Curiosities

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Polynomials

$$1 + r + r^2 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}$$

set $S = 1 + r + ... + r^n$, multiply by r, and solve

Cauchy Schwarz

$$\sum x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

look into proof

Complex: Birth of Sin, Cos

Define, for $z \in \mathbb{C}$: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$

$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} ...$

We derive Euler's formula,

$$e^{ix} = \cos(x) + i \sin(x)$$

Set x above to $2a$, then

$$\begin{aligned} e^{i2a} &= (e^{ia})^2 \\ &= (\cos(a) + i \sin(a))^2 \\ &= \cos^2(a) + 2i \cos(a) \sin(a) - \sin^2(a) \end{aligned}$$

e^{i2a} also equals $\cos(2a) + i \sin(2a)$. So real parts/imaginary parts are equal,

$$\cos(2a) = \cos^2(a) + \sin^2(a)$$

$$\sin(2a) = 2 \cos(a) \sin(a)$$

Can prove $\sin(a + b)$ identity similarly.

Complex Numbers

for $a, b \in \mathbb{C}$,

$$a * b = \text{number with length a*b, angle a + b}$$

Trig

Master Identities:
$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

*remember sin is odd $(\sin(-x) = -\sin(x))$; cos is even.

from these derive the rest.

Exercsies

1. $\tan^2(x)\sin(x) = \tan^2(x)$
   $\tan^2(x)(\sin(x) - 1) = 0$
   $\tan^2(x) = 0$ or $\sin(x) = 1$

2. $2\cos^2(x) + \sin(x) - 2 = 0$
   use $\cos^2(x) = 1 - \sin^2(x)$.

Log

$\log(ab) = \log(a) + \log(b)$
$\log(a^m) = m\log(a)$
why?

$$\begin{aligned} \log(a^m) &= \log(a ... a), \\ &= \log(a) + \log(a) + ... + \log(a), \\ &= m\log(a). \end{aligned}$$

Derivatives

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 + x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}$$
proofs?

$\frac{d}{dx}a^x = a^x ln(a)$
$\int \frac{1}{x} = ln(|x|) + c$

Integration Tricks

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

is nontrivial.
Idea is to consider $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy$ and use polar coordinates to integrate.
(see machine leanring hw1)

For **Integrating a Polynomial over a sphere** see trick in paper by Folland.

Min and Max in $\mathbb{R}$

$$max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2}$$

(a+b)/2 takes you to midpoint. |a-b|/2 adds the remaining half to the larger number

$$min(a, b) = \frac{a+b}{2} - \frac{|a-b|}{2}$$

similar idea, except goes down by the half

Geometry

area of an equilaterla triangle

$A = \sqrt{3}/4s^2$ derive by bisecting angle, using 30, 60, 90 triangle