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CORSO DI LAUREA MAGISTRALE IN ASTROFISICA E FISICA TEORICA

Titolo

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Relatore:

Prof.
Panero Marco

Candidato:

Aliberti Marco
Matricola 855766

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Abstract

This is the abstract

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QCD on the Lattice

1.1 The QCD Continuum Action

In order to write the action of QCD on the lattice, I must first recall how the theory is formulated in the continuum.

1.1.1 Spinor Fields

Let us take into consideration a (free) quantum field theory describing a fermion, such as a quark or a lepton, in a 4-dimensional spacetime with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Its action (in natural units, where $c = \hbar = 1$) can be written as:

$$S_\psi[\psi(x), \bar{\psi}(x)] = \int d^4x (\bar{\psi} \not{\partial} \psi - m \bar{\psi} \psi) \quad (1.1.1)$$

from which, upon the application of the variational principle, the Dirac equation follows:

$$(\not{\partial} - m) \psi(x) = 0 \quad (1.1.2)$$

It can now be easily checked by direct computation that this action is invariant under a rigid (global) phase transformation, also called global $U(1)$ transformation:

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{-i\alpha} \psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{i\alpha} \end{aligned} \quad (1.1.3)$$

where α is a constant that does not depend on the spacetime coordinate x , while if $\alpha = \alpha(x)$ the action (1.1.1) would not be invariant because of the kinetic term.

1.1.2 Quantum Electrodynamics

As the free field theory itself is non interacting, it does not provide any real-world prediction, so it is useful to write an interacting action where the spinor field is coupled, for instance, to a vector field A_μ , i.e. the photon. One way to implement this interaction is to ask for local, instead of global, invariance of the action (1.1.1) under the phase transformation (1.1.3), where now $\alpha = \alpha(x)$. In order to do so, one has to define the covariant derivative as follows:

$$D_\mu \equiv \partial_\mu + igA_\mu \quad (1.1.4)$$

where g is the couplig constant.¹

The vector field's kinetic term is written in terms of its field-strength, namely:

$$\begin{aligned} F_{\mu\nu} &\equiv -\frac{i}{g} [D_\mu, D_\nu] = \\ &= -\frac{i}{g} (D_\mu (\partial_\nu + igA_\nu) - D_\nu (\partial_\mu + igA_\mu)) = \\ &= -\frac{i}{g} (\cancel{\partial_\mu \partial_\nu} + ig\partial_\mu A_\nu - g^2 A_\mu A_\nu - \cancel{\partial_\nu \partial_\mu} - ig\partial_\nu A_\mu + g^2 A_\nu A_\mu) = \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \quad (1.1.5)$$

¹Usually, in QED, g is called e , the electron charge, though I will be using g in analogy to nonabelian gauge theories.

where $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu = 0$ in the abelian theory.

Two different fields A_μ and A'_μ describe the same physics if one can be obtained from another through a gauge transformation:

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \frac{1}{g} \partial_\mu \alpha(x) \\ F'_{\mu\nu} &= F_{\mu\nu} + \frac{1}{g} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \alpha(x) = F_{\mu\nu} \end{aligned} \quad (1.1.6)$$

Thus, the free action for the vector field is:

$$S_{EM} = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} \quad (1.1.7)$$

That is also gauge invariant, i.e. invariant under (1.1.6), as $F_{\mu\nu}$ is gauge invariant.

The term that broke the local phase invariance of the action (1.1.1) can now be “absorbed” by A_μ through a gauge transformation (1.1.6), thus making the full action gauge invariant:

$$\begin{aligned} S_{QED} &= \int d^4x \left(i\bar{\psi} \not{D} \psi - m\bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \\ &= \int d^4x \left(i\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi - g\bar{\psi} \not{A} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\ S_{QED} \rightarrow S'_{QED} &= \int d^4x \left(i\bar{\psi} \not{\partial} \psi + \bar{\psi} \not{\partial} \alpha \psi - m\bar{\psi} \psi - g\bar{\psi} \not{A} \psi - \bar{\psi} \not{\partial} \alpha \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \\ &= \int d^4x \left(i\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi - g\bar{\psi} \not{A} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = S_{QED} \end{aligned} \quad (1.1.8)$$

