March 23, 2015

Chapter 5: Limits

Chapter 5: 1, 7, 8, 10, 15, 16, 18, 20, 21

lim means x within δ implies expression within ϵ of limit.

Failed

1ii remember x cannot equal 2, it approaches it!
further factoring tricks:

- $a^3 + b^3 = (a+b)(a^2 - ab - b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

1iv recall polynomial factoring theorems and long division of polynomials. if a value for x is a zero, then polynomial can be factored with $(x - value)$.

8 idea of limits not existing when a product exists.

Tips

Given, $\lim_{x \to 0} \frac{f(x)}{x} = l$,

$$lim_{x \to 0} \frac{f(bx)}{x} = bl.$$

$\frac{f(bx)}{x} = \frac{bf(bx)}{bx}$
Let y be a new input of $f(x)$ then we have bl.

Chapter 6: Continuity

1, 3, 4, 7, 8, 10, 14, 15

f is continuous at a means $$\lim_{x \to a} f(x) = f(a)$$

- rational functions are continuous
  use product, sums of $f(x) = x$ are continuous
- elementary functions: c, x, log, exponents, trig are continuous
  easy to show using idea above

Failed

7. Suppose $f(x+y) = f(x) + f(y)$ and $f(x)$ is continuous at 0. Show $f(x)$ is continuous at any a.
We want $|f(x) - f(a)| < \epsilon$ for all $|x - a| < \delta$.
Tool: show $f(-a) = -f(a)$ (look at $f(a+0)$). Then problem is easy.

10. a. need inequality: $|x| - |y| \leq |x - y|$ 10. b. every continuous function f(x) written as the sum of even and odd function.
even = $\frac{f(x)+f(-x)}{2}$; odd = $\frac{f(x)-f(-x)}{2}$

Tips

graphing functions reveals continuity

Chapter 7: Three Hard Theorems

1(viii),(ix)(x), 3(ii), 5, 6, 10, 15, 18
Extreme Value Theorem and Intermediate Value Theorem and similar ideas.

Extreme Value Theorem

1. continuous $\rightarrow$ bounded
2. $\{f(x) : x \in [a,b]\}$ is non empty, so has l.u.b., say G
3. Suppose no input y generates G.

contradition: consider $g(x) = \frac{1}{G - f(x)}$ continuous, hence bounded, but since G is l.u.b., $g(x) > \frac{1}{\epsilon}$.

add proofs from next chapter

Failed

3. ii. use intermediate value theorem on $g(x) = sin(x) - x + 1$ by finding values of the function greater/less than 0.

15. also IVT

Tips

Chapter 8: Least Upper Bound

skip

$\boxed{\text{any set of reals with an upper bound has a least upper bound.}}$

used to prove three hard theorems.

a function is **uniformly continuous**

$$|x - y| < \delta \to |f(x) - f(y)| < \epsilon$$

Tips

Chapter 9:Derivatives

2,3,4,5,6,8

a function is **differentiable**, if for all a in domain,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

- differentiable implies continuous
  use $\lim_{x \to a} f(x) - f(a)$ multiply by $\frac{h}{h}$
- cont $\not\to$ diff
  $|x|$ is not differentiable at $x = 0$

Failed

2. factor cubic
3. algebra: multiply by the conjugate.
5. recall $[x]$ is the largest integer less than or equal to x

Tips

factoring cubic: either a linear and quadratic or 3 linear terms.
Search for linear factor (root). Think about factors of constant term.

To clean

$$\frac{\sqrt{a+h} - \sqrt{a}}{h}$$

multiply by conjugate $\sqrt{a+h} + \sqrt{a}$.

Chapter 10: differentiation, finding derivatives

1(v)(vi), 2(ix)(xvi), 15,16,18

- $(fg)' = f'g'$ since limits decompose over products
  (similarly with sums and $cf(x)$)
- quotient rule
  key is to write $\frac{f}{g}$ as $f * (\frac{1}{g})$
  $\frac{d}{dx}\frac{1}{g(x)}$ careful to check expression makes sense.
- Chain Rule: $f(g(a))'$ is $f'(g(a))g'(a)$
  use limit definition
  *must be careful to seperate case where $g(a+h) - g(a) = 0$.
  see analysis
-

Chain proof

use $x \to 0$ by def, $\lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} \frac{g(x)-g(a)}{x-a}$
(BIG STEP $x \to a$ becomes $g(x)$)
$= \lim_{g(x)\to g(a)} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$
need to reason that $f(x) - f(a)$ is not zero.

Failed

Good!

