Confinement of Unimodal Distributions in High Dimension and an FKG-Gaussian Correlation Inequality

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Motivation: Unimodal Probability Distributions

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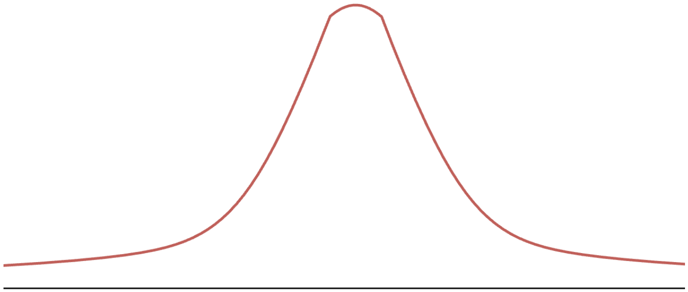
In 1-dimension, a reasonable criterion is **unimodality**



A stringent notion of unimodality is **log-concavity**. $d\mu(x) = \rho(x)dx$ is:

- **Log-concave** if $\log \rho(x)$ is concave.
- M-**log-concave** if for positive-definite M and all $x \in \mathbb{R}^N$:

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- μ inherits Poincare/LSI constants from γ_M [Bakry-Emery 85].
  - Spectral gap, isoperimetry, concentration.

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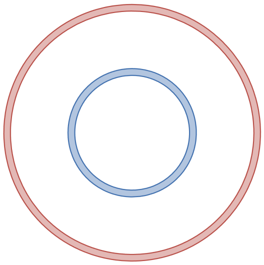
- Unimodal, but no concentration.
- Without log-concavity, we lack tools to reason about such distributions.



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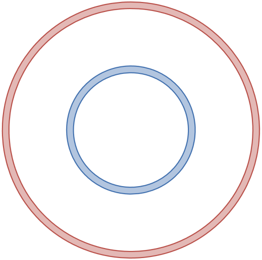

This talk will provide one such tool.

- We prove **confinement** without needing concentration.

Plan for Today

1. Ginzburg–Landau Surfaces and Main Results
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. Putting it all together

Discrete Gaussian Free Fields

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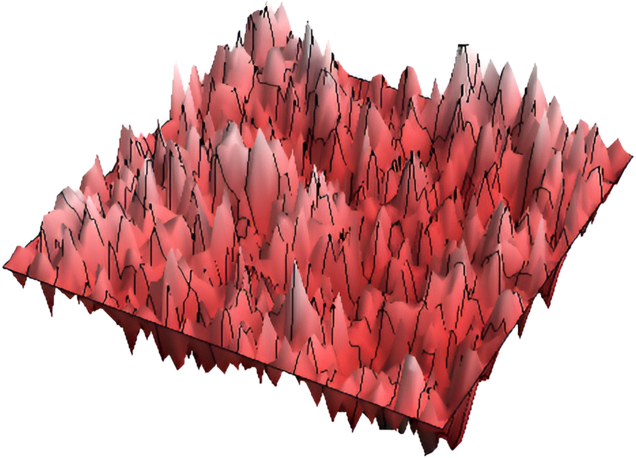
- Models fluctuations of random interfaces.
- Lots of probabilistic interest, notably on $\mathbb{Z}^2$ (extreme values, LQG).



Picture of GFF by Sam Watson. Here $\Lambda = [-L, \ldots, L]^2 \subseteq \mathbb{Z}^2$.

Discrete Gaussian Free Fields

Fundamental properties on general graphs:

- Let $R_{eff}(\cdot)$ be effective resistance on G as an electrical network. Then
$$\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda).$$
- More generally, $\mathbb{E}[(\varphi(v)-\varphi(w))^2] = R_{eff}(v \leftrightarrow w)$.

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- More generally, $\mathbb{E}[(\varphi(v) - \varphi(w))^2] = R_{eff}(v \leftrightarrow w)$.
- Definition of effective resistance: if $f : E(G) \to \mathbb{R}$ is a unit flow from $v \to w$, its energy is
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  Effective resistance equals the minimum energy of any unit flow.
- $R_{eff}(v \leftrightarrow \infty) < \infty$ iff random walk on G is transient.
- Similar for weighted edges (just think of multi-edges).

