus  $\text{Proj}_x(D) = C$ . Now define  $E = wC$  and note that the action of  $W$  respects projections and thus we have

$$E = wC = \text{Proj}_{wx}wD = \text{Proj}_{wx}C \quad C = wD = \text{Proj}_{w(w^{-1}x)}wC = \text{Proj}_xE$$

In particular, if  $\gamma$  is any positive root through  $wx$  then  $E \in \gamma$  by the convexity of  $\gamma$ .

Our goal is to apply Lemma 4 at the vertex  $wx$ . Now suppose that  $\gamma$  is a non-simple, positive root through  $wx$  and  $y$  is another vertex on  $\partial\gamma$ . We must show that  $\gamma$  is simple at  $y$ . Since  $\gamma$  is positive through  $wx$  we know that  $C, E \in \gamma$ . If we apply  $w^{-1}$  then we get the following facts. We know that  $w^{-1}\gamma$  is a root such that  $\partial(w^{-1}\gamma)$  passes through  $w^{-1}wx = x$ . We also know that  $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$  so that  $w^{-1}\gamma$  is also a positive root.

The first claim is that  $w^{-1}\gamma$  is not simple at  $x$ . Suppose that  $\delta$  is any positive root at  $wx$ . Then  $E \subset \delta$  and so applying  $w^{-1}$  we get that  $w^{-1}E = C \subset w^{-1}\delta$ . Thus  $w^{-1}$  sends positive roots at  $wx$  to positive roots at  $x$ . By Lemma 7 this means that  $w^{-1}$  sends simple roots at  $wx$  to simple roots at  $x$ . Since  $\gamma$  is not simple at  $wx$  this means that  $w^{-1}\gamma$  is not simple at  $x$ .

So  $w^{-1}\gamma$  is a non-simple positive root at  $x$ , and since  $y$  lies on  $\partial\gamma$  we also know that  $w^{-1}y$  lies on  $w^{-1}(\partial\gamma)$ . If we apply Lemma 5 we can see that  $w^{-1}\gamma$  must be simple at  $w^{-1}y$ .

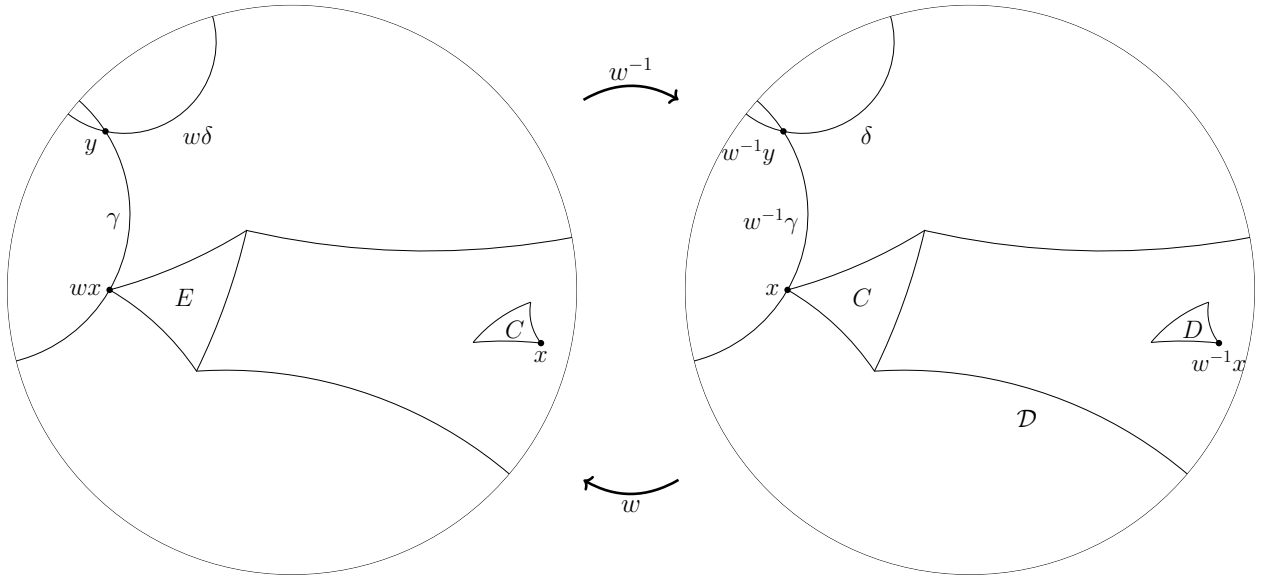


Figure 2.2: The effect of  $w$  and  $w^{-1}$  on the chambers and roots.

Recall that  $D \in \mathcal{D}$  by assumption. Now suppose that  $\delta$  is any positive root at  $w^{-1}y$ . Then by Lemma 6 we know that  $D \in \mathcal{D} \subset \delta$ . If we apply  $w$  then we get  $C = wD \in w\delta$  and  $w\delta$  is a root through  $y$ . Thus  $w\delta$  is a positive root through  $y$  and therefore  $w$  sends positive roots through  $w^{-1}y$  to positive roots through  $y$ . Again we can apply Lemma 7 to say that  $w$  must also send simple roots through  $w^{-1}y$  to simple roots through  $y$ . But  $w^{-1}\gamma$  was a simple root through  $w^{-1}y$  and thus  $\gamma$  is simple at  $y$  as desired.

We know that  $wx$  and  $x$  are both of type  $s$ . We assumed that  $[U_x : U'_x] \geq 2$  and thus  $[U_{wx} : U'_{wx}] \geq 2$  as well. We have also shown that for any positive root at  $wx$ , which is not simple at  $wx$ , and any point  $y \neq wx$  on  $\partial\gamma$  that  $\gamma$  is simple at  $y$ . Thus we can apply Lemma 4 to say that  $\tilde{\phi}_{wx}$  exists as desired.  $\square$

Now we have shown that vertices of  $\mathcal{D}$  in some way correspond to  $\tilde{\phi}_v$ . If our goal is to find infinitely many such  $v$  then there is still some work to be done. For instance, we do not yet

know if the region  $\mathcal{D}$  contains infinitely many chambers, or even if it does, if all the vertices of  $D$  lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

## 2.2 When $\mathcal{D}$ is infinite

Our first task will be to show that the region  $\mathcal{D}$  contains infinitely many unique vertices. Intuitively, this will happen if the walls for  $\beta$  and  $\beta'$  do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that  $W$  is defined by the edge labels  $a = m(s, t), b = m(s, u), c = m(t, u)$  with  $a \leq b \leq c$ . For the remainder of the section we will also add the assumption that  $b \geq 4$ . This assumption will allow us to show that the region  $\mathcal{D}$  contains infinitely many vertices.

{infmany}

**Lemma 9.** *Let  $W$  as before with diagram labels  $3 \leq a \leq b \leq c$ , and  $b \geq 4$ . Also let  $w_k = (tus)^k$  for all  $k \geq 0$ . Then the vertices  $(w_k)^{-1}x$  are all distinct from one another, and they all lie in  $\mathcal{D}$ .*

*Proof.* Note that  $(w_k)^{-1} = (sut)^k$  for all  $k$ . First we will show that  $(w_k)^{-1}x \in \mathcal{D}$  for all  $k$ . Since  $x$  is a vertex of  $C$  we know that  $(w_k)^{-1}x$  is a vertex of  $(w_k)^{-1}C$  and thus it will suffice to show  $(w_k)^{-1}C$  is contained in  $\mathcal{D}$  for all  $k$ . Since the roots  $\alpha_1, \alpha_n, \beta, \beta'$  can be identified with their corresponding subsets of  $W$ , we can use the length function to check containment in these roots.

