Calculus: single

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Relearn

Contents

1 Functions

1.1 e, sin, and cos

Euler's Formula $\boxed{e^{ix} = \cos(x) + i\sin(x)}$

for now take as a fact

Define e^x using Taylor as

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

also sometimes defined as $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

By Euler's Formula,

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) && \text{(breaking up real and imaginary)} \\
&= \cos(x) + i\sin(x).
\end{aligned}$$

1.2 Taylor

The **Taylor Series** of a function f at $x = 0$ is

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots}$$

*Taylor series can be differented term by term!

At any other point a it's

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}}{2!}(x-a)^2 + \dots$$

(remember, a must be inside the domain of convergece for the Taylor Poly)

Does Taylor exist for every function?

NO, more on this later

Geometric Series

$$1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1-r} \qquad \text{(for } |r| < 1\text{)}$$

proof: $s_n = 1 + r + r^2 + r^3 + \dots + r^n$

Then, $s_n - rs_n = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$

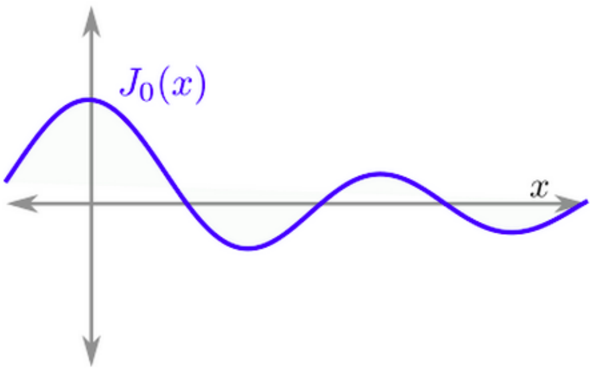
(as $n \to \infty$, this becomes $\frac{1}{1-r}$ if $|r| < 1$)

e.g., A neat example is the **Bessel function** used to describe the waves in water when a stone drops:

$$\begin{aligned}
J_0(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} \\
&= 1 - \frac{1}{2^2}x^2 + \frac{1}{2^4(2!)^2}x^4 - \frac{1}{2^6(3!)^2}x^6 + \dots
\end{aligned}$$

note the Bessel function is defined by its Taylor series.

It's a decaying cosine-like wave, since the denominator grows faster than cosine's.



Orders of Growth and Error Size

Big-O: "not faster than"

- f(x) = O(1) if $|f(x)| \leq c * 1$ as $x \to \infty$ (for some c)
- f(x) = O(g(x)) if $|f(x)| \leq c * |g(x)|$ as $x \to \infty$ (for some c)

Can be equivalently defined for $x \to 0$ to describe how quickly a function decays.
Little-o: "ultimately smaller than"

- $f(x) = o(g(x))$ means $\dfrac{f(x)}{g(x)} \to 0$ as $g(x) \to 0$.
  equivalent to $f(x) \leq c * g(x)$, for *all c*

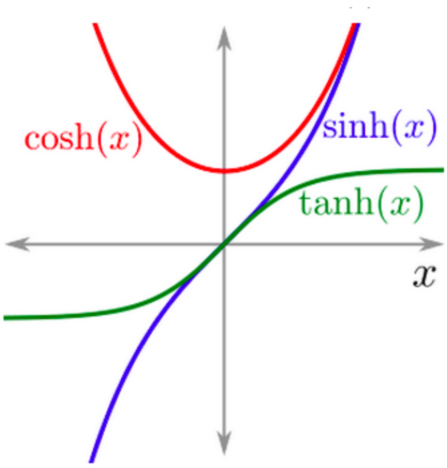
A **Linear Approximation** of a function, aka **first order Taylor** approximation about $x = a$ is $f(x) \approx f(a) + f'(a)(x - a)$.
idea: you approximate the function by its tangent.

You can use linear approximations to estimate values like $\sqrt{250}$ and π^{20}.
We know $\sqrt{256} = 16$. So, we can linearly approximate $\sqrt{250}$:
$\sqrt{250} \approx \sqrt{256} + 1/2(256)^{-1/2}(250 - 256) = 16 - 6/32 = 15.8$

1.3 Hyperbolic Trigonometric Functions

These fuctions have similar algebraic properties to $\sin, \cos$ and similarities in their taylor expansions.
(Taylor is the same without alternating signs–all +)
Define

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



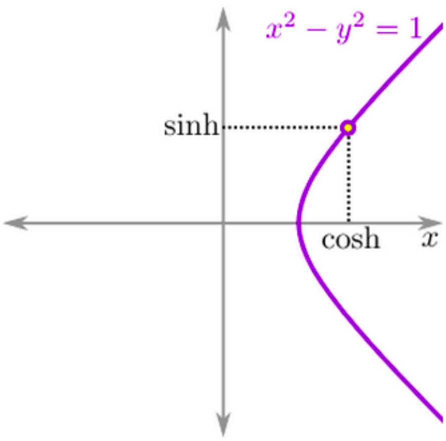
A variation of the Pythagorean Theorem holds:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Geometrically, $\cosh$ and $\sinh$ give x, y coordinates on the hyperbola $x^2 - y^2 = 1$ (similar to $\sin, \cos$ with the unit cirlce).