Non-gaussian interactions are also physically and mathematically interesting. Put a general function $U : \mathbb{R} \to \mathbb{R}$ on each edge:

$$d\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(- \sum_{e \in E} U(\nabla\varphi(e)) \right) \prod_{v \in \Lambda} d\varphi(v).$$

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- First rigorous study in [Brascamp-Lieb-Lebowitz 1975].
- Names: "Ginzburg–Landau", "$\nabla\varphi$", "anharmonic crystal".
- Free energy, dynamics, large deviations, fluctuations... [Funaki-Spohn 97, Naddaf-Spencer 97, Deuschel-Giacomin-Ioffe 00, Sheffield 03, Miller 11, Armstrong-Dario 22,...].

Localization

We will consider the question of **localization**.

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Are fluctuations of $\varphi(v_0)$ stochastically bounded on large domains $\Lambda \uparrow V$? If so, we say the model is **localized**. Otherwise **delocalized**.

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  - One can even take this as another definition of localization.
- GFF on $\mathbb{Z}^d$ localizes iff $d \geq 3$. Equivalent to transience/recurrence of simple random walk.
  - $\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda)$.
  - On $[-L, \ldots, L]^2 \subseteq \mathbb{Z}^2$, one has $\mathbb{E}[\varphi(\vec{0})^2] \approx \log L$.

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  - On $[-L, \ldots, L]^2 \subseteq \mathbb{Z}^2$, one has $\mathbb{E}[\varphi(\vec{0})^2] \approx \log L$.
- Conjecture of [Brascamp-Lieb-Lebowitz 1975]: localization is determined by the geometry of G, not the potential U.
  - Proved delocalization for general $U \in C_b^2(\mathbb{R})$ on $\mathbb{Z}^2$.

Localization of Ginzburg–Landau Random Surfaces

Localization is proven for various U (often focused on lattices):

- Strongly convex potentials with $\inf_{x\in\mathbb{R}} U''(x) \geq c > 0$.
    - $\mu_{G,\Lambda,U}$ is dominated by GFF [Brascamp-Lieb-Lebowitz 75].

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- $e^{-U(x)}$ is a mixture of centered Gaussians (will explain soon) [Biskup-Kotecky 07, Biskup-Spohn 11, Brydges-Spencer 12, Ye 19,...].
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- [Magazinov-Peled 22]: convex U with $U''(x) > 0$ for a.e. x.
- Still open for **Hammock potential** $U(x) = \infty \cdot 1_{|x|>1}$. This gives a uniformly random 1-Lipschitz $\varphi : V \to \mathbb{R}$.

Main Result: Localization for Monotone Potentials

We prove localization for **monotone** potentials.

Definition ((α, ε)-monotonicity)

U is (α, ε)-monotone if it is increasing on $\mathbb{R}^+$ and $U'(x) \geq \min\left(\varepsilon x, \frac{\alpha}{x}\right)$ for all points of differentiability $x \geq 0$.

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Theorem (Localization for (α,ε)-monotone U)

Let G be transient, and U be (α,ε)-monotone for $\alpha > 2$. Then $\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v_0)| \geq t] \leq O(t^{-\alpha})$ uniformly in $\Lambda \subseteq V$, for any $v_0 \in V$.

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- Proof will be based on unimodality of $\mu_{G,\Lambda,U}$.
- U_e can depend on edge e, as long as (α, ε) is uniform.
- If G is transient and **transitive**, tightness even for $\alpha = 1 + \varepsilon$.
    - $\approx$ minimal condition for $\int_{\mathbb{R}} e^{-U(x)} dx < \infty$ so $Z_{G,\Lambda,U} < \infty$.

Extreme Values of the Field

These bounds are often sharp enough to understand $\max_{v\in\Lambda}|\varphi(v)|$.

Theorem (Extreme Values from Polynomial Bounds)

Let U be (α,ε)-monotone with $\sup_{x\geq 0}|U(x)-\alpha\log(x+1)|<\infty$ and $\alpha>2$. As $\Lambda\subseteq\mathbb{Z}^d$ varies for $d\geq 3$, the laws of

$$|\Lambda|^{-\frac{1}{2d\alpha}}\max_{v\in\Lambda}|\varphi(v)|$$

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- Upper bound: Markov after extra tricks within the proof. Split $\mathbb{Z}^d$ into $2d$ transient subgraphs containing the origin.
- Lower bound is easy: condition outside a constant-density independent set $\mathcal{I}\subseteq\Lambda$.
- Similar for stretched exponential tails.
    - Monotonicity condition: $U'(x)\geq\min\left(\varepsilon x,\varepsilon x^{\beta-1}\right)$, for $\beta\in(0,2]$.

