NOTES ON THE MAIN CONJECTURE

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1. Preamble

These are working notes connected very broadly with the attempt to prove the Main Conjecture.

2. Towers, Forms, and Stability Groups

Let D be a square-free integer greater than 1, assumed fixed throughout this discussion. Let $K = \mathbb{Q}(\sqrt{D})$, and let

$$\Delta_0 = \begin{cases} D & D = 1 \text{ (mod } 4) \\ 4D & D \neq 1 \text{ (mod } 4) \end{cases} \tag{2.1}$$

be the discriminant of K. We will call Δ_0 the fundamental discriminant. For each positive integer f let Δ_f be the discriminant $f^2\Delta_0$, and let O_f be the order

$$\mathbb{Z}\left[f\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right)\right] \tag{2.2}$$

The set of discriminants Δ_f and orders O_f are thus in bijective correspondence. It will be convenient to have names for these sets:

$$\mathcal{D} = \{\Delta_f \colon f \in \mathbb{Z}, f \geq 1\}, \tag{2.3}$$

$$\mathcal{O} = \{O_f \colon f \in \mathbb{Z}, f \geq 1\}. \tag{2.4}$$

STF: This notation hides the implicit dependence on D (or K). Not sure if we want that baggage around, but maybe this is worth commenting? For each positive integer f, let $\mathcal{Q}_f$ be the set of primitive quadratic forms over $\mathbb{Z}$ with discriminant $f^2\Delta_0$, and let

$$\mathcal{Q} = \bigcup_{f \in \mathbb{N}} \mathcal{Q}_f. \tag{2.5}$$

I will typically identify a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with the corresponding matrix

$$Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}. \tag{2.6}$$

Sometimes I will want to consider non-primitive quadratic forms, or quadratic forms over $\mathbb{Q}$ or even $\mathbb{R}$, but in that case this will always be made explicit. In the absence of such an explicit statement it should be assumed that in the above expression a, b, c are integers whose GCD is 1.

We will denote by $d_1, d_2, \ldots$ the dimension tower [2] associated to D, which we will define in the next section. Finally, for each $Q \in \mathcal{Q}$ let $\mathcal{P}_Q$ (respectively $\mathcal{G}_Q$) be its stabilizer group in $\mathrm{GL}(2, \mathbb{Z})$ (respectively $\Gamma(d)$). **STF:** Define $\Gamma(d)$? Even if only in words? The purpose of this section is to describe the relationships between these objects. We will start by discussing the dimension tower in a bit more detail than is done in Ref. [2].

2.1. The dimension tower associated to *K*.

Let $v_1 \in \mathbb{Z}_K$ be the (unique) fundamental unit greater than 1 and define

$$u_1 = \begin{cases} v_1 & N(v_1) = 1 \\ v_1^2 & N(v_1) = -1 \end{cases} \tag{2.7}$$

and, for $r \geq 1$,

$$d_r = 1 + u_1^r + u_1^{-r} \tag{2.8}$$

$$\Delta_r = (d_r - 1)^2 - 4 \tag{2.9}$$

Proposition 2.1. *For all r*

$$\Delta_r = \begin{cases} 0 \ (\textit{mod } 4), & \textit{if } d_r \textit{ is odd,} \\ 1 \ (\textit{mod } 4), & \textit{if } d_r \textit{ is even.} \end{cases} \tag{2.10}$$

Also

$$\Delta_r = f_r^2 \Delta_0, \tag{2.11}$$

$$u_1^r = \frac{d_r - 1 + f_r\sqrt{\Delta_0}}{2}, \tag{2.12}$$

for some positive integer f_r.

Proof. The first statement is an immediate consequence of the definition. To prove the second and third statement, observe that

$$u_1^r = e_r + f_r\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.13}$$

for some $e_r, f_r \in \mathbb{Z}$. Let σ be the non-trivial automorphism of $K/\mathbb{Q}$. Then the fact that $u_1^r > 1 > u_1^{-r} = \sigma(u_1^r)$ means

$$e_r + f_r\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) > e_r + f_r\left(\frac{\Delta_0 - \sqrt{\Delta_0}}{2}\right) \tag{2.14}$$

implying $f_r > 0$. Using

$$\begin{aligned} \Delta_r &= \left(u_1^r + u_1^{-r}\right)^2 - 4 \\ &= \left(u_1^r - u_1^{-r}\right)^2 \\ &= f_r^2 \Delta_0 \end{aligned} \tag{2.15}$$

and

$$d_r - 1 = u_1^r + u_1^{-r} = 2e_r + f_r\Delta_0 \tag{2.16}$$

we deduce

$$d_r - 1 + \sqrt{\Delta_r} = 2e_r + f_r\left(\Delta_0 + \sqrt{\Delta_0}\right) = 2u_1^r. \tag{2.17}$$

