

Problem Set #1

1. Evaluate $\int_{-2}^{-1} \frac{dx}{\sqrt{-x^2 - 6x}}$.
 - (a) $\arcsin(2/3) - \arcsin(1/3)$
 - (b) $\arccos(2/3) - \arccos(1/3)$
 - (c) $\arctan(2/3) - \arctan(1/3)$
 - (d) $\arctan(2/3) + \arctan(1/3)$
 - (e) $\arccos(2/3) + \arccos(1/3)$
2. Let $f(x) = x^3 + 6x^2 - 32$. Define $T_{c,f}(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ to be the Taylor series centered at c of f . For what values of c does $a_0 = 0$?
 - (a) $c = 3, 2$
 - (b) $c = -3, 2, -2$
 - (c) $c = 4, 2$
 - (d) $c = -4, 2, -2$
 - (e) $c = -4, 2$
3. Suppose that X and Y are independent random variables with standard deviations σ_X and σ_Y , respectively. If $Z = X + Y$, then $\sigma_Z =$
 - (a) $\sigma_X + \sigma_Y$
 - (b) $\sigma_X^2 + \sigma_Y^2$
 - (c) $(\sigma_X^2 + \sigma_Y^2)^{1/2}$
 - (d) $2\sigma_X + 2\sigma_Y$
 - (e) $\sigma_X\sigma_Y$
4. Let $S = \{a, b, c, d\}$ and $X = \{f : S \rightarrow S \mid f \text{ is bijective and } f(x) \neq x \text{ for each } x \in S\}$. Then $|X| =$
 - (a) 8
 - (b) 9
 - (c) 12
 - (d) 18
 - (e) 20
5. $\int_0^1 \sqrt{e^{2x} + e^{-2x} + 2} =$
 - (a) $e - \frac{1}{e}$
 - (b) $e + \frac{1}{e}$
 - (c) $e - \frac{1}{e} + 1$

(d) $e + \frac{1}{e} - 1$

(e) $2e$

6. Which of the following graphs has two connected components?

(a) $\frac{x^2}{5} + y^2 - \frac{z^2}{2} = 1$

(b) $\frac{x^2}{5} - \frac{y^2}{5} - \frac{z^2}{5} = 1$

(c) $x^2 - 2x + y^2 + z^2 = 1$

(d) $x = y^2 - z^2$

(e) $z = x - y + 2$

7. The function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Which of the following is true?

(I) f is not continuous at $(0, 0)$.

(II) f is differentiable everywhere

(III) f as a well defined partial derivatives everywhere (i.e. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are both defined)

(IV) f is continuous at $(0, 0)$ but not differentiable at $(0, 0)$.

(a) I only

(b) II only

(c) III only

(d) IV only

(e) I and III only

8. Evaluate the limit: $\lim_{n \rightarrow \infty} (3^n + 5^n)^{2/n}$.

(a) 0

(b) 1

(c) 9

(d) 16

(e) 25

9. Let T be a linear operator on \mathbb{R}^3 equipped with the standard inner product. Which of the following must be true?

(I) If W is subspace such that $T(W) \subset W$, then $W^\perp = \{v \in V \mid \langle w, v \rangle = 0, v \in W\}$ also has the property that $T(W^\perp) \subset W^\perp$.

(II) If T is self-adjoint, then T^{-1} exists and is diagonalizable.

(III) If $T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 5 & 3 \\ -1 & 3 & -1 \end{pmatrix}$, then \mathbb{R}^3 has an orthonormal basis of eigenvectors for T .

(IV) If T^2 is diagonalizable, so is T .

- (a) I only
- (b) II only
- (c) III only
- (d) III and IV
- (e) None are true

10. Evaluate: $\int_0^{\pi/2} \frac{1}{1+\tan x} dx$ (TRICKS INVOLVED!)

- (a) $\frac{\pi}{2}$
- (b) $\frac{\pi}{3}$
- (c) $\frac{\pi}{4}$
- (d) $\frac{\pi^2}{2}$
- (e) $\frac{\pi^2}{4}$

Problem Set #2

1. The series $1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \dots =$

- (a) $\frac{\pi}{6}$
- (b) $\frac{\pi}{5}$
- (c) $\frac{\pi}{4}$
- (d) $\frac{\pi^2}{2}$

(Hint: Start playing around with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$).

2. Which of the following are true of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x+1) = f(x)$ for all x .

- (I) f achieves it's maximum and minimum
- (II) f is uniformly continuous.
- (III) There is a x_0 such that $f(x_0 + \pi) = f(x_0)$
- (IV) There is a x_1 such that $f(x_1 + \frac{1}{2}) = f(x_1)$.

- (a) I only
- (b) I, II only
- (c) I, II, III only

- (d) I, II, III, IV
3. Let S be a set with $|S| \geq 3$. How many non-surjective functions from $S \rightarrow \{1, 2, 3\}$ are there?
- (a) $3 \cdot 2^{|S|}$
 - (b) $3 \cdot 2^{|S|} - 3$
 - (c) $3^{|S|} - 3$
 - (d) $|S|^3$
 - (e) $2|S|^2 - 3$
4. Let X and Y be random variables. Which of the following is always true?
- (I) $E(X + Y) = E(X) + E(Y)$
 - (II) $\text{Var}(X) = \text{Var}(-X)$
 - (III) $P(\max(X, Y) < 2) = P(X < 2)P(Y < 2)$
 - (IV) $\text{var}(X + Y) > \text{Var}(X) + \text{Var}(Y)$
- (a) I only
 - (b) I and II only
 - (c) I, II, and III only
 - (d) I and IV only
 - (e) None are true.
5. Which of the following exist:
- (I) $f_1 : [0, 1] \rightarrow (0, 1)$ continuous and surjective.
 - (II) $f_2 : (0, 1) \rightarrow [0, 1]$ continuous and surjective
 - (III) $f_3 : (0, 1) \rightarrow [0, 1]$ continuous and bijective
- (a) I only
 - (b) II only
 - (c) II and III only
 - (d) III only
 - (e) None exist.
6. The angle between the tangent plane $x^2 + y^2 + 2z^2 = 4$ at $(1, 1, 1)$ and the xy -plane is:
- (a) $\arccos\left(\frac{\sqrt{6}}{6}\right)$
 - (b) $\arccos\left(\frac{\sqrt{6}}{2}\right)$
 - (c) $\arccos\left(\frac{\sqrt{6}}{3}\right)$

- (d) $\frac{\pi}{4}$
- (e) $\frac{\pi}{2}$

7. Which of the following are subrings of polynomials with coefficients in \mathbb{Z} ?

- (I) Even degree polynomials.
- (II) Polynomials of degree less than or equal to 3.
- (III) Polynomials all of whose coefficients are even.

- (a) I, II, and III
- (b) I and III only
- (c) II only
- (d) III only
- (e) None of the above

8. Consider the graph $y = e^x$. Rotating this graph about the x -axis for $1 \leq x \leq 2$, what is the area of this surface of revolution? (WARNING: pretty long computation)

- (a) $\pi \cdot \left[e^2 \sqrt{1 + e^4} - e \sqrt{1 + e^2} + \ln \left(\frac{\sqrt{e^4 + 1} + e^2}{\sqrt{e^2 + 1} + e} \right) \right]$.
- (b) $\pi \cdot \left[e^2 \sqrt{1 + e^4} - e \sqrt{1 + e^2} - \ln \left(\frac{\sqrt{e^4 + 1} + e^2}{\sqrt{e^2 + 1} + e} \right) \right]$
- (c) $\pi \cdot \left[e \sqrt{1 + e^2} - \sqrt{1 + e} - \ln \left(\frac{\sqrt{e^2 + 1} + e}{\sqrt{e^2 + 1} + 1} \right) \right]$
- (d) $2\pi\sqrt{2}$
- (e) $\pi \cdot \left[e \sqrt{1 + e^2} - \sqrt{1 + e} + \ln \left(\frac{\sqrt{e^2 + 1} + e}{\sqrt{e^2 + 1} + 1} \right) \right]$

9. For how many values of the parameter k , will the following system have no solutions?

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

- (a) 0
- (b) 1
- (c) 2
- (d) 4
- (e) Infinitely many

10. Evaluate $\int_0^{\pi/2} \frac{1}{1 + (\tan x)^e} dx$

- (a) $\frac{\pi}{2}$

- (b) $\frac{\pi}{3}$
- (c) $\frac{\pi}{4}$
- (d) $\frac{\pi^2}{2}$
- (e) $\frac{\pi^2}{4}$

Problem Set #3

1. Which of the following are compact sets?

- (I) $GL(2, \mathbb{R}) \subset \mathbb{R}^4$, the set of invertible 2×2 matrices over \mathbb{R} .
- (II) $SL(2, \mathbb{R}) \subset \mathbb{R}^4$, the set of 2×2 matrices over \mathbb{R} with determinant 1.
- (III) $O(2, \mathbb{R}) \subset \mathbb{R}^4$, the set of orthonormal 2×2 matrices over \mathbb{R}

- (a) I, II, III
- (b) I only
- (c) II only
- (d) III only
- (e) II and III only

2. For which values of a does $\sum_{n=1}^{\infty} \frac{(\ln n)^{2014}}{n^a}$ does the series converge?

- (a) $a = -2014$
- (b) $1 > a > 0$
- (c) $a > 1$
- (d) $a \neq 1$ and $a > 0$
- (e) For no values of a does the series converge.

