

# Quantum Field Theory on a Highly Symmetric Lattice

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In quantum field theory scattering amplitudes in the form

$$\langle f|i\rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}[\phi] e^{-S[\phi]}$$

need to be evaluated.

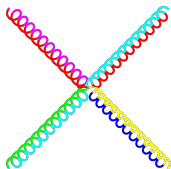
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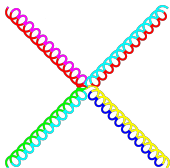
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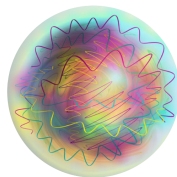
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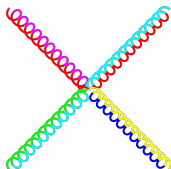


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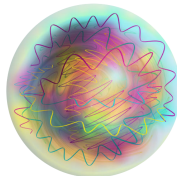


# Perturbative vs Non-Perturbative

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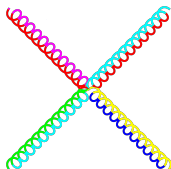
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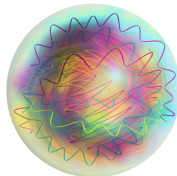
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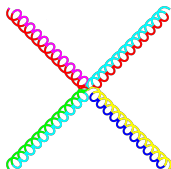
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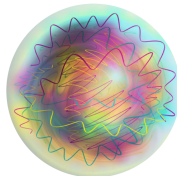
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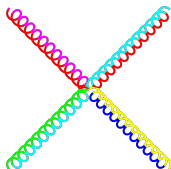
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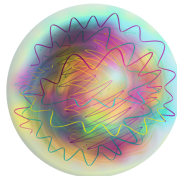
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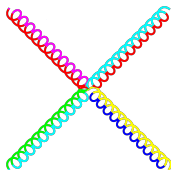


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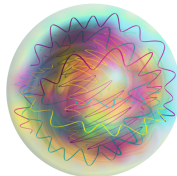
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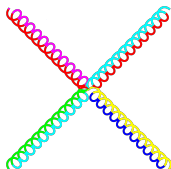
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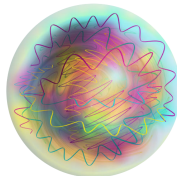
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# What is a Lattice?

## Definition: Lattice $\Lambda$

$\Lambda = \{ \sum_{i=1}^n a_i e_i \mid a_i \in \mathbb{Z} \}$ , with  $\{e_i\}$  any basis of  $\mathbb{R}^n$

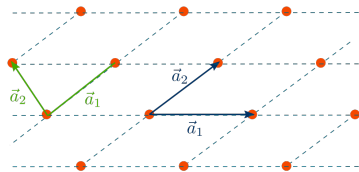


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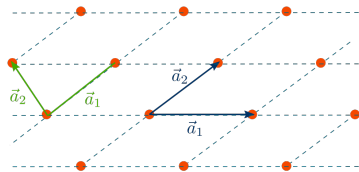


Figure: A bidimensional lattice.

## Hypercubic lattice

$\{e_i\}$  is the canonical basis of  $\mathbb{R}^n$   
 $a$  is called *lattice spacing*.

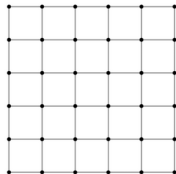


Figure: A square lattice.

## Basic idea

Fields can take values only in given parts of the lattice,  $x \rightarrow n \in \Lambda$ .

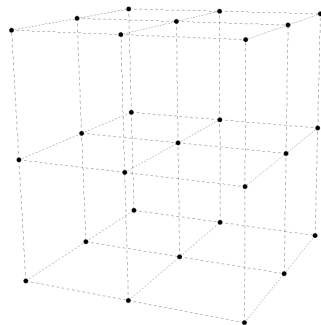


Figure: A (hyper)cubic lattice in  $\mathbb{R}^3$ .

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- **Scalar fields**  $\Phi(x) \rightarrow \Phi(n)$  on sites

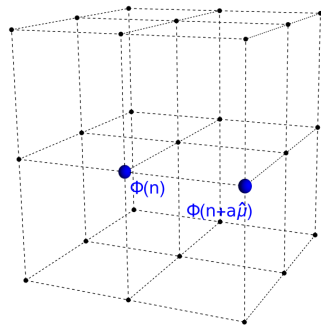


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$$U_\mu(x) = \exp(igaA_\mu(x))$$

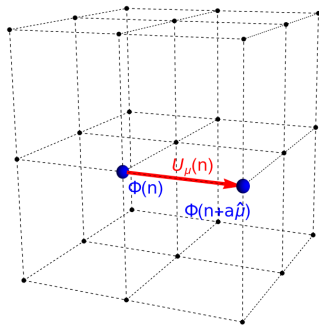


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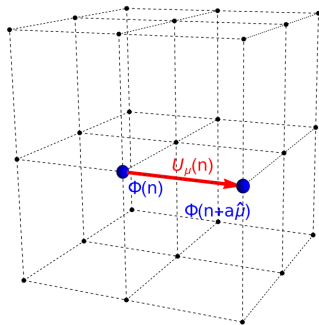


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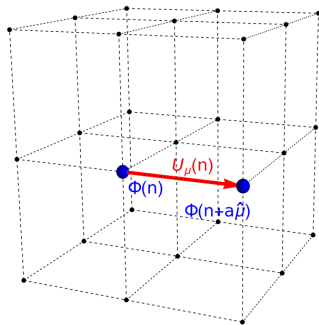
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## Beware!

