M408D-AP HOMEWORK SOLUTIONS

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ABSTRACT. I have recorded answers for homework exercises from Spivak's *Calculus*. If you see any errors please let me know! I will fix them as soon as possible.

CONTENTS

1. Homework 1 — 1
2. Homework 2 — 5
3. Homework 3 — 13
4. Homework 4 — 18
5. Homework 5 — 21
6. Homework 6 — 24
7. Homework 7 — 26
8. Homework 8 — 27
9. Homework 9 — 29
10. Homework 10 — 30
11. Homework 11 — 31
12. Homework 12 — 33

1. HOMEWORK 1

Chapter 1: 1(i) (ii), 2, 3(iii), 4(i) (iv), 25 ; 11(i) (v), 13, 14, 20

Extra Problems to think about
Chapter 1: 21, 22, 23

1(i) $ax = a$ for $a \neq 0$. Prove that $x = 1$.

Proof. Since a is nonzero we can consider its multiplicative inverse $\frac{1}{a}$. Remember this is the unique real number such that $a \cdot \frac{1}{a} = 1$. Since

$$ax = a$$

We have

$$(\frac{1}{a}) \cdot ax = (\frac{1}{a}) \cdot a$$

Hence using the associate property for multiplication on the LHS we see that we have:

$$x = 1$$

$\square$

1(ii) Prove that $x^2 - y^2 = (x-y)(x+y)$

Proof. We will work exclusively with the RHS. The idea will be to get the RHS to look like the LHS and each step we perform will be an equality. We start by using the distributive property:

$$(x-y)(x+y) = x(x+y) - y(x+y)$$

Use Distributive again on each term:

$$= x^2 + xy - yx - y^2$$

Using commutative property we know that $xy = yx$, so the middle term cancels (since $-xy$ is the additive inverse of yx) and we are left with:

$$= x^2 - y^2$$

□

2 Find the error in the proof (kind of like Where's Waldo)

Proof. Since we assume from the beginning that $x = y$, this means that $x - y = 0$. Notice how our friend writing this proof was not careful to recognize this fact. So they end up dividing by (x-y). Oops. This is where the error in this proof lies. □

3(iii) Prove the following statement: If $a, b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$

Proof. There are many ways to do this problem. I will do the most straightforward. Spivak gives us a hint on how to approach this problem. He wants you to first prove that

$$(ab)^{-1} = b^{-1}a^{-1}$$

If we can show this I claim we are done. Why you ask? Because we are working with real numbers and NOT Matrices, we can actually use the commutative property to say that

$$b^{-1}a^{-1} = a^{-1}b^{-1}$$

So we only need to prove the first statement. Just recall what $(ab)^{-1}$ means. It is the unique multiplicative inverse of (ab). So to show that it has the desired form take the RHS of the above claim and start working. In particular start by multiplying the RHS by ab and using associative property a bunch of times:

$$\begin{aligned}(ab)\cdot(b^{-1}a^{-1}) = ((ab)b^{-1})\cdot a^{-1} = (a(bb^{-1}))\cdot a^{-1}\\ = (a\cdot 1)\cdot a^{-1} = aa^{-1} = 1\end{aligned}$$

So what have we shown? We have shown that $(b^{-1}a^{-1})$ is the multiplicative inverse of (ab). But we know that the multiplicative inverse is unique so

$$(ab)^{-1} = b^{-1}a^{-1}$$

□

4(i)(iv)

Proof. For these problems, which I leave to you, the only things you need to do besides basic algebra is state which axioms of the Real Numbers you are using. (This includes using the order properties of $\mathbb{R}$). If anyone wants to see this one done, just email me. □

11(i)(v)

Proof. **11(i)** This one is very straightforward. Find all x values satisfying:

$$|x-3| = 8$$

Just unravel the definition of absolute value

$$|x-8| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases}$$

So we have two equations we have to solve:

$$x - 3 = 8$$

$$-(x-3) = 8$$

For the first equation use the fact that $a = b \implies a + c = b + c$ for c =3 to get:

$$x = 11$$

For the second equation use distributive property to get negative to both terms then use the property just mentioned for c=-3 to get:

$$x = -5$$

□

Proof. **11(v)** This one you have to again unravel the definition of absolute value. But this time you have two absolute values so since there are two cases for each absolute value, you will have a total of 4 cases to check. Check them all to convince yourself that NO x satisfies this inequality. Hence the answer is actually the empty set:

$$\varnothing$$

□

13 Verify the formula for max and min.

Proof. This one is actually done by you imposing an additional assumption. When you define the max function you are comparing two values. For the purposes of a proof we can assume that one value is greater than the other. For example we assume that $x > y$. Since this condition is symmetric (we could have equally started by assuming $y > x$) we only have to consider one case and say the theorem is true in the other case by symmetry. Confused? Lets dispel it. So assume $x > y$. Then the $max(x, y) = x$. Hence the LHS of the identity we are trying to show evaluates to x. The goal is to get x on the RHS.

$$x = \frac{x + y + |y - x|}{2}$$

Since $x > y$ by assumption, $|y - x| = -(y - x)$. So

$$x = \frac{x + y - (y - x)}{2}$$

$$x = \frac{x + y - y + x}{2}$$

$$x = \frac{2x}{2}$$

$$x = x$$

Hence we get x on the RHS as well! Convince yourself with what I said above that by symmetry it is true for the case $y > x$ and also that the same argument works for the min function as well. Finally to generalize this to the case of $max(x, y, z)$ just convince yourself that you can take the max of two elements first then compare it to the third, namely:

$$max(x, y, z) = max(x, max(y, z))$$

Let $b = max(y, z)$. Hence:

$$max(x, y, z) = \frac{x + b + |b - x|}{2} = \frac{x + (\frac{y+z+|z-y|}{2}) + |(\frac{y+z+|z-y|}{2}) - x|}{2}$$

Similar for min. □

14 This seemed to be the trickiest of the problems.

Proof. (a) We want to show that $|-a| = |a|$ for any real number a

The idea is to consider cases (how original!). Start by assuming

$$a \geq 0$$

This implies that

$$|a| = a$$

$$|-a| = -(-a)$$

This last equality follows from the fact that $-a \leq 0$ and definition of absolute value. Hence

$$|a| = a = -(-a) = |-a|$$

Now do the same argument but assuming

$$a \leq 0$$

(b) We now prove $-b \leq a \leq b \iff |a| \leq b$.

Again this just means unraveling the definition of absolute value.

$$|a| \leq b \iff a \leq b \text{ and } -a \leq b \iff a \leq b \text{ and } a \geq -b \iff -b \leq a \leq b$$

So:

$$|a| \leq b \iff -b \leq a \leq b$$

Notice that we proved both statements at once because each statement is an if and only if statement. The argument is just as good forwards as it is backwards.

(c) We now give a pretty simple proof of the triangle inequality. The following inequalities are always true:

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

Now we can add the inequalities together keeping the inequality signs pointing in the same direction:

$$-|a| - |b| \leq a + b \leq |a| + |b| = -(|a| + |b|) \leq a + b \leq |a| + |b|$$

Now notice that this is exactly of the form of the LHS of the previous problem. So using part (b) this implies that :

$$|a+b| \leq |a|+|b|$$

The Triangle Inequality!

□

20 This problem is a gimme. You just have to see it correctly.

Proof.
$$\begin{aligned} |(x+y)-(x_0+y_0)| &= |(x-x_0)+(y-y_0)| \\ &\leq |(x-x_0)|+|(y-y_0)| \end{aligned}$$
By Triangle Inequality. Now use the hypothesis
$$< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

Do the same thing for the other one.

□

25

Proof. Sooo I don't really feel like writing this one out. I'm sure most of you got this one. It was simply checking that for this funky number field all of the axioms are satisfied. The only thing to note is that the multiplicative inverse and additive inverse axioms are satisfied basically because there is only one way for each number to add to 0 or multiply to 1. Again email me if you want to talk about this one. □

2. Homework 2

Chapter 4: 1(ii) (iii) and Chapter 5: 1(ii) (iv), 3(i) (ii) (iii) (iv) (v) (vi), 8, 12, 16, 17(a), 19

Extra Problems to think about
Chapter 4: 2
Chapter 5: 20,22

Chapter 4

1(ii) (iii). I am going to assume that you guys know how to draw number lines. The point of this question is to recall the problem you proved on the first Homework Assignment:

Proof. Let us first start by recalling the lemma:

Lemma 1. *Let $a, b \in \mathbb{R}$. Then $-b \leq a \leq b \iff a \leq |b|$*

Hence the set of x satisfying $|x-3| \leq 1$ also satisfy $-1 \leq x-3 \leq 1$. We now use the fact that $a \leq b \implies a+c \leq b+c$ for any $c \in \mathbb{R}$. By choosing our $c=3$ we have the inequality:
$$2 \leq x \leq 4$$
Similarly 1 (iii) can be reduced to
$$a-\epsilon < x < a+\epsilon$$

Drawing the number line is up to you! □

Chapter 5 This is where the fun starts!

