

Quantum Field Theory on a Highly Symmetric Lattice

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Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

$$\langle f|i\rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}[\phi] e^{-S[\phi]}$$

need to be evaluated.

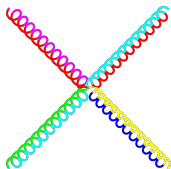
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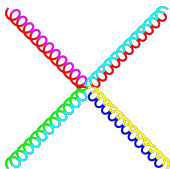
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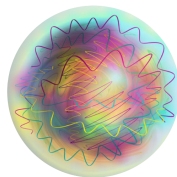
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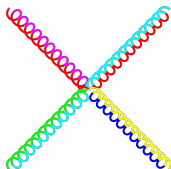


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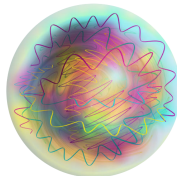


Perturbative vs Non-Perturbative

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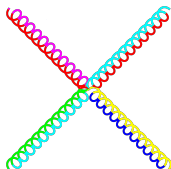
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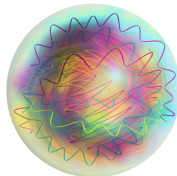
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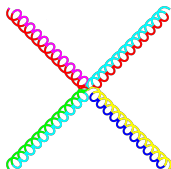
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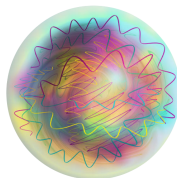
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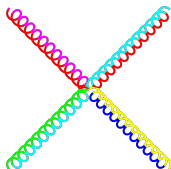
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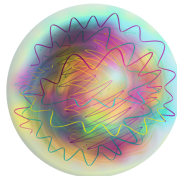
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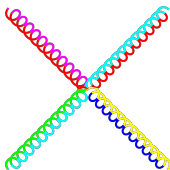
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- Straightforward series expansion in powers of small $g \Leftrightarrow$ Feynman diagrams with n loops
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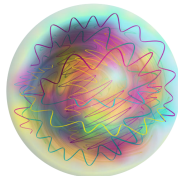
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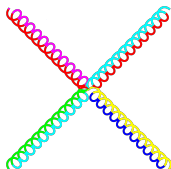
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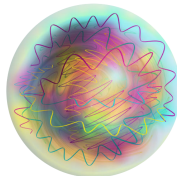
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What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{ \sum_{i=1}^n a_i e_i \mid a_i \in \mathbb{Z} \}$, with $\{e_i\}$ any basis of \mathbb{R}^n

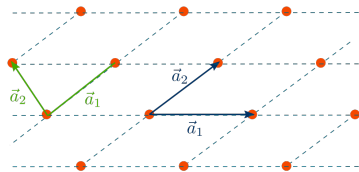


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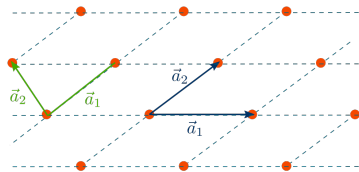


Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of \mathbb{R}^n
 a is called *lattice spacing*.

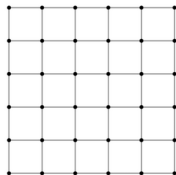


Figure: A square lattice.

Basic idea

Fields can take values only in given parts of the lattice, $x \rightarrow n \in \Lambda$.

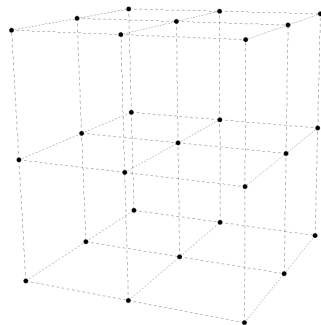


Figure: A (hyper)cubic lattice in \mathbb{R}^3 .

