Assignment 4

① Let a, b, c be points in 3 dimensions with coordinates (1,2,3), (1,3,2) and (2,-1,3).

1) Equation of the plane h:

$\vec{AB} = (0, 1, -1)$; $\vec{AC} = (1, -3, 0)$.

To find the normal to the plane,

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & -3 & 0 \end{vmatrix} = -3\vec{i} - \vec{j} - \vec{k} = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$$

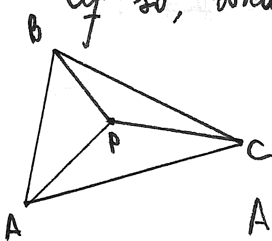
The equation of the plane is, then:

$$-3(x-1) - 1(y-2) - 1(z-3) = 0$$

$$-3x + 3 - y + 2 - z + 3 = 0$$

$$-3x - y - z = -8.$$

2) Does the point $\left(\frac{7}{6}, \frac{11}{6}, \frac{16}{6}\right)$ lie in the triangle abc? If so, what are the barycentric coordinates w.r.t a, b, and c?



$$\alpha = \frac{A_{PCB}}{A_{ABC}}, \quad \beta = \frac{A_{PCA}}{A_{ABC}}, \quad \gamma = 1 - \alpha - \beta.$$

$$A_{ABC} = \vec{n} \cdot (\vec{AB} \times \vec{AC}) = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} = 9 + 1 + 1 = 11.$$

$$A_{PBC} = \vec{n} \cdot ((B-P) \times (C-P)) = \vec{n}\left(\left(-\frac{1}{6}, \frac{7}{6}, -\frac{4}{6}\right) \times \left(\frac{5}{6}, -\frac{17}{6}, \frac{2}{6}\right)\right) = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 3 \\ -3 \end{pmatrix} = 27 - 3 + 3 = 27.$$

$$\begin{vmatrix} i & j & k \\ -\frac{1}{6} & \frac{7}{6} & -\frac{4}{6} \\ \frac{5}{6} & -\frac{17}{6} & \frac{2}{6} \end{vmatrix} = i\left(\frac{14-68}{6}\right) - j\left(\frac{-2+20}{6}\right) + k\left(\frac{17-35}{6}\right) = -\frac{54}{6}i + \frac{18}{6}j - \frac{18}{6}k = \begin{pmatrix} -9 \\ 3 \\ -3 \end{pmatrix}.$$

$$A_{PCA} = \vec{n}\cdot\left((C-P)\times(A-P)\right) = \vec{n}\cdot\left(\left(\frac{5}{6}, -\frac{17}{6}, \frac{2}{6}\right)\times\left(-\frac{1}{6}, \frac{1}{6}, \frac{2}{6}\right)\right) = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} -6 \\ -2 \\ -2 \end{pmatrix} = 18+2+2 = 22.$$

$$\begin{vmatrix} i & j & k \\ \frac{5}{6} & -\frac{17}{6} & \frac{2}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{2}{6} \end{vmatrix} = i\left(\frac{-34-2}{6}\right) - j\left(\frac{10+2}{6}\right) + k\left(\frac{5-17}{6}\right) = -6i - 2j - 2k = \begin{pmatrix} -6 \\ -2 \\ -2 \end{pmatrix}.$$

$A_{ABC} = 11$; $A_{PCB} = 27$, $A_{PCA} = 22$.

$\alpha = \frac{A_{PCB}}{A_{ABC}} = \frac{27}{11}$; $\beta = \frac{A_{PCA}}{ABC} = \frac{22}{11} = 2$; $\gamma = 1-\alpha-\beta = 1 - \frac{27}{11} = -\frac{38}{11}$.

3) Consider the ray $p_0 + t\vec{v}$, $t \geq 0$, where p_0 is the point $(1,1,1)$ and $\vec{v}$ is the vector $(0,2,-1)$. Does the ray intersect the plane h? If so, at which point?

ray: $p = p_0 + t\vec{v}$, $t \geq 0$.

plane: $\cancel{p =} (p - p_1)\cdot\vec{n} = 0$.

$(p_0 + t\vec{v} - p_1)\cdot\vec{n} = 0$

$t\vec{v}\cdot\vec{n} + (p_0 - p_1)\cdot\vec{n} = 0$

$\cancel{t\vec{v}}$ $t = \frac{-(p_0 - p_1)\cdot\vec{n}}{\vec{v}\cdot\vec{n}}$.

$$t = \frac{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)}{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}} = \frac{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}}{-1} = -\frac{1-2}{-1} = -\frac{3}{-1} = 3.$$

$$p = 3\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$$

ray intersects plane h at $(1, 7, -2)$.

② ~~Write th~~ (Write the following)

Write the 3×3 homogeneous transformation matrices for the following transformations in 2 dimensions:

1) translation by 1 along positive x axis and 2 along the positive y axis, followed by scaling by 3 along x and y axes.

$$\begin{bmatrix} x' \\ y' \\ p_0' \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{scaling}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translation}} \cdot \begin{bmatrix} \cancel{3\ 0}\ 3 & x \\ & y \\ & p_0 \end{bmatrix}.$$

$$\begin{bmatrix} x' \\ y' \\ p_0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ p_0 \end{bmatrix}$$

2) clockwise rotation by 30 degrees around the point $(-1,2)$.

$Rot_{(a,b)}\,\theta = \text{Translation}\,(a,b) * Rot(\theta) \cdot \text{Translation}\,(-a,-b)$.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rot_{(1,2)}\,30^\circ = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2} - 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2} + 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{5}{2} - \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & \sqrt{3} \\ 1 & \sqrt{3} & 5-2\sqrt{3} \\ 0 & 0 & 2 \end{bmatrix}.$$

3) reflection about the line $y = x+2$.

$ax + by + c = 0$

$1x - 1y + 2 = 0$

1. The line intersects the y-axis in the point $(0, -\frac{c}{b})$.
2. Make a translation that maps $(0, -\frac{c}{b})$ to the origin.
3. The slope of the line is $\tan\theta = -\frac{a}{b}$, where the angle θ is the angle between the line and the x-axis. Rotate the line about the origin by $-\theta$. This maps the line to the x-axis.
4. Reflect about the x-axis.
5. Rotate about the origin by θ and translate by $(0, -\frac{c}{b})$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix} = \ldots =$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & \frac{2c}{b}\sin\theta\cdot\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -\frac{2c}{b}\cos^2\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\tan\theta = -\frac{a}{b}$, $\Rightarrow \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{b^2}{a^2+b^2}$.

$\sin^2\theta = 1 - \cos^2\theta = \frac{a^2}{a^2+b^2}$. $\Rightarrow \sin\theta\cos\theta = \tan\theta\cdot\cos^2\theta = \frac{-ab}{a^2+b^2}$.

$$\Rightarrow \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} & \frac{-2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2cb}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since multiplication by a factor does not change the ~~coordinate~~ point in homogeneous (equations) coordinates.

$$Reflect_{(a,b,c)} = \begin{bmatrix} b^2-a^2 & -2ab & -2ac \\ -2ab & a^2-b^2 & -2cb \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

$a=1,\ b=-1,\ c=2.$

$$\mathrm{Ref}_{(1,-1,2)} = \begin{bmatrix} 0 & 2 & -4 \\ 2 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$