3. Which set does not have the same cardinality as the others?

- (a) $\{f : \mathbb{N} \rightarrow \mathbb{N}\}$
- (b) \mathbb{R}
- (c) $\{f : \{0, 1, \dots, 100\} \rightarrow \mathbb{R}\}$
- (d) $\{\text{solutions to polynomial equations with coefficients in } \mathbb{Z}\}$.
- (e) $\{f : \mathbb{Z} \rightarrow \{0, 1\}\}$

4. The series $\sum_{n \geq 1} (-1)^n \sin\left(\frac{\pi}{n}\right)$:

- (a) converges absolutely.
- (b) converges conditionally.
- (c) diverges to infinity.

- (d) does not approach any value in the extended real number system.
5. $\int \ln(\sqrt[3]{x})dx =$
- (a) $\frac{x}{3} \ln x + \frac{x}{3} + C$
 - (b) $\frac{x}{3} \ln x - \frac{x}{3} + C$
 - (c) $\frac{x}{3} \ln x + \frac{x}{2} + C$
 - (d) $x^3 \ln x - x + C$
 - (e) $x^3 \ln x - \frac{x}{3} + C$
6. Suppose that $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ is a curve given by $t \mapsto \vec{r}(t)$ and the dot product of the curve with its derivative is greater than zero (i.e. $\vec{r} \cdot \vec{r}' > 0$). Which of the following can we conclude?
- (a) $|\vec{r}|$ is increasing in t .
 - (b) $|\vec{r}'|$ is increasing in t .
 - (c) $|\vec{r}''|$ is increasing in t .
 - (d) None of the above
7. Let $f(x) = x^5 + 3x^3 - 1$. Which of the following is true?
- (a) f has two real zeros.
 - (b) $\lim_{x \rightarrow \infty} (f^{-1})(x) = \infty$
 - (c) f has only rational roots
 - (d) f has a local minimum
 - (e) All of the above
8. Evaluate the following limit:
- $$\lim_{x \rightarrow 0} \left[\frac{1}{x^2} \int_0^x \frac{t + t^2}{1 + \sin t} dt \right]$$
- (a) 1
 - (b) $\frac{1}{2}$
 - (c) $\frac{1}{4}$
 - (d) $-\frac{1}{2}$
 - (e) -1
9. Let $S = \{1, 2, 3, 4\}$ and $X = \{f : S \rightarrow S \mid x < y \implies f(x) \leq f(y)\}$. $|X| =$
- (a) 18
 - (b) 20
 - (c) 24
 - (d) 35
 - (e) 40

10. Suppose y_n and x_n satisfy:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

with $x_0 = y_0 = 1$. What is $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$ assuming this limit exists?

- (a) ∞
- (b) $1 - \sqrt{2}$
- (c) $1 + \sqrt{2}$
- (d) $\frac{1+\sqrt{5}}{2}$
- (e) $\frac{1-\sqrt{5}}{2}$

Problem Set #4

1. Let G be a finite group. Which of the following never happen?

- (I) A subgroup of G with index 2 that is not normal.
- (II) All the conjugacy classes of G have size 1.
- (III) G has no proper subgroups other than the subgroup containing only the identity.

- (a) I only
- (b) II only
- (c) III only
- (d) I and II only
- (e) I and III only

2. Let $(x(t), y(t))$ be a parametric curve in \mathbb{R}^2 be given by:

$$\begin{aligned} x(t) &= e^t \\ y(t) &= \sin(t) \end{aligned}$$

What is $\frac{d^2y}{dx^2}$ as a function of t ?

- (a) $\frac{-\sin t}{e^t}$
- (b) $\frac{\cos t - e^t \sin t}{e^t}$
- (c) $\frac{-\sin t - \cos t}{e^t}$
- (d) $\frac{\sin t + \cos t}{e^{2t}}$
- (e) $\frac{-\sin t - \cos t}{e^{2t}}$

3. Which of the following are true about subgroups of $(\mathbb{Q}, +)$

- (I) If a subgroup is finitely generated, then it is cyclic.
(II) There non-cyclic subgroups.
(III) There are no finite subgroups other than the trivial one.
- (a) I only
(b) II only
(c) II and III only
(d) I and III only
(e) All are true
4. Suppose $y(t) = y$ solves $y' = (y^2 - 1)e^{2012y-1}$ with initial condition $y(0) = 0$, then which of the following are true?
- (a) $\lim_{t \rightarrow \infty} y(t) = \infty$
(b) $\lim_{t \rightarrow \infty} y(t) = 1$
(c) $\lim_{t \rightarrow \infty} y(t) = -1$
(d) $-1 < y(t) < 1$ for all t
(e) Both C and D
5. Evaluate $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$ converge:
- (a) 1
(b) $\frac{3}{2}$
(c) 2
(d) $\frac{5}{2}$
(e) 3
6. Suppose y_1 and y_2 are solutions to the differential equation

$$y'' - 2y' + y = f(t)$$

Which of the following is true:

- (I) If $y_1(0) = y_2(0)$, then $y_1 \equiv y_2$
(II) If $a + b = 1$, then $ay_1 + by_2$ is another solution to the differential equation.
(III) If $f(t) = 0$, then the general solutions is $C_1e^t + C_2te^t$
- (a) I, II, III
(b) III only
(c) II only
(d) II and III only
(e) I and III only

7. 2. A tank contains 50 gal of pure water. Starting at time zero, a salt-water solution containing 0.4 lb/gal of salt enters the tank at a rate of 2 gal/min. At the same time a drain is opened at the bottom of the tank allowing the salt solution to leave at a rate of 2 gal/min. Assume that the solution is kept perfectly mixed at all times. How long will it take for the amount of salt (in lbs) in the tank to reach 10 lbs? Leave your answer in exact form (in terms of logarithms).

- (a) $2 \ln 25$
- (b) $-2 \ln \left(\frac{1}{2}\right)$
- (c) $25 \ln \left(\frac{1}{2}\right)$
- (d) $25 \ln 2$
- (e) $25 \ln 10$

8. Evaluate $\int_0^{2\pi} e^{e^{i\theta}} d\theta$.

- (a) $2\pi i$
- (b) 2π
- (c) $4\pi i$
- (d) $4\pi e$
- (e) $-2\pi i$

9. Let $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ be given by $F(t) = \int_{-\infty}^{\infty} e^{-tx^2} dx$. Given $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, then $F'(t) =$

- (a) $\frac{\sqrt{\pi}}{2t^{3/2}}$
- (b) $-\frac{\sqrt{\pi}}{t^{3/2}}$
- (c) $-\frac{\sqrt{\pi}}{t^{1/2}}$
- (d) $-\frac{\sqrt{\pi}}{2t^{3/2}}$
- (e) $\frac{\sqrt{\pi}}{t^{1/2}}$

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable, then:

$$\lim_{\varepsilon \rightarrow 0} \frac{3}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t f(x+t) dt =$$

- (a) $\frac{1}{3} f(x)$
- (b) $\frac{1}{2} f'(x)$
- (c) $\frac{1}{2} f''(x)$
- (d) $2f'(x)$
- (e) $3f''(x)$

Problem Set #5

- Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and define $\|A\|_{op} := \sup_{\|v\|_{\mathbb{R}^2}=1} \{\|Av\|_{\mathbb{R}^2}\}$. There is a vector v such that $\|A\|_{op} = \|Av\|_{\mathbb{R}^2}$. Also, note that once we find one v that maximizes the norm $\|*\|_{op}$, $-v$ also does. Find the angle v makes with the positive x -axis assuming v is to the right of the y -axis:
 - Arctan2
 - $\frac{\text{Arctan } 2}{2}$
 - $\frac{3\pi}{4}$
 - $\frac{\pi}{4}$
 - $-\frac{\pi}{4}$
- Which of the following is the largest:
 - e^π
 - π^e
 - 3^π
 - Not enough information.
- Let $F(x) = \int_{\sin x}^{\cos x} e^{t^2+xt} dt$. $F'(0) =$
 - $e \cos 1 - 2$
 - $-2 + e$
 - $\frac{e}{2} - \frac{3}{2}$
 - $\frac{e}{2} + \frac{1}{2}$
 - $\frac{e}{2} - \frac{1}{2}$
- Which functions are entire:
 - $f(z) = 2^{z^2}$
 - $g(z) = z^{2z} \sin z$
 - $h(z) = \sum_{i=0}^{\infty} z^i$
 - I, II only
 - I only
 - I and III only
 - II only
 - II and III only
- Which of the following holomorphic functions must be constant functions:
 - An entire function with positive real part.

