Analysis Final Review

Mark

Tools: set theory, functions

a **set** is a collection of things.

Powerset of A is the set of all subsets of A.

De Morgan's Law: $\complement(A \cap B) = \complement(A) \cup \complement(B)$
think of rows and columns of students in a room.
Also, $\complement(A \cup B) = \complement(A) \cap \complement(B)$

an **equivalence relation** partions a set into "equivalence classes"; it's
reflexive: $a \equiv a$
symmetric: $a \equiv b \iff b \equiv a$
transitive: $if a \equiv b\ and\ b \equiv c, then\ a \equiv c$

a **function** is a rule that sends each element of one set to a specific element of another set.

$$f : X \to Y$$

X, Y are sets: X is domain, Y co-domain

one-to-one or **injective**: "no two inputs are sent to the same output"
onto or **surjective**: "each item in Y is hit with some X"
bijective: injective and surjective

Real Numbers

Begin with numbers as objects and three axioms,
1. **Field**: $+, *$ associative, commutative, inverses and distribution
2. **Order**: $+$ is closed over $\mathbb{R}^+$ and any $a \in \mathbb{R}$ is in $\mathbb{R}^+$, is 0, or $\mathbb{R}^-$
3. **Upper Bound property**: every **nonempty** set that is bounded above has a **least** upper bound.

Field Properties

For a, b in a field,

Master Tool

$x + a = b$ and $ax = b$ have one **unique solution**.
use $-a$ and a^{-1} to show

$a0 = 0$ for all a is a consequence, as are others we expect.

Order

Trichotomy (any $a, b \in \mathbb{R}$): $a < b$, $a = b$, or $a > b$.

Archimedean Property: there is an $n \in \mathbb{Z}$ for any ε such that $\frac{1}{n} < \varepsilon$
useful
for proofs like $a^m a^n = a^{m+n}$ consider all cases $\pm m$ and n.
$|x - a| < \epsilon$ implies $a - \epsilon < x < a + \epsilon$.

Existence of Square Roots

Every positive number has a unique square root.
modify proof below to show

There is an $x \in \mathbb{R}^+$ such that $x^2 = 3$. Consider the lub, call it r, of $\{x \in \mathbb{R} : x^2 < 3\}$. Show 10 is an upper bound and that $r : r^2 = 3$ is the lub.

We negate the possibilities that $r^2 < 3$ and $r^2 > 3$.
Suppose $r^2 < 3$, then we want to show $(r+\epsilon)^2 \in$ set (for $0 < \epsilon < 1$, but is greater than r^2. KEY: $r^2 + 2r\epsilon + \epsilon^2 < r^2 + 2r\epsilon + \epsilon$. Then, it becomes obvious $\epsilon = \frac{3-r^2}{2r+1}$ to yield the result we want.
Next suppose $r^2 > 3$. Then show $(r-\epsilon)^2$ is an upper bound smaller than the least upper bound.

Triangle Inequality

For $a, b \in \mathbb{Z}$,

$$|a| - |b| \leq |a - b| \leq |a| + |b|$$

$\pm a \leq |a|,$ means both a and $-a$are $\leq a$
then, $+-a++-b \leq |a|+|b|$
we only use two of these cases: $a+b$ and $-a-b$
to get $|a+b| \leq |a|+|b|$
left part
trick $|a| = |b-(b-a)| \leq |b| + |b-a|$ using triangle

$\boxed{\text{idea: } |a|-|b| \text{ is a definitive decrease; } |a|+|b| \text{ increase}}$

Cauchy-Schwarz

For $a_1, ..., a_n$ and $b_1, ..., b_n \in \mathbb{R}$,

$$(\sum_{k=1}^{n} a_k b_k)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

In vector land,

$$(\vec{a} \cdot \vec{b}\ )^2 \leq ||\vec{a}||^2\ ||\vec{b}||^2$$

$||\vec{a}||$ is the length of a.
sometimes also written as $|\vec{a}\vec{b}| \leq ||\vec{a}||\ ||\vec{b}||$.

Complex Numbers

complex numbers $\mathbb{C}$ have no ordering since i^2 is negative.

