STAT 212 Problem Set 2.

—

Due: Friday, February 21st at 11:59PM

Instructions: Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

1. Let $\{X_n : n \geq 1\}$ be a sequence of IID integrable random variables. Define

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}.$$

   (a) Prove that the sequence $\{\bar{X}_n : n \geq 1\}$ is a UI backwards martingale relative to some backwards filtration. Hence the limit $\bar{X} = \lim_{n\to\infty} \bar{X}_n$ exists almost surely and in L^1.

   (b) Assuming $X_1 > 0$ almost surely, show that for any $\lambda > 1$,

$$\mathbb{P}[\sup_{k\geq 1} \bar{X}_k \geq \lambda \mathbb{E}[X_1]] \leq 1/\lambda.$$

   (c) Let $\bar{X}'$ be an independent copy of $\bar{X}$. Show that $\frac{\bar{X}+\bar{X}'}{2} \overset{d}{=} \bar{X}$.

   (d) Deduce the strong law of large numbers, i.e. that $\bar{X} = \mathbb{E}[X_1]$ almost surely. (Hint: if $\bar{X}$ is not a.s. constant, show $\mathbb{E}|\frac{\bar{X}+\bar{X}'}{2} - a| < \mathbb{E}|\bar{X} - a|$ for some $a \in \mathbb{R}$.)

2. Suppose $X_n \to X$ in L^1, and let $\{\mathcal{F}_n : n \geq 1\}$ be a filtration. Define $\mathcal{F}_\infty = \sigma(\cup_{n\geq 1}\mathcal{F}_n)$. Prove that $\mathbb{E}[X_n|\mathcal{F}_n] \to \mathbb{E}[X|\mathcal{F}_\infty]$ in L^1.

3. Consider a box containing a black and a red ball. At each step, we sample a ball uniformly from the box, and return this ball, along with an additional ball of the same color, to the box. Let $\{A_n : n \geq 1\}$ denote the number of black balls after n-rounds.

   (i) Prove that $M_n = \frac{A_n}{n+2}$ is a martingale.

   (ii) Show that $M_n \to M_\infty$ a.s. and in L^p for all $p \in [1, \infty)$, but not in L^∞.

4. Place N red points R_k and N blue points B_k uniformly and independently in $[0, 1]^2$. Let X be the length of the minimum matching between red and blue points, i.e. the minimum value of

$$\sum_{k=1}^{N} \|R_k - B_{\pi(k)}\|$$

   across all bijections $\pi : [N] \to [N]$. Show that

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{4N}}.$$

Optional questions (not graded)

- Let $f : \mathbb{Z}^2 \to [0, 1]$ be such that for all $a, b \in \mathbb{Z}$, we have

$$f(a,b) = \frac{f(a+1,b) + f(a-1,b) + f(a,b+1) + f(a,b-1)}{4}.$$

  Prove that f is constant. For an extra challenge, extend the result to higher dimensions.