Quantum Field Theory on a Highly Symmetric Lattice

Marco Aliberti

Università degli Studi di Torino

July 12, 2023

Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

$$\langle f | i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\left[\phi\right] e^{-S[\phi]}$$

need to be evaluated.

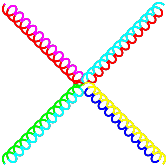
Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

$$\langle f|i\rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\left[\phi\right] e^{-S[\phi]}$$

need to be evaluated. There are two possible approaches:

Perturbative



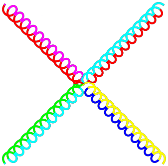
Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

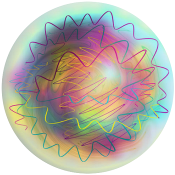
$$\langle f | i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}[\phi] \, e^{-S[\phi]}$$

need to be evaluated. There are two possible approaches:

Perturbative

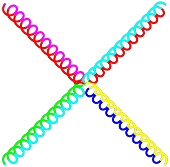


Non-Perturbative

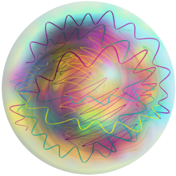


Perturbative vs Non-Perturbative

Perturbative



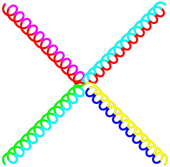
Non-Perturbative



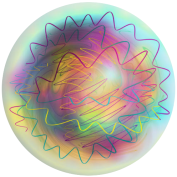
- Straightforward series expansion in powers of small $g \Leftrightarrow$ Feynman diagrams with n loops

Perturbative vs Non-Perturbative

Perturbative

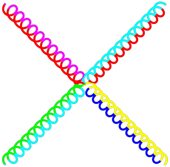


Non-Perturbative

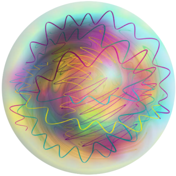


- Straightforward series expansion in powers of small g $\Leftrightarrow$ Feynman diagrams with n loops
- UV divergencies need to be eliminated

Perturbative



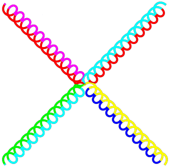
Non-Perturbative



- Straightforward series expansion in powers of small $g \Leftrightarrow$ Feynman diagrams with n loops
- UV divergencies need to be eliminated
- Fails predicting quantities with essential singularities as $g \to 0$

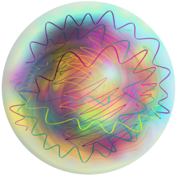
Perturbative vs Non-Perturbative

Perturbative



- Straightforward series expansion in powers of small g $\Leftrightarrow$ Feynman diagrams with n loops
- UV divergencies need to be eliminated
- Fails predicting quantities with essential singularities as $g \to 0$

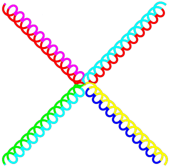
Non-Perturbative



- No straightforward approach

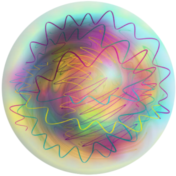
Perturbative vs Non-Perturbative

Perturbative



- Straightforward series expansion in powers of small g $\Leftrightarrow$ Feynman diagrams with n loops
- UV divergencies need to be eliminated
- Fails predicting quantities with essential singularities as $g \to 0$

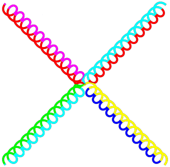
Non-Perturbative



- No straightforward approach
- Can have a natural cut-off for high momenta $\Rightarrow$ No UV divergencies

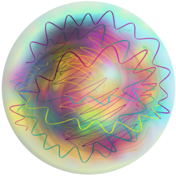
Perturbative vs Non-Perturbative

Perturbative



- Straightforward series expansion in powers of small g $\Leftrightarrow$ Feynman diagrams with n loops
- UV divergencies need to be eliminated
- Fails predicting quantities with essential singularities as $g \to 0$

Non-Perturbative



- No straightforward approach
- Can have a natural cut-off for high momenta $\Rightarrow$ No UV divergencies
- Can predict quantities with essential singularities as $g \to 0$

What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

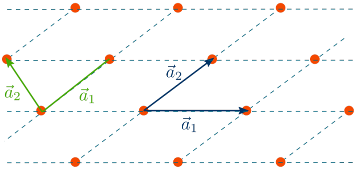


Figure: A bidimensional lattice.

