Statistics 212: Lecture 5 (February 10, 2025)

Doob's Inequality / L^p Maximal Inequality and Reverse Martingales

Instructor: Mark Sellke

Scribe: Théo Voldoire & Sarah McDonald

1 Doob's Inequality / L^p Maximal Inequality

In the previous lecture, we discussed the L^p Maximal Inequality. In this lecture, we will use Doob's Inequality to prove the L^p Maximal Inequality.

Theorem 1.1 (Doob's Inequality). *If (X_n) is a submartingale and $\lambda > 0$, then*

$$\lambda \mathbb{P}(\max_{j \le n} X_j \ge \lambda) \le \mathbb{E}\left[X_n \cdot 1_{\max_{j\le n} X_j \ge \lambda}\right]$$

The proof of Doob's Inequality uses stopping times:

Proof. Let τ be first time t with $X_t \ge \lambda$, with $\tau = +\infty$ if that never happens. For each $0 \le k \le n$,

$$\mathbb{E}[X_\tau \cdot 1_{\tau=k}] \le \mathbb{E}[X_n \cdot 1_{\tau=k}]$$

Now, summing over k,

$$\mathbb{E}[X_\tau \cdot 1_{\tau\le n}] \le \mathbb{E}[X_n \cdot 1_{\tau\le n}]$$

Note also that

$$\tau \le n \Leftrightarrow \max_{j\le n} X_j \ge \lambda$$
$$X_\tau \ge \lambda \text{ a.s. if } \tau \le +\infty$$

Now, putting it all together, we have

$$\begin{aligned}
\lambda \mathbb{P}(\max X_j \ge \lambda) &= \lambda \mathbb{P}(\tau \le n) \\
&\le \mathbb{E}[X_\tau \cdot 1_{\tau \le n}] && \text{since } X_\tau \ge \lambda \\
&\le \mathbb{E}[X_n \cdot 1_{\tau \le n}] \\
&= \mathbb{E}\left[X_n \cdot 1_{\max_{j\le n} X_j \ge \lambda}\right].
\end{aligned}$$

□

Now, we will prove the L^p Maximum Inequality from last lecture (Lecture 4 Lemma 1.4) using Doob's Maximal Inequality.

Proof. First, let $U = U_n = \max_{j\le n}|X_j|$, $V = V_n = |X_n|$. We will show $\|U_n\|_p \le \frac{p}{p-1}\|V_n\|_p$.

Deducing L^p maximal inequality: first, $U_n \uparrow X^*$ as $n \to +\infty$ so $\|U_n\|_p \uparrow \|X^*\|_p$. Then, $\|V_n\|_p \uparrow \sup_{n\ge1}\|X_n\|_p$ because $\mathbb{E}[|X_{n+1}|^p] \ge \mathbb{E}[|X_n|^p]$ as $u \mapsto |u|^p$ is convex.

Now, for fixed n,

$$\begin{aligned}
\|U\|_p^p = \mathbb{E}[U^p] &\overset{*}{=} p\int_0^\infty \lambda^{p-1}\mathbb{P}(U\ge\lambda)d\lambda \\
&\le p\int_0^\infty \lambda^{p-2}\mathbb{E}[V\cdot 1_{U\ge\lambda}]d\lambda && \text{Doob's inequality, as } |X_n| \text{ is submartingale} \\
&\overset{\dagger}{=} \frac{p}{p-1}\int_{(V,U)}\left(\int_0^\infty \lambda^{p-2}V\cdot 1_{U\ge\lambda}d\lambda\right)d\mu \\
&= \frac{p}{p-1}\mathbb{E}\left[VU^{p-1}\right]
\end{aligned}$$

where μ is the law of (V,U). Step $*$ holds because for each *fixed* $u\ge0$, we have $u^p = p\int_0^\infty \lambda^{p-1}1_{u\ge\lambda}d\lambda = p\int_0^u \lambda^{p-1}d\lambda$. In other words, we are using Fubini to exchange the integration with respect to λ and the expectation with respect to U. Step $\dagger$ is proved similarly, in reverse. (Note that the power of λ changed from $p-1$ to $p-2$ from Doob's inequality, which leads to the factor $\frac{1}{p-1}$ rather than $1/p$.)

