Quantum Field Theory on a Highly Symmetric Lattice

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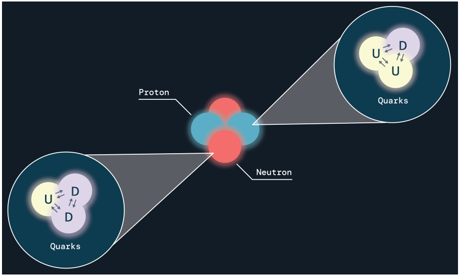
The Strong Interaction

Matter is made of Atoms

Atoms are made of Nuclei and Electrons

Nuclei are made of Protons and Neutrons,

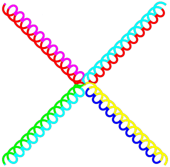
composed of Quarks and Gluons



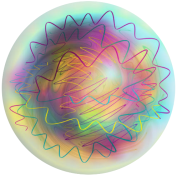
Described by Quantum Chromodynamics (QCD)

Image credits: NASA[StrongForceImg]

Perturbative



Non-Perturbative



- Straightforward series expansion in powers of small $g \Leftrightarrow$ Feynman diagrams with n loops

Perturbative vs Non-Perturbative

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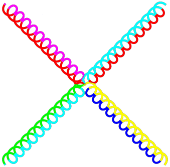

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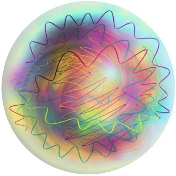

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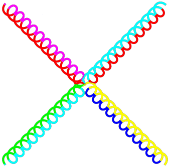

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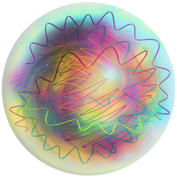

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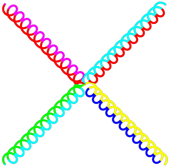

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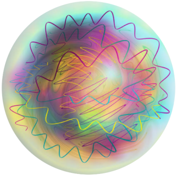

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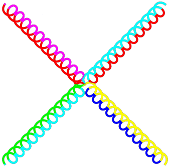

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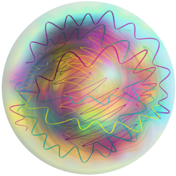

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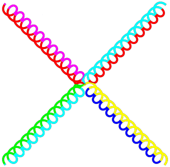

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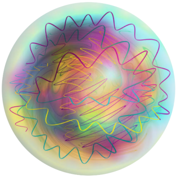

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What is a Lattice?

Definition: Lattice Λ

$\Lambda = \{\sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{Z}\}$, with $\{e_i\}$ any basis of $\mathbb{R}^n$

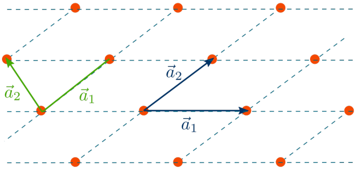

Figure: A bidimensional lattice.

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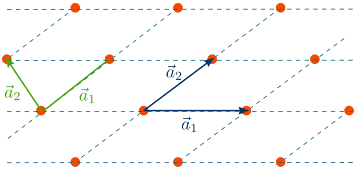

Figure: A bidimensional lattice.

Hypercubic lattice

$\{e_i\}$ is the canonical basis of $\mathbb{R}^n$

a is called *lattice spacing*.

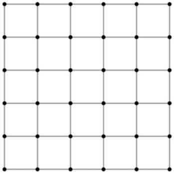

Figure: A square lattice.

Lattice Field Theory

Basic idea

Fields can take values only in given parts of the lattice, $x \to n \in \Lambda$.

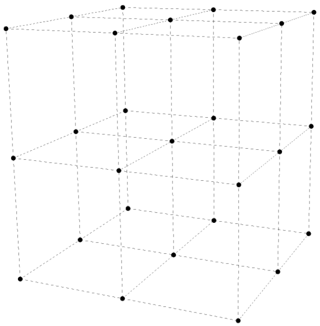

Figure: A (hyper)cubic lattice in $\mathbb{R}^3$.

