

# STAT 212 Problem Set 2.

Due: Friday, February 21st at 11:59PM

**Instructions:** Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

1. Let  $\{X_n : n \geq 1\}$  be a sequence of IID integrable random variables. Define

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}.$$

- (a) Prove that the sequence  $\{\bar{X}_n : n \geq 1\}$  is a UI backwards martingale relative to some backwards filtration. Hence the limit  $\bar{X} = \lim_{n \rightarrow \infty} \bar{X}_n$  exists almost surely and in  $L^1$ .
- (b) Show that for any  $\lambda > 1$ ,

$$\mathbb{P}[\sup_{k \geq 1} \bar{X}_k \geq \lambda \mathbb{E}[X_1]] \leq 1/\lambda.$$

- (c) Let  $\bar{X}'$  be an independent copy of  $\bar{X}$ . Show that  $\frac{\bar{X} + \bar{X}'}{2} \stackrel{d}{=} \bar{X}$ .
- (d) Deduce the strong law of large numbers, i.e. that  $\bar{X} = \mathbb{E}[X_1]$  almost surely.  
(Hint: if  $\bar{X}$  is not a.s. constant, show  $\mathbb{E}|\frac{\bar{X} + \bar{X}'}{2} - a| < \mathbb{E}|\bar{X} - a|$  for some  $a \in \mathbb{R}$ .)
2. Suppose  $X_n \rightarrow X$  in  $L^1$ , and let  $\{\mathcal{F}_n : n \geq 1\}$  be a filtration. Define  $\mathcal{F}_\infty = \sigma(\cup_{n \geq 1} \mathcal{F}_n)$ . Prove that  $\mathbb{E}[X_n | \mathcal{F}_n] \rightarrow \mathbb{E}[X | \mathcal{F}_\infty]$  a.s. and in  $L^1$ .
3. Consider a box containing a black and a red ball. At each step, we sample a ball uniformly from the box, and return this ball, along with an additional ball of the same color, to the box. Let  $\{A_n : n \geq 1\}$  denote the number of black balls after  $n$ -rounds.
- (i) Prove that  $M_n = \frac{A_n}{n+2}$  is a martingale.
- (ii) Show that  $M_n \rightarrow M_\infty$  a.s. and in  $L^p$  for all  $p \in [1, \infty)$ , but not in  $L^\infty$ .
4. Place  $N$  red points  $R_k$  and  $N$  blue points  $B_k$  uniformly and independently in  $[0, 1]^2$ . Let  $X$  be the length of the minimum matching between red and blue points, i.e. the minimum value of

$$\sum_{k=1}^N \|R_k - B_{\pi(k)}\|$$

across all bijections  $\pi : [k] \rightarrow [k]$ . Show that

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{4N}}.$$

**Optional questions (not graded)**

- Let  $f : \mathbb{Z}^2 \rightarrow [0, 1]$  be such that for all  $a, b \in \mathbb{Z}$ , we have

$$f(a, b) = \frac{f(a+1, b) + f(a-1, b) + f(a, b+1) + f(a, b-1)}{4}.$$

Prove that  $f$  is constant. For an extra challenge, extend the result to higher dimensions.