Analysis II

Mark

Chapter 5: differentiation

1 Derivatives

f is **differentiable** at c if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) \text{ exists in } \mathbb{R}$$

If $f'(c)$ exists, a function $f^*(x)$ **continuous** at c such that

$$f(x)-f(c) = (x-c)f^*(x) \text{ with } f^*(c) = f'(c).$$

Proof: follows directly from definition of $f*(x)$
–note subscript c in f_c^* emphasizes that f^* depends on c.
–$f'(c)$ exists, means $f'(c)$ can be ∞.

If f is **differentiable** at c, then it's **continuous** at c

*$f'(c)$ need not be continuous!

Note, $f(x) = f(c) + f_c^*(x)(x-c)$.
Since f_c^* is continous, the RHS is continous.

Product Rule proof idea: construct $(fg)_c^*$, show it's continous.

Quotient Rule idea: from last semester, f is non-zero at a point and continous means there exists neighborhood of inputs where f is nonzero. Use to show $\frac{1}{g(x)}$ is nonzero.

Chain Rule

For g differentiable at c and f differentiable at $g(c)$,

$$(f\circ g)'(c) = g'(c)f'(g(c))$$

Proof: $h(x) = f(g(x))$.
Write $h(x)-h(c)$ with the hopes of finding $h_c^*(x)$ continous.

Right/Left Limits

For $f : [a,b] \to \mathbb{R}$, continous,

$$f'_+(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

if limit exists (even as ∞).

e.g. $f(x) = \sqrt{x}$ on $[0,1]$. Then,
$f'_+(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x} = +\infty$,
$f'_-(1) = \frac{1}{2}$.

$\boxed{f'(c) > 0 \text{ exists } \iff f'_+(c) = f'_-(c) \text{ both exist and are } > 0}$

Key: neighborhood where $f'(c) > r > 0$. right-half of neighborhood, $f'_+(c)$ checks out (similarly for left).

Extremum

Local **max** at c means there exists neighborhood of x such that for all x, $f(c) \geq f(x)$.

$\boxed{\text{If local max/min at } c \text{ and } f'(c) \text{ exists, } f'(c) = 0}$

proof: rule out $f'(c) > 0$, since $f(c + \delta) > f(c)$ (similarly for < 0) Thus, $f'(c) = 0$.
–max can occur without $f'(c) = 0$ if $f'(c)$ doesn't exist.

2 Mean Value

Rolle's Theorem

$f : [a,b] \to \mathbb{R}$, continuous on $[a,b]$, and $f'(x)$ exists on (a,b)

$\boxed{\text{If } f(a) = f(b), \text{ there exists } c \in (a,b) \text{ such that } f'(c) = 0}$

"smooth curve with end points must a turning point" (or is constant) proof:
max/min exists by EVT since f is continous on a compact set, call $f(c)$ max/min.
Case 1: either max or min occurs at c in $(a,b) \to f'(c) = 0$.
Case 2: neither max nor min in (a,b)
implies max and min at $c = a = b$.
Thus, f is constant. $\square$

Recall, **Extreme Value Theorem** continous function on a compact set has a max/min.
continous image of compact is compact $\to$ (by Heine-Borel in $\mathbb{R}$) closed, bounded.
Thus, contains sup, inf.

Mean Value Theorem

$f : [a, b] \to \mathbb{R}$, continous on $[a, b]$, and $f'(x)$ exists on (a, b), for some $c \in (a, b)$,

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}.}$$

"line connecting end points has same slope as some point on curve"
physical: "instantaneous velocity at some point = average velocity"
proof, follows from below by letting $g(x) = x$. *f'(x) can exist both as a finite real or as infinity.

Generalized Mean Value Theorem, "Cauchy"

$f(x), g(x)$ continuous on $[a, b]$ and differentiable on (a, b) $(c)(f(b) - f(a))$.

nicely written for $g(b) - g(a), g'(c) \neq 0$ as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

proof:
Define

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

Note,
1. $h(x)$ is continuous, since all terms are.
2. $h'(x)$ exists for all x.
Furthermore $h(a) = h(b)$.
Thus, by Rolle's Theorem, there exists c such that $h'(c) = 0$.