Now introduce q s.t. $\frac{1}{p}+\frac{1}{q}=1$, i.e. $q=\frac{p}{p-1}$, thus leading to

$$\begin{aligned}
\frac{p}{p-1}\mathbb{E}\left[VU^{p-1}\right] &\le \frac{p}{p-1}\|V\|_p\|U^{p-1}\|_q && \text{Holder's} \\
&= \frac{p}{p-1}\|V\|_p\|U\|_p^{p-1}.
\end{aligned}$$

Canceling $\|U\|_p^{p-1}$ on both sides of the inequality, we obtain

$$\|U\|_p \le \frac{p}{p-1}\|V\|_p,$$

as desired. ☐

Theorem 1.2 (Lévy's Upward Theorem)**.** *Fix $X\in L^1$, consider the filtration $(\mathscr{F}_n)$, and let $X_n = \mathbb{E}[X|\mathscr{F}_n]$. Then*

$$X_n \overset{a.s.,L^1}{\to} X_\infty = \mathbb{E}[X|\mathscr{F}_\infty],$$

where $\mathscr{F}_\infty = \sigma\left(\bigcup_n \mathscr{F}_n\right)$ is the smallest σ-algebra generated by $(\mathscr{F}_n)$.

Proof. X_∞ is $\mathscr{F}_\infty$-measurable as each X_n is $\mathscr{F}_\infty$-measurable. We want to show that for all $S\in\mathscr{F}_\infty$, the following property holds:

$$\mathbb{E}[X\cdot 1_S] = \mathbb{E}[X_\infty\cdot 1_S]. \tag{1}$$

First, property (1) holds if $S\in\mathscr{F}_n$,

$$\mathbb{E}[X\cdot 1_S] = \mathbb{E}[X_m\cdot 1_S] \ \ \forall m\ge n,$$

as $X_m = \mathbb{E}[X|\mathscr{F}_m]$, $S\in\mathscr{F}_n\subseteq\mathscr{F}_m$. So, we find $\mathbb{E}[X\cdot 1_S]\to\mathbb{E}[X_\infty\cdot 1_S]$ as $m\to\infty$, since $X_m \overset{L^1}{\to} X_\infty$.

We need to use the $\pi-\lambda$-theorem. Let $\mathscr{G} = \{S\in\mathscr{F}_\infty : (1) \text{ holds for } S\}$. $\mathscr{G}$ is a $\lambda-$system (closed under complement, countable disjoint union). Then, by what preceeds, $\bigcup_n\mathscr{F}_n\subseteq\mathscr{G}$, which is a π-system. So by $\pi-\lambda$-Theorem, we find $\sigma(\cup_n\mathscr{F}_n)\subseteq\mathscr{G}$. ☐

A consequence of this result is the Kolmogorov 0-1 law.

Theorem 1.3 (Kolmogorov's 0-1 Law)**.** *Let (X_n) be a sequence of iid random variables, $A \in \bigcap_n \sigma(X_n, X_{n+1}, \dots)$. Then $\mathbb{P}(A) \in \{0, 1\}$.*

Proof. Define $Y = 1_A$, $Y_n = \mathbb{P}(A|\sigma(X_1, \dots, X_n))$. A is $\sigma(X_1, \dots, X_n, \dots)$-measurable, so

$$Y_n \longrightarrow \mathbb{P}(A|\sigma(X_1, \dots, X_n, \dots)) = 1_A$$

Thus, $Y_n = \mathbb{P}(A)$ for all n, as $A \in \sigma(X_{n+1}, X_{n+2}, \dots)$ is independent of $\sigma(X_1, \dots, X_n)$.

