Measure Theory

Mark Ibrahim
*based on Principles of Real Analysis by Aliprantis and Burkinshaw

Contents

1 Preliminaries

a function $f : A \to B$ is **continuous** $\iff f^{-1}$(open set) is an open set.

a bounded sequence a_n has a $\limsup$ defined as $\lim_{N\to\infty} \sup\{a_N, a_{N+1}, \dots\}$ "largest tail"
a_n converges if $\limsup = \liminf$.

A **Hausdorff** topological space (T2 space) is a topological space where any two points can be seperated by open sets.

$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$.
union of countably sets is countable.

2 Algebras and Measures

2.1 Semirings and Sigma-algebras of Sets (section 12)

2.1.1 semirings

a collection S of subsets of a set X is called a **semiring** if

1. $\emptyset \in S$,
2. $A \cap B \in S$, and
3. $A - B = C_1 \cup \dots C_n$ for $C_1, \dots C_n \in S$.

Any countable union in S can be written as a countable **disjoint** union.

e.g., $S = \{[a, b) | a \leq b \in \mathbb{R}\}$ is a semiring, not an algebra.
* note $[a, a) = \emptyset$.

2.1.2 algebras

a nonempty collection S of subsets of a set X is an **algebra** if

1. $A \cap B \in S$
2. and $A^c \in S$.

Nice properties of algebras are:

- $\emptyset, X \in S$
- S is closed under finite unions and finite intersections as well as subtraction

a σ-**algebra** is an algebra that is closed under countable unions.

Borel sets of a topological space (X, T) [1] is a σ-algebra generated by the open sets.

2.2 Measures on Semirings (section 13)

A function μ from a semiring S to $[0, \infty]$ is a **measure on** S if

1. $\mu(\emptyset) = 0$
2. countably additive: $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

* $\cup_{n=1}^{\infty} A_n$ must be in S and each is disjoint.
* don't need to check if S is a $\sigma-$algebra!
• If $A \subseteq B$, $(A, B \in S)$, then $\mu(A) \leq \mu(B)$.

Alternatively, can show μ is a measure if and only if "squeeze"

1. $\mu(\emptyset) = 0$

[1] (X, T) is a topological space with a set X and subsets T if $\emptyset, X \in T$, and T is closed under unions (even uncountable), finite intersections.

2. $\sum_{i=1}^{n} \mu(A_i) \leq \mu(A)$ if $\cup_{i=1}^{n} A_i \subseteq A$ and A_i are disjoint.

3. $\mu(B) \leq \sum_{n=1}^{\infty} \mu(B_n)$, "subadditive" if $B \subseteq \cup_{n=1}^{\infty} B_n$.

2.2.1 Examples of Measures on S

- **Counting Measure** $\mu(A) = |A|$
- **Dirac Measure** Fix $a \in X$, $\mu_a(A) = 0$ if $a \notin A$, else 1.
- **Lebesgue Stieltjes** For $f : \mathbb{R} \to \mathbb{R}$, increasing, left continuous and $S = \{[a, b) | a \leq b \in \mathbb{R}\}$, $\mu([a, b)) = f(b) - f(a)$.
  - **Lebesgue Measure on** S, denoted λ is defined by $\lambda([a, b)) = b - a$.

2.3 Outer Measures (section 14)

an **outer measure** is a function $\bar{\mu} : P(X) \to [0, \infty$ such that

1. $\bar{\mu}(\emptyset) = 0$
2. if $A \subseteq B$, $\bar{\mu}(A) \leq \mu(B)$
3. countably subadditive: $\bar{\mu}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$

*an outer measure is not always a measure!