Absolute Value of a + bi is defined as

$$|a + bi| = \sqrt{a^2 + b^2}$$

Triangle Inequality holds for complexy numbers.

$$e^{iy} = \cos y + i \sin y$$

use derivatives to intuit why

We can also represent complex numbers using polar coordinates with θ and d as

$$r \cos\theta + ir \sin\theta$$

Hyperbolic sin and cos are ways of extending trig to complex numbers. They are defined as

$$\sinh x = \frac{e^{ix} - e^{-ix}}{2}$$

$$\cosh x = \frac{e^{ix} + e^{-ix}}{2i}$$

Metric Spaces

To speak of points near each other, continuity or limits, we need to understand the space in which these objects live: metric spaces.

metric spaces are a generalization of the real line, plane, or 3-d space.

a **metric space** a set E together with a rule d associating a pair to elements to a real number such that for any $a, b \in E$:

1. $d(a, b) \geq 0$
2. $d(a, b) = 0$, if and only if $a = b$, definite
3. $d(a, b) = d(b, a)$, symmteric
4. $d(a, c) \leq d(a, b) + d(b, c)$, Triangle inequality

Examples

- discrete metric space: 0 if same point; 1 otherwise
- $E : d(a, b) = |a - b|$
- $E^n(\text{in } \mathbb{R}^n) : d(a, b) = \sqrt{|a_1 - b_1|^2 + ... + |a_n - b_n|^2}$
  "Euclidean": $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
- Taxicab: $d(a, b) = |a_1 - b_1| + |a_2 - b_2|$
  "distance one would travel on streets by taxi", $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$
- Sup ("max metric"): $d(a, b) = max\{|a_1 - b_1|, |a_2 - b_2|\}$
  $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

Closeness

Open Ball

Let (X, d) be a metric space. An **open** ball of radius r ($r \in \mathbb{R}^+$) and center $p \in X$ is

$$B(p, r) = \{y \in X : d(y, p) < r\}$$

Open Set G

- every point is center of open ball in G
- union of open balls

Properties

- the entire metric space is open; empty set is too!
- the union of any collection of open sets is open
- the interesection of a *finite* collection of open sets is open

Examples
Open in E^1 is just an open interval
E^2 is a circle
Taxicab in $\mathbb{R}^2$ is a square pointing up

Closed Set G

- $\complement G$ is open
- every sequence in G converges in G

Properties

- entire set is closed (both closed and open); empty set too
- intersection of closed sets is closed
- union of **finite** closed sets is closed

Subsets of a metric can be **neither closed nor open**. e.g., $[a, b)$

A subset of a metric space is **bounded** if it's contained in some ball (open or closed).

Completeness

Cauchy: elements as close as we wish beyond a certain point

Complete: every Cauchy sequence converges.

Convergent $\rightarrow$ **Cauchy** use triangle

- subsequence of Cauchy is Cauchy
  any two elements are as close as we wish, so any subcollection has to be too.

– Cauchy with convergent subsequence is convergent
  For p_m the convergent subsequence:
  $d(p, p_n) \leq d(p, p_m) + d(p_m, p_n) < \epsilon$

$\mathbb{R}^n$ is complete (that is E^n).

Any closed subset of complete space is complete
complete so cauchy converges in space. However, subset is closed, so must converge in subset.

accumulation point = limit point.

Compactness

$\boxed{\text{every open cover of } S\text{, has a } \textbf{finite subcover}.}$

subcover is a finite subset of the open cover.
open cover: union of open balls.

Any **closed subset** of compact is compact
K compact; S closed subset.
Any open cover of S can be expanded by another open cover of $\complement S$, which has a finite subcover.

closed cell in E^m set of points with each coordinate inside some interval $[a, b]$.

cluster point of a set, need not be in the set.

Bolzano Weirstrauss

Every bounded sequence in R^n has a convergent subsequence

For any sequence a_n,
$a_n \subset$ interval [a, b] since bounded
infinitely many points of a_n in either $[\frac{a+b}{2}, b]$ or $[a, \frac{a+b}{2}]$.
Continue shrinking interval to construct a convergent subsequence.

Compact is

- complete every cauchy has a convergent subsequence$\rightarrow$ every cauchy converges by theorem
- closed every convergent sequence converges to the same limit as its subsequence, so limit is in the set.

Heine-Borel

Any closed, bounded subset of $\mathbb{R}^n$ is compact
unclear, but use contradiction

Nested Cells

In compact metric space,
... $\subset S_3 \subset S_2 \subset S_1$
for S_i closed set, at least one point in all S_i.
Alternatively stated as the $\cap$ infinite nested closed cells is nonempty.

Intuitively: nested [a, b] intersect at the least at one point.

counter example: nested rays

Connectedness

Space E is connected if only subsets that are both open and closed are: E itself and empty-set.

A subset is connected if it forms a connected subsapce.

Any interval in $\mathbb{R}$ is connected (open or closed).