What are hyperbolas?
They are two parabolas approaching approaching asymptotes (facing up/down or right/left):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



(negative sign on x, instead of y, for up/down)

1.4 Tricks

Multiplying Taylor Polynomials

$$\begin{aligned}\cos(x)\cos(x) &= (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots)(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots) \\ &= 1 + (-\frac{1}{2!} - \frac{1}{2!})x^2 + (\frac{1}{4!} + \frac{1}{2!}\frac{1}{2!} + \frac{1}{4!})x^4 + \dots\end{aligned}$$

idea: what are the coefficients of x^4

Limits

General trick for limits is to factor, taylor expand, or rewrite and cancel troublesome terms.

1.5 Binomial Theorem and Series

Binomial Series

For $|x| < 1$ and any $\alpha \in \mathbb{C}$,

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.\end{aligned}$$

by computing derivatives and using Taylor Expansion

The Binomial Theorem refer to a slightly different case

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

*note here $n \in \mathbb{N}$

Exercises

1. Find the Taylor Expansion of $\ln(1+x)$ using $\ln(1+x) = \int \frac{1}{1+x}\, dx$ and the geometric series.
2. Similarly find the Taylor Expansion of $\arctan(x)$ using $\arctan(x) = \int \frac{1}{1+x^2}\, dx$ and the geometric series.

Summarizing, using Geometric + Taylor Expansion we have:
ONLY FOR $|x| < 1$, based on geometric series convergence

- $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots$
- $\arctan(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots$

1.6 Limits

The limit as x approaches a of $f(x)$ **exists** if for all x close, but NOT equal to a, $f(x)$ is close to the value of the limit.

A limit may not exist at a discontinuity, blow-up, or oscillation.
A function is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

- $\lim_{x\to a} f(x) + g(x) = l + m$
- $\lim_{x\to a} f(x)g(x) = lm.$
- "root of a limit" $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$

If a function is continuous, then you can evaluate a limit by simply plugging in the value a.

1.6.1 L'Hopital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

, the limit is equivalent to $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$, provided it exists.
*careful: l'hopital's **can't** be used to show a limit doesn't exist.

Exercises

1. Evaluate $\lim_{x\to 0} \dfrac{\sin(x)}{x}$. use Taylor (about x=0), since we are interested in $\sin(x)$ as x approaches 0

2 Differentiation

Sometimes the derivative is written as $\dot{y} = \frac{dy}{dx}$

An alternate definition of the derivative is

$$f(a+h) = f(a) + f'(a)h + O(h^2)$$

indicating the derivative gies a first-order approximation.

An **operator** is a mapping from one vector space (or module) to another.

The **curvature** of a funciton is defined as

$$\frac{f''}{(1+(f')^2)^{3/2}}$$

2.1 Newton's Method

A good way to approximate the root of a function that's differentiable.

Idea: improve guess based on tangent line (linear approximation)

Method: 1. Guess a root a_0

2. Improve by seeing where tangent intersects x-axis:
i.e., $a_1 = a_0 - \frac{f(x)}{f'(x)}$,

from $0 = f(a_0) + f'(a_0)(a_1 - a_0)$, line approximation of zero

3. Now consider a linear approximation of $f(x)$ at a_1, repeat

Caveat: If the sequence is defined, meaning $f'(a_n) \neq 0$, and it converges, then it's guaranteed to converge to a root of f. Otherwise, all bets are off!

Derivative of a^x

To find $\frac{d}{dx}a^x$:

$$y = a^x \implies ln(y) = xln(a)$$
$$\frac{d}{dx}ln(y) = 1/y\frac{dy}{dx} = ln(a)$$

Thus, $dy/dx = yln(a) = a^x ln(a)$.

2.2 Optimization

A **critical point** is an input where $f(x)$ has a derivative of zero or the *derivative* is undefined; critical points include max/min points. (max/min $\implies$ critical point)

Second Derivative Test: $f''(a) > 0 \implies$ min. ($f''(a) = 0$ says nothing)

Global max/min occur at a critical point or end points!

Exercises

1. Find $\frac{dy}{dx}$ from $y^2 - y = sin(2x)$.
   implicity differentiation–ask what's the derivative of the left hand side and the right hand side with respect to x. Then solve for $\frac{dy}{dx}$.

2. Show $\lim_{x\to\infty}(1 + a/x)^x = e^a$. Idea: take ln of both sides to work with variable x in exponent.
   $y = (1 + \frac{a}{x})^x \implies \lim_{x\to\infty} xln(y) = \lim_{x\to\infty} xln(1 + \frac{a}{x})$
   Using the Taylor expansion of $ln(1 + z)$ (note $|z| < 1$, since $x \to \infty$)
   $\lim_{x\to\infty} x(a/x - (a/x)^2/2 + (a/x)^3/3 + \dots) = \lim_{x\to\infty} a - 0 = a.$
   So, $\lim_{x\to\infty} x\ln(y) = a \implies$ solution.

3. Find $\frac{d}{dy}$ of $y = x^{x^{x^x}}$. same idea of implicity differentiation with clever redefinition of function as $y = x^y$ (recursive!).

3 Integration

4 Series

Curiosities

Striling's Approximation

Gives a good asymptotic (as n gets bigger) approximation for $n!$:

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

Typically written as $\ln(n!) = n\ln(n) - n + O(\ln(n))$.

Resources

Calculus succinctly and rigorously summarized in 10 pages