Università degli Studi di Torino

Corso di Laurea Magistrale in Astrofisica e Fisica Teorica

Titolo

Tesi di Laurea Magistrale

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Abstract

This is the abstract

Contents

QCD on the Lattice

1.1 The QCD Continuum Action

In order to write the action of QCD on the lattice, I must first recall how the theory is formulated in the continuum.

1.1.1 Spinor Fields

Let us take into consideration a (free) quantum field theory describing a fermion, such as a quark or a lepton, in a 4-dimensional spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Its action (in natural units, where $c = \hbar = 1$) can be written as:

$$S_\psi[\psi(x), \overline{\psi}(x)] = \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi\right) \tag{1.1.1}$$

from which, upon the application of the variational principle, the Dirac equation follows:

$$\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi(x) = 0 \tag{1.1.2}$$

It can now be easily checked by direct computation that this action is invariant under a rigid (global) phase transformation, also called global $U(1)$ transformation:

$$\begin{aligned} \psi(x) &\to \psi'(x) = e^{-\mathrm{i}\alpha}\psi(x) \\ \overline{\psi}(x) &\to \overline{\psi}'(x) = \overline{\psi}(x)e^{\mathrm{i}\alpha} \end{aligned} \tag{1.1.3}$$

where α is a constant that does not depend on the spacetime coordinate x, while if $\alpha = \alpha(x)$ the action (1.1.1) would not be invariant because of the kinetic term.

1.1.2 Quantum Electrodynamics

As the free field theory itself is non interacting, it does not provide any real-world prediction, so it is useful to write an interacting action where the spinor field is coupled, for instance, to a vector field A_μ, i.e. the photon. One way to implement this interaction is to ask for local, instead of global, invariance of the action (1.1.1) under the phase transformation (1.1.3), where now $\alpha = \alpha(x)$. In order to do so, one has to define the covariant derivative as follows:

$$D_\mu \equiv \partial_\mu + \mathrm{i}gA_\mu \tag{1.1.4}$$

where g is the couplig constant.[1]
The vector field's kinetic term is written in terms of its field-strength, namely:

$$\begin{aligned} F_{\mu\nu} \equiv& -\frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] = \\ =& -\frac{\mathrm{i}}{g}\left(D_\mu\left(\partial_\nu + \mathrm{i}gA_\nu\right) - D_\nu\left(\partial_\mu + \mathrm{i}gA_\mu\right)\right) = \\ =& -\frac{\mathrm{i}}{g}\left(\cancel{\partial_\mu\partial_\nu} + \mathrm{i}g\partial_\mu A_\nu - g^2A_\mu A_\nu - \cancel{\partial_\nu\partial_\mu} - \mathrm{i}g\partial_\nu A_\mu + g^2A_\nu A_\mu\right) = \\ =&\partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu] = \\ =&\partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{1.1.5}$$

[1] Usually, in QED, g is called e, the electron charge, though I will be using g in analogy to nonabelian gauge theories.

where $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu = 0$ in the abelian theory.
Two different fields A_μ and A'_μ describe the same physics if one can be obtained from another throug a gauge transformation:

$$\begin{aligned} A'_\mu(x) =& A_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x) \\ F'_{\mu\nu} = F_{\mu\nu} &+ \frac{1}{g}\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)\alpha(x) = F_{\mu\nu} \end{aligned} \tag{1.1.6}$$

Thus, the free action for the vector field is:

$$S_{EM} = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} \tag{1.1.7}$$

That is also gauge invariant, i.e. invariant under (1.1.6), as $F_{\mu\nu}$ is gauge invariant.
The term that broke the local phase invariance of the action (1.1.1) can now be "absorbed" by A_μ through a gauge transformation (1.1.6), thus making the full action gauge invariant:

$$\begin{aligned} S_{QED} &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \\ &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ S_{QED} \to S'_{QED} &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi + \overline{\psi}\not{\partial}\alpha\psi - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \overline{\psi}\not{\partial}\alpha\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \\ &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = S_{QED} \end{aligned} \tag{1.1.8}$$

