Complex Analysis

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based on Conway 's Functions of One Complex Variable

Contents

1 Complex Number System (I.1 - I.4)

Bombelli discovered it's useful to consider negative square roots after studying the equation: $x^3 = 5x + 3$

The set of complex numbers, $\mathbb{C}$, is sometimes called the **Complex Plane**. Since any complex number z can be written as $x + iy$, we can identify any complex number in $\mathbb{R}^2$.

1.1 Arithmetic in Complex Plane

Addition is akin to adding vectors in $\mathbb{R}^2$, because we add the real and imaginary parts. Multiplication is as expected; also the associative, commutative, and distributive properties hold. In fact, $\mathbb{C}$ is a field

The **modulus** of a complex number $z = x + iy$ is the length of the vector:

$$|z| = \sqrt{x^2 + y^2}$$

Modulus Properties

- |zw| = |z| |w|
  write z as $z = |z|\hat{z}$, where $\hat{z}$ is the unit vector in the direction of z then you can expand zw to obtain result.
- $|\bar{z}|$ = |z|
  because you're squaring entries to find the length

The **complex conjugate** of z, denoted $\bar{z} = x - iy$. Note $z\bar{z} = |z|^2$.

Trick: Re $z = \dfrac{z + \bar{z}}{2}$.

Note to find the multiplicative inverse of z, it's

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

(equivalently $\dfrac{\bar{z}}{|z|^2}$)

The distance between two vectors z, w is $|z - w|$.

Can show Triangle on modulus holds: $|z + w| \leq |z| + |w|$. * can extend to $|z_1 + z_2 + z_3 + \ldots| \leq |z_1| + |z_2| + \ldots$ by induction.

Note $|z + w| = |z| + |w| \iff |z\bar{w}| = Re(z\bar{w})$

A useful variant of the triangle inequality is

$$||z| - |w|| \leq |z - w|$$

1.2 Polar Coordinates

We can express a complex number z as a vector in $\mathbb{R}^2$ using Polar Coordinates: $r = |z|$ and an angle θ.
note $x = |z| \cos(\theta)$

The **principal argument** of z is θ restricted to values between $-\pi$ and π.

e.g., Arg(1 - i) = $\dfrac{-\pi}{4}$

Often we write z in terms of cos and sin: $z = r(\cos(\theta) + i\sin(\theta))$
in exponential notation, $z = re^{i\theta}$ by euler's identity.

e.g., $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$. *
note conway denotes $\cos(\theta) + i\sin(\theta)$ as $cis(\theta)$

nice properties:
arg($\bar{z}$) = - arg(z)
can see this by drawing vector in $\mathbb{R}^2$
arg(z_1z_2) = arg(z_1) + arg(z_2)

Multiplication of z_1z_2 results in: adding angles and multiplying lengths

De Moivre's Formula: $(e^{i\theta})^n = e^{in\theta}$, meaning

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

1.3 Nth Roots

An nth root of w is a number z such that $z^n = w$.

There are n distinct roots for a number $w \neq 0$. For $w = \rho e^{i\phi}$, they are $\sqrt[n]{\rho}e^{i(\frac{\phi}{n}+\frac{2kpi}{n})}$ for $k = 0, 1, \ldots, n-1$.

nth roots of unity are the nth roots of 1.

2 Metric Spaces and Topology in C

2.1 Review

a set (in a metric space) is **compact** if every open cover has a finite subcover.

A metric space is **sequentially compact** if every sequence has a convergent subsequence.

It turns out a metric space is compact $\iff$ it's sequentially compact.