Chapter 1

Coxeter Groups and Coxeter Complexes

{ch:coxeter}

The theory of RGD systems was developed for some reason. These systems describe groups with an incredible ammound of geometric structure, which allows us to say a great deal about the group theory. While we will ultimately use some geometric properties to prove results about finite generation in RGD systems, before we can do that we will need to understand the underlying geometry. This geometry starts with Coxeter groups.

1.1 Coxeter Groups

{sec:coxgrp}

Definition 1. A Coxeter system is a pair (W, S) such that S is a finite set, and W is a group with a presentation

$$W = \langle s \in S | (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

subject to the conditions that $m(s,t) \in \mathbb{N} \cup \{\infty\}$, $m(s,s) = 1$, and $m(s,t) = m(t,s) \geq 2$ if $s \neq t$. If $m(s,t) = \infty$ then we simply discard the relation $(st)^{m(s,t)} = 1$.

Through slight abuse of terminology we will refer to W as a Coxeter group, but we will always have a specifice generating set S for the Coxeter system in mind. Coxeter groups have many nice properties, and far too many to discuss here, but we will mention a few which will be of use later. The first of which is the length function. If (W, S) is a Coxeter system then we can define a function $\ell : W \to \mathbb{N}$ by $\ell(w)$ is the minimum number n such that w can be written as $w = s_1 s_2 \cdots s_n$ with $s_i \in S$ for all i. This length function is standard in group theory, and can be defined on any group with any generating set. However, in Coxeter groups this length function takes on a much richer structure which we will describe in more detail.

1.1.1 M-Operations

We say that $(s_1, s_2, \ldots, s_n)$ is a decomposition of w if $w = s_1 s_2 \cdots s_n$, and that it is a reduced decomposition if $n = \ell(w)$. Certainly decompositions, and even reduced decompositions need not be unique, and we will see some ways that we can generate new decompositions. By definition, $m(s,s) = 1$ so $s^2 = 1$ for all $s \in S$. Thus if we ever have an element of s repeated twice in a row in a decompostion, we can simply delete the copies to get another decomposition with smaller length. If $s \neq t \in S$ then $(st)^{m(s,t)} = 1$ and thus we can say

$$\underbrace{sts\cdots t(s)}_{m(s,t)} = \underbrace{tst\cdots s(t)}_{m(s,t)}$$

This again means if we have any alternating string of s and t of the right length, then we can replace it with the swapped alternating string, and get another decompostion of the same length. These two decompostion operations are immediate consequences of the definition of a Coxeter system, but as we will see in the following theorem from [1], these give us everything we need.

{thm:Mop}

Theorem 1. *If (W,S) is a coxeter system and $(s_1, s_2, \ldots, s_n)$ is a decompostion of w, then we can obtain a new decomposition of W by deleting a substring of the form (s,s), or relpacing a substring of length $m(s,t)$ of the form $(s,t,\ldots,s(,t))$ with a substring of the form $(t,s,\ldots,t(,s))$. We will call these two operations* M-Operations *of type 1 and 2 respectively. Furthermore, any decomposition of w can be transformed into a reduced decompostion by repeated application of M-Operations of type 1 and 2, and any two reduced decompostions of w can be transformed into one another by applications of M-Operations of type 2.*

There are many consequences of Theorem 1 but one of the most notable is this, we have a simple algorithm to obtain a reduced decomposition of any w, and we can always check if a decomposition is reduced. In either case we repeatedly apply any possible M-Operations, and applying those of type 1 if possible or noting if none are possible in the case of a decomposition which is already reduced. It also gives us some facts about the length function. For example, if we can write $w = s_{i_1} \cdots s_{i_k}$ then $\ell(w)$ and k are either both even, or both odd, as application of type 1 operations will always reduce the length of a decomposition by 2.

1.1.2 Standard Subgroups and Standard Cosets

Coxeter groups also have a nice subgroup structure will give rise to the rich geometry we will use later. If (W,S) is a Coxeter system then by definition W is generated by S. For any $J \subset S$ we can form a subgroup $W_J = \langle s | s \in J\rangle \leq W_S$. For example, $W_S = W$ and $W_\emptyset = \{1\}$. We will also define a standard coset to be any coset of the form wW_J for any $w \in W$ and $J \subset S$. Standard cosets also have a type function and it is the type of the associated standard subgroup.

As before, there is nothing special about these definitions, as similar definitions hold for any group, but what is special is the structure on standard subgroups. The map which sends

$J \to W_J$ is a bijection from subsets of S to standard subgroups. If H is a standard subgroup, then its J can be recovered as $H \cap S$. We can also check that (W_J, J) is also a Coxeter system.

We can use Theorem 1 to derive some basic consequences about standard subgroups. For example, we can show that $W_J \cap W'_J = W_{J\cap J'}$. One inclusion is clear, and if we take $w \in W_J \cap W'_J$ we can write two reduced decompositions of w, one of which only uses letters from J and the other only uses letters from J'. These reduced decompositions can be transformed into one another by M-Operations of type 2, but M-Operations cannot introduce new letters into a reduced decomposition, only change the order. Thus every letter in the intial decompositions must be in J and J'.

One situation which will be very useful later is when the group W is finite. We say that a Coxeter Group or Coxeter System is *sperical* if W is finite. If (W, S) is spherical then we can prove several facts. First of all, W has a unique element of maximal length, which is usually denoted w_0. It has the property that $\ell(ww_0) = \ell(w_0) - \ell(w)$ for every $w \in W$. One consequence of this fact is that for any $w \in W$ a reduced decomposition of w can be extended to a reduced decomposition of w_0. This element of maximal length will be of some interest in the geometry of W as well. In a similar fashion, we say that J is a spherical subset of S if W_J is spherical. We also say that W is 2-spherical if every subset of S of size 2 is a spherical subset. This is equivalent to saying that $m(s,t) < \infty$ for every $s, t \in S$.

Let Δ be the set of all standard subgroups of W, with a partial order given by reverse inclusion, so that $W_J \leq W_{J'}$ if and only if $J' \subset J$. Using the fact from the previous paragraph, one can check that Δ is isomorphic as a poset to the subsets of S under reverse inclusion. This fact is the basis for our definition of the Coxeter Complex.

1.2 Coxeter Complex

Definition 2. If (W, S) is a Coxeter system, let Σ be the collection of standard cosets of W, ordered by reverse inclusion. Then Σ is a simplical complex called the Coxeter Complex of W.

In the standard terminology of simplicial complexs, we will refer to each standard coset as a simplex, and A and B are simplices with $A \leq B$ then we say A is a face of B. One can check that the dimension of any simplex wW_J will be $|S| - |J| - 1$ because the ordering is by reverse inclusion. For this reason, it is sometimes more useful to refer to the rank of a simplex which is one more than the dimension, so that the rank of $wW_J = |S| - |J|$. We can also draw several conclusions from this fact. First of all, every maximal simplex of Σ has the same dimension, $|S| - 1$, and they will correspond exactly to the elements of W by $w \mapsto wW_\emptyset$. We can also see that the standard subgroup $W = W_S$ is a simplex of dimension -1 and of rank 0 which is a face of every other simplex.

The Coxeter complex is also equiped with a type function $\tau : \Sigma \to \mathcal{P}(S)$ by $\tau(wW_J) = S \setminus J$. However, for convinience, we will more often refer to the *cotype* of a simplex which is $S \setminus \tau(wW_J) = J$. For example, maximial dimensional simplices will have cotype $\emptyset$, and co-dimension 1 simplices will have type $\{s\}$ for some $s \in S$. This convention is also convinient as simplices of cotype J will have rank $|J|$ and dimension $|J|-1$.

We will call the maximal simplices of Σ *chambers* and the co-dimension 1 simplices of Σ will be called *panels*. A panel will have cotype $\{s\}$ for some $s \in S$, or just cotype s for short. If we take a look at a panel of cotype s, we see that it is a standard subgroup of the form $wW_{\{s\}} = w\{1, s\} = \{w, ws\}$. Thus each panel will contain exactly two chambers, corresponding to w and ws, and we will say that the chambers w and ws are s-adjacent. We say that two chambers are ajacent if they are s-adjacent for some $s \in S$. We will also note that there is an obvious chamber which can be distinguished, namely the chamber $W_\emptyset = \{1\}$. We will call this the *fundamental chamber* of Σ and denote it as C.

A *gallery* in Σ is a sequence of chambers $D_0, D_2, \ldots, D_n$ such that D_i and D_{i+1} are ajacent for every i. We will say that a subset $\mathcal{D}$ of chambers of Σ is gallery connected if for all chambers $D, E \in \mathcal{D}$, there is a gallery $D_0, \ldots, D_n$ in $\mathcal{D}$ such that $D_0 = D$ and $D_n = E$. Using this definition we have the following result about Σ.

{prop:gallerycon}

Proposition 1. *If Σ is the Coxeter complex of a Coxeter system (W, S) then Σ is gallery connected.*

Proof. If $D_1, \ldots, D_n$ is a gallery then $D_n, \ldots, D_1$ will also be a gallery. We can also see that if $D_1, \ldots, D_n$ and $E_1, \ldots, E_m$ are galleries with $D_n = E_1$ then $D_1, \ldots, D_n = E_1, \ldots, E_m$ will also be a gallery. Thus it will suffice to show that for any chamber D, there is a gallery between D and the fundamental chamber C.

If D is a chamber of Σ then D is a standard coset $wW_\emptyset = \{w\}$ for some $w \in W$. We will induct on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ and $D = C$ so the result is immediate. Otherwise, let $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ be a minimal decomposition of w with $n \geq 1$ so that $w = s_{i_1}s_{i_2}\cdots s_{i_n}$ and $n = \ell(w)$. Then $w' = s_{i_1}s_{i_2}\cdots s_{i_{n-1}} \in W$ with $\ell(w') = n-1 < \ell(w)$. If D' is the chamber corresponding to w' then inductively there is a gallery $C, \ldots, D'$. If we examine the panel of D' of cotype s_{i_n} then we will see it must be $w'\{1, s_{i_n}\} = \{w', w\}$ and thus D' and D are adjacent. Thus we can extend our gallery $C, \ldots, D', D$ to get a gallery from C to D as desired. □

It turns out that Σ is sufficiently nice that the geometry of the lower dimension simplices can be recovered from the chambers of Σ and from the s-ajacency relations. Thus we will rarely need to make arguments using simplices other than chambers or panels. This also means when considering subset of Σ, we will instead use the chambers of Σ, $\mathcal{C}(S)$ almost exclusively.

If D and E are chambers then a minimal gallery between D and E is a gallery of minimal length, that is, any other gallery between D and E is at least as long. Then we can turn $\mathcal{C}(\Sigma)$ into a metric space where $d(D, E)$ is the length of a minimal gallery between D and E. It is not so surprising that there is a direct link between galleries in Σ and decompositions in W. In fact, we have the following facts which can be found in [1]. If $D = w$ and $E = w'$ are chambers of Σ, then $d(D, E) = \ell(w^{-1}w')$. Furthermore, if $(s_{i_1}, \ldots, s_{i_n})$ is any decomposition of $w^{-1}w'$ then there is a gallery $D_0, \ldots, D_n$ from D to E where D_j is s_{i_j} adjacent to D_{j+1} for all j. In this case the minimal galleries will correspond to reduced decompositions.

1.2.1 Links and Stars

We saw before that if $J \subset S$ then (W_J, J) is also a Coxeter system. This structure will also carry over into the coxeter complexes. Before giving the details, we need to define a few more terms. In any simplical complex, we say that two simplices A and B are joinable if they are contained in a common maximal simplex. In term of the coxeter complex Σ, two simplices $A = wW_J$ and $B = w'W_{J'}$ are joinable if they share a common element w. We can now make two more definitions which we will use extensively through the rest of the paper.

Definition 3. If A is a simplex of Σ, then the star of A, $\mathrm{st}(A)$, is all of the simplices of Σ which are joinable to A. In terms of chambers $\mathcal{C}(\mathrm{st}(A)) = \{w \in W | w \in A = w'W_J\}$. We can also define the link of A, $\mathrm{lk}(A)$, as the set of all simplices of Σ which are joinable to A, but do not contain A.

Now we can see how the subgroup structure of W translates to the geometry of Σ.

{prop:link}

Proposition 2. *If A is a simplex of Σ of cotype J, then $\mathrm{lk}(A)$ is isomorphic as simplical complexes to the Coxeter complex Σ_J of (W_J, J).*

We can define $\Sigma_{\geq A}$ to be the set of simplices in Σ which contain A. There is a bijection from $\mathrm{lk}(A)$ to $\Sigma_{\geq A}$ given by $B \mapsto B \cup A$ which is also an isomorphism as posets. Using this fact we can check that the chambers of $\mathrm{st}(A)$ will be in 1-1 correspondence with the maximal simplices of $\mathrm{lk}(A)$ which are also the chambers of Σ_J. For a simplex A, the star and link of A will give more or less the same combinatorial information, and thus which one we use will be somewhat a matter of convinience.

Stars and links have other nice properties which we will take advantage of later. First of all $\mathcal{C}(\mathrm{st}(A))$ is gallery connected, and the galleries in $\mathrm{st}(A)$ correspond exactly to galleries in Σ_J. Furthermore, suppose that $D_0, \ldots, D_n$ is a minimal gallery between two chambers in $\mathrm{st}(A)$ where A has cotype J. Then we know that D_i and D_{i+1} are s_i adjacent for some $s_i \in S$. But in fact, $s_i \subset J$ for every i. In fact, the types of these adjacencies is exactly the same as those in the minimal gallery of Σ_J.

We say that a Coxeter complex Σ is spherical or 2-spherical if W is spherical or 2-spherical. If Σ is spherical then we will define C^{op} to be the chamber of Σ corresponding to w_0. Then C^{op} is the unique chamber of Σ at maximal distance from C, and it has the property that every chamber of Σ is part of a minimal gallery from C to C^{op}.

Now suppose that Σ is a 2-spherical coxeter complex, and let A be a simplex of Σ of co-dimension 2. Then A is a simplex of cotype $J = \{s, t\}$ for some $s, t \in S$. By definition of 2-spherical, this means W_J is spherical and thus there are finitely many chambers in $\mathrm{st}(A)$. Every chamber in $\mathrm{st}(A)$ also has a unique chamber at maximal distance away in $\mathrm{st}(A)$ which we will call opposite in $\mathrm{st}(A)$. If we examine the structure of W_J we can even see that it is the dihedral group of order $2m(s,t)$, and the simplical compelex Σ_J will be a $2m(s,t)$-gon with edges as chambers and vertices as panels. Translating to Σ this means that $\mathrm{st}(A)$ consists of $2m(s,t)$ chambers arranged in a circular patern around A, and opposite chambers in $\mathrm{st}(A)$ will be at distance $m(s,t)$ away from each other.