IDEA: define $h(x)$ with equality we want. Use Rolle's.

CAUTION: tempting to say $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)=g(a)}$, but C is different for f and g!.

Increasing Functions

$f'(x)$ exists on (a, b) and $f'(x) > 0$, then

$$f(x) \text{ is strictly increasing.}$$

Let $a < x < y < b$.
By MVT there exists c such that

$$f(y) - f(x) = f'(c)(y - x)$$

RHS > 0, so $f(y) - f(x) > 0$.

L'Hôpital's Rule

$f(x), g(x)$ continous and differentiable, $f(c) = g(c) = 0$, and $g'(x)$ never 0 in $(c - \delta, c + \delta) \setminus \{c\}$.

$$\boxed{\text{If } \lim_{x \to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}, \text{ then } L = \lim_{x \to c} \frac{f(x)}{g(x)}.}$$

WWTS $\lim_{x \to c^+} \frac{f(x)}{g(x)} = L$ (similarly for c^-).

(note $\frac{f(x)}{g(x)}$ by the contraposition of Rolle's: $g'(x) \neq 0 \to g(x)$) (1) On (c, x) with $x < c + \delta$, by GMVT there exists α such that

$$f'(\alpha)(g(x) - g(c)) = g'(\alpha)(f(x) - f(c))$$

$$\to$$

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha)}{g'(\alpha)} \text{ since } g(c) = f(c) = 0$$

Intermediate Value Theorem

f continuous on $[a, b]$.

For $f(a) > c > f(b)$, there is $x \in [a, b]$ such that $f(x) = c$.

proof only given for IVT for Derivs below.

Intermediate Value Theorem for Derivatives

$f : [a, b] \to \mathbb{R}$ differentiable on (a, b).

If $f'_+(a)$ exists and is < 0,

$f'_-(b)$ exists, > 0.

Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

*Wilson diverges from Apostle's proof.

f is continous on $[a, b]$ (on (a, b) since differentiable and a since $f'_+(a)$ exists, implicitly indicating continuity).

There there exists a least value for the function, say at c:

$$f(c) \leq f(x) \text{ for all } x \in [a, b]$$

If $c \neq a : f'_+(a) < 0 \to f(x) < f(c)$ in $(a, a + \delta)$.

If $c \neq b : f'_-(b) > 0 \to f(x) < f(b)$ in $(b - \delta, b)$. ?

Generalization

For t between $f'_+(a), f'_-(b)$, there exists $c \in [a, b]$ such that $f'(c) = t$.

(or if $a = c$ ($b = c$), then $f'_+(a) = f'_+(t)$)

*careful, this doesn't imply derivative is continuous. e.g.,

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$$

but $\lim_{x \to 0} f'(x)$ doesn't exists, hence the derivative is not continuous.
IVT for derivatives revisit as well as discontinuity idea below

Discontinuity of Continuous Increasing

f on (a, b) is increasing.
Then,
$f(x^+)$ exists for all x such that $f(x) \leq f(x^+)$ and f is continous at x

$$\iff$$

$$f(x^-) = f(x^+)$$

For any fixed x_0.
$f(x_0)$ is a lower bound of $\{f(x) : x > x_0\}$.
Let α be the infimum, then
$f(x_0) \leq \alpha$ (since $f(x_0)$ is a lower bound.
If $x_0 < x' < x$, $\alpha \leq f(x') \leq f(x) \leq \alpha + \epsilon$.
Thus, $|f(x)\alpha - \alpha| < \epsilon$.

$$\boxed{\text{Increasing function can only have jump discontinuities.}}$$

$f'(x)$ exists and $f'(x)$ is monotonic, then $f'(x)$ is continuous.
If it wasn't, we'd have a jump discontinuity, violating the IVT.

3 Cardinality

Equinumerous sets A, B means there is a bijective map between the sets.
denoted, $A \sim B$.