General Statement without Graphs

The graph structure is irrelevant in the main result!

Let U be (α, ε)-monotone, and $\ell_1, \ldots, \ell_j : \mathbb{R}^d \to \mathbb{R}$ be linear.

Take φ as in Ginzburg-Landau, and $\widetilde{\varphi}$ as in GFF:

$$\varphi \sim \exp\Big(\sum_{i=1}^{j} -U(\ell_i(\varphi))\Big)/Z_{\vec{\ell},U}\, d\varphi,$$

$$\widetilde{\varphi} \sim \exp\Big(\sum_{i=1}^{j} -\ell_i(\widetilde{\varphi})^2\Big)/Z_{\vec{\ell},Gaus}\, d\widetilde{\varphi},$$

Fix any other linear function $\ell_* : \mathbb{R}^d \to \mathbb{R}$. Then $\ell_*(\varphi)$ is bounded **on the same scale** as the centered Gaussian $\ell_*(\widetilde{\varphi})$, with α-power tails.

- Recovering GFF/Ginzburg-Landau: set $\ell_e(\varphi) = \varphi(v) - \varphi(v')$.

Preview of the Proof

The core proof idea has two components:

1. Handle the case that $U = V$ takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

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2. Reduce to this case via the FKG-Gaussian correlation inequality. This gives a notion of **domination by Gaussian mixtures**.
   - Dominating Gaussian mixtures must have special structure.
   - Perfectly suited for products of 1-dimensional Gaussian mixtures.

Plan for The Remainder

1. Results on Ginzburg–Landau Model
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
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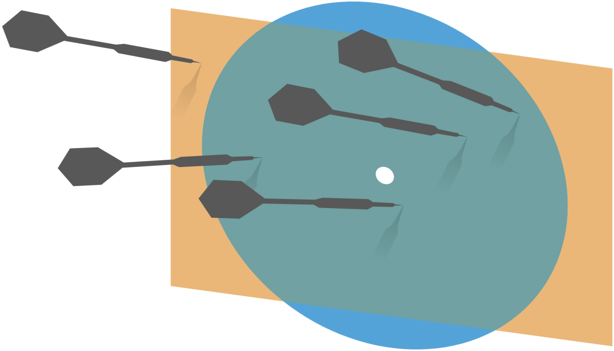
Royen's Gaussian Correlation Inequality

Theorem (Royen 2014)

Let γ be a centered Gaussian measure on $\mathbb{R}^d$, and $K_1, K_2 \subseteq \mathbb{R}^d$ symmetric convex sets (i.e. $K_i = -K_i$). Then 1_{K_1} and 1_{K_2} have non-negative correlation under γ, i.e.

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$



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History (see 2017 Quanta article):

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Proof idea: for $x, y \overset{i.i.d.}{\sim} \gamma$, equivalent to

$$\mathbb{P}[x \in K_1 \wedge x \in K_2] \geq \mathbb{P}[x \in K_1, y \in K_2].$$

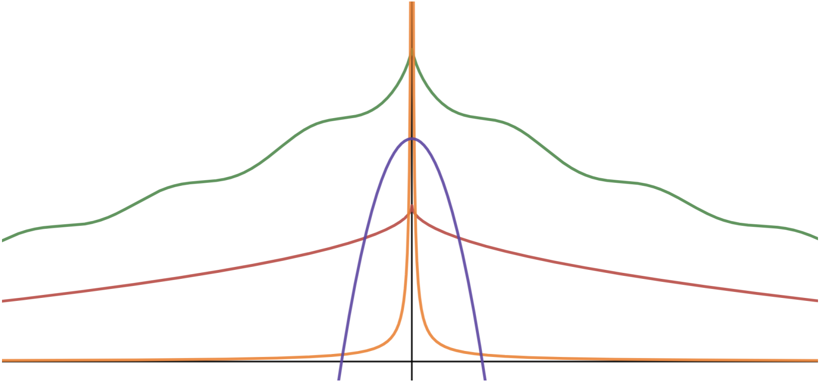
Royen showed $f(t) = \mathbb{P}[x \in K_1 \ \wedge \ \sqrt{1-t}x + \sqrt{t}y \in K_2]$ is decreasing.