1.2.2 Projections

Another useful tool for studying the geometry of Σ is the concept of projections.

Theorem 2. *If A is a simplex of Σ, and D is a chamber of Σ, then there is a chamber $E \in \mathrm{st}(A)$ such that $d(D, E) \le d(D, E')$ for all $E' \in \mathrm{st}(A)$. Additionally, the chamber E is unique and we define the projection of D on to A, or $\mathrm{Proj}_A(D)$ to be the chamber E. The projection E is also characterized by the property that $d(D, E') = d(D, E) + d(E, E')$ for all $E' \in \mathrm{st}(A)$.*

The property $d(D, E') = d(D, E) + d(E, E')$ is known as the gate property because it means for any $E' \in \mathrm{st}(A)$, there is a minimal gallery from D to E' which passes through E. Projections also allow us to define a notion of convexity in a Coxeter complex.

{defn:convex}

Definition 4. We say that a subcomplex Δ of Σ is convex, if $\mathrm{Proj}_A(D) \in \Delta$ whenever A is a simplex of Δ and D is a chamber of Δ.

Convexity also has another interpretation, which can be taken as the definition if desired. A chamber subcomplex Δ of Σ is convex if for any chambers D, E of Δ, any minimial gallery from D to E in Σ is contained in Δ. This means that we can look for minimal gallerys in a convex chamber subcomplex of Σ, and still be sure that it will be minimal in all of Σ. One of the most common uses for this is to apply the result to the convex chamber subcomplex $\mathrm{st}(A)$ for some simplex A. If $D, E \in \mathrm{st}(A)$ for some simplex A, then any minimal gallery from D to E will be contained in $\mathrm{st}(A)$, which is very easy to understand based on our earlier remarks.

1.2.3 Roots

Intuitively we should think of Coxeter groups as reflection groups in some space. A reflection should divide a space into two halves, which are sswitched by a reflection. We will formalize this notion with the concept of roots.

{defn:root}

Definition 5. For any adjacent chambers D, D', let $\alpha_{D,D'}$ be the subcomplex of Σ defined by $\mathcal{C}(\alpha_{D,D'}) = \{E \in \Sigma | d(E, D) < d(E, D')\}$. Then $\alpha_{D,D'}$ is called a root, and the collection of all $\alpha_{D,D'}$ for adjacent chambers D and D' are called the roots of Σ.

We will denote the set of all roots of Σ by Φ. A consequence of Theorem **??** is that $d(E, D) \neq d(E, D')$ for every chamber E of Σ, so we cna think about $\alpha_{D,D'}$ as the chambers which are closer to D than to D'. This also means that for any chamber E, we have either $d(E, D) > d(E, D')$ or $d(E, D) < d(E, D')$. If D and D' are adjacent chambers then both $\alpha_{D,D'}$ and $\alpha_{D',D}$ will be roots, and we will have $\mathcal{C}(\alpha_{D,D'}) \cap \mathcal{C}(\alpha_{D',D}) = \emptyset$ and $\mathcal{C}(\alpha_{D,D'}) \cup \mathcal{C}(\alpha_{D',D}) = \mathcal{C}(\Sigma)$.

The roots $\alpha_{D,D'}$ and $\alpha_{D',D}$ are very closely related, and roughly correspond to the two half spaces defined by a reflection. To differentiate between these roots, we say a root is *positive* if it contains the fundamental chamber C. This choice is of course arbitrary, but the chamber C is a convinient choice. Similarly, we say a root is negative if it does not contain C, and we say that $\alpha_{D,D'}$ and $\alpha_{D',D}$ are opposite roots. We will also denote this with the notation $\alpha_{D',D} = -\alpha_{D,D'}$.

If roots are roughly analagous to the half spaces defined by a reflection, then we should also have some notion of the reflection line. If α is a root of Σ then we define the *wall* of α, denoted by $\partial\alpha$ or $\mathcal{H}_\alpha$, to be $\alpha \cap (-\alpha)$. Then certainly $\partial\alpha$ will contain no chambers, but will not be non-empty, as the panel contained in D and D' will be in $\partial\alpha$ if $\alpha = \alpha_{D,D'}$.

There are several facts about roots and walls which we will use later. Every root is gallery connected, and is also a convex chamber subcomplex of Σ. What is even more interesting is the interaction between roots and links. Suppose A is a simplex of Σ of cotype J. Then we can recall that $\mathrm{lk}(A) \cong \Sigma_J$ where Σ_J is is a Coxeter complex for (W_J, J). Then there is a natual corespondence between roots in Σ to roots in $\mathrm{lk}(A)$. The map $\alpha \to \alpha \cap \mathrm{lk}(A)$ is a bijection between the roots of Σ such that $A \in \alpha$, and the roods of $\mathrm{lk}(A)$ viewed as a Coxeter complex in its own right. Furthermore, this map is also a bijection between walls as well. These results further reiterate the fact that when working in $\mathrm{lk}(A)$, we can essentially forget about the rest of Σ and consider only the Coxeter complex for (W_J, J). This will be especially useful when discussing links of co-dimension 2 simplices.

Thus far we have discussed many properties and attributes of Σ, but we have not really described how the group theory of W interacts with Σ besides in the notion of galleries. In the next section we will see that we can say much more about the interaction of the group W and the Coxeter complex Σ.

1.3 W-Action

{prop:wact}

Proposition 3. *There is a well defined action of W on Σ by $w'(wW_J) = w'wW_J$. Then each $w \in W$ induces an isomorphism of Σ which also preserves (co)type of each simplex.*

As Σ is built directly from W, it is unsurprising that this W action plays very nicely with the geometry of Σ, and we will briefly collect the more relevant facts. The W-action sents galleries to galleries, minimal galleries to minimal galleries, and thus $d(D, E) = d(wD, wE)$ for all $D, E \in \mathcal{C}(\Sigma)$ and $w \in W$.

Because of how natural our definition of the W action is, we can also check relatively easily that W interacts nicely with all of the concepts we have defined so far. If A is a simplex and D is a chamber then we have $\mathrm{Proj}_{wA}(wD) = w\mathrm{Proj}_A(D)$ for all $w \in W$. If α is a root of Σ then $w\alpha$ is also a root with wall $w\partial\alpha$. Furthermore, if $\partial\alpha$ is a wall which separates D and D' then $w\partial\alpha$ will separate wD and wD'. This also means $w\alpha_{D,D'} = \alpha_{wD,wD'}$.

It will also be useful to provie some properties of this action.

{thm:stabW}

Theorem 3. *The action of W is transitive on simplices of Σ of cotype J. Furthermore, suppose A is a simplex of W of cotype J which is a face of $w = wW_\emptyset$. Then* $\mathrm{stab}_W(A) = wW_Jw^{-1}$.

An immediate result is that W acts simply transitively on the chambers of Σ, which is no surprise given the definition of the action. An application is that when working with links or galleries, it is almost always good enough to assume that a simplex A is a face of the fundamental chamber C.

Chapter 2

Buildings and BN-Pairs

{ch:building}

In Chapter 1 we saw that for a Coxeter system (W, S), we can define a simplical complex Σ which will encapsulate the group theoretic structure of W in its geometry. This allows us to understand, W very well, but is somewhat limited as Coxeter groups are very specific. In this chapter we will see how we can generalize some of these notions to other simplical complexes, and then use geometry to study groups which act on them.

{defn:building}

Definition 6. A *building* is a simplical complex Δ which can be expressed as a union of subcomplexes Σ, called Apartments, such that

(B0) Every apartment Σ is a Coxeter complex

(B1) For any two simplices $A, B \in \Delta$, there is an apartment containing A and B.

(B2) For any two apartments Σ, Σ', there is an isomorphism from Σ to Σ' which fixes $\Sigma \cap \Sigma'$ pointwise.

We are using much of the same notation and terminology as [1] but we have changed (B2). When introducing the theory of buildings, we can weaken (B2) to another property which is actually equivalent. However, for our purposes it will be easier to simply state the stronger result as an axiom.

As buildings are defined as unions of Coxeter complex, it should come as no surprise that many of the properties of Chapter 1 will still hold, possibly with some slight modification. In fact, a Coxeter complex Σ is an example of a building with a single apartment, so nearly every result about buildings in general will also hold for Coxeter complexes.

First of all, we will note that every maximal simplex of Δ will have the same dimension as any two maximal simplices will lie in some apartment Σ, and appartments, which are Coxeter complexes, have the property that every maximal simplex has the same dimension. As with any simplical complex, we will say the dimenion of Δ is the dimension of a maximal simplex. We will call these maximal simplices Chambers, and we will call co-dimension 1 simplices panels.

As with Coxeter complexes, we will say that two chambers are adjacent if they share a panel. One key difference between buildings and Coxeter complexes is that in a Coxeter complex, exactly two chambers will be adjacent on every panel, where in a building, there can be any number of chambers sharing the same panel, possibly infinitely many. As in the previous chapter, a sequence of chambers $D_0, \ldots, D_n$ is called a gallery if D_i and D_{i+1} are adjacent for all i. A building Δ will be gallery connected as any two chambers will be contained in an apartment, and apartments are gallery connected. We can use galleries to define a metric on the set of chambers of Δ, where $d(D, E)$ is the length of a minimal gallery from D to E. Even though we know that any two chambers can be connected through an apartment, there is no guarantee a priori that such a gallery would be minimal, or that a minimal gallery can even be contained in a single apartment. However, the following lemma from [1] shows that we can focus our attention to apartments.

{lem:dist}

Lemma 1. *Suppose Δ is a building with chambers D and E. If Σ is an apartment of Δ which contains D and E, then any minimal gallery connecting D and E in Σ will also be minimal in Δ.*

A consequence of the previous lemma is that when trying to determine the distance between any two chambers of Δ, it is enough consider any appartment between the two chambers. When working with Coxeter complexes, we had a stronger notion of adjacency coming from the type function on Σ. We were able to say that two chambers D and E were s-adjacent if they shared a panel of cotype s. It turns out that we can construct a type function for buildings as well. We will state the result found in [1]

{thm:type}

Theorem 4. *If Δ is a building of rank n, and S is a set of size n, then there is a type function τ which takes values in S.*

We will not include the proof of this theorem, but the idea is as follows. If we fix a chamber C then each apartment containing C will have a type function with values in S. By some permutation of S, we can choose all of these type functions so that they agree on C. Then we glue these type functions together to get a type function on the union of apartments containing C, where compatibility is ensured by (B2). But (B1) ensures that the union of apartments containing C is all of Δ so we have a well defined type function. This also allows us to define the types and cotypes of simplices, and refer to s-adjacent chambers as we did with Coxeter complexes. It also means if we have any gallery $D_0, \ldots, D_k$, they we can define the type of this gallery to be a tuple $(s_1, \ldots, s_k)$ such that D_{i-1} and D_i are s_i-adjacent for all i.

Using S-adjacency, and gallery types, we can introduce the notion of residues on Δ. Assume Δ has a type function taking values in S, $J \subset S$, and D is a chamber of Δ. Then we can define the J-residue of Δ containing D, denoted $\mathcal{R}_J(D)$, to be the chamber sub-complex of Δ where the chambers are those which can be connected to D through galleries consisting of only J-adjacencies. More precicely, a chamber E is in $\mathcal{R}_J(D)$ if and only if there is a gallyer $D = D_0, \ldots, D_k = E$ of type $(s_1, \ldots, s_k)$ where $s_i \in J$ for all i. There are two ideas which should be discussed before developing more of the general theory started in Coxeter complexes.

If Δ is a building, then each apartment Σ is a Coxeter complex for some Coxeter group W. Since every apartment is isomorphic, and Coxeter complexes are isomorphic if and only if their associated Coxeter groups are isomorphic, we can assign to each building Δ a well defined (up to isomorphism) Coxeter group W. In this case we say that W is the Weyl group of Δ, and Δ is a building of type W. It is also worth mentioning that W can be recovered purely from the combinatorial information of Δ. If τ is a type function on Δ taking values in S, then we can define a Coxeter group W generated by S with $m(s,t) = \mathrm{diam}(\mathcal{R}_J(D))$ where D is any chamber of Δ and $J = \{s,t\}$. It can be shown that every apartment of Δ will be isomorphic to Σ_W for this W, and thus Δ is a building of type W.

Finally, we will discuss the multiple ways in which we can treat buildings, a topic that we have mostly glossed over thus far. Our definition of a buildings involved with simpliclial compexes, where we view lower dimensional simplices as being contained in chambers. However, residues give another point of view. For example, suppose we have a panel P of Δ of cotype s, and a chamber D containing P. Then $\mathrm{st}(P)$ will be all of the simplices of Δ joinable to P, and the chambers of $\mathrm{st}(P)$ will be exactly those containing P. But if two chambers both contain P, then they are s-adjacent, and they also lie in the same $\{s\}$-residue of Δ. A similar idea holds for simplices of lower rank, and motivates the following theorem.

{thm:sim-cham}

Theorem 5. *Suppose Δ is a building. Then the poset of residues of Δ, ordered by reverse inclusion, defines a simplical complex which is isomorphic to Δ.*

The previous theorem allows us to ignore lower dimensional simplices all together, and instead focus on chambers and J-residues. In practice, we will not devote completely to one approach or the other, but use whichever is more convinient at the time. The biggest difference between the two aproaches is language. For example, in the simplical viewpoint we think of a panel as a co-dimension 1 simplex, and chambers contain a panel, where in the residue viewpoint, we think of a panel as a J residue where $|J| = 1$, and we say that a panel contains a chamber. This mixing of terminology will not be confusing in context however, as it will be clear what approach is being used at any given time.

2.1 Links, Projections, and Roots

Throughout this section, assume that Δ is a building with a type function taking values in S. In this section we will examine some of the ideas introduced in the previous chapter.

Suppose that $J \subset S$ and A is a simplex of Δ of cotype J. As with Coxeter complexes, we define $\mathrm{st}(A)$ to be the set of all simplices which are joinable to A, and $\mathrm{lk}(A)$ is the set of all simplices in $\mathrm{st}(A)$ which are disjoint from A. As aluded to before, the set of all chambers in $\mathrm{st}(A)$ will form a J-residue of Δ. It is also shown in [1] that $\mathrm{lk}(A)$ is a chamber complex as well, and it is also a building of type W_J. We also get a nice description of the apartments of $\mathrm{lk}(A)$. Suppose $\mathcal{A}$ is a set of apartments for Δ. Then $\{\Sigma \cap \mathrm{lk}(A) | A \in \Sigma\}$ is a set of apartments for $\mathrm{lk}(A)$. Furthermore, we know that $\Sigma \cap \mathrm{lk}(A) = \mathrm{lk}_\Sigma(A)$ where $\mathrm{lk}_\Sigma(A)$ simply denotes the link in the apartment Σ.