A subset E of X is **measurable** if for all $A \subseteq X$,

$$\bar{\mu}(A) = \bar{\mu}(A \cap E) + \bar{\mu}(A \cap E^c)$$

A nicer equivalent way to show E is measurable is by considering all A in S with $\mu^*(A) < \infty$ and showing

$$\mu(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Nice Properites

- every A in S is μ^*-measurable
- if $\bar{\mu}(E) = 0$, E is measurable
- for E_i measurable and any $A \subseteq X$, $\bar{\mu}(\cup_{i=1}^{n} A \cap E_i) = \sum_{i=1}^{n} \bar{\mu}(A \cap E_i)$

the collection of measurable subsets is denoted by Λ. This collection is a σ-algebra!

Remarkably, the outer measure $\bar{\mu}$ restricted to Λ is a measure!

2.4 Outer Measures generated by a measure (section 15)

The outer measure μ^* generated by a measure μ is defined for any subset A of X,

$$\mu^*(A) =$$

$$\inf\{\sum_{n=1}^{\infty} \mu(A_n) : A \subseteq \cup_{n=1}^{\infty} A_n \text{ for } A_n \in S\}$$

μ^* is called the Cathéodory extension of μ. By convention $\mu^*(A) = \infty$ if no cover exits in S.

On semiring S, $\mu* = \mu$.

For E_n measurable, if $E_n \uparrow E$, then $\mu^*(E_n) \uparrow \mu^*(E)$. * $E_n \uparrow E$ means:

1) $E_1 \subseteq E_2 \subseteq \dots$
2) $\cup_{n=1}^{\infty} E_n = E$

* note E must be measurable since it's the union of measurable sets

For B_n measurable with $\mu^*(B_n) < \infty$, if $B_n \downarrow B$, then $\mu^*(B_i) \downarrow \mu^*(B)$.

a measure space if **finite** if $\mu^*(X) < \infty$.

For X a **finite measure** space E is measurable, if and only if

$$\mu^*(E) + \mu^*(E^c) = \mu^*(X)$$

For all $A \subseteq X$, there is a measurable set E such that $A \subseteq E$ and $\mu^*(A) = \mu^*(E)$.

2.4.1 Cantor Set

Cantor set $C = \cap_{n=1}^{\infty} c_n$, where
$c_1 = [0, 1] - (1/3, 2/3)$
$c_2 = c_1 - ((1/9, 2/9) \cup (7/9, 8/9))$

each c_n is closed, because it's a closed set minus open sets.

- C has measure 0
- $|C| = |\mathbb{R}|$
- every point of C is an accumulation point of C

Vitali set is an example of a **non-measurable** subset of $\mathbb{R}$.

2.5 Lebesgue Measure (section 18)

Outer Lebesgue measure λ^* is defined as $\lambda^*(A) = \inf\{\sum_{i=n}^{\infty} \lambda(a_n, b_n) : A \subset \cup_{n=1}^{\infty}(a_n, b_n)\}$
* note $\lambda(a, b) = b - a$.
* often, we say Lebesgue measure instead of outer Lebesgue measure.

By result about $E_n \uparrow E$ from section 15, we can show (a, b), $[a, b]$, and $(a, b]$ are all measurable with same measure.

$E \subseteq \mathbb{R}$ is **Lebesgue measurable** $\iff$ there is open $O \subseteq \mathbb{R}$ for each ϵ such that $E \subseteq O$ and $\lambda(O - E) < \epsilon$.

Every Borel set in $\mathbb{R}$ is $\lambda-$measurable

2.5.1 What are the Borel sets in the reals?

By definition, it's the $\sigma-$algebra generated by open sets in $\mathbb{R}$. (Borel $\sigma-$algebra is generated by intervals of the form $(-\infty, a]$, for $a \in \mathbb{Q}$).
Borel sets contain:

- all closed sets
- union of all open sets or closed sets
- intersection of all open/closed sets

* we can write any open set in $\mathbb{R}$ as disjoint countable union of open intervals!