Now we recall the two  $M$  operations on words in a Coxeter group are as follows:

1. Delete a subword  $ss$  for some  $s \in S$
2. Replace a subword of the form  $stst \cdots st(s)$  by a subword of the form  $ts ts \cdots ts(t)$  where each of these strings has length  $m(s, t)$ .

Also recall that any word in a Coxeter group can be reduced to its minimum length by repeated application of these operations, and any two reduced words can be converted each other by application of operations of type 2. Therefore, in order to check that the length relations are satisfied, it will be enough to show that we can never perform an  $M$  operation of type 1 as this is the only way to reduce length.

It is immediate from the definition that  $\ell((w_k)^{-1}) = 3k$  for all  $k$ . We can also see that  $\ell(t(w_k)^{-1}) = 3k + 1$  and thus  $(w_k)^{-1} \in \alpha_1$  for all  $k$ . Similarly,  $u(w_k)^{-1} = u(sutsut \cdots)$ , and no reduction operations can be done as we assumed  $m(s, u) \geq 4$ . Thus  $\ell(u(w_k)^{-1}) = 3k + 1$  which means  $(w_k)^{-1} \in \alpha_n$  as well.

Now consider the element  $st(w_k)^{-1}$ . If we write this element out in terms of the generators

and apply the only possible Coxeter relations we get

$$\begin{aligned}
st(w_k)^{-1} &= st(sutsut \cdots) \\
&= (sts)(utsuts \cdots) \\
&= (tst)(utsuts \cdots) \\
&= (ts)(tut)(sutsut \cdots)
\end{aligned}$$

and none of these can be reduced as  $m(t, u) \geq 4$ . Note that the commutation relation  $sts = tst$  may not be possible if  $m(s, t) \geq 4$ , but it is the only relation possible in  $st(w_k)^{-1}$  and even if it does exist then it does not allow  $st(w_k)^{-1}$  to be reduced in length. We previously showed  $\ell(t(w_k)^{-1}) = 3k + 1$  and now we see  $\ell(st(w_k)^{-1}) = 3k + 2$  and so  $(w_k)^{-1} \in \beta$ .

Now we can consider  $su(w_k)^{-1}$  in a similar manner. Writing  $su(w_k)^{-1}$  out as a word in the generators and applying Coxeter relations gives us

$$\begin{aligned}
su(w_k)^{-1} &= su(sutsut \cdots) \\
&= (susu)(tsutsu \cdots) \\
&= (usus)(tsutsu \cdots) \\
&= (usu)(sts)(utsuts \cdots) \\
&= (usu)(tst)(utsuts \cdots)
\end{aligned}$$

Note once again that not all of these relations may be possible if  $m(s, u) = 6$  or  $m(s, t) \geq 4$ . However, these are the only possible relations, and since  $su(w_k)^{-1}$  cannot be reduced under these assumptions, it cannot be reduced at all. Thus  $\ell(su(w_k)^{-1}) = 3k + 2$  which means  $su(w_k)^{-1} \in \beta'$  as well.

Now it only remains to show that  $v_m \neq v_n$  for  $m \neq n$ . Suppose  $(w_m)^{-1}x = (w_n)^{-1}x$  for  $m > n$ . Then we would have  $x = w_m(w_n)^{-1}x = w_{m-n}$ . Thus it will suffice to show  $w_k x \neq x$  for any  $k \geq 1$ . But we know that  $\text{stab}_W(x) = \langle u, t \rangle$  which does not contain  $w_k$  for any  $k \geq 1$  and thus  $(w_k)^{-1}x \neq x$  so that  $(w_m)^{-1}x \neq (w_n)^{-1}x$  as desired.

□

We now know that each of the  $(w_k)^{-1}x$  is distinct and each of them lies in  $\mathcal{D}$ . By Lemma 8 we know that  $\tilde{\phi}_{w_k x}$  exists for each  $k \geq 0$ . Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the  $k$  at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices  $w_n x$  distinct, but also no two lie on the same wall.

{samewall}

**Lemma 10.** *Let  $w_k = (tus)^k$  for all  $k \geq 0$  and  $x$  the vertex of  $C$  of type  $s$ . If  $W$  as in the rest of this section then  $w_m x$  and  $w_n x$  do not lie on the same wall of  $\Sigma$  if  $m > n \geq 0$ .*

*Proof.* Suppose  $w_m x$  and  $w_n x$  do lie on the same wall with  $m > n$ . Then we also know that  $w_n w_m^{-1}x = w_{n-m}x$  and  $x$  will lie on the same wall. Since  $m > n$  we can let  $k = m - n$  and thus it will suffice to show that  $(w_k)^{-1}x$  and  $x$  do not lie on the same wall for any  $k \geq 1$ .

We know from Lemma 9 that  $(w_k)^{-1}x \in \mathcal{D}$ . Thus if  $(w_k^{-1})x$  and  $x$  lie on the same wall, it must be a wall through  $x$  and thus it must be  $\partial\alpha_i$  for some  $i$ . We know that  $(w_k^{-1})x \in \alpha_1 \cap \alpha_n$  since  $\mathcal{D} \subset \alpha_1 \cap \alpha_n$  by definition. But we can also recall that  $\partial\alpha_j \cap \alpha_1 \cap \alpha_n = \{x\}$  for  $2 \leq j \leq n-1$ . Thus we have  $i = 1$  or  $i = n$  so that  $(w_k^{-1})x$  either lies on  $\partial\alpha_1$  or  $\partial\alpha_n$ . Therefore, we either have  $u(w_k)^{-1}x = (w_k)^{-1}x$  or  $t(w_k)^{-1}x = (w_k)^{-1}x$  which implies that either  $w_k u w_k^{-1}$  or  $w_k t w_k^{-1}$  is contained in  $\text{stab}_W(x) = \langle u, t \rangle$ . However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced. The only possible relations we have are

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots t u s t u s) t (s u t s u t \cdots) \\ &= (\cdots t u s t u) (s t s) (u t s u t \cdots) \\ &= (\cdots t u s t u) (t s t) (u t s u t \cdots) \quad m(t, u) \geq 4 \end{aligned}$$

or

$$\begin{aligned} w_k u w_k^{-1} &= (\cdots s t u s t u s) u (s u t s u t s \cdots) \\ &= (\cdots s t u s t) (u s u s u) (t s u t s \cdots) \\ &= (\cdots s t u s t) (s u s) (t s u t s \cdots) \\ &= (\cdots s t u) (s t s) u (s t s) (u t s \cdots) \\ &= (\cdots s t u) (t s t) u (t s t) (u t s \cdots) \end{aligned}$$

Similarly as before, even these relations are only possible if  $m(s, u) = 4$ , but even in that case we cannot eliminate every instance of  $s$  in  $w_k u w_k^{-1}$ . In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in  $\langle u, t \rangle$ . Thus the  $w_n x$  all lie on distinct walls as desired.  $\square$

We now have all the ingredients and are ready to prove the main theorem.