Symmetric Quasi-Concave Functions

Definition

$f : \mathbb{R}^N \to \mathbb{R}$ is <u>symmetric quasi-concave</u> (SQC) if:

- $f(x) = f(-x)$ for all $x \in \mathbb{R}^N$.
- All super-level sets $\{x \in \mathbb{R}^N \ : \ f(x) \geq \lambda\}$ are convex.



GCI Yields Confinement

GCI: if $K_1, K_2 \subseteq \mathbb{R}^d$ are symmetric convex, then

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If $K_1, \ldots, K_{m+1} \subseteq \mathbb{R}^d$ are symmetric convex:

$$\gamma(K_1 \cap \cdots \cap K_{m+1}) \geq \gamma(K_1 \cap \cdots \cap K_m) \cdot \gamma(K_{m+1}),$$

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By level sets, if $f_1, \dots, f_{m+1} : \mathbb{R}^d \to \mathbb{R}^+$ are symmetric quasi-concave,

$$\mathbb{E}^\gamma[f_1 f_2 \dots f_{m+1}] \geq \mathbb{E}^\gamma[f_1 f_2 \dots f_m] \cdot \mathbb{E}^\gamma[f_{m+1}].$$

(Products of SQC functions need not be SQC, hence the middle step.)

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Suppose γ is centered Gaussian and $\frac{d\nu}{d\gamma} = f_1 f_2 \ldots f_m$ is a product of SQC functions. Then

$$\nu(K) = \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma} \cdot 1_K\right] \overset{\text{GCI}}{\geq} \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma}\right] \cdot \gamma(K) = \gamma(K)$$

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We say $\nu \preceq_{\text{con}} \gamma$ if $\gamma(K) \leq \nu(K)$ for all symmetric convex sets K.

Application: an Easy Case of Localization

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Since $U' \geq U'_\varepsilon$, the ratio

$$W(x) = e^{-U(x)+U_\varepsilon(x)}$$

is SQC. The Radon–Nikodym derivative is a product of these:

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Conclusion: $\mu_{G,\Lambda,U} \preceq_{\text{con}} \gamma_\varepsilon$. Localization on all transient G.

However this proof **cannot** work if U diverges subquadratically. μ must have subgaussian tails to be dominated by a single Gaussian.

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Does GCI extend to **mixtures** of centered Gaussians? If so, we could use them as the dominating measures and have more flexibility.

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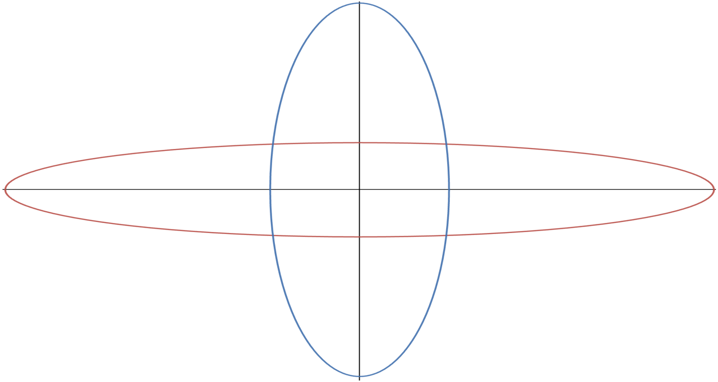
- Counterexample for two gaussians in $\mathbb{R}^2$:



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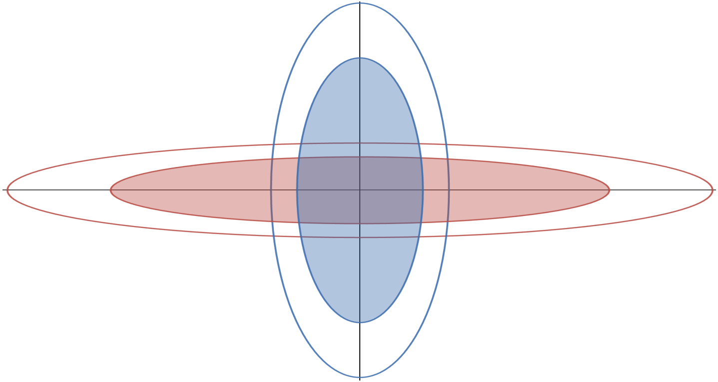


GCI For Gaussian Mixtures

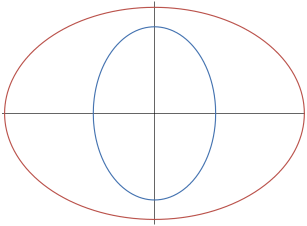
If the Gaussians have **comparable covariance**, GCI extends!!

