

STAT 212 Problem Set 2.

Due: Friday, February 21st at 11:59PM

Instructions: Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

1. Let $\{X_n : n \geq 1\}$ be a sequence of IID integrable random variables. Define

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}.$$

- (a) Prove that the sequence $\{\bar{X}_n : n \geq 1\}$ is a UI backwards martingale relative to some backwards filtration. Hence the limit $\bar{X} = \lim_{n \rightarrow \infty} \bar{X}_n$ exists almost surely and in L^1 .
- (b) Assuming $X_1 > 0$ almost surely, show that for any $\lambda > 1$,

$$\mathbb{P}[\sup_{k \geq 1} \bar{X}_k \geq \lambda \mathbb{E}[X_1]] \leq 1/\lambda.$$

- (c) Let \bar{X}' be an independent copy of \bar{X} . Show that $\frac{\bar{X} + \bar{X}'}{2} \stackrel{d}{=} \bar{X}$.
- (d) Deduce the strong law of large numbers, i.e. that $\bar{X} = \mathbb{E}[X_1]$ almost surely.
(Hint: if \bar{X} is not a.s. constant, show $\mathbb{E}|\frac{\bar{X} + \bar{X}'}{2} - a| < \mathbb{E}|\bar{X} - a|$ for some $a \in \mathbb{R}$.)
2. Suppose $X_n \rightarrow X$ in L^1 , and let $\{\mathcal{F}_n : n \geq 1\}$ be a filtration. Define $\mathcal{F}_\infty = \sigma(\cup_{n \geq 1} \mathcal{F}_n)$. Prove that $\mathbb{E}[X_n | \mathcal{F}_n] \rightarrow \mathbb{E}[X | \mathcal{F}_\infty]$ a.s. and in L^1 .
3. Consider a box containing a black and a red ball. At each step, we sample a ball uniformly from the box, and return this ball, along with an additional ball of the same color, to the box. Let $\{A_n : n \geq 1\}$ denote the number of black balls after n -rounds.
- (i) Prove that $M_n = \frac{A_n}{n+2}$ is a martingale.
- (ii) Show that $M_n \rightarrow M_\infty$ a.s. and in L^p for all $p \in [1, \infty)$, but not in L^∞ .
4. Place N red points R_k and N blue points B_k uniformly and independently in $[0, 1]^2$. Let X be the length of the minimum matching between red and blue points, i.e. the minimum value of

$$\sum_{k=1}^N \|R_k - B_{\pi(k)}\|$$

across all bijections $\pi : [N] \rightarrow [N]$. Show that

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{4N}}.$$

Optional questions (not graded)

- Let $f : \mathbb{Z}^2 \rightarrow [0, 1]$ be such that for all $a, b \in \mathbb{Z}$, we have

$$f(a, b) = \frac{f(a+1, b) + f(a-1, b) + f(a, b+1) + f(a, b-1)}{4}.$$

Prove that f is constant. For an extra challenge, extend the result to higher dimensions.