1.1.3 Nonabelian Gauge Theories

Let us now consider a theory with N fermions all with the same mass m , described by the spinorial fields $\psi_i(x)$ with $i = 1, \dots, N$. Its free action is:

$$S_\psi[\psi_i(x), \bar{\psi}_i(x)] = \sum_{i=1}^N \int d^4x (i\bar{\psi}_i \not{\partial} \psi_i - m\bar{\psi}_i \psi_i) \quad (1.1.9)$$

From now on, the sum over i (and all other repeated latin indexes) will be omitted, unless specified differently. This action is invariant under the global transformation:

$$\begin{aligned} \psi_i(x) &\rightarrow \psi'_i(x) = U_{ij} \psi_j(x) \\ \bar{\psi}_i(x) &\rightarrow \bar{\psi}'_i(x) = \bar{\psi}_j(x) U_{ji}^\dagger \end{aligned} \quad (1.1.10)$$

if U is any (constant) $N \times N$ matrix such that $UU^\dagger = U^\dagger U = \mathbf{1} \Leftrightarrow U^\dagger = U^{-1}$, or in other words, if $U \in U(N)$. For this reason, this transformation is also called a global $U(N)$ transformation. The phase transformation (1.1.3) is the particular case where $U = e^{-i\alpha} \in U(1)$, that is the only abelian (commutative) unitary group.

As $U(N) = SU(N) \otimes U(1) \forall N > 1$, $U \in SU(N)$ instead of $U \in U(N)$ can be imposed, and will be from now on, without loss of generality.

In an analogous way to what has been done in Section 1.1.2, this invariance can be made local by implementing a proper covariant derivative, similar to (1.1.4). In order to do so, the infinitesimal $SU(N)$ transformation has to be considered:

$$U_{ij}(x) = \delta_{ij} + i\theta^a(x) (T^a)_{ij} + O(\theta^2) \quad (1.1.11)$$

where the indices i and j run from 1 to N (as before) and the index a runs from 1 to $N^2 - 1$ (the dimension of the group $SU(N)$). The matrixes T^a are the $N^2 - 1$ generators of $SU(N)$, thus they are $N \times N$ hermitean and traceless, which obey the commutation relations:

$$[T^a, T^b] = if^{abc}T^c \quad (1.1.12)$$

where f^{abc} are called *structure constants* of $SU(N)$. The normalization of these matrices can be chosen such that they obey the condition:

$$\text{Tr}\{T^a T^b\} = \frac{1}{2}\delta^{ab} \quad (1.1.13)$$

some examples are:

- for $N = 2$, $T^a = \frac{\sigma^a}{2}$, with σ^a the Pauli matrices;
- for $N = 3$, $T^a = \frac{\lambda^a}{2}$, with λ^a the Gell-Mann matrices.

In both those examples, the structure constants are $f^{abc} = \epsilon^{abc}$, the completely antisymmetric Levi-Civita symbol.

The covariant derivative, therefore, is written as:

$$D_\mu \equiv \partial_\mu + ig\mathbf{A}_\mu(x) \quad (1.1.14)$$

where an $N \times N$ identity matrix $\mathbb{1}$ multiplying ∂_μ has to be understood, and $\mathbf{A}_\mu(x)$ is a gauge field of $SU(N)$, i. e. a traceless, hermitean $N \times N$ matrix.

The covariant derivative can be written more explicitly acting on the set of spinors ψ_i :

$$D_\mu \psi_i = \partial_\mu \mathbb{1}_{ij} \psi_j + ig\mathbf{A}_\mu(x)_{ij} \psi_j$$

In order for the action to be gauge invariant, the field \mathbf{A}_μ must satisfy the gauge transformation property

$$\mathbf{A}_\mu(x) \rightarrow \mathbf{A}'_\mu(x) = U(x)\mathbf{A}_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \quad (1.1.15)$$

Computer Simulation of Gauge Theories

Gauge Theories Simulation on non-hypercubic lattice F4

Simulation Results

Conclusions