Connected $\rightarrow$ can't be written as **two disjoint open** sets.

Two disjoint open, imply a hole, so space is not connected.

Continuous Functions

$$f : E \rightarrow E' \text{ is } \textbf{continuous}$$

$$\Longleftrightarrow$$

$$\text{for every } \textbf{open} \text{ set } S \text{ in } E', f^{-1}(S) \textbf{ is open}$$

$$***$$

"inverse image of open is open if and only if f is continuous" (also works with closed sets)

Continuity on Compact Sets

$$f : E \rightarrow E' \text{ continuous}$$

$$E \text{ compact}$$

Big:

$$\text{image of compact is compact}$$

Let U be an open cover of $f(E)$.
Then, $f^{-1}(U)$ is open, as f is continuous.

Hence, there is a finite cover $f^{-1}(U)$ containing E
Thus,

$$f(E) \subset f(\text{ finite cover of } f^{-1}(U)\ ) \subset \text{ finite cover of } U$$

consequences,

Extreme Value Theorem

$f([a,b])$ compact $\rightarrow$ closed, bounded.
Thus, l.u.b. exists.
For K a compact set,

1. continuous on $K \rightarrow$ uniformly continuous
2. there is a neast point in K to any point
3. f continuous and bijective in $K \rightarrow f^{-1}$ continuous

Continuity on Connected Sets

$$f : E \rightarrow E' \text{ connected}$$
$$E \text{ connected}$$

Big:

$$\text{image of connected is connected}$$

Consequences,

1. convex set $\rightarrow$ connected
2. f_i a sequence of **uniformly** continuous functions
   $\rightarrow f$ the **limit is continuous** proof: $\epsilon/3$ theorem
3. Cauchy Criterion
   f_n is **uniformly Cauchy** if there is N such that for all $m, n > N$,

   $$d(f_n(x), f_m(x) < \epsilon \text{ for all x}$$

   f_n uniformly cauchy in a complete space $\rightarrow f_n$ converges uniformly

Curiosities: "Coffee Cup Theorem," Brower Fixed Point:
for $f([0,1]) \rightarrow [0,1]$ continuous, there is p such that $f(p) = p$.

Series

series with terms a_k **converges** if $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists

Geomteric Series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \text{ for } |r| < 1$$

compute $s_n - rs_n$ where s_n is the nth partial sum for proof

Cauchy Sequence Criterion

series converges $\iff |S_m - S_n| < \epsilon, \quad$ for all $m, n >$ some N.

consequence: if series converges, $\lim_{n\to\infty} a_n = 0$.

Comparison Test

if $|b_n| < a_n$ for all n and series of a_n converges, so does $\sum b_n$.

Alternating Series Test

The $\sum(-1)^k a_k$ converges if,

$$1. a_k > 0$$

$$2. a_k > a_{k+1}$$

$$3. \lim_{n\to\infty} a_k = 0$$

conditional convergence: series $|a_n|$ diverges, but series of a_n converges.

Rearrangement

For an absolutely convergent series, any rearrangement conveges to same value.

Conditional convergent series, may have a divergent rearrangement!

Ratio Test

Let $R = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.
Then,

$$\text{series converges for } R < 1$$
$$\text{diverges for } R > 1.$$
$$\text{*no info if } R = 1.$$

Series of Functions

Weierstrauss M-test

For a series of real-valued function on a set S, if

$$|f_k(x)| \leq M_k \quad \text{for all x } \in S$$

where M_k is a **convergent** series of constants,

$$\sum f_k(x) \text{ converges uniformly (and absolutely)}$$

* if each f_k is continuous, the series converges to a continuous function.

look at apostle for sequences
index cards
practice problems
add proofs?

Exercises From Apostle

http://www.math.ucla.edu/ hendricks/Math131B.html

Apostle Problems: 2.9, 2.15, 2.18, 2.19, 3.27, 3.28, and 3.29
3.26, 3.30, 3.31, 3.2(a)-(d),(f),(g), 3.12(b),(c),(f), 3.43, and 3.46

apostle 2.15 a number is called **algebraic** if it's the root of a polynomial with integer coefficients. Algebraic numbers are coutably infinite, since an nth degree polynomial has at most n roots, thus the total is countable. (non-algebraic numbers like π are **transcendental**)

apostle 2.18 Is the set of sequences made up of 0 or 1 countable? No. Suppose you were, then you can create a list of all sequences in the set. Then construct a sequence whose terms differ from each in the list...(see solution); it's a bit weird