1 (ii) (iv) Note: For these two problems you **DO NOT** need to specify the value of δ that makes the limit true. These problems test your skill with Algebra and applying Theorem 2.

1(ii): Evaluate

$$\lim_{x\to 2} \frac{x^3-8}{x-2}$$

Proof. The first step is to do a little bit of algebraic manipulation. In particular assuming $x \neq 2$ we can apply polynomial division to get:

$$\lim_{x\to 2} \frac{x^3-8}{x-2}$$
$$= \lim_{x\to 2}(x^2+2x+4)$$

This step is justified because we are evaluating limits and the definition of limits tells us that we are interested in all values of x close to 2 but not equaling 2. Hence the above two limits are identical and we can evaluate the second limit instead of the first one and get an equivalent answer. (Think about this step carefully). Looking at the second limit we rewrite it as follows (to make the application of Theorem 2 easier):

$$= \lim_{x\to 2}(x\cdot x + (x+x) + 4)$$

We know that

$$\lim_{x\to 2} x = 2$$
$$\lim_{x\to 2} 4 = 4$$

So applying Theorem 2:

$$\lim_{x\to 2} x\cdot x = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2\cdot 2 = 4$$
$$\lim_{x\to 2}(x+x) = (2+2) = 4$$

Hence:

$$\lim_{x\to 2}(x^2+2x+4) = \lim_{x\to 2}(x^2) + \lim_{x\to 2}(2x) + \lim_{x\to 2}(4) = 4+4+4$$
$$= 12$$

□

1(iv): Evaluate

$$\lim_{x\to y} \frac{x^n-y^n}{x-y}$$

Proof. The idea is similar to the last problem. Assuming $x \neq y$, the limit becomes after polynomial division:

$$\lim_{x\to y} \frac{x^n-y^n}{x-y}$$
$$= \lim_{x\to y}(x^{n-1}+x^{n-2}y+\cdots+y^{n-1})$$

Applying Theorem 2 in exactly the same way we did in the last problem we see that:

$$\lim_{x \to y}(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$$
$$= y^{n-1} + y^{n-1} + \cdots + y^{n-1}$$
$$= ny^{n-1}$$

□

3 (i) -(vi): In these problems you have to prove the limit using the $\epsilon - \delta$ technique.

3(i) Find $\lim_{x \to a} f(x)$ when $a = 0$ and $f(x) = x[3 - cos(x^2)]$

Proof. A way to guess at the limit is to plug in the "a = 0" value into the function and see what it evaluates to. Since there is no division by 0 and we don't encounter any infinities, it is reasonable to think that $\lim_{x \to 0} f(x) = 0$. We will now show this to be true. Remember that in order to prove that this is in fact the correct limit, we must show that given any $\epsilon > 0$ we can find a $\delta > 0$ such that if $x \neq 0$ and $|x| < \delta$ then $|x[3 - cos(x^2)]| < \epsilon$. Hence we start by fixing an arbitrary $\epsilon > 0$. Consider the following string of inequalities:

$$|x[3 - cos(x^2)]| = |x||3 - cos(x^2)|$$
$$\leq |x|(|3| + |cos(x^2)|) \quad \text{(Triangle Inequality)}$$
$$\leq |x|(3 + 1) \qquad (|cos(x^2)| \leq 1)$$
$$= 4|x|$$

If we choose $\delta = \frac{\epsilon}{4}$ then we will have the inequality

$$|x[3 - cos(x^2)]| = |x||3 - cos(x^2)| < \epsilon$$

So we have shown that for each $\epsilon > 0$ there does in fact exist a $\delta > 0$ (in this case $\delta = \frac{\epsilon}{4}$) satisfying the definition of the limit. Hence we have proved

$$\lim_{x \to 0} f(x) = 0$$

□

3(ii) Find $\lim_{x \to a} f(x)$ when $a = 2$ and $f(x) = x^2 + 5x - 2$

Proof. We will not be as verbose as the last problem. If you feel uncomfortable with what we are about to show, look back at the last problem for a guided approach to evaluating limits. We want to find a $\delta > 0$ such that

$$if\ 0 < |x - 2| < \delta \text{ ,then } |x^2 + 5x - 2 - (2^2 + 5(2) - 2)| < \epsilon$$

We have guessed that the limit is 12. Fix $\epsilon > 0$. We start by trying to find δ_1, δ_2, and $\delta_3 > 0$ such that for all x:

$$if\ 0 < |x - 2| < \delta_1 \text{ ,then } |x^2 - 2^2| < \frac{\epsilon}{3}$$
$$if\ 0 < |x - 2| < \delta_2 \text{ ,then } |5x - 10| < \frac{\epsilon}{3}$$
$$if\ 0 < |x - 2| < \delta_3 \text{ ,then } |2 - 2| < \frac{\epsilon}{3}$$

Note: We can choose any $\delta_3 > 0$ to satisfy the third inequality. Hence we need to only find: δ_1, δ_2 such that for all x:

$$if\ 0 < |x-2| < \delta_1 \text{ ,then } |x^2 - 2^2| < \frac{\epsilon}{2}$$

$$if\ 0 < |x-2| < \delta_2 \text{ ,then } |5x - 10| < \frac{\epsilon}{2}$$

Why we consider these particular combinations will be clear shortly.

Using the result from the textbook we know that we should set:

$$\delta_1 = min(1, \frac{\frac{\epsilon}{2}}{2|2|+1}) = min(1, \frac{\epsilon}{10})$$

$$\delta_2 = \frac{\epsilon}{10}$$

Hence setting

$$\delta = min(\frac{\epsilon}{10}, min(1, \frac{\epsilon}{10})) = \delta_1$$

We see that $|x-2| < \delta$ implies, after rearranging the terms and using the triangle inequality:

$$|x^2 + 5x - 2 - (2^2 + 5(2) - 2)| \leq |x^2 - 4| + |5x - 10| \leq \frac{\epsilon}{2} + 5\frac{\epsilon}{10} = \epsilon$$

□

3(iii) Find $\lim_{x \to a} f(x)$ when $a = 1$ and $f(x) = \frac{100}{x}$

Proof. We want to find a $\delta > 0$ such that

$$if\ 0 < |x-1| < \delta \text{ ,then } |\frac{100}{x} - 100| < \epsilon$$

Note: We guess that the limit is 100. We now prove this claim. Start by fixing $\epsilon > 0$. Then:

$$|\frac{100}{x} - 100| = 100|\frac{1}{x} - 1|$$

$$= 100 \cdot |x-1| \cdot \frac{1}{|x|}$$

We first require $|x-1| < \frac{1}{2}$. This ensures that $\frac{1}{2} < x < \frac{3}{2}$ so in particular $x \neq 0$ and $\frac{1}{|x|} < 2$. This implies

$$100 \cdot |x-1| \cdot \frac{1}{|x|} < 100 \cdot |x-1| \cdot 2 = 200 \cdot |x-1|$$

Choosing $\delta = min(\frac{1}{2}, \frac{\epsilon}{200})$ We see that:

$$0 < |x-1| < \delta \implies |\frac{100}{x} - 100| < \epsilon$$

□

3(iv) Find $\lim_{x \to a} f(x)$ when a arbitrary and $f(x) = x^4$

Proof. It should now be clear that we first guess the limit to be a^4. We will now prove this to be true. Fix $\epsilon > 0$.The textbook tells us that:

$$|x^2 - a^2| < \epsilon$$

whenever

$$|x - a| < \delta_1 = min(1, \frac{\epsilon}{2|a| + 1})$$

Applying this result to our situation we have:

$$|x^4 - a^4| = |(x^2)^2 - (a^2)^2| < \epsilon$$

whenever

$$|x^2 - a^2| < \delta_1 = min(1, \frac{\epsilon}{2|a^2| + 1})$$

We now have to find a $\delta > 0$ such that

$$|x - a| < \delta \implies |x^2 - a^2| < \delta_1$$

Hence we choose

$$\delta = min(1, \frac{\delta_1}{2|a| + 1})$$

□

3(v) Find $\lim_{x\to 1} f(x)$ when $a = 1$ and $f(x) = x^4 + \frac{1}{x}$

Proof. Hint: Use the last two problems and the triangle inequality □

3(vi) Find $\lim_{x\to 0} f(x)$ when $a = 0$ and $f(x) = \frac{x}{2-sin^2 x}$

Proof. We guess that the limit is 0. This problem is rather easy once you recognize that the denominator can be bounded trivially:

$$|2 - sin^2 x| \geq |2| - |sin^2 x| \geq 2 - 1 = 1$$

Hence:

$$|\frac{x}{2 - sin^2 x}| \leq |x|$$

If we fix $\epsilon > 0$, choosing $\delta = \epsilon$ completes the proof.