What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

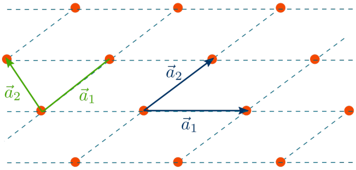


Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$
a is called *lattice spacing*.

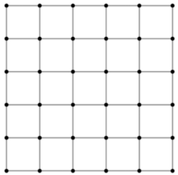


Figure: A square lattice.

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

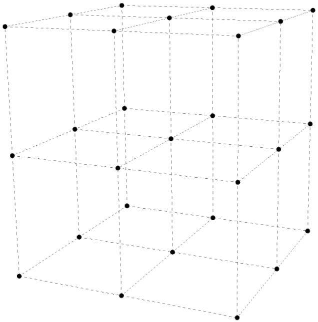


Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

Examples:

- Scalar fields $\Phi(x) \to \Phi(n)$ on sites

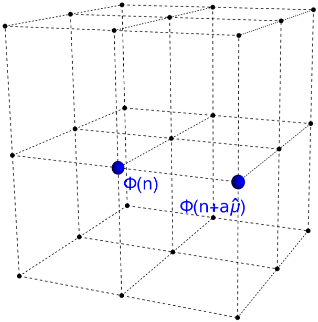


Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

Examples:

- Scalar fields $\Phi(x) \to \Phi(n)$ on sites
- Vector fields $U_\mu(x) \to U_\mu(n)$ on links

Parallel Transporter

$$U_\mu(x) = \exp(igaA_\mu(x))$$

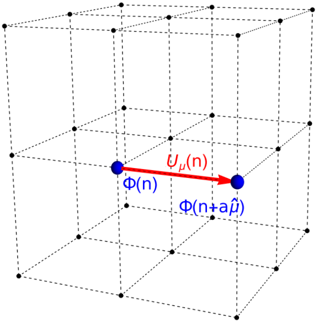


Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

Examples:

- Scalar fields $\Phi(x) \to \Phi(n)$ on sites
- Vector fields $U_\mu(x) \to U_\mu(n)$ on links
- Object with k indices on k-symplexes

Parallel Transporter

$$U_\mu(x) = \exp(igaA_\mu(x))$$

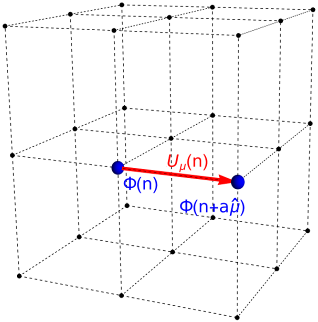


Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

Examples:

- Scalar fields $\Phi(x) \to \Phi(n)$ on sites
- Vector fields $U_\mu(x) \to U_\mu(n)$ on links
- Object with k indices on k-symplexes

Beware!

Spinorial fields are trickier to be discretized.

Parallel Transporter

$$U_\mu(x) = \exp(igaA_\mu(x))$$

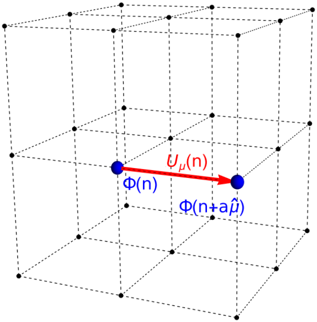


Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is $S_E = \frac{1}{4}\int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

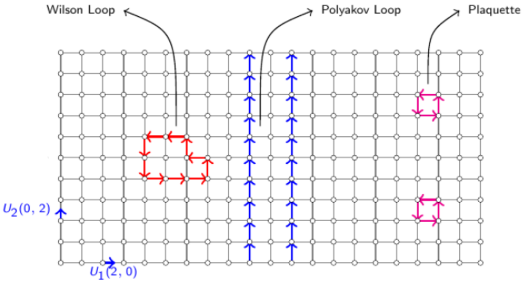


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is $S_E = \frac{1}{4}\int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

Definition: Plaquette $U_{\mu\nu}(n)$

$U_\mu(n) U_\nu(n+\mu) U^\dagger_\mu(n+\nu) U^\dagger_\nu(n)$

On the lattice, every closed path is gauge-invariant.