Spinorial fields are trickier to be discretized.

## Parallel Transporter

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**Figure:** A (hyper)cubic lattice in  $\mathbb{R}^3$ .

# Gauge-Invariant Observables and Wilson Action

The Yang-Mills continuum action is  
$$S_E = \frac{1}{4} \int d^4x F^{a\mu\nu}(x) F_{\mu\nu}^a(x).$$

On the lattice, every closed path is gauge-invariant.

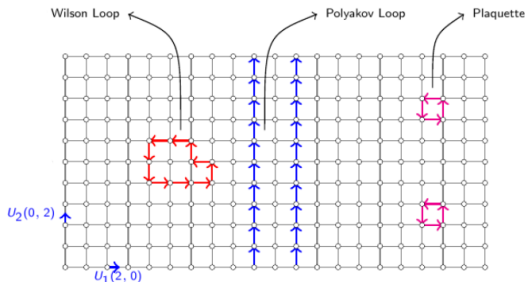


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

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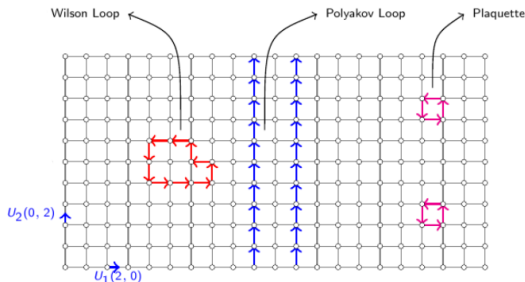


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**Wilson's Idea**

$$S = \frac{\beta}{2N} \sum_{n,\mu,\nu} \Re \text{Tr} (1 - U_{\mu\nu}(n))$$

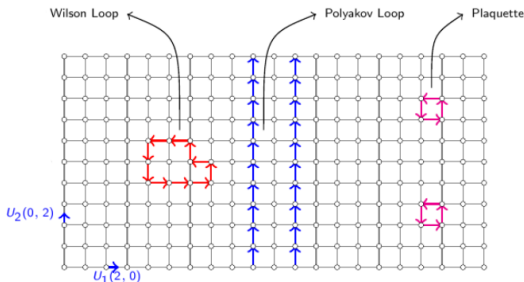


Figure: Gauge-invariant paths on a bidimensional lattice.[1]

# Polyakov Loops and Potential

If the time coordinate is taken to be periodic, more closed paths arise.

## Polyakov Loop

$$P(n) = \text{Tr} \prod_{t=0}^{T-1} U_t(n)$$

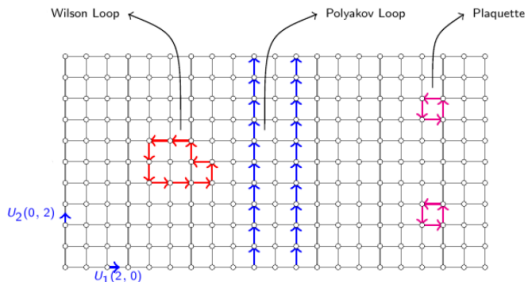


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The expectation value of two Polyakov loops is the potential.

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$$V(R) = -\frac{1}{T} \log \langle P(0) P^\dagger(R) \rangle$$

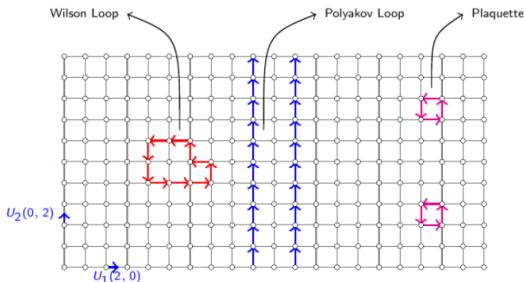


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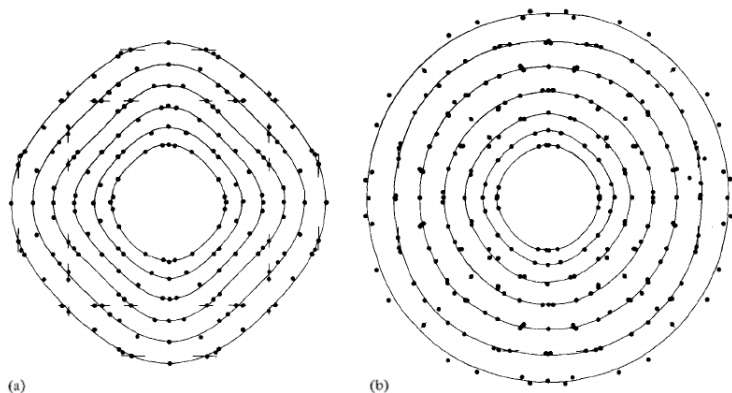
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**Important:**

Rotational invariance seems to be broken.