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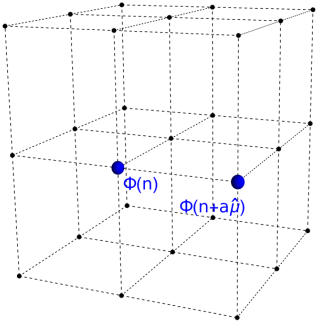

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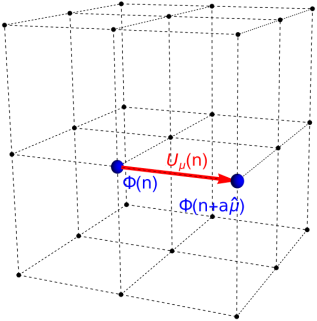
Parallel Transporter

$U_\mu(x) = \exp(igaA_\mu(x))$



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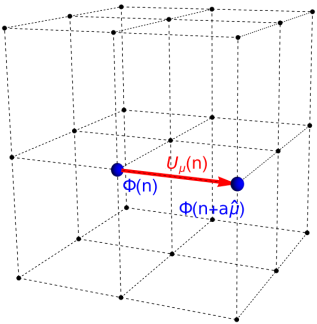

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Beware!

Spinorial fields are trickier to be discretized.

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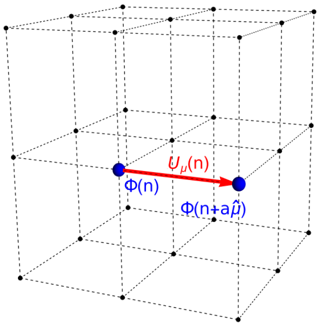

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Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is $S_E = \frac{1}{4}\int \mathrm{d}^4x F^{a\mu\nu}(x) F^a_{\mu\nu}(x)$.

On the lattice, every closed path is gauge-invariant.

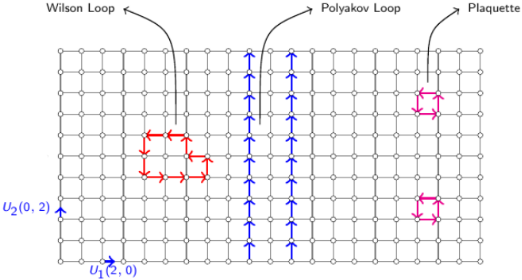

Figure: Gauge-invariant paths on a bidimensional lattice.[Sigdel, 2016]

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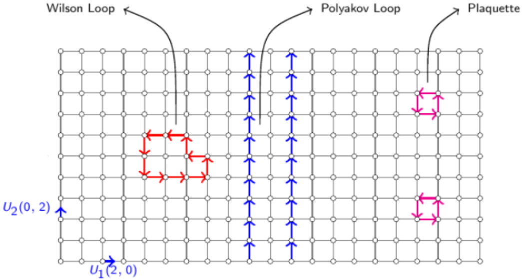

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Wilson's Idea

$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \mathfrak{Re}\, \mathrm{Tr}\left(\mathbb{1} - U_{\mu\nu}(n)\right)$

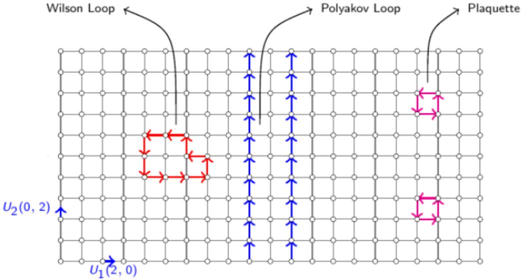


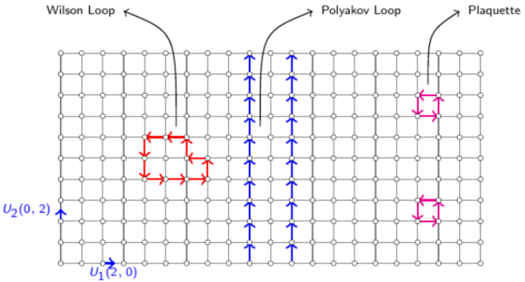
Figure: Gauge-invariant paths on a bidimensional lattice.[Sigdel, 2016]

Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

Polyakov Loop

$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$



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The expectation value of two Polyakov loops is the potential.

