Quantum Field Theory on a Highly Symmetric Lattice

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Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

$$\langle f | i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\,[\phi]\, e^{-S[\phi]}$$
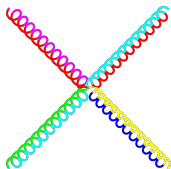
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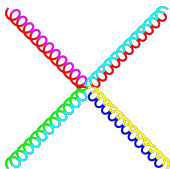need to be evaluated. There are two possible approaches:

Perturbative

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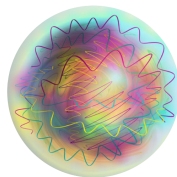In quantum field theory scattering amplitudes in the form

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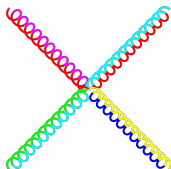need to be evaluated. There are two possible approaches:

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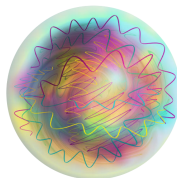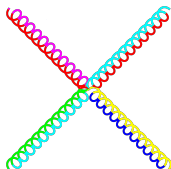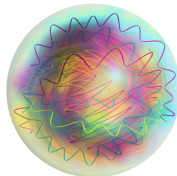- UV divergencies need to be eliminated

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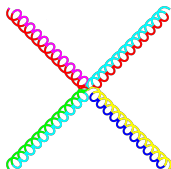# Perturbative vs Non-Perturbative
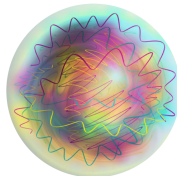
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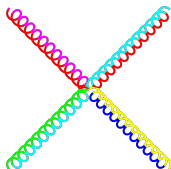
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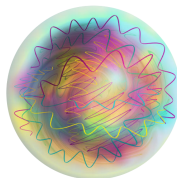- Fails predicting quantities with essential singularities as $g \to 0$

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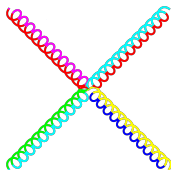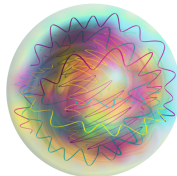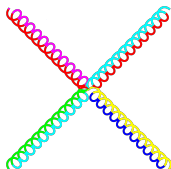- Can have a natural cut-off for high momenta $\Rightarrow$ No UV divergencies
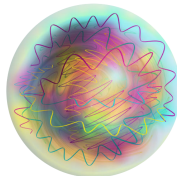- Can predict quantities with essential singularities as $g \to 0$

What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$
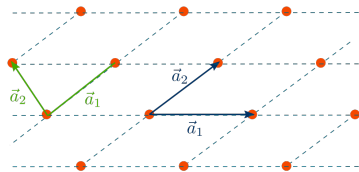
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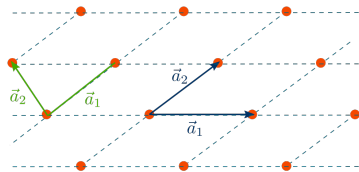
Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$
a is called *lattice spacing*.
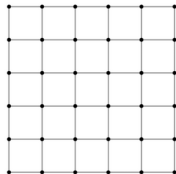
Figure: A square lattice.
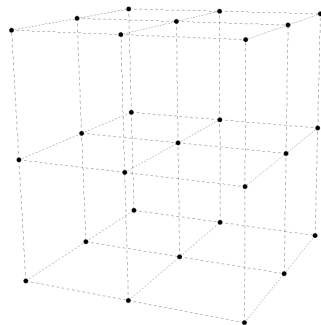
Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