2.5.2 Regular Borel Measure

For X, a Hausdorff topological space and B the borel sets in X, a measure μ on B is called a **regular borel measure** if

1. $\mu(K) < \infty$ if K is compact
2. for B a borel set, $\mu(B) = \inf\{\mu(O) | O \text{ is open } B \subseteq O\}$
3. for O open, $\mu(O) = \sup\{\mu(K) | K \text{ is compact and } K \subseteq O\}$

1. λ is a regular borel measure
2. Dirac measure is a regular borel measures
3. Counting measure is not
   for example $[0, 1]$ is compact, but has infinite measure
4. any **translation invariant** regular borel measure on $\mathbb{R}$ is $c\lambda$ for some $c \in \mathbb{R}^+$

3 Integration: functions

3.1 Measurable Functions (section 16)

a relation holds **almost everywhere** if set where it fails has measure 0.

$f : X \to \mathbb{R}$ is a **measurable function** if

- $f^{-1}(O)$ is measurable, for all open sets O
- $f^{-1}(a, \infty)$ is measurable, for all a in $\mathbb{R}$

If $f, g : X \to \mathbb{R}$, $f = g$ **almost everywhere** and f is measurable, then g is measurable too!
"= a.e. means measurability carries over"

If $f, g : X \to \mathbb{R}$ are **measurable** then $\{x \in X | f(x) > g(x)\}$ is measurable.

Sum, product, constant multiple, $||$, $\max$, and f^+ [2] of measurable functions is also measurable!

3.1.1 Sequences of Functions and Measurability

recall (from analysis): $f_n \to f$ **uniformly** means $|f_n(x) - f(x)| < \epsilon$ for all x if you go out far enough in the sequence.

Key Theorem: If $f_n \to f$ **uniformly** and f_n are continuous, then f is continous.

We can define $\limsup$ ($\liminf$) for any **bounded** sequence.

For a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$

- If $f_n \to f$ a.e., then f is measurable func.
- If $\{f_n\}_{n=1}^{\infty}$ is bounded, then $\limsup$ is a measurable function (so is $\liminf$)

A sequence of functions, $\{f_n\}_{n=1}^{\infty}$ ($f_n : X \to \mathbb{R}$) converges **almost uniformly** on X if for any ϵ, there exists a measurable set F where $\mu(F) < \epsilon$ and $\{f_n\} \to f$ **uniformly** on $X - F$.

If $f_n \to f$ **almost uniformly** on X and $\mu(X) < \infty$ then, $|f_n(x) - f(x)| < \epsilon$ for all $n >$ some $N \in \mathbb{N}$, and all x in a set J where $\mu(J^c) < \delta$.

3.1.2 Ergov's Theorem (16.7)

If $f_n \to f$ **almost uniformly** on X, then $f_n \to f$ pointwise **almost everywhere** on X.

Also, if $\mu(X) < \infty$ and $f_n \to f$ pointwise on X, then $f_n \to f$ uniformly on X.

[2] $f^+ = f(x)$ if $f(x) \geq 0$ or 0 otherwise.

counter example: if $\mu(X)$ is not finite, consider $X = \mathbb{R}$, $\mu = \lambda$ and $f_n = \chi_{[n,n+1)}$. Then, $f_n \to 0$, but not almost uniformly

3.2 Simple and step functions (section 17)

nice properties of χ_A

- $A \subseteq B \iff \chi_A \leq \chi_B$
- $\chi_{A \cap B} = \chi_A \chi_B$ (equivalently $\min\{\chi_A, \chi_B\}$)
- $\chi_{A \cap B} = \chi_A + \chi_B - \chi_{A \cap B}(= \max\{\chi_A, \chi_B\})$

a measurable function $f : X \to \mathbb{R}$ is a **simple function** if it takes on finitely many values.
the standard representation of a simple function is

$$\sum_i^n a_i \chi_{Ai}$$

where a is are distinct nonzero outputs and A inputs

If each A_i has finite measure, then f is called a **step function**.

The **integral** of a step function ϕ is

$$\int \phi du = \sum_i^n a_i \mu^*(A_i)$$

*it turns out any representation, even when A_i are not disjoint (or a_i distinct) yield the same integral value

addition and scalar multiplication can be split over integrals as expected.