□

8(a) $\lim_{x\to a}(f(x) + g(x))$ can exist. Take $f(x) = \frac{1}{x-a}$ and $g(x) = -\frac{1}{x-a}$. For the second question, $\lim_{x\to a}(f(x)g(x))$ can also exist. Take $f(x) = sin(\frac{1}{x})$ and $g(x) = \frac{1}{sin(\frac{1}{x})}$.

8(b) Yes

Proof.

$$\lim_{x\to a} g(x) = \lim_{x\to a}(g(x) + f(x) - f(x))$$
$$= \lim_{x\to a}(f(x) + g(x)) - \lim_{x\to a} f(x)$$

The second equality follows from the assumptions as well as Theorem 2. The RHS is finite, since both limits exist, this also implies that the LHS is finite, hence, $\lim_{x\to a} g(x)$ exists. □

8(c) No

Proof. This is a proof by contradiction. Suppose $\lim_{x\to a}(f(x)+g(x))$ does exist. Then

$$g(x) = f(x) + g(x) - f(x) = (f(x) + g(x)) - f(x)$$

Taking the limit of the RHS we have a finite number. But taking the limit of the LHS we have a nonexistent limit. This is a contradiction. □

8(d) No

Proof. Notice that if we try to argue like before we would have to consider a quantity like $g(x) = \frac{f(x)g(x)}{f(x)}$. The problem here is that $f(x)$ could be zero. This points us in the direction of what our counterexample should look like. Let $f(x) = x - a$. We set $a = 0$. This ensures that

$$\lim_{x\to 0} f(x) = 0$$

Let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

We will prove in question 19 that

$$\lim_{x\to a} g(x)$$

does not exist for any a. We take this on faith for now. But notice that this implies,

$$f(x)g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Hence

$$\lim_{x\to 0} f(x)g(x) = 0$$

□

12(a) Assume that $f(x) \le g(x)\ \forall x$. Prove that $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$ provided that these limits exist.

Proof. The idea is to argue by contradiction. We want to show that if the inequality does not hold in the limit this would somehow contradict the assumption that $f(x) \le g(x)\ \forall x$. Remember finding even one point where this fails to hold is a contradiction. Suppose:

$$k = \lim_{x\to a} f(x) > \lim_{x\to a} g(x) = m$$

We set $\epsilon = k - m$. By the definition of limits we know that there exists a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - k| < \frac{\epsilon}{2} \text{ and } |g(x) - m| < \frac{\epsilon}{2}$$

This implies that,

$$g(x) < m + \frac{\epsilon}{2} = k - \frac{\epsilon}{2} < f(x)$$

Hence on the interval

$$a - \delta < x < a + \delta$$

g(x) is strictly less then f(x). This is a contradiction. □

12(b) Since limits are local one can weaken the hypothesis to: There exists some $\delta > 0$ such that

$$f(x) \leq g(x)$$

for all x satisfying

$$|x - a| < \delta$$

12(c) No. Let $f(x) = 0$, $g(x) = \frac{1}{|x|}$ and $a = \infty$. We have for all x

$$f(x) < g(x)$$

But,

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} f(x) = 0$$

16(a) Prove the following statement

$$\lim_{x \to a} f(x) = l \implies \lim_{x \to a} |f|(x) = |l\,|$$

Proof. From the triangle inequality one can show that $\forall a, b \in \mathbb{R}$

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |a - b|$$

This implies that

$$||a| - |b|| \leq |a - b|$$

Let $a = f(x)$ and $b = l$. We have:

$$||f(x)| - |l|| \leq |f(x) - l|$$

Hence the convergence of $f(x) \to l$ as $x \to a$ implies the convergence of $|f|(x) \to |l\,|$ as $x \to a$.

□

16(b) Prove that if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

$$\lim_{x \to a} max(f, g)(x) = max(l, m)$$

Similar for min.

Proof. This proof merely requires you to recall the max and min formulas you proved in the first homework.

$$max(f, g)(x) = \frac{f(x) + g(x) + |g(x) - f(x)|}{2}$$

$$min(f, g)(x) = \frac{f(x) + g(x) - |g(x) - f(x)|}{2}$$

We consider the max function. The argument for the min function is identical. Using Theorem 2 and part a of this problem:

$$\lim_{x \to a} \frac{f(x) + g(x) + |g(x) - f(x)|}{2} = \frac{l + m + |m - l\,|}{2} = max(l, m)$$

□

17(a) Prove that the following limit does not exist:

$$\lim_{x\to 0} \frac{1}{x}$$

Proof. The idea for this proof is to argue by contradiction. Suppose there exists $l < \infty$ such that $\frac{1}{x} \to l$ as $x \to 0$. This is equivalent to saying that $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$|x| < \delta \implies |\frac{1}{x} - l\,| < \epsilon$$

Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ be the corresponding value in the definition of the limit. We define:

$$\delta_1 = max(\delta, \frac{1}{\epsilon + |\,l\,|})$$

Note: $\delta \leq \delta_1$. We also have:

$$|\frac{1}{x} - l\,| < \epsilon \implies \frac{1}{|x|} < |\,l\,| + \epsilon$$

$$|x| < \delta \leq \delta_1 \implies \frac{1}{\delta_1} \leq \frac{1}{\delta} < \frac{1}{|x|}$$

We have two distinct cases to consider. Suppose $\delta_1 = \frac{1}{|\,l\,|+\epsilon}$ We have:

$$\frac{1}{\delta_1} = \frac{1}{\frac{1}{|\,l\,|+\epsilon}} = |\,l\,| + \epsilon < \frac{1}{|x|} < |\,l\,| + \epsilon$$

This is a contradiction.

The other case is when $\delta_1 = \delta$. This implies that $\delta \geq \frac{1}{|\,l\,|+\epsilon}$. Since in the definition of limit we consider all x values satisfying $|x| < \delta$, we pick an x value that satisfies $|x| < \frac{1}{|\,l\,|+\epsilon}$ (because these x values also satisfy $|x| < \delta$). Hence for these particular x values we have

$$\frac{1}{\frac{1}{|\,l\,|+\epsilon}} = |\,l\,| + \epsilon < \frac{1}{|x|} < |\,l\,| + \epsilon$$

This is also a contradiction.

□

19 Consider the following function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Prove that

$$\lim_{x\to a} f(x)$$

does not exist for any a.

Proof. To prove this theorem we need to state two results that have not been proven in class up to now. We will take these two results as given and not concern ourselves with a proof of them here.

Lemma 2. *Suppose $x < y$. There exists a rational number r such that $x < r < y$*

Lemma 3. *Suppose $x < y$. There exists an irrational number s such that $x < s < y$*

Now we argue by contradiction. Suppose there exists $m < \infty$ such that $\forall \epsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - m| < \epsilon$$

Let us consider

$$\epsilon = \frac{1}{2}$$

Using the two lemmas above we know that there exists a rational number, r, and an irrational number, s, in the interval

$$a - \delta < x < a + \delta$$

. Hence in this interval we have the contradictory inequalities

$$|m| < \frac{1}{2}$$

$$|1 - m| < \frac{1}{2}$$

□

3. Homework 3

Chapter 5: 29,32,36,38

29 Prove the following statement:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \implies \lim_{x \to a} f(x) \text{ exists}$$

Proof. This problem tests your understanding of the definition of limits and absolute value. The first of the two hypothesis means:

$$\forall \epsilon > 0 \ \exists \delta_1 > 0 \ s.t.$$

$$0 < x - a < \delta_1 \implies |f(x) - L| < \epsilon$$

The second hypothesis means:

$$\forall \epsilon > 0 \ \exists \delta_2 > 0 \ s.t.$$

$$0 < a - x < \delta_2 \implies |f(x) - L| < \epsilon$$

One important observation: The deltas are different! This is because it is not clear from the start that the same epsilon in each statement will give the same delta. Lets look at the second statement:

$$0 < a - x < \delta_2 \implies 0 > x - a > -\delta_2$$

We now set $\delta = max(\delta_1, \delta_2)$ The reason for this is the following two inequalities:

$$0 < x - a < \delta_1 \leq \delta$$

$$0 > x - a > -\delta_2 \geq -\delta$$

This implies that both

$$0 < x - a < \delta$$

$$0 < -(x - a) < \delta$$

So in particular

$$|x - a| < \delta$$

. Hence for each $\epsilon > 0$ we have found a $\delta > 0$ such that

$$|x-a| < \delta \implies |f(x) - L| < \epsilon$$

So in particular $\lim_{x\to a} f(x)$ exists and is equal to L. □

32 Prove the following statement:

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0} \text{ exists} \iff m \geq n$$

