Quantum Field Theory on a Highly Symmetric Lattice

Marco Aliberti

Università degli Studi di Torino

23rd October, 2023

The Strong Interaction

Matter is made of Atoms

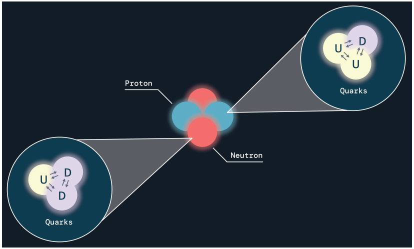

Image credits: NASA[NASA, 2022]

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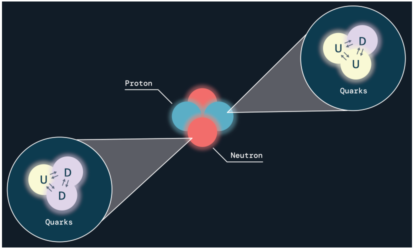

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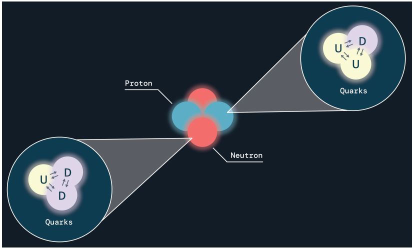

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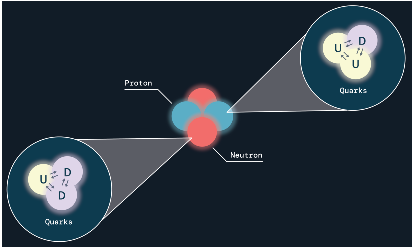

Quarks and Gluons

Described by Quantum Chromodynamics (QCD)

Image credits: NASA[NASA, 2022]

Quantum Chromodynamics (QCD)

Described by an $SU(3)$ Yang-Mills theory

$$S = \frac{1}{4}\int \mathrm{d}^4 x F^a_{\mu\nu}(x) F^{a\mu\nu}(x)$$

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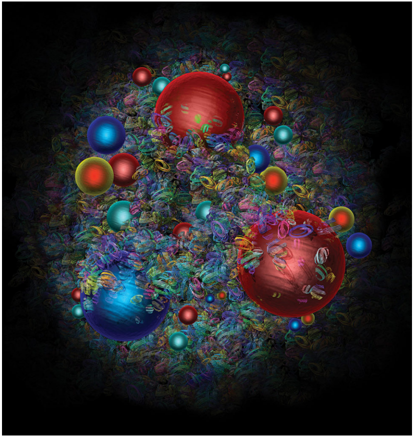

Figure: An artist's representation of a proton[CERN, 2019].

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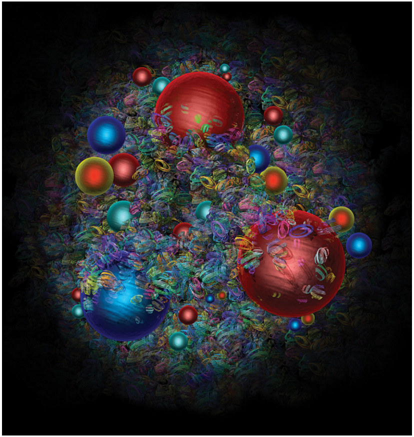

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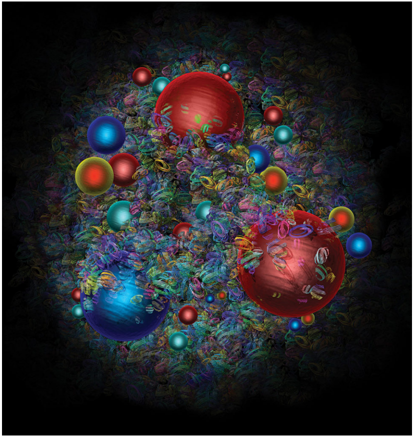
- Interesting purely-gluonic physics



