Exam Corrections

Measure Theory
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3. If μ is a function from S to $[0, \infty]$ such that $\mu(\emptyset) = 0$ and μ is countably additive, meaning if $\cup_{i=1}^{\infty} A_i$ is in S with all A_i pairwise disjoint, then $\sum_{i=1}^{\infty} \mu(A_i) = \mu(\cup_{i=1}^{\infty} A_i)$.

11. b) Since $\emptyset$ is finite, $\mu(\emptyset) = 0$ as needed. Next consider any pairwise disjoint sets $A_1, A_2, \ldots$ in S such that $\cup_{i=1}^{\infty} A_i$ is in S. Then $\cup_{i=1}^{\infty} A_i$ must be finite or $\cap_{i=1}^{\infty} A_i^c$ must be finite. If $\cup_{i=1}^{\infty} A_i$ is finite, then

$$\mu(\cup_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} \mu(A_i),$$

because each A_i must be finite if the union of all A_i is finite. If $\cap_{i=1}^{\infty} A_i^c$ is finite, then at least one A_i must be infinite, because $\cap_{i=1}^{\infty} A_i^c$ is finite (otherwise $\cup_{n=1}^{\infty} A_i$ would be at most countably infinite, implying $\cap_{i=1}^{\infty} A_i^c$ is infinite, a contradiction). If some A_i is infinite, then A_i^c is finite, because A_i is in S, implying all other $A_j \neq A_i$ are finite. Thus, $\sum_{i=1}^{\infty} \mu(A_i) = 1 = \mu(\cup_{i=1}^{\infty} A_i)$.

11. c) If B is countable or finite, then $B = \cup_{i=1}^{\infty} \{a_i\}$, where a_i are single points or empty sets. Thus, $0 \leq \mu^*(B) \leq \sum_{i=1}^{\infty} \mu(a_i) = 0$, because $B \subseteq \cup_{i=1}^{\infty} \{a_i\}$, implying $\mu^*(B) = 0$. Otherwise, B is uncountable, meaning B can't be covered by a union of countable sets in $\mathbb{R}$. Thus, $\mu^*(B) > 0$. Furthermore, $B \subseteq \mathbb{R}$, meaning $\mu^*(B) \leq \mu(\mathbb{R}) + \mu(\emptyset) + \cdots = 1$. Since $\mu-$values are in $\mathbb{N} \cup 0$,

$$\mu^*(B) = \begin{cases} 0 & \text{if B is countable or finite} \\ 1 & \text{if B is uncountable} \end{cases}$$

are the possible measures of an arbitrary subset B.

14. The set of points in infinitely sets E_n, $E = \lim_{n \to \infty} E_n$. Each $\lambda(E_n) > 0$ implying $\sum_{n=k}^{\infty} E_n = 0$ as $k \to \infty$, because $\sum_{n=1}^{\infty} \lambda(E_n) < \infty$. For any $k \in \mathbb{N}$, $E \subseteq \cup_{n=k}^{\infty} E_n$. Thus as $k \to \infty$,

$$\lambda(E) \leq \sum_{n=k}^{\infty} \lambda(E_n) = 0.$$

Since $\lambda(E) \geq 0$, $\lambda(E) = 0$ as desired.

15. Let $B = \{x | f(x) \geq \epsilon\}$. Note $A \subseteq B$. Furthermore, $f(x) = |f(x)|$ except on a set of measure 0, call it C, because $f(x) \geq 0$ a.e.. Therefore, $B \subseteq \{x| \ |f(x)| \geq \epsilon\} \cup C$. Since f

is an integrable function, by Theorem 22.5, $\{x| \ |f(x)| \geq \epsilon\}$ has finite measure. Thus,

$$\begin{aligned}
\mu(A) &\leq \mu(B) \\
&\leq \mu(\{x| \ |f(x)| \geq \epsilon\} \cup C) \\
&\leq \mu(\{x| \ |f(x)| \geq \epsilon\} + \mu(C) \\
&= \mu(\{x||f(x) \geq \epsilon\}) \\
&< \infty.
\end{aligned}$$

16. Let $f_n(x) = n(e^{x/n} - 1)$. Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots,$

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} n(\frac{x}{n} + \frac{x^2}{n^2 2!} + \frac{x^3}{n^3 3!} + \dots) \\
&= \lim_{n\to\infty} x + \frac{x^2}{n2!} + \frac{x^3}{n^2 3!} + \dots
\end{aligned}$$

Define $g_m(x) = \dfrac{x^m}{n^{m-1} m!}$. Then on $[0, 1]$ each $|g_m(x)|$ is bounded by $\dfrac{1}{m!}$. Since $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m!}$ converges (to e), by the Weierstrass M-test, $\displaystyle\sum_{m=1}^{\infty} g_m(x)$ converges uniformly. Thus,

$$\begin{aligned}
\lim_{n\to\infty} x + \frac{x^2}{n2!} + \frac{x^3}{n^2 3!} + \dots &= \lim_{n\to\infty} x + \lim_{n\to\infty} \frac{x^2}{n2!} + \lim_{n\to\infty} \frac{x^3}{n^2 3!} + \dots \\
&= x.
\end{aligned}$$

Furthermore, on $[0, 1]$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
|f_n(x)| &= \left| x + \frac{x^2}{n2!} + \frac{x^3}{n^2 3!} + \dots \right| \\
&= x + \frac{x^2}{n2!} + \frac{x^3}{n^2 3!} + \dots \\
&\leq x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots && \text{(since } n \in \mathbb{N}\text{)} \\
&\leq 1 + x + \frac{x^2}{2!} + \dots = e^x.
\end{aligned}$$

Note each $f_n(x)$ is measurable on $[0, 1]$ because $f_n(x)$ is continuous. Since e^x is Riemann integrable, by Lebesgue Dominated Convergence,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}.$$