If $\phi \geq \psi$ a.e., then $\int \phi \geq \int \psi$
*holds if $\psi = 0$ or $\geq$ is $=$

If ϕ_n is a **sequence of step functions** with $\phi_n \downarrow 0$ a.e., then $\int \phi_n \downarrow 0$.
(similarly if $\phi_n \uparrow \psi$ a.e.)
*careful, $\uparrow \psi$, but $\downarrow 0$
* also $\phi_n \to \phi$ isn't good enough!

If $\phi_n \uparrow f$ a.e. and $\psi_n \uparrow f$ a.e., then

$$\int \phi_n = \int \psi_n \text{ as } n \to \infty$$

We can show A is **measurable** if we can find step functions $\phi_n \uparrow \chi_A$. In this case, $\mu^*(A) = \lim \int \phi_n$

For any measurable $f \geq 0$, there exists **simple** ψ_n such that

$$0 \geq \psi_n \uparrow f$$

3.2.1 sigma-finite

X is a $\sigma-$**finite measure space** if there exists E_i such that $\cup_{i=1}^{\infty} E_i = X$, $\mu(E_i) < \infty$, and $E_1 \subseteq E_2 \subseteq \dots$.

Who cares? Well if, X is $\sigma-$finite then for a **measurable** $f \geq 0$ a.e., then there exists **step** $\phi_n \uparrow f$ a.e.

4 Lebesgue Integral

4.1 Upper Functions (section 21)

$f : X \to \mathbb{R}$ is an **upper function** if there exist step ϕ_n such that

- $\phi_n \uparrow f$ a.e.
- $\lim \int \phi_n du < \infty$

ϕ_n is called a **generating sequence** for f.

* all step functions are upper functions
* f upper does **not** imply $-f$ is upper

The integral of f an **upper function** is defined as

$$\int f du = \lim \int \phi_n du$$

* the value is independent of our choice of ϕ_n because if any other $\psi_n \uparrow f$ too, then $\int \phi_n = \int \psi_n$ as $n \to \infty$

sums, scalar multiples, maxes of upper functions are upper functions.

If $f \geq g$ a.e. (both upper) then $\int f \geq g$ (same for g = 0)

If a **sequence of upper** functions $f_n \uparrow f$ a.e. and $\lim \int f_n < \infty$ then f is upper and $\int f = \lim \int f_n$ (similarly if $f_n \downarrow 0$)

4.2 Integrable Functions (section 22)

a function f is **integrable** if $f = u - v$, both upper functions.

We define $\int f$ as $\int u - \int v$

* well-defined no matter the representation of f

4.2.1 How does integrable relate to other properties?

- **upper** functions are integrable
- **step** functions are integrable (b/c step are upper)
- integrable implies **measurable**
  - measurable does **not** imply integrable
    e.g., constant functions are measurable, but only integrable when $\mu(X) < \infty$.

Canoncial way to write integrable

$$f = f^+ - f^-$$

b/c: both f^+ and f^- are upper if f is integrable

4.2.2 When is f integrable?

If integrable $f = g$ a.e., then g is integrable (and integrals are equal).

sums, scalar multiples, max, || of integrable are integrable.

* $|f|$ integrable does **not** imply f is integrable.

If f is measurable and $h \le f \le g$ a.e. for h, g integrable, then f is **integrable**. "measurable sandwiched between integrable is integrable"

nice properties of f integrable:

- if $f \ge 0$ a.e. then f is **upper**
- $A = \{x | |f(x)| \ge \epsilon\}$ has **finite measure** (A is also measurable)
  b/c: $|f|$ is measurable so $|f|^{-1}(\epsilon, \infty)$

For f, g integrable,

1. $\int |f| = 0 \iff f = 0$ a.e.
2. If $f \ge g$ a.e., then $\int f \ge \int g$
3. $\int |f| \ge \left| \int f \right|$

