Finite Generation of RGD systems with Exceptional Links

Mark Schrecengost

Department of Mathematics
University of Virginia

April 23, 2020

Coxeter Systems

Definition

A Coxeter System is a pair (W, S), consisting of a group W and a set $S \subset W$ such that W is generated by S, and W admits a presentation of the form

$$W = \langle s | s \in S,\ s^2 = 1,\ (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

where $m(s, t) \in \mathbb{N} \cup \{\infty\}$.

Coxeter Systems

Definition

A Coxeter System is a pair (W, S), consisting of a group W and a set $S \subset W$ such that W is generated by S, and W admits a presentation of the form

$$W = \langle s | s \in S,\ s^2 = 1,\ (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

where $m(s, t) \in \mathbb{N} \cup \{\infty\}$.

Common examples include:

1. S_n
2. D_{2n}
3. Reflection groups

Some basic facts about Coxeter Systems:

Some basic facts about Coxeter Systems:

- For every $J \subset S$ we can form a subgroup $W_J = \langle s \in J \rangle$

Coxeter Systems

Some basic facts about Coxeter Systems:

- For every $J \subset S$ we can form a subgroup $W_J = \langle s \in J \rangle$
- The subgroups W_J are all distinct, and the set J can be recovered as $J = W_J \cap S$

Some basic facts about Coxeter Systems:

- For every $J \subset S$ we can form a subgroup $W_J = \langle s \in J \rangle$
- The subgroups W_J are all distinct, and the set J can be recovered as $J = W_J \cap S$
- We will call W_J standard subgroups, and cosets of the form wW_J standard cosets

Coxeter Systems

Some basic facts about Coxeter Systems:

- For every $J \subset S$ we can form a subgroup $W_J = \langle s \in J \rangle$
- The subgroups W_J are all distinct, and the set J can be recovered as $J = W_J \cap S$
- We will call W_J standard subgroups, and cosets of the form wW_J standard cosets
- The length function ℓ on W has nice properties, and there is an algorithm to write any element of w as a minimal length word.

Definition (Coxeter Complex)

For each Coxeter system (W, S), there is a simplicial complex $\Sigma = \Sigma(W, S)$, where the simplices of Σ correspond to standard cosets of W, and the face relation is given by reverse inclusion.

The Coxeter Complex

Definition (Coxeter Complex)

For each Coxeter system (W, S), there is a simplicial complex $\Sigma = \Sigma(W, S)$, where the simplices of Σ correspond to standard cosets of W, and the face relation is given by reverse inclusion.

- The maximal simplices, called chambers, of Σ are all the same dimension, and correspond to the elements of W.

The Coxeter Complex

Definition (Coxeter Complex)

For each Coxeter system (W, S), there is a simplicial complex $\Sigma = \Sigma(W, S)$, where the simplices of Σ correspond to standard cosets of W, and the face relation is given by reverse inclusion.

- The maximal simplices, called <u>chambers</u>, of Σ are all the same dimension, and correspond to the elements of W.
- Each simplex has a type given by $\tau(wW_J) = S \setminus J$, and a cotype J.

Definition (Coxeter Complex)

For each Coxeter system (W, S), there is a simplicial complex $\Sigma = \Sigma(W, S)$, where the simplices of Σ correspond to standard cosets of W, and the face relation is given by reverse inclusion.

- The maximal simplices, called <u>chambers</u>, of Σ are all the same dimension, and correspond to the elements of W.
- Each simplex has a type given by $\tau(wW_J) = S \setminus J$, and a cotype J.
- A <u>panel</u> is a co-dimension 1 simplex, which will have cotype s for some $s \in S$. We say that two chambers are s-adjacent if they share a common s panel. Two chambers are adjacent if they are s-adjacent for some $s \in S$.

Some useful facts and definitions about the Coxeter Complex

Some useful facts and definitions about the Coxeter Complex

1. W acts on Σ by left multiplication, and acts simply transitively on Chambers

The Coxeter Complex

Some useful facts and definitions about the Coxeter Complex

1. W acts on Σ by left multiplication, and acts simply transitively on Chambers
2. We call $C = \{1\} = W_\emptyset$ the fundamental chamber

The Coxeter Complex

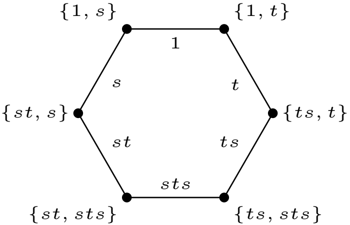
Some useful facts and definitions about the Coxeter Complex

1. W acts on Σ by left multiplication, and acts simply transitively on Chambers
2. We call $C = \{1\} = W_\emptyset$ the fundamental chamber
3. A gallery is a sequence of chambers $D_0, \ldots, D_n$ such that D_i and D_{i+1} are adjacent for all i.

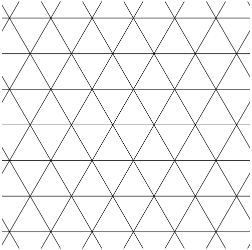
Some useful facts and definitions about the Coxeter Complex

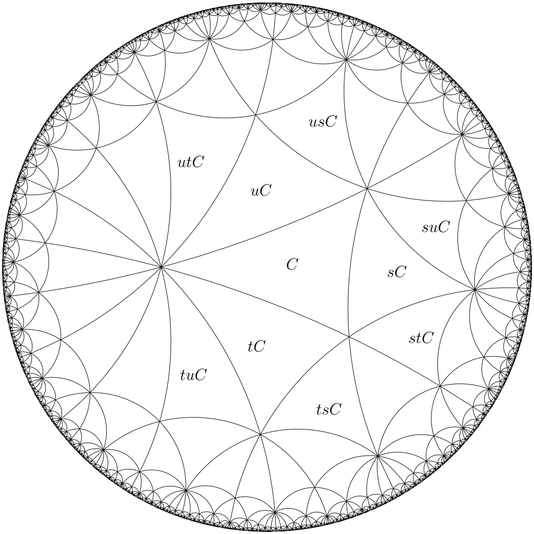
1. W acts on Σ by left multiplication, and acts simply transitively on Chambers
2. We call $C = \{1\} = W_\emptyset$ the fundamental chamber
3. A gallery is a sequence of chambers $D_0, \ldots, D_n$ such that D_i and D_{i+1} are adjacent for all i.
4. We can define a metric d on the chambers $\mathcal{C}(\Sigma)$ where $d(D, E)$ is the length of the shortest gallery from D to E.

