

# Refinement for Symbolic Trajectory Evaluation

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**Abstract.** Model refinement such that it preserves symbolic trajectory evaluations.

**Keywords:** STE · Refinement · ?

## 1 Symbolic Trajectory Evaluation

*Symbolic trajectory evaluation* [5] (STE) is a high-performance model checking technique based on *symbolic simulation* extended with a temporal *next-time* operator to describe circuit behaviour over time. In its simplest form, STE tests the validity of an *assertion* of the form  $A \Rightarrow C$ , where both the *antecedent*  $A$  and *consequent*  $C$  are formulas in the following logic:

$$f ::= p \mid f \wedge f \mid P \rightarrow f \mid \mathbf{N} f$$

Here,  $p$  is a simple predicate over “values” in a circuit and  $P$  is a Boolean propositional formula, and the operators  $\wedge$ ,  $\rightarrow$  and  $\mathbf{N}$  are conjunction, domain restriction and the next-time operator, respectively.

If the circuit contains Boolean signals,  $p$  is typically drawn from the following two predicates:  $n \text{ is } 1$  and  $n \text{ is } 0$ , where  $n$  ranges over the signals (or nodes) in a circuit. For example, suppose we have a unit-delayed, two-input AND-gate, then it is reasonable to assume that the assertion  $(in_1 \text{ is } 1 \wedge in_2 \text{ is } 1) \Rightarrow \mathbf{N}(out \text{ is } 1)$  is true. Indeed, STE efficiently validates such statements for us.

While the truth semantics of an assertion in STE is defined as the satisfaction of its “defining” trajectory (bounded sequence of states) relative to a model structure of the circuit, what the STE algorithm computes is exactly the solution of a data-flow equation [1] in the classic format [4]. . . .

### 1.1 Set-theoretic STE

Consider an arbitrary, but fixed, digital circuit  $M$  operating in discrete time. A *configuration* of  $M$ , denoted by  $\mathbb{C}$ , is non-empty and finite set that represents a snapshot of  $M$  at a discrete point in time. If the circuit  $M$  has  $m$  boolean signals, then its set of configurations is typically represented as a sequence  $\mathbb{B}^m$ , where  $\mathbb{B} = \{0, 1\}$  is the set of boolean values.

**Circuit Model** A simple conceptual model of  $M$  is a *transition relation*,  $M_R \subseteq \mathbb{C} \times \mathbb{C}$ , where  $(c, c') \in M_R$  means that  $M$  can move from  $c$  to  $c'$  in one step<sup>1</sup>. The power set of  $\mathbb{C}$ , denoted by  $\wp(\mathbb{C})$ , can be viewed as a the set of *predicates* on configurations, where  $\cap$ ,  $\cup$ , and  $\subseteq$  correspond to conjunction, disjunction and implication, respectively. We denote by  $\cap S$  and  $\cup S$  the intersection and union of all members of any  $S \subseteq \wp(\mathbb{C})$ .

$M_R$  induces a *predicate transformer*  $M_F \in \wp(\mathbb{C}) \rightarrow \wp(\mathbb{C})$  using the relational image operation:

$$M_F(C) = \{c' \in \mathbb{C} \mid \exists c \in C : (c, c') \in M_R\}$$

It is intuitively obvious that if  $M$  is in one of the configurations in  $C \in \wp(\mathbb{C})$ , then in one time step it must be in one of the configurations in  $M_F(p)$ . We also see that  $M_F$  distributes over arbitrary unions:

$$M_F(\cup S) = \cup \{M_F(C) \mid C \in S\}$$

for all  $S \subseteq \wp(\mathbb{C})$ . In general, any  $M_F$  that satisfies this distributive property also defines a  $M_R$  through the equivalence  $(c, c') \in M_R \Leftrightarrow c' \in M_F(\{c\})$ , that is to say, there is no loss of information going from  $M_R$  to  $M_F$  or vice versa. We adopt this functional model of  $M$  and drop its subscript.

