Quantum Field Theory on a Highly Symmetric Lattice

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Why Lattice Quantum Chromodynamics?

In quantum field theory scattering amplitudes in the form

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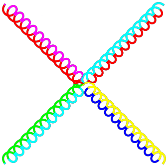

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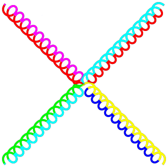
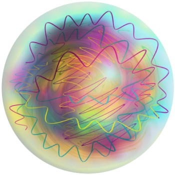


Non-Perturbative



Perturbative vs Non-Perturbative

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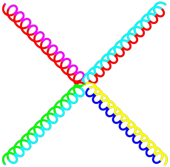

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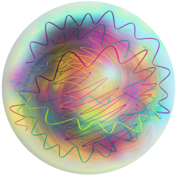

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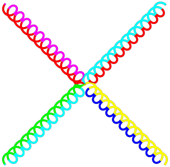

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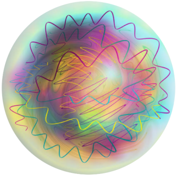

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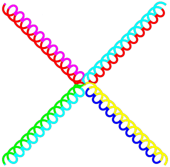

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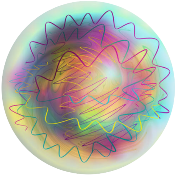

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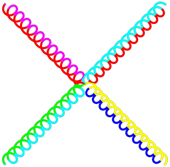

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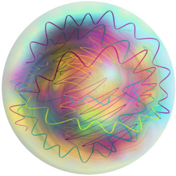

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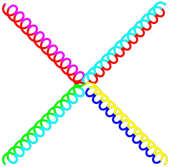

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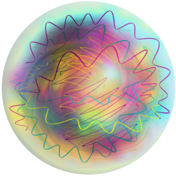

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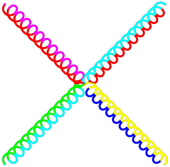

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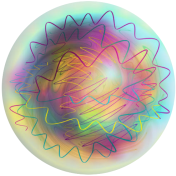

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What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

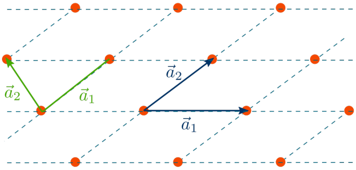

Figure: A bidimensional lattice.

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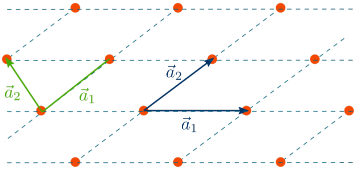
Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$
a is called *lattice spacing*.



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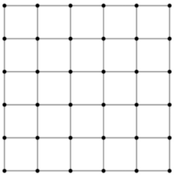

Figure: A square lattice.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

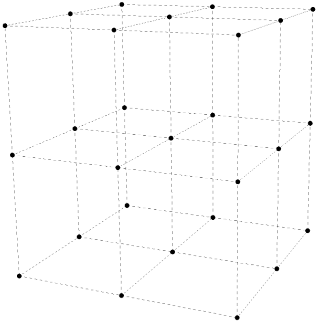

Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

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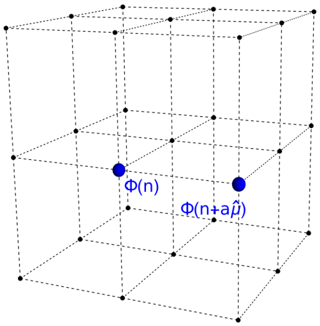

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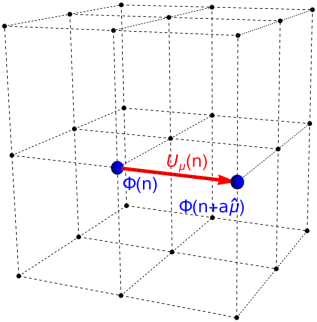
Parallel Transporter

$$U_\mu(x) = \exp(igaA_\mu(x))$$



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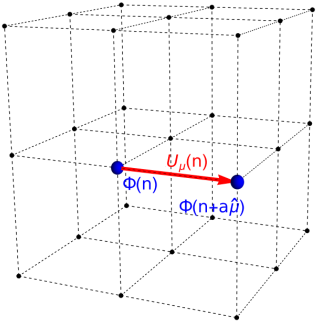

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Beware!

