# Calculus: single

The Robert Grist Approach
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Relearn

# 1 Functions

## 1.1 e, sin, and cos

Euler's Formula

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

for now take as a fact

Define $e^x$ using Taylor as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \dots$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

By Euler's Formula,

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) \qquad \text{(breaking up real and imaginary)}$$
$$= \cos(x) + i\sin(x).$$

## 1.2 Taylor

The **Taylor Series** of a function $f$ at $x = 0$ is

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots}$$

*Taylor series can be differented term by term!
At any other point $a$ it's

$$f(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}}{2!}(x - a)^2 + \dots$$

(remember, $a$ must be inside the domain of convergece for the Taylor Poly)

**Does Taylor exist for every function?**

NO, more on this later

> **Geometric Series**
>
> $$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \qquad \text{(for } |r| < 1\text{)}$$
>
> proof: $s_n = 1 + r + r^2 + r^3 + \cdots + r^n$
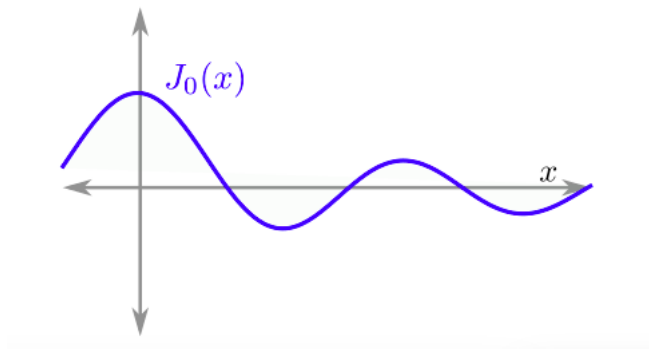> Then, $s_n - r s_n = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$
> (as $n \to \infty$, this becomes $\frac{1}{1-r}$ if $|r| < 1$)

e.g., A neat example is the **Bessel function** used to describe the waves in water when a stone drops:

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$$

$$= 1 - \frac{1}{2^2}x^2 + \frac{1}{2^4(2!)^2}x^4 - \frac{1}{2^6(3!)^2}x^6 + \cdots$$

note the Bessel function is defined by its Taylor series.
It's a decaying cosine-like wave, since the denominator grows faster than cosine's.



## Orders of Growth and Error Size

**Big-O**: "same order as"

- f(x) = O(1) if $f(x) \to c * 1$ as $x \to \infty$

- f(x) = O(g(x)) if $f(x) \to c * g(x)$ as $x \to \infty$

**Little-o**: "ultimately smaller than"

- $f(x) = o(g(x))$ means $\dfrac{f(x)}{g(x)} \to 0$ as $g(x) \to 0$.

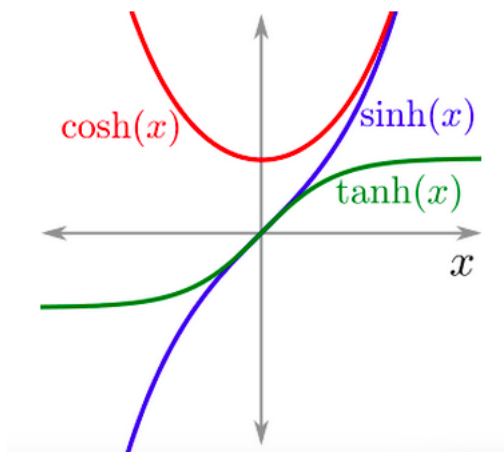## 1.3   Hyperbolic Trigonometric Functions

These fuctions have similar algebraic properties to $\sin, \cos$ and similarities in their taylor expansions.
(Taylor is the same without alternating signs–all +)

Define

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
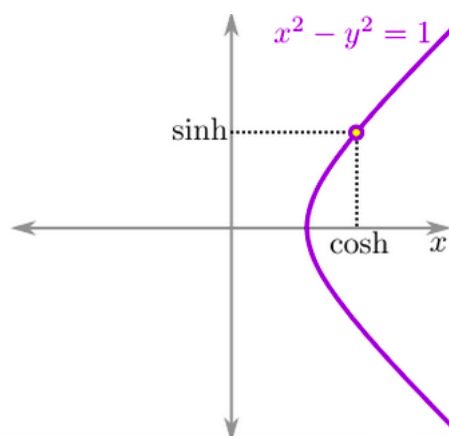


A variation of the Pythagorean Theorem holds:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Geometrically, $\cosh$ and $\sinh$ give $x, y$ coordinates on the hyperbola $x^2 - y^2 = 1$ (similar to $\sin, \cos$ with the unit cirlce).

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**What are hyperbolas?**
They are two parabolas approaching approaching asymptotes (facing up/down or right/left):

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$



(negative sign on x, instead of y, for up/down)

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## Multiplying Taylor Polynomials

$$\cos(x)\cos(x) = (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots)(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots)$$
$$= 1 + (-\frac{1}{2!} - \frac{1}{2!})x^2 + (\frac{1}{4!} + \frac{1}{2!}\frac{1}{2!} + \frac{1}{4!})x^4 + \dots$$

idea: what are the coefficients of $x^4$

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**Binomial Series** (for $|x| < 1$)

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \dots$$
$$= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

by computing derivatives and using Taylor Expansion

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## Exercises

1. Find the Taylor Expansion of $\ln(1 + x)$ using $\ln(1 + x) = \int \frac{1}{1+x} \, dx$ and the geometric series.

2. Similarly find the Taylor Expansion of $\arctan(x)$ using $\arctan(x) = \int \frac{1}{1+x^2} \, dx$ and the geometric series.

Summarizing, using Geometric + Taylor Expansion we have:
ONLY FOR $|x| < 1$, based on geometric series convergence

- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

# 2 Differentiation

# 3 Integration

# 4 Series