Now suppose that A is a simplex of cotype J and D is any chamber of Δ. Then there is a unique chamber $E \in \mathrm{st}(A)$ such that $d(D,E) \leq d(D,E')$ for all chambers $E' \in \mathrm{st}(A)$. In this case we call E the projection of D onto A and denote it $\mathrm{Proj}_A(D)$. If we use the chamber complex point of view then we say $\mathrm{Proj}_R(D)$ where R is the J-residue coresponding to A. The projection still posesses the gate property so that $d(D, E') = d(D, E) + d(E, E')$ for all chambers $E' \in \mathrm{st}(A)$. Since distances and minimal galleries can be computed in suitable apartments, it is of no surprise that projections can also be computed in apartments. To be more precise, if Σ is an apartment of Δ containing A and D, then $\mathrm{Proj}_A^{\Delta}(D) = \mathrm{Proj}_A^{\Sigma}(D)$.

A subset of $\mathcal{M}$ of Δ is called convex if for every simplex A and chamber D of M, we have that $\mathrm{Proj}_A(D) \in M$. The condition that A is contained in $\mathcal{M}$ is replaced by the assumption that the residue R meets $\mathcal{M}$ in the chamber complex viewpoint. Similar to the case for Coxeter complexes, the condition that $\mathcal{M}$ is convex is eqivalent to ensuring that for any chambers D, E of $\mathcal{M}$, any minimal gallery connecting D and E will be completely contained in $\mathcal{M}$. Through some of the comments made earlier, we have more or less shown that residues and apartments of a building are both convex subcomplexes.

Recall that in a Coxeter complex, for every pair of adjacent chambers D, D', we define the root $\alpha_{D,D'}$ to be the chambers which are closer to D than to D'. In a a building Δ, a subset α is called a root if it is a root of some apartment Σ of Δ.

We have mentioned it before but it is worth reiterating, most of the properties and definitions for buildings can be defined in terms of apartments. For this reason, you will see throughout the remainder our our work that we will rarely reference the building Δ at all, but will instead choose appropriate apartments and work there. This also makes our lives easier as panels contain only 2 chambers, the the interaction between W and Σ is much more straightforward than that between W and Δ.

2.2 Spherical Buildings

We say a building Δ is spherical if the Weyl group W is spherical, or equivalently if each apartment Σ is a spherical Coxeter complex. As stated before, this means that W has a unique element of maximal length and the diameter of any apartment will also be this length. We say that two chambers C and D are opposite if $d(C, D)$ is maximal, and we write $C \mathrm{op} D$. If C and C' are opposite chambers of Δ, then there is a unique apartment containing C and C', and this apartment is the minimal convex subset of Δ containing C and C'.

Results about spherical buildings will be especially useful when consider 2-spherical buildings, where $m(s, t) < \infty$ for all $s, t \in S$. In this case, every codimension 2 link will be a spherical building and we can use facts about opposition to study local properties of Δ.

Chapter 3

Known Results on Finite Generation

{ch:known}

Throughout this chapter $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) with the following assumptions:

$$\begin{aligned}
&W \text{ has rank } 3,\ S = \{s, t, u\},\ a = m(s,t), b = m(s,u), c = m(t,u)\\
&3 \le a \le b \le c \\
&[U_\alpha, U_\beta] = 1 \text{ when } \alpha, \beta \text{ are nested}
\end{aligned} \tag{3.1}$$

Let Σ be the Coxeter complex of W with fundamental chamber C, and Φ_+ be the positive roots of Σ. We will also let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ be the subgroup of G generated by the positive root groups. We will also note that properties of RGD systems tell us that $a, b, c \in \{2, 3, 4, 6, 8\}$ and thus by (A) we know that $a, b, c \in \{3, 4, 6, 8\}$.

For any vertex v of Σ, there will be some walls of Σ which pass through v, and for each of these walls we have a unique *positive* root. We will call these the **positive roots at** v and denote them by Φ_+^v. Recall that $\mathrm{st}(v)$ is defined as all the chambers containing v as a vertex. If there are n positive roots at v then $|\mathrm{st}(v)| = 2n$. Furthermore, it is possible to label the positive roots at v as $\alpha_1, \dots, \alpha_n$ in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \le i \le k \le j \le n$. This ordering is unique up to a reversal of the form $\alpha_i \mapsto \alpha_{n+1-i}$. This possible reversal will not matter in most cases and if it does then a choice of α_1 will be specified. It does however allow us to unambigiously define α_1 and α_n as the **simple** roots at v. They are the unique positive roots at v whose intersection is contained in all other positive roots at v.

Now we can define U_v to be the subgroup of G generated by all of the root groups of the positive roots at v. That is

$$U_v = \langle U_\alpha | \alpha \text{ is a positive root at } v\rangle = \langle U_\alpha | \alpha \in \Phi_+^v\rangle$$

If $\alpha_1, \alpha_2, \dots, \alpha_n$ is a standard ordering of the positive roots at v then we can simplify notation by letting $U_i = U_{\alpha_i}$ for all α_i through v. Since v is a simples of Σ of co-dimension 2, we know from the theory of RGD systems that U_v will also have the structure of a spherical, rank 2 RGD system as well. Let $U_v' = \langle U_1, U_n\rangle$ be the subgroup of U_v generated by the simple root groups, where $|\mathrm{st}(v)| = 2n$. Then it is known that $U_v = U_v' = \langle U_1, U_n\rangle$ with the exception of a few cases which we will explictly state in the following Lemma.

{lem:index}

Lemma 2. *Let v be a vertex of Σ with $|\mathrm{st}(v)| = 2n$ and let $U'_v = \langle U_1, U_n \rangle$ where U_1, U_n are the root groups of the simple roots at v. Then the group U_v is has the structure of a spherical, rank 2 RGD system and $U_v = U'_v$ unless U_v is isomorphic to one of the following groups:*

$$C_2(2) \qquad G_2(2) \qquad G_2(3) \qquad {}^2F_4(2)$$

In fact, we also know the index $[U_v : U'_v]$ in each of these cases which is summarized in the following table.

U_v	$[U_v : U'_v]$
$C_2(2)$	2
$G_2(2)$	4
$G_2(3)$	3
${}^2F_4(2)$	2

We can see from the previous lemma that even when $U'_v \neq U_v$, it is still a fairly large subgroup and in some cases it will even be normal. This will allow us to construct helpful homomorphisms later, but before we do so we will explicitly state the desired result.

{lem:normal}

Lemma 3. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U'_v] \geq 2$. If U_v is isomorphic to $C_2(2), G_2(3)$, or ${}^2F_4(2)$ then U'_v is a normal subgroup of U_v. If $U_v \cong G_2(2)$ then U'_v is not a normal subgroup of U_v, but there is a standard labeling of the positive roots through v so that $U''_v = \langle U_1, U_5, U_6 \rangle$ is a normal subgroup of U_v with $[U_v : U''_v] = 2$.*

Proof. If $U_v \cong C_2(2)$ or ${}^2F_4(2)$ then U'_v is a subgroup of index 2 and thus it is normal. If $U_v \cong G_2(3)$ then U_v is a 3-group and thus 3 is the smallest prime dividing $|U_v|$ and we know that U'_v is normal in this case as well.

Now suppose $U_v \cong G_2(2)$. Need to add this proof later □

{cor:phiv}

Using Lemma 3 and elementary group theory, we get the following result.

Corollary 1. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U'_v] \geq 2$. Then there is a cyclic group H and a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$ where U_1 and U_n are the simple root groups at v.*

Proof. If $[U_v : U'_v] \geq 2$ then U_v must be isomorphic to one of $C_2(2), G_2(2), G_2(3), {}^2F_4(2)$. If $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$ then we can apply Lemma 3 to let $H = U_v/U'_v$ and ϕ_v be the quotient map which certainly will be surjective and send U_1 and U_n to $\{1\}$ by the definition of U'_v. The group H is cyclic because it has prime order.

If $U_v \cong G_2(2)$ then we know that $U'_v \subset U''_v = \langle U_1, U_5, U_6 \rangle$ for an appropreate standard labeling, and we again apply Lemma 3 to set $H = U_v/U''_v$ and ϕ_v as the quotient map. The group H is again cyclic because it has prime order. □

The following corollary will show that we do not have very much wiggle room when defining ϕ_v, and thus if we can write any function which "looks like" ϕ_v then they must be esentially the same.

{cor:uniquephiv}

Corollary 2. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ such that $[U_v : U_v'] \geq 2$ and let ϕ_v be defined as in the previous corollary. Then $\ker \phi_v$ is the unique, proper, normal subgroup of U_v which contains U_1 and U_n.*

Proof. If $U_v \cong C_2(2), G_2(3), {}^2F_4(2)$ then U_v' is normal, it is generated by U_1 and U_n,, and it has prime index so there cannot be another proper subgroup containing U_v'. By the construction of ϕ_v, we also know that $\ker \phi_v = U_v'$ so that $\ker \phi_v$ is the unique proper, normal subgroup of U_v containing U_1 and U_n.

If $U_v \cong G_2(2)$ then $\ker \phi_v = U_v'' = \langle U_1, U_5, U_6 \rangle$ under a standard labeling. If N is any normal subgroup containing U_1 and U_n then we can apply the commutator relations in $G_2(2)$ to get add proof later ☐

So far we have only considered each vertex v and U_v separately. But in the Coxeter complex Σ, we have not only a collection of vertices, but an action of the group W on the vertices which behaves nicely with properties like the type of a vertex. We will show that the W action also interacts nicely with U_v and ϕ_v in a similar way.

{lem:responder}

Lemma 4. *Suppose v is a vertex of Σ of type s, $|\mathrm{st}(v)| = 2n$, and $[U_v : U_v'] \geq 2$. Also suppose that w is an element of W such that $w\gamma$ is a positive root at wv for every positive root γ at v. Then there are standard labelings $\alpha_1, \ldots, \alpha_n$ and $\alpha_1', \ldots, \alpha_n'$ of the positive roots through v and wv respectively such that $\alpha_i' = w\alpha_i$ for all i. In particular, w sends roots at v which are simple to roots at v' which are also simple. Furthermore, if v' is any vertex of Σ of type s then there is a $w \in W$ such that $wv = v'$ and $w\gamma$ is a positive root at v' for any positive γ at v.*

Proof. Recall a standard labeling is on of the form $\alpha_1, \ldots, \alpha_n$ where $\alpha_i \cap \alpha_j \subset \alpha_k$ for all $1 \leq i \leq k \leq j \leq n$. If w sends all of the positive roots at v to the positive roots at wv then w induces a bijection on the positive roots at v and wv. Now we can define a labeling of the positive roots at wv by $\alpha_i' = w\alpha_i$ for all i. It only remains to check that this is a standard labeling. If $1 \leq i \leq k \leq j \leq n$ then $\alpha_i \cap \alpha_j \subset \alpha_k$ and thus $\alpha_i' \cap \alpha_j' = w\alpha_i \cap w\alpha_j \subset w\alpha_k = \alpha_k'$ so this is a standard labeling as desired.

Now it suffices to show that such a w exists for any vertex v' in Σ. Since the W action on Σ is transitive on vertices of the same type, it will suffice to show the result when v is a vertex of the fundamental chamber C. Let $D = \mathrm{Proj}_{v'}(C)$ so that $d(D, C)$ is minimal among all chambers of $\mathrm{st}(v')$. Then we know that no walls through v' can separate D and C, because crossing one of these walls would produce a chamber in $\mathrm{st}(v)$ which is closer to C. Therefore, a root at v' is positive if and only if it contains D.

Now choose the unique $w \in W$ such that $D = wC$. We claim that w satisfies the desired properties. First of all, v is a vertex of C of type s and thus wv is a vertex of $wC = D$ of type s. But we know that v' is a vertex of D of type s by definition and thus $wv = v'$ as desired. Now suppose γ is any positive root at v. Then $C \in \gamma$ and thus $D = wC \in w\gamma$ and thus $C \in w\gamma$ so $w\gamma$ is positive at $wv = v'$. Now this w sends positive roots at v to positive roots at v' as desired.

$\square$

Before moving on it is worth clarifying that the type s of the vertex v in the previous lemma can by any type, not just the literal type s in the definition of W.

The previous result can also be used to show that the W action on Σ also behaves nicely with respect to the group U_v and the homomorphisms ϕ_v when they exit.

{cor:respectphiv}

Corollary 3. *Suppose v is a vertex of Σ with $|\mathrm{st}(v)| = 2n$ and $[U_v : U'_v] \geq 2$ and v' is any other vertex of Σ of the same type. Then there is an isomorphism between U_v and $U_{v'}$ which sends U'_v to $U'_{v'}$. Consequently, $[U_v : U'_v] = [U_{v'} : U'_{v'}]$, ϕ_v exists if and only if $\phi_{v'}$ exists, and if ϕ_v exists then this isomorphism sends $\ker \phi_v$ to $\ker \phi_{v'}$. If w is any element of W such that $wv = v'$ and $w\gamma$ is positive for all positive γ at v, then this isomorphism can be defined by the property that U_γ is sent to $U_{w\gamma}$ for every γ at v.*

Proof. Let w be any element of W with $wv = v'$ which sends positive roots at v to positive roots at v'. Such a w is guaranteed to exist by Lemma 4. By the theory of RGD systems there is an element $\tilde{w} \in G$ such that $\tilde{w} U_\alpha (\tilde{w})^{-1} = U_{w\alpha}$ for all $\alpha \in \Phi$. Let $f_w : G \to G$ be the isomorphism of conjugation by $\tilde{w}$. Since $w\gamma$ is positive at v' for every positive root γ at v we know that $f_w(U_\gamma) = U_{w\gamma} \subset U_{v'}$ and thus f_w restricts to a homomorphism $\bar{f}_w : U_v \to U_{v'}$ which is necessesarily injective. But w also give a bijection on positive roots at v and v', and $U_{v'}$ is generated by positive root groups at v' so $\bar{f}_w$ is surjective and thus an isomorphism. Now it remains to check it statisfies the rest of the properties.

Since w preserves standard labelings at v and v' we know that it also preserves simple roots. Thus $\bar{f}_w(U_{\alpha_1}) = U_{\alpha'_1}$ for a standard labeling, and similarly for U_{α_n} and $U_{\alpha'_n}$. Since $U'_v = \langle U_{\alpha_1}, U_{\alpha_n} \rangle$ and $U'_{v'} = \langle U_{\alpha'_1}, U_{\alpha'_n} \rangle$ we can also see that $\bar{f}_w$ sends U'_v to $U'_{v'}$. Since $\bar{f}_w$ is an isomorphism it also preserves index so $[U_v : U'_v] = [U_{v'} : U'_{v'}]$.