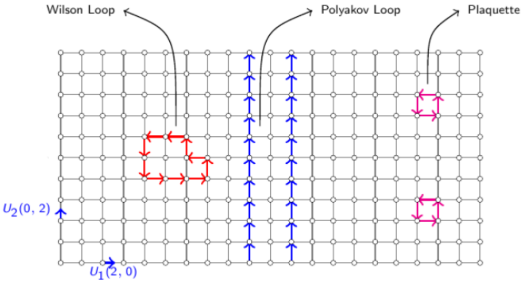


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is $S_E = \frac{1}{4}\int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

Definition: Plaquette $U_{\mu\nu}(n)$

$$U_\mu(n)U_\nu(n+\mu)U^\dagger_\mu(n+\nu)U^\dagger_\nu(n)$$

Wilson's Idea

$$S = \frac{\beta}{2N}\sum_{n,\mu,\nu} \mathfrak{Re}\,\mathrm{Tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right)$$

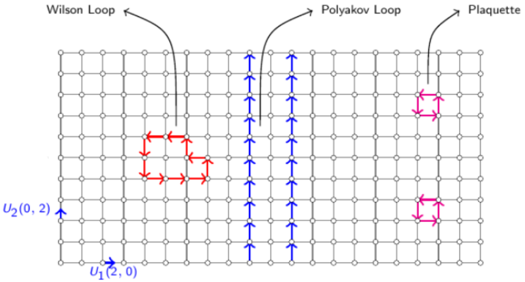


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

Polyakov Loop

$$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$$

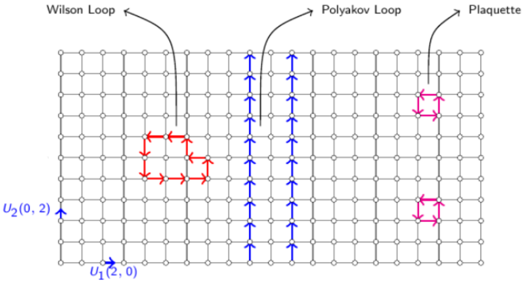


Figure: Gauge-invariant paths on a bidimensional lattice. [1]

Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

The expectation value of two Polyakov loops is the potential.

Polyakov Loop

$$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$$

Potential

$$V(R) = -\frac{1}{T} \log < P(0)P^{\dagger}(R) >$$

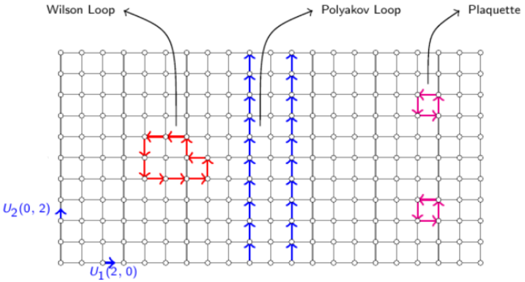


Figure: Gauge-invariant paths on a bidimensional lattice. [1]

Lattice Symmetries

Poincaré Group can be divided in:

Translations	Rotations

Lattice Symmetries

Poincaré Group can be divided in:

Translations

$$x^\mu \rightarrow x^\mu + \varepsilon^\mu$$

$$\Downarrow$$

$$n \rightarrow n + a\hat{\mu}$$

Rotations

Lattice Symmetries

Poincaré Group can be divided in:

Translations

$$x^\mu \to x^\mu + \varepsilon^\mu$$
$$\Downarrow$$
$$n \to n + a\hat{\mu}$$

Rotations

$$x^\mu \to R^\mu_\nu x^\nu \quad R \in SO(4)$$
$$\Downarrow$$
$$n \to \Gamma n \quad \Gamma \in T$$

T: group of rotations of multiples of 90° around any axis.