□

Proposition 2.2. *Let d_r, Δ_r, f_r be as defined in Eqs.* (2.8), (2.9), *and Proposition 2.1 respectively. Then for all r*

$$4 \leq d_1 < d_2 < \cdots < d_r < \ldots, \tag{2.18}$$

$$5 \leq \Delta_0 \leq \Delta_1 < \Delta_2 < \cdots < \Delta_r < \ldots, \tag{2.19}$$

$$1 \leq f_1 < f_2 < \cdots < f_r < \ldots. \tag{2.20}$$

Proof. The fact that $u_1 > 1$ means

$$d_1 = 1 + u_1 + u_1^{-1} > 3 \tag{2.21}$$

implying $d_1 \geq 4$. If $D \geq 5$ it is immediate that $\Delta_0 \geq 5$. If $D = 2$ (respectively $D = 3$) then $\Delta_0 = 8$ (respectively $\Delta_0 = 12$). So $\Delta_0 \geq 5$ in every case. The fact that f_1 is a positive integer means $f_1 \geq 1$ and consequently $\Delta_1 \geq \Delta_0$. To prove the remaining inequalities observe that the fact that $u_1 > 1$ means $u_1^{r+1} > u_1^r$ for all r and, consequently,

$$d_{r+1} = 1 + u_1^{r+1} + u_1^{-r-1} > 1 + u_1^r + u_1^{-r} = d_r \tag{2.22}$$

for all r. So the sequence $d_1 < d_2 < \dots$ is strictly increasing. Since $d_1 > 1$ this means

$$\Delta_{r+1} = (d_{r+1} - 1)^2 - 4 > (d_r - 1)^2 - 4 = \Delta_r \tag{2.23}$$

It follows that the sequences $\Delta_1 < \Delta_2 <$ and $f_1 < f_2 < \dots$ are also strictly increasing. □

Proposition 2.3. *Let $T_n(x)$, $U_n(x)$ be the Chebyshev polynomials of the first and second kinds. Then for all $r, k \in \mathbb{N}$*

$$d_{kr} = 1 + 2T_k\left(\frac{d_r - 1}{2}\right) \tag{2.24}$$

$$f_{kr} = f_r U_{k-1}\left(\frac{d_r - 1}{2}\right) \tag{2.25}$$

Proof. Let

$$\theta = \ln(u_1^r) \tag{2.26}$$

Then

$$\cosh k\theta = \frac{u_1^{kr} + u_1^{-kr}}{2} = \frac{d_{kr} - 1}{2} \tag{2.27}$$

$$\sinh k\theta = \frac{u_1^{kr} - u_1^{-kr}}{2} = \frac{f_{kr}\sqrt{\Delta_0}}{2} \tag{2.28}$$

In view of the relations

$$T_k(\cosh r\theta) = \cosh kr\theta \tag{2.29}$$

$$U_{k-1}(\cosh r\theta) = \frac{\sinh kr\theta}{\sinh r\theta} \tag{2.30}$$

it follows that

$$T_k\left(\frac{d_r - 1}{2}\right) = \frac{d_{kr} - 1}{2}, \tag{2.31}$$

$$U_{k-1}\left(\frac{d_r - 1}{2}\right) = \frac{f_{kr}}{f_r}. \tag{2.32}$$

□

Proposition 2.4. *Let n be a positive integer. Then the following statements are equivalent:*