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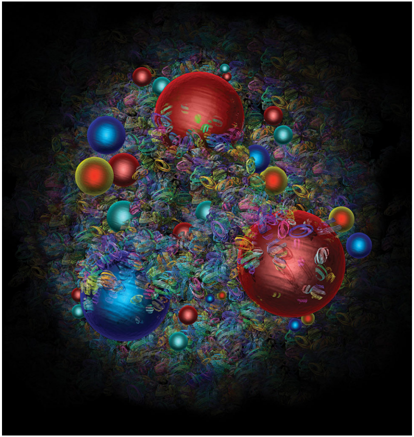
- Interesting purely-gluonic physics
- Heavily non-perturbative nature (except for specific regimes)



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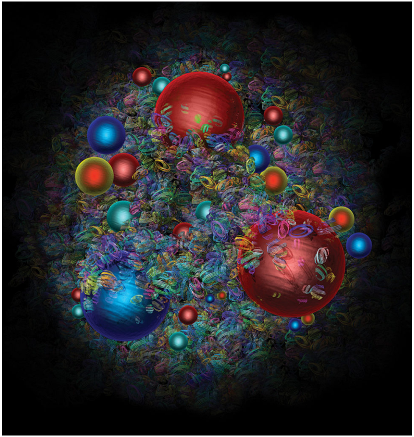
$$\Downarrow$$

Lattice Field Theory



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What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

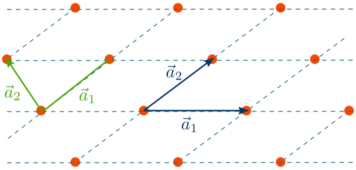

Figure: A bidimensional lattice.

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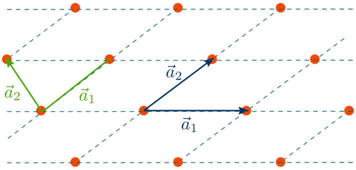

Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$
a is called *lattice spacing*.

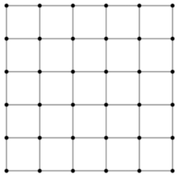

Figure: A square lattice.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

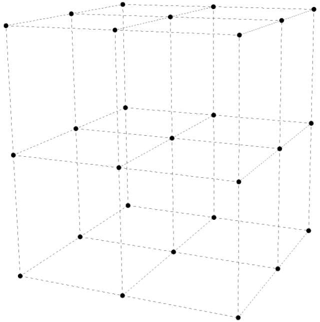

Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

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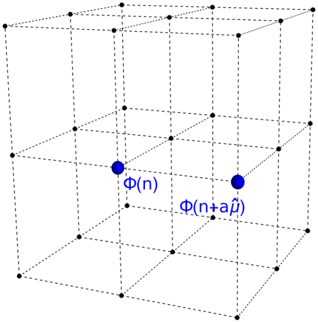
Examples:

- Scalar fields $\Phi(x) \to \Phi(n)$ on sites



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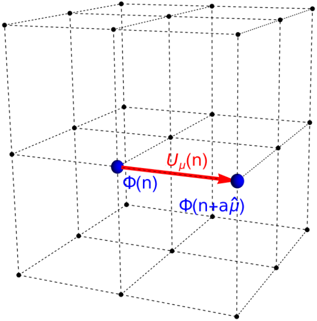
Parallel Transporter

$U_\mu(x) = \exp(igaA_\mu(x))$



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- Object with k indices on k-symplexes

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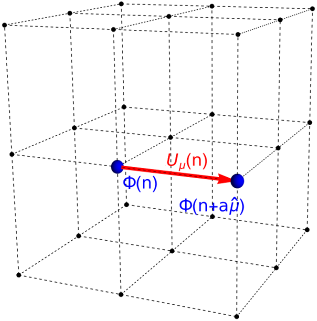

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Beware!

Spinorial fields are trickier to be discretized.

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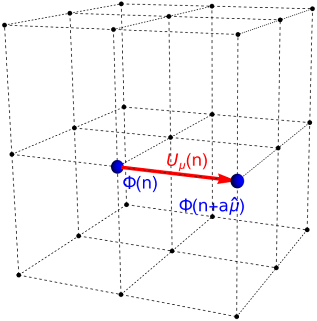

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The Yang-Mills continuum action is
$S_E = \frac{1}{4}\int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

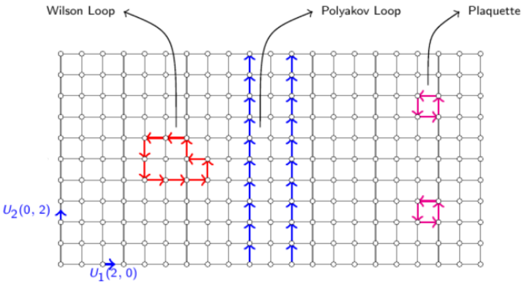

Figure: Gauge-invariant paths on a bidimensional lattice[Sigdel, 2016].