1.1.3 Nonabelian Gauge Theories

Let us now consider a theory with N fermions all with the same mass m, described by the spinorial fields $\psi_i(x)$ with $i = 1, \ldots, N$. Its free action is:

$$S_\psi[\psi_i(x), \overline{\psi}_i(x)] = \sum_{i=1}^{N}\int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}_i\not{\partial}\psi_i - m\overline{\psi}_i\psi\right) \tag{1.1.9}$$

From now on, the sum over i (and all other repeated latin indexes) will be omitted, unless specified differently. This action is invariant under the global transformation:

$$\begin{aligned} \psi_i(x) &\to \psi'_i(x) = U_{ij}\psi_j(x) \\ \overline{\psi}_i(x) &\to \overline{\psi}'_i(x) = \overline{\psi}_j(x)U^\dagger_{ji} \end{aligned} \tag{1.1.10}$$

if U is any (constant) $N \times N$ matrix such that $UU^\dagger = U^\dagger U = \mathbb{1} \Leftrightarrow U^\dagger = U^{-1}$, or in other words, if $U \in U(N)$. For this reason, this transformation is also called a global $U(N)$ transformation. The phase transformation (1.1.3) is the particular case where $U = e^{-\mathrm{i}\alpha} \in U(1)$, that is the only abelian (commutative) unitary group.
As $U(N) = SU(N) \otimes U(1)\ \forall N > 1$, $U \in SU(N)$ instead of $U \in U(N)$ can be imposed, and will be from now on, without loss of generality.

In an analogous way to what has been done in Section 1.1.2 , this invariance can be made local by implementing a proper covariant derivative, similar to (1.1.4). In order to do so, the infinitesimal $SU(N)$ transformation has to be considered:

$$U_{ij}(x) = \delta_{ij} + \mathrm{i}\theta^a(x)\,(T^a)_{ij} + O(\theta^2) \tag{1.1.11}$$

where the indices i and j run from 1 to N (as before) and the index a runs from 1 to N^2-1 (the dimension of the group $SU(N)$). The matrixes T^a are the N^2-1 generators of $SU(N)$, thus they are $N\times N$ hermitean and traceless, which obey the commutation relations:

$$\left[T^a, T^b\right] = \mathrm{i}f^{abc}T^c \tag{1.1.12}$$

where f^{abc} are called *structure constants* of $SU(N)$. The normalization of these matrices can be chosen such that they obey the condition:

$$\mathrm{Tr}\left\{T^aT^b\right\} = \frac{1}{2}\delta^{ab} \tag{1.1.13}$$

some examples are:

- for $N=2$, $T^a = \frac{\sigma^a}{2}$, with σ^a the Pauli matrices;
- for $N=3$, $T^a = \frac{\lambda^a}{2}$, with λ^a the Gell-Mann matrices.

In both those examples, the structure constants are $f^{abc} = \epsilon^{abc}$, the completely antisymmetric Levi-Civita symbol.
The covariant derivative, therefore, is written as:

$$D_\mu \equiv \partial_\mu + \mathrm{i}g\mathbf{A}_\mu(x) \tag{1.1.14}$$

where an $N\times N$ identity matrix $\mathbb{1}$ multiplying ∂_μ has to be understood, and $\mathbf{A}_\mu(x)$ is a gauge field of $SU(N)$, i. e. a traceless, hermitean $N\times N$ matrix.
The covariant derivative can be written more explicitly acting on the set of spinors ψ_i:

$$D_\mu\psi_i = \partial_\mu\mathbb{1}_{ij}\psi_j + \mathrm{i}g\mathbf{A}_\mu(x)_{ij}\psi_j$$

In order for the action to be gauge invariant, the field $\mathbf{A}_\mu$ must satisfy the gauge transformation property