In particular what is the limit when $m > n$ and when $m = n$

Proof. To prove this theorem we need the following result for $k > 0$:

$$\lim_{x\to\infty} \frac{1}{x^k} = 0$$

To prove this result we need to show that

$$\forall \epsilon > 0 \ \ \exists N > 0 \ \ s.t. \ \ \forall x > N \ \ |\frac{1}{x^k}| < \epsilon$$

So we fix $\epsilon > 0$. Let us choose

$$N > \frac{1}{\epsilon^{\frac{1}{k}}}$$

Hence for each $x > N$:

$$0 < |\frac{1}{x^k}| = \frac{1}{x^k} < \frac{1}{N^k} < (\epsilon^{\frac{1}{k}})^k = \epsilon$$

Next we turn to the limit we are interested in:

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0}$$

Let us first assume that $m \geq n$. We have the following:

$$\lim_{x\to\infty} (\frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0})$$

$$= \lim_{x\to\infty} (\frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0} \cdot \frac{\frac{1}{x^n}}{\frac{1}{x^n}})$$

$$= \lim_{x\to\infty} (\frac{a_n + \cdots + \frac{a_0}{x^n}}{a_m x^{m-n} + \cdots + \frac{b_0}{x^n}})$$

Notice that if $m = n$ then after repeated applications of Theorem 2 and the limit we just proved above (in this problem) we see that

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0} = \frac{a_n}{b_n}$$

Now if $m > n$ then using Theorem 2 again and the trivial limit for $k > 0$

$$\lim_{x\to\infty} x^k = \infty$$

we see that

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0} = 0$$

To prove the second statement in our if and only if statement we will argue by showing that the contrapositive of the second statement is true. In particular assume $m < n$. Then arguing as before:

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{a_m x^m + \cdots + b_0}$$
$$= \lim_{x\to\infty} \left(\frac{a_n + \cdots + \frac{a_0}{x^n}}{\frac{a_m}{x^{n-m}} + \cdots + \frac{b_0}{x^n}} \right)$$

Let the numerator be denoted by $f(x)$ and the denominator be $g(x)$. We now claim that this limit does not exist. We will argue by contradiction so lets assume that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

Then,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{f(x)g(x)}{g(x)}\right) = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \lim_{x\to\infty} g(x) = L \cdot 0 = 0$$

But notice that

$$\lim_{x\to\infty} f(x) = a_n \neq 0$$

This is a contradiction. □

36 Start by defining what the limit as x approaches negative infinity means. Particularly give a meaning to this statement:

$$\lim_{x\to-\infty} f(x) = L$$

This can be defined as follows:

$$\forall \epsilon > 0 \;\; \exists N < 0 \;\; s.t. \;\; \forall x < N \;\; |f(x) - L| < \epsilon$$

(a) Evaluate :

$$\lim_{x\to-\infty} \frac{a_n x^n + \ldots a_0}{a_m x^m + \ldots b_0}$$

Proof. I will not do the nitty gritty here since we already did this for **32**. But one thing to point out is that you need to show $\forall k > 0$

$$\lim_{x\to-\infty} \frac{1}{x^k} = 0$$

The proof proceeds along the same way as before but be careful with negative signs! The sign of $\frac{1}{x^k}$ changes depending on if k is odd or even. You need to say something about why things still work out fine.

□

(b) Prove:

$$\lim_{x\to-\infty} f(-x) = \lim_{x\to\infty} f(x)$$

Proof. I will call the limit of the RHS L. The point of this problem is to show that the limit of the LHS is also L. Lets fix $\epsilon > 0$. The definition of limit tells me that for the RHS:

$$\exists M > 0 \;\; s.t. \;\; x > M \implies |f(x) - L| < \epsilon$$

We now do a change of variable. Set

$$y = -x < 0$$

So
$$-y = x > 0$$
Hence we have
$$-y > M \implies |f(-y) - L| < \epsilon$$
In particular
$$y < -M < 0 \implies |f(-y) - L| < \epsilon$$
Setting $N = -M$ we see that this is exactly the definition of the limit in the case that $y \to -\infty$.

□

(c) Prove:
$$\lim_{x \to 0^-} f(\frac{1}{x}) = \lim_{x \to -\infty} f(x)$$

Proof. We will argue in an analogous manner to the part (b). Let the limit of the LHS be called L. Fix $\epsilon > 0$. The definition of limits for the LHS tells me that there exists $\delta > 0$ such that:

$$0 < -x < \delta \implies |f(\frac{1}{x}) - L| < \epsilon$$
This is equivalent to requiring
$$0 > x > -\delta \implies |f(\frac{1}{x}) - L| < \epsilon$$
Which is equivalent to
$$\frac{1}{x} < -\frac{1}{\delta} < 0 \implies |f(\frac{1}{x}) - L| < \epsilon$$
Now we make a change of variables:
$$y = \frac{1}{x} < 0$$
So we have:
$$y < -\frac{1}{\delta} < 0 \implies |f(y) - L| < \epsilon$$
Finally to conclude set
$$N = -\frac{1}{\delta} < 0$$
So our statement reads
$$y < N < 0 \implies |f(y) - L| < \epsilon$$
This is now the correct statement for the limit in the case that $y \to -\infty$.

□

38 (a) Start by defining the following limits:
$$\lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = \infty$$
translates to
$$\forall N > 0 \ \exists \delta > 0 \ \ s.t. \ \ 0 < x - a < \delta \implies f(x) > N$$

And

$$\lim_{x \to a^+} f(x) = \infty$$

translates to

$$\forall N > 0 \ \ \exists \delta > 0 \ \ s.t. \ \ 0 < a - x < \delta \implies f(x) > N$$

Question: Why no absolute value in either definition?

(b) Prove:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Proof. Using our newly formed definitions we want to show that by first fixing $N > 0$ there exists $\delta > 0$ such that

$$0 < x < \delta \implies \frac{1}{x} > N$$

Set

$$\delta = \frac{1}{N}$$

Then we have

$$0 < x < \frac{1}{N} \implies \frac{1}{x} > N$$

This is what we were after. □

(c) Prove:

$$\lim_{x \to 0^+} f(x) = \infty \iff \lim_{x \to \infty} f(\frac{1}{x}) = \infty$$

Proof. The strategy for this question is to show that each limit implies the other. Lets start by assuming the LHS is true. Using the definition of limit let us fix $M > 0$. This means that there exists $\delta > 0$ such that

$$0 < x < \delta \implies f(x) > M$$

Lets make a change of variables:

$$y = \frac{1}{x}$$

Our equation becomes:

$$\frac{1}{y} < \delta \implies f(\frac{1}{y}) > M$$

This is equivalent to the following statement

$$0 < \frac{1}{\delta} < y \implies f(\frac{1}{y}) > M$$

Now define

$$N = \frac{1}{\delta}$$

We now have the definition of the limit for the RHS. To show that the RHS implies the LHS argue similarly.

□

4. Homework 4

Chapter 6: 1,3,4,7,8,10,14,15

1 For which of the following functions f is there a continuous function F with domain $\mathbb{R}$ such that $F(x) = f(x)$ for all x in the domain f?

Proof. As we pointed out in the class, for this problem you have to check if you can find a GLOBAL function F(x) that is defined for every x value, but coincides with f(x) for the x-values in the domain of f(x). Looking at the list of functions and drawing a graph for each we see that only (i) and (iii) satisfy this condition. For (i) we can take $F(x) = x + 2$ and for (iii) we can take $F(x) = 0$. Convince yourself that you cannot do this for any of the other parts. $\square$

3 Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x.

(a) Show that $f(x)$ is continuous at 0

Proof. Lets start with the statement we are trying to prove. We want to show that $\forall \epsilon > 0\ \exists \delta > 0$ such that $0 < |x| < \delta \implies |f(x)| < \epsilon$. Fix $\epsilon > 0$. By assumption we know that

$$|f(x)| \leq |x|$$

Choose $\delta = \epsilon$. This implies

$$|f(x)| \leq |x| < \delta = \epsilon$$

$\square$

(b) Give an example of a function which is not continuous at any $a \neq 0$.

Proof. Try the following function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

It clearly satisfies the condition $|f(x)| \leq |x|$. Also you can check that it is not continuous at any nonzero value. $\square$

(c) Suppose $g(x)$ is continuous at 0, $g(x) = 0$, and $|f(x)| \leq |g(x)|$. Prove that $f(x)$ is continuous at 0.

