

# Calculus: single

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Relearn

## 1 Functions

### 1.1 e, sin, and cos

Euler's Formula

for now take as a fact

$$e^{ix} = \cos(x) + i \sin(x)$$

Define  $e^x$  using Taylor as

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

also sometimes defined as  $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$ .

By Euler's Formula,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \quad (\text{breaking up real and imaginary}) \\ &= \cos(x) + i \sin(x). \end{aligned}$$

### 1.2 Taylor

The **Taylor Series** of a function  $f$  at  $x = 0$  is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

\*Taylor series can be differentiated term by term!

At any other point  $a$  it's

$$f(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}(a)}{2!}(x - a)^2 + \dots$$

(remember,  $a$  must be inside the domain of convergence for the Taylor Poly)

## Does Taylor exist for every function?

NO, [more on this later](#)

### Geometric Series

$$1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1-r} \quad (\text{for } |r| < 1)$$

proof:  $s_n = 1 + r + r^2 + r^3 + \dots + r^n$

Then,  $s_n - r s_n = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$

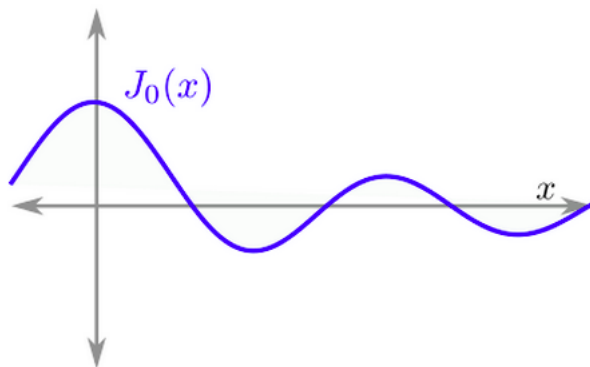
(as  $n \rightarrow \infty$ , this becomes  $\frac{1}{1-r}$  if  $|r| < 1$ )

e.g., A neat example is the **Bessel function** used to describe the waves in water when a stone drops:

$$\begin{aligned} J_0(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2} \\ &= 1 - \frac{1}{2^2} x^2 + \frac{1}{2^4 (2!)^2} x^4 - \frac{1}{2^6 (3!)^2} x^6 + \dots \end{aligned}$$

note the Bessel function is defined by its Taylor series.

It's a decaying cosine-like wave, since the denominator grows faster than cosine's.



## Orders of Growth and Error Size

**Big-O:** "not faster than"

- $f(x) = O(1)$  if  $|f(x)| \leq c * 1$  as  $x \rightarrow \infty$  (for some  $c$ )
- $f(x) = O(g(x))$  if  $|f(x)| \leq c * |g(x)|$  as  $x \rightarrow \infty$  (for some  $c$ )

Can be equivalently defined for  $x \rightarrow 0$  to describe how quickly a function decays.

**Little-o:** "ultimately smaller than"

- $f(x) = o(g(x))$  means  $\frac{f(x)}{g(x)} \rightarrow 0$  as  $g(x) \rightarrow 0$ .

### 1.3 Hyperbolic Trigonometric Functions

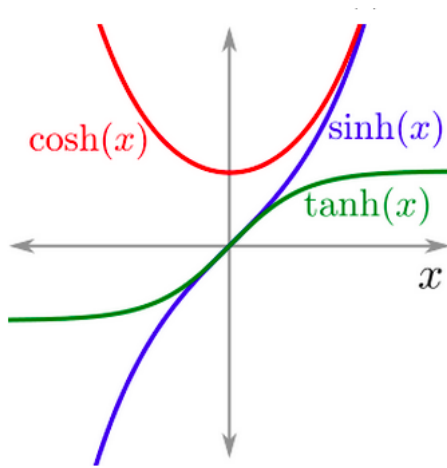
These functions have similar algebraic properties to  $\sin$ ,  $\cos$  and similarities in their Taylor expansions.