**Theorem 2.** *Let  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  be an RGD system of type  $(W, S)$ . Assume  $W$  is defined by a Coxeter diagram with edge labels  $3 \leq a \leq b \leq c$  and also assume that  $b \geq 4$ . Let  $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$  and suppose that  $[U_x : U'_x] \geq 2$  where  $x$  is the vertex of  $C$  of type  $s$ . Then  $U_+$  is not finitely generated.*

*Proof.* Suppose that  $U_+$  is finitely generated. Then there is some finite set of roots  $\beta_1, \dots, \beta_m$  such that  $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m \rangle$ . Now no two of the vertices  $(tus)^{-k}x$  lie on the same wall and thus we can choose  $k$  so that  $v = (tus)^{-k}x$  does not lie on  $\partial\beta_i$  for any  $i$ . By Lemma 9 and Lemma 8 we know that  $\tilde{\phi}_v$  exists, and by definition it is a surjective map from  $U_+ \rightarrow H$  where  $H$  is a cyclic group. However, we can also see by definition that  $\tilde{\phi}_v(U_{\beta_i}) = 1$  for all  $i$ , since none of these walls meet  $v$ . But this means  $\tilde{\phi}_v$  sends all of the generators of  $U_+$  to the identity and thus it must be the trivial map which is a contradiction. Thus  $U_+$  is not finitely generated as desired.  $\square$

A remark worth noting is that the previous proof actually shows something a bit stronger. Since  $H$  is abelian, the map  $\tilde{\phi}_v$  will factor through the abelianization  $(U_+)_{\text{ab}}$ . Then the same arguments as before also show that  $(U_+)_{\text{ab}}$  cannot be finitely generated either.

# Chapter 3

## Exceptional Cases

In the previous chapter we were able to show that  $U_+$  is not finitely generated for a large family of Coxeter groups  $W$  with labels  $a \leq b \leq c$ . These results were based on assuming  $b \geq 4$  which allowed us to show that  $\mathcal{D}$  was infinite and proceed from there. In fact, we didn't even describe all of the chambers in  $\mathcal{D}$ , just an infinite family. However, the same approach will not work in the remaining cases because of the following lemma.

**Lemma 11.** *If  $W$  is a Coxeter group with labels  $a \leq b \leq c$  as before, then  $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$  as defined in the previous chapter is infinite if and only if  $b \geq 4$ .*

*Proof.* We know by Lemma 9 that  $\mathcal{D}$  is infinite if  $b \geq 4$ . Thus it remains to show that  $\mathcal{D}$  is finite if  $b = 3$ . If  $b = 3$  then  $a = 3$  also, and by definition of  $a, b, c$  this means  $m(s, t) = m(s, u) = 3$ . We will also recall the definition of  $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$  where

$$\begin{aligned}\alpha_1 &= \{D \in \Sigma \mid d(D, C) < d(D, tC)\} = \{w \in W \mid \ell(w) < \ell(tw)\} \\ \alpha_n &= \{D \in \Sigma \mid d(D, C) < d(D, uC)\} = \{w \in W \mid \ell(w) < \ell(uw)\} \\ \beta &= \{D \in \Sigma \mid d(D, tC) < d(D, tsC)\} = \{w \in W \mid \ell(tw) < \ell(stw)\} \\ \beta' &= \{D \in \Sigma \mid d(D, uC) < d(D, usC)\} = \{w \in W \mid \ell(uw) < \ell(suw)\}\end{aligned}$$

Let  $w \in W$  and suppose  $\ell(w) \geq 2$ . Then we can write  $w = s_1 s_2 w'$  where  $\ell(w') = \ell(w) - 2$ . If  $s_1 = t$  then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows  $w \notin \alpha_1$  and thus  $w \notin \mathcal{D}$ . A similar argument shows that  $w \notin \mathcal{D}$  if  $s_1 = u$ .

Now we assume  $s_1 = s$  and so we can also assume  $s_2 = t, u$ . First let  $s_2 = t$  so that  $w = stw'$ . If  $w \notin \alpha_1$  then  $w \notin \mathcal{D}$  and so we will suppose  $w \in \alpha_1$ . Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \leq \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus  $w \notin \mathcal{D}$ . A similar argument shows that  $w \notin \mathcal{D}$  if  $s_2 = u$ .

We have shown that if  $\ell(w) \geq 2$  then  $w \notin \mathcal{D}$  and thus  $\mathcal{D}$  must be finite as desired. In fact, if  $a = b = 3$  then we can check relatively easily that  $\mathcal{D} = \{C, sC\}$  which proves the desired result.  $\square$



The previous lemma shows that generating results for the remaining cases is not just a matter of being slightly more clever when we look for vertices, but changing the strategy as a whole. In fact, in some cases our proof strategy needs to switch entirely since  $U_+$  will be finitely generated in some cases as we will see. First, we will show which of the remaining cases are not finitely generated.

All of the remaining rank 3 cases have the property that  $m(s, u) = m(s, t) = 3$ . If  $x$  is the vertex of  $C$  of type  $s$  then  $x$  is the only possible vertex of type  $C$  with the property that  $[U_x : U'_x] \geq 2$ . With two edge labels of 3 it is impossible for  $U_x \cong {}^2F_4(2)$  and so the only remaining possibilities are  $U_x \cong C_2(2)$ ,  $G_2(2)$ , and  $G_2(3)$ . We will enumerate through each of these cases individually.

### 3.1 Case: $U_x \cong G_2(2)$

We saw in the previous chapter that a vertex contained in  $\mathcal{D}$  was a sufficient condition to construct a corresponding map  $\tilde{\phi}_v$ . However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct  $\tilde{\phi}_v$  for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statement similar to Lemma 4.

For the remainder of the section we will assume that  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  is an RGD system of type  $(W, S)$  where  $S = \{s, t, u\}$  and

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^6 = 1 \rangle$$

Furthermore, let  $x$  be the vertex of  $C$  of type  $s$  and assume that  $U_x \cong G_2(2)$ . Recall that this means  $[U_x : U'_x] = 4$  and  $[U_v : U'_v] = 4$  for all vertices  $v$  of type  $s$  by Lemma ??.

Recalling from the previous chapter, we know that there is a presentation of  $U_+$  generated by  $U_\alpha$  for all  $\alpha \in \Phi_+$ . Again, there are several types of relations we need to consider. There are relations among the  $U_\alpha$  and there are relations between  $U_\alpha$  and  $U_\beta$  when  $\{\alpha, \beta\}$  is a prenilpotent pair. By (A) we know that  $[U_\alpha, U_\beta] = \{1\}$  if  $\alpha$  and  $\beta$  are nested. We also know that when  $\partial\alpha \cap \partial\beta \neq \emptyset$  that  $[u, u'] = w$  for some word  $w \in U_{(\alpha, \beta)}$  where  $u \in U_\alpha$  and  $u' \in U_\beta$ .

Now recall from Chapter 1 that there is a surjective homomorphism  $\phi_x : U_x \rightarrow H$  where  $H$  is a cyclic group. We can also choose a standard labeling  $\alpha_1, \dots, \alpha_6$  of the positive roots through  $x$  in such a way that  $\ker \phi_x = U''_x = \langle U_1, U_5, U_6 \rangle$ . Similarly to the last chapter, if  $v$  is any vertex of type  $s$ , our goal is to construct an extension of the form  $\tilde{\phi}_v$  in such a way that

$$\tilde{\phi}_v(U_\alpha) = \begin{cases} \phi_v(U_\alpha) & v \in \partial\alpha \\ 1 & \text{otherwise} \end{cases}$$

If we can do this for enough vertices  $v$  then we will be able to show that  $U_+$  is not finitely generated in the same way as the previous chapter. Our first step is to prove an analagous result to Lemma 4 in the current context.