For any vertex v, the map ϕ_v exists if and only if $[U_v : U'_v] \geq 2$ and thus ϕ_v will exist exactly when $\phi_{v'}$ exists. By Corollary 2 we know that $\ker \phi_v$ is a proper normal subgroup of U_v contaning U'_v and thus $\bar{f}_w(\ker \phi_v)$ will be a proper, normal subgroup of $U_{v'}$ containing $U'_{v'}$. By Corollary 2 again this means $\bar{f}_w(\ker \phi_v) = \ker \phi_{v'}$ which completes the result.

$\square$

The general theory gives us the following result

{thm:knownfg}

Theorem 6. *Let $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ be an RGD system of type (W, S) of any rank. If $U_v = U'_v$ for every vertex v of Σ then U_+ is finitely generated.*

Remark: In fact, we can make an even stronger statement. Let α_s be the positive root defined by the wall which separates C and sC and similarly define α_t and α_u. If $U'_v = U_v$ for all $v \in \Sigma$ then U is generated by $U_{\alpha_s}, U_{\alpha_t}$, and U_{α_u}.

Chapter 4

Conditions for Infinite Generation

{ch:general}

4.1 Extension of ϕ_v

Throughout this chapter $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ will be an RGD system of type (W, S) with the following assumptions:

$$\begin{aligned}&W \text{ has rank } 3,\ S = \{s, t, u\},\ a = m(s, t), b = m(s, u), c = m(t, u)\\&3 \le a \le b \le c,\ 4 \le c\\&[U_\alpha, U_\beta] = 1 \text{ when } \alpha, \beta \text{ are nested}\end{aligned} \tag{A}$$

Let Σ be the Coxeter complex of W with fundamental chamber C, and Φ_+ be the positive roots of Σ. We will also let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ be the subgroup of G generated by the positive root groups. We will also note that properties of RGD systems tell us that $a, b, c \in \{2, 3, 4, 6, 8\}$ and thus by (A) we know that $a, b, c \in \{3, 4, 6, 8\}$. We assume that $c \ge 4$ because otherwise Lemma 2 and Theorem 6 tell us that U_+ is finitely generated and there is nothing to show.

We can also recall some terminology from the last chapter. We will say that α is a positive root at v if α is positive and the wall $\partial\alpha$ passes through v and we will denote the positive roots at v as Φ_+^v. Then we can define $U_v = \langle U_\alpha | \alpha \in \Phi_+^v\rangle$. We can also label the roots of Φ_+^v as $\alpha_1, \ldots, \alpha_n$, where $2n = |\mathrm{st}(v)|$ in Σ, in such a way that $\alpha_i \cap \alpha_j \subset \alpha_k$ for $1 \le i \le k \le j \le n$. With this labeling we will call α_1, α_n the simple roots at v and we will note that they do not depend on the labeling. We will use this labeling many times throughout the section and we will refer to it as the standard labeling. This definition is a slight abuse as this labeling scheme is not unique, however, the only other possible labeling is given by flipping the order and sending $\alpha_i \mapsto \alpha_{n+1-i}$. In practice, this ambiguity will not matter and so most of the time we can simply refer to the standard labeling without any further detail.

We say that two distinct positive roots α, β are a *pre-nilpotent* pair if $\alpha\cap\beta$ and $(-\alpha)\cap(-\beta)$ both contain a chamber. There is a very nice characterization of pre-nilpotent roots which we will use in the remainder of the chapter. Two roots α, β form a pre-nilpotent pair if and only if one of the following holds:

$$\text{(i) } \partial\alpha \cap \partial\beta \neq \emptyset \qquad \text{(ii) } \alpha, \beta \text{ are nested}$$

where we say α, β are nested if $\alpha \subset \beta$ or vice versa. By definition, $\partial\alpha \cap \partial\beta = \emptyset$ if α, β are nested so only one of the previous conditions can be satisfied.

We will also briefly recall the definitions of open and closed intervals of roots. If α, β are two pre-nilpotent, positive roots then we define the closed interval

$$[\alpha, \beta] = \{\gamma \in \Phi_+ | \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset -\gamma\}$$

and the open interval $(\alpha, \beta) = [\alpha, \beta] \setminus \{\alpha, \beta\}$. In a similar manner as before, we will define $U_{(\alpha,\beta)} = \langle U_\gamma | \gamma \in (\alpha, \beta) \rangle$.

One feature of the standard labeling is that it allows us to describe some of these intervals in a very natural way. If v is some vertex of Σ and $\alpha_1, \ldots, \alpha_n$ are the positive roots through v with the standard labeling, then $[\alpha_i, \alpha_j] = \{\alpha_k | i \leq k \leq j\}$ whenever $i \leq j$. Similarly we get $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ whenever $i < j$.

By definition, U_+ is generated by the U_α for all positive roots α. However we can say a little bit more about U_+. Each U_α will have its own set of relations $\mathcal{R}_\alpha$. The theory of RGD systems tells us that we have a presentation of U_+ of the following form

$$U_+ = \langle U_\alpha, \alpha \in \Phi_+ | \mathcal{R}_\alpha, \alpha \in \Phi_+, [u, u'] = v, u \in U_\alpha, u' \in U_\beta, \{\alpha, \beta\} \text{ a pre-nilpotent pair} \rangle$$

where v is a word in $U_{(\alpha,\beta)}$ which depends on u, u'. Furthermore, by condition (A) we know that $[u, u'] = 1$ if α and β are nested. Therefore, the only non-trivial commutator relations will occur when $\partial\alpha \cap \partial\beta \neq \emptyset$.

Let $U'_v = \langle U_1, U_n \rangle$ for any vertex $v \in \Sigma$, where U_1 and U_n are the simple roots at v. By Theorem 6 we know that U is finitely generated if $U'_v = U_v$ for all $v \in \Sigma$. What we will show in the rest of the chapter is that if $U'_v \neq U_v$ for some $v \in \Sigma$, then most of the time U will not be finitely generated. Our general strategy will be as follows. If v is some vertex of Σ such that $U'_v \neq U_v$ then Corollary 1 shows the existence of a surjective group homomorphism $\phi_v : U_v \to H$ where H is a cyclic group of the appropriate order. If we can extend this map to all of U_+ in a certain way then we will be able to show certain root groups must be in any generating set of U_+. If we can do this for enough v then we will be able to show that U_+ is not finitely generated.

Our first lemma will define our notion of extending ϕ_v, and give a sufficient condition for this extension to exist.

{lem:existence}

Lemma 5. *Suppose that v is a vertex of Σ such that $U'_v = \langle U_1, U_n \rangle \neq U_v$, where U_1, U_n are the simple roots at v. Then there is a surjective group homomorphism $\phi_v : U_v \to H$ with the property that $\phi_v(U_1) = \phi_v(U_n) = \{1\}$, where H is a cyclic group. Also suppose that for any positive root γ with $v \in \partial\gamma$ which is not simple at v, that γ is simple at y for all $y \in \partial\gamma$ with $y \neq v$. Then the map $\tilde{\phi}_v : \cup_{\gamma \in \Phi_+} U_\gamma \to H$ defined by*

$$\tilde{\phi}_v(u) = \begin{cases} \phi_v(u) & \textit{if } u \in U_\gamma \textit{ and } v \textit{ lies on } \partial\gamma \\ 1 & \textit{otherwise} \end{cases}$$

Extends uniquely to a well defined group homormoprhism $\tilde{\phi}_v : U_+ \to H$.

Proof. Since $U'_v \neq U_v$ we know that the map ϕ_v exists by Corollary 1. We have a presentation for U_+ and we have defined $\tilde{\phi}_v$ on the generators of U_+, so in order to check that it is well defined we will need to verify that the relations of U_+ are satisfied in the image.

There are three types of relations in the presentation for U_+. There are relations within the same root group U_α for all positive roots α. There are also relations between root groups of pre-nilpotent pairs where either the walls intersect or the roots are nested.

Let R_α be a relation for U_α where R_α is considered as a word with letters in U_α. If v lies on $\partial\alpha$ then $\tilde{\phi}_v(R_\alpha) = \phi_v(R_\alpha) = 1$ since ϕ_v is a well defined homomorphism. Otherwise, every element of U_α is sent to 1 and thus $\tilde{\phi}_v(R_\alpha) = 1$ as well so that R_α is mapped to the identity as desired.

Now suppose that α and β are any two positive roots. If α, β nested, then (A) tells us that $[U_\alpha, U_\beta] = 1$. Since the codomain of $\tilde{\phi}_v$ is an abelian group, then any relation of the form $[x, y] = 1$ will be satisfied by the image.

Now suppose that $\partial\alpha$ and $\partial\beta$ meet at a point y and consider any relation of the form $[u_\alpha, u_\beta] = w$ where $u_\alpha \in U_\alpha$, $u_\beta \in U_\beta$, and w is a word in $U_{(\alpha,\beta)} \subset U_y$. Again, note that the image of the left side of this equation will always be the identity as the codomain is still abelian. If $y = v$ then $U_y = U_v$ and thus $\tilde{\phi}_v(w) = \phi_v(w) = 1$ because ϕ_v is well defined.

Now suppose that $y \neq v$. Then we can label the positive roots passing through y as $\gamma_1, \cdots, \gamma_n$ in such a way that $(\gamma_i, \gamma_j) = \{\gamma_r | i < r < j\}$ whenever $i < j$. In this case we can can say without loss of generality that $\alpha = \gamma_l$ and $\beta = \gamma_m$ with $l < m$. There can be at most one root whose wall passes through y and v, which we will call γ_k if it exists. If γ_k does not exist, or $k \leq l$ or $k \geq m$ then the root γ_k is not contained in (α, β) and thus $\tilde{\phi}_v(U_\delta) = 1$ for all $\delta \in (\alpha, \beta)$. This means $\tilde{\phi}_v(w) = 1$ and the relation is satisfied.

Now we suppose that γ_k exists and $l < k < m$. Then γ_k is not simple at y and thus γ_k must be simple at v by assumption. This means $\tilde{\phi}_v(U_{\gamma_k}) = \phi_v(U_{\gamma_k}) = 1$ by the construction of ϕ_v. Since $\tilde{\phi}_v(U_{\gamma_i}) = 1$ for all $i \neq k$ by definition, this means that $\tilde{\phi}_v(w) = 1$ showing the relation is satisfied and giving the desired result. □

Now Lemma 5 gives a sufficient condition for the existence of $\tilde{\phi}_v$ which is fairly easy to check. This will be the main tool we use in the remainder of the section.

Recall our assumptions in (A) that (W, S) is a rank 3 Coxeter system with $S = \{s, t, u\}$. We also assumed that $a = m(s, t)$, $b = m(s, u)$, and $c = m(t, u)$ with $3 \leq a \leq b \leq c$. Let x be the vertex of C of type s and assume that $[U_x : U'_x] \geq 2$. Our first step in the main proof will be to show that $\tilde{\phi}_x$ exists. We will do this by applying Lemma 5 and to do this we need to prove the following result about roots through x.

{lem:xpos}

Lemma 6. *Let x be the vertex of C of type s. If γ is any positive root at x, and y is any other vertex on $\partial\gamma$, then γ is simple at y.*

Proof. Suppose that γ is not simple at y. Then we can label the positive roots at y as $\delta_1, \ldots, \delta_m$ in such a way that $\delta_i \cap \delta_j \subset \delta_k$ for $1 \leq i \leq k \leq j$. In this case we have δ_1, δ_m

are simple at y and $\gamma = \delta_r$ for some $1 < r < m$. But x is a vertex of C and $C \in \delta_1 \cap \delta_m$ and thus $x \in \delta_1 \cap \delta_m$ as well. We know that x lies on $\partial\delta_r$ by assumption and thus x is an element of $\partial\delta_r \cap \delta_1 \cap \delta_m$. But this is impossible as we can observe from the geometry of Σ that $\partial\delta_i \cap \delta_1 \cap \delta_m = \{y\}$ for all $1 < i < m$. Thus γ is simple at y as desired. □

Despite some of the technical details the previous result should be intuitively clear. The walls through y will divide Σ into $2m$ regions, and the region which contains C will be bounded by the two simple roots. Since x lies on $\partial\gamma$, it is impossible for any other roots through y to be any "closer" to C and thus γ must be simple at y as we proved.

{cor:phix}

Corollary 4. *Let x be the vertex of C of type s, and assume that $[U_x : U'_x] \geq 2$. Then the map $\tilde{\phi}_x$ as defined in Lemma 5 is well defined.*

Proof. Let γ be any non-simple, positive root through x and let y be another vertex on $\partial\gamma$. Then by the previous lemma, γ is simple at y and thus $\tilde{\phi}_x$ exists by Lemma 5. □

The remainder of the section will be used to show that we can use $\tilde{\phi}_x$ and the W action on Σ to construct a large family of vertices for which $\tilde{\phi}_v$ exists.

We can label the roots through x as $\alpha_1, \ldots, \alpha_n$ so that α_1 and α_n are the simple roots at x. Also note that $n = c$. The ordering on these roots is chosen so that $\alpha_i \cap \alpha_j \subset \alpha_k$ for any $1 \leq i \leq k \leq j \leq n$. This is equivalent to the condition that $(\alpha_i, \alpha_j) = \{\alpha_k | i < k < j\}$ for any $i < j$.

We can describe any root in terms of a pair of adjacent chambers. We can also identify $\mathcal{C}(\Sigma)$ with W where the chamber wC is associated to w. If we use this identification then we can describe the roots as follows

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\}
\end{aligned}$$

In a similar way we can define two more roots

$$\begin{aligned}
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Now we can define $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$. These roots are chosen and $\mathcal{D}$ is defined in such a way to give the following lemma:

{lem:containD}

Lemma 7. *Let x be the vertex of C of type s and assume $W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^a = (su)^b = (tu)^c = 1\rangle$. Let $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where $\alpha_1, \alpha_n, \beta, \beta'$ are roots of Σ defined by*

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D, uC) < d(D, usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

{fig:defineD}

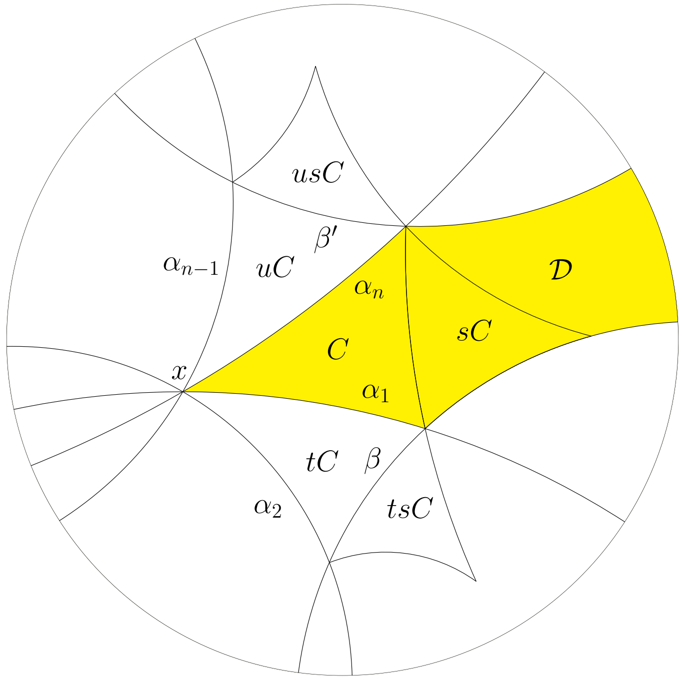

Figure 4.1: The Roots $\alpha_1, \alpha_n, \beta, \beta'$ with the region $\mathcal{D}$ in yellow.