- (II) An entire function whose image does not intersect a line $L \subset \mathbb{C}$.
- (III) An entire function such that $f(0) = 0$ and $f(z)/z \rightarrow 0$ as $|z| \rightarrow \infty$. (Hint: $f(z)/z$ has a removable singularity at 0, so can be extended to an entire function).
- (IV) A holomorphic function whose range misses two points.
- (a) I, II, III, IV only
 (b) II, III, IV only
 (c) I, II only
 (d) III only
 (e) I, II, III only
6. A basketball player decides to practice his free throws in a gym by promising to not stop until he makes 20 free throws (not necessarily consecutively). If his free-throw percentage is 80 percent. What's the probability that he needs exactly 50 tries to complete his goal?
- (a) $\binom{50}{20} (.8)^{20} (.2)^{30}$
 (b) $(.8)^{20} (.2)^{30}$
 (c) $\binom{50}{20} (20 \cdot .8)(30 \cdot .2)$
 (d) $\binom{49}{19} (.8)^{20} (.2)^{30}$
 (e) $(20 \cdot .8)(30 \cdot .2)$
7. Let R be a ring. Which of the following are always true?
- (I) The set of idempotents $\{x \in R \mid x^2 = x\}$ is a subring.
 (II) If $x^2 = x$ for all $x \in R$, then R is commutative.
 (III) The only idempotents of an integral domain R are 0 and 1.
- (a) I only
 (b) II only
 (c) III only
 (d) II and III only
 (e) None are true.
8. What is the characteristic polynomial of an $n \times n$ matrix all of whose entries are 1?
- (a) $x(x-1)(x-2)\dots(x-n)$
 (b) $x^{n-1}(x-n)$
 (c) $x^{n-1}(x-1)$
 (d) $x(x+1)(x+2)\dots(x+n)$

(e) $x^{n-2}(x-1)(x+1)$

9.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \dots + \frac{1}{2n} \right) =$$

(a) 1

(b) $\frac{3}{4}$

(c) $\frac{1}{2}$

(d) $\ln 2$

(e) $\ln 3$

10. How many times do the hands of a analog clock (i.e. one with a hour and minute hand) form perpendicular angles in a 24 hour time cycle?

(a) 48

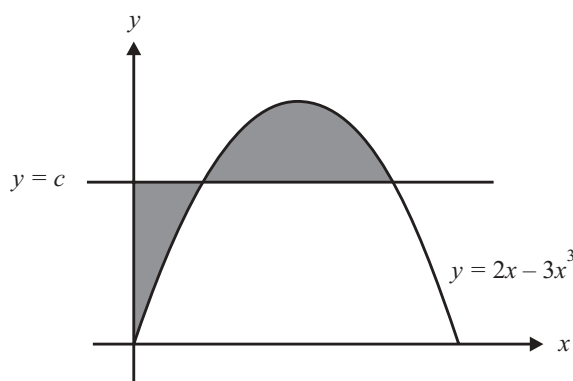
(b) 47

(c) 46

(d) 45

(e) 44

Problem Set #6



1.

The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the the area of the two shaded regions are equal.

(a) $\frac{2}{9}$

(b) $\frac{1}{3}$

(c) $\frac{4}{9}$

(d) $\frac{1}{2}$

(e) $\frac{5}{9}$

2. Determine which of the following statements are true:

(I) There is a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that:

$$\text{grad } f = \langle y, x + z \cos y, \sin y \rangle$$

(II) There is a vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that:

$$\text{curl } F = \langle z, y, x \rangle$$

(III) If $u, v : \mathbb{R} \rightarrow \mathbb{R}^3$ satisfy

$$u' = v - u$$

$$v' = v + u$$

then $u \times v$ is constant.

- (a) I, II, and III
- (b) II and III only
- (c) I and III only
- (d) II and III only
- (e) I only

3. Suppose that A and B are 3×3 invertible matrices. Which of the following are always true?

- (I) The eigenvalues of AB are the same as BA .
- (II) The eigenvectors of AB are the same as BA .
- (III) If A and B both only have 3 as an eigenvalue, then they are similar.

- (a) I only
- (b) II only
- (c) III only
- (d) I and III only
- (e) None of them are true

4. Find the volume enclosed by $(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2)$.

- (a) $\frac{\pi}{40}$
- (b) $\frac{3\pi}{40}$
- (c) $\frac{\pi}{15}$
- (d) $\frac{2\pi}{15}$
- (e) $\frac{\pi}{2}$

5. Suppose you have two cards: one is painted black on both sides and the other is painted black on one side and orange on the other. You select a card at random and view one side. You notice it is black. What is the probability the other side is orange?
- (a) $\frac{1}{2}$
 - (b) $\frac{1}{3}$
 - (c) $\frac{1}{4}$
 - (d) $\frac{1}{5}$
 - (e) 1
6. Suppose that $A^n = \text{Id}$ for some $n \in \mathbb{Z}$ where A is an $m \times m$ matrix over \mathbb{C} and Id is the $m \times m$ identity matrix. Which of the following are true:
- (I) A can be diagonalized
 - (II) If λ is an eigenvalue that is not real, then $1 + \lambda + \dots + \lambda^{n-1} = 0$.
 - (III) The characteristic polynomial has coefficients in \mathbb{R} .
- (a) I only
 - (b) II only
 - (c) III only
 - (d) I and II only
 - (e) I, II, and III
7. For a continuously differentiable vector field $\vec{F} : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which of the following are true?
- (a) $\text{curl } \vec{F} = 0$ implies \vec{F} is conservative.
 - (b) $\text{curl } \vec{F} = \vec{0}$ and $\text{div } F = 0$ implies \vec{F} is conservative
 - (c) $\text{curl } \vec{F} = \vec{0}$ and D is simply connected imply \vec{F} is conservative.
 - (d) $\text{div } \vec{F} = \vec{0}$ and D is simply connected imply \vec{F} is conservative.
 - (e) No statement is correct
8. A right circular cone has a base of radius 1 and height 3. A cube is inscribed in the cone so that one face is contained in the base. What is the length of the cube? (Hint: Cut the cone along the diagonal of the cube and look at the planar figure)
- (a) 2
 - (b) $\frac{3\sqrt{2}-5}{7}$
 - (c) $\frac{3\sqrt{2}-5}{7}$
 - (d) $\frac{9\sqrt{2}-6}{7}$
 - (e) $\frac{9\sqrt{2}+6}{7}$

9. Assume $b > a > 0$. Calculate the integral:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$$

(Hint: Write the integrand as an integral from a to b)

- (a) $\ln a - \ln b^2$
 - (b) $\ln b - \ln a^2$
 - (c) $\ln a^2 - \ln b^2$
 - (d) $\ln a - \ln b$
 - (e) $\ln b - \ln a$
10. 100 people have bought tickets for the sold-out showing of Casablanca playing at the NuArt. The theater has precisely the right amount of seats for this size audience, and so to prevent any confusion, has assigned each person a seat. The people arrive one by one and respectfully sit in the seat they have been assigned. One complication: the first person to arrive is a drunkard who has lost his ticket and does not know his assigned seat. He randomly chooses one. As the 99 other people arrive, they may face the uncomfortable fact that their assigned seat is already filled. If it is, they avoid confrontation and randomly select an open seat. Finally the last person arrives. What is the probability he sits in his assigned seat?
- (a) $1/100$
 - (b) $2/77$
 - (c) $1/2$
 - (d) $99/100$
 - (e) $3/4$

Problem Set #7

1. Which of the following apply to the vector field $\vec{F} = \langle (x^2 - y^2), -2xy, z^2 \rangle$ defined on all of \mathbb{R}^2 .
- (a) $\text{div } \vec{F} = 0$
 - (b) Conservative and curl vanishes.
 - (c) Conservative, but curl does not vanish
 - (d) Curl vanishes, but not conservative
2. Suppose you wish to make a bouquet of 10 flowers and give them to Professor Bertozzi. You go to the flower store and there are 4 types of flowers you think fit would fit her personality. How many bouquets can you make using only these flowers?
- (a) 4^{10}

(b) 40

(c) $\binom{9}{4}^{10}$

(d) $\binom{13}{3}$

(e) $\binom{13}{9}$

3. Which of the following matrices is not diagonalizable over \mathbb{C} ?

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$

4. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and that u is related to t and x by the equation:

$$u = f(x - ut)$$

Which of the following differential equations does u satisfy?