(1) *D is the square-free part of $(n-1)^2 - 4$,*
(2) *$n = d_r$ for some r.*

Proof. To prove 1 $\implies$ 2, suppose D is the square free part of $(n-1)^2-4$. So

$$(n-1)^2 - 4 = g^2 D \tag{2.33}$$

for some positive integer g.
Case 1: $D = 1$ (mod 4). Then $D = \Delta_0$ so if we set $h = g$

$$(n-1)^2 = h^2 \text{ (mod 4)} \tag{2.34}$$

implying $n - 1 = h$ (mod 2). Consequently

$$w = \frac{n-1+h\sqrt{\Delta_0}}{2} \tag{2.35}$$

is a unit.
Case 2: $D = 2$ or 3 (mod 4). Then

$$(n-1)^2 = 2g^2 \text{ or } 3g^2 \tag{2.36}$$

implying $n-1$, g are both even. So if we define $h = g/2$ then

$$w = \frac{n-1+h\sqrt{\Delta_0}}{2} \tag{2.37}$$

is a unit.

In both cases the unit w is greater than 1, which means $w = u_1^r$ for some r, implying $n = d_r$. The reverse implication is immediate. □

Proposition 2.5. *The following statements are equivalent*

(1) $N(v_1) = -1$,
(2) $d_r - 3$ *is a perfect square for all odd values of* r,
(3) $d_r - 3$ *is a perfect square for one odd value of* r.

In that case

$$v_1^r = \frac{\sqrt{d_r - 3} + \sqrt{d_r + 1}}{2} \tag{2.38}$$

for all r*, odd or even.*

Irrespective of the value of $N(v_1)$*:*

(1) $d_r - 3$ *is not a perfect square for any even value of* r,
(2) $d_r + 1$ *is a perfect square if and only if* r *is even.*

Proof. 1 $\implies$ 2. Let r be any odd integer. We have

$$v_1^r = \alpha + \beta\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.39}$$

for some α, $\beta \in \mathbb{Z}$. Then

$$v_1^{2r} = \frac{d_r - 1 + f_r\sqrt{\Delta_0}}{2} \tag{2.40}$$

implying

$$\left(\alpha + \frac{\beta\Delta_0}{2}\right)^2 + \frac{\beta^2\Delta_0}{4} = \frac{d_r - 1}{2} \tag{2.41}$$

while the fact that $N(v_1) = -1$ means

$$\left(\alpha + \frac{\beta\Delta_0}{2}\right)^2 - \frac{\beta^2\Delta_0}{4} = -1. \tag{2.42}$$

Putting these facts together we deduce

$$(2\alpha + \beta\Delta_0)^2 = d_r - 3, \tag{2.43}$$
$$\beta^2\Delta_0 = d_r + 1. \tag{2.44}$$

implying $d_r - 3$ is a perfect. These equations also prove Eq. (2.38) for the case r odd.

2 $\implies$ 3. Immediate.

3 $\implies$ 1. Suppose

$$d_r = n^2 + 3 \tag{2.45}$$

for positive integers r, n. Then

$$\Delta_r = (n^2 + 2)^2 - 4 = n^2(n^2 + 4) \tag{2.46}$$

implying

$$n^2 + 4 = g^2 D \tag{2.47}$$

for some positive integer g. Also define

$$w = \frac{n + g\sqrt{D}}{2}. \tag{2.48}$$

We claim that $w \in \mathbb{Z}_k$. Indeed, this is immediate if n is even, while if n is odd then g is also odd and $D = 1$ (mod 4), implying

$$w = \frac{n-g}{2} + g\left(\frac{1+\sqrt{D}}{2}\right) \in \mathbb{Z}_K. \tag{2.49}$$

If σ is the non-trivial automorphism of $K/\mathbb{Q}$ then

$$w\sigma(w) = \frac{n^2 - g^2 D}{4} = -1 \tag{2.50}$$

implying w is a negative norm unit, which means v_1 must be negative norm.

We have already proved Eq. (2.38) for the case r odd. To prove it for r even write

$$v_1^r = \alpha + \beta\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.51}$$

as before and observe that Eq. (2.41) holds unchanged, while Eq. (2.42) must be replaced with

$$\left(\alpha + \frac{\beta\Delta_0}{2}\right)^2 - \frac{\beta^2\Delta_0}{4} = 1. \tag{2.52}$$

Putting these two facts together we conclude

$$(2\alpha + \beta\Delta_0)^2 = d_r + 1, \tag{2.53}$$
$$\beta^2\Delta_0 = d_r - 3, \tag{2.54}$$

from which Eq. (2.38) follows for the case r even.