Examples



$\{1, s\}$
$\{1, t\}$
1
s
t
$\{st, s\}$
$\{ts, t\}$
st
ts
sts
$\{st, sts\}$
$\{ts, sts\}$





usC
utC
uC
suC
C
sC
stC
tC
tuC
tsC

Definition (Links and Stars)

If A is a simplex of Σ then the star of A, denoted $\mathrm{st}(A)$, is the set of all chambers containing A. The link of A, denoted $\mathrm{lk}(A)$, is all the faces of the chamber of $\mathrm{st}(A)$ which are disjoint from A.

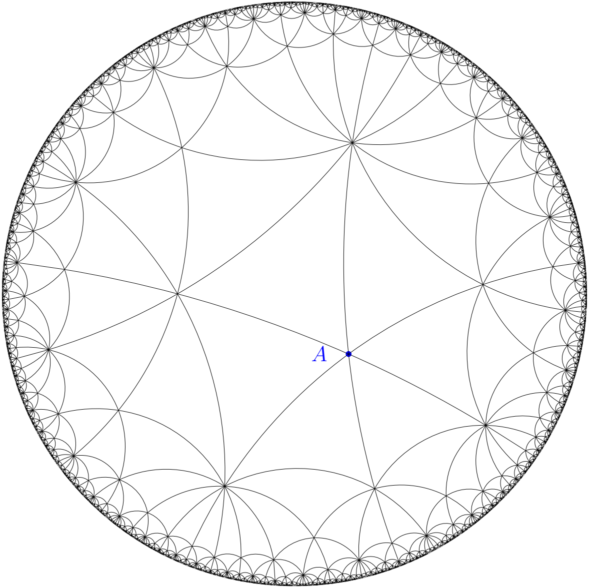
Links and Stars

Definition (Links and Stars)

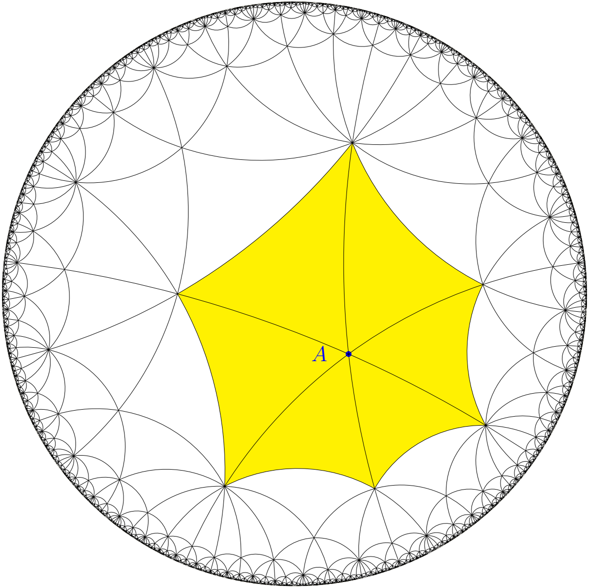
If A is a simplex of Σ then the star of A, denoted $\mathrm{st}(A)$, is the set of all chambers containing A. The link of A, denoted $\mathrm{lk}(A)$, is all the faces of the chamber of $\mathrm{st}(A)$ which are disjoint from A.

If A is a simplex of cotype J then $\mathrm{lk}(A)$ is canonically isomorphic to $\Sigma(W_J, J)$.

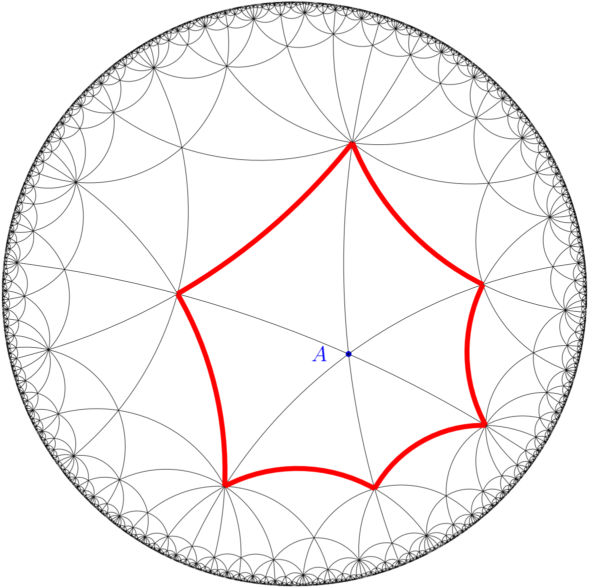
Links and Stars



Links and Stars



Links and Stars



Definition (Projections)

If A is a simplex of Σ, and D is a chamber of Σ then there is a unique chamber E in $\mathrm{st}(A)$ such that $d(D,E') = d(D,E) + d(E,E')$ for all $E' \in \mathrm{st}(A)$. The chamber E is called the projection of D onto A and is denoted $\mathrm{proj}_A(D)$.