{336f2existence}

**Lemma 12.** *Let  $v$  be a vertex of  $\Sigma$  of type  $s$ , meaning  $|\text{st}(v)| = 12$ . Assume  $\gamma_1, \dots, \gamma_6$  is a standard ordering of the positive roots through  $v$  such that  $U_{\gamma_5} \subset \ker \phi_v$ . If  $\gamma_2, \gamma_3$ , and  $\gamma_4$  are simple at all other vertices they meet, then  $\tilde{\phi}_v$  as defined in Lemma 4 exists.*

*Proof.* To check  $\tilde{\phi}_v$  is well defined is a matter of checking the relations are satisfied by the images under  $\tilde{\phi}_v$ . Since  $\tilde{\phi}_v$  has a cyclic group as its codomain, we can see immediately that the first two types of relations will be satisfied regardless of  $\alpha$  and  $\beta$ . Now to check the third type.

Suppose  $\alpha$  and  $\beta$  are any two positive roots with  $y = \partial\alpha \cap \partial\beta$ . Then there is a relation in  $U_+$  of the form  $[u_\alpha, u_\beta] = w$  where  $w \in U_{(\alpha, \beta)}$ . Since  $[u_\alpha, u_\beta]$  must be mapped to the identity then we just need to check that  $w$  is also mapped to the identity. If  $y = v$  then  $u_\alpha, u_\beta, w$  all lie in  $U_v$  and  $\tilde{\phi}_v(w) = \phi_v(w)$  which must be the identity because  $\phi_v$  is a well defined homomorphism.

Now suppose  $y \neq v$ . Let  $\delta_1, \dots, \delta_n$  be the positive roots through  $y$ , with a standard labeling, and assume that  $\alpha = \delta_i$  and  $\beta = \delta_j$  with  $i < j$ . There is at most one positive root whose wall can pass through both  $v$  and  $y$ , call it  $\delta_k$  if it exists. If  $\delta_k$  does not exist, then no positive roots through  $y$  pass through  $v$  and so  $\tilde{\phi}_v(u_{\delta_m}) = 1$  for all  $m$ . Thus  $\tilde{\phi}_v(w) = 1$  as desired.

Now suppose  $\delta_k$  does exist and  $\delta_k = \gamma_r$  for  $r \in \{1, 5, 6\}$ . Then we know  $\tilde{\phi}_v(u_{\delta_m}) = 1$  for all  $m \neq k$  and  $\tilde{\phi}_v(u_{\delta_k}) = \tilde{\phi}_v(u_{\gamma_r}) = \phi_v(u_{\gamma_r}) = 1$  by the construction of  $\phi_v$ . Thus  $\tilde{\phi}_v(u_{\delta_m}) = 1$  for all  $m$  and so  $\tilde{\phi}_v(w) = 1$  as well.

Now suppose  $\delta_k$  does exist and  $\delta_k = \gamma_r$  for  $r \in \{2, 3, 4\}$ . Then by assumption,  $\delta_k$  is simple at  $y$  and thus  $k = 1, n$ . Thus  $\tilde{\phi}_v(u_{\delta_m}) = 1$  for all  $2 \leq m \leq n - 1$ . But  $w$  is a word in  $U_{(\alpha, \beta)} \subset U_{(\delta_2, \delta_{n-1})}$  and thus  $\tilde{\phi}_v(w) = 1$  again, which gives the result.  $\square$

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma 4, and so we have a hope of constructing more  $\tilde{\phi}_v$  than the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Let  $x$  be the vertex of  $C$  of type  $s$  as in the previous chapter and let  $\alpha_1, \dots, \alpha_6$  be the positive roots through  $x$ , labeled as usual. Also assume without loss of generality that  $\phi_x(U_{\alpha_5}) = \{1\}$ . Now let  $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$  where  $\beta$  is defined as in the previous chapter. We can now prove a lemma similar to Lemma 6.

picture of  $\mathcal{D}'$

**Lemma 13.** *Let  $x$  be the vertex of  $C$  of type  $s$  so that  $|\text{st}(x)| = 12$ . Let  $\alpha_1, \dots, \alpha_6$  be the positive roots at  $x$  with the standard ordering. Also assume that  $\phi_x(U_{\gamma_5}) = 1$ . Suppose  $\gamma = \alpha_i$  for  $i \in \{2, 3, 4\}$ . If  $\delta$  is any positive root with  $\partial\gamma \cap \partial\delta \neq \emptyset$  then  $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta \subset \gamma \cap \delta$  where*

$$\beta = \{D \in \Sigma \mid d(D, tC) < d(D, tsC)\} = \{w \in W \mid \ell(tw) < \ell(stw)\}$$

*as in the previous chapter.*

*Proof.* By assumption,  $\gamma$  is a positive root through  $x$  and thus we have  $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$ . Thus it remains to show that  $\mathcal{D}' \subset \delta$ .

Let  $y = \partial\gamma \cap \partial\delta$ . If  $y = x$  then  $\delta$  is also a positive root through  $x$  and so  $\mathcal{D}' \subset \delta$  as desired. Now suppose  $y \neq x$ . Then there are two cases to consider. First suppose that  $\partial\delta$  does not meet  $\partial\alpha_1$  or  $\partial\alpha_6$ . **need to fix this proof similar to the other one**

□

We now have a condition for  $\tilde{\phi}_v$  to exist which we can check and so it remains to find potential candidates to use at  $v$ . We know by Lemma 3 that  $\phi_v$  will exist for all vertices  $v$  of type  $s$ . We also know from Lemma 3 that there is a compatibility of standard orderings, which we can use to check the hypothesis in Lemma ???. We now prove the analogue of Lemma 8.

**Lemma 14.** *Let  $x$  be the vertex of  $C$  of type  $s$  and label the positive roots at  $x$  as  $\alpha_1, \dots, \alpha_6$  with the standard ordering in such a way that  $\phi_x(U_{\alpha_5}) = 1$ . If  $v = w^{-1}x \in \mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$  with then  $\tilde{\phi}_{wx}$  as defined in Lemma 4 exists. Recall from the previous chapter that*

$$\beta = \{D \in \Sigma \mid d(D, tC) < d(D, tsC)\} = \{w \in W \mid \ell(tw) < \ell(stw)\}$$

{336f2Dexists}

*Proof.* The proof will proceed in a manner very similar to the proof of Lemma 8. Let  $D = \text{Proj}_{w^{-1}x}(C)$  and let  $D = (w')^{-1}C$ . By the definition of projections,  $w^{-1}x$  is a vertex of  $D$  of type  $s$ , but  $(w')^{-1}x$  is also a vertex of  $D$  of type  $s$ , and thus  $(w')^{-1}x = w^{-1}x$ . Thus we can set  $w = w'$  and simply use  $w$  from now on. Again, the definition of projections means that  $D$  is the closest vertex to  $C$  which has a vertex of  $w^{-1}x$ . Since  $\mathcal{D}$  is convex, and  $w^{-1}x$  and  $C$  both lie in  $\mathcal{D}$ , we also know that  $D = \text{Proj}_{w^{-1}x}(C)$  lies in  $\mathcal{D}$  as well. By a similar argument we know that  $\text{Proj}_x(D)$  must lie in  $\mathcal{D} \subset \alpha_1 \cap \alpha_n$  and thus  $\text{Proj}_x(D) = C$ . Now define  $E = wC$  and note that the action of  $W$  respects projections and thus we have

$$E = wC = \text{Proj}_{wx}wD = \text{Proj}_{wx}C \quad C = wD = \text{Proj}_{w(w^{-1}x)}wC = \text{Proj}_xE$$

In particular, if  $\gamma$  is any positive root through  $wx$  then  $E \in \gamma$  by the properties of projections.