If γ is a positive root at x which is not simple at x, and δ is any other positive root such that $\partial\gamma \cap \partial\delta \neq \emptyset$, then $\mathcal{D} \subset \gamma \cap \delta$.

Proof. By assumption, γ is a positive root through x so $\gamma = \alpha_i$ for some i. Furthermore, we assumed that γ was not simple which means $2 \leq i \leq n-1$. Since α_1 and α_n are simple at x we can see that $\mathcal{D} \subset \alpha_1 \cap \alpha_n \subset \alpha_i = \gamma$. Thus it will suffice to prove that $\mathcal{D} \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_n \subset \alpha_j = \delta$ and thus $\mathcal{D} \subset \delta$ so that $\mathcal{D} \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_n = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_n$. Thus the point y will lie in exactly one of α_1 or α_n.

First suppose that $y \in \alpha_n$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilites would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D} \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_n$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_n$. Since neither y or y' lies on $\partial\alpha_n$ this means that $y' \in \alpha_n$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_n$. If $E = D'$ then $D' \in \alpha_n$ since x, y, y' all lie in α_n. But this is a contradiction as α_n cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D} \subset \beta = \delta$ and $\mathcal{D} \subset \gamma \cap \delta$ as desired.

If we assume insteach that $y \in \alpha_1$ so that $y \notin \alpha_n$ then identical arguments show that $\delta = \beta'$ and we can again conclude that $\mathcal{D} \subset \gamma \cap \delta$ as desired.

□

We are now ready to construct a large family of vertices $\{v\}$ for which $\tilde{\phi_v}$ will exist. The idea is as follows. If we take any chamber in $\mathcal{D}$ and treat it as a new "C" then $\tilde{\phi_x}$ would exist for this "C." So what we do is apply elements of W which map the chambers of $\mathcal{D}$ to C, and use these choices of w to get new vertices v. We can use Lemma 4 to show that this W action will play nicely with the map ϕ_v.

{lem:Dexists}

Lemma 8. *Let x be the vertex of C of type s, and assume $U'_x \neq U_x$. If v is a vertex in $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ of type s then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. Let $D = \mathrm{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 3.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \mathrm{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\mathrm{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\mathrm{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \mathrm{Proj}_{wx} wD = \mathrm{Proj}_{wx} C \qquad C = wD = \mathrm{Proj}_{w(w^{-1}x)} wC = \mathrm{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 5 at the vertex wx. Now suppose that γ is a non-simple, positive root through wx and y is another vertex on $\partial\gamma$. We must show that γ is simple at y. Since γ is positive through wx we know that $C, E \in \gamma$. If we apply w^{-1} then we get the following facts. We know that $w^{-1}\gamma$ is a root such that $\partial(w^{-1}\gamma)$ passes through $w^{-1}wx = x$. We also know that $w^{-1}C = D, w^{-1}E = C \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is also a positive root. Since w^{-1} sends positive roots at wx to positive roots at x we can apply Lemma 4 when necessary.

The first claim is that $w^{-1}\gamma$ is not simple at x. Suppose that δ is any positive root at wx. Then $E \subset \delta$ and so applying w^{-1} we get that $w^{-1}E = C \subset w^{-1}\delta$. By Lemma 4 this means that w^{-1} sends simple roots at wx to simple roots at x. Since γ is not simple at wx this means that $w^{-1}\gamma$ is not simple at x.

So $w^{-1}\gamma$ is a non-simple positive root at x, and since y lies on $\partial\gamma$ we also know that $w^{-1}y$ lies on $w^{-1}(\partial\gamma)$. If we apply Lemma 6 we can see that $w^{-1}\gamma$ must be simple at $w^{-1}y$.

{fig:mappicture}

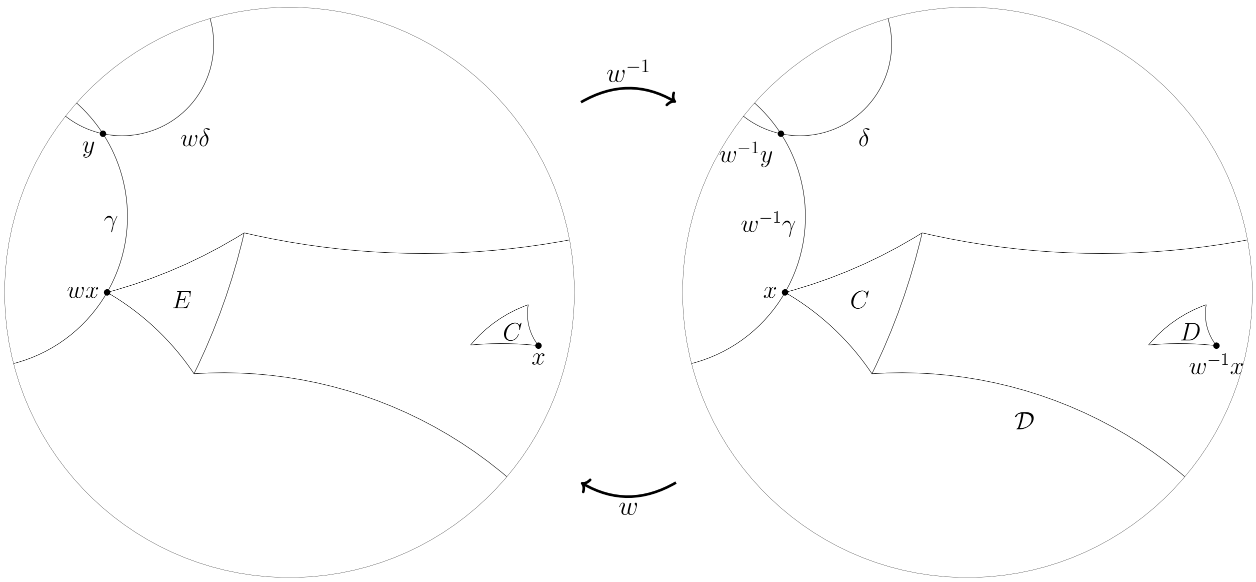

Figure 4.2: The effect of w and w^{-1} on the chambers and roots.

Recall that $D \in \mathcal{D}$ by assumption. Now suppose that δ is any positive root at $w^{-1}y$. Then by Lemma 7 we know that $D \in \mathcal{D} \subset \delta$. If we apply w then we get $C = wD \in w\delta$ and $w\delta$ is a root through y. Thus $w\delta$ is a positive root through y and therefore w sends positive roots through $w^{-1}y$ to positive roots through y. Again we can apply Lemma 4 to say that w must also send simple roots through $w^{-1}y$ to simple roots through y. But $w^{-1}\gamma$ was a simple root through $w^{-1}y$ and thus γ is simple at y as desired.

We now have a vertex wx where $[U_{wx} : U'_{wx}] = [U_x : U'_x] \geq 2$ and the positive roots at wx which are not simple at wx are simple everywhere else. Thus we can apply Lemma 5 to say that $\tilde{\phi}_{wx}$ exists as desired. □

Now we have shown that vertices of $\mathcal{D}$ in some way correspond to $\tilde{\phi}_v$. If our goal is to find infinitely many such v then there is still some work to be done. For instance, we do not yet know if the region $\mathcal{D}$ contains infinitely many chambers, or even if it does, if all the vertices of D lie on finitely many walls. We will show in the next section that these issues are not a problem in most cases.

4.2 When $\mathcal{D}$ is infinite

Our first task will be two show that the region $\mathcal{D}$ contains infinitely many unique vertices. Intuitively, this will happen if the walls for β and β' do not meet, and we will first give a sufficient (and necessary) condition for this.

Recall that W is defined by the edge labels $a = m(s,t), b = m(s,u), c = m(t,u)$ with $a \leq b \leq c$. For the remainder of the section we will also add the assumption that $b \geq 4$. This assumption will allow us to show that the region $\mathcal{D}$ contains infinitely many vertices.

{lem:infmany}

Lemma 9. *Let W as before with diagram labels $3 \leq a \leq b \leq c$, and $b \geq 4$. Also let $w_k = (tus)^k$ for all $k \geq 0$. Then the vertices $(w_k)^{-1}x$ are all distinct from one another, and they all lie in $\mathcal{D}$.*

Proof. Note that $(w_k)^{-1} = (sut)^k$ for all k. First we will show that $(w_k)^{-1}x \in \mathcal{D}$ for all k. Since x is a vertes of C we know that $(w_k)^{-1}x$ is a vertex of $(w_k)^{-1}C$ and thus it will suffice to show $(w_k)^{-1}C$ is contained in $\mathcal{D}$ for all k. Since the roots $\alpha_1, \alpha_n, \beta, \beta'$ can be identified with their corresponding subsets of W, we can use the length function to check containment in these roots.

Now we recall the two M operations on words in a Coxeter group are as follows:

1. Delete a subword ss for some $s \in S$
2. Replace a subword of the form $stst \cdots st(s)$ by a subword of the form $tsts \cdots ts(t)$ where each of these strings has length $m(s,t)$.

Also recall that any word in a Coxeter group can be reduced to its minium length by repeated application of these operations, and any two reduced words can be converted each other by application of operations of type 2. Therefore, in order to check that the lenght relations are satisfied, it will be enough to show that we can never perform an M operation of type 1 as this is the only way to reduce length.

It is immediate from the definition that $\ell((w_k)^{-1}) = 3k$ for all k. We can also see that $\ell(t(w_k)^{-1}) = 3k+1$ and thus $(w_k)^{-1} \in \alpha_1$ for all k. Similarly, $u(w_k)^{-1} = u(sutsut \cdots)$, and no reduction operations can be done as we assumed $m(s,u) \geq 4$. Thus $\ell(u(w_k)^{-1}) = 3k+1$ which means $(w_k)^{-1} \in \alpha_n$ as well.

Now consider the element $st(w_k)^{-1}$. If we write this element out in terms of the generators and apply the only possible Coxeter relations we get

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut \cdots) \\ &= (sts)(utsuts \cdots) \\ &= (tst)(utsuts \cdots) \\ &= (ts)(tut)(sutsut \cdots) \end{aligned}$$

and none of these can be reduced as $m(t,u) \geq 4$. Note that the commutation relation $sts = tst$ may not be possible if $m(s,t) \geq 4$, but it is the only relation possible in $st(w_k)^{-1}$ and even if it does exists then it does not allow $st(w_k)^{-1}$ to be reduced in length. We previously showed $\ell(t(w_k)^{-1}) = 3k+1$ and now we see $\ell(st(w_k)^{-1}) = 3k+2$ and so $(w_k)^{-1} \in \beta$.

Now we can consider $su(w_k)^{-1}$ in a similar manner. Writing $su(w_k)^{-1}$ out as a word in the generators and applying Coxeter relations gives us

$$\begin{aligned} su(w_k)^{-1} &= su(sutsut \cdots) \\ &= (susu)(tsutsu \cdots) \\ &= (usus)(tsutsu \cdots) \\ &= (usu)(sts)(utsuts \cdots) \\ &= (usu)(tst)(utsuts \cdots) \end{aligned}$$

Note once again that not all of these relations may be possible if $m(s,u) = 6$ or $m(s,t) \geq 4$. However, these are the only possible relations, and since $su(w_k)^{-1}$ cannot be reduced under these assumptions, it cannot be reduced at all. Thus $\ell(su(w_k)^{-1}) = 3k+2$ which means $su(w_k)^{-1} \in \beta'$ as well.

Now it only remains to show that $v_m \neq v_n$ for $m \neq n$. Suppose $(w_m)^{-1}x = (w_n)^{-1}x$ for $m > n$. Then we would have $x = w_m(w_n)^{-1}x = w_{m-n}$. Thus it will suffice to show $w_k x \neq x$ for any $k \geq 1$. But we know that $\mathrm{stab}_W(x) = \langle u, t\rangle$ which does not contain w_k for any $k \geq 1$ and thus $(w_k)^{-1}x \neq x$ so that $(w_m)^{-1}x \neq (w_n)^{-1}x$ as desired.

□

We now know that each of the $(w_k)^{-1}x$ is distinct and each of them lies in $\mathcal{D}$. By Lemma 9 we know that $\tilde{\phi}_{w_k x}$ exists for each $k \geq 0$. Our idea is still to use each of these vertices to give a root which must be contained in any generating set. However, there is still one possible issue. If almost all of these vertices lie on the same wall, then an inclusion of that root in a generating set could satisfy infinitely many of the k at once, which would not allow us to prove infinite generation. So it remains to show that not only are the vertices $w_n x$ distinct, but also no two lie on the same wall.

{lem:samewall}

Lemma 10. *Let $w_k = (tus)^k$ for all $k \geq 0$ and x the vertex of C of type s. If W as in the rest of this section then $w_m x$ and $w_n x$ do not lie on the same wall of Σ if $m > n \geq 0$.*

Proof. Suppose $w_m x$ and $w_n x$ do lie on the same wall with $m > n$. Then we also know that $w_n w_m^{-1} x = w_{n-m} x$ and x will lie on the same wall. Since $m > n$ we can let $k = m - n$ and thus it will suffice to show that $(w_k)^{-1}x$ and x do not lie on the same wall for any $k \geq 1$.

We know from Lemma 9 that $(w_k)^{-1}x \in \mathcal{D}$. Thus if $(w_k^{-1})x$ and x lie on the same wall, it must be a wall through x and thus it must be $\partial\alpha_i$ for some i. We know that $(w_k^{-1})x \in \alpha_1 \cap \alpha_n$ since $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ by defintion. But we can also recall that $\partial\alpha_j \cap \alpha_1 \cap \alpha_n = \{x\}$ for $2 \leq j \leq n-1$. Thus we have $i = 1$ or $i = n$ so that $(w_k^{-1})x$ either lies on $\partial\alpha_1$ or $\partial\alpha_n$. Therefore, we either have $u(w_k)^{-1}x = (w_k)^{-1}x$ or $t(w_k)^{-1}x = (w_k)^{-1}x$ which implies that either $w_k u w_k^{-1}$ or $w_k t w_k^{-1}$ is contained in $\mathrm{stab}_W(x) = \langle u, t\rangle$. However, by a similar argument as before, we can simply write out these elements and show that they cannot be reduced. The only possible relations we have are

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \qquad m(t,u) \geq 4 \end{aligned}$$

or

$$\begin{aligned} w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\ &= (\cdots stust)(ususu)(tsuts\cdots) \\ &= (\cdots stust)(sus)(tsuts\cdots) \\ &= (\cdots stu)(sts)u(sts)(uts\cdots) \\ &= (\cdots stu)(tst)u(tst)(uts\cdots) \end{aligned}$$

Similarly as before, even these relations are only possible if $m(s,u) = 4$, but even in that case we cannot eliminate every instance of s in $w_k u w_k^{-1}$. In both cases we can see that there is no further reduction possible and thus neither of these conjugates can possibly lie in $\langle u, t\rangle$. Thus the $w_n x$ all lie on distinct walls as desired. $\square$

We now have all the ingredients and are ready to prove the main theorem.

