Dana Abu Ali and
Mark Surnin

Assignment 5.

① Rotation of a point in 3-dimensional space

(1) Rotate space about the x axis so that the rotation axis lies in the xz plane.

(2) Rotate space about the y axis so that the rotation axis lies along the z axis.

(3) Perform the desired rotation by theta about the z axis.

(4) Apply the inverse of step (2).

(5) Apply the inverse of step (1).

Step 1.

Let $\hat{u} = (u_1, u_2, u_3)$ be the unit vector along the rotation axis.
Let $d = \sqrt{u_2^2 + u_3^2}$ be the length of the projection onto the yz plane. We need to rotate the rotation axis so that it lies in the xz plane. The rotation angle is the angle between the projection of rotation axis in the yz plane and the z axis.

This can be calculated from the dot product of the z component of $\hat{u}$ – $(0, 0, u_3)$ and its yz projection – $(0, u_2, u_3)$.

$$\cos(t) = \frac{\begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix}}{u_3 \cdot d} = \frac{(u_3)^2}{u_3 \cdot d} = \frac{u_3}{d}.$$

$$\sin(t) = \frac{\|a \times b\|}{ab}$$

$$\sin(t) = \frac{\left\| \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix} \right\|}{u_3 \cdot d} = \frac{\left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & u_3 \\ 0 & u_2 & u_3 \end{matrix} \right\|}{u_3 d} = \frac{\|-u_2 u_3\|}{u_3 d} = \frac{u_2}{d}$$

Rotation matrice $R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{d} & -\frac{u_2}{d} & 0 \\ 0 & \frac{u_2}{d} & \frac{u_3}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$R_x^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{d} & \frac{u_2}{d} & 0 \\ 0 & -\frac{u_2}{d} & \frac{u_3}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2.

Rotate space around the y axis so that the rotation axis along the z axis. Similarly to (1), $\cos(t') = d$, $\sin(t') = u_1$. Therefore,

$$R_y = \begin{bmatrix} d & 0 & -u_1 & 0 \\ 0 & 1 & 0 & 0 \\ u_1 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_y^{-1} = \begin{bmatrix} d & 0 & u_1 & 0 \\ 0 & 1 & 0 & 0 \\ -u_1 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3.

Rotation about the z axis by theta is R_z.

$$R_z = \begin{bmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the complete transformation is:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = R_x^{-1} R_y^{-1} R_z R_y R_x \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

$$R_x^{-1} \cdot R_y^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{d} & \frac{u_2}{d} & 0 \\ 0 & -\frac{u_2}{d} & \frac{u_3}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d & 0 & u_1 & 0 \\ 0 & 1 & 0 & 0 \\ -u_1 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} d & 0 & u_1 & 0 \\ -\frac{u_1 \cdot u_2}{d} & \frac{u_3}{d} & u_2 & 0 \\ -\frac{u_1 \cdot u_3}{d} & -\frac{u_2}{d} & u_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cancel{d^2\cos\theta + u_1^2 \quad -u_1u_2\cos\theta - u_3\cdot\sin\theta + u_1u_2 \quad -u_1u_3\cos\theta + u_2\sin\theta + u_1 + u_3 \quad 0}$$

