Support Vector Machine (SVM)

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NYU / CUSP - GX 5006

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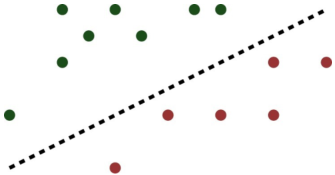
Linearly Separable Data

Suppose a given data $\{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\}$, $\mathbf{x}_i \in \mathbb{R}^d$, $y_i \in \{-1, +1\}$ is linearly separable.



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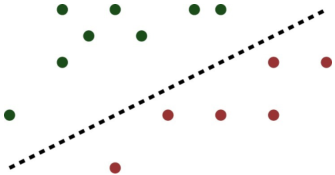

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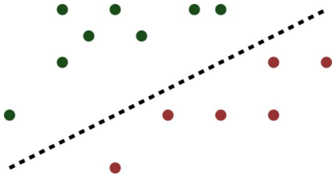

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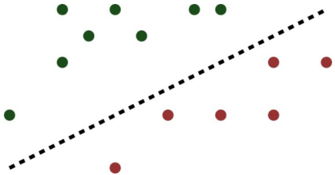

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The hyperplane is not unique, there a multitude of parameters $\mathbf{w}$ and b from which $\mathbf{w}^\top \mathbf{x} + b = 0$ is a separating hyperplane.

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Let $\mathbf{x}_j^+$ and $\mathbf{x}_j^-$ be the data points from classes $+1$ and -1 that are closer to a given separating hyperplane.
We can tune $\mathbf{w}$ and b such that

$$\mathbf{w}^\top \mathbf{x}_j^+ + b = 1 \quad \mathbf{w}^\top \mathbf{x}_j^- + b = -1$$

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Equations above can be written as:

$$y_j(\mathbf{w}^\top \mathbf{x}_j + b) = 1$$

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Therefore, the sought hyperplane should maximize those distances, which is equivalent to minimizing $\|\mathbf{w}\|^2$.

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$$\min \mathbf{w}^\top \mathbf{w} \qquad \textit{subject to} \quad y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1$$

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This optimization can be handled by Lagrange multipliers

$$L(\mathbf{w}, b, \boldsymbol{\alpha}) = \mathbf{w}^\top \mathbf{w} - \sum_{i=1}^{n} \alpha_i \left(y_i(\mathbf{w}^\top \mathbf{x}_i + b) - 1 \right)$$

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The dual form of the optimization problem is given by

$$L_D(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^\top \mathbf{x_j}$$

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only the α_i corresponding to the *support vectors* are not zero.

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New data points can be classified by analyzing the sign of

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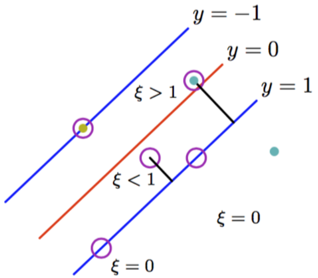
We can use slack variables.

$$y_i(\mathbf{w}^\top \mathbf{x} + b) \geq 1 - \boxed{\epsilon_i}, \quad \epsilon_i \geq 0$$



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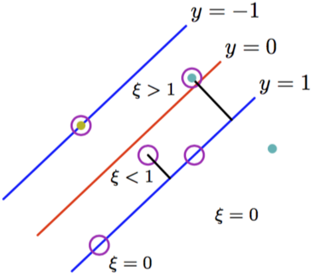

Using Lagrange multipliers, the optimization becomes:

$$L(\mathbf{w}, b, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{w}^\top \mathbf{w} - \sum_{i=1}^{n} \alpha_i \left(y_i(\mathbf{w}^\top \mathbf{x}_i + b) - 1 + \epsilon_i \right) + C \sum_{i=1}^{n} \epsilon_i - \sum_{i=1}^{n} \epsilon_i \beta_i$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are Lagrange multipliers and C controls the trade-off between slack variables and the margin.

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Interestingly, the dual formulation is identical to the separable case, with a change only on the constraints

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$\alpha_i = C \longrightarrow$ points inside the margin (correctly or misclassified).

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- Why so much sophistication to derived a classifier with linear decision boundary?

The answer for both questions comes from the concept of kernels.

Kernels

Let $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a function that assigns to any pair of points $\mathbf{x}_i, \mathbf{x}_j$ a scalar value $\kappa(\mathbf{x}_i, \mathbf{x}_j)$ (a measure of similarity between those points).