Exactly what  $\mathbb{C}$  and its signals are, is not important in this section. In practice, however, signals are typically divided into external, such as “inputs” and “outputs”, and internal parts. While an input signal is generally controlled by the external environment, and thus unconstrained by  $M$  itself, non-input signals are determined by the circuit topology and functionality. For example, supposed  $M$  is the earlier example of a unit-delayed two-input AND gate, we could then define its model  $M \in \wp(\mathbb{B}^3) \rightarrow \wp(\mathbb{B}^3)$  as follows:

$$M(C) = \{\langle b_1, b_2, i_1 \wedge i_2 \rangle \in \mathbb{B}^3 \mid \langle i_1, i_2, o \rangle \in C\}$$

Here  $i_1$  and  $i_2$  refer to the two inputs of the AND gate,  $o$  the ignored output, and  $b_1$  and  $b_2$  are unconstrained inputs for the new configurations.

**Assertions and satisfaction** A *trajectory assertion* for  $M$  is quintuple  $A = (S, s_0, R, \pi_a, \pi_c)$ , where  $S$  is a finite set of *states*,  $s_0 \in S$  is an *initial state*,  $R \subseteq S \times S$  is a *transition relation*,  $\pi_a \in S \rightarrow \wp(\mathbb{C})$  and  $\pi_c \in S \rightarrow \wp(\mathbb{C})$  label each state  $s$  with an *antecedent*  $\pi_a(s)$  and a *consequent*  $\pi_c(s)$ . We assume that  $(s, s_0) \notin R$  for all  $s \in S$  without any loss of generality.

The circuit model  $M$  intuitively *satisfies* an assertion  $A$  if, for every *trajectory*  $\tau$  through  $M$  and every *run*  $\rho$  through  $A$ ,  $\tau$  satisfying the antecedents of  $\rho$  entails that  $\tau$  also satisfies the consequents of  $\rho$ . To be specific, a *trajectory* of  $M$  is a non-empty sequences of configurations,  $\tau \in \mathbb{C}^+$ , such that  $\tau_n \in M(\{\tau_{n-1}\})$  for all  $n \in \mathbb{N} : 0 < n < |\tau|$ . And a *run* of  $A$  is a non-empty sequence of states,  $\rho \in S^+$ , such that  $\rho_0 = s_0$  and  $(\rho_{n-1}, \rho_n) \in R$  for all  $n \in \mathbb{N} : 0 < n < |\rho|$ .

<sup>1</sup> Mention how this affects circuits with zero-delays?

A  $\tau$  satisfies the antecedents of  $\rho$ , denoted by  $\tau \models_a \rho$ , iff  $\tau_n \in \pi_a(\rho_n)$  for all  $n \in \mathbb{N} : n < |\tau| = |\rho|$ ; satisfaction of consequents is defined similarly with  $\pi_c$  and denoted by  $\tau \models_c \rho$ .

That  $M$  satisfies  $A$ , denoted by  $M \models A$ , can then be formalized as follows:

$$\forall \tau \in \text{Traj}(M) : \forall \rho \in \text{Runs}(A) : |\tau| = |\rho| \Rightarrow (\tau \models_a \rho \Rightarrow \tau \models_c \rho)$$

where  $\text{Traj}(M)$  and  $\text{Runs}(A)$  denote the sets of all trajectories of  $M$  and runs of  $A$ , respectively.

**Refinement** Set-theoretic refinement should go here.

## 1.2 Lattice-theoretic STE

Manipulating subsets of  $\mathbb{B}^m$  is impractical for even moderately large  $m$ , which leads us to one of the key insights of STE. Namely, instead of manipulating subsets of  $\mathbb{B}^m$  directly, one can use sequences of ternary values  $\mathbb{T} = \mathbb{B} \cup \{X\}$  to approximate them, whose sizes are only linear in  $m$ . Here the 1 and 0 from  $\mathbb{B}$  denotes specific, defined values whereas  $X$  denotes an “unknown” value that could be either 1 or 0. This intuition induces a partial order  $\sqsubseteq$  on  $\mathbb{T}$ , where  $0 \sqsubseteq X$  and  $1 \sqsubseteq X^2$ . For any  $m \in \mathbb{N}$ , this ordering on  $\mathbb{T}$  is lifted component-wise to  $\mathbb{T}^m$ .

