Quantum Field Theory on a Highly Symmetric Lattice

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Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

$$\langle f|i\rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\left[\phi\right] e^{-S[\phi]}$$

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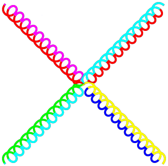

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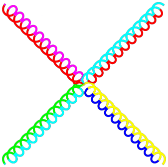
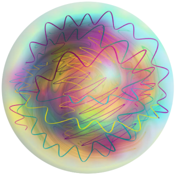


Non-Perturbative



Perturbative vs Non-Perturbative

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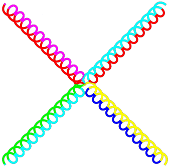

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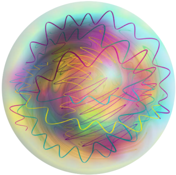

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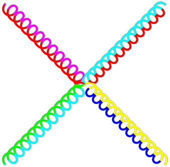

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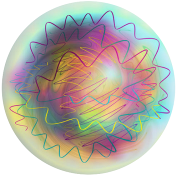

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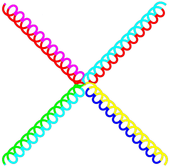

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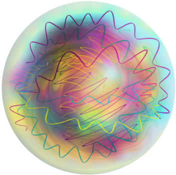

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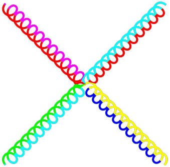

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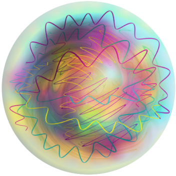

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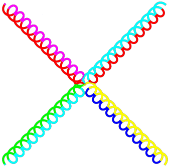

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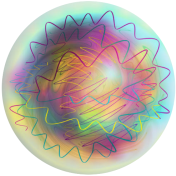

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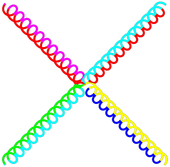

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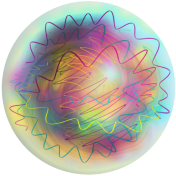

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What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

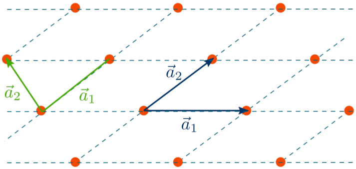

Figure: A bidimensional lattice.

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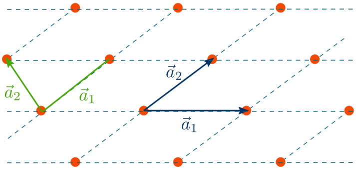

Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$
a is called *lattice spacing*.

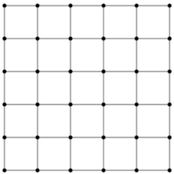

Figure: A square lattice.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

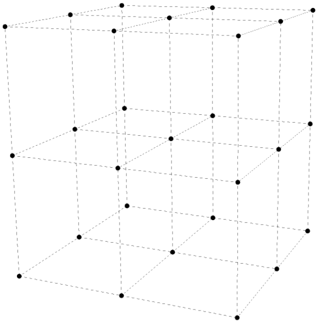

Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

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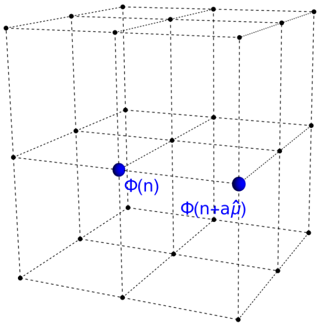
- Scalar fields $\Phi(x) \to \Phi(n)$ on sites



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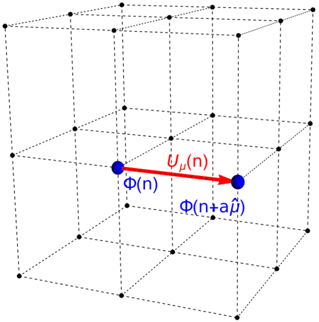
Parallel Transporter

$$U_\mu(x) = \exp(igaA_\mu(x))$$



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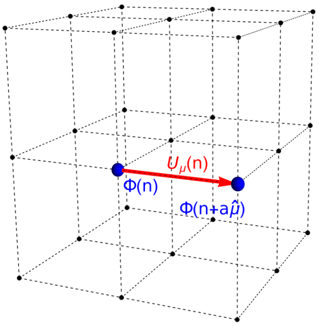

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Beware!