Definition (Projections)

If A is a simplex of Σ, and D is a chamber of Σ then there is a unique chamber E in $\mathrm{st}(A)$ such that $d(D, E') = d(D, E) + d(E, E')$ for all $E' \in \mathrm{st}(A)$. The chamber E is called the projection of D onto A and is denoted $\mathrm{proj}_A(D)$.

We want to think about Coxeter groups as Reflection groups, and roots give us the half spaces that any reflection should define.

Definition (Projections)

If A is a simplex of Σ, and D is a chamber of Σ then there is a unique chamber E in $\mathrm{st}(A)$ such that $d(D, E') = d(D, E) + d(E, E')$ for all $E' \in \mathrm{st}(A)$. The chamber E is called the projection of D onto A and is denoted $\mathrm{proj}_A(D)$.

We want to think about Coxeter groups as Reflection groups, and roots give us the half spaces that any reflection should define.

Definition (Roots)

If D and D' are adjacent chambers of Σ, then the set of chambers $\alpha = \{E \in \Sigma | d(D, E) < d(D', E)\}$ is called a root of Σ. The boundary of a root is called a wall, and is denoted $\partial\alpha$. We say a root is positive if it contains the fundamental chamber C.

Links, Projections, and Roots

All of these objects interact nicely with each other:

Links, Projections, and Roots

All of these objects interact nicely with each other:

- The action of W also acts on the set of links, projections and roots.

All of these objects interact nicely with each other:

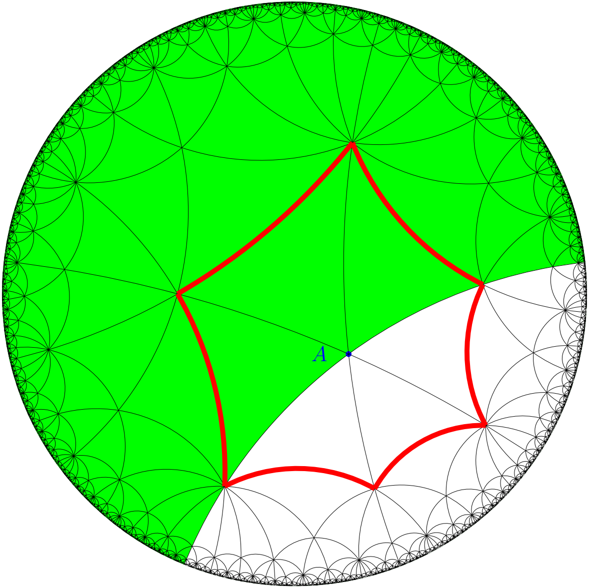
- The action of W also acts on the set of links, projections and roots.
- A root through A is positive if and only if it contains the projection of C onto A.

Links, Projections, and Roots

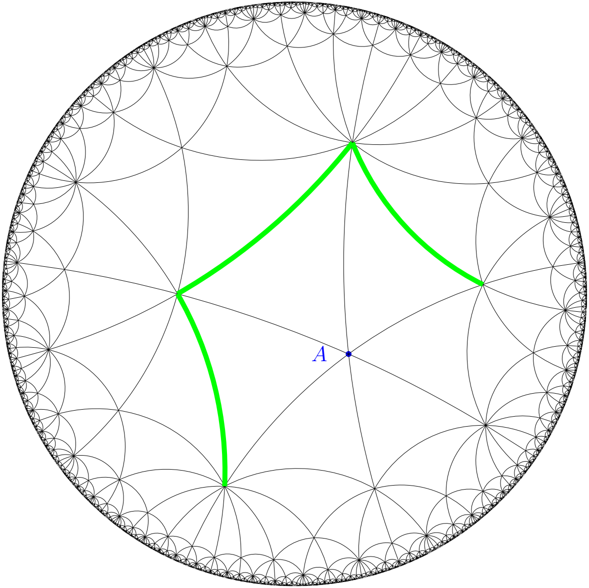
All of these objects interact nicely with each other:

- The action of W also acts on the set of links, projections and roots.
- A root through A is positive if and only if it contains the projection of C onto A.
- The roots of $\mathrm{lk}(A)$ are given by intersections of roots with $A \in \partial\alpha$. We will call these the roots through A.

Links, Projections, and Roots



Links, Projections, and Roots



Definition (Building)

A building Δ is a simplical complex which is a union of a family of sub-complexes $\mathcal{A}$, called apartments, such that

(B0) Each $\Sigma \in \mathcal{A}$ is a Coxeter Complex

(B1) Any two simplices are contained in a common apartment

(B2) For any two apartments, there is an isomorphism between them which fixes the intersection

Definition (Building)

A building Δ is a simplical complex which is a union of a family of sub-complexes $\mathcal{A}$, called apartments, such that

(B0) Each $\Sigma \in \mathcal{A}$ is a Coxeter Complex

(B1) Any two simplices are contained in a common apartment

(B2) For any two apartments, there is an isomorphism between them which fixes the intersection

Buildings were introduced by Tits to study groups of Lie type.

When working in buildings, we have many of the same definitions as in Coxeter complexes, and in most cases it will suffice to work in suitable choices of apartments instead of the entire building.

Groups Acting on Buildings

Groups acting on buildings in certain ways yield nice algebraic properties

Groups acting on buildings in certain ways yield nice algebraic properties

- Strongly transitive actions give BN-Pairs

Groups acting on buildings in certain ways yield nice algebraic properties

- Strongly transitive actions give BN-Pairs
- A stronger condition called Moufang gives RGD systems, which means that G is generated by some family of groups U_α for all the roots of Δ, and another group T.

Groups Acting on Buildings

Groups acting on buildings in certain ways yield nice algebraic properties

- Strongly transitive actions give BN-Pairs
- A stronger condition called Moufang gives RGD systems, which means that G is generated by some family of groups U_α for all the roots of Δ, and another group T.
- Most common examples are groups of Lie type and Kac-Moody groups over fields. Specific examples include $\mathrm{SL}_n(k)$ and $\mathrm{SL}_n(k[t, t^{-1}])$.