Note that  $\mathbb{T}^m$  does not quite form a complete lattice because it lacks a bottom: both  $0 \sqsubseteq X$  and  $1 \sqsubseteq X$  but  $0$  and  $1$  are equally defined. A special bottom element  $\perp$  is therefore introduced, such that  $\perp \sqsubseteq t$  and  $\perp \neq t$  for all  $t \in \mathbb{T}^m$ . The extended  $\mathbb{T}_\perp^m = \mathbb{T}^m \cup \{\perp\}$  then becomes a complete lattice. We denote the top element  $\langle X, \dots, X \rangle$  of  $\mathbb{T}_\perp^m$  by  $\top$ .

**Ternary lattices** Generalising from any specific domain, let  $(\hat{\mathbb{P}}, \sqsubseteq)$  be a finite, complete lattice of *abstract predicates* in which the meet  $\sqcap$  and join  $\sqcup$  of any subset  $\hat{S} \subseteq \hat{\mathbb{P}}$  exists. Similar to the previous set operations for power sets,  $\sqcap$ ,  $\sqcup$  and  $\sqsubseteq$  correspond to conjunction, disjunction and implication for abstract predicates, respectively. Furthermore, for any  $\hat{S} \subseteq \hat{\mathbb{P}}$ , we denote by  $\sqcap \hat{S}$  and  $\sqcup \hat{S}$  the meet and join of all members of  $\hat{S}$ .

Let there be a Galois connection relating “concrete” predicates  $\wp(\mathbb{C})$  and abstract predicates  $\hat{\mathbb{P}}$ . The usual definition of a Galois connection is in terms of an *abstraction*  $\alpha \in \wp(\mathbb{C}) \rightarrow \hat{\mathbb{P}}$  and a *concretisation*  $\gamma \in \hat{\mathbb{P}} \rightarrow \wp(\mathbb{C})$  function, such that  $\alpha(C) \sqsubseteq \hat{p} \Leftrightarrow C \subseteq \gamma(\hat{p})$  for all  $C \in \wp(\mathbb{C})$  and  $\hat{p} \in \hat{\mathbb{P}}$ . For example, a Galois connection from  $\wp(\mathbb{B}^m)$  to  $\mathbb{T}_\perp^m$  for any  $m \in \mathbb{N}$  can be defined in a natural way by its concretisation function  $\gamma \in \mathbb{T}_\perp^m \rightarrow \wp(\mathbb{B}^m)$ :

$$\begin{aligned} \gamma(\langle t_0, \dots, t_{m-1} \rangle) &= \{ \langle b_0, \dots, b_{m-1} \rangle \in \mathbb{B}^m \mid \forall i < m : t_i \neq X \Rightarrow b_i = t_i \} \\ \gamma(\perp) &= \emptyset \end{aligned}$$

<sup>2</sup> We use the reverse ordering of what is originally used in STE.

Listing each concrete predicate approximated by a given abstract predicate. Its abstraction function  $\alpha \in \wp(\mathbb{B}^m) \rightarrow \mathbb{T}_{\perp}^m$  instead finds the most precise abstract predicate for a set of concrete predicates:

$$\begin{aligned}\alpha(C) &= \sqcup \{ \langle t_0, \dots, t_{m-1} \rangle \in \mathbb{T}_{\perp}^m \mid \langle b_0, \dots, b_{m-1} \rangle \in C, \forall i < m : b_i = t_i \} \\ \alpha(\emptyset) &= \perp\end{aligned}$$