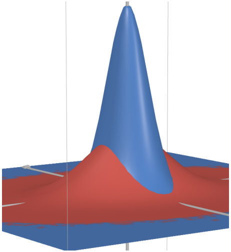
Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ be symmetric matrices. Let $d\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ and $d\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. Then GCI holds for $\mu = p\gamma_1 + (1-p)\gamma_2$:

$$\mu(K \cap K') \geq \mu(K)\mu(K')$$

for any symmetric convex sets K, K' and $0 \leq p \leq 1$.





GCI For Gaussian Mixtures

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- Can show $\gamma_1(K) \geq \gamma_2(K)$ (by GCI or otherwise). Thus $q \geq p$:

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$$\begin{aligned}\widetilde{\mu}(K') = q\widetilde{\gamma}_1(K') + (1-q)\widetilde{\gamma}_2(K') &\overset{GCI}{\geq} q\gamma_1(K') + (1-q)\gamma_2(K') \\ &\geq p\gamma_1(K') + (1-p)\gamma_2(K') = \mu(K').\end{aligned}$$

GCI For Totally Ordered Gaussian Mixtures

We can generalize further! Suppose:

1. $\mu = p_1\gamma_1 + \cdots + p_j\gamma_j$, with totally ordered inverse covariances $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} \cdots \succeq_{PSD} \Sigma_j$.
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We need these two functions on $\{1, 2, \ldots, j\}$ to be positively correlated with respect to the probability measure $\mathbb{P}[i] = p_i$.

This is the rearrangement inequality, **a special case of FKG**.

The FKG-Gaussian Correlation Inequality

We say the probability measure $d\nu = f(x)dx$ on $\mathbb{R}^k$ is log-supermodular if

$$f(\xi)f(\xi') \leq f(\xi \wedge \xi')f(\xi \vee \xi'), \quad \forall \xi, \xi' \in \mathbb{R}^k.$$

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Let ν be a log-supermodular probability measure on $\mathbb{R}_+^k$, and $\Sigma : \mathbb{R}_+^k \to \mathcal{S}_+^n$ be **order-reversing** from $\preceq_{\text{coord}}$ to $\preceq_{\text{PSD}}$. For each $\xi \in \mathbb{R}_+^k$ let $d\gamma_\xi(x) \propto e^{-\langle x, \Sigma(\xi) x\rangle}$ on $x \in \mathbb{R}^n$. The associated **log-supermodular Gaussian mixture** (LSGM) is

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Then $\nu((i,j)) = p_{i,j}$ is log-supermodular, so

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satisfies GCI, i.e. $\mu(K_1 \cap K_2) \geq \mu(K_1)\mu(K_2)$.

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For any LSGM $\Gamma_{\nu,\Sigma}$, suppose for symmetric quasi-concave $f_1, \dots, f_m$:

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For Ginzburg-Landau, need to express $\mu_{G,\Lambda,U}$ as some $\widetilde{\Gamma}$ above.

Plan for The Talk

1. Ginzburg–Landau Surfaces and Main Results
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. **Putting it all together**

Dominating LSGMs will be $\mu_{G,\Lambda,V}$ where V takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

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Then the Gibbs measure is a mixture of **GFFs with edge weights**.

$$d\mu_{G,\vec{\xi},GFF}(\varphi) = \frac{1}{Z_{G,GFF}} \exp\left(-\sum_{e\in E} \frac{1}{2\xi_e^2} \cdot |\nabla\varphi(e)|^2 \right) \prod_{v\in V} d\varphi(v).$$

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Encoding: inverse covariance $\Sigma(\vec{\xi})$ given by

$$\left\langle \varphi, \Sigma(\vec{\xi})\varphi \right\rangle = \sum_{e\in E(G)} (\varphi(v) - \varphi(v'))^2/\xi_e{}^2.$$

- Clearly Σ is decreasing from $\preceq_{coord}$ to $\preceq_{PSD}$.

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Then the Ginzburg-Landau measure has mixture representation of the form:

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- Elementary fact: if $A, B, C \succeq_{PSD} 0$, then

$$\det(A)\det(A+B+C) \leq \det(A+B)\det(A+C).$$

- This yields log-supermodularity, since $\Sigma(\vec{\xi}) = \sum_{e \in E} F(\xi_e)$ is additive.

