STAT 212 Problem Set 1.

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Due: Friday, February 7th at 11:59PM

Instructions: Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

1. Let $C([0,1]) = \{f : [0,1] \to \mathbb{R}\,\text{continuous}\}$. For $f, g \in C([0,1])$, define

$$d_{\sup}(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

   - Prove that $d_{\sup}(\cdot,\cdot)$ defines a metric on $C([0,1])$.
   - A metric space (X,d) is defined to be *complete* if every Cauchy sequence is convergent. Prove that $(C([0,1]), d_{\sup})$ is a complete metric space.
   - A metric space (X,d) is defined to be *separable* if there exists a *countable* set $S \subseteq X$ which is *dense* in (X,d). (A subset S is dense if for all $x \in X$ and $\varepsilon > 0$, there exists $y \in S$ such that $d(x,y) < \varepsilon$. Prove that $(C([0,1]), d_{\sup})$ is separable.

     Hint: Construct functions with rational values at points of the form i/n, and interpolate linearly otherwise.

   A metric space that is both *complete* and *separable* is called a *Polish space*. For the purposes of probability theory, Polish spaces enjoy many of the nice properties of $\mathbb{R}$. This will be very useful in our later study of *Brownian Motion*, which can be defined as a "$C([0,1])$-valued random variable".

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

   - Let $\{\mathcal{F}_\alpha : \alpha \in I\}$ be any collection of sub-σ-algebras of $\mathcal{F}$. Prove that $\cap_{\alpha\in I}\mathcal{F}_\alpha$ is a σ-algebra.
   - Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. By definition, X is measurable with respect to a σ-algebra $\mathcal{G}$ if $\{X \in A\} \in \mathcal{G}$ for all Borel subsets A of $\mathbb{R}$. Consider the collection

$$\mathcal{C} = \{\mathcal{G} \subset \mathcal{F} : \mathcal{G}\,\text{is a}\,\sigma-\text{algebra}, X\,\text{is}\,\mathcal{G}-\text{measurable}\}.$$

     It follows from the previous part that $\cap_{\mathcal{G}\in\mathcal{C}}\mathcal{G}$ is a σ-algebra. It is called the 'σ-algebra generated by X', and is often denoted $\sigma(X)$. By definition, it is the *smallest* sigma algebra with respect to which X is measurable.

     Prove that $\sigma(X)$ consists **exactly** of the sets $\{X \in A\}$ for Borel $A \subseteq \mathbb{R}$.

3. Let $\{X_n : n \geq 1\}$, X be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $X_n \xrightarrow{P} X$ if and only if any subsequence of X_n has a further subsequence converging to X almost surely.

4. We showed in class that if ν, μ are finite measures on the same sigma-algebra such that

$$0 \leq \nu(S) \leq \mu(S) \tag{1}$$

for all measurable sets S, then there exists a $[0,1]$-valued measurable function f such that $\nu(S) = \int_S f d\mu$ for all S. Note that given a general pair (ν, μ) of finite measures, the pair $(\nu, \nu + \mu)$ always obeys (1). By applying the result from class to this setup, show the following stronger forms of the Radon–Nikodym theorem:

   (a) If $\nu \ll \mu$ (i.e. $\nu(S) = 0$ whenever $\mu(S) = 0$), then there exists a non-negative integrable function f such that $\nu(S) = \int_S f d\mu$ for all S.

   (b) In complete generality, there exists a non-negative integrable function f and finite measure θ such that

   $$\nu(S) = \theta(S) + \int_S f d\mu \tag{2}$$

   for all measurable sets S. Furthermore, one can arrange that θ and μ are *mutually singular*: there exists a measurable set S_* with $\theta(S_*) = 0$ and $\mu(S_*^c) = 0$. (Hint: consider the set where the function f coming from $(\nu, \nu + \mu)$ equals 1.)