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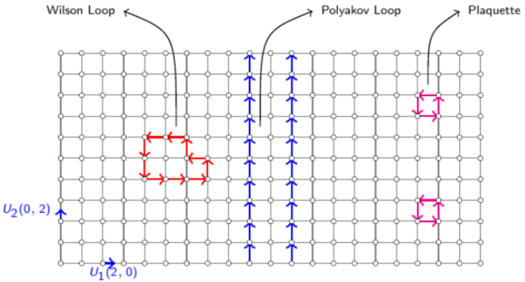
$P(n) = \mathrm{Tr} \prod_{t=0}^{T-1} U_t(n)$

Potential

$V(R) = -\frac{1}{T} \log < P(0)P^\dagger(R) >$



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Important:

Rotational invariance seems to be broken.

Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached.

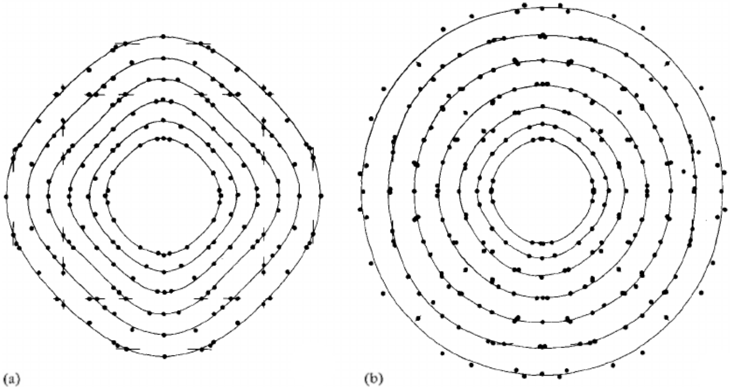

Figure: Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the curves represent equipotential curves.
[Lang and Rebbi, 1982]

Values of β are slightly different from Lang and Rebbi's because $a(\beta) \approx \Lambda e^{-b_0\beta}$, with Λ, $b_0 > 0$[1].

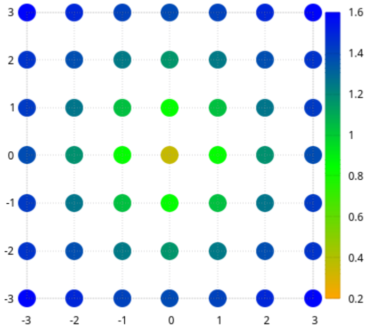

Figure: Potential from $\beta = 2.20$, $n_s = 8$, $n_t = 4$.

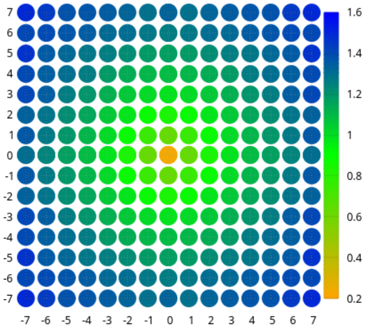

Figure: Potential from $\beta = 2.35$, $n_s = 16$, $n_t = 6$.

[1] The simulation code is based on the code presented in refs. [Panero, 2009; Mykkänen, Panero, and Rummukainen, 2012].

Higher Symmetry Lattices

Other, more rotational-symmetric, lattices have been used:

Body Centered Tesseract
[Celmaster, 1982]

- 24 nearest neighbours
- 1152-element symmetry group

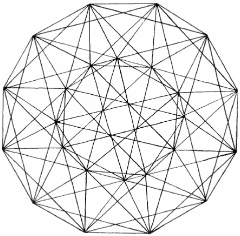

Figure: Two-dimensional projection of a BCT. [Celmaster, 1982]

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

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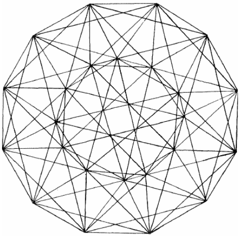
F_4 coroots lattice [Neuberger, 1987]

- 48 nearest neighbours
- 2304-element symmetry group



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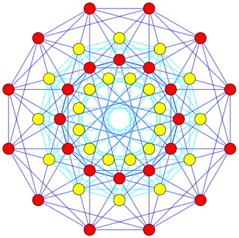

Figure: Two-dimensional projection of a F_4 coroots lattice. [Wikipedia, 2023]

