Calculus: single

The Robert Ghrist Approach
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Relearn

1 Functions

1.1 e, sin, and cos

Euler's Formula $\boxed{e^{ix} = \cos(x) + i\sin(x)}$

for now take as a fact

Define e^x using Taylor as

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

By Euler's Formula,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) && \text{(breaking up real and imaginary)} \\ &= \cos(x) + i\sin(x). \end{aligned}$$

1.2 Taylor

The **Taylor Series** of a function f at $x = 0$ is

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots}$$

*Taylor series can be differented term by term!

At any other point a it's

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}}{2!}(x-a)^2 + \dots$$

(remember, a must be inside the domain of convergece for the Taylor Poly)

Does Taylor exist for every function?

NO, more on this later

Geometric Series

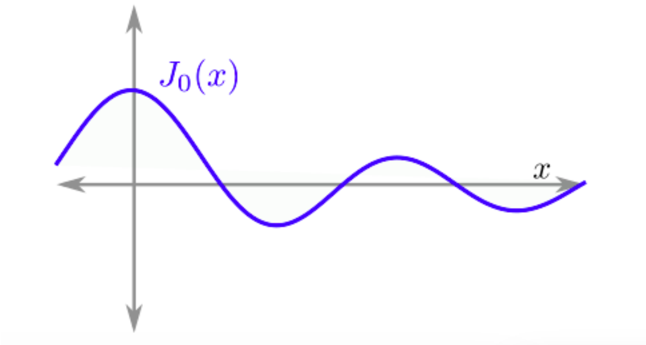
$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \qquad \text{(for } |r| < 1\text{)}$$

proof: $s_n = 1 + r + r^2 + r^3 + \cdots + r^n$
Then, $s_n - rs_n = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$
(as $n \to \infty$, this becomes $\frac{1}{1-r}$ if $|r| < 1$)

e.g., A neat example is the **Bessel function** used to describe the waves in water when a stone drops:

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$$
$$= 1 - \frac{1}{2^2}x^2 + \frac{1}{2^4(2!)^2}x^4 - \frac{1}{2^6(3!)^2}x^6 + \ldots$$

note the Bessel function is defined by its Taylor series.
It's a decaying cosine-like wave, since the denominator grows faster than cosine's.



Orders of Growth and Error Size

Big-O: "not faster than"

- f(x) = O(1) if $|f(x)| \leq c * 1$ as $x \to \infty$ (for some c)
- f(x) = O(g(x)) if $|f(x)| \leq c * |g(x)|$ as $x \to \infty$ (for some c)

Can be equivalently defined for $x \to 0$ to describe how quickly a function decays.
Little-o: "ultimately smaller than"

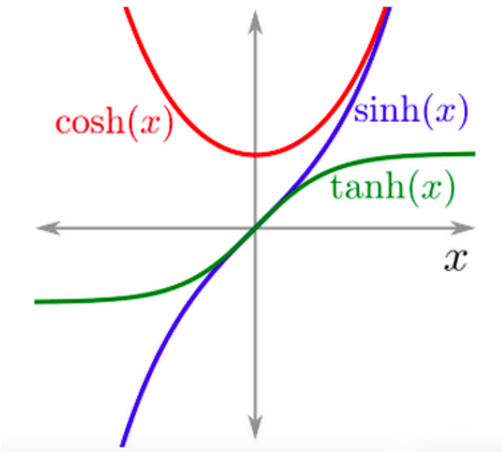
- $f(x) = o(g(x))$ means $\dfrac{f(x)}{g(x)} \to 0$ as $g(x) \to 0$.

1.3 Hyperbolic Trigonometric Functions

These fuctions have similar algebraic properties to $\sin, \cos$ and similarities in their taylor expansions.

(Taylor is the same without alternating signs–all +)
Define

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



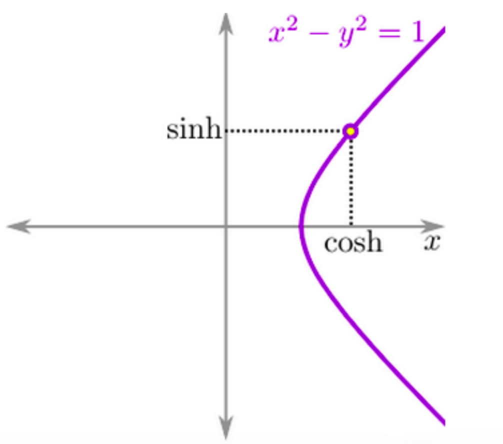
A variation of the Pythagorean Theorem holds:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Geometrically, $\cosh$ and $\sinh$ give x, y coordinates on the hyperbola $x^2 - y^2 = 1$ (similar to $\sin, \cos$ with the unit cirlce).

What are hyperbolas?
They are two parabolas approaching approaching asymptotes (facing up/down or right/left):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



(negative sign on x, instead of y, for up/down)

Tricks

Multiplying Taylor Polynomials

$$\begin{aligned}\cos(x)\cos(x) &= (1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots)(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots)\\ &= 1+(-\frac{1}{2!}-\frac{1}{2!})x^2+(\frac{1}{4!}+\frac{1}{2!}\frac{1}{2!}+\frac{1}{4!})x^4+\dots\end{aligned}$$

idea: what are the coefficients of x^4

Binomial Series (for $|x|<1$)

$$\begin{aligned}(1+x)^\alpha &= 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\dots\\ &= \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k.\end{aligned}$$

by computing derivatives and using Taylor Expansion

Exercises

1. Find the Taylor Expansion of $\ln(1+x)$ using $\ln(1+x)=\int\frac{1}{1+x}\,dx$ and the geometric series.
2. Similarly find the Taylor Expansion of $\arctan(x)$ using $\arctan(x)=\int\frac{1}{1+x^2}\,dx$ and the geometric series.

Summarizing, using Geometric + Taylor Expansion we have:
ONLY FOR $|x|<1$, based on geometric series convergence

- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$
- $\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$

1.4 Limits

The limit as x approaches a of $f(x)$ **exists** if for all x close, but NOT equal to a, $f(x)$ is close to the value of the limit.

A limit may not exist at a discontinuity, blow-up, or oscillation.

A function is **continuous** at a if $\lim_{x\to a} f(x)=f(a)$.

If $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$, then

- $\lim\limits_{x\to a} f(x)+g(x)=l+m$
- $\lim\limits_{x\to a} f(x)g(x)=lm$.
- "root of a limit" $\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}$.

If a function is continuous, then you can evaluate a limit by simply plugging in the value a.

Exercises

1. Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$. use Taylor (about x=0), since we are interested in $\sin(x)$ as x approaches 0

2 Differentiation

3 Integration

4 Series

Curiosities

Striling's Approximation

Gives a good asymptotic (as n gets bigger) approximation for $n!$:

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

Typically written as $\ln(n!) = n\ln(n) - n + O(\ln(n))$.