(a) $u_t - uu_x = 0$

(b) $u_x - u^2 u_t = 0$

(c) $u_x + t u_t = 0$

(d) $u_t + u_x (u_t)^2 = 0$

(e) $u_t + uu_x = 0$

5. Find the point on the plane $x + 2y + 3z = 14$ closest to the origin.

(a) $(7, 2, 1)$

(b) $(2, 2, \frac{8}{3})$

(c) $(-3, -2, 7)$

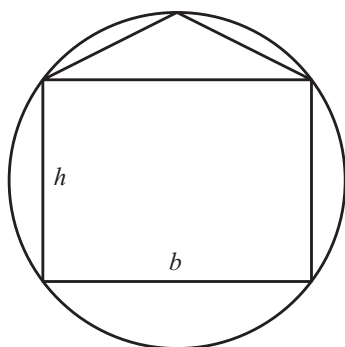
(d) $(14, 0, 0)$

(e) $(1, 2, 3)$

6. A bead slides down the curve $xy = 10$. Find the bead's horizontal velocity at time $t = 2$ if its height at time t seconds is $y = 400 - 16t^2$.

- (a) $\frac{14}{9}$
- (b) $\frac{5}{3}$
- (c) $\frac{16}{9}$
- (d) $\frac{17}{9}$
- (e) 2

7. Inscribe a rectangle and isosceles triangle as shown below in a circle of radius one. For what values of h do the triangle and rectangle have the same area?



- (a) $\frac{1}{5}$
- (b) $\frac{2}{5}$
- (c) $\frac{3}{5}$
- (d) $\frac{4}{5}$
- (e) 1

8. What is the last digit of $12^{12^{12^{12}}}$?

- (a) 2
- (b) 4
- (c) 6
- (d) 8
- (e) 0

9. Evaluate the integral $\int_0^1 \int_0^1 e^{\max(x^2, y^2)} dx dy$ where the “max” of two real numbers is their maximum.

- (a) $\frac{1}{2}(e + 1)$
- (b) $e + 1$
- (c) $1 - e$
- (d) $\frac{1}{2}(e - 1)$

(e) $e - 1$

10. Let

$$f(x) = g(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and

$$h(y) = \int_{-\infty}^{\infty} f(y-x)g(x)dx.$$

If $y < 0$, then $h(y) =$

- (a) $4ye^{-2y}$
- (b) $4y^2e^{-2y}$
- (c) $2y^2e^{-2y}$
- (d) $2ye^{-2y}$
- (e) 0

Problem Set #8

1. Let $u, v, w \in \mathbb{R}^3$ such that:

- (i) $u \times v \neq 0$
- (ii) $u \times w \neq 0$
- (iii) $(u \times v) \cdot (u \times w) = 0$

Which of the following is true?

- (a) $v \perp w$
- (b) $u \perp w$ and $u \perp v$
- (c) u, v , and w lie in the same plane
- (d) The plane spanned by u and v is orthogonal to the plane spanned by u and w
- (e) None of the above.

2. If $x^2 + x = 3$, then $x^4 + x =$

- (a) $x^3 + 3$
- (b) $4(3 - x)$
- (c) $3(x - 1)$
- (d) $6(2 - x)$
- (e) $3x$

3. $\frac{d}{dx} \Big|_{x=e} \ln(\ln x^{x^2+x}) =$

- (a) $\frac{3e-1}{e^2+e}$
(b) $\frac{3e+2}{e^2}$
(c) $\frac{2e+1}{e^2+e}$
(d) $\frac{e+2}{e^2+e}$
(e) $\frac{3e+2}{e^2+e}$
4. At how many points does the graph $y = x^5 + x^3 + x - 2000$ cross the x -axis.
- (a) 1
(b) 2
(c) 3
(d) 4
(e) 5
5. Let C be the circle in the yz -plane whose equation is $(y - 3)^2 + z^2 = 1$. If C is revolved around the z -axis, the surface generated is a torus. What is the equation of the torus.
- (a) $x^2 + y^2 + z^2 = 6y$
(b) $(x^2 + y^2 + z^2)^2 = 8 + 36(x^2 + z^2)$
(c) $(x^2 + y^2 + z^2 + 8)^2 = 36(x^2 + z^2)$
(d) $(x^2 + y^2 + z^2 + 8)^2 = 36(x^2 + y^2)$
(e) $(x^2 + y^2 + z^2 + 8)^2 = 36(y^2 + z^2)$
6. How many distinct abelian groups have order 96?
- (a) 4
(b) 5
(c) 6
(d) 7
(e) 8
7. If x is a real number and P is a polynomial function, then
- $$\lim_{h \rightarrow 0} \frac{P(x + 3h) + P(x - 3h) - 2P(x)}{h^2} =$$
- (a) 0
(b) $6P'(x)$
(c) $3P''(x)$
(d) $9P''(x)$
(e) ∞

8. If C is the circle in the complex plane whose equation is $|z| = \pi$, oriented counterclockwise, find the value of the integral:

$$\oint_C \left(\cos z - z \cos \frac{1}{z} \right) dz$$

- (a) $-2\pi i$
 - (b) $-\pi i$
 - (c) 0
 - (d) πi
 - (e) $2\pi i$
9. Let R be the rectangle bounded by $x - y = 0$, $x - y = 2$, $x + y = 0$ and $x + y = 3$. Evaluate

$$\iint_R (x + y)e^{x^2 - y^2} dA$$

- (a) $e^6 - 7$
 - (b) $e^6 + 7$
 - (c) $\frac{1}{2}(e^6 - 7)$
 - (d) $\frac{1}{4}(e^6 - 7)$
 - (e) $\frac{1}{2}(e^6 + 7)$
10. How many binary digits are between 101110 and 11 (inclusive of the endpoints)?
- (a) 41
 - (b) 42
 - (c) 43
 - (d) 44
 - (e) 45

Problem Set #9

1. The function $f(x, y) = x^2 + 3xy + y^2 + y^4$:
- (a) has a global minimum at $(0, 0)$
 - (b) has a local minimum at $(0, 0)$, but not a global minimum
 - (c) has a local max at $(0, 0)$
 - (d) has a saddle at $(0, 0)$
 - (e) has countably many local minima

2. Find the smallest degree polynomial $p(x)$ such that:

$$\begin{aligned}p(0) &= -1 \\p(-1) &= -2 \\p(1) &= 4\end{aligned}$$

- (a) $p(x) = x - 1$
 - (b) $p(x) = 3x + 1$
 - (c) $p(x) = 2x^2 - 3x - 1$
 - (d) $p(x) = -2x^2 + 3x - 1$
 - (e) $p(x) = 2x^2 + 3x - 1$
3. Let C be the boundary, oriented counterclockwise, of the triangle in the (x, y) -plane whose vertices are $(0, 1)$, $(1, 1)$ and $(1, 3)$. What is the value of the following integral?

$$\oint_C (\sqrt{2x+1} - y)dx + (2x + \sqrt{y^2 - y}) dy$$

- (a) 1
 - (b) 2
 - (c) 3
 - (d) $2\sqrt{2} - 3$
 - (e) $2(\sqrt{2} - 1)$
4. Which of the following statements are true about sequences of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly on a set E ?

- (I) $\{f_n + g_n\}$ converge uniformly on E
- (II) $\{f_n g_n\}$ converge uniformly on E
- (III) $\{\sin f_n\}$ converge uniformly on E

- (a) I only
 - (b) I and III only
 - (c) II and III only
 - (d) I and IV only
 - (e) III and IV only
5. Find the smallest positive $a \in \mathbb{N}$ such that there is a $b \in \mathbb{Z}$ with $13a + 35b = 1$.
- (a) 8
 - (b) 3
 - (c) 27
 - (d) 22

(e) 10

6. If $w = f(x, y)$ is a solution of the partial differential equation:

$$2\frac{\partial w}{\partial x} - 3\frac{\partial w}{\partial y} = 0$$

then w could equal:

(a) $\log(\log(2x - 3y))^6$

(b) $\sin(3x - 2y)$

(c) $e^{\arctan(3x+2y)}$

(d) $(\arccos(2y - 3x))^2$

(e) $\sqrt{2x + 3y}$

7. Up to isomorphism, how many additive abelian groups G of order 16 have the property that $x + x + x + x = 0$ for each $x \in G$?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

8. $\sum_{n=1}^{\infty} (-1)^n (1 - \cos(\frac{1}{n}))$ is a(n)

(a) conditionally convergent series

(b) absolutely convergent series

(c) divergent series

(d) cannot be determined

9. Suppose that f is differentiable function such that $f(x) \geq 0$ and $\int_0^{\infty} f(x) < \infty$. Which of the following statements must always be true?