This defines an equivalence relation.

$A \sim B$, $B \sim C \to A \sim C$ by composition of bijective functions.

countable means finite or countably infinite.

*for countable sets, we can assume without loss of generality the set is $\mathbb{N}$.

A, B countable $\rightarrow A \times B$ is countable.

The output is never the same for any two inputs, based on the prime factorization of integers.

Idea of counting $\mathbb{N}$ through sieve.

A, B countable $\rightarrow A \cup B$ countable.

For $x \in A \cup B$,

$$f(x) = \begin{cases} 2^{f_1(x)} & x \in A \\ 3^{f_2(x)} & x \in B \setminus A \end{cases}$$

second line accounts for elements in both sets.

$\boxed{\mathbb{Q} \text{ is countable.}}$

$\mathbb{Q} = \{\pm \frac{m}{n}\}$.

For $x \in \mathbb{Q}$,

$$f(x) = \begin{cases} 23^m 5^n & x > 0 \\ 2^2 3^m 5^n & x < 0 \end{cases}$$

$\cup_1^\infty A_i$ is countable for A_i countable

$k(x) = i$ for i the smallest A_i containing x.

Then, $f(x) = 2^{k(x)} 3^{f_{k(x)}(x)}$

$\boxed{\text{No set is equinumerous to its powerset}}$

Suppose between S and $P(S)$, the powerset of S, there exists a bijective map $f(S)$.

KEY: $R = \{a \in S : a \notin f(a)\}$

(a) R is a subset of S, hence in range of $f(S)$: $R = f(\alpha)$

(b) α is in R, hence $\alpha \notin f(\alpha)$.

idea: $R = \{a : a \notin a\}$. Ask is $a \in R$?

"Russel's Paradox"

Godel talked about a similar notion for sets: "I am a false statement". Prove the statement.

$\boxed{\text{Power set of } \mathbb{N} \text{ is uncountable}}$

$\mathbb{N}$ is not equinumerous with its powerset by above.

$\boxed{\text{R is uncountable}}$

(1) $[0, 1)$ is uncoutable

proof: suppose $f : \mathbb{N} \rightarrow [0, 1)$

$f(1) = a_{11}a_{12}a_{13}...$, some number.

$f(2) = a_{11}a_{12}a_{13}...$, some number
Then, defined b_k to differ at the last digit from all possible outputs.
Hence, $[0,1)$ uncountable.

(2) A subset of $\mathbb{R}$ is uncountable, hence $\mathbb{R}$ is too.

Let $E \subset (0, \infty)$.

$$M = Sup\{\sum_{x \in F} x : F \subset E \text{ and finite}\}$$

If $M < \infty$, E is countable.

By way of contradiction, suppose E is uncountable and $M < \infty$. Then idea unclear of proof and what we're trying to prove.

Tips

- "epsilon the sup": $s = \sup\{A\}$ implies there is $x \in A$ such that $x < \sup -\epsilon$.

Chapter 6
Functions of Bounded Variation

Let $\mathcal{P}[a,b]$ be the collection of all possible **partitions** of $[a,b]$.

f is of **bounded variation** on $[a,b]$ if for *any* partition,

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

*M need not be fixed, as long as < ∞ (follows from BV $\iff$ $V_f < \infty$).

Sweet consequences of BV

- f increasing on $[a,b] \implies$ BV
- f BV $\implies$ f is bounded.
- f cont on $[a,b]$, f' exists, and $|f'(x)| \leq R \implies f$ BV

$$\boxed{\textbf{continuous} \not\Longrightarrow \text{BV}}$$

counterexample with $x\cos(\pi/x)$ on $[0,1]$ see book for deets

Useful idea:

f increasing $\implies \sum f(x_k^+) - f(x_k^-) \leq f(b) - f(a)$ for any partition.
"degree of discontinuity, or jump"

proof: pick $y_i \in (x_i, x_{i+1})$
So,

$$f(x_i) \leq f(y_i) \leq f(x_{i+1})$$

then, idea: $f(y_i)$ and $f(y_{i-1})$ surround $f(x_i)$.
So, $\sum f(y_i) \geq \sum f(x_i^{\pm})$, but sum using y_i is bounded by $f(b) - f(a)$. $\square$