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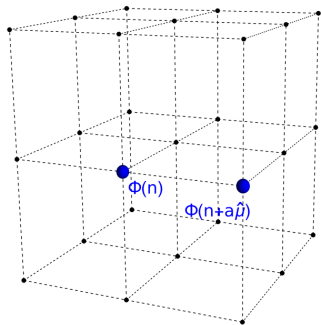


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$$U_\mu(x) = \exp(igaA_\mu(x))$$

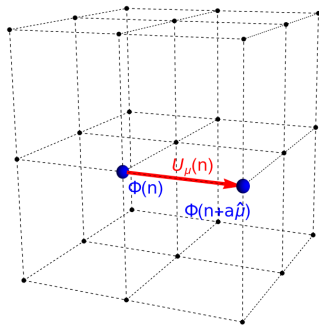


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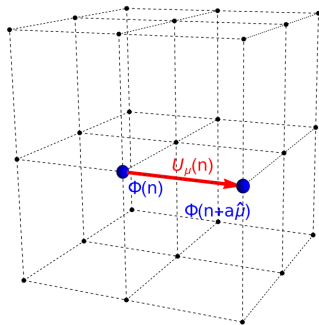


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Beware!

Spinorial fields are trickier to be discretized.

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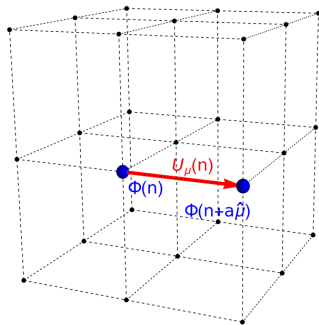


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Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is
$$S_E = \frac{1}{4} \int d^4x F^{a\mu\nu}(x) F_{\mu\nu}^a(x).$$

On the lattice, every closed path is gauge-invariant.

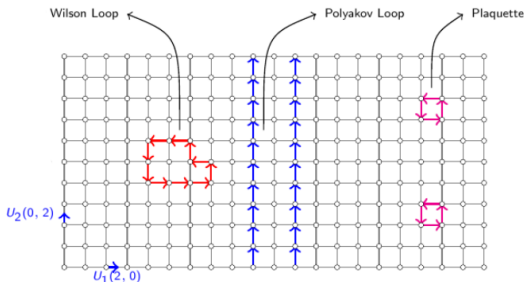


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

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$$U_\mu(n) U_\nu(n + \mu) U_\mu^\dagger(n + \nu) U_\nu^\dagger(n)$$

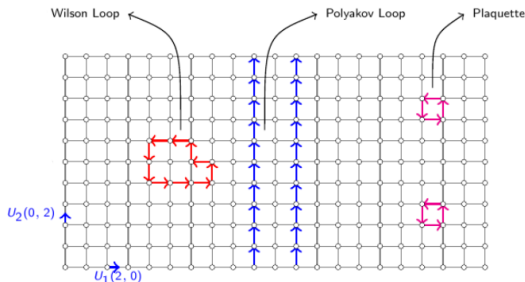


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Wilson's Idea

$$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \Re \text{Tr} (1 - U_{\mu\nu}(n))$$

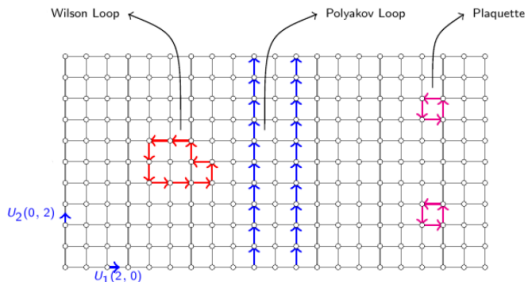


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

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$$P(n) = \text{Tr} \prod_{t=0}^{T-1} U_t(n)$$

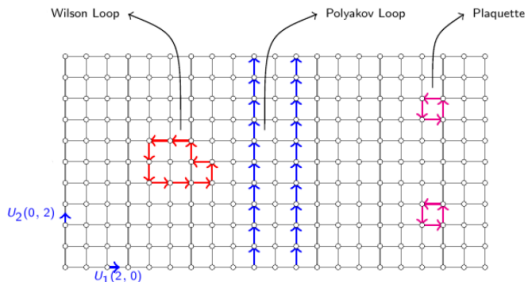


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The expectation value of two Polyakov loops is the potential.