(a) $\lim_{x \rightarrow \infty} f(x) = 0$

(b) $\lim_{x \rightarrow \infty} f'(x) = 0$

(c) For every $\varepsilon > 0$, there is an M such that $\int_{x>M} f(x)dx < \varepsilon$

(d) f is bounded

(e) A, C, and D

10. Evaluate $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right)$

(a) $\pi/4$

(b) $\pi/3$

(c) $\pi/2$

(d) $3\pi/3$

(e) $\frac{\sqrt{2}}{2}$

Problem Set #10

1. Which of the following statements are false?

- (I) The set of holomorphic functions are dense in the set of continuous on $D = \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| \leq 1\}$ functions with respect to the metric $d(f, g) = \sup_{\vec{x} \in D} |f(\vec{x}) - g(\vec{x})|$.
- (II) If $\{f_n\}$ converges pointwise on a compact set, then it also converges uniformly.
- (III) Let $\{f_n\}$ is a sequence of infinitely differentiable functions on $[0, 1]$. If there is a function g such that $f'_n \rightarrow g$ uniformly and $f_n(0) = 1$, then there is a function G such that $G(0) = 1$ and $f_n \rightarrow G$ uniformly.

- (a) I only
- (b) II only
- (c) III only
- (d) I and III only
- (e) II and III only

2. Let A be a real 2×2 matrix. Which of the following statements must be true?

- (I) All of the entries of A^2 are nonnegative
- (II) The determinant of A^2 is nonnegative
- (III) If A has two distinct eigenvalues, then A^2 has two distinct eigenvalues.

- (a) I, II, and III
- (b) II and III only
- (c) III only
- (d) II only
- (e) I and II only

3. Consider the initial value problem:

$$\begin{cases} y' = y^{1/2} \\ y(0) = a \end{cases}$$

where a is a constant. Which of the following is true?

- (I) If $a = 0$, the unique solution to the above is $y(t) = 0$.
- (II) If $a = 0$, there are exactly 2 solutions.
- (III) If $a = 0$, there are infinitely many solutions.
- (IV) If $a \neq 0$, there is a unique solution.

- (a) I and IV only
- (b) II and IV only

- (c) III and IV only
- (d) III only
- (e) I only

4. Evaluate:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n}}{(2n+1)!}$$

- (a) $\frac{1}{2} \cos 2$
 - (b) $\sin^2 1$
 - (c) $\frac{1}{2} \sin 1$
 - (d) $\sin 2$
 - (e) $\cos 1 \sin 1$
5. Let C_1, C_2 be the circles of radius 1 centered at $(1, 0)$ and $(0, 1)$ respectively. Find the area on which the interiors of the circles overlap.
- (a) $4 - 2\sqrt{2}$
 - (b) $\frac{\pi+1}{2}$
 - (c) $\frac{\pi-2}{2}$
 - (d) $\frac{\pi+8}{2}$
 - (e) $\frac{\pi+2}{4}$
6. Consider a hemisphere of radius R . Consider the portion of sphere that lies height h above it's center. What is the surface area of this portion?
- (a) $2\pi R^{1/2}(R-h)^{3/2}$
 - (b) $2\pi R(R-h)$
 - (c) $2\pi R(R^2-h^2)^{1/2}$
 - (d) $3\pi Rh$
 - (e) $\pi R(R-h)$
7. Let X and Y be independent uniformly distributed random variables on $[0, 1]$. $P(X^2 \leq Y) =$
- (a) $\frac{1}{4}$
 - (b) $\frac{1}{3}$
 - (c) $\frac{1}{2}$
 - (d) $\frac{2}{3}$
 - (e) $\frac{3}{4}$
8. Evaluate: $\lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right)^{1/x^2}$.

- (a) Does not exist
 - (b) $\frac{1}{3}$
 - (c) $\frac{e}{2}$
 - (d) $e^{-\frac{1}{3}}$
 - (e) $e^{-\frac{1}{6}}$
9. The radius of convergence of $\sum_{n=0}^{\infty} \frac{n!}{2^{n^2}} x^n$ is:
- (a) 0
 - (b) 1
 - (c) 3
 - (d) e^2
 - (e) ∞
10. How many zeros does the function $f(x) = 3 \sin(100x) + 1$ have on $[0, 1]$?
- (a) 30
 - (b) 31
 - (c) 32
 - (d) 33
 - (e) 34

Linear Algebra #1

1. If the matrices

$$\begin{pmatrix} 3 & -2 & -2 \\ 1 & -1 & -1 \\ 3 & -1 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & a & 0 \\ -1 & b & 1 \\ 2 & c & -1 \end{pmatrix}$$

are inverses of each other, what is the value of c ?

- (a) -3
 - (b) -2
 - (c) 0
 - (d) 2
 - (e) 3
2. If A is a square matrix of order $n \geq 4$, and $a_{ij} = i + j$ represents the entry in row i and column j , then the rank of A is always
- (a) 1

- (b) 2
(c) $n - 2$
(d) $n - 1$
(e) n
3. Let A be a 2 by 2 matrix with characteristic polynomial $p(x) = x^2 + 2x + 1$, which of the following must be true:
- (I) A has an eigenvalue.
(II) If A is diagonalizable, then A is the identity matrix.
(III) A is invertible
- (a) I only
(b) II only
(c) III only
(d) I and III only
(e) I, II, III
4. A square $n \times n$ matrix A is said to be skew symmetric if $A = -A^T$. The vector space of antisymmetric matrices has dimension:
- (a) $\frac{n(n+1)}{2}$
(b) $n^2 - 1$
(c) $\frac{n^2}{2}$
(d) $\frac{n^2+1}{2}$
(e) $\frac{n(n-1)}{2}$
5. For what value of x will the following matrix be non-invertible?

$$\begin{pmatrix} 7 & 6 & 0 & 1 \\ 5 & 4 & x & 0 \\ 8 & 7 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

- (a) -1
(b) 0
(c) 1
(d) 3
(e) 9
6. What is the Jordan form of the transformation $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $(w, z) \mapsto (-z, w)$?

(a) $\begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$

- (b) $\begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$
- (c) $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
- (d) $\begin{pmatrix} -i & 1 \\ 0 & -i \end{pmatrix}$
- (e) $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

7. What is the Jordan form of the transformation $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $T(x, y) = (4x + y, -x + 2y)$?

- (a) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$
- (b) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$
- (c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- (d) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$
- (e) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

8. A square matrix is said to be nilpotent if $A^n = 0$ for some $n \in \mathbb{N}$. Which of the following are true?

- (I) A is nilpotent implies the only eigenvalue of A is 0.
- (II) If A is nilpotent and normal, then A is the zero matrix.
- (III) If A and B commute, then A is nilpotent if and only if B is
- (IV) If ST is nilpotent, then so is TS .

- (a) I only
- (b) I and II only
- (c) I and III only
- (d) I, II, and III only
- (e) I, II, and IV only

9. Let V be the vector space of all polynomials of degree at most 3 with coefficients in \mathbb{R} . Let W be subspace of V such that $p(0) = p(-1) = p(1) = 0$. Then $\dim V + \dim W =$

- (a) 4
- (b) 5
- (c) 6

- (d) 7
- (e) 8

10. Let V be the set of polynomials $p(x)$ with coefficients in \mathbb{R} . Define linear operators T and S by:

$$T(p(x)) = xp(x)$$

$$S(p(x)) = p'(x) = \frac{d}{dx} [p(x)]$$

We denote ST and TS as the expected composition. Which of the following is true

- (a) $ST = 0$
- (b) $ST = T$
- (c) $ST = TS$
- (d) $ST - TS$ is the identity map of V
- (e) $ST + TS$ is the identity map of V

Linear Algebra #2

1. Which of the following are true for square matrices A and B ?

- (I) If AB is invertible, then A and B are.
- (II) If A has characteristic polynomial $x^3 - x - 1$, then A is invertible.
- (III) If A has eigenvalue λ , then $3A^2 - \lambda A$ has eigenvalue $2\lambda^2$.