Proof. By definition of continuity we know that $\forall \epsilon > 0\ \exists \delta_1 > 0$ such that $|x| < \delta_1 \implies |g(x)| < \epsilon$. By assumption we know that $|f(x)| \leq |g(x)|$. Hence by fixing $\delta_1 = \delta$, we see that $\forall \epsilon > 0\ \exists \delta > 0$ such that $|x| < \delta \implies |f(x)| \leq |g(x)| < \epsilon$. $\square$

4 Give an example of a function $f(x)$ such that $f(x)$ is continuous nowhere but $|f(x)$ is continuous everywhere.

Proof. Try the following function

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

It is clearly not continuous anywhere (supply a proof!). But the $|f(x)| = 1$ which is continuous everywhere! □

7 Suppose that $f(x+y) = f(x) + f(y)$ and that $f(x)$ is continuous at 0. Prove that $f(x)$ is continuous at a for all a.

Proof. Start by stating what you want to show. You are trying to prove that $\forall\epsilon > 0\ \exists\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - f(a)| < \epsilon$. First thing to do is find out what $f(0)$ is equal to. This is because we know that $f(x)$ is continuous at 0.

$$f(a) = f(0+a) = f(0) + f(a)$$

This implies that $f(0) = 0$. The continuity of the function at 0 means specifically $\forall\epsilon > 0\ \exists\delta > 0$ such that $0 < |y| < \delta \implies |f(y)| < \epsilon$ Now we do a simple substitution: $y = x - a$. The statement now reads: $\forall\epsilon > 0\ \exists\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x-a)| < \epsilon$ But remember that $f(x-a) = f(x) + f(-a)$. We are almost there. Last step is find out what $f(-a)$ equals. So we do the following:

$$0 = f(0) = f(a-a) = f(a) + f(-a)$$

This implies that $f(-a) = -f(a)$. Hence going back to our limit statement we now have the following: $\forall\epsilon > 0\ \exists\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - f(a)| < \epsilon$ This is the statement we were trying to prove!

□

8 Suppose that $f(x)$ is continuous at a and $f(a) = 0$. Prove that if $\alpha \neq 0$ then $f + \alpha$ is nonzero in some open interval containing a.

Proof. Our hypothesis tells us that: $\forall\epsilon > 0\ \exists\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x)| < \epsilon$

Because $f(a) = 0$. Now we consider the case where $\alpha > 0$. The negative case is similar and I leave it to you to work out the details. We set $\epsilon = \alpha$. Then we know there exists some $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \implies |f(x)| < \alpha$. Unraveling the definition of absolute value this means that for all x values living δ_1 away from a

$$-\alpha < f(x) < \alpha$$

In particular adding α to each term:

$$0 < f(x) + \alpha < 2\alpha$$

So we see that for all of these x values

$$f(x) + \alpha > 0$$

. □

10

(a) Prove that if $f(x)$ is continuous then so is $|f(x)|$.

Proof. This problem is just a restatement of a previous problem you guys proved. (16b). Just remember the inequality

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)|$$

□

(b)Prove that every continuous function $f(x)$ can be written as $f(x) = E(x) + O(x)$ where $E(x)$ is an even continuous function and $O(x)$ is an odd continuous function.

Proof. Let us define $E(x) = \frac{f(-x)+f(x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Note that $E(-x) = E(x)$ and $O(-x) = -O(x)$. Also notice that $E(x) + O(x) = f(x)$. Since $E(x)$ and $O(x)$ are sums and quotients of continuous functions we know that they are continuous. Hence we have found the desired decomposition.

□

(c) Prove that if $f(x)$ and $g(x)$ are continuous then so are $max(f, g)$ and $min(f, g)$.

Proof. Recall that

$$max(f, g) = \frac{f(x) + g(x) + |g(x) - f(x)|}{2}$$

Similar definition for $min(f, g)$. Notice that we have a function that is the sum and quotient of continuous functions. The fact that the numerator is continuous follows from part(a). Hence the maximum function is continuous as well. Similar argument works for minimum function.

□

(d) Prove that every continuous function $f(x)$ can be written as $f(x) = g(x) - h(x)$ where g and h are non-negative continuous functions

Proof. Define the following:

$$g(x) = max(f, 0)$$
$$h(x) = -min(f, 0)$$

Notice that $g(x), h(x) \geq 0$. Moreover

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

$$h(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ f(x) & \text{if } f(x) < 0 \end{cases}$$

After thinking for a bit you see that $f(x) = g(x) - h(x)$.

□

14(a) Check the book for all of the assumptions

Proof.

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = g(a)$$

The last equality follows from the continuity of $g(x)$ at a. Similarly

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} h(x) = h(a)$$

Since by assumption we know that

$$f(a) = g(a) = h(a)$$

This implies that

$$\lim_{x \to a^+} f(x) = f(a)$$
$$\lim_{x \to a^-} f(x) = f(a)$$

Since the right and left limits are equal from a previous HW assignment you know that the total limit is equal to the same value. So:

$$\lim_{x\to a} f(x) = f(a)$$

Hence $f(x)$ is continuous at a. □

(b)

Proof. This is similar to (a). Just notice that the only place you have to check that the function is continuous is at b. □

15 Prove that if $f(x)$ is continuous at a, then for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|x-a| < \delta$ and $|y-a| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Proof. By the definition of continuity we know that $\exists \delta_1 > 0$ such that $|x-a| < \delta_1 \implies |f(x) - f(a)| < \frac{\epsilon}{2}$. Now if we actually choose this δ_1 to be our δ then whenever $|x-a| < \delta$ and $|y-a| < \delta$ we know that

$$|f(x)-f(y)| = |f(x)-f(a)+f(a)-f(y)| \leq |f(x)-f(a)|+|f(a)-f(y)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

□

5. Homework 5

Chapter 7: 1(viii),(ix)(x), 3(ii), 5, 6, 9, 10, 15, 18

1 For the functions below decide which are bounded above, below on the indicated interval, and which take on their maximum and minimum value.

Proof. (viii) After graphing the function you should convince yourself that function is bounded from above by 1 and from below by 0. The max is obtained at every irrational point. The minimum however is never obtained. Since no rational point evaluates to 0.

(ix) For this function it is exactly the same as (viii) except that all the rational values are now reflected across the x-axis. The bound from above is still 1, but the minimum point is now -1/2. The max is again achieved at every rational point, and now the minimum is also obtained at the rational point 1/2.

(x) In this case the function is bounded from above by 1 and from below by 0. The max is obtained at the rational point 1, and the minimum is obtained at every irrational number and the rational point 0. □

3(ii) Prove that there is some number x such that $\sin x = x - 1$.

Proof. Whenever you see a problem like this you need to create your own function and then apply the Intermediate Value Theorem. So for this particular case the function you should consider is

$$F(x) = x - 1 - \sin x$$

Notice if there exists an x value such that $F(x) = 0$ then we will have found an x value where $\sin x = x - 1$. Consider the following:

$$x = 0 \implies F(0) = -1$$

$$x = \pi \implies F(\pi) = \pi - 1 > 0$$

By the Intermediate Value Theorem there exists an $x \in (0, \pi)$ such that $F(x) = 0$. We can apply intermediate value theorem because F(x) is continuous on the interval $[0, \pi]$. $\square$

5 Suppose you have a continuous function $f(x)$ on $[a, b]$ such that $f(x)$ is rational for every $x \in [a, b]$. What can you say about the function $f(x)$?

Proof. Assuming the hypothesis the claim is that $f(x)$ can only be the constant function. We argue by contradiction. Suppose $f(x)$ is not constant. This means that there exists $x \neq y$ such that $f(x) \neq f(y)$. Without loss of generality we assume that $f(y) > f(x)$. We know from our lemma about rational and irrational numbers in an interval that $\exists z \in (f(x).f(y))$ such that z is an irrational number. Hence by the intermediate value theorem there exists $c \in [x, y]$ such that $f(c) = z$. But this contradicts the fact that the function $f(x)$ takes only rational values. Hence $f(x)$ must be constant. $\square$

6

Proof. This question is about carefully reading what you are being asked. The claim is that $f(x) = \sqrt{1 - x^2}\ \forall x$ or $f(x) = -\sqrt{1 - x^2}\ \forall x$. To prove this statement we have to argue by contradiction. Assume this is not the case. In particular assume there exists $x_1, x_2 \in (0, 1)$ such that $f(x_1) = \sqrt{1 - x_1^2}$ and $f(x_1) = -\sqrt{1 - x_2^2}$. This tells us that

$$f(x_1) > 0$$

$$f(x_2) < 0$$

Since $f(x)$ is continuous on the entire interval $[-1, 1]$ it is definitely continuous on the interval $[x_1, x_2]$. So there exists an $x \in (x_1, x_2)$ such that $f(x) = 0$. But recall the fact that $(x, f(x))$ is a point on the unit circle.