idea: surround x_i with points y_i whose sum is less than $f(b) - f(a)$

If f is **monotonic** on $[a, b]$, then set of discontinuities is **countable**.
Proof: discontinuity at x means: $f(x^+) > f(x^-)$
Look at all discontinuities with jump greater than $1/n$
Let m be the number of discontinuities, then

$$m\frac{1}{n} \leq \sum f(x_k^+) - f(x_k^-) \leq f(b) - f(a)$$

so, $m \leq n(f(b) - f(a))$.
Let $n \to \infty$, to get countably many. $\square$

4 Total Variation

The **total variation** of f on $[a, b]$ is

$$V_f(a, b) = sup\{\sum |\Delta f_k| \text{ of all partition}\}$$

Properties

- $V_f = 0 \iff f$ is **constant**
- f is of **BV** $\iff V_f$ is **finite**.
- $\frac{1}{f}$ is of **BV** if $0 < m \leq |f(x)|$ for all x.
  condition ensures $\frac{1}{f}$ is never zero

Finer partition $\implies \sum |\Delta f_k|$ **increases**.
look at difference $\sum_{p'} - \sum_p$ where p' is finer.

Algebra of Total Variation

- $V_{f+g} = V_f + V_g$ trianlge inequality with sums

- $V_{fg} \leq \sup(|g(x)|)V_f + \sup(|f(x)|)V_g$
- $V_f(a,b) = V_f(a,c) + V_f(c,b)$ "total variation breaks up over interval sums"
  First $V_f(a,c) + V_f(c,b) \leq V_f(a,b)$, follows by taking union of partiions, since V_f is supremum over any partition.
  Second inequality follows by adding c to partiion of (a,b).
  - f BV on $(a,b) \implies f$ BV on (a,c) and (c,b).

$V_f - f$ is **increasing**
for $x < y$, consider $V(a,y) - f(y) - (V(a,x) - f(x))$ use $f(y) - f(x) \leq V(x,y)$.

f on $[a,b]$ is of **bounded variation**

$$\iff$$

f can be expressed as the difference of two increasing functions.

*representation as two increasing functions is not unique.
$\rightarrow f = f_1 - f_2$ use algebra above.
$\leftarrow f = V - (V - f)$, with $V - f$ is increasing and V increasing.

Chapter 7
Riemann-Stieltjes Integral

We begin with a more general concept than traditional Riemann Intergal (cutting up into rectangles) using **two functions** of x, $f(x)$ and $\alpha(x)$.
Allows us to compute integral of **partly continuous** functions (useful in physics)

Notation

For a partition P, $||P||$ is called the norm of P and is the **length of the largest subinterval.**

A partition A is **finer** than P if A contains all the points of P.

For Riemann-Stieltjes Integration we **assume** $f(x), \alpha(x)$ are **real-valued**, **bounded** functions.

Riemann-Stieltjes Integral

The **Riemann-Stieltjes sum** of f with respect to α and a partition P is

$$S(P,f,\alpha) = \sum_{k=1}^{n} f(t_k)[\alpha(x_k) - \alpha(x_{k-1})]$$

for any choice of t_k in $[x_{k-1}, x_k]$.

f is **Riemann-Stieltjes Integrable** if for all partitions P finer than P_ϵ,

$$|S(P,f,\alpha) - A| < \epsilon.$$

The value A, denoted $\int_a^b f(x)d\alpha(x)$, is **unique**.
proof of uniquess:
If $A_1 \neq A_2$ both satisfy integral, then, for P_ϵ finer than both P_1, P_2,

$$|A_1 - A_2| < 2\epsilon \implies A_1 = A_2.\square$$

Properties of the Riemann-Stieltjes Integral

c_1, c_2 constants

- "sum/constant multiple": $\int_a^b (c_1 f(x) + c_2 g(x))d\alpha(x) = c_1 \int_a^b f(x)d\alpha(x) + c_2 \int_a^b g(x)d\alpha(x)$.
  proof follows direclty by manipulating sums
- "sum/multiple over $\alpha(x)$": $\int_a^b f(x)d(c_1\beta(x) + c_2\gamma(x)) = c_1 \int_a^b f(x)d\beta(x) + c_2 \int_a^b f(x)d\gamma(x)$.
- "split over interval": $\int_a^b = \int_a^c + \int_c^b$, if two of the three integrals exist.
  *can't be used to prove $\int_a^c$ exists