Recall that the positive roots through  $x$  are  $\alpha_1, \dots, \alpha_6$  and we assumed that  $\phi_x(U_{\alpha_5}) = 1$ . For any positive root through  $x$ , say  $\alpha_i$ , we know that  $D \in \alpha_i$  and thus  $C = wD \in w\alpha_i$ . We also know  $w\alpha_i$  will be a root through  $wx$  and thus  $w\alpha_i$  is a positive root through  $x$ . Since  $w$  sends positive roots at  $x$  to positive roots at  $wx$  we can use Lemma 7 and Lemma ??.

Now we can label the positive roots at  $wx$  as  $\gamma_1, \dots, \gamma_6$  in such a way that  $\gamma_i = w\alpha_i$  for all  $i$ . We need to check that this labeling satisfies all of the properties we normally use for labeling the positive roots through a vertex. If  $1 \leq i \leq k \leq j \leq 6$  then we know  $\alpha_i \cap \alpha_j \subset \alpha_k$  and thus  $w\alpha_i \cap w\alpha_j \subset w\alpha_k$  which shows  $(\gamma_i, \gamma_j) = \{\gamma_k \mid i < k < j\}$  as desired. We also know by Lemma ?? that  $\phi_{wx}(u_{\gamma_5}) = 1$ .

Now we can try to apply Lemma 12 to show  $\tilde{\phi}_{wx}$  exists. Consider  $\gamma_i$  for  $2 \leq i \leq 4$ . Let  $y \neq wx$  be any other vertex on  $\partial\gamma_i$ . If we apply  $w^{-1}$  we get that  $w^{-1}y \neq x$  is a vertex on  $\alpha_i$  and thus  $\alpha_i$  is simple at  $w^{-1}y$  by Lemma 5. Now suppose  $\delta$  is any positive root at  $w^{-1}y$ .

Then  $D \in \mathcal{D}' \subset \delta$  by Lemma 13 and so  $C, D \in \delta$ . But this means that  $E, C \in w\delta$  and thus  $w\delta$  is a positive root at  $y$ . So  $w$  sends positive roots at  $w^{-1}y$  to positive roots at  $y$ , and so by Lemma 7 it must also send simple roots at  $w^{-1}y$  to simple roots at  $y$ . Since  $\alpha_i$  is simple at  $w^{-1}y$  then  $\gamma_i$  is simple at  $y$  as desired, and  $\phi_{wx}$  exists by Lemma 12.

this proof may need some work based on the changes in Chapter 1 □

As in the previous chapter, we now have a potentially large class of vertices for which  $\tilde{\phi}_v$  exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. In fact, we can even use the same vertices as in the previous chapter. Let  $w_k = (tus)^k$  for all  $k \geq 0$  and let  $v_k = w_k x$ . Recall in our current setup that  $m(t, u) = 6$  and  $m(s, u) = m(s, t) = 3$ .

**Lemma 15.** *Let  $w_k = (tus)^k$  for all  $k \geq 0$  and let  $x$  be the vertex of  $C$  of type  $s$ . Then the vertices  $(w_k)^{-1}x$  are all distinct, and they all lie in  $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$  as defined previously.*

*Proof.* Many of the proofs will be identical to those in the proof of Lemma 9 and so work will not be repeated when unnecessary. Also note that  $w_k^{-1} = (sut)^k$  for all  $k$ . We can check that  $\ell((w_k)^{-1}) = 3k$  and  $\ell(t(w_k)^{-1}) = 3k + 1$  by identical arguments as before. We can also check that

$$\begin{aligned} u(w_k)^{-1} &= u(sutsut \cdots) \\ &= (usu)(tsutsu \cdots) \\ &= (sus)(tsutsu \cdots) \\ &= (su)(sts)(utsuts \cdots) \\ &= (su)(tst)(utsuts \cdots) \\ &= (su)(ts)(tut)(sutsut \cdots) \end{aligned}$$

We have exhausted all possible Coxeter relations in  $u(w_k)^{-1}$  and none of them led to a reduction in length so we can conclude that  $\ell(u(w_k)^{-1}) = 3k + 1$  also so that  $(w_k)^{-1} \in \alpha_1 \cap \alpha_6$ .

Now we do the same analysis for  $st(w_k)^{-1}$  to see

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut \cdots) = (sts)(utsuts \cdots) \\ &= (tst)(utsuts \cdots) = (ts)(tut)(sutsut \cdots) \end{aligned}$$

and since no reductions can be performed we also get  $\ell(st(w_k)^{-1}) = 3k + 2$  so that  $(w_k)^{-1} \in \beta$  as well. Thus each  $(w_k)^{-1}x$  lies in  $\mathcal{D}'$  as desired. We also know that  $(w_m)^{-1}x \neq (w_n)^{-1}x$  if  $m > n$  by the same argument as in Lemma 9. □

The last major step is to show that the  $w_k x$  cannot somehow lie on only finitely many walls. The analysis here will be slightly more complicated, but ultimately similar to that done in the previous chapter.

336f2finitewalls}

**Lemma 16.** *Let  $x$  be the vertex of  $C$  of type  $s$  and let  $w_k = (tus)^k$  for all  $k \geq 0$ . Any wall of  $\Sigma$  can contain only finitely many  $w_k x$ .*

*Proof.* By arguments identical to those in Lemma 10,  $w_m x$  and  $w_n x$  will lie on the same wall if and only if  $x$  and  $w_{n-m} x$  lie on the same wall. If we assume  $m > n$  then it will suffice to show that a wall containing  $x$  can contain  $(w_k)^{-1} x$  for only finitely many  $k > 0$ . Using the argument of Lemma 10 again we know that  $x$  and  $(w_k)^{-1} x$  will lie on the same wall if and only if  $w_k t w_k^{-1}$  or  $w_k u w_k^{-1}$  lies in  $\text{stab}_W(x) = \langle u, t \rangle$ . If we check these two conjugates we see

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \\ &= (\cdots tus)(tut)(s)(tut)(sut\cdots) \end{aligned}$$

and then we see also

$$\begin{aligned} w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(s)(tsuts\cdots) \\ &= (\cdots stu)(ststs)(uts\cdots) \\ &= (\cdots stu)(t)(uts\cdots) \\ &= (\cdots stustu)(t)(utsuts\cdots) \\ &= (\cdots stus)(tutut)(suts\cdots) \end{aligned}$$

In the first case, no reduction is possible and thus there will always be an  $s$  in any reduced word for  $w_k t w_k^{-1}$  and thus  $w_k t w_k^{-1} \notin \langle u, t \rangle$ . In the second case, We are able to do two reductions in length but then are unable to continue. If we check the relations applied, we will see that the relations cannot continue if  $k \geq 3$ . For completion we will also note that  $w_1 u w_1^{-1} = tst \notin \langle u, t \rangle$  but  $w_2 u w_2^{-1} = tutut \in \langle u, t \rangle$ . Regardless, we know that  $w_m x$  and  $w_n x$  cannot lie on the same wall if  $|m - n| \geq 3$  so that any wall can contain only finitely many  $w_k x$  as desired.