**Abstract circuit model** An *abstract predicate transformer*  $\hat{M} \in \hat{\mathbb{P}} \rightarrow \hat{P}$  is an *abstract interpretation* [2] of  $M \in \wp(\mathbb{C}) \rightarrow \wp(\mathbb{C})$  iff:  $\hat{M}$  preserves  $\perp$ , i.e.  $\hat{M}(\perp) = \perp$ ;  $\hat{M}$  is monotonic, i.e.  $\hat{p} \sqsubseteq \hat{q} \Rightarrow \hat{M}(\hat{p}) \sqsubseteq \hat{M}(\hat{q})$  for all  $\hat{p}, \hat{q} \in \hat{\mathbb{P}}$ ; and  $\alpha$ , or  $\gamma$ , form a *simulation relation* between  $\wp(\mathbb{C})$  and  $\hat{P}$ , i.e.  $M(\gamma(\hat{p})) \subseteq \gamma(\hat{M}(\hat{p}))$  for all  $\hat{p} \in \hat{\mathbb{P}}$ , or  $\alpha(M(C)) \sqsubseteq \hat{M}(\alpha(C))$  for all  $C \in \wp(\mathbb{C})$ .

Unlike its concrete model,  $\hat{M}$  does not distribute over arbitrary join because information is potentially discarded by the ternary logic during a join. As an example, let the following  $\hat{M}$  abstract an unit-delayed two-input AND gate:

$$\begin{aligned}\hat{M}(\langle 1, 1, \hat{p}_2 \rangle) &= \langle X, X, 1 \rangle & \hat{M}(\langle 0, 0, \hat{p}_2 \rangle) &= \langle X, X, 0 \rangle \\ \hat{M}(\langle 0, X, \hat{p}_2 \rangle) &= \langle X, X, 0 \rangle & \hat{M}(\langle X, 0, \hat{p}_2 \rangle) &= \langle X, X, X \rangle \\ \hat{M}(\langle \hat{p}_0, \hat{p}_1, \hat{p}_2 \rangle) &= \langle X, X, X \rangle\end{aligned}$$

where the last, most general matching is overlapped by the more concrete ones. If we apply  $\hat{M}$  to the join of  $\langle 0, 1, X \rangle$  and  $\langle 1, 0, X \rangle$ , or if we apply  $\hat{M}$  to them individually and then join, we get two different results:

$$\begin{aligned}\hat{M}(\langle 0, 1, X \rangle \sqcup \langle 1, 0, X \rangle) &= \hat{M}(\langle X, X, X \rangle) = \langle X, X, X \rangle \\ \hat{M}(\langle 0, 1, X \rangle) \sqcup \hat{M}(\langle 1, 0, X \rangle) &= \langle X, X, 0 \rangle \sqcup \langle X, X, 0 \rangle = \langle X, X, 0 \rangle\end{aligned}$$

The inequality  $\sqcup \{ \hat{M}(\hat{p}) \mid \hat{p} \in \hat{S} \} \sqsubseteq \hat{M}(\sqcup \hat{S})$  for all  $\hat{S} \sqsubseteq \hat{P}$  does however hold, since it is implied by the monotonicity of  $\hat{M}$ .

**Assertions and satisfaction** A trajectory assertion for an abstract model  $\hat{M}$  is a quintuple  $\hat{A} = (S, s_0, R, \hat{\pi}_a, \hat{\pi}_c)$ , where  $S$ ,  $s_0$ , and  $R$  are as in section 1.1 and  $\hat{\pi}_a \in S \rightarrow \hat{\mathbb{P}}$  and  $\hat{\pi}_c \in S \rightarrow \hat{\mathbb{P}}$  label each state  $s$  with an abstract predicate for its antecedent and consequent, respectively.

