UNIVERSITÀ DEGLI STUDI DI TORINO

CORSO DI LAUREA MAGISTRALE IN ASTROFISICA E FISICA TEORICA

Titolo

TESI DI LAUREA MAGISTRALE

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Università degli Studi di Torino

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Abstract

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Quantum Chromodynamics on the Lattice

1.1 The Yang-Mills Continuum Action

The aim of this chapter is to discretize the Yang-Mills action on a hypercubic lattice in 4 dimensions. In order to do so, the action is obtained firstly in the continuum, beginning from the simplest case, Quantum Electrodynamics.
This first section is mostly taken from Srednicki [1], personal notes and computations.

1.1.1 Dirac Spinor Fields

Let us take into consideration a (free) quantum field theory describing a fermion, such as a quark or a lepton, in a 4-dimensional spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Its action (in natural units, where $c = \hbar = 1$) can be written as:

$$S_\psi[\psi(x), \overline{\psi}(x)] = \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi \right) \tag{1.1.1}$$

from which, upon the application of the variational principle, the Dirac equation follows:

$$\left(\mathrm{i}\not{\partial} - m\right)\psi(x) = 0 \tag{1.1.2}$$

It can now be easily checked by direct computation that this action is invariant under a rigid (global) phase transformation, also called a global $U(1)$ transformation:

$$\begin{aligned}\psi(x) &\to \psi'(x) = e^{-\mathrm{i}\alpha}\psi(x) \\ \overline{\psi}(x) &\to \overline{\psi}'(x) = \overline{\psi}(x)e^{\mathrm{i}\alpha}\end{aligned} \tag{1.1.3}$$

where α is a constant that does not depend on the spacetime coordinate x, because if α was a function of x, the kinetic term of the action (1.1.1) would not be invariant under such transformation.

1.1.2 Quantum Electrodynamics

As the free field theory itself is non interacting, it does not provide any real-world prediction, so it is useful to write an interacting action where the spinor field is coupled, for instance, to a vector field A_μ, i.e. the photon. One way to implement this interaction is to ask for local, instead of global, invariance of the action (1.1.1) under the phase transformation (1.1.3), where now $\alpha = \alpha(x)$. In order to do so, the covariant derivative has to be defined as follows:

$$D_\mu \equiv \partial_\mu + \mathrm{i}gA_\mu \tag{1.1.4}$$

where g is the couplig constant.[1]
The vector field's kinetic term is written in terms of its field-strength, namely:

$$\begin{aligned}F_{\mu\nu} \equiv& -\frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] = \\ =& -\frac{\mathrm{i}}{g}\left(D_\mu\left(\partial_\nu + \mathrm{i}gA_\nu\right) - D_\nu\left(\partial_\mu + \mathrm{i}gA_\mu\right)\right) = \\ =& -\frac{\mathrm{i}}{g}\left(\cancel{\partial_\mu\partial_\nu} + \mathrm{i}g\partial_\mu A_\nu - g^2 A_\mu A_\nu - \cancel{\partial_\nu\partial_\mu} - \mathrm{i}g\partial_\nu A_\mu + g^2 A_\nu A_\mu\right) = \\ =&\partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu] = \\ =&\partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned} \tag{1.1.5}$$

where $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu = 0$ in the abelian theory.
Two different fields A_μ and A'_μ describe the same physics if one can be obtained from another throug a gauge transformation:

$$\begin{aligned}A'_\mu(x) =& A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x) \\ F'_{\mu\nu} = F_{\mu\nu} &+ \frac{1}{g}\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)\alpha(x) = F_{\mu\nu}\end{aligned} \tag{1.1.6}$$

[1] Usually, in QED, g is called e, the electron charge, though g will be used in analogy to nonabelian gauge theories.

Thus, the free action for the vector field is:

$$S_{EM} = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} \tag{1.1.7}$$

That is also gauge invariant, i.e. invariant under (1.1.6), as $F_{\mu\nu}$ is gauge invariant.
The term that broke the local phase invariance of the action (1.1.1) can now be "absorbed" by A_μ through a gauge transformation (1.1.6), thus making the full action gauge invariant:

$$\begin{aligned} S_{QED} &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \\ &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ S_{QED} \to S'_{QED} &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi + \cancel{\overline{\psi}\not{\partial}\alpha\psi} - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \cancel{\overline{\psi}\not{\partial}\alpha\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = \\ &= \int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - g\overline{\psi}\not{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = S_{QED} \end{aligned} \tag{1.1.8}$$

1.1.3 Nonabelian Gauge Theories

Let us now consider a theory with N fermions, all with the same mass m, described by the spinorial fields $\psi_i(x)$ with $i = 1, \ldots, N$. Its free action is:

$$S_\psi[\psi_i(x), \overline{\psi}_i(x)] = \sum_{i=1}^{N}\int \mathrm{d}^4x \left(\mathrm{i}\overline{\psi}_i\not{\partial}\psi_i - m\overline{\psi}_i\psi_i\right) \tag{1.1.9}$$