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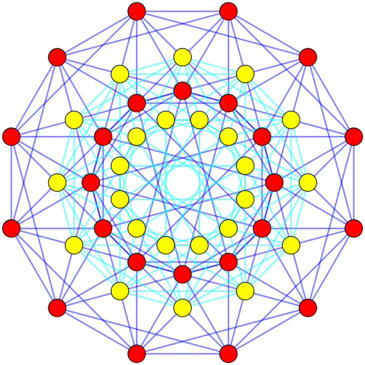
F_4 Coroots Lattice

- Obtained from the roots lattice of the exceptional Lie algebra F_4 and its dual;



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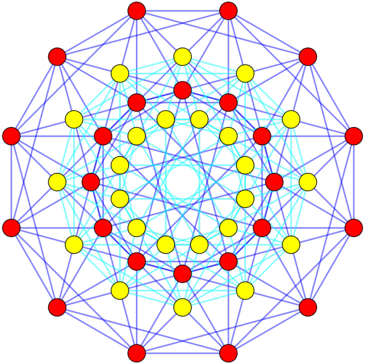

Figure: The 24 roots (red) of the F_4 lattice, projected on a bidimensional plane.

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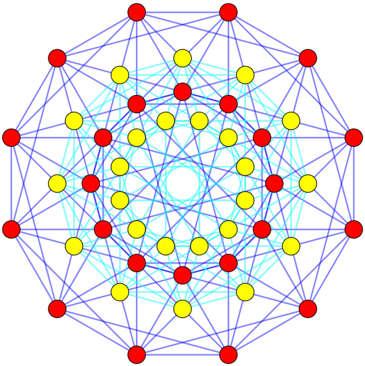

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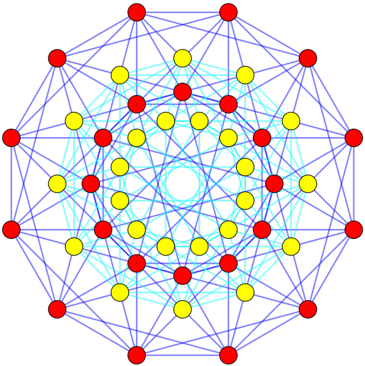

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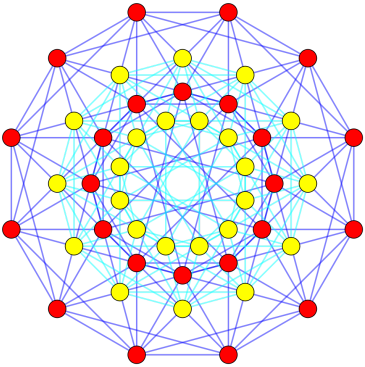

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- Exists only in 4 dimensions;
- Contains the Simple Hypercubic lattice and the BCT;
- Has been used only to simulate scalar fields, in [Neuberger, 1987].

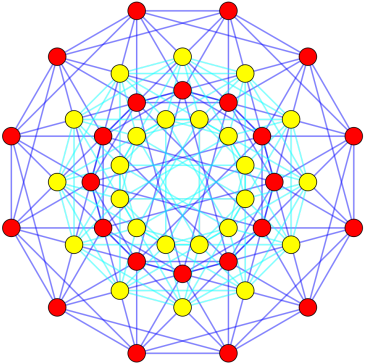
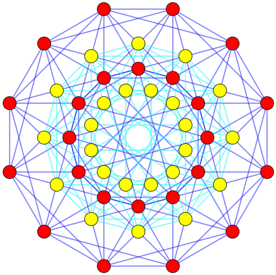


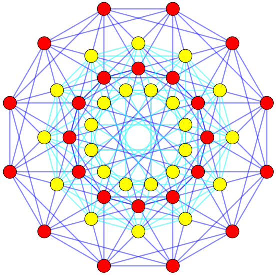
Figure: The 24 roots (red) and the 24 coroots (yellow) of the F_4 lattice, projected on a bidimensional plane.

- Implement the F_4 lattice in the simulation program and make efficiency studies;

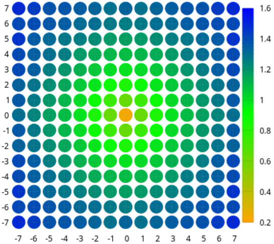


Work in Progress

- Implement the F_4 lattice in the simulation program and make efficiency studies;



- Make a rotational invariance study on the new lattice, hoping to get better results than the Simple Hypercubic lattice.



Thank you for your attention

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