(Taylor is the same without alternating signs—all +)

Define

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



A variation of the Pythagorean Theorem holds:

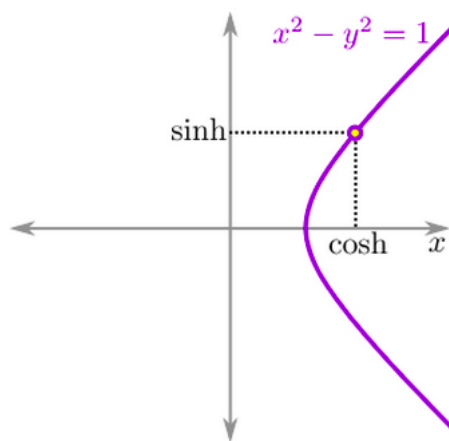
$$\cosh^2(x) - \sinh^2(x) = 1$$

Geometrically,  $\cosh$  and  $\sinh$  give  $x, y$  coordinates on the hyperbola  $x^2 - y^2 = 1$  (similar to  $\sin, \cos$  with the unit circle).

### What are hyperbolas?

They are two parabolas approaching approaching asymptotes (facing up/down or right/left):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



(negative sign on x, instead of y, for up/down)

## Tricks

### Multiplying Taylor Polynomials

$$\begin{aligned}\cos(x) \cos(x) &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) \\ &= 1 + \left(-\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \left(\frac{1}{4!} + \frac{1}{2!} \frac{1}{2!} + \frac{1}{4!}\right)x^4 + \dots\end{aligned}$$

idea: what are the coefficients of  $x^4$

### Binomial Series

For  $|x| < 1$  and any  $\alpha \in \mathbb{C}$ ,

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.\end{aligned}$$

by computing derivatives and using Taylor Expansion

The Binomial Theorem refer to a slightly different case

### Binomial Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

\*note here  $n \in \mathbb{N}$

## Exercises

1. Find the Taylor Expansion of  $\ln(1+x)$  using  $\ln(1+x) = \int \frac{1}{1+x} dx$  and the geometric series.
2. Similarly find the Taylor Expansion of  $\arctan(x)$  using  $\arctan(x) = \int \frac{1}{1+x^2} dx$  and the geometric series.

Summarizing, using Geometric + Taylor Expansion we have:

ONLY FOR  $|x| < 1$ , based on geometric series convergence

- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

## 1.4 Limits

The limit as  $x$  approaches  $a$  of  $f(x)$  **exists** if for all  $x$  close, but NOT equal to  $a$ ,  $f(x)$  is close to the value of the limit.

A limit may not exist at a discontinuity, blow-up, or oscillation.

A function is **continuous** at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

If  $\lim_{x \rightarrow a} f(x) = l$  and  $\lim_{x \rightarrow a} g(x) = m$ , then

- $\lim_{x \rightarrow a} f(x) + g(x) = l + m$
- $\lim_{x \rightarrow a} f(x)g(x) = lm$ .
- "root of a limit"  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ .

If a function is continuous, then you can evaluate a limit by simply plugging in the value  $a$ .

## Exercises

1. Evaluate  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ . use Taylor (about  $x=0$ ), since we are interested in  $\sin(x)$  as  $x$  approaches 0

## 2 Differentiation

Sometimes the derivative is written as  $\dot{y} = \frac{dy}{dx}$

An alternate definition of the derivative is

$$f(a+h) = f(a) + f'(a)h + O(h^2)$$

indicating the derivative gives a first-order approximation.

An **operator** is a mapping from one vector space (or module) to another.

The **curvature** of a function is defined as

$$\frac{f''}{(1 + (f')^2)^{3/2}}$$

## Newton's Method

A good way to approximate the root of a function that's differentiable.

1. Guess a root  $a_0$
2. Improve:  $a_1 = a_0 - \frac{f(x)}{f'(x)}$   
from  $0 = f(a_0) + f'(a_0)(a_1 - a_0)$ , line approximation of zero
3. Repeat

## Derivative of $a^x$

To find  $\frac{d}{dx}a^x$ :

$$y = a^x \implies \ln(y) = x \ln(a)$$
$$\frac{d}{dx} \ln(y) = 1/y \frac{dy}{dx} = \ln(a)$$

Thus,  $dy/dx = y \ln(a) = a^x \ln(a)$ .

## Optimization

A **critical point** is an input where  $f(x)$  has a derivative of zero or the *derivative* is undefined; critical points include max/min points. (max/min  $\implies$  critical point)

Second Derivative Test:  $f''(a) > 0 \implies$  min. ( $f''(a) = 0$  says nothing)

**Global max/min** occur at a critical point or end points!

## 3 Integration

## 4 Series

## Curiosities

### Striling's Approximation

Gives a good asymptotic (as  $n$  gets bigger) approximation for  $n!$ :

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Typically written as  $\ln(n!) = n \ln(n) - n + O(\ln(n))$ .