From now on, the sum over i (and all other repeated latin indexes) will be omitted, unless differently specified. This action is invariant under the global transformation:

$$\begin{aligned} \psi_i(x) &\to \psi'_i(x) = U_{ij}\psi_j(x) \\ \overline{\psi}_i(x) &\to \overline{\psi}'_i(x) = \overline{\psi}_j(x)U^\dagger_{ji} \end{aligned} \tag{1.1.10}$$

if U is any (constant) $N \times N$ matrix such that $UU^\dagger = U^\dagger U = \mathbb{1} \Leftrightarrow U^\dagger = U^{-1}$, or in other words, if $U \in U(N)$. For this reason, this transformation is also called a global $U(N)$ transformation. The phase transformation (1.1.3) is the particular case where $U = e^{-\mathrm{i}\alpha} \in U(1)$, that is the only abelian (commutative) unitary group.
As $U(N) = SU(N) \otimes U(1)\ \forall N > 1$, $U \in SU(N)$ instead of $U \in U(N)$ can be imposed, and will be from now on, without loss of generality.
In an analogous way to what has been done in Section 1.1.2 , this invariance can be made local by implementing a proper covariant derivative, similar to (1.1.4). In order to do so, the infinitesimal $SU(N)$ transformation has to be considered:

$$U_{ij}(x) = \delta_{ij} + \mathrm{i}\theta^a(x)\left(T^a\right)_{ij} + O(\theta^2) \tag{1.1.11}$$

where the indices i and j run from 1 to N (as before) and the index a runs from 1 to $N^2 - 1$ (the dimension of the group $SU(N)$). The matrixes T^a are the $N^2 - 1$ generators of $\mathfrak{su}(N)$ (the Lie algebra of $SU(N)$), thus they are $N \times N$ hermitean and traceless, which obey the commutation relations:

$$\left[T^a, T^b\right] = \mathrm{i} f^{abc} T^c \tag{1.1.12}$$

where f^{abc} are called *structure constants* of $\mathfrak{su}(N)$. The normalization of these matrices can be chosen such that they obey the condition:

$$\mathrm{Tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab} \tag{1.1.13}$$

some examples are:

- $N = 2$, $T^a = \frac{\sigma^a}{2}$, with σ^a the Pauli matrices;
- $N = 3$, $T^a = \frac{\lambda^a}{2}$, with λ^a the Gell-Mann matrices.

In both these examples, the structure constants are $f^{abc} = \varepsilon^{abc}$, the completely antisymmetric Levi-Civita symbol.
The covariant derivative, therefore, is written as:

$$D_\mu \equiv \partial_\mu + \mathrm{i} g \mathbf{A}_\mu(x) \tag{1.1.14}$$

where an $N \times N$ identity matrix $\mathbb{1}$ multiplying ∂_μ has to be understood, and $\mathbf{A}_\mu(x)$ is a gauge field of $SU(N)$, i.e. a traceless, hermitean $N \times N$ matrix, or, in other words, $\mathbf{A}_\mu(x) \in \mathfrak{su}(N)$.
The covariant derivative can be written more explicitly acting on the set of spinors ψ_i:

$$\left(D_\mu\right)_{ij} \psi_j = \partial_\mu \mathbb{1}_{ij} \psi_j + \mathrm{i} g \left(\mathbf{A}_\mu(x)\right)_{ij} \psi_j$$

In order for the action to be gauge invariant, the field $\mathbf{A}_\mu$ must satisfy the gauge transformation property

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) = U(x)\mathbf{A}_\mu(x)U^\dagger(x) - \frac{\mathrm{i}}{g}U(x)\partial_\mu U^\dagger(x) \tag{1.1.15}$$

This expression is a little more complicated than (1.1.6), due to the fact that $\mathbf{A}_\mu$ is now a non-commuting matrix. However if the abelian case $U(1)$ is taken into consideration, where $U(x) = e^{-\mathrm{i}\alpha(x)}$, (1.1.6) follows directly from (1.1.15).
Now, it can be easily checked that the kinetic term of the Lagrangian

$$\mathcal{L}_K = \mathrm{i}\overline{\psi}_i \not{D} \psi_i = \mathrm{i}\overline{\psi}_i \not{\partial} \psi_i - g\overline{\psi}_i \not{\mathbf{A}} \psi_i$$

is gauge invariant (i.e. invariant under (1.1.10) and (1.1.15)) through direct computation:

$$\begin{aligned}
\mathcal{L}_K \to \mathcal{L}'_K =& \mathrm{i}\overline{\psi}_i U^\dagger \not{\partial}\left(U\psi_i\right) - g\overline{\psi}_i \underbrace{U^\dagger U}_{\mathbb{1}} \not{\mathbf{A}} \underbrace{U^\dagger U}_{\mathbb{1}} \psi_i + \mathrm{i}\overline{\psi}_i \underbrace{U^\dagger U}_{\mathbb{1}} \left(\not{\partial} U^\dagger\right) U \psi_i = \\
=& \mathrm{i}\overline{\psi}_i U^\dagger \left(\not{\partial} U\right) \psi_i + \underbrace{\mathrm{i}\overline{\psi}_i \not{\partial} \psi_i - g\overline{\psi}_i \not{\mathbf{A}} \psi_i}_{\mathcal{L}_K} + \mathrm{i}\overline{\psi}_i \left(\not{\partial} U^\dagger\right) U \psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \left(U^\dagger \partial_\mu U + \partial_\mu U^\dagger U\right) \psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \partial_\mu \left(U^\dagger U\right) \psi_i = \\
=& \mathcal{L}_K + \mathrm{i}\overline{\psi}_i \gamma^\mu \underbrace{\partial_\mu \left(\mathbb{1}\right)}_{=0} \psi_i = \mathcal{L}_K
\end{aligned}$$