Spinorial fields are trickier to be discretized.

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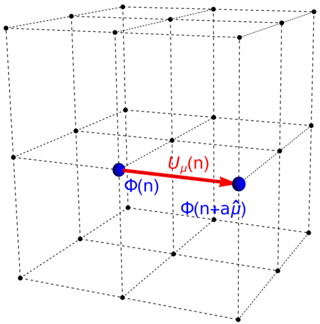

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Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is
$S_E = \frac{1}{4} \int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

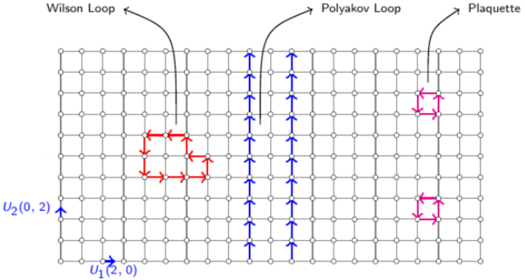

Figure: Gauge-invariant paths on a bidimensional lattice.[Sigdel, 2016]

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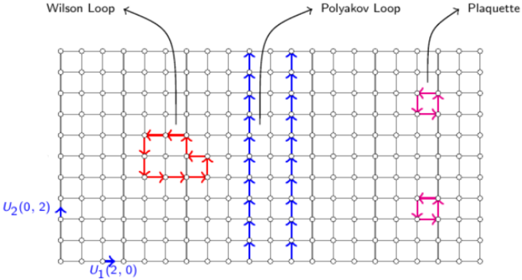

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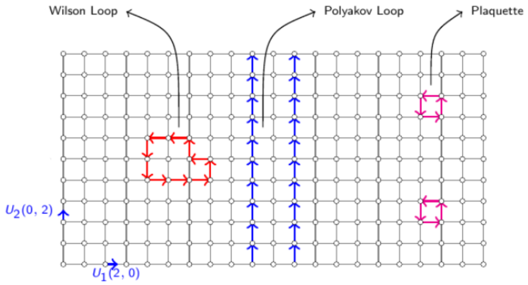
Wilson's Idea

$$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \mathfrak{Re}\,\mathrm{Tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right)$$



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If the time coordinate is taken to be periodic, more closed paths arise.

Polyakov Loop

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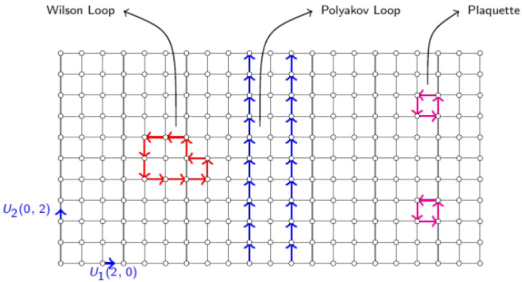

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The expectation value of two Polyakov loops is the potential.

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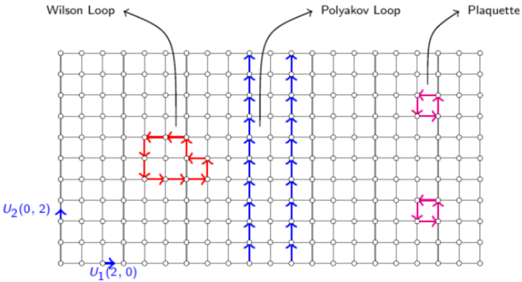
Potential

$V(R) = -\frac{1}{T} \log < P(0)P^{\dagger}(R) >$



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Translations	Rotations

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T: group of rotations of multiples of 90° around any axis.