If E is **measurable**, f is **integrable**, then

$$\int_X f = \int_E f + \int_{X-E} f$$

4.2.3 Big: Levi, Fatou, and Lebesgue Dominated Convergence

Levi's Theorem

For f_n a sequence of **integrable** functions such that $f_n \le f_{n+1}$ a.e. for all n and $\lim \int f_n < \infty$, then there exists f integrable such that $f_n \uparrow f$ a.e.
(and $\lim \int f_n = \int f$)
"an integrable function waits at the top of an increasing sequence"

* f is defined a.e. on X

nice consequence: If integrable $f_n > 0$ a.e., with $\sum_{n=1}^{\infty} \int f_n < \infty$, then $\sum f_n$ defined an integrable function and

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

*not true in Riemann land!
* trick: when $f_1 \le f_2 \le \dots$ can make a positive sequence by considering $f_1 - f_1, f_2 - f_1, \dots$

Fatou's Lemma

For integrable $f_n \ge 0$ a.e. for all n and $\liminf \int f_n < \infty$, then

$$\int \liminf f_n \le \liminf \int f_n$$

where $\liminf f_n$ defines an integrable function a.e. on X.
"lim inf of integrable is integrable and less than integral of parts"

Lebesgue Dominated Convergence

1. **measurable** $f_n \to f$ a.e.
2. $|f_n| \le g$ a.e. for g **integrable**

then

$$\int f = \lim_{n\to\infty} \int f_n$$

where f_n and f are integrable (for all n)
"interchange lim and $\int$ for measurable functions bounded by an integrable function"

4.3 Riemann Integrals (section 23)

a partitions P is just a collection of points inside an interval.
A second partition Q **refines** P if $P \subseteq Q$.

The **Upper** Riemann sum is

$$U(f,P)=\sum_i^n M_i(x_i-x_{i-1})$$

where M_i is the sup of f on $[x_{i-1},x_i]$. (similarly lower sum is defined with m_i, the inf on the interval)

If Q refines P, then $U(f,Q) \leq U(f,P)$.
$(L(f,Q) \geq L(f,P))$.

f is **Riemann integrable** if

$$\lim_{||P_i||\to 0} U(f,P_i) = \lim_{||P_i||\to 0} L(f,P_i) = \int f(x)dx$$

where $||P_i||$ the length of the largest subinterval.

* The Riemann integral of f can also be defined when $\sup L = \inf S$; this value is said to be the integral of f.

* **Riemann's Critereon** f is integrable if if L and U can be made arbitrarily close by selecting a sufficiently fine partition.

Every **Riemann** integrable function is **Lebesgue** integrable.

A **bounded function** $f:[a,b]\to\mathbb{R}$ is **Riemann** integrable

$$\iff$$

f is **continuous** a.e.

4.4 Product Measures and Iterated Integrals (section 26—only a sketch)

If S and T are semirings, then their cross-product: $S\times T$ is also a semiring.
(similarly, for measures μ and v)

a function $f: X\times Y$ is $\mu\times v$ integrable by computing cross-sections:

$$\int_{X\times Y} fd(\mu\times v) = \int_X\int_Y fd\mu dv$$

"Fubini" says the order of $\int_X\int_Y$ doesn't matter!

Tonelli's Theorem f is $\mu\times v$-measurable and $\int_X\int_Y |f|dvd\mu$ exists (or other order) then $\int\int f$ exists.