We define $\int_a^b f(x)d\alpha(x) = -\int_b^a f(x)d\alpha(x)$.

*Careful: $S(P,f,\alpha) - S(P,f,\alpha) \neq 0$, depends on choice of t_k.

Wilson's "Cauchy-Criterion" like Result

recall Cauchy Convergence Criterion: x_n converges $\iff |x_i - x_j| < \epsilon$ for any $i > N$.
For all P, Q finer than some P_ϵ,

$$f(x) \in R(\alpha) \iff |S(P,f,\alpha) - S(Q,f,\alpha)| < \epsilon.$$

Proof: $\to$) triangle.
$\leftarrow$) construct sequence of partitions $P_1 \subseteq P_2 \dots$.
Then, $S(P_k, f, \alpha)$ satisfies Cauchy Criterion, thus converges to a limit (there's a bit more to it). $\square$

up to exam 1

Integration by Parts

f, α bounded.

$$\boxed{f \in R(\alpha) \iff \alpha \in R(f)}$$

and

$$\boxed{\int_a^b f d\alpha + \int_a^b \alpha df = f\alpha\Big|_a^b}$$

proof: $f\alpha\Big|_a^b = \sum_P \Delta(f\alpha)$ since it telescopes.
Then, $f\alpha\Big|_a^b - S(P, \alpha, f) = S(P', f, \alpha)$
for some P' which includes additional points
implying $|S(P', f, \alpha) - \int_a^b f d\alpha| < \epsilon$. $\square$
*easy to misread sum with respect to α as f! careful!

5 Lower and Upper Riemann-Stieltjes Integral

Notation

- $M_k(f) = \sup f(x)$ for $x \in [x_{k-1}, x_k]$
  - $m_k(f)$ for inf
- **Upper Stieltjes Sum**: $U(P, f, \alpha) = \sum_P M_k(f)\Delta\alpha$
  - lower, $L(P, f, \alpha)$ is with m_k
- For $\alpha \nearrow$, **Upper Stieltjes Integral** $\int_a^{\bar{b}} f d\alpha = \bar{I} =$ inf of $U(P, f, \alpha)$ over all partitions.
  *CAREFUL: Upper -> Inf

Properties when $\alpha \nearrow$

- $L(P, f, \alpha) \leq S \leq U$

- For $P' \supseteq P$, $U(P') \leq U(P)$ idea: sup f on larger interval $\geq$ on smaller interval prove using only one additional point, then generalize
  - $L(P') \geq L(P)$
- For any two partiions, $L(P_1) \leq U(P_2)$ by above $L(P_1) \leq L(P_1 \cup P_2) \leq U(P_1 \cup P_2) \leq U(P_2)$
- $\underline{I} \leq \bar{I}$ key: $U \geq L$. So $\inf U \geq u \geq l > supL - \epsilon$

Triangle for Upper and lower

$$\bar{\int_a^b} = \bar{\int_c^b} + \bar{\int_c^b}$$

However,

$$\bar{\int_a^b} f + g d\alpha \leq \bar{\int_a^b} f d\alpha + \bar{\int_a^b} g d\alpha$$

(similarly with $\geq$ for lower integral)

6 Riemann's Condition

f satisfies **Riemann's condition** if for all P finer than P_ϵ

$$0 \leq U - L \leq \epsilon$$

For $\alpha \nearrow$, below are equivalent

1. $f \in R(\alpha)$
2. f satisfies **Riemann's condition**
3. $\underline{I} = \bar{I}$