- (a) I only
- (b) I and II only
- (c) I and III only
- (d) II and III only
- (e) I, II, and III

2. If $A = \begin{pmatrix} t & 1-t \\ 1 & 2t \end{pmatrix}$ has 1 as an eigenvalue, then another eigenvalue could be:

- (a) -3
- (b) -2
- (c) 2
- (d) 3
- (e) 4

3. Which of the following sets are dense in the set of square $n \times n$ square matrices over \mathbb{C} ?

- (I) Invertible matrices
 - (II) Unitary Matrices
 - (III) Symmetric Matrices
 - (IV) Diagonalizable Matrices
- (a) I and II only
 - (b) II and III only
 - (c) I and IV only
 - (d) III and IV only
 - (e) IV only
4. Let A be an $n \times n$ matrix all of whose diagonal entries are 1 and whose other entries are b , with $b > 1$. Which of the following are true:
- (I) A has eigenvalue $nb - b - 1$
 - (II) A has eigenvalue $nb - b + 1$
 - (III) A has an eigenvalue $b - 1$ with multiplicity $n - 1$.
- (a) I only
 - (b) II only
 - (c) III only
 - (d) I and II only
 - (e) II and III only
5. Describe the following matrix transformation geometrically:
- $$\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$
- (a) A reflection through the plane $x = y$ plane followed by a rotation through the same plane by 45 degrees
 - (b) A rotation through the xz -plane by 135 degrees followed by a reflection through the same plane.
 - (c) A rotation in the yz -plane by 45 degrees followed by reflection through the same plane
 - (d) A rotation in the xy -plane by 135 degrees followed by a reflection through the same plane
6. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the function defined by the equation $f(\vec{x}) = A\vec{x}$, where:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The range of f is:

- (a) A point
 - (b) A line through the origin
 - (c) A plane through the origin
 - (d) all of \mathbb{R}^3
7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(a\vec{x}) = af(\vec{x})$ for every $a \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$. Which of the following are true?
- (a) f is linear.
 - (b) $f(\vec{x}) = 0$ if and only if $\vec{x} = 0$.
 - (c) If $f(a) + f(b) = f(a + c)$ for $a, b \in \mathbb{Z}$, then f is surjective precisely when f is non-zero.
 - (d) All of the above
 - (e) A and C
8. Which of the following are subspaces?
- (I) $\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + ix_2 + 3x_3 = 0\}$
 - (II) $\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + 2x_2 + 3x_3 = 4 - 5i\}$
 - (III) $\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1x_2x_3 = 0\}$
 - (IV) $\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = 10x_3\}$
- (a) All the above
 - (b) I and IV only
 - (c) I, III, and IV only
 - (d) I, II, and III only
 - (e) I, II, IV only

Abstract Algebra

1. Two subgroups H and K of a group G have orders 12 and 30 respectively. Which of the following could not be the order of the subgroup of G generated by H and K ?
- (a) 30
 - (b) 60
 - (c) 120
 - (d) 360
 - (e) Countably infinite
2. Let G_n denote the cyclic group of order n . Which of the following are not cyclic?

- (a) $G_{11} \times G_7$
 - (b) $G_{24} \times G_{17}$
 - (c) $G_{11} \times G_3 \times G_8$
 - (d) $G_{33} \times G_{11}$
 - (e) $G_{49} \times G_{121}$
3. A group such that $(ab)^2 = a^2b^2$:
- (a) Finite
 - (b) Cyclic
 - (c) of order
 - (d) Abelian
 - (e) None of the above
4. Let G be a group and $f : G \rightarrow G$ be given by $f(x) = axa^2$. a is a homomorphism if and only if:
- (a) G is abelian
 - (b) $G = \{e\}$
 - (c) $a = e$
 - (d) $a^2 = a$
 - (e) $a^3 = e$
5. Let G be a group of order 9, and let e denote the identity of G . Which one of the statements about G cannot be true:
- (a) There exists an element x in G such that $x \neq e$ and $x^{-1} = x$
 - (b) There exists an element x in G such that $x, y \neq e$ and $\langle x \rangle \cap \langle y \rangle = \{e\}$
 - (c) There exists an element $x \in G$ such that $\langle x \rangle$ has order 3.
 - (d) G is cyclic
 - (e) G is Abelian.
6. Which of the following subsets are subrings of the ring of real numbers?
- (I) $\{a + b\sqrt{\frac{2}{3}} \mid a, b \in \mathbb{Q}\}$
 - (II) $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$
 - (III) $\{\frac{n}{3^m} \mid n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}_{\geq 0}\}$
 - (IV) $\{a + b\sqrt{5} \mid a, b \in \mathbb{R} \text{ such that } a^2 + b^2 \leq 1\}$
- (a) I only
 - (b) I and II only
 - (c) III , IV only
 - (d) II, III, IV

- (e) I, II, III
7. If the finite group G contains a subgroup of order seven but no element other than the identity is its own inverse, then the order of G could be:
- (a) 27
 - (b) 28
 - (c) 35
 - (d) 37
 - (e) 42
8. Let p and q be distinct primes. How many mutually non-isomorphic Abelian groups are there of order p^2q^4 ?
- (a) 6
 - (b) 8
 - (c) 10
 - (d) 12
 - (e) 16
9. Let R be a ring with multiplicative identity. If U is an additive subgroup of R such that $ur \in U$ for all $u \in U$ and for all $r \in R$, then U is said to be a right ideal of R . If R has a exactly two right ideals, which of the following must be true?
- (I) R is commutative
 - (II) R is a division ring (every nonzero element has an inverse)
 - (III) R is infinite
- (a) I only
 - (b) I, II only
 - (c) II, III only
 - (d) II only
 - (e) III only
10. If S is a ring with the property that $s = s^2$ for each $s \in S$, which of the following must be true?
- (I) $s + s = 0$ for each $s \in S$
 - (II) $(s + t)^2 = s^2 + t^2$ for each $s, t \in S$
 - (III) S is commutative (i.e. multiplication is commutative)
- (a) III only
 - (b) I and II only
 - (c) I and III only
 - (d) II and III only
 - (e) I, II, and III

Number Theory

1. How many generators does the group \mathbb{Z}_{24} have?
 - (a) 2
 - (b) 6
 - (c) 8
 - (d) 10
 - (e) 12
2. Let p and q be distinct primes. There is a proper subgroup J of the additive group of integers which contains exactly three elements of the set: $\{p, p + q, pq, p^q, q^p\}$. Which three elements are in J ?
 - (a) pq, p^q, q^p
 - (b) $p + q, pq, p^q$
 - (c) $p, p + q, pq$
 - (d) p, p^q, q^p
 - (e) p, pq, p^q
3. Let H be the set of all group homomorphisms $\phi : \mathbb{Z}_3 \rightarrow \mathbb{Z}_6$. How many functions does H contain?
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
 - (e) 6
4. Let x and y be positive integers such that $3x + 7y$ is divisible by 11. Which of the following is also divisible by 11?
 - (a) $4x + 6y$
 - (b) $x + y + 5$
 - (c) $9x + 4y$
 - (d) $4x - 9y$
 - (e) $x + y - 1$
5. What is the units digit in 7^{25} ?
 - (a) 1
 - (b) 3
 - (c) 5
 - (d) 7

- (e) 9
6. Let P_1 be the set of all primes $\{2, 3, 5, 7, \dots\}$ and for each n , define P_n to be the set of all prime multiples of n , i.e. $\{2n, 3n, 5n, 7n, \dots\}$. Which of the following intersections is non-empty?
- (a) $P_1 \cap P_{23}$
 - (b) $P_7 \cap P_{21}$
 - (c) $P_{12} \cap P_{20}$
 - (d) $P_{20} \cap P_{24}$
 - (e) $P_5 \cap P_{25}$
7. For how many positive integers k does the ordinary decimal representation of the integer $k!$ end in exactly 99 zeros?
- (a) None
 - (b) One
 - (c) Four
 - (d) Five
 - (e) Twenty Four

Real Analysis and Advanced Calculus

1. Let K be a nonempty subset of \mathbb{R}^n , where $n > 1$. Which of the following statements must be true?
- (I) If K is compact, then every continuous real valued-function defined on K is bounded.
 - (II) If every continuous real valued function defined on K is bounded, then K is compact.
 - (III) If K is compact, then K is connected.
- (a) I only
 - (b) II only
 - (c) III only
 - (d) I and II only
 - (e) I, II, and III only
2. For each positive integer n , let f_n be the function defined on $[0, 1]$ by

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Which of the following statements are true?