Lattice Symmetries

Poincaré Group can be divided in:

Translations

$$x^\mu \to x^\mu + \varepsilon^\mu$$

$$\Downarrow$$

$$n \to n + a\hat{\mu}$$

$a\hat{\mu} \to \varepsilon^\mu$ for $a \to 0$

Rotations

$$x^\mu \to R^\mu_\nu x^\nu \quad R \in SO(4)$$

$$\Downarrow$$

$$n \to \Gamma n \quad \Gamma \in T$$

T: group of rotations of multiples of 90° around any axis.

Lattice Symmetries

Poincaré Group can be divided in:

Translations

$$x^\mu \to x^\mu + \varepsilon^\mu$$

$$\Downarrow$$

$$n \to n + a\hat{\mu}$$

$a\hat{\mu} \to \varepsilon^\mu$ for $a \to 0$

Rotations

$$x^\mu \to R^\mu_\nu x^\nu \quad R \in SO(4)$$

$$\Downarrow$$

$$n \to \Gamma n \quad \Gamma \in T$$

T: group of rotations of multiples of 90° around any axis.

$\Gamma \not\to R$ for $a \to 0$

Lattice Symmetries

Poincaré Group can be divided in:

Translations

$$x^\mu \to x^\mu + \varepsilon^\mu$$

$$\Downarrow$$

$$n \to n + a\hat{\mu}$$

Rotations

$$x^\mu \to R^\mu_\nu x^\nu \quad R \in SO(4)$$

$$\Downarrow$$

$$n \to \Gamma n \quad \Gamma \in T$$

T: group of rotations of multiples of 90° around any axis.

$a\hat{\mu} \to \varepsilon^\mu$ for $a \to 0$

$\Gamma \not\to R$ for $a \to 0$

Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached.

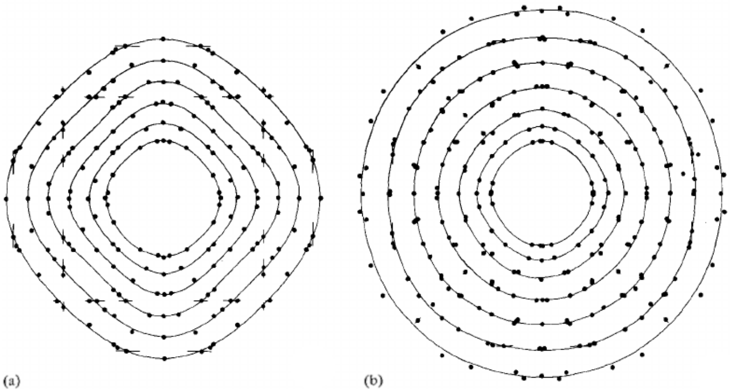


Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves. [2]

Values of β are slightly different from Lang and Rebbi's because $a(\beta) \approx \Lambda e^{-b_0 \beta}$, with Λ, $b_0 > 0$[1].

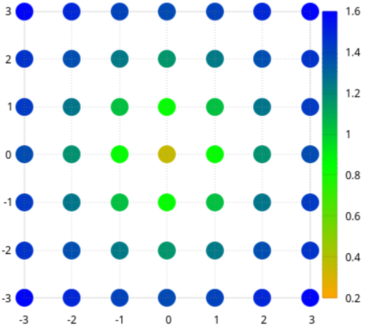


Figure: Potential from $\beta = 2.20$, $n_s = 8$, $n_t = 4$.

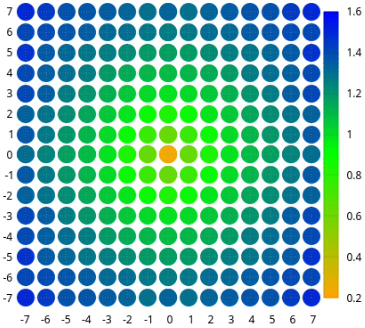


Figure: Potential from $\beta = 2.35$, $n_s = 16$, $n_t = 6$.