**Here follows the definition in [1].** For all functions  $\hat{\Phi} \in S \rightarrow \hat{\mathbb{P}}$  and states  $s \in S$ , define  $\hat{F} \in S \rightarrow (\hat{\mathbb{P}} \rightarrow \hat{\mathbb{P}})$  and  $\hat{\mathcal{F}} \in (S \rightarrow \hat{\mathbb{P}}) \rightarrow (S \rightarrow \hat{\mathbb{P}})$  as follows:

$$\hat{F}(s)(\hat{p}) = \hat{M}(\pi_a(s) \sqcap \hat{p}) \tag{1}$$

$$\hat{\mathcal{F}}(\hat{\Phi})(s) = \text{if } (s = s_0) \text{ then } \top \text{ else } \sqcup \{ \hat{F}(s')(\hat{\Phi}(s')) \mid (s', s) \in R \} \tag{2}$$

$\hat{F}$  preserves  $\perp$ , and both  $\hat{F}$  and  $\hat{\mathcal{F}}$  are monotonic; two  $\hat{\Phi}, \hat{\Phi}' \in S \rightarrow \hat{\mathbb{P}}$  are ordered as  $\hat{\Phi} \sqsubseteq \hat{\Phi}' \Leftrightarrow \forall s \in S : \hat{\Phi}(s) \sqsubseteq \hat{\Phi}'(s)$ . Let  $\hat{\Phi}_* \in S \rightarrow \hat{\mathbb{P}}$  be the least fixpoint of the

equation  $\hat{\Phi} = \hat{\mathcal{F}}(\hat{\Phi})$  [3]. Since both  $S$  and  $\hat{\mathbb{P}}$  are finite,  $\hat{\Phi}_*$  is given by  $\lim \hat{\Phi}_n(s)$ , where  $\hat{\Phi}_n$  is defined as follows:

$$\hat{\Phi}_n = \text{if } (n = 0) \text{ then } (\lambda s \in S : \perp) \text{ else } \hat{\mathcal{F}}(\hat{\Phi}_{n-1}) \quad (3)$$

We say that  $\hat{M}$  satisfies a trajectory assertion<sup>3</sup>  $\hat{A}$ , denoted by  $\hat{M} \models_{\text{lat}} \hat{A}$ , iff  $\hat{\Phi}_*(s) \sqcap \pi_\alpha(s) \subseteq \pi_c(s)$  for all  $s \in S$ .

**Refinement** Lattice-theoretic refinement should go here.

## 2 Refinement

### 2.1 Set-Theoretic refinement

Consider another fixed, but arbitrary, circuit model  $N \in \wp(\mathbb{D}) \rightarrow \wp(\mathbb{D})$ , where  $\mathbb{D}$  is a non-empty and finite set of configurations. Further, let there be a Galois connection between predicates  $\wp(\mathbb{C})$  and  $\wp(\mathbb{D})$  ordered by set inclusion and given by means of a binary relation  $\ll \subseteq \wp(\mathbb{C}) \times \wp(\mathbb{D})$ , where  $C \ll D$  reads “C can be approximated by D”. We define  $\ll$  from either of the usual functions for abstraction  $\alpha \in \wp(\mathbb{C}) \rightarrow \wp(\mathbb{D})$  and concretisation  $\gamma \in \wp(\mathbb{D}) \rightarrow \wp(\mathbb{C})$ :

$$C \ll D \Leftrightarrow \alpha(C) \subseteq D \quad C \ll D \Leftrightarrow C \subseteq \gamma(D)$$

Here  $\subseteq$  on the  $\alpha$ -derivation side is the inclusion order of  $\wp(\mathbb{D})$ , and on the  $\gamma$ -derivation side  $\subseteq$  is the inclusion order of  $\wp(\mathbb{C})$ . Intuitively, the relation  $\ll$  acts as an extension of the orderings inside  $\wp(\mathbb{C})$  and  $\wp(\mathbb{D})$  to one between them.

Exactly what configurations such as  $\mathbb{C}$  and  $\mathbb{D}$  are, were not important previously. But to reason about refinement, which relates the external behaviour of circuits, we make a distinction between the elements of a configuration. Let  $\sim \subseteq \mathbb{C} \times \mathbb{C}$  be an equivalence relation on  $\mathbb{C}$ . The equivalence class of a  $c \in \mathbb{C}$  under  $\sim$ , denoted by  $[c]$ , is defined as  $[c] = \{c' \in \mathbb{C} \mid c' \sim c\}$ . With a slight abuse of notation, we extend  $[\cdot]$  to sequences component-wise and overload both  $\sim$  and  $[\cdot]$  to accept configurations in  $\mathbb{D}$ .