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$$U_\mu(x) = \exp(iga A_\mu(x))$$
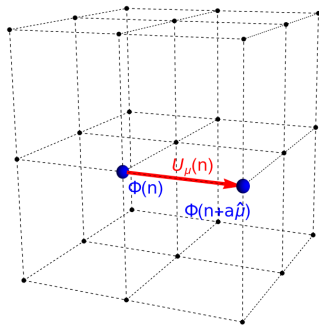
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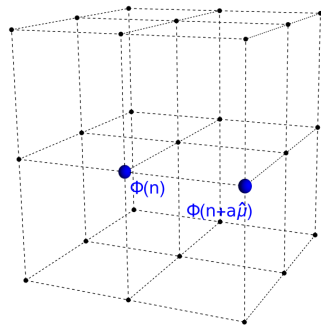
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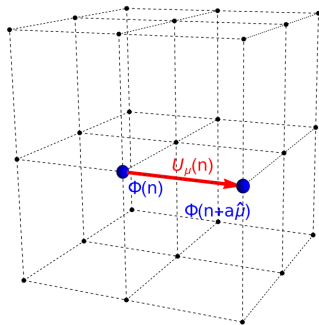Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

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Beware!

Spinorial fields are trickier to be discretized.

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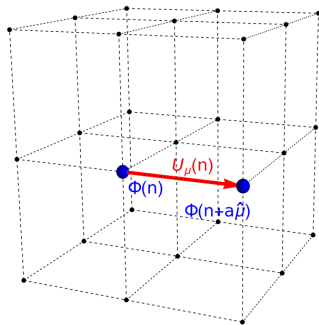
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Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is
$S_E = \frac{1}{4} \int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

Figure: Gauge-invariant paths on a bidimensional lattice.[1]

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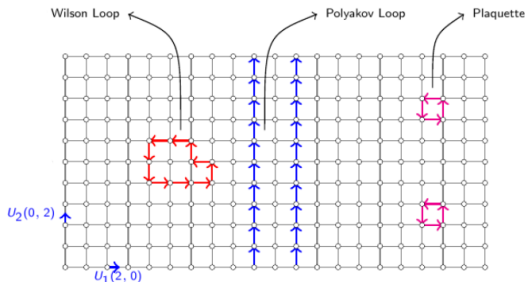$U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)$

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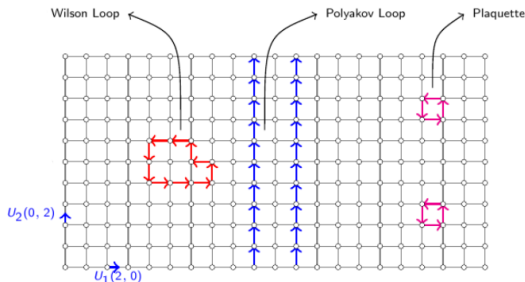## Wilson's Idea

$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \mathfrak{Re} \operatorname{Tr} \left(\mathbb{1} - U_{\mu\nu}(n) \right)$

Figure: Gauge-invariant paths on a bidimensional lattice.[1]

Polyakov Loops and Potential

If the time coordinate is taken to be
periodic, more closed paths arise.

Polyakov Loop

$$P(n) = \text{Tr} \prod_{t=0}^{T-1} U_t(n)$$

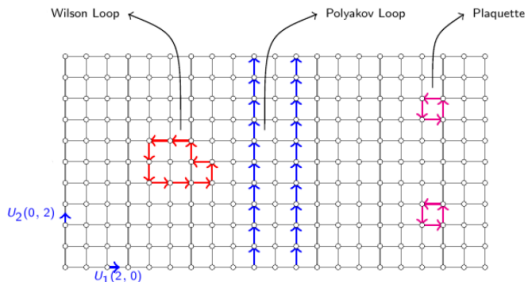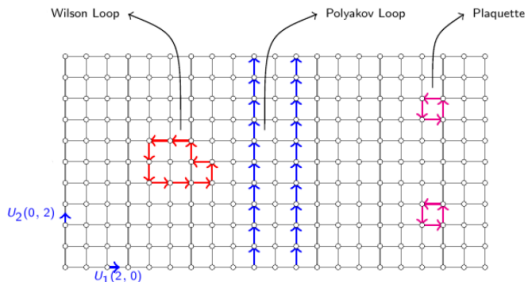Figure: Gauge-invariant paths on a bidimensional lattice. [1]

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The expectation value of two Polyakov loops is the potential.