- (I) The sequence $\{f_n\}$ converges pointwise to a limit function f .
 (II) The sequence $\{f_n\}$ converges uniformly to a limit function f .
 (III) $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 \left(\lim_{n \rightarrow \infty} f_n(x) \right) dx$.
- (a) I only
 (b) II only
 (c) I and II only
 (d) I and III only
 (e) II and III only
3. The set of real numbers for which the series $\sum_{n=1}^{\infty} \frac{n!x^{2n}}{n^n(1+x^{2n})}$
- (a) $\{0\}$
 (b) $\{x : -1 < x < 1\}$
 (c) $\{x : -1 \leq x \leq 1\}$
 (d) $\{x : |x| \leq \sqrt{e}\}$
 (e) \mathbb{R}
4. Let n be an integer greater than 1. Which of the following guarantee that the equation $x^n = \sum_{i=0}^{n-1} a_i x^i$ has at least one root in the interval $(0, 1)$?
- (I) $a_0 > 0$ and $\sum_{i=0}^{n-1} a_i < 1$
 (II) $a_0 > 0$ and $\sum_{i=0}^{n-1} a_i > 1$
 (III) $a_0 < 0$ and $\sum_{i=0}^{n-1} a_i > 1$
- (a) None
 (b) I only
 (c) II only
 (d) III only
 (e) I and III only
5. Let f be a continuous, strictly decreasing, real valued function such that $\int_0^{+\infty} f(x) dx$ is finite. In terms of f^{-1} (the inverse function of f), $\int_0^{\infty} f(x) dx$ is:
- (a) Less than $\int_1^{\infty} f^{-1}(y) dy$
 (b) Greater than $\int_0^1 f^{-1}(y) dy$
 (c) Equal to $\int_1^{\infty} f^{-1}(y) dy$
 (d) Equal to $\int_0^1 f^{-1}(y) dy$
 (e) Equal to $\int_0^{\infty} f^{-1}(y) dy$

6. If f and g are real valued differentiable functions and if $f'(x) \geq g'(x)$ for all $x \in [0, 1]$, which of the following must be true?
- (a) $f(0) \geq g(0)$
 - (b) $f(1) \geq g(1)$
 - (c) $f(1) - g(1) \geq f(0) - g(0)$
 - (d) $f - g$ has no maximum on $[0, 1]$
 - (e) $\frac{f}{g}$ is a non-decreasing function on $[0, 1]$

Topology

1. How many topologies are possible on a set of 2 points?
- (a) 5
 - (b) 4
 - (c) 3
 - (d) 2
 - (e) 1
2. Let $X = \{a, b, c\}$. Which of the following classes of subsets of X do not form a topology?
- (a) $\{X, \emptyset\}$
 - (b) $\{X, \emptyset, \{a\}\}$
 - (c) $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$
 - (d) $\{X, \emptyset, \{a, b\}, \{a, c\}, \{b, c\}\}$
 - (e) $P(X)$, the power set of X .
3. Which of the following does NOT define a metric on the set of real numbers?
- (a) $\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y \end{cases}$
 - (b) $\rho(x, y) = \min\{|x - y|, 1\}$
 - (c) $\sigma(x, y) = \frac{1}{3} \cdot |x - y|$
 - (d) $\tau(x, y) = \frac{|x - y|}{|x - y| + 1}$
 - (e) $\omega(x, y) = (x - y)^2$.
4. An ordinary polyhedron with 12 faces and 17 edges has how many vertices?
- (a) 5
 - (b) 7

- (c) 9
 - (d) 11
 - (e) 13
5. Suppose X is a compact and Y is Hausdorff such that $f : X \rightarrow Y$ is a continuous bijection. Which of the following are true?
- (I) f is open.
 - (II) f is a local homeomorphism.
 - (III) f^{-1} is continuous.
- (a) I only
 - (b) II only
 - (c) III only
 - (d) II and III only
 - (e) All of them
6. If A is a subset of the real line \mathbb{R} and A contains each rational number, which of the following must be true?
- (a) If A is open, then $A = \mathbb{R}$.
 - (b) If A is closed, then $A = \mathbb{R}$.
 - (c) If A is uncountable, then $A = \mathbb{R}$.
 - (d) If A is uncountable, then A is open.
 - (e) If A is countable, then A is closed.
7. Let S be a compact topological space, let T be a topological space, and let f be a function from S onto T . Which of the following conditions on f is the weakest sufficient condition to guarantee compactness of T .
- (a) f is a homeomorphism
 - (b) f is continuous and injective
 - (c) f is continuous
 - (d) f is injective
 - (e) f is bounded
8. Let R be the set of real numbers with the topology generated by the basis $\{[a, b) : a < b, a, b \in \mathbb{R}\}$. If X is the subset $[0, 1]$ of R , which of the following must be true?
- (I) X is compact
 - (II) X is Hausdorff
 - (III) X is connected

- (a) I only
- (b) II only
- (c) III only
- (d) I and II only
- (e) II and III only

9. Which of the following are equivalent to continuity?

- (I) $f^{-1}(A^o) = (f^{-1}(A))^o$ where S^o denotes interior of a set S
- (II) $f^{-1}(\overline{A}) = \overline{(f^{-1}(A))}$
- (III) $f^{-1}(A^c) = (f^{-1}(A))^c$

- (a) I only
- (b) II only
- (c) III only
- (d) II and III only
- (e) None

Combinatorics

1. Sixteen players participated in a round-robin tennis tournament. Each of them won a different number of games. How many games did the player finishing sixth win?

- (a) 14
- (b) 13
- (c) 12
- (d) 9
- (e) 10

2. A circular region is divided by 5 radii into five equal area sectors. Twenty one points are chosen in the circular region none along the circular radii that divide each region. Which of the following statements are true?

- (I) Some sector contains at least 5 points
- (II) Some sector contains at most 3 points
- (III) Some pair of adjacent sectors contains a total of at least 9 of the points.

- (a) I only
- (b) III only
- (c) I and II only

- (d) I and III only
(e) I, II, and III only
3. From the set of integers $\{1, 2, \dots, 9\}$, how many nonempty subset sum to an even integer?
- (a) 512
(b) 255
(c) $\frac{9!}{2! \cdot 7!}$
(d) $\frac{9!}{2!}$
(e) None of the above
4. $\sum_{k=0}^n \binom{n}{k} 4^k (-1)^{n-k} =$
- (a) $\binom{2n}{n}$
(b) $\binom{n^2}{n}$
(c) 3^n
(d) 2^n
(e) $4^n \binom{2n}{n}$
5. A city has square blocks formed by a grid of north-south and east-west streets. One possible way to get from City Hall to the Main Fire House is first travel 5 blocks east and then 7 blocks north. How many different routes from City Hall to the main firehouse that travel exactly 12 city blocks?
- (a) $5 \cdot 7$
(b) $\frac{7!}{5!}$
(c) $\frac{12!}{7!5!}$
(d) 2^{12}
(e) $7!5!$
6. Same question as 5 only now we must avoid the car accident that occurred 3 blocks east and 5 blocks north from City Hall. This will not be a multiple choice question because it would give the previous one away!
7. How many trees are there with 5 vertices? A tree is a graph with no cycles.
- (a) 5
(b) 6
(c) 7
(d) 8
(e) 9

8. Alan and Barbara play a game in which they take turns filling entries of an initially empty 10×10 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?
- (a) Barbara
 - (b) Alan
 - (c) Neither
9. For each $n \in \mathbb{N}$, we let $[n]$ denote the set $\{1, \dots, n\}$. How many injective functions are there from $[n] \rightarrow [n + k]$ where $k \geq 0$?
- (a) $\binom{n}{k} k!$
 - (b) $\binom{n+k}{k} k!$
 - (c) $\binom{n+k}{n-k} k!$
 - (d) $(n-k)!$
 - (e) $\binom{n+k}{k} n!$
10. 4 distinct positive integers are randomly chosen between 1 and 101, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 3?
- (a) $1/5$
 - (b) $1/4$
 - (c) $1/2$
 - (d) $3/4$
 - (e) 1

Probability

1. Two people play Russian Roulette with a six shooter spinning with one bullet in the chamber. They spin the chamber each time before they pull the trigger. The probability the first person dies is:
- (a) $\frac{2}{3}$
 - (b) $\frac{3}{5}$
 - (c) $\frac{4}{7}$
 - (d) $\frac{6}{11}$

(e) $\frac{7}{13}$

2. Suppose X is a discrete random variable on the set of positive integers such that for each positive integer n , the probability that $X = n$ is $\frac{1}{2^n}$. If Y is a random variable with the same distribution and X and Y are independent, what is the probability that the value of at least one of the variables X and Y is greater than 3?

- (a) $1/64$
- (b) $15/64$
- (c) $1/4$
- (d) $3/4$
- (e) $3/8$
- (f) $4/9$

3. A fair coin is to be tossed 100 times, with each toss resulting in a head or a tail. If H is the number of heads and T is the number of tails, which of the following events has the greatest probability?

- (a) $H = 50$
- (b) $T \geq 60$
- (c) $51 \leq H \leq 55$
- (d) $H \geq 48$ and $T \geq 48$
- (e) $H \leq 5$ or $H \geq 95$

4. Let X be a random variable with probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the standard deviation of X ?

