Small Loss Regret Bounds for Thompson Sampling

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August 29, 2018

Overview

1. Learning from Experts
2. Analysis of Thompson Sampling for Full Feedback
3. Small Loss Bound for Full Feedback
4. Bandit and Semibandit Cases
5. Open Problems

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- Later we also consider the *semibandit* case: player picks m of n actions each round.

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- As we observe losses, we update our distribution to obtain a new posterior distribution each round.
- Given initial prior distribution, finding optimal play is a complicated, deterministic computation.
- Thompson Sampling is a simple strategy for any probability distribution. Not exactly optimal, but it does very well and is feasible in practice.

Thompson Sampling

Thompson Sampling Procedure

At each time t, compute the posterior distribution p_t for the best coordinate $i^\star = \arg\min_{i\in[n]} L_{i,T}$. Then pick the next action i_t according to the distribution p_t.

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- In the full-information case, this strategy intuitively hedges by softly following the leader. In the bandit case, this strategy intuitively balances explore/exploit similarly to multiplicative weights or upper confidence bound algorithms.
- Unlike the exact optimal strategy, Thompson Sampling is often efficient to simulate, and is amenable to analysis.

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- However, if $L^\star$ is very small, not so impressive. Why should we do worse when the same loss comes more slowly?
- Slow accumulation of small losses? But we can always assume losses are binary, either 0 or 1.
- Hence, interest in showing more refined $O(\sqrt{L^\star})$ regret bounds.

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- (*) Thompson Sampling for semibandits was only analyzed when different coordinate losses are independent of each other.

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- We also address the semibandit case with arbitrary priors.
- To achieve full T-independence in the bandit/semibandit settings, we have to modify Thompson Sampling by never playing low probability actions. Otherwise upper bounds have $\log(T)$ terms.

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- New goal: estimate ℓ^1 movement of $\vec{p}_t$.

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- Pinsker's Inequality controls the movement:

$$||\vec{p}_t - \vec{p}_{t+1}||_{\ell^1}^2 \leq 2 \cdot Ent[\vec{p}_{t+1}; \vec{p}_t]$$

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- Goal: estimate

$$\mathbb{E}\left[\sum_{t=0}^{T-1} ||\vec{p}_t - \vec{p}_{t+1}||_{\ell^1}\right].$$

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- Cauchy-Schwarz:

$$\mathbb{E}\left[\sum_{t=0}^{T-1} ||\vec{p}_t - \vec{p}_{t+1}||_{\ell^1}\right] \leq \sqrt{T \cdot \mathbb{E}\left[\sum_{t=0}^{T-1} ||\vec{p}_t - \vec{p}_{t+1}||_{\ell^1}^2\right]}$$

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Information Theoretic Analysis for Full-Feedback

Theorem [Russo and Van Roy '16]

Thompson Sampling gives expected regret $\mathbb{E}[R_T] \leq \sqrt{\frac{T \cdot H(\vec{p}_0)}{2}} \leq \sqrt{\frac{T \log(n)}{2}}$.

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Thompson Sampling gives expected regret
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- Another way to understand this:

Information Ratio

For time-step t, define the *information ratio* between squared regret and information obtained to be

$$\Gamma_t := \frac{\mathbb{E}^{p_t}[r_t]^2}{I_t(i^\star, (i_t, \ell_t(i_t)))}.$$

Here the mutual information I_t is the average amount of new information we obtain about the best coordinate $i^\star$ on round t.

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- The goal is to upper-bound Γ_t; then regret implies learning.
- Pinsker tells us that $\Gamma_t \leq \frac{1}{2}$ for Thompson Sampling.

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Here the mutual information I_t is the average amount of new information we obtain about the best coordinate $i^\star$ on round t.

- The goal is to upper-bound Γ_t; then regret implies learning.
- Pinsker tells us that $\Gamma_t \leq \frac{1}{2}$ for Thompson Sampling.
- In general, if $\mathbb{E}[\Gamma_t] \leq a_t$, we obtain:

$$\mathbb{E}[R_T]^2 = \left(\mathbb{E}\left[\sum_t \mathbb{E}^{p_t}[r_t]\right]\right)^2 \leq \mathbb{E}\left[\sum_t I_t\right]\mathbb{E}\left[\sum_t \Gamma_t\right] \leq H(\vec{p}_0)\sum_t a_t$$

Obtaining a Small Loss Bound

Theorem

Thompson Sampling satisfies

$$\mathbb{E}[R_T] = O(\sqrt{\mathbb{E}[L^\star]H(\vec{p}_0)}).$$

- To prove this, we could try to prove $\Gamma_t = O(\bar{\ell}_t)$ where $\bar{\ell}_t = \mathbb{E}^{p_t}[\ell_t(i_t)] = \sum_i p_t(i)\bar{\ell}_t(i)$. Then $\mathbb{E}[\Gamma_t] = O(\mathbb{E}^{p_0}[\ell_t])$ and we'd obtain:

$$\mathbb{E}[L_T - L^\star] = \mathbb{E}[R_T] \leq O\left(\sqrt{H(\vec{p}_0)\sum_t \mathbb{E}^{p_0}[\ell_t]}\right) = O\left(\sqrt{H(\vec{p}_0)\mathbb{E}[L_T]}\right).$$

- Simple algebra would now yield the result.