Dominating Monotone Potentials by Gaussian Mixtures

If $U'(x) \geq V'(x)$ on $\mathbb{R}_+$, then Radon–Nikodym derivative

$$\frac{\mathrm{d}\mu_{G,\Lambda,U}}{\mathrm{d}\mu_{G,\Lambda,V}} \propto \prod_{e \in E} e^{-U(\nabla\varphi(e))+V(\nabla\varphi(e))}$$

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Dominating Monotone Potentials by Gaussian Mixtures

If $U'(x) \geq V'(x)$ on $\mathbb{R}_+$, then Radon–Nikodym derivative

$$\frac{d\mu_{G,\Lambda,U}}{d\mu_{G,\Lambda,V}} \propto \prod_{e \in E} e^{-U(\nabla\varphi(e))+V(\nabla\varphi(e))}$$

is a product of symmetric quasi-concave functions. Thus

$$\mu_{G,\Lambda,U} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

We can reduce further! By more FKG, $\mu_{G,\Lambda,V}$ is dominated by the "naive independent" LSGM with mixing measure

$$d\widehat{\nu}(\xi) = \prod_{e \in E} d\rho(\xi_e).$$

This reduces localization to GFFs with IID edge weights.

Dominating Monotone Potentials by Gaussian Mixtures

Lemma

There exist potentials $V(\rho)$ in centered Gaussian mixture form such that:

1. $V'(x) \leq \min\left(\varepsilon x, \frac{1+\varepsilon}{x}\right), \quad \forall x \geq 0.$
2. $V'(x) \leq \min\left(\varepsilon x, \frac{\alpha}{x}\right)$ *and* $\rho([t,\infty)) \leq O(t^{-\alpha}), \quad \forall t \geq 0.$
3. $V'(x) \leq \min\left(\varepsilon x, \varepsilon x^{\beta-1}\right)$ *and* $\rho([t,\infty)) \leq e^{-\Omega(t^{\beta})}, \quad \forall t \geq 0.$

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Proof Idea: Explicit construction. Match tail of ρ to the decay rate.

1. $\rho([t,\infty)) \asymp t^{-\varepsilon}$.
2. $\rho([t,\infty)) \asymp t^{-\alpha}$.
3. $\rho([t,\infty)) \asymp e^{-t^{\beta}}$. $\square$

Proof of Main Localization Result

Main result from before:

Theorem

Fix $\alpha > 2$ and transient G. Let U be (α, ε)-monotone. Then

$$\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v)| \geq t] \leq O(t^{-\alpha})$$

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Proof: we need to bound the tail of the weighted effective resistance $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$. This is the variance of $\varphi(v)$ in a weighted GFF.

Consider the energy-minimizing unit flow $v \to \infty$ in the **unweighted** graph G. $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$ is at most its (random) **weighted** energy. This is $\sum_e a_e \xi_e^2$, where $\sum_e a_e = R_{eff}(v \leftrightarrow \infty) < \infty$. Bound tails by Jensen. $\square$

Tightness from ε-Monotonicity

Theorem

Suppose U is ε-monotone, and p-bond percolation on G has transient infinite cluster for $p \in [1-\delta, 1]$. Then $Law(\varphi(v))$ is tight as $\Lambda \uparrow \infty$.

(The condition holds for all transient transitive G [Hutchcroft 23].)

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Proof: take $\xi_e \sim \rho$ independent. Consider edges with $\xi_e \leq M$, where

$$\rho([0, M]) \geq 1 - \delta.$$

By definition, these edges form a transient infinite cluster $\mathcal{C}$. Let $w \in \mathcal{C}$ be the closest point to v. Then both $R^{(\xi)}_{eff}(v \leftrightarrow w)$ and $R^{(\xi)}_{eff}(w \leftrightarrow \partial\Lambda)$ are tight. Hence $R^{(\xi)}_{eff}(v \leftrightarrow \partial\Lambda)$ is also tight. $\square$

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- ε-monotonicity can't imply good tail bounds. U may diverge slowly.

Another Application: the Fröhlich Polaron

Let $d\mathbb{Q}(B)$ be the law of 3-dimensional Brownian motion.