(1 $\rightarrow$ 2)
L and U can be considered partitions; use def so that $|L - U| < \epsilon$
(1 $\rightarrow$ 3)
$\int_a^{\bar{b}} f d\alpha = inf(U) < sup(L) = \int_{\underline{a}}^b f d\alpha + \epsilon$

Comparison Theorems

For $\alpha \nearrow$, $f, g \in R(\alpha)$,

- If $f(x) \leq g(x)$ for all $x \in [a,b]$,

$$\int_a^b f(x)d\alpha(x) \leq \int_a^b g(x)d\alpha(x)$$

  proof: $U(P,f,\alpha) \leq U(P,g,\alpha)$

- $|f| \in R(\alpha)$ and

$$\left|\int_a^b f(x)d\alpha(x)\right| \leq \int_a^b |f(x)|d\alpha(x)$$

- $f^2 \in R(\alpha)$

$$\begin{aligned} f(t)^2 - f(s)^2 &= (f(t)+f(s))(f(t)-f(s)) \\ &\leq 2Msup(f(t)-f(s)) && \text{(M is bound of f)} \\ &\leq 2M(M_k(f)-m_k(f)) \\ &\leq 2M * U - L \leq 2M\epsilon \leq \epsilon && \text{(with adjustment)} \end{aligned}$$

- product $f(x)g(x) \in R(\alpha)$

  *Careful: $|f| \in R(\alpha) \not\Longrightarrow f \in R(\alpha)$

Example,

$$f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ -1 & : x \notin \mathbb{Q} \end{cases}$$

U(|f|) = L(|f|) = 1, but U(f) = 1 and L(f) = -1.

7 Integrators of Bounded Variation

Assume α is of **bounded variation**.
Let $V(x)$ be the total variation of α on $[a,b]$ with $V(a) = 0$.
Then for f **bounded** on $[a,b]$,

$$f \in R(\alpha) \implies f \in R(V)$$

in class proved: $\alpha \in BV, f \in R(\alpha), V_f = V_\alpha \implies f \in R(\alpha)$
For α of **bounded variation**, $f \in R(\alpha)$ on $[a, b]$

$$\boxed{f \in R(\alpha) \text{ on every subinterval of } [a, b]}$$

For $f, g \in R(\alpha)$ with $\alpha \nearrow$

$$\int_a^b f(x)g(x)d\alpha(x) = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$$

proof ???

8 When does Riemann-Stieltjes exist?

Big Theorem

$$\boxed{\alpha \in \text{BV}, f \text{ continuous} \implies f \in R(\alpha)}$$

proof: only consider $\alpha \nearrow$ since BV implies α can be written as difference of increasing functions.
f cont $\implies$ uniformly continuous (by last semester)
Choose partition P such that $||P|| < \delta$

$$\begin{aligned} U - L &= \sum (M_k - m_k)\Delta\alpha \\ &\leq \epsilon \sum \Delta\alpha && \text{(by uniform cont)} \\ &\leq \epsilon \end{aligned}$$

Consequences:
1. $\int_a^b f(x)dx$ exists for f continous!
2. $f \in BV, \alpha cont \implies f \in R(\alpha)$
by Integration by Parts $\alpha \in R(f) \iff f \in \alpha(f)$

Exercises

1. Find $f(x)$, $\alpha(x)$ such that the Riemann-Stieltjes integral does not exist. On $[-1, 1]$,

$$f(x) = \alpha(x) \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0 \end{cases}$$

Select partition to include point 0. Then, somehow contradicts?

Fundamental Theorems of Calculus

$\alpha \nearrow, f \in R(\alpha)$, and there exists m, M such that $m \leq f(x) \leq M$, then

$$\int_a^b f(x)\, d\alpha = u(\alpha(b) - \alpha(a) \qquad \text{(some } u \in [m, M])$$

proof: choose u = $\frac{\int_a^b f(x) d\alpha}{\alpha(b)-\alpha(a)}$, with $m \leq U \leq M$ since $m \leq L(P) \leq U(P) \leq M$.