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Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached.

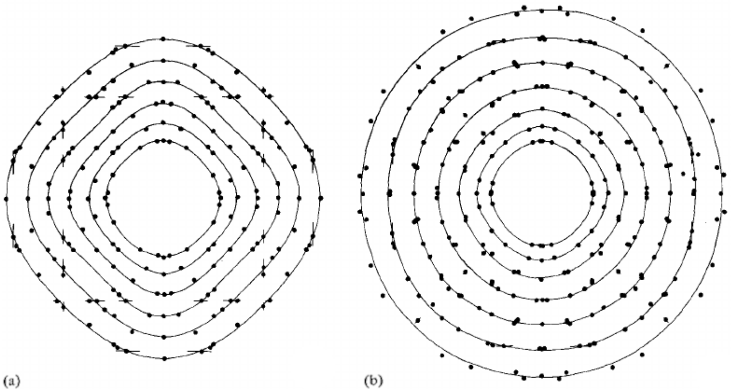

Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves.
[Lang and Rebbi, 1982]

Rotational Invariance Restoration

Values of β are slightly different from Lang and Rebbi's because $a(\beta) \approx \Lambda e^{-b_0 \beta}$, with $\Lambda,\ b_0 > 0$[1].

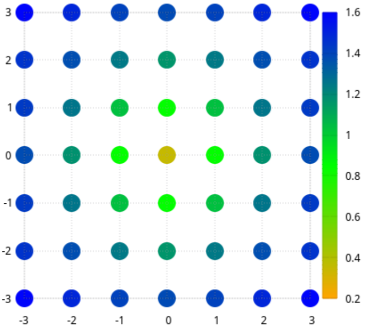

Figure: Potential from $\beta = 2.20$, $n_s = 8,\ n_t = 4$.

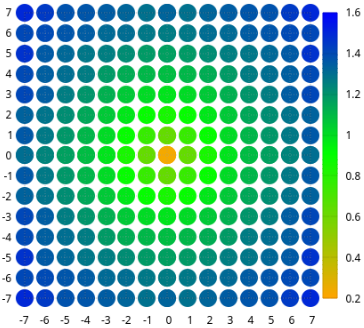

Figure: Potential from $\beta = 2.35$, $n_s = 16,\ n_t = 6$.

[1] The simulation code is based on the code presented in refs. [Panero, 2009; Mykkänen, Panero, and Rummukainen, 2012].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract
[Celmaster, 1982]

- 24 nearest neighbours
- 1152-element symmetry group

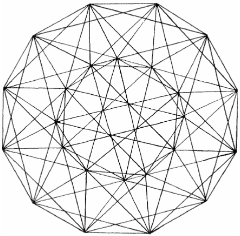

Figure: Two-dimensional projection of a BCT. [Celmaster, 1982]

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

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F_4 coroots lattice [Neuberger, 1987]

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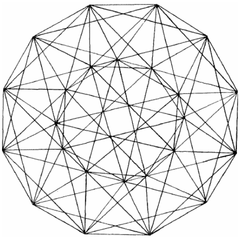

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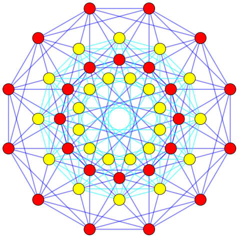

Figure: Two-dimensional projection of a F_4 coroots lattice. [Wikipedia, 2023]

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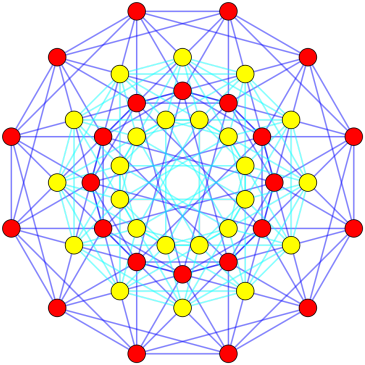
F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;



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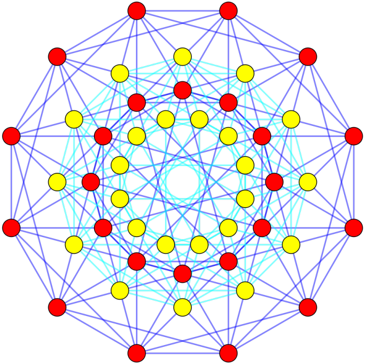

Figure: The 24 roots (red) of the F_4 lattice, projected on a bidimensional plane.