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The function κ is called a kernel if there is a mapping $\phi : \mathbb{R}^d \to \mathcal{H}$ such that:

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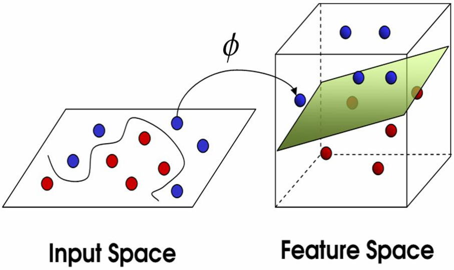
Typically, ϕ and $\mathcal{H}$ are implicitly defined from the kernel.



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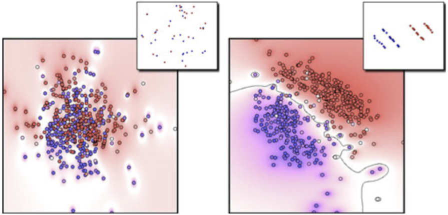

(a) (b)

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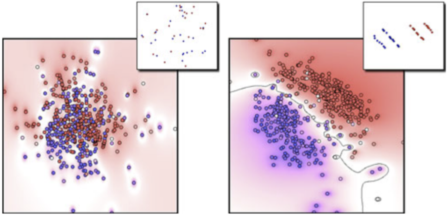
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Examples:

$\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^\top \mathbf{x}_j$ (Identity)

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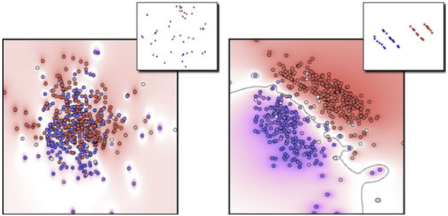

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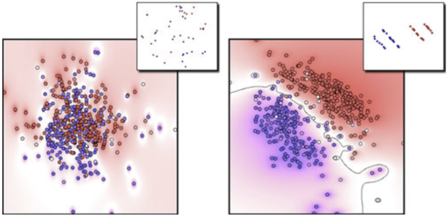

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$$\kappa(\mathbf{x}_i, \mathbf{x}_j) = \exp(-\gamma \|\mathbf{x}_i - \mathbf{x}_j\|^2), \quad \gamma > 0 \text{ (RBF)}$$

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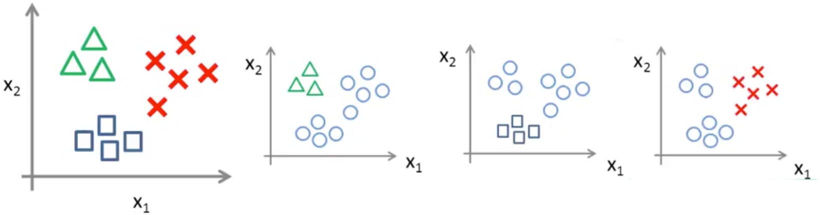

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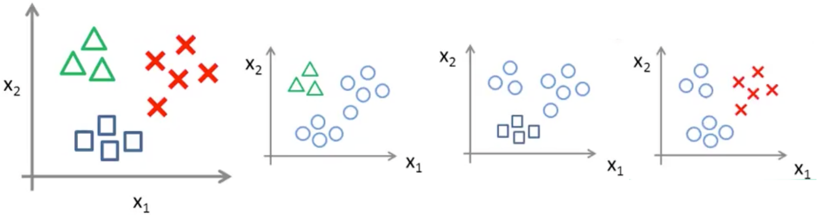
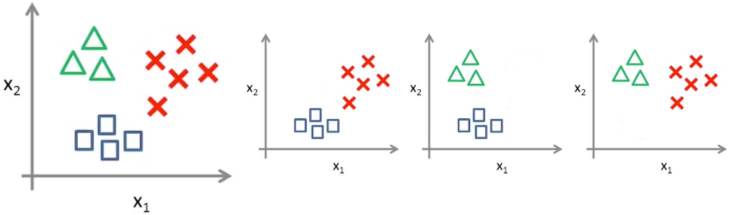


- One-vs-One: $k(k-1)/2$ classifiers $\boxed{y = \arg\max_i (\sum_j f_{ij}(x))}$



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- Support vectors are found based on the coefficients α_i
- The dual formulation allows for using the kernel trick
- With kernels SVM can perform non-linear classification