Groups acting on buildings in certain ways yield nice algebraic properties

- Strongly transitive actions give BN-Pairs
- A stronger condition called Moufang gives RGD systems, which means that G is generated by some family of groups U_α for all the roots of Δ, and another group T.
- Most common examples are groups of Lie type and Kac-Moody groups over fields. Specific examples include $\mathrm{SL}_n(k)$ and $\mathrm{SL}_n(k[t,t^{-1}])$.

The axioms of an RGD system are quite strong, and they imply certain commutator relations $[U_\alpha, U_\beta]$ for pairs of roots.

Presentation of U_+

If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S), then we have a root group U_α for every root of the Coxeter complex $\Sigma(W, S)$.

Presentation of U_+

If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S), then we have a root group U_α for every root of the Coxeter complex $\Sigma(W, S)$.

The subgroup $U_+ = \langle U_\alpha | \alpha \in \Phi_+\rangle$ has a nice presentation where the only relations are those in the root groups U_α, and commutator relations between pre-nilpotent pairs of roots. As long as W is infinite, this presentation is in terms of infinitely many generators, but we still have the following question

If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system of type (W, S), then we have a root group U_α for every root of the Coxeter complex $\Sigma(W, S)$.

The subgroup $U_+ = \langle U_\alpha | \alpha \in \Phi_+ \rangle$ has a nice presentation where the only relations are those in the root groups U_α, and commutator relations between pre-nilpotent pairs of roots. As long as W is infinite, this presentation is in terms of infinitely many generators, but we still have the following question

Question

When is U_+ finitely generated?

Finite Generation of U_+

Theorem (Abramenko, Van Maldeghem)

If Δ satisfies (co) then U_+ is generated $\{U_{\alpha_s}\}$ where α_s is the root separating C and sC.

Finite Generation of U_+

Theorem (Abramenko, Van Maldeghem)

If Δ satisfies (co) then U_+ is generated $\{U_{\alpha_s}\}$ where α_s is the root separating C and sC.

As a Corollary, U_+ is finitely generated if all root groups are finitely generated, in particular, if they are finite.

Finite Generation of U_+

Theorem (Abramenko, Van Maldeghem)

If Δ satisfies (co) then U_+ is generated $\{U_{\alpha_s}\}$ where α_s is the root separating C and sC.

As a Corollary, U_+ is finitely generated if all root groups are finitely generated, in particular, if they are finite.

Definition (Condition (co))

A building satisfies (co) if the set of chambers opposite a given chamber is gallery connected.

Condition (co)

A building will satisfy condition (co) if all of its co-dimension 2 links satisfy (co).

Condition (co)

A building will satisfy condition (co) if all of its co-dimension 2 links satisfy (co).

Theorem

Every rank 2 Moufang building satisfies (co) except the following finite (twisted) Chevalley groups

$$C_2(2) \quad G_2(2) \quad G_2(3) \quad {}^2F_4(2)$$

Condition (co)

A building will satisfy condition (co) if all of its co-dimension 2 links satisfy (co).

Theorem

Every rank 2 Moufang building satisfies (co) except the following finite (twisted) Chevalley groups

$$C_2(2) \quad G_2(2) \quad G_2(3) \quad {}^2F_4(2)$$

We want to look at buildings and RGD systems where one of the co-dimension 2 links are one of the 4 exceptional cases.

Assumptions

Assume that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ *is an RGD system of type* (W, S) *with associated building* Δ*. Additionally assume*

$$\left.\begin{aligned} &S = \{s,t,u\},\ a = m(s,t), b = m(s,u), c = m(t,u) \\ &3 \le a,b,c < \infty \\ &U_\alpha \text{ is finitely generated for all } \alpha\in\Phi \\ &[U_\alpha, U_\beta] = 1 \text{ when } \alpha,\beta \text{ are nested} \end{aligned}\right. \qquad \text{(A)}$$

and also assume that Δ *has a vertex of type* s *with an exceptional link.*

Heuristic for the Proof

1. Each vertex v has a set of positive roots $\alpha_1, \ldots, \alpha_n$ which pass through v, and subgroups $U_v = \langle U_i \rangle$ and $U'_v = \langle U_1, U_n \rangle$.

Heuristic for the Proof

1. Each vertex v has a set of positive roots $\alpha_1, \ldots, \alpha_n$ which pass through v, and subgroups $U_v = \langle U_i \rangle$ and $U_v' = \langle U_1, U_n \rangle$.
2. If v is exceptional then $U_v \neq U_v'$ and there is a surjective homomorphism $\phi_v : U_v \to H$ where H is a non-trivial cyclic group. This homomorphism sends U_v' to the identity.

Heuristic for the Proof

1. Each vertex v has a set of positive roots $\alpha_1, \ldots, \alpha_n$ which pass through v, and subgroups $U_v = \langle U_i \rangle$ and $U_v' = \langle U_1, U_n \rangle$.
2. If v is exceptional then $U_v \neq U_v'$ and there is a surjective homomorphism $\phi_v : U_v \to H$ where H is a non-trivial cyclic group. This homomorphism sends U_v' to the identity.
3. We can sometimes extend this homomorphism to a map $\tilde{\phi}_v : U_+ \to H$ so that $\tilde{\phi}_v(U_\beta) = \{1\}$ for all other roots β.