{thm:notfg}

Theorem 7. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S). Assume W is defined by a Coxeter diagram with edge labels $3 \leq a \leq b \leq c$ and also assume that $b \geq 4$. Let $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ and suppose that $[U_x : U_x'] \geq 2$ where x is the vertex of C of type s. Then U_+ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m\rangle$. Now no two of the vertices $(tus)^{-k}x$ lie on the same wall and thus we can choose k so that $v = (tus)^{-k}x$ does not lie on $\partial\beta_i$ for any i. By Lemma 9 and Lemma 8 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$ where H is a cyclic group. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. $\square$

A remark worth noting is that the previous proof actually shows something a bit stronger. Since H is abelian, the map $\tilde{\phi}_v$ will factor through the abelianization $(U_+)_{\mathrm{ab}}$. Then the same arguments as before also show that $(U_+)_{\mathrm{ab}}$ cannot be finitely generated either.

Chapter 5

Exceptional Cases

{exceptional}

In the previous chapter we were able to show that U_+ is not finitely generated for a large family of Coxeter groups W with labels $a \leq b \leq c$. These results were based on assuming $b \geq 4$ which allowed us to show that $\mathcal{D}$ was infinite and proceed from there. In fact, we didn't even describe all of the chambers in $\mathcal{D}$, just an infinite family. However, the same approach will not work in the remaining cases because of the following lemma.

{lem:infD} **Lemma 11.** *If W is a Coxeter group with labels $a \leq b \leq c$ as before, then $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ as defined in the previous chapter is infinite if and only if $b \geq 4$.*

Proof. We know by Lemma 9 that $\mathcal{D}$ is infinite if $b \geq 4$. Thus it remains to show that $\mathcal{D}$ is finite if $b = 3$. If $b = 3$ then $a = 3$ also, and by definition of a, b, c this means $m(s,t) = m(s,u) = 3$. We will also recall the definition of $\mathcal{D} = \alpha_1 \cap \alpha_n \cap \beta \cap \beta'$ where

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D,C) < d(D,tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D,C) < d(D,uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D,tC) < d(D,tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\} \\
\beta' &= \{D \in \Sigma | d(D,uC) < d(D,usC)\} = \{w \in W | \ell(uw) < \ell(suw)\}
\end{aligned}$$

Let $w \in W$ and suppose $\ell(w) \geq 2$. Then we can write $w = s_1 s_2 w'$ where $\ell(w') = \ell(w) - 2$. If $s_1 = t$ then we have

$$\ell(tw) = \ell(s_2 w') = \ell(w) - 1 < \ell(w)$$

which shows $w \notin \alpha_1$ and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_1 = u$.

Now we assume $s_1 = s$ and so we can also assume $s_2 = t, u$. First let $s_2 = t$ so that $w = stw'$. If $w \notin \alpha_1$ then $w \notin \mathcal{D}$ and so we will suppose $w \in \alpha_1$. Now we can see

$$\ell(stw) = \ell(ststw') = \ell(sstsw') = \ell(tsw') \leq \ell(w') + 2 = \ell(w) < \ell(tw)$$

and thus $w \notin \mathcal{D}$. A similar argument shows that $w \notin \mathcal{D}$ if $s_2 = u$.

We have shown that if $\ell(w) \geq 2$ then $w \notin \mathcal{D}$ and thus $\mathcal{D}$ must be finite as desired. In fact, if $a = b = 3$ then we can check relatively easily that $\mathcal{D} = \{C, sC\}$ which proves the desired result. $\square$

The previous lemma shows that generating results for the remaining cases is not just a matter of being slightly more clever when we look for vertices, but changing the strategy as a whole. In fact, in some cases our proof strategy needs to switch entirely since U_+ will be finitely generated in some cases as we will see. First, we will show which of the remaining cases are not finitely generated.

All of the remaining rank 3 cases have the property that $m(s,u) = m(s,t) = 3$. If x is the vertex of C of type s then x is the only possible vertex of type C with the property that $[U_x : U_x'] \geq 2$. With two edge labels of 3 it is impossible for $U_x \cong {}^2F_4(2)$ and so the only remaining possibilites are $U_x \cong C_2(2), G_2(2)$, and $G_2(3)$. We will enumerate through each of these cases individually.

5.1 Case: $U_x \cong G_2(2)$

We saw in the previous chapter that a vertex contained in $\mathcal{D}$ was a sufficient condition to construct a corresponding map $\tilde{\phi}_v$. However, it is not a necessary condition, and we will see in this section we can relax a few conditions to still construct $\tilde{\phi}_v$ for infinitely many vertices. Our first step is to make some general observations about this case and then prove a statment similar to Lemma 5.

For the remainder of the section we will assume that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) where $S = \{s, t, u\}$ and

$$W = \langle s, t, u | s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^6 = 1\rangle$$

Furthermore, let x be the vertex of C of type s and assume that $U_x \cong G_2(2)$. Recall that this means $[U_x : U_x'] = 4$ and $[U_v : U_v'] = 4$ for all vertices v of type s by Lemma 2 and Lemma 4.

Recalling from the previous chapter, we know that there is a presentation of U_+ generated by U_α for all $\alpha \in \Phi_+$. Again, there are several types of relations we need to consider. There are relations among the U_α and there are relations between U_α and U_β when $\{\alpha, \beta\}$ is a prenilpotent pair. By (A) we know that $[U_\alpha, U_\beta] = \{1\}$ if α and β are nested. We also know that when $\partial\alpha \cap \partial\beta \neq \emptyset$ that $[u, u'] = w$ for some word $w \in U_{(\alpha,\beta)}$ where $u \in U_\alpha$ and $u' \in U_\beta$.

Now recall from Chapter 3 that there is a surjective homormorphism $\phi_x : U_x \to H$ where H is a cyclic group. We can also choosea standard labeling $\alpha_1, \ldots, \alpha_6$ of the positive roots through x in such a way that $\ker \phi_x = U_x'' = \langle U_1, U_5, U_6\rangle$. Similarly to the last chapter, if v is any vertex of type s, our goal is to construct an extension of the form $\tilde{\phi}_v$ in such a way that

$$\tilde{\phi}_v(U_\alpha) = \begin{cases} \phi_v(U_\alpha) & v \in \partial\alpha \\ 1 & \text{otherwise} \end{cases}$$

If we can do this for enough vertices v then we will be able to show that U_+ is not finitely generated in the same way as the previous chapter. Our first step is to prove an analagous result to Lemma 5 in the current context.

{lem:336f2ex}

Lemma 12. *Let v be a vertex of Σ of type s, meaning $|\mathrm{st}(v)| = 12$. Assume $\gamma_1, \ldots, \gamma_6$ is a standard ordering of the positive roots through v such that $U_{\gamma_5} \subset \ker \phi_v$. If γ_2, γ_3, and γ_4 are simple at all other vertices they meet, then $\tilde{\phi}_v$ as defined in Lemma 5 exists.*

Proof. To check $\tilde{\phi}_v$ is well defined is a matter of checking the relations are satisfied by the images under $\tilde{\phi}_v$. Since $\tilde{\phi}_v$ has a cyclic group as its codomain, we can see immediately that the first two types of relations will be satisfied regardless of α and β. Now to check the third type.

Suppose α and β are any two positive roots with $y = \partial\alpha \cap \partial\beta$. Then there is a relation in U_+ of the form $[u, u'] = w$ where $u \in U_\alpha, u' \in U_\beta$, and $w \in U_{(\alpha,\beta)}$. Since $[u_\alpha, u_\beta]$ must be mapped to the identity then we just need to check that w is also mapped to the identity. If $y = v$ then u_α, u_β, w all lie in U_v and $\tilde{\phi}_v(w) = \phi_v(w)$ which must be the identity because ϕ_v is a well defined homomorphism.

Now suppose $y \neq v$. Let $\delta_1, \ldots, \delta_n$ be the positive roots through y, with a standard labeling, and assume that $\alpha = \delta_i$ and $\beta = \delta_j$ with $i < j$. There is at most one positive root whose wall can pass through both v and y, call it δ_k if it exists. If δ_k does not exist, then no positive roots through y pass through v and so $\tilde{\phi}_v(u_{\delta_m}) = 1$ for all m. Thus $\tilde{\phi}_v(w) = 1$ as desired.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{1, 5, 6\}$. Then we know $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all $m \neq k$ and $\tilde{\phi}_v(U_{\delta_k}) = \tilde{\phi}_v(U_{\gamma_r}) = \phi_v(U_{\gamma_r}) = \{1\}$ by the construction of ϕ_v. Thus $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all m and so $\tilde{\phi}_v(w) = \{1\}$ as well.

Now suppose δ_k does exist and $\delta_k = \gamma_r$ for $r \in \{2, 3, 4\}$. Then by assumption, δ_k is simple at y and thus $k = 1, n$. Thus $\tilde{\phi}_v(U_{\delta_m}) = \{1\}$ for all $2 \leq m \leq n-1$. But w is a word in $U_{(\alpha,\beta)} \subset U_{(\delta_2,\delta_{n-1})}$ and thus $\tilde{\phi}_v(w) = 1$ again, which gives the result. □

It is worth noting that the hypotheses of this Lemma are weaker than those of Lemma 5, and so we have a hope of constructing more $\tilde{\phi}_v$ then the theory of the previous chapter would allow us to. However, many of the ideas will still be similar and the proofs in this section will run parallel to those in the previous chapter.

Let x be the vertex of C of type s as in the previous chapter and let $\alpha_1, \ldots, \alpha_6$ be the positive roots through x, labeled as usual. Recall from the previous chapter that

$$\begin{aligned}
\alpha_1 &= \{D \in \Sigma | d(D, C) < d(D, tC)\} = \{w \in W | \ell(w) < \ell(tw)\} \\
\alpha_n &= \{D \in \Sigma | d(D, C) < d(D, uC)\} = \{w \in W | \ell(w) < \ell(uw)\} \\
\beta &= \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}
\end{aligned}$$

Also assume without loss of generality that $\phi_x(U_{\alpha_5}) = \{1\}$. Now let $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$. We can now prove a lemma similar to Lemma 7.

picture of $\mathcal{D}'$

{lem:336f2D}

Lemma 13. *Let x be the vertex of C of type s so that $|\mathrm{st}(x)| = 12$. Let $\alpha_1, \ldots, \alpha_6$ be the positive roots at x with the standard ordering. Also assume that $\phi_x(U_{\gamma_5}) = 1$. Suppose $\gamma = \alpha_i$*

for $i \in \{2, 3, 4\}$. If δ is any positive root with $\partial\gamma \cap \partial\delta \neq \emptyset$ then $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta \subset \gamma \cap \delta$ where

$$\beta = \{D \in \Sigma | d(D, tC) < d(D, tsC)\} = \{w \in W | \ell(tw) < \ell(stw)\}$$

as in the previous chapter.

Proof. Since γ is a positive root at x, and α_1, α_6 are the simple roots at x, we know that $\mathcal{D}' \subset \alpha_1 \cap \alpha_6 \subset \gamma$ and thus it will suffice to show that $\mathcal{D}' \subset \delta$.

Let $y = \partial\gamma \cap \partial\delta$. If $y = x$ then δ is also a root which passes through x and so $\delta = \alpha_j$ for some $j \neq i$. Then as before we get $\alpha_1 \cap \alpha_6 \subset \alpha_j = \delta$ and thus $\mathcal{D}' \subset \delta$ so that $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

Now suppose that $\partial\gamma \cap \partial\delta = y \neq x$. From the local geometry of Σ around x we can see the following facts. For any α_i with $2 \leq i \leq n-1$ we know that $\partial\alpha_i \cap \alpha_1 \cap \alpha_6 = \{x\}$ and $\partial\alpha_i \subset \alpha_1 \cup \alpha_6$. Thus the point y will lie in exactly one of α_1 or α_6.

First suppose that $y \in \alpha_6$ so that $y \notin \alpha_1$. If $\partial\alpha_1 \cap \partial\delta = \emptyset$ then there are exactly 3 possibilities. Either $\alpha_1 \subset \delta$, $\delta \subset \alpha_1$, or $-\delta \subset \alpha_1$. But the last two possibilites would contradict our assumption that $y \notin \alpha_1$ and thus we get $\alpha_1 \subset \delta$ and thus $\mathcal{D}' \subset \alpha_1 \subset \gamma \cap \delta$ as desired.

Alternatively, assume that $\partial\alpha_1 \cap \partial\delta = y'$. Then the points x, y, y' will form a triangle with sides on walls of Σ. Then by the triangle condition, these three vertices must form a chamber, call it E. The points x, y lie on $\partial\gamma = \partial\alpha_i$ and the points x, y' lie on $\partial\alpha_1$. Since y and y' are adjacent this means that either $\gamma = \alpha_2$ or $\gamma = \alpha_6$. The latter is a contradiction of our assumptions and thus $\gamma = \alpha_2$. We know that y and y' are adjacent and $y \in \alpha_6$. Since neither y or y' lies on $\partial\alpha_6$ this means that $y' \in \alpha_6$ as well.

We know that E is a chamber in $\mathrm{st}(x)$ with a side on $\partial\alpha_1$ and $\partial\alpha_2$. let $D = tC$ and D' be the chamber opposite D in $\mathrm{st}(x)$. Then either $E = D$ or $E = D'$. By definition, α_1 is the only wall separating C and tC which means $D = tC \in \alpha_6$. If $E = D'$ then $D' \in \alpha_6$ since x, y, y' all lie in α_6. But this is a contradiction as α_6 cannot contain two opposite chambers in $\mathrm{st}(x)$. Thus $E = D = tC$ and $\delta = \beta$ by definition. Thus $\mathcal{D}' \subset \beta = \delta$ and $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

If we assume insteach that $y \in \alpha_1$ so that $y \notin \alpha_6$ then we have the same two possibilites. If $\partial\alpha_6 \cap \partial\delta = \emptyset$ then by similar arguments we get $\mathcal{D}' \subset \alpha_6 \subset \delta$ and thus $\mathcal{D}' \subset \gamma \cap \delta$ as desired. If $\partial\alpha_6 \cap \partial\delta = y'$ then the vertices x, y, y' form a chamber with y' on α_6. Again, by similar arguments as before, this would imply that $\gamma = \alpha_5$ or α_1, both of which are impossible.