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) = U(x)\mathbf{A}_\mu(x)U^\dagger(x) - \frac{\mathrm{i}}{g}U(x)\partial_\mu U^\dagger(x) \tag{1.1.15}$$

This expression is a little more complicated than (1.1.6), due to the fact that $\mathbf{A}_\mu$ is now a non-commuting matrix. However if the abelian case $U(1)$ is taken into consideration, where $U(x) = e^{-\mathrm{i}\alpha(x)}$, (1.1.6) follows directly from (1.1.15).
It can now be easily checked that the kinetic term of the Lagrangian

$$\mathcal{L}_K = \mathrm{i}\overline{\psi}_i\not{D}\psi_i = \mathrm{i}\overline{\psi}_i\not{\partial}\psi_i - g\overline{\psi}_i\not{\mathbf{A}}\psi_i$$

is gauge invariant (i. e. invariant under (1.1.10) and (1.1.15)) through direct computation:

$$\begin{aligned}
\mathcal{L}_K \to \mathcal{L}'_K =& \mathrm{i}\overline{\psi}_i U^\dagger \not{\partial} (U\psi_i) - g\overline{\psi}_i \underbrace{U^\dagger U}_{\mathbb{1}} \not{\mathbf{A}} \underbrace{U^\dagger U}_{\mathbb{1}} \psi_i + \mathrm{i}\overline{\psi}_i \underbrace{U^\dagger U}_{\mathbb{1}} \left(\not{\partial} U^\dagger\right) U\psi_i = \\
=& \mathrm{i}\overline{\psi}_i U^\dagger \left(\not{\partial} U\right) \psi_i + \underbrace{\mathrm{i}\overline{\psi}_i \not{\partial} \psi_i - g\overline{\psi}_i \not{\mathbf{A}} \psi_i}_{\mathcal{L}_K} + \mathrm{i}\overline{\psi}_i \left(\not{\partial} U^\dagger\right) U\psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \left(U^\dagger \partial_\mu U + \partial_\mu U^\dagger U\right) \psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \partial_\mu \left(U^\dagger U\right) \psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \underbrace{\partial_\mu \left(\mathbb{1}\right)}_{=0} \psi_i = \mathcal{L}_K
\end{aligned}$$

Because of this fact, it is directly implied that the covariant derivative (1.1.14) must transform, under a gauge transformation, as:

$$D_\mu \to D'_\mu = U D_\mu U^\dagger \tag{1.1.16}$$

The field-strength for the field $\mathbf{A}_\mu$ is obtained, as for the abelian case, through the commutator of two covariant derivatives (the computation is the same as (1.1.5), but this time the commutator term is not $= 0$):

$$F_{\mu\nu} \equiv -\frac{\mathrm{i}}{g}[D_\mu, D_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + \mathrm{i}g[\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{1.1.17}$$

this expression can be simplified a little by considering that $\mathbf{A}_\mu$ and $F_{\mu\nu}$ are elements of $SU(N)$, thus writing them in the basis T^a:

$$\mathbf{A}_\mu(x) = A^a_\mu(x) T^a \tag{1.1.18}$$

$$F_{\mu\nu}(x) = F^a_{\mu\nu}(x) T^a \tag{1.1.19}$$

and considering the relations (1.1.12) and (1.1.13):

$$\begin{aligned}
F^a_{\mu\nu} T^a =& \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) T^a + \mathrm{i}g\left[A^b_\mu T^b, A^c_\nu T^c\right] = \\
=& \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) T^a + \mathrm{i}g A^b_\mu A^c_\nu \underbrace{\left[T^b, T^c\right]}_{if^{bca}T^a} = \\
=& \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right) T^a \\
F^a_{\mu\nu} =& \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu
\end{aligned}$$

Computer Simulation of Gauge Theories

Gauge Theories Simulation on non-hypercubic lattice F4

Simulation Results

Conclusions