$$x^2 + (f(x))^2 = 1 \implies x^2 = 1$$

But $x < 1$. This is a contradiction. $\square$

9

Proof. Email me about this one if you want to see it. Its a bit tedious to write out, but very straightforward. $\square$

10

Proof. Same idea as a previous problem. Set $F(x) = g(x) - f(x)$. Notice that $F(a) > 0$ and $F(b) < 0$. Hence by the intermediate value theorem $\exists c \in (a, b)$ such that $F(c) = 0$. In particular $f(c) = g(c)$. $\square$

15 Suppose that $\phi(x)$ is continuous and

$$\lim_{x \to \infty} \frac{\phi(x)}{x^n} = 0 = \lim_{x \to -\infty} \frac{\phi(x)}{x^n}$$

Proof. (a) We want to show that if n is odd, then there exists a number x such that $x^n + \phi(x) = 0$. Start with some algebra.

$$f(x) = x^n + \phi(x) = x^n(1 + \frac{\phi(x)}{x^n})$$

By the fact that $\lim_{x\to\infty} \frac{\phi(x)}{x^n} = 0$ we know that $\forall \epsilon > 0\ \exists N > 0$ such that $\forall x > N \implies |\frac{\phi(x)}{x^n}| < \epsilon$. We pick an $\epsilon < 1$ and let M be the large constant we need in the definition of limit. So we have for these values of x:

$$-\epsilon < \frac{\phi(x)}{x^n} < \epsilon$$

This implies

$$0 < 1 - \epsilon < 1 + \frac{\phi(x)}{x^n}$$

Pick $x_1 > M > 0$. Note that $x_1^n > 0$. So we have:

$$0 < (x_1)^n(1-\epsilon) < (x_1)^n(1 + \frac{\phi(x_1)}{x_1^n}) = f(x_1)$$

Now we use the assumption regarding the limit in the negative infinity direction, $\lim_{x\to-\infty} \frac{\phi(x)}{x^n} = 0$. This tells us in particular that: $\forall \epsilon > 0\ \exists N < 0$ such that $\forall x < N \implies |\frac{\phi(x)}{x^n}| < \epsilon$. We pick an $\epsilon < 1$ and let K be the large constant we need in the definition of limit. So we have for these values of x:

$$-\epsilon < \frac{\phi(x)}{x^n} < \epsilon$$

Again we have;

$$0 < 1 - \epsilon < 1 + \frac{\phi(x)}{x^n}$$

Choose $x_2 < K$. Since n is odd we know that $x_2^n < 0$. Hence:

$$0 > (x_2)^n(1-\epsilon) > (x_2)^n(1 + \frac{\phi(x_2)}{x_2^n}) = f(x_2)$$

Now we apply the intermediate value theorem on the interval $[x_2, x_1]$. This means we can find an $x \in [x_2, x_1]$ such that

$$f(x) = x^n + \phi(x) = 0$$

□

Proof. (b) We now want to show that if n is even, then there exists y such that $y^n + \phi(y) \leq x^n + \phi(x)\ \forall x \in \mathbb{R}$. Arguing as in part(a) we can take $\bar{N} = max(|N|, |M|)$. From this fact we have $\forall |x| \geq \bar{N}$ that

$$x^n(1-\epsilon) \leq f(x) \leq x^n(1+\epsilon)$$

In particular this tells us that $f(\bar{N})$ is a finite number. Now choose $b > 0$ such that $b > \bar{N}$ and $b^n(1-\epsilon) \geq f(\bar{N})$. To obtain this we simply set $b^n \geq \frac{\bar{N}^n(1+\epsilon)}{(1-\epsilon)}$. The point is that $[-\bar{N}, \bar{N}] \subseteq [-b, b]$. Now we know that on the interval $[-b, b]$ there exists a y such that $y^n + \phi(y) \leq x^n + \phi(x)$. In particular

$$f(y) \leq f(\bar{N})$$

Now we consider the other two intervals:

$$x \geq b \implies f(\bar{N}) \leq b^n(1-\epsilon) \leq x^n(1-\epsilon) \leq f(x)$$

$$x \leq -b \implies f(\bar{N}) \leq b^n(1-\epsilon) = (-b)^n(1-\epsilon) \leq x^n(1-\epsilon) \leq f(x)$$

Hence $\forall x \in \mathbb{R}$

$$f(y) \leq f(x)$$

. So we have found a global minimum. □

18 Suppose that $f(x) > 0 \ \forall x$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$

Prove $\exists y$ such that $f(y) \geq f(x) \ \forall x$.

Proof. Fix $\epsilon > 0$. By the assumption on limits $\exists N_1 > 0$ and $\exists M_1 < 0$ such that $\forall x > N_1$ and $\forall x < M_1 \implies 0 < f(x) < \epsilon$. First pick $\bar{N}_1 = max(|N_1|, |M_1|)$. Hence on the symmetric interval $[-\bar{N}_1, \bar{N}_1]$ we have $0 < f(x) < \epsilon$. Furthermore on the interval $[-\bar{N}_1, \bar{N}_1]$ there exists y_1 such that $f(y_1) \geq f(x)$.

Now either $f(y_1) \geq f(x) \ \forall x$ or $\exists x_1 \in (-\infty, -\bar{N}_1] \cup [\bar{N}_1, \infty)$ such that $f(y_1) < f(x_1)$. If the first case occurs then we are done and the theorem is proven. But if we are in the second case then this implies that $0 < f(x) < \epsilon$ for all x. This last statement follows from the fact that y_1 is the maximum on the interval $[-\bar{N}_1, \bar{N}_1]$, so all the x values there satisfy this inequality. And furthermore the rest of the x-values satisfy this inequality from the choice of ϵ. Let us now define a new epsilon for the problem. We consider $\frac{\epsilon}{2}$. Then as above, $\exists \bar{N}_2$ such that $f(y_2) \geq f(x)$ $\forall x \in [-\bar{N}_2, \bar{N}_2]$ and $0 < f(x) < \frac{\epsilon}{2}$ for the other x values. Again either $f(y_2) \geq f(x)$ $\forall x$ or $\exists x_2 \in (-\infty, -\bar{N}_2] \cup [\bar{N}_2, \infty)$ such that $f(y_2) < f(x_2)$. Again if the first scenario occurs then we are done. But if we are in the second scenario then again $0 < f(x) < \frac{\epsilon}{2}$ for all x. Suppose we keep doing this construction then we claim that at some point the first scenario must occur. If it does not then we will have the alternative statement $\forall n$:

$$f(x) < \frac{\epsilon}{2^n}$$

Hence this implies that $\forall x$

$$f(x) = 0$$

But this contradicts the fact that $0 < f(x)$ for all x. So the construction above realizes the first scenario at some k^{th} step. Then that y_k is the maximum point we desire. In particular

$$f(y_k) \geq f(x) \ \ \forall x$$

□

6. Homework 6

Chapter 8: 1,2,3

1

(i) Least upper bound = 1 (in the set). Greatest Lower bound = 0 (not in the set)
(ii) Least upper bound = 1 (in the set). Greatest Lower bound = -1 (in the set)
(iii) Lest upper bound = 1 (in the set). Greatest Lower bound = 0 (in the set)
(iv) Least upper bound = $\sqrt{2}$ (not in the set). Greatest Lower bound = 0 (in the set)
(v) Least upper bound = ∞. Greatest Lower bound = $-\infty$
(vi) Least upper bound = $-\frac{1}{2} + \frac{\sqrt{5}}{2}$ (not in the set). Greatest Lower bound =

$-\frac{1}{2} - \frac{\sqrt{5}}{2}$ (not in the set)
(vii) Least upper bound = 0 (not in the set). Greatest Lower bound = $-\frac{1}{2} - \frac{\sqrt{5}}{2}$ (not in the set)
(viii) Least upper bound = 1.5 (in the set). Greatest Lower bound = -1 (not in the set)

2a

Proof. Since $a \in A$ this implies that $-A \neq \varnothing$, since $-a \in -A$ by definition of the set. Furthermore let L be the lower bound for the set A. This means in particular that

$$L \leq a \ \ \forall a \in A \quad \Longrightarrow$$
$$-L \geq -a \ \ \forall a \in A \quad \Longrightarrow$$

$-M$ is an upper bound for the set $-A$. Finally we note that A satisfies the greatest lower bound property (P13). Notice also that $-A$ satisfies the greatest lower bound property (P13). Hence $\exists\ \alpha = \inf A$ and $\exists\ \beta = \sup -A$. Our task is to show that

$$\beta = -\alpha$$

Let $-M$ be arbitrary upper bound of $-A$.

$$-M \geq -a \ \ \forall -a \in -A \quad \Longrightarrow$$
$$M \leq a \ \ \forall a \in A \quad \Longrightarrow$$