Therefore, regardless of case we have $\mathcal{D}' \subset \gamma \cap \delta$ as desired.

□

We now have a condition for $\tilde{\phi}_v$ to exist which we can check and so it remains to find potential candidates to use at v. We know by Lemma 4 that ϕ_v will exist for all vertices v of type s. We will us a strategy similar to that of the previous chapter which relies on the definition of D' to show $\tilde{\phi}_v$ exists for certain v. To this end we now prove the analogue of Lemma 8.

{lem:336f2Dex}

Lemma 14. *Let x be the vertex of C of type s and suppose that v is any vertex in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ of type s. Then there is a $w \in W$ such that $w^{-1}x = v$ and $\tilde{\phi}_{wx}$ exists.*

Proof. Let $D = \text{Proj}_v(C)$ and define w so that $D = w^{-1}C$. By definition, v is a vertex of D of type s and $w^{-1}x$ is also a vertex of D of type s and thus $w^{-1}x = v$. The claim is that this w will satisfy the desired properties. First we mention that wx is also a vertex of Σ of type s and thus $[U_{wx} : U'_{wx}] \geq 2$ and ϕ_{wx} exists by Corollary 3.

Again, the definition of projections means that D is the closest vertex to C which has a vertex of $w^{-1}x$. Since $\mathcal{D}$ is convex, and $w^{-1}x$ and C both lie in $\mathcal{D}$, we also know that $D = \text{Proj}_{w^{-1}x}(C)$ lies in $\mathcal{D}$ as well. By a similar argument we know that $\text{Proj}_x(D)$ must lie in $\mathcal{D} \subset \alpha_1 \cap \alpha_n$ and thus $\text{Proj}_x(D) = C$. Now define $E = wC$ and note that the action of W respects projections and thus we have

$$E = wC = \text{Proj}_{wx} wD = \text{Proj}_{wx} C \qquad C = wD = \text{Proj}_{w(w^{-1}x)} wC = \text{Proj}_x E$$

In particular, a root through wx is positive if and only if it contains E.

Our goal is to apply Lemma 12 at the vertex wx. Let $\gamma_1, \ldots, \gamma_6$ be a standard labeling of the positive roots through wx such that $U_{\gamma_5} \subset \ker \phi_{wx}$. We need to check that if $y \neq wx$ is on $\partial\gamma_i$ for $i \in \{2, 3, 4\}$ then γ_i is simple at y. First we will show that w^{-1} sends positive roots at wx to positive roots at x. Suppose γ is any positive root at wx. Then we know that $E \in \gamma$ and thus $C = w^{-1}E \in w^{-1}\gamma$ so that $w^{-1}\gamma$ is positive, and thus w^{-1} sends positive roots at wx to positive roots at x.

If we apply Lemma 4 then we know that $w^{-1}\gamma_1 = \alpha_1, \ldots, w^{-1}\gamma = \alpha_6$ is a standard labeling of the of the positive roots at x. If we apply this isomorphism given by Corollary 3 then we know that $U_{w^{-1}\gamma_5} = U_{\alpha_5} \subset \ker \phi_x$ since $U_{\gamma_5} \subset \ker \phi_{wx}$.

Now we fix $i \in \{2, 3, 4\}$ and we need to check γ_i is simple at all vertices $y \neq v$ on $\partial\gamma_i$. If we apply w^{-1} we get that $w^{-1}y \neq x$ is a vertex on $\partial\alpha_i$. Thus by Lemma 6 we know that α_i is simple at $w^{-1}y$. Now suppose that δ is any positive root at $w^{-1}y$. Recall that $D \in \mathcal{D}'$ and we can apply Lemma 13 to see that $D \in \mathcal{D}' \subset \delta$. If we apply w we get $C = wD \in w\delta$ where $w\delta$ is a positive root through $w(w^{-1}y) = y$. Thus w sends positive roots at $w^{-1}y$ to positive roots at y. We can apply Lemma 4 again to say that w sends the simple roots at $w^{-1}y$ to the simple roots at y. Since α_i is simple at $w^{-1}y$ we know that $w\alpha_i = \gamma_i$ is simple at y as desired. We now for all positive roots γ_i for $i \in \{2, 3, 4\}$ at wx that γ_i is simple at all other vertices, and thus we can apply Lemma 12 to say that $\tilde{\phi}_{wx}$ exists as desired.

□

As in the previous chapter, we now have a potentially large class of vertices for which $\tilde{\phi}_v$ exists, but we still must show there are infinitely many, and that they do not lie on finitely many walls. In fact, we can even use the same vertices as in the previous chapter. Let $w_k = (tus)^k$ for all $k \geq 0$ and let $v_k = w_k x$. Recall in our current setup that $m(t, u) = 6$ and $m(s, u) = m(s, t) = 3$.

{lem:336f2inf}

Lemma 15. *Let $w_k = (tus)^k$ for all $k \geq 0$ and let x be the vertex of C of type s. Then the vertices $(w_k)^{-1}x$ are all distinct, and they all lie in $\mathcal{D}' = \alpha_1 \cap \alpha_6 \cap \beta$ as defined previously.*

Proof. Many of the proofs will be identical to those in the proof of Lemma 9 and so work will not be repeated when unnecessary. Also note that $w_k^{-1} = (sut)^k$ for all k. We can check that $\ell((w_k)^{-1}) = 3k$ and $\ell(t(w_k)^{-1}) = 3k+1$ by identical arguments as before. We can also check that

$$\begin{aligned} u(w_k)^{-1} &= u(sutsut\cdots) \\ &= (usu)(tsutsu\cdots) \\ &= (sus)(tsutsu\cdots) \\ &= (su)(sts)(utsuts\cdots) \\ &= (su)(tst)(utsuts\cdots) \\ &= (su)(ts)(tut)(sutsut\cdots) \end{aligned}$$

We have exhausted all possible M-Operations in $u(w_k)^{-1}$ and none of them led to a reduction in length so we can conclude that $\ell(u(w_k)^{-1}) = 3k+1$ also so that $(w_k)^{-1} \in \alpha_1 \cap \alpha_6$.

Now we do the same analysis for $st(w_k)^{-1}$ to see

$$\begin{aligned} st(w_k)^{-1} &= st(sutsut\cdots) = (sts)(utsuts\cdots) \\ &= (tst)(utsuts\cdots) = (ts)(tut)(sutsut) \end{aligned}$$

and since no reductions can be performed we also get $\ell(st(w_k)^{-1}) = 3k+2$ so that $(w_k)^{-1} \in \beta$ as well. Thus each $(w_k)^{-1}x$ lies in $\mathcal{D}'$ as desired. We also know that $(w_m)^{-1}x \neq (w_n)^{-1}x$ if $m > n$ by the same argument as in Lemma 9. □

The last major step is to show that the $w_k x$ cannot somehow lie on only finitely many walls. The analysis here will be slightly more complicated, but ultimately similar to that done in the previous chapter.

{lem:336f2walls}

Lemma 16. *Let x be the vertex of C of type s and let $w_k = (tus)^k$ for all $k \geq 0$. Any wall of Σ can contain only finitely many $w_k x$.*

Proof. By arguments identical to those in Lemma 10, $w_m x$ and $w_n x$ will lie on the same wall if and only if x and $w_{n-m}x$ lie on the same wall. If we assume $m > n$ then it will suffice to show that a wally containing x can contain $(w_k)^{-1}x$ for only finitely many $k > 0$. Using the argument of Lemma 10 again we know that x and $(w_k)^{-1}x$ will lie on the same way if and only if $w_k t w_k^{-1}$ or $w_k u w_k^{-1}$ lies in $\mathrm{stab}_W(x) = \langle u, t\rangle$. If we recall that $m(s,t) = m(s,u) = 3$ and $m(t,u) = 6$ we can check these two conjugates we see

$$\begin{aligned} w_k t w_k^{-1} &= (\cdots tustus)t(sutsut\cdots) \\ &= (\cdots tustu)(sts)(utsut\cdots) \\ &= (\cdots tustu)(tst)(utsut\cdots) \\ &= (\cdots tus)(tut)(s)(tut)(sut\cdots) \end{aligned}$$

and then we see also

$$\begin{aligned}
w_k u w_k^{-1} &= (\cdots stustus)u(sutsuts\cdots) \\
&= (\cdots stust)(ususu)(tsuts\cdots) \\
&= (\cdots stust)(s)(tsuts\cdots) \\
&= (\cdots stu)(ststs)(uts\cdots) \\
&= (\cdots stu)(t)(uts\cdots) \\
&= (\cdots stustu)(t)(utsuts\cdots) \\
&= (\cdots stus)(tutut)(suts\cdots)
\end{aligned}$$

In the first case, no reduction is possible and thus there will always be an s in any reduced word for $w_k t w_k^{-1}$ and thus $w_k t w_k^{-1} \notin \langle u, t\rangle$. In the second case, We are able to do two reductions in length but then are unable to continue. If we check the relations applied, we will se that the relations cannot continue if $k \geq 3$. For completion we will also note that $w_1 u w_1^{-1} = tst \notin \langle u, t\rangle$ but $w_2 u w_2^{-1} = tutut \in \langle u, t\rangle$. Regardless, we know that $w_m x$ and $w_n x$ cannot lie on the same wall if $|m - n| \geq 3$ so that any wall can contain only finitely many $w_k x$ as edesired.

□

Now we are ready to prove the main result of the section, which is nearly identical to the proof of Theorem 7.

{thm:336f2notfg}

Theorem 8. *Let $(G, (U_\alpha)_{\alpha \in Phi}, T)$ be an RGD system of type (W, S) with assumptions as in* (A). *Suppose that $a = m(s,t) = b = m(s,t) = 3$ and $U_x \cong G_2(2)$ where x is the vertex of C of type S. Then $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ is not finitely generated.*

Proof. Suppose that U_+ is finitely generated. Then there is some finite set of roots $\beta_1, \ldots, \beta_m$ such that $U_+ = \langle U_{\beta_i} | 1 \leq i \leq m\rangle$. Let $w_k = (tus)^k)$ for all $k \geq 0$. Now only finitely many of the vertices $w_k x$ lie on the same wall and thus we can choose k so that $v = w_k x$ does not lie on $\partial\beta_i$ for any i. By Lemma 15 we know that $\tilde{\phi}_v$ exists, and by definition it is a surjective map from $U_+ \to H$. However, we can also see by definition that $\tilde{\phi}_v(U_{\beta_i}) = 1$ for all i, since none of these walls meet v. But this means $\tilde{\phi}_v$ sends all of the generators of U_+ to the identity and thus it must be the trivial map which is a contradiction. Thus U_+ is not finitely generated as desired. □

5.2 Finite Generation in the Exceptional Cases

Now there are two cases left to consider, and no ammount of modification to our previous strategies will work since we will see that these remaining cases are finitely generated.

For any positive root γ, we say that a chamber D borders γ if a panel of D lies on $\partial\gamma$. This allows us to define

$$d(\gamma, C) = \min_{D \text{ borders } \gamma} \{d(D, C)\}$$

It is worth noting that if $d(\gamma, C) = k$ then there is a chamber D which borders γ and $d(\gamma, C) = d(D, C)$. Furthermore, the chamber D must lie in γ since, otherwise, the chamber adjacent to D across $\partial\gamma$ would be closer to C.

We can now define $U_n = \langle U_\gamma | \gamma \in \Phi^+, d(\gamma, C) \leq n \rangle$ which is a subgroup of U_+ for all n. We also have a few facts which are immediate from the definition of U_n. We can see that $U_1 \subset U_2 \subset U_3 \subset \cdots$ and $U_+ = \cup_n U_n$ as any positive root will be some finite distance from C.

Slightly less obvious is the fact that U_n is finitely generated for all n. If $d(\gamma, C) \leq n$ then there must be a chamber D which borders γ with $d(D, C) \leq n$. There are only finitely many such chambers, and each of these chambers borders at most 3 roots, so U_n is finitely generated.

The idea of the remaining proofs will be to use the following lemma

{lem:fgcond}

Lemma 17. *For any positive root γ we define $d(\gamma, C) = \min\{d(D, C) | D$ has a panel on $\partial\gamma\}$. Let $U_n = \langle U_\gamma | d(\gamma, C) \leq n \rangle$ for all $n \geq 0$ where $d(\gamma, C)$. If there is some N such that $U_n \subset U_{n-1}$ for $n > N$ then U_+ is finitely generated.*

Proof. If $U_n = U_{n-1}$ for all $n > N$ then inductively we know that $U_n = U_N$ for all $n > N$. Thus

$$U_+ = \cup_{n=N}^{\infty} U_n = \cup_{n=N}^{\infty} U_N = U_N$$

which is fintely generated as desired. □

By the results of Chapter 4 and the previous section, we know that the only cases remaining to consider are when W has a Coxeter diagram defined by edge labels 334, 336, or 338. The 338 case is impossible. And we have already covered the 336 case when Σ has a vertex x with $U_x \cong G_2(2)$. Thus we only need to consider when Σ has a vertex x with $U_x \cong C_2(2)$ or $G_2(3)$.

5.2.1 Case: $U_x \cong C_2(2)$

Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system of type (W, S) where $S = \{s, t, u\}$ and $m(s, t) = m(s, u) = 3$ and $m(t, u) = 6$. Let x be the vertex of the fundamental chamber C of type s and assume $U_x \cong C_2(2)$. We will show that U_+ is finitely generated.

{thm:334f2fg}

Theorem 9. *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ of type (W, S) as above. If x is the vertex of C of type s and $U_x \cong C_2(2)$ then $U_n \subset U_{n-1}$ for all $n > 2$ where $U_n = \langle U_\gamma | d(\gamma, C) \leq n \rangle$.*

Proof. Let γ be any positive root with $d(\gamma, C) = n > 2$. Then choose a chamber D_1 which borders γ such that $d(D_1, C) = d(\gamma, C)$. Now there is another chamber D_2 such that D_1 and D_2 are adjacent and $d(D_2, C) = d(D_1, C) - 1$. Then D_1 and D_2 will share exactly one vertex which lies on $\partial\gamma$, call it v. Recall that $\mathrm{st}(v)$ is the set of chambers of Σ for which v is a vertex. Then we have $|\mathrm{st}(v)| = 6$ or 8.

