

Calculus: single

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Relearn

1 Functions

1.1 e, sin, and cos

Euler's Formula

for now take as a fact

$$e^{ix} = \cos(x) + i \sin(x)$$

Define e^x using Taylor as

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

also sometimes defined as $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$.

By Euler's Formula,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \quad (\text{breaking up real and imaginary}) \\ &= \cos(x) + i \sin(x). \end{aligned}$$

1.2 Taylor

The **Taylor Series** of a function f at $x = 0$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

*Taylor series can be differentiated term by term!

At any other point a it's

$$f(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}(a)}{2!}(x - a)^2 + \dots$$

(remember, a must be inside the domain of convergence for the Taylor Poly)

Does Taylor exist for every function?

NO, [more on this later](#)

Geometric Series

$$1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1-r} \quad (\text{for } |r| < 1)$$

proof: $s_n = 1 + r + r^2 + r^3 + \dots + r^n$

Then, $s_n - r s_n = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$

(as $n \rightarrow \infty$, this becomes $\frac{1}{1-r}$ if $|r| < 1$)

e.g., A neat example is the **Bessel function** used to describe the waves in water when a stone drops:

$$\begin{aligned} J_0(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2} \\ &= 1 - \frac{1}{2^2} x^2 + \frac{1}{2^4 (2!)^2} x^4 - \frac{1}{2^6 (3!)^2} x^6 + \dots \end{aligned}$$

note the Bessel function is defined by its Taylor series.

It's a decaying cosine-like wave, since the denominator grows faster than cosine's.

Orders of Growth and Error Size

Big-O: "not faster than"

- $f(x) = O(1)$ if $|f(x)| \leq c * 1$ as $x \rightarrow \infty$ (for some c)
- $f(x) = O(g(x))$ if $|f(x)| \leq c * |g(x)|$ as $x \rightarrow \infty$ (for some c)

Can be equivalently defined for $x \rightarrow 0$ to describe how quickly a function decays.

Little-o: "ultimately smaller than"

- $f(x) = o(g(x))$ means $\frac{f(x)}{g(x)} \rightarrow 0$ as $g(x) \rightarrow 0$.

1.3 Hyperbolic Trigonometric Functions

These functions have similar algebraic properties to sin, cos and similarities in their Taylor expansions.

(Taylor is the same without alternating signs—all +)

Define

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} \end{aligned}$$

A variation of the Pythagorean Theorem holds:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Geometrically, cosh and sinh give x, y coordinates on the hyperbola $x^2 - y^2 = 1$ (similar to sin, cos with the unit circle).

What are hyperbolas?

They are two parabolas approaching asymptotes (facing up/down or right/left):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

(negative sign on x, instead of y, for up/down)

Tricks

Multiplying Taylor Polynomials

$$\begin{aligned} \cos(x) \cos(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) \\ &= 1 + \left(-\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \left(\frac{1}{4!} + \frac{1}{2!}\frac{1}{2!} + \frac{1}{4!}\right)x^4 + \dots \end{aligned}$$

idea: what are the coefficients of x^4

Binomial Series

For $|x| < 1$ and any $\alpha \in \mathbb{C}$,

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k. \end{aligned}$$

by computing derivatives and using Taylor Expansion

The Binomial Theorem refer to a slightly different case

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

*note here $n \in \mathbb{N}$

Exercises

1. Find the Taylor Expansion of $\ln(1+x)$ using $\ln(1+x) = \int \frac{1}{1+x} dx$ and the geometric series.
2. Similarly find the Taylor Expansion of $\arctan(x)$ using $\arctan(x) = \int \frac{1}{1+x^2} dx$ and the geometric series.

Summarizing, using Geometric + Taylor Expansion we have:
 ONLY FOR $|x| < 1$, based on geometric series convergence

- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

1.4 Limits

The limit as x approaches a of $f(x)$ **exists** if for all x close, but NOT equal to a , $f(x)$ is close to the value of the limit.

A limit may not exist at a discontinuity, blow-up, or oscillation.

A function is **continuous** at a if $\lim_{x \rightarrow a} f(x) = f(a)$.

If $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$, then

- $\lim_{x \rightarrow a} f(x) + g(x) = l + m$
- $\lim_{x \rightarrow a} f(x)g(x) = lm$.
- "root of a limit" $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$.

If a function is continuous, then you can evaluate a limit by simply plugging in the value a .

Exercises

1. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$. use Taylor (about $x=0$), since we are interested in $\sin(x)$ as x approaches 0

2 Differentiation

Sometimes the derivative is written as $\dot{y} = \frac{dy}{dx}$

An alternate definition of the derivative is

$$f(a+h) = f(a) + f'(a)h + O(h^2)$$

indicating the derivative gives a first-order approximation.

An **operator** is a mapping from one vector space (or module) to another.

The **curvature** of a function is defined as

$$\frac{f''}{(1 + (f')^2)^{3/2}}$$

Newton's Method

A good way to approximate the root of a function that's differentiable.

1. Guess a root a_0
2. Improve: $a_1 = a_0 - \frac{f(x)}{f'(x)}$
from $0 = f(a_0) + f'(a_0)(a_1 - a_0)$, line approximation of zero
3. Repeat

Derivative of a^x

To find $\frac{d}{dx}a^x$:

$$y = a^x \implies \ln(y) = x \ln(a)$$
$$\frac{d}{dx} \ln(y) = 1/y \frac{dy}{dx} = \ln(a)$$

Thus, $dy/dx = y \ln(a) = a^x \ln(a)$.

Optimization

A **critical point** is an input where $f(x)$ has a derivative of zero or the *derivative* is undefined; critical points include max/min points. (max/min \implies critical point)

Second Derivative Test: $f''(a) > 0 \implies$ min. ($f''(a) = 0$ says nothing)

Global max/min occur at a critical point or end points!

3 Integration

4 Series

Curiosities

Striling's Approximation

Gives a good asymptotic (as n gets bigger) approximation for $n!$:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Typically written as $\ln(n!) = n \ln(n) - n + O(\ln(n))$.