Intermediate Value Theorem for Integrals

f cont, $\alpha \nearrow$, then there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)\, d\alpha(x) = f(x_0)(\alpha(b) - \alpha(a))$$

proof: since f is cont., use IVT

First Fundamental Theorem of Calculus

*slightly less general than apostol
$\alpha \nearrow, f \in R(\alpha)$. Define, $F(x) = \int_a^x f(t) d\, \alpha(t)$.

1. $F(x) \in BV$
2. $F(x)$ is continuous where α is
3. $F'(x)$ exists where $\alpha'(x)$ exists and f is cont.
   (a) $F'(x) = f(x)\alpha'(x)$ when it exists.

proof

Second Fundamental Theorem of Calculus

*differs from Apostol's
$f \in R$, g is cont on $[a, b]$, $g' = f(x)$ exists on (a, b)
(continuity assumption ensures g cont on endpoints)
Then,

$$\int_a^b f(x)\, dx = \int_a^b g'(x)\, dx = g(x)\Big|_a^b$$

*baby version assumes f is continuous, which this version doesn't!

Extensions of the FTC

$\alpha \nearrow$, f, g $\in R(\alpha)$. Then,

$$f \in R(G), g \in R(F)$$

and

$$\int_a^b f dG = \int_a^b g dF = \int_a^b f(t)g(t)d\alpha(t)$$

*F, G are antiderivatives
proof
For $f \in R, \alpha$ cont. and $\alpha' \in R$,

$$f \in R(\alpha) \text{ and } \int_a^b f \, d\alpha = \int_a^b f(t)\alpha'(t)dt$$

by previous theorem

Change of Variables

Part I

Chapter 8: Infinite Produtcs (and series review)

Infinite Series

Infinite Products

Part II

Chapter 9: Sequences of Functions

goal: sequence f_n has a property say cont., integrable etc. does $\lim_{n\to\infty} f_n$?

A sequence of functions $\{f_n\}_1^\infty$ is **uniformly Cauchy** if

$$|f_n - f_m| < \epsilon \qquad \text{(for all m, n} \geq \text{some N)}$$

$$\boxed{f_n \text{ converges uniformly to some } f \iff \{f_n\}_1^\infty \text{ is uniformly Cauchy}}$$

proof later

copy application

Weierstrass M-Test for Infinite Products

$f_n : A \to \mathbb{C}, |f_n| \leq M_n$ for all $x \in A$, and $\sum_1^\infty M_n = B < \infty$, then

$$\prod_1^\infty (1 + f_n) \text{ converges uniformly to some } f \qquad \text{(f also on A} \to \mathbb{C})$$

proof to come

copy application

Uniform Convergence of Integrals and Derivatives

BIG

$\alpha \in$ BV on $[a, b]$ in $\mathbb{R}$, f_n converges uniformly, and $f_n \in R(\alpha)$, then

$$f \in R(\alpha) \text{ and } \int_a^b f_n \, d\alpha \to \int_a^b f \, d\alpha$$

"uniformly convergent sequence, lim $\int f_n$ = $\int$ lim f_n"

copy application

for derivatives

*slightly different from Apostol's who assume $\lim_{n\to\infty} f_n(x_0)$ exists

BIG

If $f_n \to \mathbb{R}$ and f_n' exists; $f_n' \to g$ uniformly (for some g); $f_n(x_0) = 0$ for some x_0 and all n.
THEN,

1. $f_n \to f$ **uniformly** (some f)
2. f' exists with $f' = g$

Power Series

A **power series** in $z - z_0$ is of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converg with $a_n, z, z_0 \in \mathbb{C}$.

If power series converges at a **single point**, it converges **uniformly**.

A power series **converges absolutely** if

$$|z - z_0| < \frac{1}{\lim_{n\to\infty} \sup \sqrt[n]{|a_n|}}$$

The values of z satisfying is the above forms the **disk of convergence** with center z_0. The series **converges uniformly** on every **compact** subset of the disk of convergence.

proof by root test

see next file