Chapter 11: Significance of the Derivative

max, min, critical, l'hopital's rule
26,35,36,37,38, 48, 52, 63

Finding Max/Min

- max/min at a with $f'(a)$ defined $\rightarrow f'(a) = 0$
- $f'(x)$ can equal 0 even when $f(x)$ is not at a local max or min. consider $f(x) = x^3$ with $f'(0) = 0$.

  $$\boxed{f'(x) = 0 \not\rightarrow \text{ local max/min}}$$

  instead, x such that $f'(x) = 0$ is a **critical point**

  - partial converse: $f'(a) = 0$ and $f''(a) > 0$ then min at a

- to find max/min, consider:
  1. critical points
  2. end points
  3. points where $f'(x)$ doesn't exist

  *works for continuous functions by Extreme Value Theorem; can't be sure min/max exist for non-continuous functions.

- **Mean Value Theorem** For f continuous and differentiable on (a,b), there is x such that
$$f'(x) = \frac{f(b)-f(a)}{b-a}$$
"instantaneous rate of change equals the average rate of change at some x"

- $f'(x) = 0$ for all $x \to f(x) = c$
idea: take any interval $[a,b]$, then by IVT $f(b) = f(a)$

- **Cauchy Mean Value Theorem** (generalization of MVT) for f, g continuous on $[a,b]$ and differentiable on (a,b), there is x such that
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}$$
*careful $g'(x)$, $f'(x) \neq 0$.

- **L'Hopital's Rule** For
$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} = 0$$
and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Then,
$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{ exists}$$
and
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
*note applies if $\lim x \to af(x)($ and $g(x)) = \pm\infty$.

Mean Value Theorem proof uses Rolle's Theorem.

Rolle's Theorem: For f continuous and differentiable on (a,b) with $f(a) = f(b)$,

$$\text{there is } x \text{ such that } f'(x) = 0$$

idea there exists min, max by EVT, somewhere inside or at endpoints implying $f'(x) = 0$ for some x

Mean Value Theorem
define $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then use Rolle's Theorem on $h(x)$ to show $h'(y) = 0$ for some y.

Failed

26: try plugging in number, consider 2 cases.

Tips

Chapter 12: Inverse Functions

1;5;6;7i)-iv),vii);8;11a)-c);22;23;26;33;37

something

Failed

Tips

- to find inverse of functions: f(x) = x + 1.
  Let $y = f^{-1}(x)$. Then y + 1 = x , meaing $y = f^{-1}(x) = x - 1$.

Curiosities

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Polynomials

$$1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$$

set $S = 1 + r + ... + r^n$, multiply by r, and solve

Cauchy Schwarz

$$\sum x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

look into proof

Complex: Birth of Sin, Cos

Define, for $z \in \mathbb{C}$: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$

$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} ...$

We derive Euler's formula,

$$e^{ix} = \cos(x) + i\sin(x)$$

Set x above to $2a$, then

$$\begin{aligned} e^{i2a} &= (e^{ia})^2 \\ &= (\cos(a) + i\sin(a))^2 \\ &= \cos^2(a) + 2i\cos(a)\sin(a) - \sin^2(a) \end{aligned}$$

e^{i2a} also equals $\cos(2a) + i\sin(2a)$. So real parts/imaginary parts are equal,

$$\cos(2a) = \cos^2(a) + \sin^2(a)$$

$$\sin(2a) = 2\cos(a)\sin(a)$$

Can prove $\sin(a+b)$ identity similarly.

Complex Numbers

for $a, b \in \mathbb{C}$,

$$a * b = \text{number with length a*b, angle a + b}$$

Trig

Master Identities:
$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$
$\cos(x + y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

*remember sin is odd $(\sin(-x) = -\sin(x))$; cos is even.

from these derive the rest.

Exercsies

1. $\tan^2(x)\sin(x) = \tan^2(x)$

2. $2\cos^2(x) + \sin(x) - 2 = 0$
   use $\cos^2(x) = 1 - \sin^2(x)$.

Log

$\log(ab) = \log(a) + \log(b)$
$\log(a^m) = m\log(a)$
why?

$$\begin{aligned}\log(a^m) &= \log(a...a),\\ &= \log(a) + \log(a) + ... + \log(a),\\ &= m\log(a).\end{aligned}$$

Derivatives

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

proofs?

$\frac{d}{dx}a^x = a^x ln(a)$
$\int \frac{1}{x} = ln(|x|) + c$

Integration Tricks

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

is nontrivial.
Idea is to consider $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy$ and use polar coordinates to integrate. (see machine leanring hw1)

For **Integrating a Polynomial over a sphere** see trick in paper by Folland.

Min and Max in $\mathbb{R}$

$$max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2}$$

(a+b)/2 takes you to midpoint. |a-b|/2 adds the remaining half to the larger number

$$min(a, b) = \frac{a+b}{2} - \frac{|a-b|}{2}$$

similar idea, except goes down by the half