# Rotational Invariance Restoration - Lang and Rebbi

Equipotential surfaces become spheres as the continuum limit is approached.



**Figure:** Restoration of rotational invariance from (a)  $\beta = 2$ ,  $n_s = 8$ ,  $n_t = 4$  to (b)  $\beta = 2.25$ ,  $n_s = 16$ ,  $n_t = 6$ ; the curves represent equipotential curves. [2]

# Rotational Invariance Restoration

Values of  $\beta$  are slightly different from Lang and Rebbi's because  $a(\beta) \approx \Lambda e^{-b_0 \beta}$ , with  $\Lambda, b_0 > 0$ .<sup>1</sup>

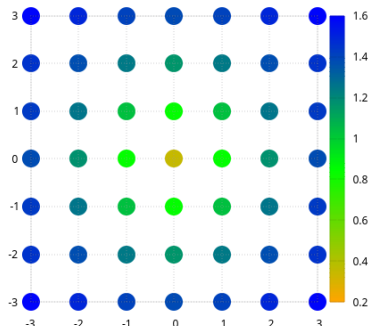


Figure: Potential from  $\beta = 2.20$ ,  
 $n_s = 8$ ,  $n_t = 4$ .

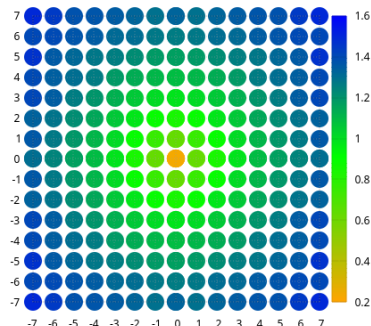


Figure: Potential from  $\beta = 2.35$ ,  
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<sup>1</sup>The simulation code is based on the code presented in refs. [3, 4].

# Higher Symmetry Lattices

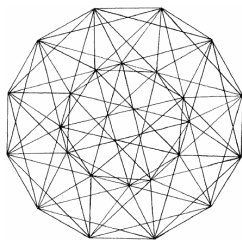
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## Body Centered Tesseract [5]

- 24 nearest neighbours
- 1152-element symmetry group



**Figure:** Two-dimensional projection of a BCT. [5]

The SH lattice has 8 nearest neighbours and a 384-element symmetry group.

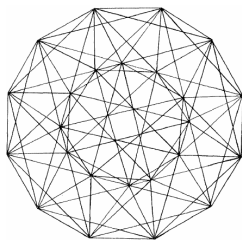


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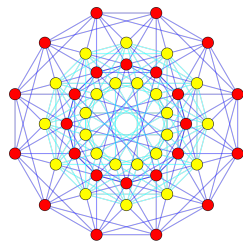
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## $F_4$ coroots lattice [6]

- 48 nearest neighbours
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**Figure:** Two-dimensional projection of a  $F_4$  coroots lattice. [7]

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- [1] Dibakar Sigdel. “Two Dimensional Lattice Gauge Theory with and without Fermion Content”. In: *FIU Electronic Theses and Dissertations* 3224 (2016). DOI: 10.25148/etd.FIDC001748. URL: [https://digitalcommons.fiu.edu/etd/3224?utm\\_source=digitalcommons.fiu.edu%2Fetd%2F3224&utm\\_medium=PDF&utm\\_campaign=PDFCoverPages](https://digitalcommons.fiu.edu/etd/3224?utm_source=digitalcommons.fiu.edu%2Fetd%2F3224&utm_medium=PDF&utm_campaign=PDFCoverPages).
- [2] C. B. Lang and C. Rebbi. “Potential and Restoration of Rotational Symmetry in SU(2) Lattice Gauge Theory”. In: *Phys. Lett. B* 115 (1982). [, 322 (1982)], p. 137. DOI: 10.1016/0370-2693(82)90813-9.
- [3] Marco Panero. “Thermodynamics of the QCD plasma and the large-N limit”. In: *Phys. Rev. Lett.* 103 (2009), p. 232001. DOI: 10.1103/PhysRevLett.103.232001. arXiv: 0907.3719 [hep-lat].

- [4] Anne Mykkänen, Marco Panero, and Kari Rummukainen. “Casimir scaling and renormalization of Polyakov loops in large- $N$  gauge theories”. In: *JHEP* 1205 (2012), p. 069. DOI: 10.1007/JHEP05(2012)069. arXiv: 1202.2762 [hep-lat].
- [5] William Celmaster. “Gauge Theories on the Body - Centered Hypercubic Lattice”. In: *Phys. Rev. D* 26 (1982), p. 2955. DOI: 10.1103/PhysRevD.26.2955.
- [6] Herbert Neuberger. “SPINLESS FIELDS ON  $F(4)$  LATTICES”. In: *Phys. Lett. B* 199 (1987), pp. 536–540. DOI: 10.1016/0370-2693(87)91623-6.
- [7] URL: [https://en.wikipedia.org/wiki/F4\\_%28mathematics%29](https://en.wikipedia.org/wiki/F4_%28mathematics%29).