Spinorial fields are trickier to be discretized.

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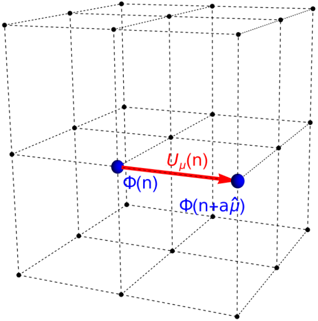

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The Yang-Mills continuum action is $S_E = \frac{1}{4}\int \mathrm{d}^4 x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

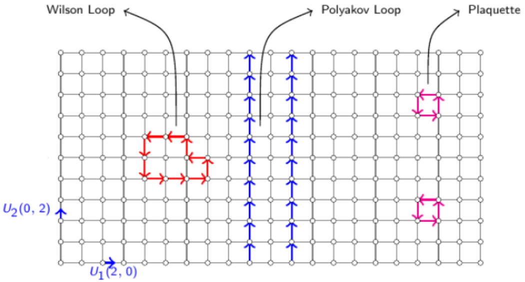

Figure: Gauge-invariant paths on a bidimensional lattice.[1]

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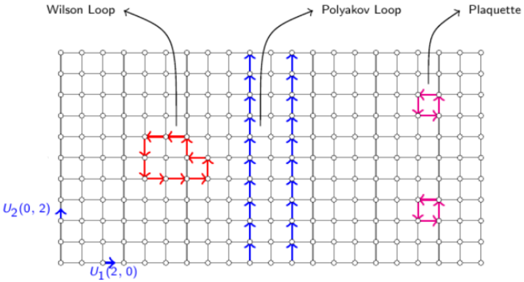

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Gauge-Invariant Observables and Wilson Action

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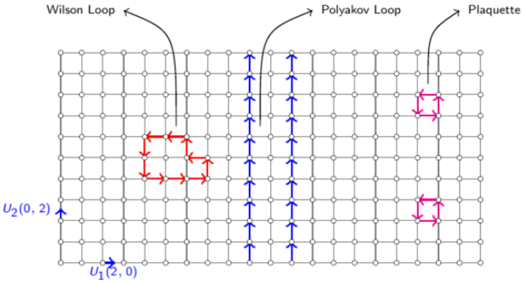
Wilson's Idea

$$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \mathfrak{Re}\,\mathrm{Tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right)$$



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Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

Polyakov Loop

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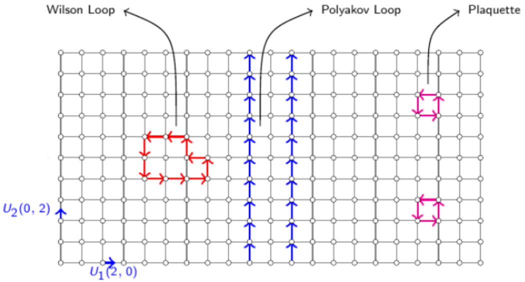

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The expectation value of two Polyakov loops is the potential.

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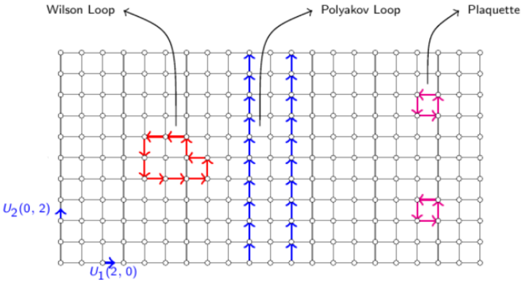
Potential

$V(R) = -\frac{1}{T} \log < P(0)P^{\dagger}(R) >$



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T: group of rotations of multiples of 90° around any axis.

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Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration

Bibliography

[1] Dibakar Sigdel. "Two Dimensional Lattice Gauge Theory with and without Fermion Content". In: *FIU Electronic Theses and Dissertations* 3224 (2016). DOI: 10.25148/etd.FIDC001748. URL: https://digitalcommons.fiu.edu/etd/3224?utm_source=digitalcommons.fiu.edu%2Fetd%2F3224&utm_medium=PDF&utm_campaign=PDFCoverPages.