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$$V(R) = -\frac{1}{T} \log \langle P(0) P^\dagger(R) \rangle$$

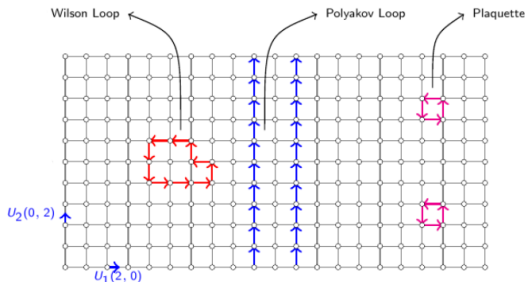


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$$x^\mu \rightarrow R^\mu_\nu x^\nu \quad R \in SO(4)$$

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$$n \rightarrow \Gamma n \quad \Gamma \in T$$

T : group of rotations of multiples of 90° around any axis.

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Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached.

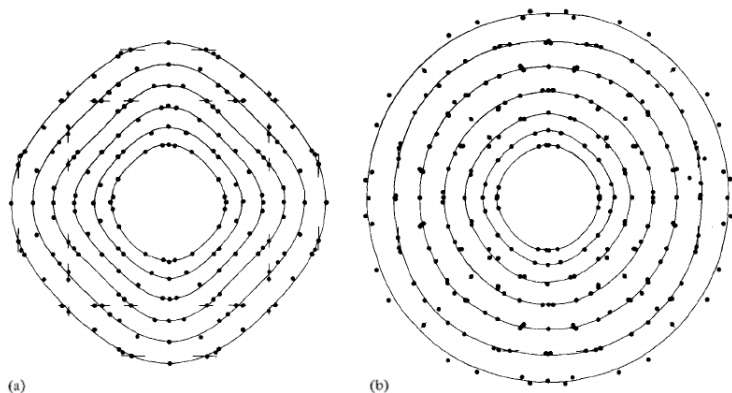


Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves. [2]

Rotational Invariance Restoration

Values of β are slightly different from Lang and Rebbi's because $a(\beta) \approx \Lambda e^{-b_0 \beta}$, with $\Lambda, b_0 > 0$ ¹.

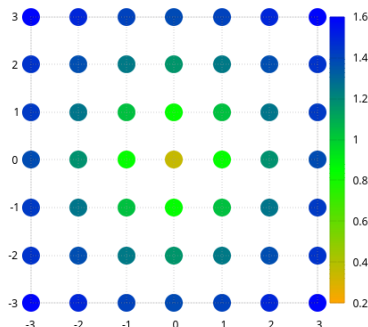


Figure: Potential from $\beta = 2.20$, $n_s = 8$, $n_t = 4$.

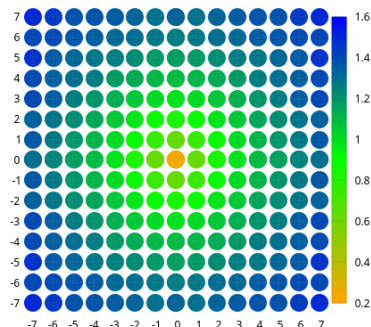


Figure: Potential from $\beta = 2.35$, $n_s = 16$, $n_t = 6$.

¹The simulation code is based on the code presented in refs. [3, 4].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract [5]

- 24 nearest neighbours
- 1152-element symmetry group

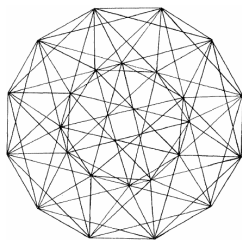


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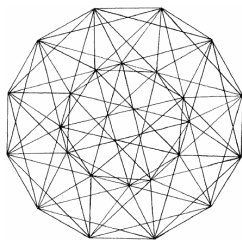


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F_4 coroots lattice [6]

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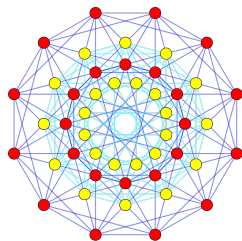


Figure: Two-dimensional projection of a F_4 coroots lattice. [7]

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F_4 Coroots Lattice

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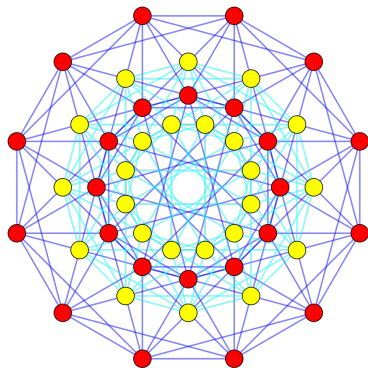