- (a) 0
- (b) $\frac{1}{5}$
- (c) $\frac{\sqrt{30}}{15}$
- (d) $\frac{\sqrt{5}}{5}$
- (e) 1

5. Let X and Y be uniformly distributed random variables on $[0, 1]$. Find $P(|X - Y| < \frac{1}{2})$.

- (a) $\frac{1}{4}$
- (b) $\frac{1}{3}$
- (c) $\frac{2}{3}$
- (d) $\frac{3}{4}$

- (e) $\frac{4}{5}$
6. Suppose you answer quiz questions until you get one wrong. If you answer 99% of the questions correctly, how many questions on average do you expect to answer?
- (a) 99
(b) 100
(c) 80
(d) $100/3$
(e) 110
7. Suppose you flip a coin until you get two consecutive heads. How many times do you expect to flip until you stop?
- (a) 5
(b) 6
(c) 7
(d) 8
(e) 9

Complex Analysis

1. Let $r > 0$ and let C be the circle $|z| = r$ in the complex plane. If P is a polynomial function, then $\int_C P(z) dz =$
- (a) 0
(b) πr^2
(c) $2\pi i$
(d) $2\pi P(0)i$
(e) $P(r)$
2. Let G be the solutions to $x^4 - 1$. G is a group. Which of the following are true about bijective homomorphisms of G into itself (also known as an automorphism)?
- (I) $z \mapsto \bar{z}$ defines a homomorphism, where \bar{z} is the complex conjugate
(II) $z \mapsto z^2$ defines a homomorphism
(III) For every homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.
- (a) I only
(b) I and II only

- (c) I, II, and III
 (d) II and III only
 (e) III only
3. If f is a function defined by a complex power series expansion in $z - a$ which converges for $|z - a| < 1$ and diverges for $|z - a| > 1$, which of the following must be true?
- (a) $f(z)$ is analytic in the open unit disk with center at a
 (b) The power series for $f(z + a)$ converges for $|z + a| < 1$
 (c) $f'(a) = 0$
 (d) $\int_C f(z) dz = 0$ for any circle C in the plane
 (e) $f(z)$ has a pole of order 1 at $z = a$.
4. If $z = e^{2\pi i/5}$, then $1 + z + z^2 + z^3 + 5z^4 + 4z^5 + 4z^6 + 4z^7 + 4z^8 + 5z^9 =$
- (a) 0
 (b) $4e^{3\pi i/5}$
 (c) $5e^{4\pi i/5}$
 (d) $-4e^{2\pi i/5}$
 (e) $-5e^{3\pi i/5}$
5. What is the fractional transformation $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f(0) = 1$, $f(1) = 5$, $f(\infty) = 3$?
- (a) $\frac{9z+1}{3z+1}$
 (b) $\frac{-6z+1}{z-2}$
 (c) $\frac{14z+1}{z+2}$
 (d) $\frac{9z+1}{z+1}$
 (e) $\frac{6z-1}{2z-1}$

Euclidean Geometry and Miscellaneous Problems

1. At how many points in the xy -plane do the graphs of $y = x^{2012}$ and $y = 5^x$ intersect?
- (a) 0
 (b) 1
 (c) 2
 (d) 3
 (e) 4
2. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^{n-1}} =$

- (a) $-1/2$
- (b) $-4/5$
- (c) $-4/25$
- (d) $-4/9$
- (e) $-5/16$



3. The group of symmetries of the regular pentagram shown above is isomorphic to the:

- (a) Symmetric group S_5
- (b) Alternating group A_5
- (c) cyclic group of order 5
- (d) cyclic group of order 10
- (e) dihedral group of order 10

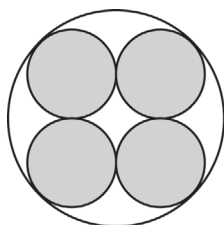


Figure 1

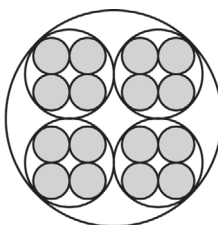
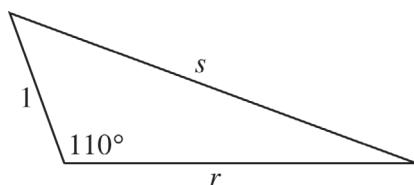


Figure 2

4. The four shaded circles in Figure 1 above are congruent and each is tangent to the large circle and to two of the other shaded circles. Figure 2 is the result of replacing each of the shaded circles in figure 1 by a figure that is geometrically similar to Figure 1. What is the ratio of the shaded portion of Figure 2 to the area of shaded portion of Figure 1?

- (a) $\frac{1}{2\sqrt{2}}$
- (b) $\frac{1}{1+\sqrt{2}}$
- (c) $\frac{4}{1+\sqrt{2}}$
- (d) $\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^2$
- (e) $\left(\frac{2}{1+\sqrt{2}}\right)^2$



5. In the figure above, as r and s increase the third side remains at 1 and the measure of the obtuse angle remains at 110 degrees. What is $\lim_{r,s \rightarrow \infty} (s - r)$?
- (a) 0
 - (b) A positive number less than 1
 - (c) 1
 - (d) A finite number greater than 1
 - (e) ∞ .

Multivariable Calculus

1. The plane $y = 1$ slices the surface $z = \arctan \frac{x+y}{1-xy}$ in a curve C . Find the slope of the tangent line to C at the point where $x = 2$.
 - (a) -3
 - (b) -1
 - (c) $\frac{1}{5}$
 - (d) $\frac{1}{3}$
 - (e) $\frac{1}{2}$
2. Find the volume of the solid in the first octant in \mathbb{R}^3 bounded below by the coordinate axes and the unit circle, and bounded above by the surface $z = 8xy$
 - (a) $\frac{1}{2}$
 - (b) 1
 - (c) 2
 - (d) 4
 - (e) 8
 - (f) 9

3. Let \vec{F} be a constant unit force that is parallel to the vector $(-1, 0, 1)$ in \mathbb{R}^3 and let $\gamma(t) = (t, t^2, t^3)$ for $t \in [0, 1]$. The work done by \vec{F} along γ is:
- (a) $-\frac{1}{4}$
 - (b) $-\frac{1}{4\sqrt{2}}$
 - (c) 0
 - (d) $\sqrt{2}$
 - (e) $3\sqrt{2}$
4. What is the minimum value of the expression $x + 4z$ as a function defined on \mathbb{R}^3 subject to the constraint $x^2 + y^2 + z^2 \leq 2$?
- (a) 0
 - (b) -2
 - (c) $-\sqrt{34}$
 - (d) $-\sqrt{35}$
 - (e) $-5\sqrt{2}$
5. If a is a positive number, what's the value of the following double integral?

$$\int_0^{2a} \int_{-\sqrt{2ay-y^2}}^0 \sqrt{x^2 + y^2} dx dy$$

- (a) $\frac{16}{9}a^3$
- (b) $\frac{32}{9}a^3$
- (c) $\frac{\pi}{2}a^2$
- (d) $\frac{8\pi}{3}a^2$
- (e) $2a^4$

Differential Equations

1. Find the family of all nonzero solutions to the equation:

$$y'' = (y')^2$$

- (a) $y = \frac{1}{x+C_1} + C_2$
- (b) $y = C_2 + \log(C_1 - x)$
- (c) $y = C_2 - \log(C_1 - x^2)$
- (d) $y = C_2 + \log(C_1 + x)$
- (e) $y = C_2 - \log(C_1 + x)$

2. Let $y = f(x)$ be a solution of the differential equation $xdy + (y - xe^x)dx = 0$ such that $y = 0$ when $x = 1$. What is $f(2)$?

- (a) $\frac{1}{2e}$
- (b) $\frac{1}{e}$
- (c) $\frac{e^2}{2}$
- (d) $2e$
- (e) $2e^2$

3. Let $y = f(x)$ be a solution of the equation:

$$y' + \frac{y}{x} = \sin x$$

such that $f(\pi) = 1$. What is the value $f(\pi/2)$?

- (a) $\frac{2}{\pi} - 1$
- (b) $\frac{2}{\pi}$
- (c) $\frac{2}{\pi} + 1$
- (d) $\frac{\pi}{2}$
- (e) $\frac{\pi}{2} + 1$

4. Find the general solution of the differential equation:

$$\frac{dy}{dx} = \frac{x + y}{x}$$

- (a) $e^{y/x} = cx$
- (b) $e^{y/x} = cy$
- (c) $e^{x/y} = cx$
- (d) $e^{x/y} = cy$
- (e) $e^{-x/y} = cx$

5. What is the general solution to the differential equation:

$$y'' - 2y' + y = te^t$$

- (a) $C_1e^t + C_2t^2e^t$
- (b) $C_1e^t + C_2te^t$
- (c) $C_1e^t + C_2te^t + t^3e^t$
- (d) $C_1e^t + C_2te^t + \frac{t}{2}e^t$
- (e) $C_1e^t + C_2te^t + \frac{t^2}{2}e^t$

References:

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