To prove the last pair of statements, suppose $d_{2r} - 3 = n^2$, for positive integers n, r. Define

$$w = \frac{\sqrt{d_{2r} - 3} + \sqrt{d_{2r} + 1}}{2}. \tag{2.55}$$

By construction $N(w) = -1$ and $w^2 = u_1^{2r}$. But that would mean $u_1^r = \pm w$, implying u_1^r is negative norm, which is a contradiction.

Finally, observe that for all r

$$d_{2r} - 1 = 2T_2\left(\frac{d_r - 1}{2}\right) = (d_r + 1)^2 \tag{2.56}$$

is a perfect square. Conversely, suppose $d_r = n^2 - 1$ for some n. We can write $e_s = n + 1$ for some s and tower $e_1, e_2, \dots$. Then

$$d_r = 1 + T_2\left(\frac{e_s + 1}{2}\right) = e_{2s} \tag{2.57}$$

implying that the towers $d_1, d_2, \dots$ and $e_1, e_2, \dots$ are the same, and that $r = 2s$ is even. □

In the sequel we will need the following lemma.

Lemma 2.6. *Suppose $N(v_1) = -1$ and let $r \in \mathbb{N}$ be odd. Then*

$$d_r + 1 = m^2 \Delta_0 \tag{2.58}$$

for some $m \in \mathbb{N}$.

Proof. We have $d_r - 3 = n^2$ for some $n \in \mathbb{N}$.
Case 1: n is odd. Then

$$\Delta_r = n^2(n^2 + 4) = 1 \text{ (mod } 4). \tag{2.59}$$

It follows that $\Delta_0 = 1$ (mod 4), and consequently that $\Delta_0 = D$. Since D is the square-free part of Δ_r this means $d_r + 1 = n^2 + 4 = m^2 \Delta_0$ for some m.
Case 2: n is even. So $n = 2l$ for some $l \in \mathbb{N}$. We then have

$$16l^2(l^2 + 1) = \Delta_r = \begin{cases} f_r^2 D & D = 1 \text{ (mod 4)} \\ 4f_r^2 D & D \neq 1 \text{ (mod 4)} \end{cases} \tag{2.60}$$

Since D is the square-free part of Δ_r it follows that $l^2 + 1 = k^2 D$ for some $k \in \mathbb{N}$, and consequently that

$$\begin{aligned} d_r + 1 = 4(l^2 + 1) &= \begin{cases} 4k^2 \Delta_0 & D = 1 \text{ (mod 4)} \\ k^2 \Delta_0 & D \neq 1 \text{ (mod 4)} \end{cases} \\ &= m^2 \Delta_0. \end{aligned} \tag{2.61}$$

for some m. □

Proposition 2.7. *If d_1 is odd then d_r is odd for all r. If, on the other hand, d_1 is even then d_r is odd (respectively even) if $r = 0$ mod 3 (respectively $r \neq 0$ mod 3).*

Remark. In particular d_{3r} is odd for all r and all D.

Proof. Let $T_r(x)$ be the Chebyshev polynomial of the first kind, and define [2]

$$T_r^*(x) = 1 + 2T_r\left(\frac{x-1}{2}\right) \tag{2.62}$$

Then

$$d_r = T_r(d_1) \tag{2.63}$$

We have

$$T_1^*(x) = x \tag{2.64}$$
$$T_2^*(x) = x(x-2) \tag{2.65}$$
$$T_3^*(x) = x^3 - 3x^2 + 3 \tag{2.66}$$

and, for all $r > 3$,

$$T_r^*(x) = xT_{r-1}^*(x) - xT_{r-2}^*(x) + T_{r-3}^*(x). \tag{2.67}$$

Reduced modulo 2 these equations become

$$d_2 = d_1, \qquad d_3 = 1, \qquad d_r = d_1(d_{r-1} + d_{r-2}) + d_{r-3} \tag{2.68}$$

from which the result is easily seen to follow. □

2.2. **Sub-tower corresponding to an order.** We have seen that to each dimension d_r in the tower there is associated the discriminant Δ_r. This gives an injective (but not surjective) mapping of the tower into $\mathcal{D}$, the set of discriminants with square-free part D. We now describe a mapping which goes the other way, and associates to each discriminant in $\mathcal{D}$ an infinite subsequence of $d_1, d_2, \dots$.