(by the definition of greatest lower bound)

$$M \leq \alpha \quad \Longrightarrow$$
$$-M \geq -\alpha$$

Since $-M$ was an arbitrary upper bound and $-\alpha$ is an upper bound by definition we have shown that $-\alpha$ is a least upper bound for $-A$. □

2b

Proof. Since $A \neq \varnothing$ Let $a \in A$. Also by assumption we know that A is bounded below, hence $\exists M$ such that $M \leq a \ \ \forall a \in A$. This implies that $B \neq \varnothing$ since $M \in B$. Furthermore $a \geq b \ \ \forall b \in B$, this follows from the definitions of the sets A and B. Hence B is bounded above. These facts about B are enough to apply (P13) and conclude that $\sup B$ exists. If we let $\beta = \sup b$ and $\alpha = \inf A$. Our task is to show

$$\beta = \alpha$$

Let us start by picking an arbitrary element of A. Denote this element by a. We recall that

$$a \geq b \ \ \forall b \in B \quad \Longrightarrow$$

a is an upper bound for B. Hence by definition of the least upper bound

$$a \geq \beta$$

Hence we have shown that $\beta \in B$. Furthermore if we take any $\gamma > \beta$ by the greatest lower bound property and the fact we just proved γ cannot be an element of B. Hence what we have shown is that β is a lower bound of A and any $\gamma > \beta$ is not a lower bound. This precisely shows that $\beta = \alpha$.

□

3a

Proof. No. Just take the interval $[-1,2]$ and the function

$$f(x) = \begin{cases} -x^2 & \text{if } x \in [-1,0] \\ 0 & \text{if } x \in [0,1] \\ (x-1)^2 & \text{if } x \in [1,2] \end{cases}$$

We do not have a second smallest zero for this function. To show that there is a largest $x \in [a,b]$ with $f(x) = 0$, redo Theorem 7-1, with the function $g(x) = -f(x)$ and the new set

$$\bar{A} = \{x \ : \ a \leq x \leq b \ \text{ and g is negative on } \ [x,b] \ \}$$

All the sup's in the proof will become inf's. □

3b

Proof. If you redo the proof in Theorem 7-1 with this new set, you will find that the zero that is located is the largest zero. I will leave you the details of redoing the proof. To see how A and B can be different. Consider the interval $[0,2\pi]$, and the function $f(x) = -\sin x$. □

7. Homework 7

Chapter 8: 12,13,14

12a

Proof. We want to show that

$$\sup A \leq y \ \ \forall y \in B$$

Let $y \in B$ arbitrary. By definition of the sets we know that $y \geq x \ \forall x \in A$. Hence By definition of sup we know that $\sup A \leq y$. Since y was an arbitrary element this implies that $\sup A \leq y \ \ \forall y \in B$. □

12b

Proof. In part (a) it was shown that $\sup A \leq y \quad \forall y \in B$. Hence $\sup A$ is a lower bound for the set B. By definition of the greatest lower bound we know that $\sup A \leq \inf B$. □

13

Proof. Let $z \in A + B$. This implies that $z = x + y$ for some $x \in A$ and $y \in B$. By definition $x \leq \sup A$ and $y \leq \sup B$. So we have that $z \leq \sup A + \sup B$. Since z was an arbitrary element of $A + B$ this implies that $\sup A + \sup B$ is an upper bound. So by definition $\sup A + B \leq \sup A + \sup B$. To show the other direction we argue using the hint in the book. Choose $x \in A$ and $y \in B$ with $\sup A - x < \frac{\epsilon}{2}$ and $\sup B - y < \frac{\epsilon}{2}$. Hence we have:

$$\sup A + \sup B < (x + y) + \epsilon$$

Now arguing as in the first part this implies that

$$\sup A + \sup B < \sup{(A + B)} + \epsilon$$

Since $\epsilon > 0$ was arbitrary this actually means that

$$\sup A + \sup B \leq \sup{(A + B)}$$

(Think about this last step carefully. Convince yourself that this is true) ☐

14a

Proof. If we define $A = \{a_n\}$ and $B = \{b_n\}$ we can use **12a** to conclude that

$$a_n \leq \sup A \leq b_n \ \ \forall n \in \mathbb{N}$$

Hence $\sup A$ is the point we desire. ☐

14b

Proof. Consider the interval $(0, \frac{1}{n})$. Here $a_n = 0\ \forall n$ and $b_n = \frac{1}{n}$. ☐

8. Homework 8

Chapter 9: 2,3,4,5,6,8

2a

Proof. This one is a simple application of the definition of derivative. Use the definition and then plug in a. ☐

2b

Proof. Now that you know that the slope of the tangent line to our function at a is $\frac{-2}{a^3}$. Use the point slope form to find that the equation for the tangent line is

$$y = \frac{-2}{a^3}x + \frac{3}{a^2}$$

Now set this line equal to the original function

$$f(x) = \frac{1}{x^2}$$

. Finally look for the points of intersection. ☐

3

Proof. Same as number 2 just some more algebra. Let me know if you get stuck on this one and need help. ☐

4

Proof. Our conjectured formula is everyone's favorite derivative formula:

$$S_n'(x) = nx^{n-1}$$

To prove this use the Binomial theorem in the numerator:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$= \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

Now just expand and see that the leading order term vanishes. Divide by h and take the limit to get the right answer. ☐

5 Recall that $[x] =$ Largest integer that is $\leq x$. Start by drawing a graph of this function. This is to give us a bit of intuition about the problem. The picture seems to indicate that the only points where we will run into problems are at the integer points. Here the function jumps. In particular the function is not continuous at these integer points. Hence if a function is not continuous at a point it is definitely not differentiable there. Everywhere else we have a constant function. So the derivative should be 0 at every point that is not an integer. I leave the details for you to work out.

6a

Proof.
$$\frac{g(x+h)-g(x)}{h} = \frac{f(x+h)-f(x)}{h}$$
This is an identity for every x. Take the limit as h goes to 0 of both sides to conclude.

□

6b

Proof.
$$\frac{g(x+h)-g(x)}{h} = c\,\frac{f(x+h)-f(x)}{h}$$
This is an identity for every x. Take the limit as h goes to 0 of both sides and recall the theorem that allows you to take constants out of the limit operation to conclude.

□

8a

Proof. Let $y = x + c$. Then we have:
$$\frac{g(x+h)-g(x)}{h} = \frac{f(y+h)-f(y)}{h}$$
This is an identity for every x. Take the limit as h goes to 0 of both sides. We have:
$$g'(x) = f'(y) = f'(x+c)$$

□

8b

Proof. We have the following:
$$\frac{g(x+h)-g(x)}{h} = \frac{f(cx+ch)-f(cx)}{h} = c\,\frac{f(cx+ch)-f(cx)}{ch}$$
Let $y = cx$ and $ch = \bar{h}$. So we have:
$$\frac{g(x+h)-g(x)}{h} = c\,\frac{f(y+\bar{h})-f(y)}{\bar{h}}$$
Since $\bar{h} \to 0$ when $h \to 0$, we have the following equalities:
$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} c\,\frac{f(y+\bar{h})-f(y)}{\bar{h}} = \lim_{\bar{h}\to 0} c\,\frac{f(y+\bar{h})-f(y)}{\bar{h}}$$

So we have:

$$g'(x) = cf'(y) = cf'(cx)$$

□

9. Homework 9

Chapter 10: 1(v)(vi), 2(ix)(xvi), 15,16,18 I trust you can do the chain rule. So questions 1 and 2 I leave to you.

15a

Proof. No. Take

$$f(x) = \frac{1}{x^2}$$
$$g(x) = -f(x)$$

They sum to 0 and thus the sum is differentiable at 0. But each individual function is not differentiable at 0. □

15b

Proof. As long as fg is differentiable at a, f is differentiable at a, and $f(a) \neq 0$ we are good. Then we have the quotient rule. □

16a

Proof. $f(a) \neq 0 \implies \exists \delta_1 > 0$ such that $f(x) > 0$ for all $x \in (a - \delta_1, a + \delta_1)$. Of course it could be that $f(x)$ has the opposite sign. But the argument we present can be used in this case as well. Just put in the appropriate negative signs where they should be. Hence in this neighborhood around a,

$$|f(x)| = f(x)$$

. By the differentiability of $f(x)$ $\forall \epsilon > 0$ there exists a $\bar{\delta}$ such that

$$|h| < \bar{\delta} \implies |\frac{f(a+h) - f(a)}{\bar{h}} - f'(a)| < \epsilon$$

Let

$$\delta = \min(\delta_1, \bar{\delta})$$

Hence we have shown that $\forall \epsilon > 0$ $\exists \delta > 0$ such that

$$|h| < \delta \implies |\frac{|f|(a+h) - |f|(a)}{\bar{h}} - f'(a)| = |\frac{f(a+h) - f(a)}{\bar{h}} - f'(a)| < \epsilon$$

□

16b

Proof. Let

$$f(x) = x - a$$

□

16c

Proof. Use the definition of max and min functions and part a □

16d

Proof.

$$f(x) = x - a$$

$$g(x) = 0$$

□

18a

Proof. I believe in you. You can do this. □

18b

Proof. This one too. □

18c

Proof. The assumption that $f(x) > 0$ is very important. Since

$$(f'(x))^2 = f(x) + \frac{1}{f(x)^3}$$

We know that we can take the square root of both sides and hence

$$f'(x) > 0$$

Now differentiate both sides and use part(b) to get the derivative on the LHS. Since $f'(x) > 0$ you can divide(!!) by $f'(x)$ on both sides. (Actually $2f'(x)$). And we are done. □

10. Homework 10

Chapter 11: 1(ii)(iv)(vi), 4(a), Worksheet Problems (They will not be presented here. I really think you will benefit more from reading the solutions in the textbook. Especially number 3. Make sure you understand uniform continuity).