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Given coupling strength $\alpha \gg 1$ and time-horizon $T \gg \alpha$, the **Polaron path measure** $\widehat{\mathbb{Q}}_{\alpha, T}$ is the reweighted law on paths $B : [0, T] \to \mathbb{R}^3$:

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha, T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha, T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|} \, \mathrm{d}t \, \mathrm{d}s \right) \mathrm{d}\mathbb{Q}(\mathrm{B}), .$$

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Obtained by Feynman's path integral applied to a quantum operator (models an electron in crystal). The inverse "effective mass" is

$$m_{eff}(\alpha)^{-1} = \lim_{T\to\infty} \frac{\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}} \|B_T\|^2}{3T} \overset{?}{\approx} C_* \alpha^{-4}.$$

[Fröhlich 37, Landau-Pekar 48, Feynman 55, Lieb 77, Donsker-Varadhan 83, Spohn 87, Lieb-Thomas 97, Lieb-Seiringer 17, Mukherjee-Varadhan 18 & 20, Dybalski-Spohn 20, Betz-Polzer 22 & 23, Brooks-Seiringer 22...]

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In fact, the Coulomb interaction is a mixture of centered Gaussians:

$$\frac{1}{x} = \sqrt{2/\pi} \int_0^\infty e^{-u^2 x^2/2} du.$$

Hence $\widehat{\mathbb{Q}}_{\alpha,T}$ is an (infinite dimensional) LSGM and obeys GCI.

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Now the mixture comes **inside** the exponent. Resulting Gaussian mixture representation of $\widehat{\mathbb{Q}}_{\alpha,T}$ is indexed by a deformed Poisson process on weighted time-intervals $([s,t];u)$ [Mukherjee-Varadhan 20].

Another Application: the Polaron

Theorem (Mukherjee-Varadhan 20)

The Polaron path measure $\widehat{\mathbb{Q}}_{\alpha,T}$ *has a mixture-of-Gaussian representation*

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha,T}(B_{[0,T]}) = \int \mathrm{Q}_{\xi}(B_{[0,T]})\, \widehat{\Theta}_{\alpha,T}\, \mathrm{d}\xi.$$

Here $\xi = \{([s_i, t_i], u_i)\}_{i=1}^{n}$ *is a point process of weighted intervals, and*

$$\mathrm{dQ}_{\xi}(B_{[0,T]}) \propto e^{\sum_{i=1}^{n} u_i^2 \|B(t_i)-B(s_i)\|^2} \mathrm{d}\mathbb{Q}(B_{[0,T]}).$$

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- Can show $(\widehat{\Theta}_{\alpha,T}, \widehat{\mathbb{Q}}_{\alpha,T}) \to (\widehat{\Theta}_{\alpha,\infty}, \widehat{\mathbb{Q}}_{\alpha,\infty})$ as $T \to \infty$.
- Functional CLT for $\widehat{\mathbb{Q}}_{\alpha,\infty}$ [Mukherjee-Varadhan 20, Betz-Polzer 22].
  - Ergodicity and LLN for $\widehat{\Theta}_{\alpha,\infty}$.
  - Rigorizes Feynman path integral [Spohn 87, Dybalski-Spohn 20].
- Mass lower bound $m_{eff}(\alpha) \gtrsim \alpha^{2/5}$ [Betz-Polzer 23].
  - [Lieb-Seiringer 17]: $\lim_{\alpha\to\infty} m_{eff}(\alpha) = \infty$ without explicit rate.

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Theorem (Bazaes-Mukherjee-S-Varadhan 24; predicted in Landau-Pekar 1948)

If $T^{0.1} \geq \alpha \gg 1$, then $\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}} \|\mathsf{B}_T\|^2 \leq O(T\alpha^{-4})$. I.e. $m_{eff}(\alpha) \geq \Omega(\alpha^4)$.

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FKG-GCI was one of several ingredients. It also has direct consequences:

- $m_{eff}(\alpha)$ is strictly increasing.
- Increments negatively correlated, e.g. $\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}}[\langle B_2 - B_1, B_1 - B_0\rangle] < 0$.
- Universality: can replace $1/x$ with "more monotone" interaction.

Conclusion

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- Sharp bounds for Ginzburg–Landau and Polaron models:

$$\mathrm{d}\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(- \sum_{e \in E(G)} U\big(|\nabla \varphi(e)|\big) \right) \prod_{v \in \Lambda} \mathrm{d}\varphi(v),$$

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- Can origin-symmetry be relaxed? E.g. non-zero tilts.
- Faster than Gaussian tail decay? Happens in Ginzburg–Landau when $U(x) = |x|^p$ for $p > 2$ [Magazinov-Peled 22].

Let me know if you have ideas for another application!