Heuristic for the Proof

1. Each vertex v has a set of positive roots $\alpha_1, \ldots, \alpha_n$ which pass through v, and subgroups $U_v = \langle U_i \rangle$ and $U'_v = \langle U_1, U_n \rangle$.
2. If v is exceptional then $U_v \neq U'_v$ and there is a surjective homomorphism $\phi_v : U_v \to H$ where H is a non-trivial cyclic group. This homomorphism sends U'_v to the identity.
3. We can sometimes extend this homomorphism to a map $\tilde{\phi}_v : U_+ \to H$ so that $\tilde{\phi}_v(U_\beta) = \{1\}$ for all other roots β.
4. Since $\tilde{\phi}_v$ is still surjective, at least 1 root passing through v must be in any generating set of U_+.

Theorem (S.)

If $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ *is an RGD system satisfying* (A), *with an exceptional link, then* U_+ *is not finitely generated if at least 2 of* a, b, c *is greater than or equal to 4.*

Theorem (S.)

If $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ *is an RGD system satisfying* (A), *where* $a = b = 3$, *and* Δ *has a vertex with link associated to* $G_2(2)$ *then* U_+ *is not finitely generated.*

These results cover all but 3 cases.

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.
- Check that this map is well defined

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.
- Check that this map is well defined

What do the relations in U_+ look like?

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.
- Check that this map is well defined

What do the relations in U_+ look like?

1. Relations in U_α

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.
- Check that this map is well defined

What do the relations in U_+ look like?

1. Relations in U_α
2. Nested root groups commute, so that $[U_\alpha, U_\beta] = 1$

Proof Sketch

- Fix a chamber C and an apartment Σ containing C. There will be a vertex x of C with an exceptional link and a non-trivial homomorphism $\phi_x : U_x \to H$.
- Extend this map to $\tilde{\phi}_x : U_+ \to H$ by sending all other roots to the identity.
- Check that this map is well defined

What do the relations in U_+ look like?

1. Relations in U_α
2. Nested root groups commute, so that $[U_\alpha, U_\beta] = 1$
3. $[U_\alpha, U_\beta] \subset U_{(\alpha,\beta)}$ for non-nested pre-nilpotent pairs

Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

Proof Sketch (cont.)

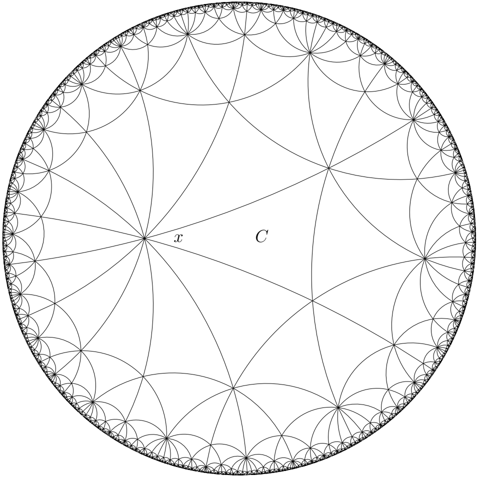
We need to check that the right side of every commutator relation of the third type is sent to the identity.

It is enough to check that the non-simple roots at x are simple everywhere else.

Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

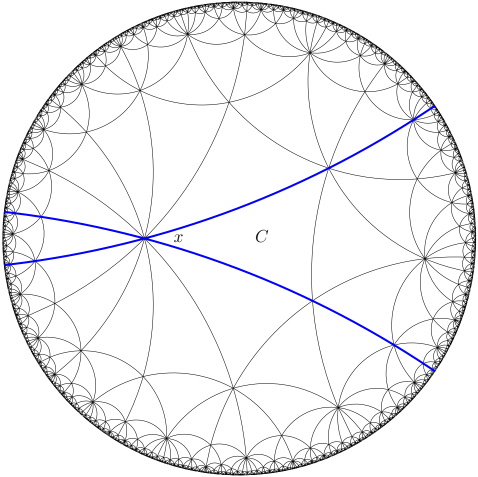
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

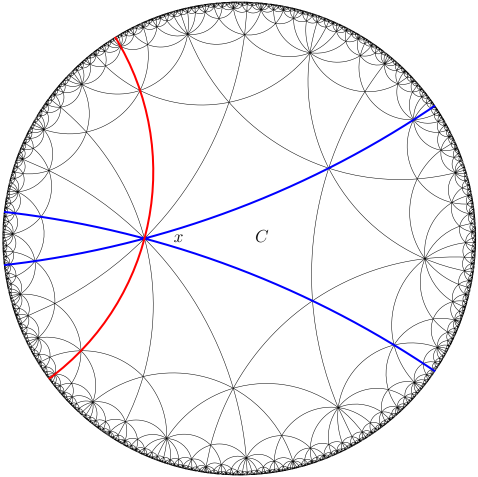
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

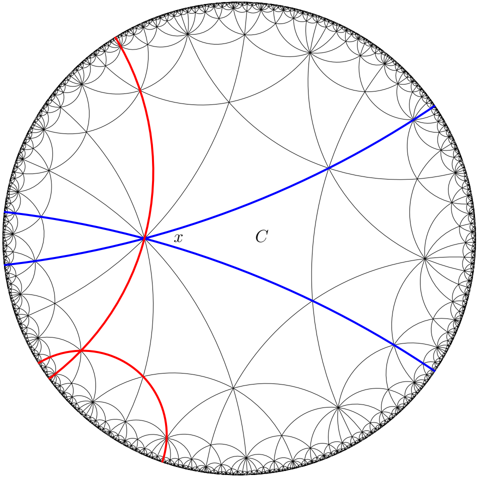
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

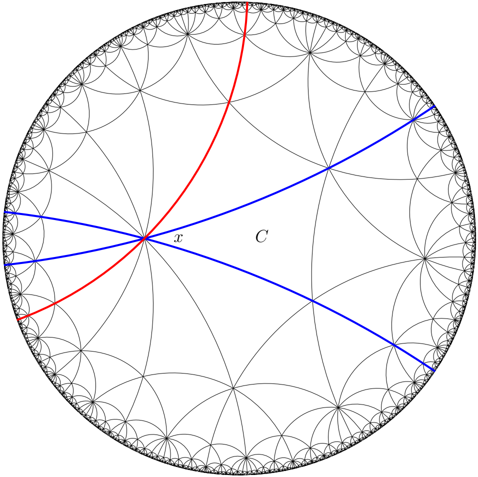
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