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- No! In fact $\chi^2[p_{t+1}; p_t] \geq 2Ent[p_{t+1}; p_t]$ always holds.

Obtaining a Small-Loss Bound

Lemma

For any probability distributions p_{t+1}, p_t we have

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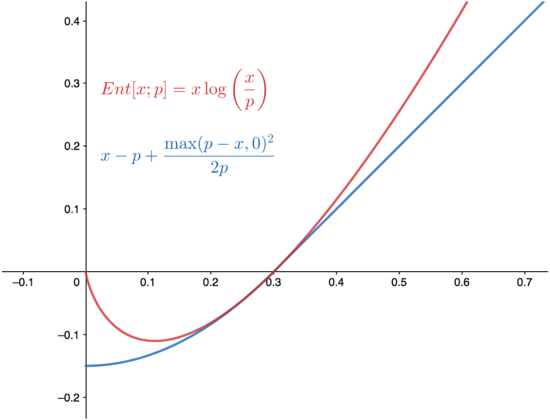
$$Ent[\vec{q};\vec{p}] = \sum_i q(i)\log\left(\frac{q(i)}{p(i)}\right).$$

- The first-order terms cancel because p_t, p_{t+1} are both probability distributions. The second-order derivatives are given by

$$\partial^2_{q(i)}\left[q(i)\log\left(\frac{q(i)}{p(i)}\right)\right] = \frac{1}{q(i)}.$$

Obtaining a Small-Loss Bound

- Since the 2nd derivative is decreasing and positive, lower bound $x \log\left(\frac{x}{p(i)}\right)$ quadratically for $x < p(i)$ and linearly for $x > p(i)$.



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Remark

We can also show that $\Gamma_t = O(\bar{\ell}_t + \bar{\ell}_t^\star)$ by using a cruder entropy inequality.

The Bandit Case

- In the bandit case, the expected regret for time t is

$$\mathbb{E}[r_t] = \sum_i p_t(i) \cdot (\overline{\ell}_t(i) - \overline{\ell}_t(i,i))$$

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- A similar Pinsker's inequality argument shows that the expected regret is $O(\sqrt{nT \cdot H(\vec{p}_0)})$. Optimal is $O(\sqrt{nT})$ and most methods (e.g. multiplicative weights) give $O(\sqrt{nT \log(n)})$.

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- Ordinary regret estimate:

$$\mathbb{E}^{p_t}[r_t]^2 = \left(\mathbb{E}^{p_t}\left[\sum_i p_t(i)\cdot(\bar{\ell}_t(i) - \bar{\ell}_t(i,i))\right]\right)^2$$

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Theorem [Russo and Van Roy '16]

Thompson Sampling for bandits satisfies $\mathbb{E}[R_T] = O(\sqrt{nT\cdot H(\vec{p}_0)})$.

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- If the player could track this sum, then $p_t(i) = 0$ once $\sum_{s \leq t} \ell_s(i) \geq L^\star$. After that, ignore coordinate i. This would result in $\mathbb{E}[R_T] \leq \sqrt{nL^\star \cdot H(\vec{p}_0)}$.

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- Thresholded Thompson Sampling avoids the bad case above. It also parallels the thresholded EXP3 algorithm which circumvents the same issue for multiplicative weights.

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- Since $p_t(i) \geq \gamma$ the unbiased estimator is a sum of bounded random variables in $[0, \frac{1}{\gamma}]$, so it is concentrated near the true value.
- Hence for each i, the player has a good estimate for $\sum_{s\leq t:p_s(i)\geq\gamma} \ell_s(i)$. When this sum gets significantly above $L^\star$, $p_t(i)$ will usually be very small.

The Bandit Case

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Theorem

Ordinary Thompson Sampling achieves regret

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The Semibandit Case

- Now we play a subset $A = (i_1, \ldots, i_m)$ of size m from a given collection $\mathcal{A} \subseteq \binom{[n]}{m}$. We observe and pay all m losses $\ell_t(i_k)$ each turn.

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- Again need to fix $L^\star$.

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- For small loss, new $\tilde{O}(\sqrt{nL^\star})$ bound! No contradiction, $L^\star \leq mT$.

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- Detail: we need to threshold separately for each $i_k^\star$.

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- Essentially, $A_S^\star$ has entropy $|S| \log n$. For each i the sum

$$\sum_{t: p_t(i \in A_S^\star) \geq \gamma} \ell_t(i)$$

reaches $\frac{L^\star}{\min_{s \in S} s}$ before $p_t(i)$ becomes small and the sum freezes.

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- As a result, if we partition $[m]$ into subsets $S_1, \ldots, S_k$ we get a regret bound like

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Theorem

A variant of thresholded Thompson Sampling achieves T-independent $\tilde{O}(\sqrt{nL^\star})$ regret in the semibandit setting. Without thresholding the regret is $\tilde{O}(\sqrt{nL^\star})$ with T-dependence.

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- Can we turn our semibandit analysis into a non-Bayesian algorithm achieving the optimal small loss bound?
- Any formal connections between Thompson Sampling and other algorithms?

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Thank You