[1] The simulation code is based on the code presented in refs. [3, 4].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract [5]

- 24 nearest neighbours
- 1152-element symmetry group

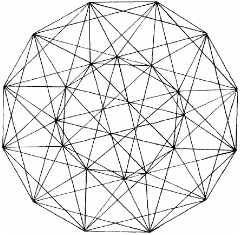


Figure: Two-dimensional projection of a BCT. [5]

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract [5]

- 24 nearest neighbours
- 1152-element symmetry group

F_4 coroots lattice [6]

- 48 nearest neighbours
- 2304-element symmetry group

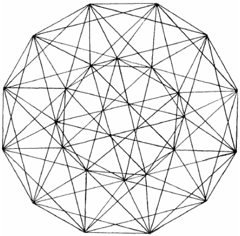


Figure: Two-dimensional projection of a BCT. [5]

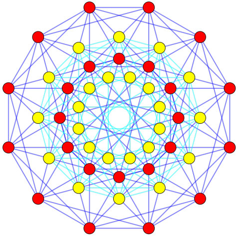


Figure: Two-dimensional projection of a F_4 coroots lattice. [7]

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;

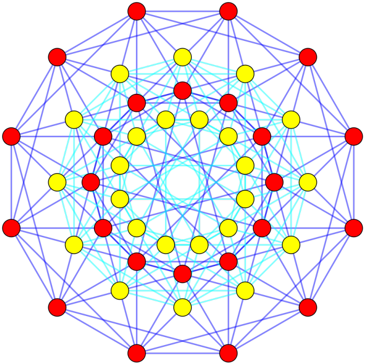


Figure:

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm1, \pm1, 0, 0)$

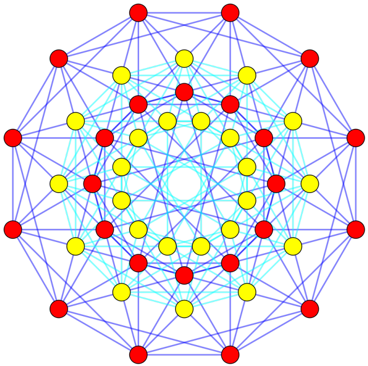


Figure: The 24 roots (red) of the F_4 lattice, projected on a bidimensional plane.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm 1, \pm 1, 0, 0)$
  - The 24 dual roots (coroots) are:
    - The 8 possible permutations of $(\pm 1, 0, 0, 0)$
    - The 16 possible permutations of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$

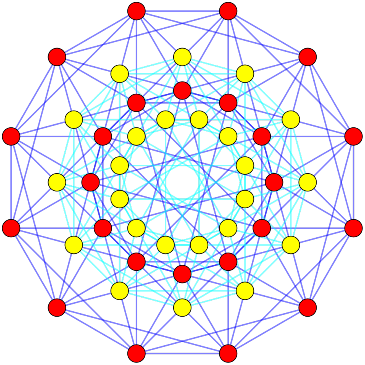


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm 1, \pm 1, 0, 0)$
  - The 24 dual roots (coroots) are:
    - The 8 possible permutations of $(\pm 1, 0, 0, 0)$
    - The 16 possible permutations of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$
- Exists only in 4 dimensions;

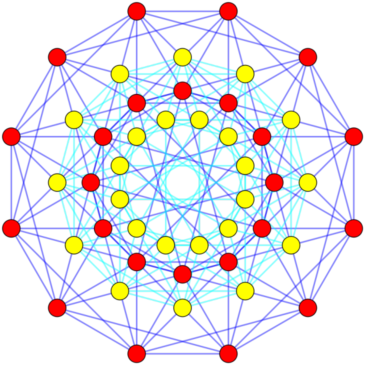


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm 1, \pm 1, 0, 0)$
  - The 24 dual roots (coroots) are:
    - The 8 possible permutations of $(\pm 1, 0, 0, 0)$
    - The 16 possible permutations of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$
- Exists only in 4 dimensions;
- Contains the Simple Hypercubic lattice and the BCT;