{fig:334deg6}

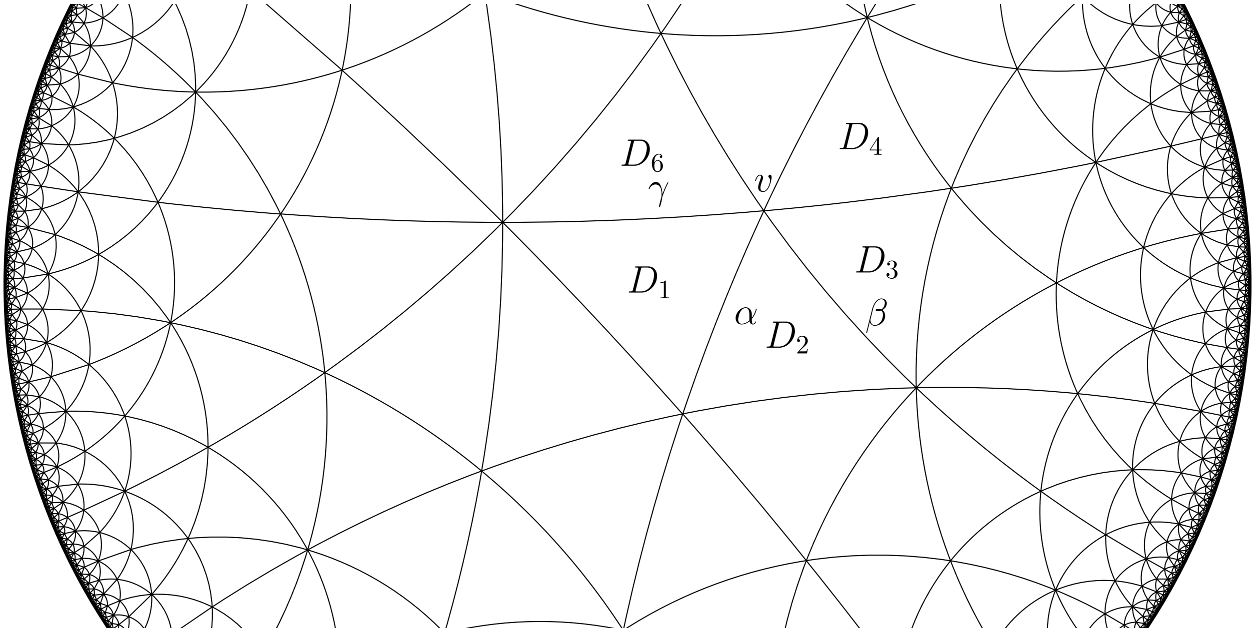

Figure 5.1: Case: $|\mathrm{st}(v)| = 6$

First suppose $|\mathrm{st}(v)| = 6$. In Σ, we can see that $\mathrm{st}(v)$ consists of the 6 chambers "surrounding" v which each have a vertex on v. Since we have already defined D_1 and D_2 we may label the other 4 chambers in $\mathrm{st}(v)$ as $D_3, \dots, D_6$ by going in a circular order around v. Equivalently this means that D_i is ajacent to D_{i+1} for $1 \leq i \leq 5$ and D_6 is also adjacent to D_1. We also know that each positive root will contain exactly 3 of these chambers, and those three chambers will be D_i, D_{i+1}, and D_{i+2} for some i, where addition is done modulo 6.

By construction, D_2 and D_1 are not adjacent along $\partial\gamma$, but a panel of D_1 lies on $\partial\gamma$, and thus D_1 and D_6 must be adjacent along $\partial\gamma$. Since $D_6 \notin \gamma$, this means that γ must contain D_1, D_2, D_3. Let α and β be the other two positive roots through v. We know that $\partial\gamma$ cannot separate D_2 and D_1 or D_2 and D_3 so we can say again without loss of generality that $\partial\alpha$ separates D_2 and D_1 while $\partial\beta$ separates D_2 and D_3.

Now $D_3 \in \gamma$ but $D_4 \notin \gamma$ which means that D_3 has a panel on $\partial\gamma$. By our choice of D_1 we know that $d(D_3, C) \geq d(D_1, C) > d(D_2, C)$. But D_1 and D_3 are the two chambers adjacent to D_2 in $\mathrm{st}(v)$ and thus D_2 must be the closest chamber to C in $\mathrm{st}(v)$. But this means $D_2 = \mathrm{Proj}_v(C)$ and thus the positive roots at v which border D_2 must be the simple roots at v. These roots are α and β by construction so we know that α and β are simple at v. Since $|\mathrm{st}(v)| = 6$ we know that U_v cannot be an exceptional rank 2 RGD system and thus U_v is generated by the simple root groups through v. Thus $U_x = \langle U_\alpha, U_\beta \rangle$. But α, β border D_2 and $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $d(\alpha, C), d(\beta, C) \leq n - 1$ so that $U_\alpha, U_\beta \subset U_{n-1}$. This means $U_x \subset U_{n-1}$ as well and thus $U_\gamma \subset U_{n-1}$.

Now suppose $|\mathrm{st}(v)| = 8$. Then we will use the same labeling scheme as before except there will be 8 chambers, and each positive root will contain exactly 4 consecutive chambers from $\mathrm{st}(v)$. The same logic as before will still tell us that γ will contain exactly the chambers D_1, D_2, D_3, D_4. Our first claim is that $D_2 = \mathrm{Proj}_v(C)$.

We know that $\mathrm{Proj}_v(C)$ must lie in any positive root through v and thus it can only be D_1, D_2, D_3, D_4. We also know it is the chamber A in $\mathrm{st}(v)$ which minimizes $d(A, C)$. Since $d(D_1, C) > d(D_2, C)$ we know that D_1 cannot be the projection. By a similar argument as before we know that D_4 borders γ and thus $d(D_4, C) \geq d(D_1, C)$ by our choice of D_1. Thus D_4 cannot be the projection. Finally, if D_3 were the projection then $d(D_4, C) = d(D_3, C) + 1 < d(D_3, C) + 2 = d(D_1, C)$ which is also a contradiction and thus $D_2 = \mathrm{Proj}_v(C)$.

Let α be the positive root separating D_1 and D_2, β the positive root separating D_2 and D_3 and δ the positive root separating D_3 and D_4. Recall that γ is the positive root separating D_8 and D_1 as well as D_4 and D_5. We know that D_2 borders α and β with $d(D_2, C) = d(D_1, C) - 1 = n - 1$ and thus $U_\alpha, U_\beta \subset U_{n-1}$. We also know that D_2 lies in all positive roots through v by convexity so $D_2 \in \alpha, \beta, \gamma, \delta$. Since D_2 is bordered by α and β we also know that α and β are the simple roots at v.

{fig:334deg8}

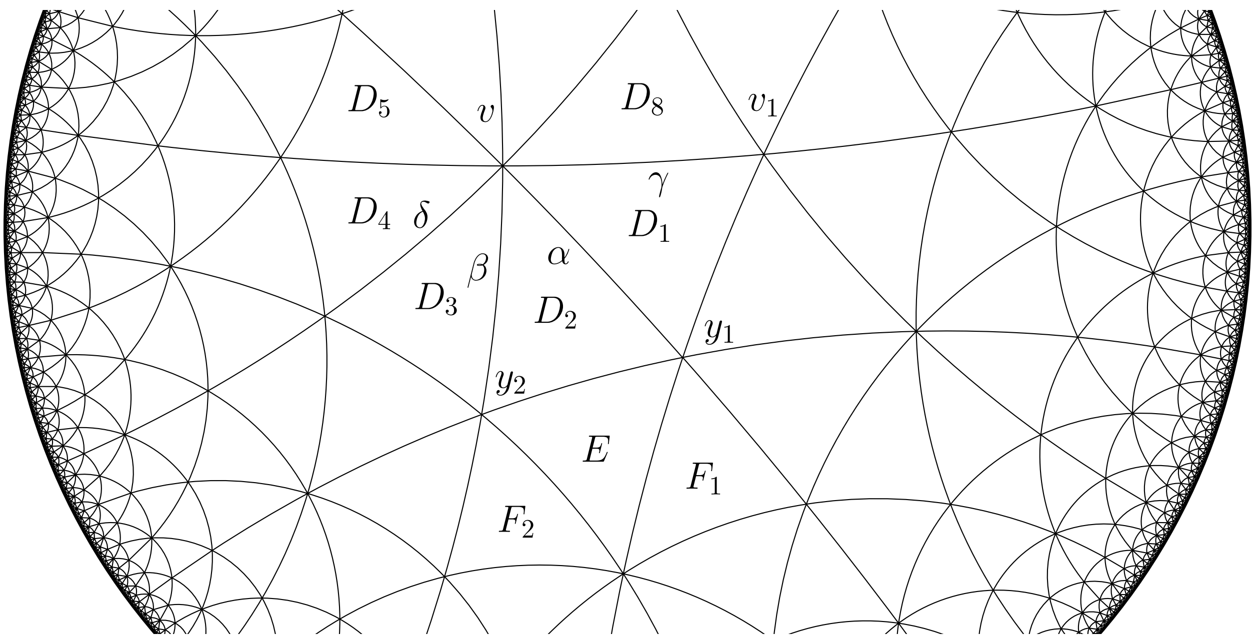

Figure 5.2: Case: $|\mathrm{st}(v) = 8|$

Let E be the third chamber adjacent to D_2. Every chamber must have an adjacent chamber which is closer to C and thus we have $d(E, C) < d(D_2, C)$. We can check that $d(E, C) = d(D_1, C) - 2 \geq 1$ by our choice of γ and thus E is not the fundamental chamber C. We know that D_1 and D_2 share two vertices, and D_2 and E share two vertices, so necessarily we have that D_1, D_2, and E must share at least one, and thus exactly one vertex, call it y_1. By a similar argument, the chambers D_3, D_2, and E will also share a vertex y_2. Let F_1 be the other chamber adjacent to E that has y_1 as a vertex, and let F_2 be the other chamber adjacent to E that has y_2 as a vertex. Note that $|\mathrm{st}(y_1)| = |\mathrm{st}(y_2)| = 6$ since v is the other vertex of D_2. The appropriate labeling can be seen in Figure 5.2.1, and the given diagram is unique up to a mirror image flip, which does not affect any of the following arguments. The labeling of these chambers could have simply been defined by the diagram, but the previous explanation seeks to convince the reader that no choices have been made and this diagram is unique.

Since $d(E,C) < d(D_2,C) < d(D_1,C)$ we know that there is some minimal gallery from D_1 to C which passes through E. If we fix such a minimal gallery we can see that it must pass through either F_1 or F_2. First suppose that it passes through F_1. Then $d(F_1,C) = d(D_1,C)-3$ and so F_1 and D_1 are distance 3 from one another. Since they are both in $\mathrm{st}(y_1)$, this means that D_1 and F_1 are opposite in $\mathrm{st}(y_1)$. Then there is another minimal gallery from D_1 to F_1 which does not pass through D_2 and can also be extended to a minimal gallery from D_1 to C. Let G_1 be the chamber adjacent to D_1 in this new minimal gallery. Then D_1 and G_1 have exactly two vertices in common, on of which is y_1, and the other cannot be v as this would imply $G_1 = D_2$ which contradicts our assumption. Let v_1 be the common vertex which is not y_1. We assumed that v was the unique vertex shared by D_1 and D_2 which lies on $\partial\gamma$. Since y_1 is also shared by D_1 and D_2 this means that y_1 does not lie on $\partial\gamma$. We assumed that D_1 has a panel on $\partial\gamma$ and thus it has two vertices on $\partial\gamma$ which means v_1 must lie on $\partial\gamma$.

Now we have the following situation. We still know that D_1 borders γ with $d(\gamma,C) = d(D_1,C)$ and G_1 is an adjacent chamber such that $d(G_1,C) < d(D_1,C)$. We know that v_1 is a common vertex which lies on $\partial\gamma$ and thus it is the only common vertex which lies on $\partial\gamma$. Finally, v is the uniqe vertex of D_1 with 8 chambers in its star. Thus $|st(v_1)| = 6$. Now we may apply the $|\mathrm{st}(v)| = 6$ case with G_1 as our new choice of D_2 and v_1 the new v. This shows that $U_\gamma \subset U_{n-1}$ as desired.

Now suppose the fixed minimal gallery from before passes through F_2. The arguments made here will be very similar to those made in the previous paragraphs, as there is an obvious symmetry in the Coxeter complex, but we will explain the arguments again, if a little more briefly. There is also a minimal gallery from D_3 to C which passes through F_2 as well. But then $d(F_2,C) = d(D_3,C) - 3$ which means F_2 and D_3 are opposite in $\mathrm{st}(y_2)$. Then there is another minimal gallery in $\mathrm{st}(y_2)$ from $D_3 \to F_2$ which does not pass through D_2. Let G_2 be the chamber adjacent to D_3 in this new minimal gallery. Then G_2 and D_3 will have two vertices in common and one of them will be y_2. Let v_2 be the other vertex in common. Then v_2 must lie on $\partial\delta$ since y_2 is the only vertex of D_3 which does not lie on $\partial\delta$. We also know that $|\mathrm{st}(v_2)| = 6$ as v is the only vertex of D_3 with $|\mathrm{st}(v)| = 12$.

Since D_3 borders δ we know that $d(\delta,C) \leq n$. If $d(\delta,C) < n$ then $U_\delta \subset U_{n-1}$. If $d(\delta,C) = n$ then we have the following situation: D_3 is a chamber which borders δ and $d(\delta,C) = d(D_3,C)$. Furthermore, G_2 is a chamber adjacent to D_3 with $d(G_2,C) = d(D_3,C) - 1$. The unique, common vertex which lies on $\partial\delta$ is v_2 and it has $|\mathrm{st}(v_2)| = 6$. Thus we can apply that $|st(v) = 6|$ case to see that $U_\delta \subset U_{n-1}$. But $U_v = \langle U_\alpha, U_\beta, U_\delta\rangle$ and thus $U_\gamma \subset U_{n-1}$ as desired.

Now we have shown that $U_\gamma \subset U_{n-1}$ in any case, and since the choice of γ such that $d(\gamma,C) = n > 2$ was arbitrary, we know that $U_n \subset U_{n-1}$ for $n > 2$ as desired. □

{cor:334f2fg}

Corollary 5. *Suppose $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W,S) with $S = \{s,t,u\}$. If $m(s,t) = m(s,u) = 3$ and $U_x \cong C_2(2)$ for the vertex x of C of type s then U_+ is finitely generated.*

5.2.2 Case: 336 over $\mathbb{F}_3$

Now we consider the last exceptional case. In this section we assume $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S) with $S = \{s, t, u\}$. Assume that $m(s,t) = m(s,u) = 3$ and $U_x \cong G_2(3)$ where x is the vertex of the fundamental chamber C of type s. We will show that U_+ is finitely generated.

Bibliography

[1] Peter Abramenko and Kenneth S Brown. *Buildings: theory and applications*, volume 248. Springer Science & Business Media, 2008.