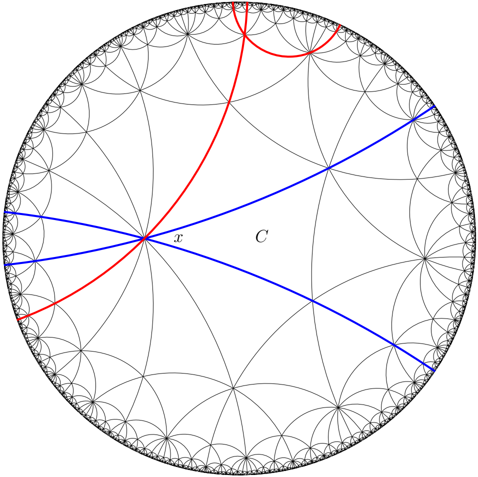
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

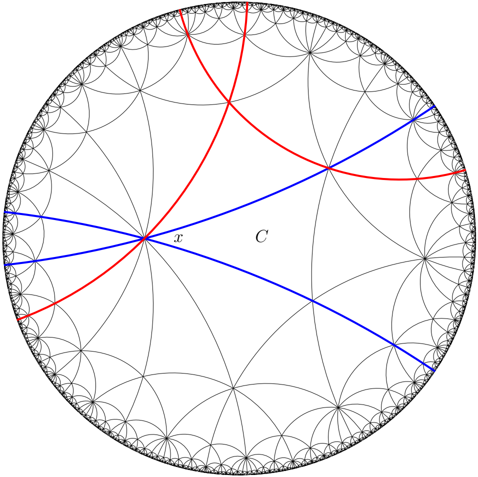
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

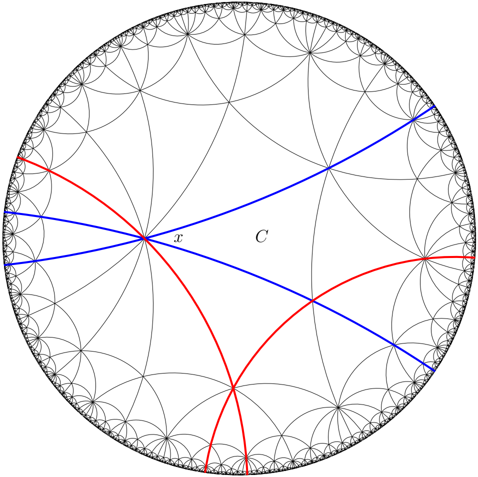
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

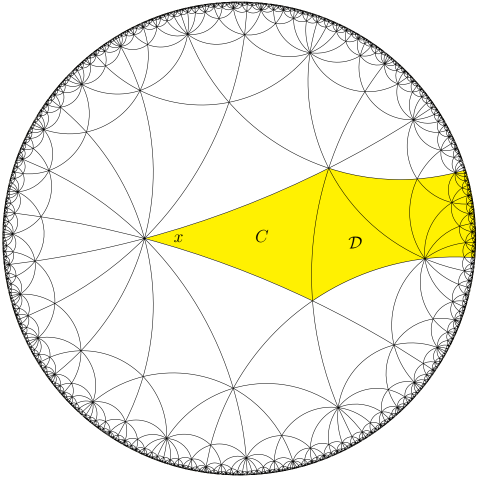
It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

We need to check that the right side of every commutator relation of the third type is sent to the identity.

It is enough to check that the non-simple roots at x are simple everywhere else.



Proof Sketch (cont.)

- The simple root condition shows that $\tilde{\phi}_x$ would exist if C lies anywhere in $\mathcal{D}$.

Proof Sketch (cont.)

- The simple root condition shows that $\tilde{\phi}_x$ would exist if C lies anywhere in $\mathcal{D}$.
- Apply the action of W with appropriate group elements to get a family of $\tilde{\phi}_v$ for chambers of $\mathcal{D}$.

Proof Sketch (cont.)

- The simple root condition shows that $\tilde{\phi}_x$ would exist if C lies anywhere in $\mathcal{D}$.
- Apply the action of W with appropriate group elements to get a family of $\tilde{\phi}_v$ for chambers of $\mathcal{D}$.
- Check these maps are distinct, and the vertices v do not lie on finitely many walls.

Proof Sketch (cont.)

- The simple root condition shows that $\tilde{\phi}_x$ would exist if C lies anywhere in $\mathcal{D}$.
- Apply the action of W with appropriate group elements to get a family of $\tilde{\phi}_v$ for chambers of $\mathcal{D}$.
- Check these maps are distinct, and the vertices v do not lie on finitely many walls.

The last two steps can be checked by switching back to the group theory of W. We have candidates $(sut)^k x$, and these last two properties can be phrased in terms of length conditions.

Proof Sketch (cont.)

- The simple root condition shows that $\tilde{\phi}_x$ would exist if C lies anywhere in $\mathcal{D}$.
- Apply the action of W with appropriate group elements to get a family of $\tilde{\phi}_v$ for chambers of $\mathcal{D}$.
- Check these maps are distinct, and the vertices v do not lie on finitely many walls.

The last two steps can be checked by switching back to the group theory of W. We have candidates $(sut)^k x$, and these last two properties can be phrased in terms of length conditions.

For example, to check that $(sut)^k x$ lies in $\mathcal{D}$, one thing we must check is that $\ell((sut)^k) \leq \ell(t(sut)^k)$ which we can do by finding a reduced decomposition.