□

Now we are ready to prove the main result of the section, which is nearly identical to the proof of Theorem 2.

{336f2notfg}

**Theorem 3.** *Let  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  be an RGD system of type  $(W, S)$  with assumptions as in (A). Suppose that  $a = m(s, t) = b = m(s, t) = 3$  and  $U_x \cong G_2(2)$  where  $x$  is the vertex of  $C$  of type  $S$ . Then  $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$  is not finitely generated.*

*Proof.* Suppose that  $U_+$  is finitely generated. Then there is some finite set of roots  $\beta_1, \dots, \beta_m$  such that  $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m \rangle$ . Let  $w_k = (tus)^k$  for all  $k \geq 0$ . Now only finitely many

of the vertices  $w_k x$  lie on the same wall and thus we can choose  $k$  so that  $v = w_k x$  does not lie on  $\partial\beta_i$  for any  $i$ . By Lemma 15 we know that  $\tilde{\phi}_v$  exists, and by definition it is a surjective map from  $U_+ \rightarrow H$ . However, we can also see by definition that  $\tilde{\phi}_v(U_{\beta_i}) = 1$  for all  $i$ , since none of these walls meet  $v$ . But this means  $\tilde{\phi}_v$  sends all of the generators of  $U_+$  to the identity and thus it must be the trivial map which is a contradiction. Thus  $U_+$  is not finitely generated as desired.  $\square$

## 3.2 Finite Generation in the Exceptional Cases

Now there are two cases left to consider, and no ammount of modification to our previous strategies will work since we will see that these remaining cases are finitely generated.

For any positive root  $\gamma$ , we say that a chamber  $D$  borders  $\gamma$  if a panel of  $D$  lies on  $\partial\gamma$ . This allows us to define

$$d(\gamma, C) = \min_{D \text{ borders } \gamma} \{d(D, C)\}$$

It is worth noting that if  $d(\gamma, C) = k$  then there is a chamber  $D$  which borders  $\gamma$  and  $d(\gamma, C) = d(D, C)$ . Furthermore, the chamber  $D$  must lie in  $\gamma$  since, otherwise, the chamber adjacent to  $D$  across  $\partial\gamma$  would be closer to  $C$ .

We can now define  $U_n = \langle U_\gamma | \gamma \in \Phi^+, d(\gamma, C) \leq n \rangle$  which is a subgroup of  $U_+$  for all  $n$ . We also have a few facts which are immediate from the definition of  $U_n$ . We can see that  $U_1 \subset U_2 \subset U_3 \subset \dots$  and  $U_+ = \cup_n U_n$  as any positive root will be some finite distance from  $C$ .

Slightly less obvious is the fact that  $U_n$  is finitely generated for all  $n$ . If  $d(\gamma, C) \leq n$  then there must be a chamber  $D$  which borders  $\gamma$  with  $d(D, C) \leq n$ . There are only finitely many such chambers, and each of these chambers borders at most 3 roots, so  $U_n$  is finitely generated.

The idea of the remaining proofs will be to use the following lemma

**Lemma 17.** *For any positive root  $\gamma$  we define  $d(\gamma, C) = \min\{d(D, C) | D \text{ has a panel on } \partial\gamma\}$ . Let  $U_n = \langle U_\gamma | d(\gamma, C) \leq n \rangle$  for all  $n \geq 0$  where  $d(\gamma, C)$ . If there is some  $N$  such that  $U_n \subset U_{n-1}$  for  $n > N$  then  $U_+$  is finitely generated.*

{fgcond}

*Proof.* If  $U_n = U_{n-1}$  for all  $n > N$  then inductively we know that  $U_n = U_N$  for all  $n > N$ . Thus

$$U_+ = \cup_{n=N}^{\infty} U_n = \cup_{n=N}^{\infty} U_N = U_N$$

which is finitely generated as desired.  $\square$

By the results of Chapter ?? and the previous section, we know that the only cases remaining to consider are when  $W$  has a Coxeter diagram defined by edge labels 334, 336, or 338. The 338 case is impossible. And we have already covered the 336 case when  $\Sigma$  has a vertex  $x$  with  $U_x \cong G_2(2)$ . Thus we only need to consider when  $\Sigma$  has a vertex  $x$  with  $U_x \cong C_2(2)$  or  $G_2(3)$ .

### 3.2.1 Case: $U_x \cong C_2(2)$

Let  $(G, (U_\alpha)_{\alpha \in \Phi^+}, T)$  be an RGD system of type  $(W, S)$  where  $S = \{s, t, u\}$  and  $m(s, t) = m(s, u) = 3$  and  $m(t, u) = 6$ . Let  $x$  be the vertex of the fundamental chamber  $C$  of type  $s$  and assume  $U_x \cong C_2(2)$ . We will show that  $U_+$  is finitely generated.

**Theorem 4.** *Let  $W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^4 = 1 \rangle$ . If  $x$  is the vertex of  $C$  of type  $s$  then  $U_n \subset U_{n-1}$  for all  $n > 2$ .*

*Proof.* Let  $\gamma$  be any positive root with  $d(\gamma, C) = n > 2$ . Then choose a chamber  $D_1$  which borders  $\gamma$  such that  $d(D_1, C) = d(\gamma, C)$ . Now there is another chamber  $D_2$  such that  $D_1$  and  $D_2$  are adjacent and  $d(D_2, C) = d(D_1, C) - 1$ . Then  $D_1$  and  $D_2$  will share exactly one vertex which lies on  $\partial\gamma$ , call it  $v$ . Recall that  $\text{st}(v)$  is the set of chambers of  $\Sigma$  for which  $v$  is a vertex. Then we have  $|\text{st}(v)| = 6$  or  $8$ .

First suppose  $|\text{st}(v)| = 6$ . In  $\Sigma$ , we can see that  $\text{st}(v)$  consists of the 6 chambers “surrounding”  $v$  which each have a vertex on  $v$ . Since we have already defined  $D_1$  and  $D_2$  we may label the other 4 chambers in  $\text{st}(v)$  as  $D_3, \dots, D_6$  by going in a circular order around  $v$ . Equivalently this means that  $D_i$  is adjacent to  $D_{i+1}$  for  $1 \leq i \leq 5$  and  $D_6$  is also adjacent to  $D_1$ . We also know that each positive root will contain exactly 3 of these chambers, and those three chambers will be  $D_i, D_{i+1}$ , and  $D_{i+2}$  for some  $i$ , where addition is done modulo 6.

By construction,  $D_2$  and  $D_1$  are not adjacent along  $\partial\gamma$ , but a panel of  $D_1$  lies on  $\partial\gamma$ , and thus  $D_1$  and  $D_6$  must be adjacent along  $\partial\gamma$ . Since  $D_6 \notin \gamma$ , this means that  $\gamma$  must contain  $D_1, D_2, D_3$ . Let  $\alpha$  and  $\beta$  be the other two positive roots through  $v$ . We know that  $\partial\gamma$  cannot separate  $D_2$  and  $D_1$  or  $D_2$  and  $D_3$  so we can say again without loss of generality that  $\partial\alpha$  separates  $D_2$  and  $D_1$  while  $\partial\beta$  separates  $D_2$  and  $D_3$ .