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$V(R) = -\frac{1}{T} \log < P(0)P^\dagger(R) >$
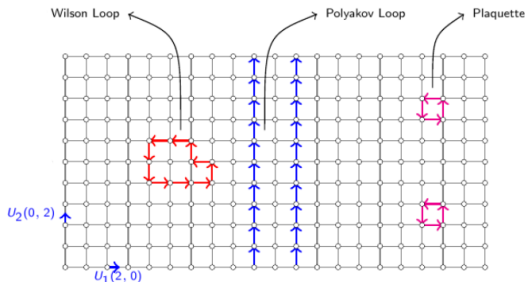
Figure: Gauge-invariant paths on a bidimensional lattice. [1]

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T: group of rotations of multiples of $90°$ around any axis.

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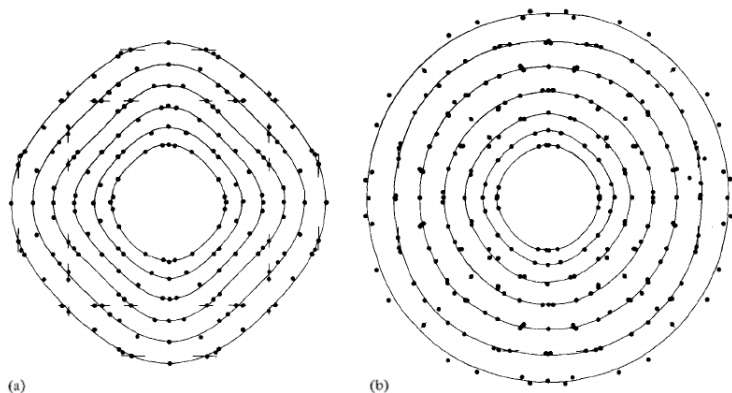T: group of rotations of multiples of $90°$ around any axis.

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Important:

Rotational invariance seems to be broken.

Equipotential surfaces become spheres as the continuum limit is approached.

Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves. [2]

Rotational Invariance Restoration

Values of β are slightly different from Lang and Rebbi's because $a(\beta) \approx \Lambda e^{-b_0 \beta}$, with Λ, $b_0 > 0$. [1]
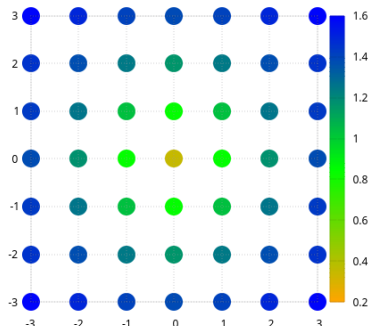

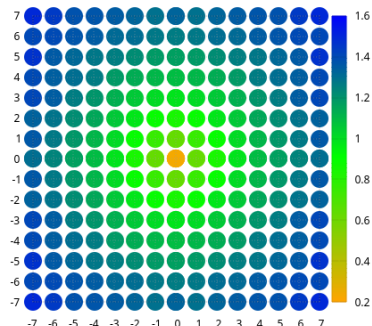Figure: Potential from $\beta = 2.20$, $n_s = 8$, $n_t = 4$.

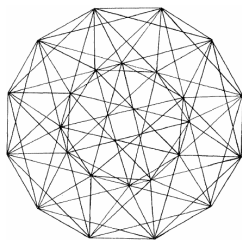Figure: Potential from $\beta = 2.35$, $n_s = 16$, $n_t = 6$.

[1] The simulation code is based on the code presented in refs. [3, 4].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract [5]

- 24 nearest neighbours
- 1152-element symmetry group
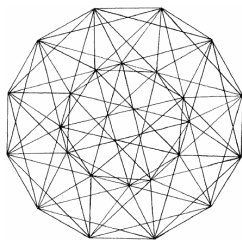

Figure: Two-dimensional projection of
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The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

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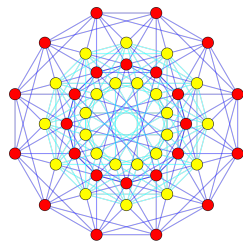
Figure: Two-dimensional projection of a F_4 coroots lattice. [7]

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