Gauge-Invariant Observables and Wilson Action

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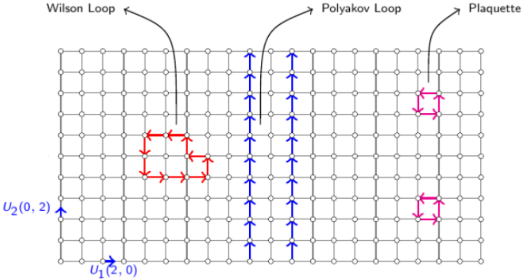
$$U_\mu(n) U_\nu(n+\mu) U^\dagger_\mu(n+\nu) U^\dagger_\nu(n)$$



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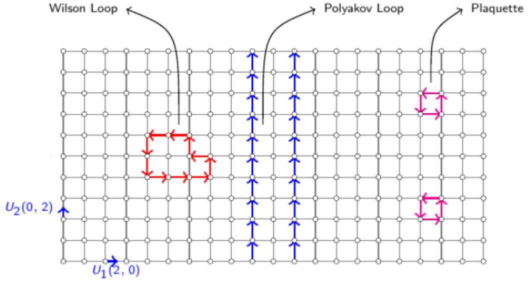

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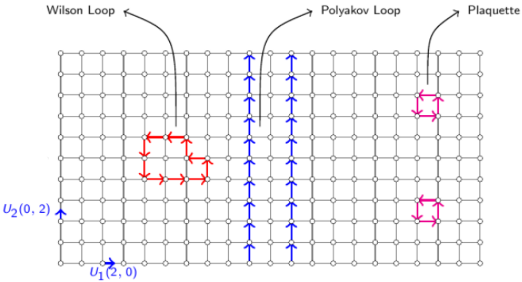
Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

Polyakov Loop

$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$



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Polyakov Loops and Potential

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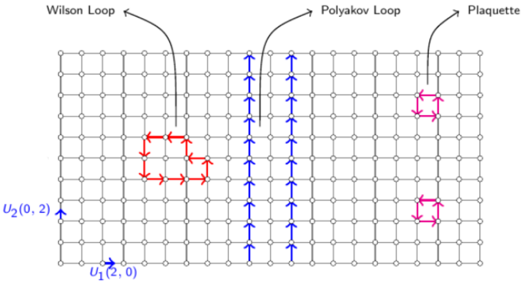
The expectation value of two Polyakov loops is the potential.

Polyakov Loop

$$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$$

Potential

$$V(R) = -\frac{1}{T} \log < P(0) P^{\dagger}(R) >$$



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Monte Carlo Simulations

Computers are used to simulate Lattice Field Theories

Figure: A rendering of the CINECA Leonardo supercomputer[Wikipedia, 2022].

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Monte Carlo Simulations

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- A great number of observables is evaluated and then their mean value is computed.

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Lattice Symmetries

Poincaré Group can be divided in:

Translations

Rotations

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$$\Downarrow$$

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$$x^\mu \to R^\mu_\nu x^\nu \quad R \in SO(4)$$
$$\Downarrow$$
$$n \to \Gamma n \quad \Gamma \in G_{\Lambda_{SH}}$$

$G_{\Lambda_{SH}}$: group of rotations of multiples of 90° around any axis.

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Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached[Lang and Rebbi, 1982].
The gauge group used was the discrete icosahedral subgroup $\tilde{Y} \subset SU(2)$.

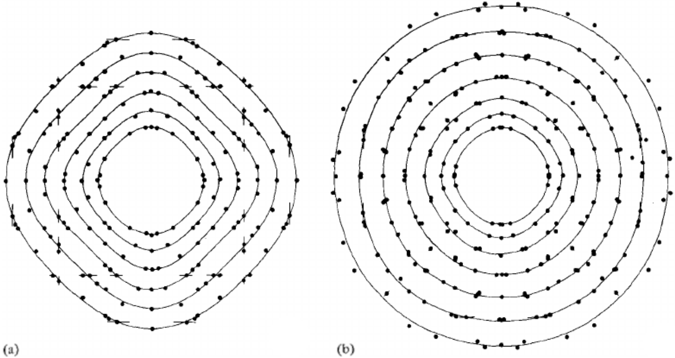

Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves.