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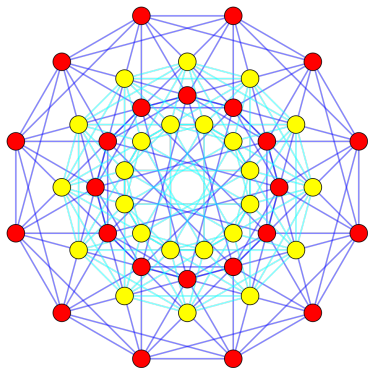


Figure: The 24 roots (red) of the F_4 lattice, projected on a bidimensional plane.

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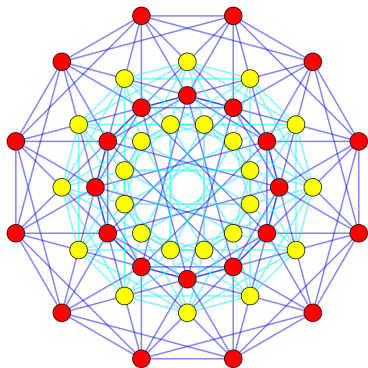


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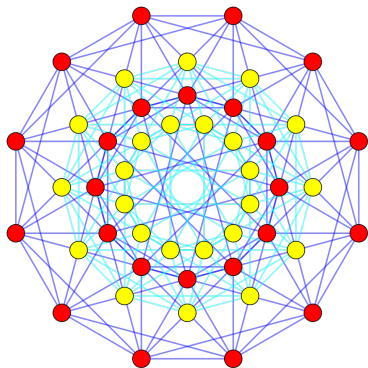


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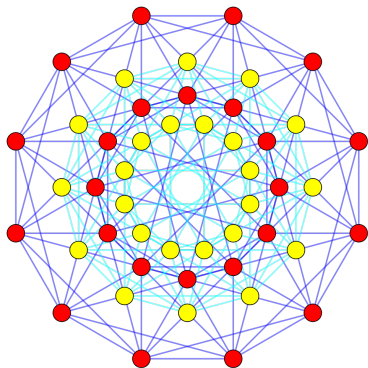


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- Contains the Simple Hypercubic lattice and the BCT;
- Has been used only to simulate scalar fields, in [6].

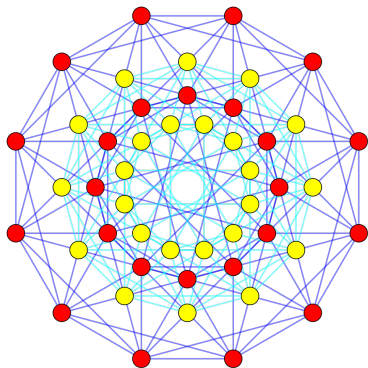
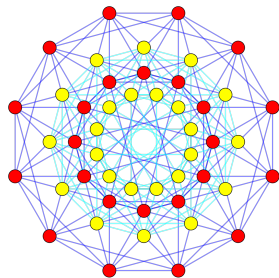
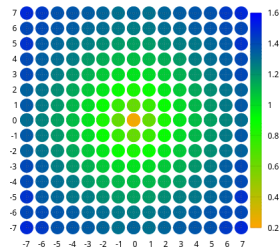
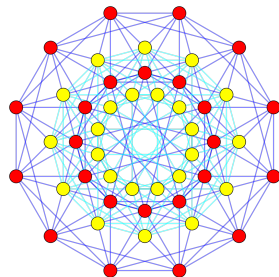


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- Make a rotational invariance study on the new lattice, hoping to get better results than the Simple Hypercubic lattice.



Grazie per l'attenzione!

- [1] Dibakar Sigdel. “Two Dimensional Lattice Gauge Theory with and without Fermion Content”. In: *FIU Electronic Theses and Dissertations* 3224 (2016). DOI: 10.25148/etd.FIDC001748. URL: https://digitalcommons.fiu.edu/etd/3224?utm_source=digitalcommons.fiu.edu%2Fetd%2F3224&utm_medium=PDF&utm_campaign=PDFCoverPages.
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