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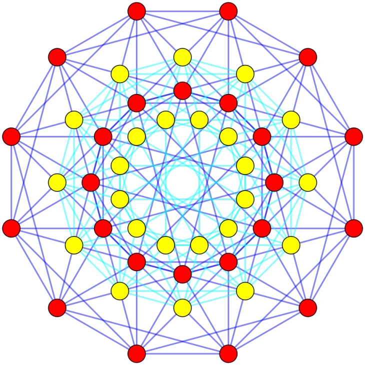
    - The 16 vectors of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$



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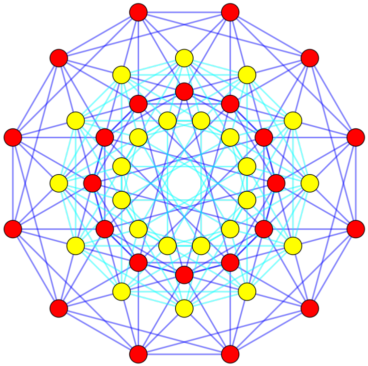

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- Contains the Simple Hypercubic lattice and the BCT;

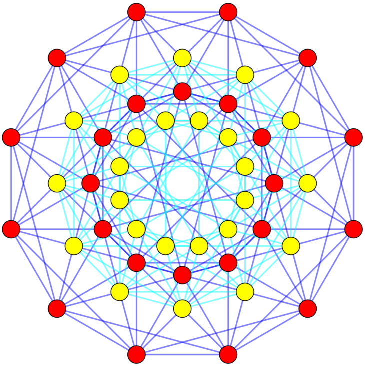


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;
- Has 48 nearest neighbours:
  - The 24 roots are all possible permutations of coordinate positions of $(\pm 1, \pm 1, 0, 0)$
  - The 24 dual roots (coroots) are:
    - The 8 possible permutations of $(\pm 1, 0, 0, 0)$
    - The 16 vectors of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$
- Exists only in 4 dimensions;
- Contains the Simple Hypercubic lattice and the BCT;
- Has been used only to simulate scalar fields, in [Neuberger, 1987].

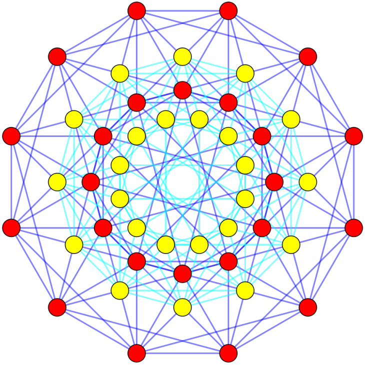
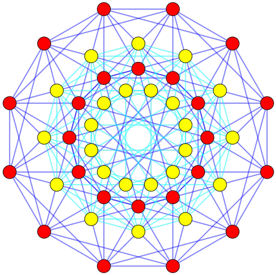


Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

Work in Progress

- Implement the F_4 lattice in the simulation program and make efficiency studies;



Work in Progress

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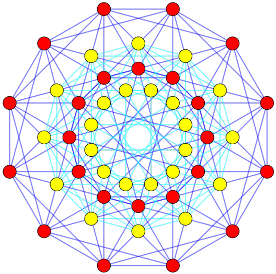
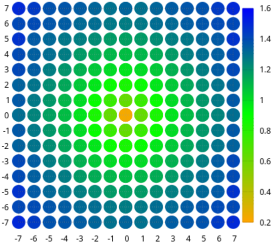


- Make a rotational invariance study on the new lattice, hoping to get better results than the Simple Hypercubic lattice.



Thank you for your attention

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