Hence, $\mathbb{P}(A) \xrightarrow{L^1} 1_A$. □

In the context of Brownian motion, we get the Blumenthal 0-1 law, so this construction is useful to understand hitting times, e.g. to show that hitting times are stopping times.

2 Reverse martingales

Definition 2.1 (Reverse Martingale)**.** A reverse filtration is a sequence $\mathscr{F}_{-1} \supseteq \mathscr{F}_{-2} \supseteq \dots$ of decreasing filtrations. A reverse martingale is then a sequence of random variables $X_{-1}, X_{-2}, \dots$ such that

(a) $X_{-i} \in L^1$

(b) X_{-i} is $\mathscr{F}_{-i}$ measurable

(c) $\mathbb{E}[X_{-i}|\mathscr{F}_{-i-1}] = X_{-i-1}$

Observation 2.2. *Any reverse martingale is uniformly integrable (UI), as*

$$X_{-i} = \mathbb{E}[X_{-i-1}|\mathscr{F}_{-i}].$$

Namely, recall that $\{\mathbb{E}[X|\mathscr{G}] : \mathscr{G} \subseteq \mathscr{F}\}$ is UI if $X \in L^1$. In this case, the time-reversal means every random variable in the sequence is a conditional expectation of X_{-1}.

Theorem 2.3 (Lévy's Downward Theorem)**.** *Let $(X_{-i}, \mathscr{F}_{-i})$ be a reverse martingale. Then*

$$X_{-n} \longrightarrow X_{-\infty} = \mathbb{E}\left[X_{-1}|\mathscr{F}_{-\infty}\right], \text{ a.s. and in } L^1$$

with $\mathscr{F}_{-\infty} = \bigcap_n \mathscr{F}_{-n}$.

Proof. Almost sure convergence has the same proof as forward direction, using up-crossing inequalities. We then obtain L^1 convergence using UI.

Now, let's show that $X_{-\infty} = \mathbb{E}[X_{-1}|\mathscr{F}_{-\infty}]$.
Observe $X_{-\infty}$ is $\mathscr{F}_{-n}$-measurable $\forall n$ as $X_{-n}, X_{-n-1}, \dots$ are F_{-n}-measurable, thus $X_{-\infty}$ is $\mathscr{F}_{-\infty}$.
Next, we want that if $S \in \mathscr{F}_{-\infty}$, then $\mathbb{E}\left[X_{-\infty} \cdot 1_S\right] = \mathbb{E}[X_{-1} \cdot 1_S]$
We see that

$$\mathbb{E}[X_{-1} \cdot 1_S] = \mathbb{E}\left[X_n \cdot 1_S\right] \forall n,$$

as $S \in \mathscr{F}_{-n}$ and $X_{-n} = \mathbb{E}[X_{-1}|\mathscr{F}_{-n}]$.
Finally, we find $\mathbb{E}\left[X_{-n} \cdot 1_S\right] \to \mathbb{E}[X_{-1} \cdot 1_S]$ by L^1 convergence. □

We can prove the SLLN using reverse martingales.

Theorem 2.4 (Strong Law of Large Numbers)**.** *Let* $X_1, X_2, \ldots$ *iid sequence of* L^1 *random variables. Then*

$$\bar{X}_n = \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s., L^1} \mathbb{E}[X_1].$$

Proof Sketch. $\bar{X}_{-n} = \mathbb{E}[X_1 | S_n, S_{n+1}, \ldots]$, that is, $(\bar{X}_n)$ is a reverse martingale, and so

$$\bar{X}_{-n} \xrightarrow{\text{a.s.}, L^1} \bar{X}_\infty = \mathbb{E}\left[X_1 \,\middle|\, \bigcap_{n \ge 1} \sigma(S_n, S_{n+1}, \ldots)\right] \stackrel{?}{=} \mathbb{E}[X_1].$$

To show the last equality, use the Hewitt-Savage 0-1 Law (see Durrett) or Homework 2, Question 1. $\square$