

Curiosities

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Polynomials

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

set $S = 1 + r + \dots + r^n$, multiply by r , and solve

Cauchy Schwarz

$$\sum x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

look into proof

Complex: Birth of Sin, Cos

Define, for $z \in \mathbb{C}$: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \dots$$

We derive Euler's formula,

$$e^{ix} = \cos(x) + i \sin(x)$$

Set x above to $2a$, then

$$\begin{aligned} e^{i2a} &= (e^{ia})^2 \\ &= (\cos(a) + i \sin(a))^2 \\ &= \cos^2(a) + 2i \cos(a) \sin(a) - \sin^2(a) \end{aligned}$$

e^{i2a} also equals $\cos(2a) + i \sin(2a)$. So real parts/imaginary parts are equal,

$$\cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(2a) = 2 \cos(a) \sin(a)$$

Can prove $\sin(a+b)$ identity similarly.

Complex Numbers

for $a, b \in \mathbb{C}$,

$a * b$ = number with length $a*b$, angle $a + b$

Trig

Master Identities:

$$\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x)$$

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

*remember sin is odd ($\sin(-x) = -\sin(x)$); cos is even.

from these derive the rest.

Exercises

$$\begin{aligned} 1. \quad & \tan^2(x) \sin(x) = \tan^2(x) \\ & \tan^2(x) (\sin(x) - 1) = 0 \\ & \tan^2(x) = 0 \text{ or } \sin(x) = 1 \end{aligned}$$

$$\begin{aligned} 2. \quad & 2 \cos^2(x) + \sin(x) - 2 = 0 \\ & \text{use } \cos^2(x) = 1 - \sin^2(x). \end{aligned}$$

Log

$$\log(ab) = \log(a) + \log(b)$$

$$\log(a^m) = m \log(a)$$

why?

$$\begin{aligned} \log(a^m) &= \log(a \dots a), \\ &= \log(a) + \log(a) + \dots + \log(a), \\ &= m \log(a). \end{aligned}$$

Derivatives

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

proofs?

$$\frac{d}{dx} a^x = a^x \ln(a)$$

$$\int \frac{1}{x} = \ln(|x|) + c$$

Integration Tricks

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

is nontrivial.

Idea is to consider $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy$ and use polar coordinates to integrate.
(see machine learning hw1)

For **Integrating a Polynomial over a sphere** see trick in paper by Folland.

Min and Max in \mathbb{R}

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}$$

$(a+b)/2$ takes you to midpoint. $|a-b|/2$ adds the remaining half to the larger number

$$\min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}$$

similar idea, except goes down by the half

Geometry

area of an equilateral triangle

$A = \sqrt{3}/4 s^2$ derive by bisecting angle, using 30, 60, 90 triangle

Number of diagonals in an n-sided figures

$$\frac{n(n-3)}{2}$$