Now  $D_3 \in \gamma$  but  $D_4 \notin \gamma$  which means that  $D_3$  has a panel on  $\partial\gamma$ . By our choice of  $D_1$  we know that  $d(D_3, C) \geq d(D_1, C) > d(D_2, C)$ . But  $D_1$  and  $D_3$  are the two chambers adjacent to  $D_2$  in  $\text{st}(v)$  and thus  $D_2$  must be the closest chamber to  $C$  in  $\text{st}(v)$ . But this means  $D_2 = \text{Proj}_v(C)$  and thus the positive roots at  $v$  which border  $D_2$  must be the simple roots at  $v$ . These roots are  $\alpha$  and  $\beta$  by construction so we know that  $\alpha$  and  $\beta$  are simple at  $v$ . Since  $|\text{st}(v)| = 6$  we know that  $U_v$  cannot be an exceptional rank 2 RGD system and thus  $U_v$  is generated by the simple root groups through  $v$ . Thus  $U_x = \langle U_\alpha, U_\beta \rangle$ . But  $\alpha, \beta$  border  $D_2$  and  $d(D_2, C) = d(D_1, C) - 1 = n - 1$  and thus  $d(\alpha, C), d(\beta, C) \leq n - 1$  so that  $U_\alpha, U_\beta \subset U_{n-1}$ . This means  $U_x \subset U_{n-1}$  as well and thus  $U_\gamma \subset U_{n-1}$ .

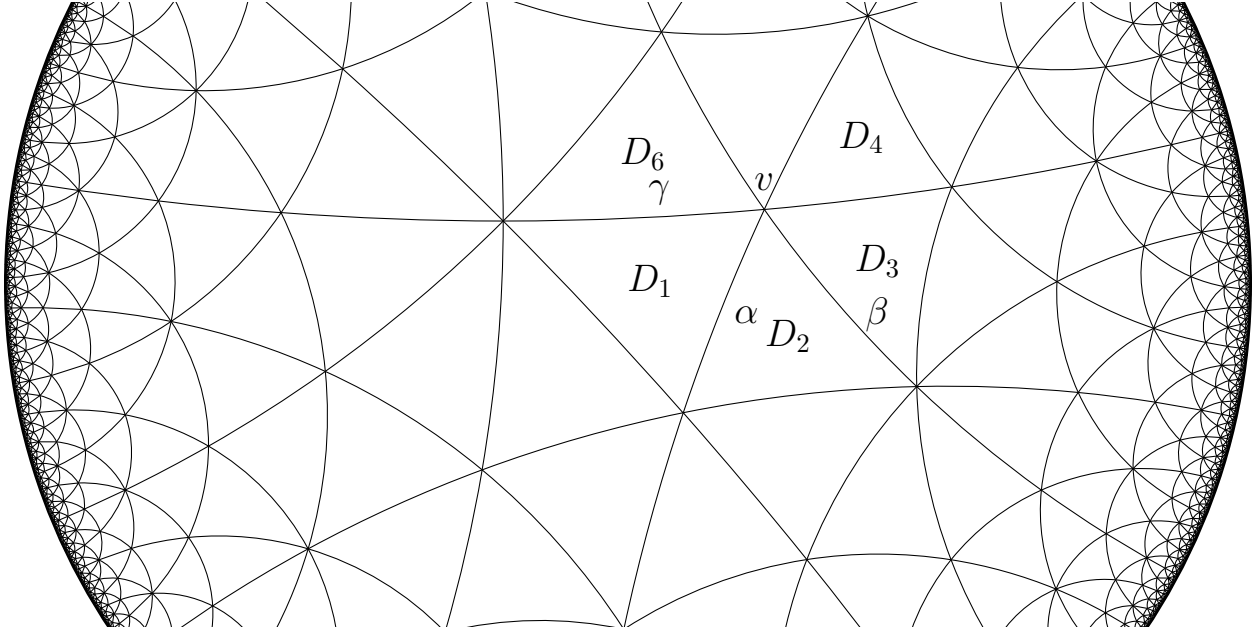
**need to tweak this proof and make some edits**

Now suppose  $|\text{st}(v)| = 8$ . Then we will use the same labeling scheme as before except there will be 8 chambers, and each positive root will contain exactly 4 consecutive chambers from  $\text{st}(v)$ . The same logic as before will still tell us that  $\gamma$  will contain exactly the chambers  $D_1, D_2, D_3, D_4$ . Our first claim is that  $D_2 = \text{Proj}_v(C)$ .

We know that  $\text{Proj}_v(C)$  must lie in any positive root through  $v$  and thus it can only be  $D_1, D_2, D_3, D_4$ . We also know it is the chamber  $A$  in  $\text{st}(v)$  which minimizes  $d(A, C)$ . Since  $d(D_1, C) > d(D_2, C)$  we know that  $D_1$  cannot be the projection. By a similar argument as before we know that  $D_4$  borders  $\gamma$  and thus  $d(D_4, C) \geq d(D_1, C)$  by our choice of  $D_1$ . Thus  $D_4$



{deg6433f2}

Figure 3.1: Case:  $|\text{st}(v)| = 6$ 

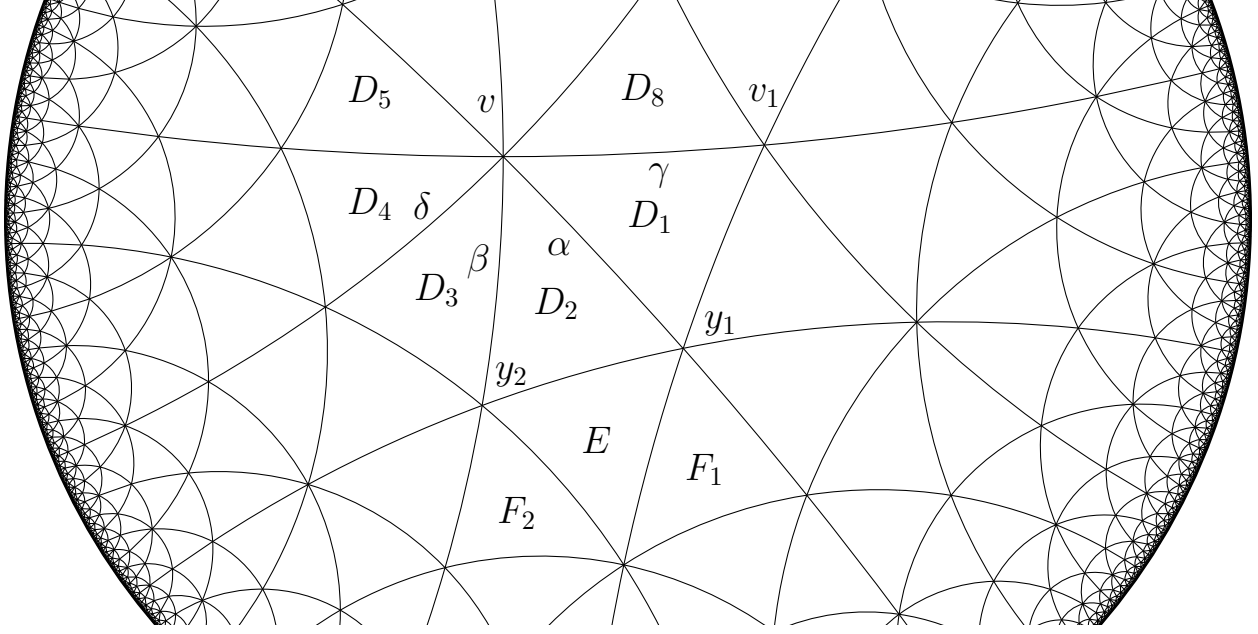
cannot be the projection. Finally, if  $D_3$  were the projection then  $d(D_4, C) = d(D_3, C) + 1 < d(D_3, C) + 2 = d(D_1, C)$  which is also a contradiction and thus  $D_2 = \text{Proj}_v(C)$ .