   (c) Recall that in class, we showed f is unique in the absolutely continuous case. Show the decomposition (2) is also unique, i.e. if $(\tilde{\theta}, \tilde{f})$ is another such decomposition then $\theta = \tilde{\theta}$ as measures, and $f = \tilde{f}$ holds almost everywhere.

5. Let $(X_i, \mathcal{F}_i)_{i \in \mathbb{N}}$ be an adapted process of real-valued random variables. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that for some fixed constants A, B we have

$$|f(x)| \leq A|x| + B, \quad \forall x \in \mathbb{R}. \tag{3}$$

Show that:

- If $(X_i, \mathcal{F}_i)_{i \in \mathbb{N}}$ is a martingale then $(f(X_i), \mathcal{F}_i)_{i \in \mathbb{N}}$ is a submartingale.
- If $(X_i, \mathcal{F}_i)_{i \in \mathbb{N}}$ is a submartingale and if f is non-decreasing then $(f(X_i), \mathcal{F}_i)_{i \in \mathbb{N}}$ is also a submartingale.
- Both of the previous statements may be false, if the assumption (3) is dropped.

6. A *Galton-Watson* branching process models the growth of a population, and can be formally described as follows. Let $\{X(i,t) : i \geq 1, t \geq 1\}$ be an array of iid $\mathbb{Z}_{\geq 0}$ valued random variables satisfying $m := \mathbb{E}[X(1,1)] \in (0, \infty)$. Define $Z_0 = 1$ and

$$Z_{t+1} = \sum_{i=1}^{Z_t} X(i, t+1).$$

- Prove that $M_t = Z_t / m^t$ is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(Z_0, \cdots, Z_t)$. Further, prove that $\mathbb{E}[Z_t] = m^t$.
- Conclude that M_t converges to a non-negative random variable M_∞ almost surely, with $\mathbb{E}[M_\infty] \leq 1$.
- Show by example that both $\mathbb{E}[M_\infty] = 1$ and $\mathbb{E}[M_\infty] < 1$ are possible.

Optional questions (not graded)

- Let A, B be two events such that $P(B > 0)$. Denote $\mathcal{G}$ to be the sigma-algebra generated by B. Prove that

$$P(A|\mathcal{G})(\omega) = \begin{cases} \frac{P(A \cap B)}{P(B)} & \omega \in B \\ \frac{P(A \cap B^c)}{P(B^c)} & \text{otherwise} \end{cases}$$

- Let X be a square-integrable random variable. Let $\mathcal{F}$ be a sub-algebra. Prove that

$$\mathrm{Var(X)} = \mathbb{E}(\mathrm{Var(X}|\mathcal{F})) + \mathrm{Var}(\mathbb{E}(\mathrm{X}|\mathcal{F})).$$

- If $S \subseteq \mathbb{R}$ is a Borel set, prove that for any $\epsilon > 0$, there exists a compact set $K \subseteq S$ such that $\mu(K) \geq \mu(S) - \epsilon$. (Hint: consider the family of S such that both S and its complement have this property. Prove this family itself forms a σ-algebra.)

- A related proof of the Radon–Nikodym theorem goes by considering the quadratic objective
$$V(f) = 2\int f d\nu - \int f^2 d\mu.$$
As in class, let's assume $\nu(S) \leq \mu(S)$ for all measurable sets S, and aim to find a $[0,1]$-valued Radon–Nikodym derivative.

  1. Explain why if one **assumes** the Radon–Nikodym theorem, then V is maximized by the Radon–Nikodym derivative $f = d\nu/d\mu$.
  2. Show that $\sup_{f:\Omega\to[0,1]} V(f)$ is attained, by considering a rapidly convergent sequence of approximate maximizers. (Hint: it may help to prove that if $V(f_n) \approx V(f_m)$ are near-maximal, then $f_n \approx f_m$ in L^2, by considering $V\left(\frac{f_n+f_m}{2}\right)$. The intuition here is that V is strictly concave.)
  3. Letting f_* attain the maximum value of V, show that f_* yields a Radon–Nikodym derivative.