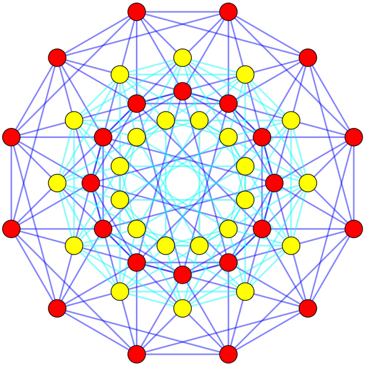


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm 1, \pm 1, 0, 0)$
  - The 24 dual roots (coroots) are:
    - The 8 possible permutations of $(\pm 1, 0, 0, 0)$
    - The 16 possible permutations of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$
- Exists only in 4 dimensions;
- Contains the Simple Hypercubic lattice and the BCT;
- Has been used only to simulate scalar fields, in [6].

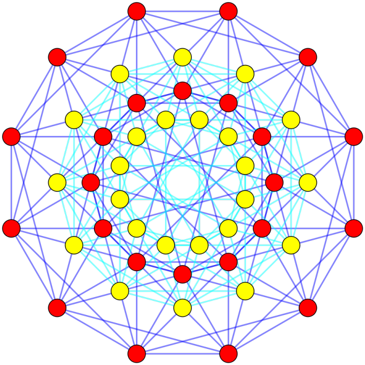
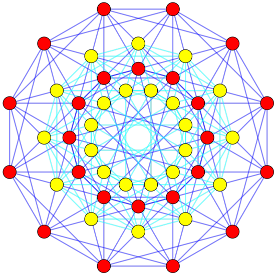


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

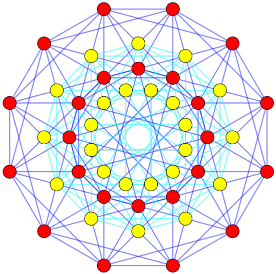
Work in Progress

- Implement the F_4 lattice in the simulation program and make efficiency studies;

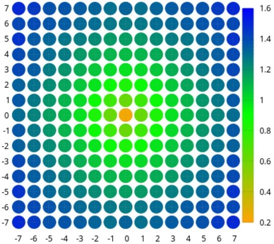


Work in Progress

- Implement the F_4 lattice in the simulation program and make efficiency studies;



- Make a rotational invariance study on the new lattice, hoping to get better results than the Simple Hypercubic lattice.



Thank you for your attention

Bibliography I

[1] Dibakar Sigdel. “Two Dimensional Lattice Gauge Theory with and without Fermion Content”. In: *FIU Electronic Theses and Dissertations* 3224 (2016). DOI: 10.25148/etd.FIDC001748. URL: https://digitalcommons.fiu.edu/etd/3224?utm_source=digitalcommons.fiu.edu%2Fetd%2F3224&utm_medium=PDF&utm_campaign=PDFCoverPages.

[2] C. B. Lang and C. Rebbi. “Potential and Restoration of Rotational Symmetry in SU(2) Lattice Gauge Theory”. In: *Phys. Lett.* B115 (1982). [, 322 (1982)], p. 137. DOI: 10.1016/0370-2693(82)90813-9.

[3] Marco Panero. “Thermodynamics of the QCD plasma and the large-N limit”. In: *Phys. Rev. Lett.* 103 (2009), p. 232001. DOI: 10.1103/PhysRevLett.103.232001. arXiv: 0907.3719 [hep-lat].

Bibliography II

[4] Anne Mykkänen, Marco Panero, and Kari Rummukainen. "Casimir scaling and renormalization of Polyakov loops in large-N gauge theories". In: *JHEP* 1205 (2012), p. 069. DOI: 10.1007/JHEP05(2012)069. arXiv: 1202.2762 [hep-lat].

[5] William Celmaster. "Gauge Theories on the Body - Centered Hypercubic Lattice". In: *Phys. Rev.* D26 (1982), p. 2955. DOI: 10.1103/PhysRevD.26.2955.

[6] Herbert Neuberger. "SPINLESS FIELDS ON F(4) LATTICES". In: *Phys. Lett. B* 199 (1987), pp. 536–540. DOI: 10.1016/0370-2693(87)91623-6.

[7] URL: https://en.wikipedia.org/wiki/F4_%28mathematics%29.