Let  $\alpha$  be the positive root separating  $D_1$  and  $D_2$ ,  $\beta$  the positive root separating  $D_2$  and  $D_3$  and  $\delta$  the positive root separating  $D_3$  and  $D_4$ . Recall that  $\gamma$  is the positive root separating  $D_8$  and  $D_1$  as well as  $D_4$  and  $D_5$ . We know that  $D_2$  borders  $\alpha$  and  $\beta$  with  $d(D_2, C) = d(D_1, C) - 1 = n - 1$  and thus  $U_\alpha, U_\beta \subset U_{n-1}$ . We also know that  $D_2$  lies in all positive roots through  $v$  by convexity so  $D_2 \in \alpha, \beta, \gamma, \delta$ . Since  $D_2$  is bordered by  $\alpha$  and  $\beta$  we also know that  $\alpha$  and  $\beta$  are the simple roots at  $v$ .

Let  $E$  be the third chamber adjacent to  $D_2$ . Every chamber must have an adjacent chamber which is closer to  $C$  and thus we have  $d(E, C) < d(D_2, C)$ . We can check that  $d(E, C) = d(D_1, C) - 2 \geq 1$  by our choice of  $\gamma$  and thus  $E$  is not the fundamental chamber  $C$ . We know that  $D_1$  and  $D_2$  share two vertices, and  $D_2$  and  $E$  share two vertices, so necessarily we have that  $D_1, D_2$ , and  $E$  must share at least one, and thus exactly one vertex, call it  $y_1$ . By a similar argument, the chambers  $D_3, D_2$ , and  $E$  will also share a vertex  $y_2$ . Let  $F_1$  be the other chamber adjacent to  $E$  that has  $y_1$  as a vertex, and let  $F_2$  be the other chamber adjacent to  $E$  that has  $y_2$  as a vertex. Note that  $|\text{st}(y_1)| = |\text{st}(y_2)| = 6$  since  $v$  is the other vertex of  $D_2$ . The appropriate labeling can be seen in Figure 3.2.1, and the given diagram is unique up to a mirror image flip, which does not affect any of the following arguments. The labeling of these chambers could have simply been defined by the diagram, but the previous explanation seeks to convince the reader that no choices have been made and this diagram is unique.

Since  $d(E, C) < d(D_2, C) < d(D_1, C)$  we know that there is some minimal gallery from  $D_1$  to  $C$  which passes through  $E$ . If we fix such a minimal gallery we can see that it must pass through either  $F_1$  or  $F_2$ . First suppose that it passes through  $F_1$ . Then  $d(F_1, C) = d(D_1, C) - 3$

{deg8433f2}

Figure 3.2: Case:  $|\text{st}(v)| = 8$ 

and so  $F_1$  and  $D_1$  are distance 3 from one another. Since they are both in  $\text{st}(y_1)$ , this means that  $D_1$  and  $F_1$  are opposite in  $\text{st}(y_1)$ . Then there is another minimal gallery from  $D_1$  to  $F_1$  which does not pass through  $D_2$  and can also be extended to a minimal gallery from  $D_1$  to  $C$ . Let  $G_1$  be the chamber adjacent to  $D_1$  in this new minimal gallery. Then  $D_1$  and  $G_1$  have exactly two vertices in common, one of which is  $y_1$ , and the other cannot be  $v$  as this would imply  $G_1 = D_2$  which contradicts our assumption. Let  $v_1$  be the common vertex which is not  $y_1$ . We assumed that  $v$  was the unique vertex shared by  $D_1$  and  $D_2$  which lies on  $\partial\gamma$ . Since  $y_1$  is also shared by  $D_1$  and  $D_2$  this means that  $y_1$  does not lie on  $\partial\gamma$ . We assumed that  $D_1$  has a panel on  $\partial\gamma$  and thus it has two vertices on  $\partial\gamma$  which means  $v_1$  must lie on  $\partial\gamma$ .

Now we have the following situation. We still know that  $D_1$  borders  $\gamma$  with  $d(\gamma, C) = d(D_1, C)$  and  $G_1$  is an adjacent chamber such that  $d(G_1, C) < d(D_1, C)$ . We know that  $v_1$  is a common vertex which lies on  $\partial\gamma$  and thus it is the only common vertex which lies on  $\partial\gamma$ . Finally,  $v$  is the unique vertex of  $D_1$  with 8 chambers in its star. Thus  $|\text{st}(v)| = 6$ . Now we may apply the  $|\text{st}(v)| = 6$  case with  $G_1$  as our new choice of  $D_2$  and  $v_1$  the new  $v$ . This shows that  $U_\gamma \subset U_{n-1}$  as desired.

Now suppose the fixed minimal gallery from before passes through  $F_2$ . Then there is also a minimal gallery from  $D_3$  to  $C$  which passes through  $F_2$  as well. But then  $d(F_2, C) = d(D_3, C) - 3$  which means  $F_2$  and  $D_3$  are opposite in  $\text{st}(y_2)$ . Since  $D_3$  borders  $\delta$ , we can use similar arguments as in the previous two paragraphs to show that  $U_\delta \subset U_{n-1}$ . However, by Lemma ?? we know that  $U_v = \langle U_\alpha, U_\beta, U_\delta \rangle$  and thus  $U_\gamma \subset U_{n-1}$  as well. Thus for any root  $\gamma$  with  $d(\gamma, C) = n \geq 3$  we have  $U_\gamma \subset U_{n-1}$  and thus  $U_n \subset U_{n-1}$  as desired.  $\square$

**Corollary 5.** Suppose  $(G, (U_\alpha)_{\alpha \in \Phi}, T)$  is an RGD system of type  $(W, S)$  with  $S = \{s, t, u\}$ .

*If  $m(s, t) = m(s, u) = 3$  and  $U_x \cong C_2(2)$  for the vertex  $x$  of  $C$  of type  $s$  then  $U_+$  is finitely generated.*

### **3.2.2 Case: 336 over $\mathbb{F}_3$**

Now we consider the last exceptional case. In this section we assume  $(G, (U_\alpha)_{\alpha \in \text{Phi}}, T)$  is an RGD system of type  $(W, S)$  with  $S = \{s, t, u\}$ . Assume that  $m(s, t) = m(s, u) = 3$  and  $U_x \cong G_2(3)$  where  $x$  is the vertex of the fundamental chamber  $C$  of type  $s$ . We will show that  $U_+$  is finitely generated.