Rotational Invariance Restoration

Results of simulations for gauge group $SU(2)$ with 20000 measurements each[1]. Approach slightly different than Lang and Rebbi's.

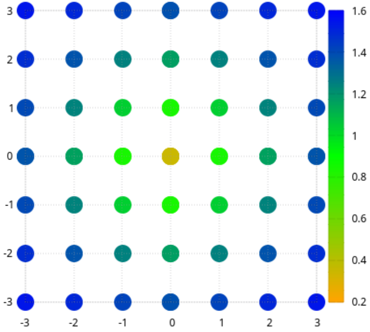

Figure: Potential from $\beta = 2.20$, $n_s = 8$, $n_t = 4$.

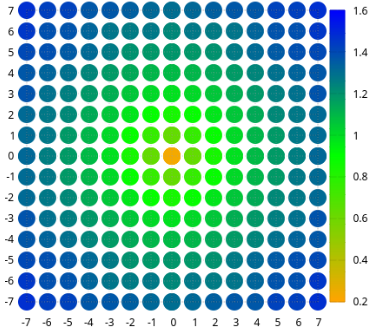

Figure: Potential from $\beta = 2.35$, $n_s = 16$, $n_t = 6$.

[1] The simulation code is based on the code presented in refs. [Panero, 2009; Mykkänen, Panero, and Rummukainen, 2012].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract

- 24 nearest neighbours
- 1152-element symmetry group

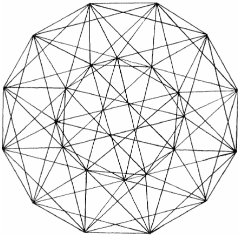

Figure: Two-dimensional projection of a BCT[Celmaster, 1982].

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

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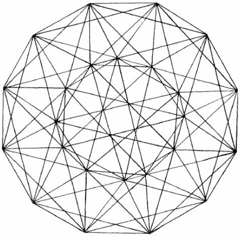
F_4 coroots lattice[Neuberger, 1987]

- 48 nearest neighbours
- 2304-element symmetry group



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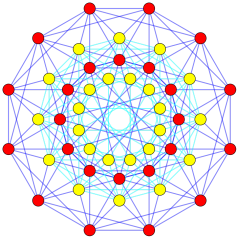

Figure: Two-dimensional projection of a F_4 coroots lattice[Wikipedia, 2010].

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Wilson Action:

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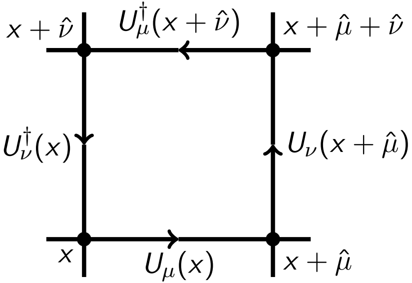

Simulations on SH Lattice

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Plaquette:

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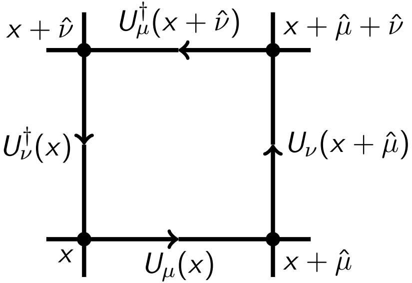
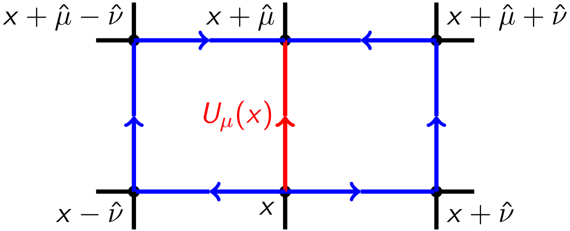


6 staples for each link



Simulations on BCT Lattice

Thank you for your attention

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