The Region $\mathcal{D}$

- When two of a, b, c are at least 4, this region $\mathcal{D}$ has infinitely many chambers, and we can prove U_+ is not finitely generated.

The Region $\mathcal{D}$

- When two of a, b, c are at least 4, this region $\mathcal{D}$ has infinitely many chambers, and we can prove U_+ is not finitely generated.
- When $a = b = 3$, then the region $\mathcal{D}$ is finite, and this approach must be modified.

The Region $\mathcal{D}$

- When two of a, b, c are at least 4, this region $\mathcal{D}$ has infinitely many chambers, and we can prove U_+ is not finitely generated.
- When $a = b = 3$, then the region $\mathcal{D}$ is finite, and this approach must be modified.
- If the exceptional vertex has link associated to $G_2(2)$ then we can relax the condition on simple roots, and enlarge $\mathcal{D}$ to get another region which is still infinite.

The Region $\mathcal{D}$

- When two of a, b, c are at least 4, this region $\mathcal{D}$ has infinitely many chambers, and we can prove U_+ is not finitely generated.
- When $a = b = 3$, then the region $\mathcal{D}$ is finite, and this approach must be modified.
- If the exceptional vertex has link associated to $G_2(2)$ then we can relax the condition on simple roots, and enlarge $\mathcal{D}$ to get another region which is still infinite.

What about the remaining cases?

Theorem (S.)

If $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ *is an RGD system satisfying* (A) *such that* $a = b = 3$, *and the vertex of type* s *has link associated to* $C_2(2)$ *or* $G_2(3)$, *then* U_+ *is finitely generated.*

Note: The case with ${}^2F_4(2)$ is impossible.

Sketch of Proof

- Define $d(\alpha, C)$ to be the minimal distance to a chamber which borders $\partial\alpha$.

Sketch of Proof

- Define $d(\alpha, C)$ to be the minimal distance to a chamber which borders $\partial\alpha$.
- Define subgroups U_n generated by all roots within a certain distance from C.

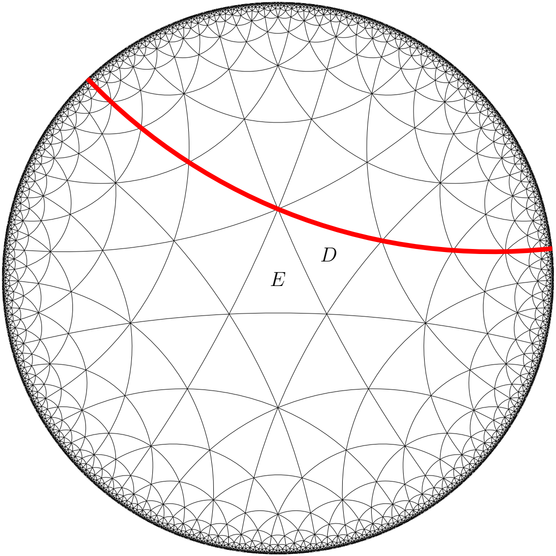
Sketch of Proof

- Define $d(\alpha, C)$ to be the minimal distance to a chamber which borders $\partial\alpha$.
- Define subgroups U_n generated by all roots within a certain distance from C.
- Each U_n is finitely generated, and their union is U_+.

Sketch of Proof

- Define $d(\alpha, C)$ to be the minimal distance to a chamber which borders $\partial\alpha$.
- Define subgroups U_n generated by all roots within a certain distance from C.
- Each U_n is finitely generated, and their union is U_+.
- When we hit vertices with exceptional links, we can "go the other way around"