1(ii)(iv)(vi)

Proof. Baby Calculus problem: Easy Peasey Brocolli squeezey □

4a

Proof. Start this problem like all of them by taking the derivative and noticing that the critical point is

$$x = \frac{2a_1 + ... + 2a_n}{2n}$$

But how do we know that this is the minimum? The point is that the original function is a polynomial of even degree. We proved that these specific polynomials always have a minimum point. Now go back to the beginning of the chapter and read the part where it says that if a function has a minimum then it must be a critical point. Hence the critical point we found is the minimum. □

11. Homework 11

Chapter 11: 23,26,35,36,37,38

23 I'm boycotting this problem. I hope many of you can join me. If someone is realllyyyy interested in this one, email me and I'll give you a partial solution.

26

Proof. Define

$$g(x) = f(x) + f'(x) + ... + f^n(x)$$

Assume by contradiction that $\exists \bar{x}$ such that $g(\bar{x}) < 0$. Since n is even it is clearly true that $g(x)$ is also a polynomial of even degree. This implies that there exists a minimum point for $g(x)$ on the entire real line. Call this minimum point α. By definition we then have the scenario where $g(\alpha) < 0$. Now α being a minimum point also tells you that it is a critical point:

$$g'(\alpha) = f'(\alpha) + ... + f^n(\alpha) = 0$$

. Hence: $g(\alpha) = f(\alpha) + f'(\alpha) + ... + f^n(\alpha) = f(\alpha)$. But this implies that

$$f(\alpha) < 0$$

This is a contradiction to the positivity of the function $f(x)$.

□

35

Proof. Using the assumption that $g(a) \neq 0$ we know that this actually holds in a larger interval $(a - \delta, a + \delta)$. This follows from the continuity of the function $g(x)$. Furthermore we can differentiate $\frac{f(x)}{g(x)}$ using the quotient rule in this interval. The point is that the numerator in the quotient rule is exactly $fg' - f'g$. So we know this is zero. Hence

$$\frac{f}{g} = C$$

in the interval $(a - \delta, a + \delta)$ where C is a constant. But we also know that $\frac{f(a)}{g(a)} = 0$. Hence

$$0 = \frac{f(a)}{g(a)} = C$$

So the constant is 0. We have proved that

$$\frac{f(x)}{g(x)} = 0$$

for each x in an interval around a. □

36

Proof. Let us define $h = x - y$ for arbitrary x and y. By assumption we have that

$$|f(y + h) - f(y)| \leq |h|^n$$

Divide each side by $|h|$:

$$\frac{|f(y + h) - f(y)|}{|h|} \leq |h|^{n-1}$$

Note: $n-1>0$ by assumption. We now have the following inequalities:

$$0 \leq \frac{|f(y+h)-f(y)|}{|h|} \leq |h|^{n-1}$$

Take the limit as h goes to 0 of the outer terms. We then have:

$$\lim_{h\to 0} \frac{|f(y+h)-f(y)|}{|h|} = \lim_{h\to 0} |\frac{f(y+h)-f(y)}{h}| = 0$$

Recall the following result about limits: If f is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x)) = f(L)$. Applying this result and recalling that $|x|$ is a continuous function we know that

$$\lim_{h\to 0} |\frac{f(y+h)-f(y)}{h}| = |\lim_{h\to 0} \frac{f(y+h)-f(y)}{h}|$$

The term on the RHS is exactly:

$$|f'(y)|$$

Hence we have shown that for an arbitrary y

$$|f'(y)| = 0$$

This implies that

$$f'(y) = 0$$

Hence by the arbitrariness of y, $f(y)$ is constant. □

37a

Proof. We want to show at a fixed point x: $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$|y-x| < \delta \implies |f(y)-f(x)| < \epsilon$$

We choose $\delta = \frac{\epsilon^{1/\alpha}}{C^{1/\alpha}}$ Then we have:

$$|f(y)-f(x)| \leq C|x-y|^\alpha \leq C(\frac{\epsilon^{1/\alpha}}{C^{1/\alpha}})^\alpha \leq \epsilon$$

□

37b

Proof. To show uniform continuity we argue exactly the same way except we let x, y be arbitrary points. Then as in part(a) using the Holder Continuity assumption we have:

$$|f(y)-f(x)| \leq C|x-y|^\alpha \leq C(\frac{\epsilon^{1/\alpha}}{C^{1/\alpha}})^\alpha \leq \epsilon$$

□

37c

Proof. Using the fact that the function is differentiable we resort to the definition of derivative:

$$\forall \epsilon > 0\ \exists \delta > 0\ s.t.\ |h| < \delta \implies |\frac{f(x+h)-f(x)}{h} - f'(x)| < \epsilon$$

Set $\epsilon = 1$. Then in the appropriate delta neighborhood we know:

$$|\frac{f(x+h)-f(x)}{h}| < |f'(x)| + 1$$
$$\implies |f(x+h)-f(x)| < (|f'(x)|+1)|h|$$

Setting $y = x + h$, we see that:

$$|f(y) - f(x)| < (|f'(x)| + 1)|y - x|$$

So our $C = (|f'(x)| + 1)$. □

37d

Proof. No. The classic counter-example is

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin \frac{1}{x^2} & otherwise \end{cases}$$

□

37e

Proof. Same as 36 □

38

Proof. Consider

$$F(x) = a_0 x + \frac{a_1 x^2}{2} + ... + \frac{a_n x^n}{n+1}$$

Notice that $F(1) = 0$ by assumption and $F(0) = 0$. So we can apply $Rolle's Theorem$ to this function which tells us that there exists $x \in (0, 1)$ such that $F'(x) = 0$. Notice that

$$F'(x) = a_0 + a_1 x + ... + a_n x^n$$

□

12. Homework 12

Chapter 13: 5,6,14,16,33,36

5(i)

Proof. Notice that the integrand is odd and we are evaluating over a symmetric interval. So the integral is 0 □

5(ii)

Proof.

$$\begin{aligned} \int_{-1}^{1} (x^5 + 3)\sqrt{1 - x^2} &= \int_{-1}^{1} x^5 \sqrt{1 - x^2} + \int_{-1}^{1} 3\sqrt{1 - x^2} \\ &= 0 + \frac{3}{2}\pi \end{aligned}$$

First term follows from part(a) the second term is because we are finding the area of a semi-circle. □

6, 14, 16

Proof. These are a bit tedious to write up. If anyone wants to see these let me know. I'll send them to you. □

33a

Proof. L(f,P) = 0 □

33b

Proof. $\frac{1}{2}$ Look at page 261 $\square$

36a

Proof. You have to break this one up into different cases. Fix an interval $[t_i, t_{i+1}]$ Start by first considering $f(x) \geq 0$ on this interval. Then

$$M_i' = M_i$$

and

$$m_i' = m_i$$

. Similarly check when $f(x) \leq 0$ on this interval. Finally you have to check when $f(x)$ changes sign on this interval. In this last case you have to compare the relative sizes of m_i to M_i. $\square$

36b

Proof. This follows from considering

$$U(|f|, P) - L(||f|, P)$$

and using part(a). Since the function f is integrable that follows that the absolute value function is integrable. $\square$

36c

Proof. Use part(b) and the definition of max and min function. $\square$

36d

Proof. If f is integrable then part(c) implies the result. Assume that we have $max(f, 0)$ and $min(f, 0)$ are integrable. Then $f(x)$ is integrable because $f(x) = max(f, 0) + min(f, 0)$. $\square$

Unfortunately due to time constraints I have not been able to type up anymore solutions. I will mention that if anyone has questions about any of the other problems please email me and I will do my best to explain it to you. Lastly as a bonus the extra problems assigned in problem set 9 and 10, the solutions can be found in the appendix of chapter 19. Happy Reading!