**Example of how equiv. classes are predicates.**

**Equiv. classes to be covered:**  $\forall P \in \mathbb{D}/\sim : \exists Q \in \mathbb{C}/\sim : Q \ll P$ .

We can now formalize that  $M$  refines  $N$ , denoted by  $M \leq N$ , as follows:

$$\forall \tau \in \text{Traj}(M) : \exists v \in \text{Traj}(N) : |\tau| = |v| \wedge [\tau] \ll [v]$$

where  $[\tau] \ll [v]$  iff  $[\tau_n] \ll [v_n]$  for all  $n \in \mathbb{N} : n < |\tau| = |v|$ . In other words, for every sequence of configurations  $\tau$  permitted by  $M$ , there must exist a sequence  $v$  for  $N$  which approximates the visual behaviour of  $\tau$ .

Recall that a trajectory assertion for  $N$  is a quintuple  $A = (S, s_0, R, \pi_a, \pi_c)$ , where  $\pi_a \in S \rightarrow \wp(\mathbb{D})$  and  $\pi_c \in S \rightarrow \wp(\mathbb{D})$  label each  $s \in S$  with its antecedents

<sup>3</sup> That  $\hat{M}$  satisfies  $\hat{A}$  implies that a concretisation of  $\hat{A}$  can also be satisfied by the original, set-based model  $M$  [1].

and consequents, respectively. If  $\pi_a$  and  $\pi_c$  accept partitions of  $\mathbb{D}/\sim$ , such that  $d \in \pi_a(s) \Leftrightarrow [d] \subseteq \pi_a(s)$  and  $d \in \pi_c(s) \Leftrightarrow [d] \subseteq \pi_c(s)$  for all  $s \in S$ , then we refer to  $A$  as an *name trajectory assertion* and suffix it as  $A_n$ . Furthermore, we define  $\gamma(A) = (S, s_0, R, \gamma(\pi_a), \gamma(\pi_c))$ , where  $\gamma(\pi_a) = \lambda s \in S : \gamma(\pi_a(s))$  and  $\gamma(\pi_c) = \lambda s \in S : \gamma(\pi_c(s))$ .

**Theorem 1.**  $M \leq N \Rightarrow (N \models A_n \Rightarrow M \models \gamma(A_n))$

The above definition of refinement can be equivalently formulated as a simulation relation. We say that  $M$  refines  $N$  by *set-theoretic refinement*, denoted by  $M \leq_{\text{set}} N$ , iff (1)  $\mathbb{C}$  is approximated by  $\mathbb{D}$ , i.e.  $\{\mathbb{C}\} \ll \{\mathbb{D}\}$ ; (2)  $\ll$  is a *name*, i.e.  $C \ll D \Rightarrow \cup\{[c] \in \wp(\mathbb{C}) \mid c \in C\} \ll \cup\{[d] \in \wp(\mathbb{D}) \mid d \in D\}$ ; and  $\ll$  is a *simulation relation* from  $\wp(\mathbb{C})$  to  $\wp(\mathbb{D})$ , i.e.  $C \ll D \Rightarrow M(C) \ll N(D)$ .

**Theorem 2.**  $M \leq N \Leftrightarrow M \leq_{\text{set}} N$

## A Appendices

### A.1 Theorem 1

We first prove a few lemmas.