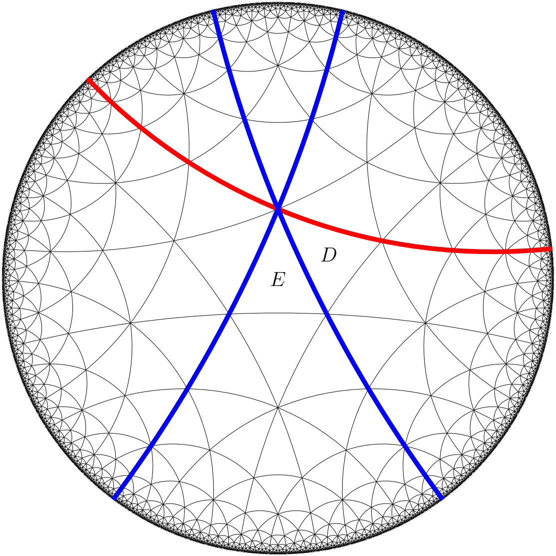
Sketch of Proof



U_n

U_{n-1}

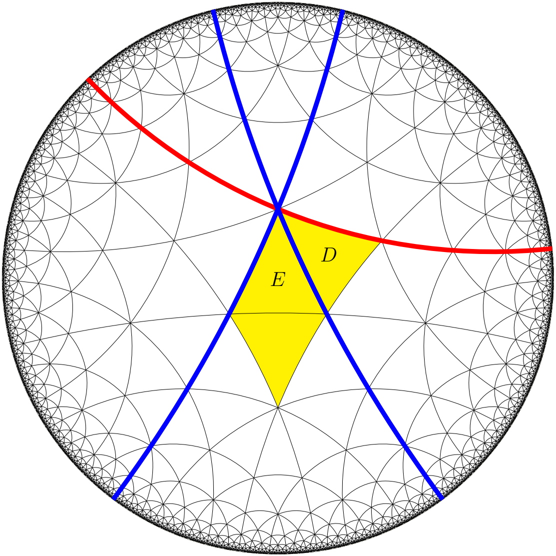
Sketch of Proof



U_n

U_{n-1}

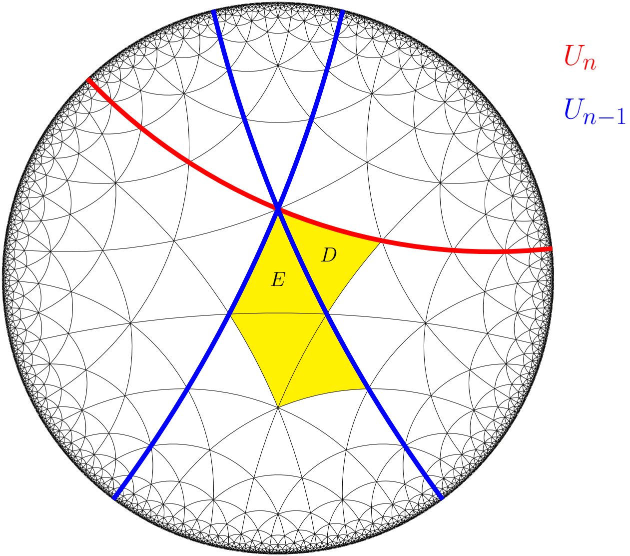
Sketch of Proof



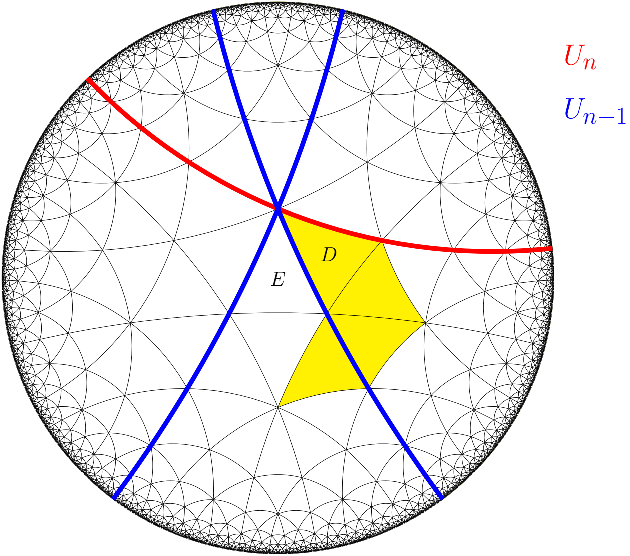
U_n

U_{n-1}

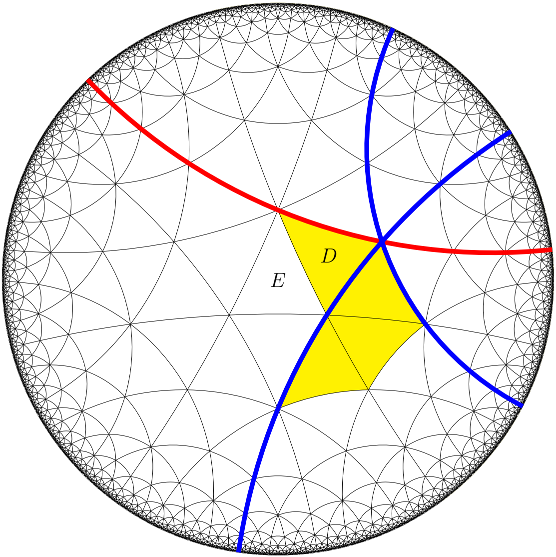
Sketch of Proof



Sketch of Proof



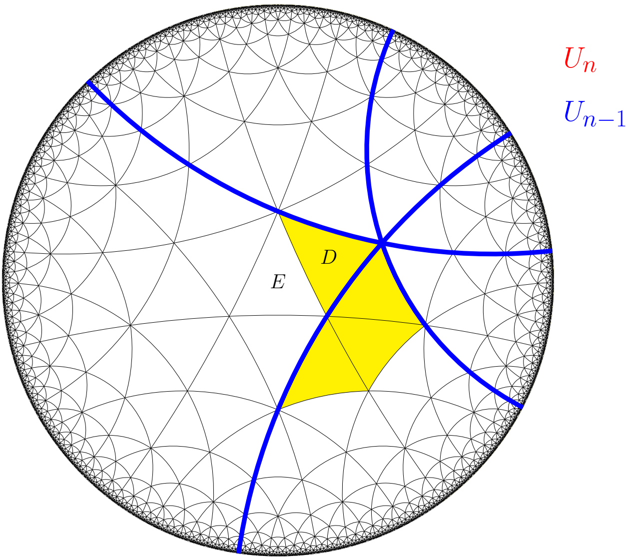
Sketch of Proof



U_n

U_{n-1}

Sketch of Proof



Sketch of Proof (cont.)

- For every chamber $D \neq C$, there is a vertex v so that $D \neq \mathrm{proj}_v(C)$.

Sketch of Proof (cont.)

- For every chamber $D \neq C$, there is a vertex v so that $D \neq \mathrm{proj}_v(C)$.
- Inductively, this shows that $U_k = U_+$ for large enough k, and in fact $U_+ = U_2$.

Sketch of Proof (cont.)

- For every chamber $D \neq C$, there is a vertex v so that $D \neq \mathrm{proj}_v(C)$.
- Inductively, this shows that $U_k = U_+$ for large enough k, and in fact $U_+ = U_2$.

This technique also yields another proof of the result when Δ satisfies (co).

Sketch of Proof (cont.)

- For every chamber $D \neq C$, there is a vertex v so that $D \neq \mathrm{proj}_v(C)$.
- Inductively, this shows that $U_k = U_+$ for large enough k, and in fact $U_+ = U_2$.

This technique also yields another proof of the result when Δ satisfies (co).

This generating set of U_2 is not minimal, and there are generating sets (consisting of root groups) with 5 roots, compared to 3 roots when condition (co) is satisfied.

Questions?

Thank you.

Any Questions?