$$(R_x^{-1}\cdot R_y^{-1})\cdot R_z = \begin{bmatrix} d & 0 & u_1 & 0 \\ -\frac{u_1u_2}{d} & \frac{u_3}{d} & u_2 & 0 \\ -\frac{u_1u_3}{d} & -\frac{u_2}{d} & u_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} d\cos\theta & -d\sin\theta & u_1 & 0 \\ \frac{u_3\sin\theta}{d} - \frac{u_1u_2\cos\theta}{d} & \frac{u_3\cos\theta}{d} + \frac{u_1u_2\sin\theta}{d} & u_2 & 0 \\ -\frac{u_1u_3\cos\theta}{d} - \frac{u_2\sin\theta}{d} & \frac{u_1u_3\sin\theta}{d} - \frac{u_2\cos\theta}{d} & u_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y\cdot R_x = \begin{bmatrix} d & 0 & -u_1 & 0 \\ 0 & 1 & 0 & 0 \\ u_1 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{d} & -\frac{u_2}{d} & 0 \\ 0 & \frac{u_2}{d} & \frac{u_3}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} d & \frac{-u_1u_2}{d} & -\frac{u_1u_3}{d} & 0 \\ 0 & \frac{u_3}{d} & -\frac{u_2}{d} & 0 \\ u_1 & u_2 & u_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{R_x^{-1}\cdot R_y^{-1}\cdot R_z}\cdot\underbrace{R_y\cdot R_x} = M$$

$$M_{11} = d^2\cos\theta + u_1^2 = (u_2^2+u_3^2)\cos\theta + u_1^2$$

$$M_{12} = -u_1u_2\cos\theta - u_3\sin\theta + u_1u_2 = u_1u_2(1-\cos\theta) - u_3\sin\theta$$

$$M_{13} = -u_1u_3\cos\theta + u_2\sin\theta + u_1u_3 = u_1u_3(1-\cos\theta) + u_2\sin\theta$$

$$M_{21} = u_3\sin\theta - u_1u_2\cos\theta + u_1u_2 = u_1u_2(1-\cos\theta) + u_3\sin\theta$$

$$M_{22} = \frac{(u_3-u_1u_2)(u_3\cos\theta + u_1u_2\sin\theta)}{d^2} + u_2^2 = \frac{(u_3-u_1u_2)(u_3\cos\theta + u_1u_2\sin\theta)}{u_2^2+u_3^2} + u_2^2$$

$$M_{23} = \frac{-u_1\sin\theta(u_3^2+u_2^2) + u_2u_3\cos\theta(u_1^2-1)}{d^2} + u_2u_3 = \frac{-u_1\sin\theta(u_3^2+u_2^2) + u_2u_3\cos\theta(u_1^2-1)}{u_2^2+u_3^2} + u_2u_3$$

$$M_{31} = -u_1u_3\cos\theta - u_2\sin\theta + u_1u_3$$

$$M_{32} = \frac{u_2u_3\cos\theta(u_1^2-1) + u_1\sin\theta(u_3^2-u_2^2)}{d^2} + u_3u_2 = \frac{u_2u_3\cos\theta(u_1^2-1) + u_1\sin\theta(u_3^2-u_2^2)}{u_2^2+u_3^2} + u_3u_2$$

$$M_{33} = \frac{-2u_1u_2u_3\sin\theta + \cos\theta(u_2^2+u_1^2u_3^2)}{d^2} + u_3^2 = \frac{-2u_1u_2u_3\sin\theta + \cos\theta(u_2^2+u_1^2u_3^2)}{u_2^2+u_3^2}$$

We can omit the 4th row and the 4th column (homogeneous matrtices are only used when dealing with transformations).

3. Prove that in any rotation matrix, the column vectors are of unit length and are pairwise orthogonal.

Any rotation is a **rigid** transformation that preserves the lengths of the vectors. Thus, a rotation cannot alter the shape or the volume of an object, it can only change the object's location and orientation.

Therefore, we need to prove that the standard basis vectors are of unit length and pairwise orthogonal.

The standard basis vectors are: (1,0,0), (0,1,0), (0,0,1). Their lengths are sqrt(1 + 0 + 0) = 1. To prove that they are pairwise orthogonal, let us compute the dot products of each pair of vectors.

(1,0,0) * (0,1,0) = 0,
(1,0,0) * (0,0,1) = 0,
(0,1,0) * (0,0,1) = 0.

Thus, the standard basis vectors are of unit length and pairwise orthogonal. Any rotation does not alter either of these properties. Therefore, in any rotation matrix, the column vectors are of unit length and are pairwise orthogonal.