**Lemma 1.**  $v \in \pi_c(\rho) \wedge [\tau] \ll [v] \Rightarrow \tau \in \gamma(\pi_c(\rho))$

By property *name* of  $\pi_c$ ,  $v \in \pi_c(\rho) \Rightarrow [v] \subseteq \pi_c(\rho)$ . By the monotonicity of  $\gamma$ ,  $[v] \subseteq \pi_c(\rho) \Rightarrow \gamma([v]) \subseteq \gamma(\pi_c(\rho))$ . By definition,  $[\tau] \ll [v] \Rightarrow [\tau] \subseteq \gamma([v])$ , thus  $[\tau] \subseteq \gamma(\pi_c(\rho))$ . ■

**Lemma 2.**  $\tau \in \gamma(\pi_a(\rho)) \wedge [\tau] \ll [v] \Rightarrow v \in \pi_a(\rho)$

By property *name* of  $\pi_a$  and the definition of  $\gamma(\pi_a)$ ,  $\tau \in \gamma(\pi_a(\rho)) \Rightarrow [\tau] \subseteq \gamma(\pi_a(\rho))$ . And by the monotonicity of  $\alpha$ ,  $\alpha(\tau) \subseteq \alpha(\gamma(\pi_a(\rho))) \subseteq \pi_a(\rho)$ . By definition,  $[\tau] \ll [v] \Rightarrow \alpha([\tau]) \subseteq [v]$ . Because  $\forall x \in \alpha([\tau]) : [x] = [v]$ , property *name* of  $\pi_a$  and implies that  $[v] \subseteq \pi_a(\rho) \Rightarrow v \in \pi_a(\rho)$ . ■

Given a  $\tau \in \text{Traj}(M)$  and  $\rho \in \text{Runs}(\gamma(A))$ , such that  $|\tau| = |\rho|$  and  $\tau_n \in \gamma(\pi_a(\rho_n))$  for all  $n \in \mathbb{N} : n < |\tau|$ , we must show that  $\tau_n \in \gamma(\pi_c(\rho_n))$ . By the refinement assumption, there must exist a  $v \in \text{Traj}(N)$  such that  $|\tau| = |v| = |\rho|$  and  $[\tau_n] \ll [v_n]$ . By lemma 2,  $v_n \in \pi_a(\rho_n)$ , and by the assumption,  $v_n \in \pi_c(\rho_n)$ . Finally, lemma 1 then states that  $\tau_n \in \gamma(\pi_c(\rho_n))$ . ■

### A.2 Theorem 2

We prove each direction separately.

( $\Rightarrow$ ) : Text.

( $\Leftarrow$ ) : Given some  $\tau \in \text{Traj}(M)$ , we must show there exists a  $v \in \text{Traj}(N)$  such that  $|\tau| = |v|$  and  $[\tau] = [v]$ . We do so by induction on  $|\tau|$ . For the base case  $|\tau| = 1$ , it must be that  $\tau = \langle C \rangle$ . The only choice for  $v$  is  $\langle D \rangle$ . By property 1

of  $\ll$ , we know that  $\{C\} \ll \{D\}$ , which implies that  $\cup[C] = [C] \ll \cap[D] = [D]$  by property 2. For the inductive step  $|\tau| = n + 1$ , we know  $\tau = \tau' \frown \langle \tau_{n+1} \rangle$  such that  $|\tau'| = n$  and  $\tau_{n+1} \in M(\{\tau'_n\})$ . Assume there exists a  $v$  such that  $|\tau'| = |v|$  and  $[\tau'] \ll [v]$ . We must find a  $d \in \mathbb{D}$  such that  $d \in N(\{v_n\})$  and  $[\tau_{n+1}] \ll [d]$ . First,  $[\tau'_n] \ll [v_n] \Rightarrow M([\tau'_n]) \ll N([v_n])$  by property 3 of  $\ll$ , and then  $\cup[M(\tau'_n)] \ll \cap[N(v_n)]$  and by property 2. By definition of  $\cup$  and  $\cap$ , we know this is equivalent to  $\forall c \in M(\tau'_n) : \forall d \in N(v_n) : c \ll d$ . Since  $M$  is monotone and  $\sim$  reflexive, we know that  $M(\{\tau'_n\}) \subseteq M([\tau'_n])$ . Thus  $\tau_{n+1} \in M([\tau'_n])$  and  $\dots$

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