Because of this fact, it is directly implied that the covariant derivative (1.1.14) must transform, under a gauge transformation, in the adjoint representation:

$$D_\mu \to D'_\mu = U D_\mu U^\dagger \tag{1.1.16}$$

The field-strength for the field $\mathbf{A}_\mu$ is obtained, as for the abelian case, through the commutator of two covariant derivatives. The computation is the same as (1.1.5), but this time the commutator term is not $= 0$:

$$F_{\mu\nu} \equiv -\frac{\mathrm{i}}{g}[D_\mu, D_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + \mathrm{i}g[\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{1.1.17}$$

This expression can be simplified a little by considering that $\mathbf{A}_\mu$ and $F_{\mu\nu}$ are elements of $\mathfrak{su}(N)$, thus writing them in terms of their components w. r. t. the basis T^a:

$$\mathbf{A}_\mu(x) = A^a_\mu(x) T^a \tag{1.1.18}$$
$$F_{\mu\nu}(x) = F^a_{\mu\nu}(x) T^a \tag{1.1.19}$$

and by considering the relation (1.1.12):

$$\begin{aligned} F^a_{\mu\nu} T^a &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) T^a + \mathrm{i}g\left[A^b_\mu T^b, A^c_\nu T^c\right] = \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) T^a + \mathrm{i}g A^b_\mu A^c_\nu \underbrace{\left[T^b, T^c\right]}_{if^{bca}T^a} = \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right) T^a \end{aligned}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{1.1.20}$$

In order to write a kinetic action for the field $\mathbf{A}_\mu$, a term proportional to $F_{\mu\nu}F^{\mu\nu}$, like in (1.1.7), is not enough: because of (1.1.16) and the definition (1.1.17), it must transform as $F_{\mu\nu}F^{\mu\nu} \to U F_{\mu\nu} F^{\mu\nu} U^\dagger$, therefore it would not be gauge invariant. In fact a gauge invariant action, called Yang-Mills action, is:

$$S_{YM} = -\frac{1}{2}\int \mathrm{d}^4x \, \mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\} \tag{1.1.21}$$

because of the ciclic property of the trace.[2] This action can be written in components, using (1.1.19) and the trace property (1.1.13):

$$S_{YM} = -\frac{1}{2}\int \mathrm{d}^4x \, \mathrm{Tr}\left\{F^a_{\mu\nu}F^{b\mu\nu}T^aT^b\right\} = -\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^{a\mu\nu} \tag{1.1.22}$$

Here, there are two remarks that need to be done. The first one is that, if the gauge group is taken to be $U(1)$, the action (1.1.22) reduces to (1.1.7), as $a = 1$ because the group $U(1)$ has only 1 generator. The second one is that, if nonabelian gauge groups are taken into consideration, this action naturally introduces self-interacting cubic and quartic terms, because the structure constants f^{abc} are non 0. This is, for example, the case for the group $SU(2)$, that is used to describe isospin, and for the group $SU(3)$, that

[2] Actually, a term proportional to $\det\left\{F^a_{\mu\nu}F^{a\mu\nu}\right\}$ would be gauge invariant as well, but it would not be a suitable kinetic term as it would involve terms of higher order than 2 in the components $F^a_{\mu\nu}$

is used to describe gluon interaction, i.e. Quantum Chromodynamics (QCD). These self-interactions make the the Yang-Mills action interesting to be studied even alone, without any other fermionic or bosonic interacting field, as it will be shown later.
Everything that has been said for $SU(N)$ can be also extended to $SO(N)$ by replacing *unitary* with *orthogonal* and *traceless* with *antisymmetric*. In fact, this discussion can be made for every compact[3] group, such as the symplectic group $Sp(2N)$ and the five exceptional Lie groups $G(2)$, $F(4)$, $E(6)$, $E(7)$ and $E(8)$.

1.2 Lattice Field Theory

[3]Compactness, i.e. $\mathrm{Tr}\{T^a T^b\}$ positive defined, is required in order to have a bounded from below Hamiltonian.

Computer Simulation of Gauge Theories

Gauge Theories Simulation on non-hypercubic lattice F4

Simulation Results

Conclusions

Bibliography

[1] Mark Srednicki. *Quantum Field Theory*. Cambridge: Cambridge University Press, 2007. ISBN: 9781139462761. URL: https://books.google.com/books?id=5OepxIG42B4C.