5 Function Spaces (Chapter 5)

5.1 norms on vector spaces (section 27)

A real valued function || || on a vector space V is a **norm** if
for all v in V,

1. $||v||>0$ and $||v||=0 \iff v=0$
2. $||\alpha v|| = |\alpha|||v||$, for all $\alpha\in\mathbb{R}$
3. $||v+w||\leq ||v||+||w||$ (triangle)

* a norm space $\implies$ a metric space (but not the converse)

Can show $|||v||-||w||\leq ||v-w|||$. by triangle

Examples of norms in different spaces:

- "Euclidean norm": $\sqrt{v_1^2+v_2^2+\dots}$
- "sup norm": $||f||_{sup} = sup|f(x)|$ over all x.
  (only valid in space of bounded, real-valued functions)
- "L^p" norm: $||f||_p = \left(\int |f|^p\right)^{1/p}$
  only valid in $L^P(X)$ space = $\{f|f$ is measurable and $|f|^p$ is integrable$\}$
  * note on $\mathbb{R}^n$, the L^p norm is: $||(a_1,\dots,a_n)||_p = (|a_1|^p+\cdots+|a_n|^p)^{1/p}$

A **bounded normed** space is one where $||v|| \leq M$ for some constant M.

A normed space (a vector space with a norm) is a **Banach** space if every Cauchy sequence converges (aka **complete**).

Two norms are equivalent if there are $K, M > 0$: $K||x||_1 \leq ||x||_2 \leq M||x||_1$ for all x.

*In a finite dimensional vector space, all norms are equivalent

5.2 Linear Operators (section 28)

A **Linear Operator** (or transformation) is a map T between two vector spaces V and W such that:

$$T(aV + bW) = aT(V) + bT(W)$$

The **Operator norm** of T, $||T||$ is

$$\sup\{||T(v)|| : ||v|| = 1\}$$

we say T is bounded if $||T||$ is finite.

What's is equivalent to T being **bounded**?

1. $||T(v)|| \leq M||v||$ for all v ($M \geq 0$)
2. T is continuous at zero
3. T is continous

5.3 Lp Spaces (section 31)

L^p is the collection functions f such that
1) f is measurable
2) $|f|^p$ is integrable

This collection, L^p forms a space.
We can define a norm on L^p $||||_P$ by

$$||f||_p = (\int |f|^p)^{1/p}$$

Proof of triangle inequality is called **Minkowski's Inequality** only holds for finite $p > 1$,

$$||f + g||_p \leq ||f||_p + ||g||_p$$

for $f, g \in L^p$.

Holder's Inequality says if $1/p + 1/q = 1$ (called "conjugate exponents") and $f \in L^p, g \in L^q$ then

$$\int |fg| \leq ||f||_p ||g||_q$$

Risz-Fischer L^p is complete (every Cauchy seq converges) for all $p \geq 1$ (with respect to L^p-norm)

5.3.1 Essentially Bounded Functions

If $|fg| \leq h$, some integrable function, then $fg \in L^1$.

a function f is **essentially bounded** if

$$|f(x)| \leq M$$

for almost all x.

The **essential supremum**, denoed by $||f||_\infty$ is

$$||f||_\infty = \inf M|\ |f(x)| \leq M \text{for almost all x}$$

A function $f : \mathbb{R} \to \mathbb{R}$ has **compact support** if the closure of f $(\{x|f(x) \neq 0\})$ is compact.

Any **continuous** function with **compact support** is in $L^p(\mathbb{R})$ ($p \geq 1$)

$$1/x^a \in L^p \iff ap < 1$$

In a finite measure space, $L^q \subseteq L^p$ if $1 \leq p \leq q$.

5.3.2 Dense Functions in Lp

* The collection of step functions is **dense** in L^p (for $1 < p < \infty$).

For μ a **regular Borel measure** on a Hausdorff locally compact topological space X, the collection of continuous functions with compact support is **dense** in L^p (for $1 \leq p < \infty$).

* Hausdorff space is one where two points can be separated by open sets.

* remember a regular borel measure requires additional requirements on compact and borel sets see: 2.5.2.

notation: $C_c(X)$ is the set of continuous real-valued functions on X with compact support

6 Hilbert Spaces (Chapter 6)

6.1 Inner Product Spaces

7 Questions

1. $f : X \to \mathbb{R}$ is Lebesgue integrable $\iff$ $|f|$ is integrable. Is this true? (can't pin in the book, no proof in my notes)
   * we know $|f|$ doesn't imply f is integrable, right?