In our definitions of v_1, u_1 at the beginning of Subsection 2.1 we relied on the fact that the unit group of the maximal order $\mathbb{Z}_K$ is isomorphic to the direct product $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$. We now appeal to the fact (see, for example, ref. [8]) that Dirichlet's unit theorem generalizes to an arbitrary order. For all $f \in \mathbb{N}$ we accordingly define $\mathcal{U}_f$ to be the unit group of the order

$$O_f = \mathbb{Z}\left[f\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right)\right] \tag{2.69}$$

and v_f to be the unique fundamental unit of O_f which is greater than 1. So

$$\mathcal{U}_f = \{\pm v_f^n \colon n \in \mathbb{Z}\}. \tag{2.70}$$

Also define

$$u_f = \begin{cases} v_f, & N(v_f) = 1, \\ v_f^2, & N(v_f) = -1. \end{cases} \tag{2.71}$$

Since the unit group of O_f is a subgroup of the unit group of $\mathbb{Z}_K$ we must have

$$u_f = u_1^{r_{\min}(f)} \tag{2.72}$$

for some $r_{\min}(f) \in \mathbb{N}$ (the reason for this notation will become clear in the sequel).

Proposition 2.8. *The following statements are equivalent*

(1) $N(v_f) = -1$,

(2) $d_1 - 3$ *is a perfect square,* $r_{\min}(f)$ *is odd, and* f *divides* $\sqrt{\frac{d_{r_{\min}(f)}+1}{\Delta_0}} = \frac{f_{r_{\min}(f)}}{\sqrt{d_{r_{\min}(f)}-3}}$.

In that case $v_f = v_1^{r_{\min}(f)}$.

Remark. Note that it follows from Proposition 2.5 and Lemma 2.6 that if $d_1 - 3$ is a perfect square and $r_{\min}(f)$ is odd then $\sqrt{\frac{d_{r_{\min}(f)}+1}{\Delta_0}} \in \mathbb{N}$. The significance of this result is that we can use it to characterize those SICs having an anti-unitary symmetry.

Proof. 1 $\implies$ 2. Suppose $N(v_f) = -1$. Then $d_1 - 3$ is a perfect square. Also $v_f = v_1^s$ for some odd integer s, implying $u_f = v_f^2 = u_1^s$, and consequently $r_{\min}(f) = s$. So $r_{\min}(f)$ is odd and

$$v_f = v_1^{r_{\min}(f)}. \tag{2.73}$$

It follows from Proposition 2.5 that $d_{r_{\min}(f)} - 3$ is a perfect square. Let $m = \sqrt{\frac{d_{r_{\min}(f)}+1}{\Delta_0}}$ and $n = \sqrt{d_{r_{\min}(f)} - 3}$. Then

$$v_f = v_1^{r_{\min}(f)} = \frac{\sqrt{d_{r_{\min}(f)} - 3} + \sqrt{d_{r_{\min}(f)} + 1}}{2} = \frac{n + m\sqrt{\Delta_0}}{2} \tag{2.74}$$

The fact that $m^2\Delta_0 = n^2 \pmod 4$ means $m\Delta_0 = n \pmod 2$. So

$$v_f = \frac{n - m\Delta_0}{2} + m\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right), \qquad \frac{n - m\Delta_0}{2} \in \mathbb{Z}. \tag{2.75}$$

By assumption $v_f \in O_f$. So $f \mid m$.

2 $\implies$ 1. Suppose $d_1 - 3$ is a perfect square, $r_{\min}(f)$ is odd and f divides $m = \sqrt{\frac{d_{r_{\min}(f)}+1}{\Delta_0}}$. Then $N(v_1^{r_{\min}(f)}) = -1$. It follows from Proposition 2.5 that $d_{r_{\min}(f)} = n^2 + 3$ for some $n \in \mathbb{N}$, and

$$v_1^{r_{\min}(f)} = \frac{n + m\sqrt{\Delta_0}}{2} = \frac{n - m\Delta_0}{2} + m\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.76}$$

The fact that $f \mid m$ means $v_1^{r_{\min}(f)} \in O_f$. Since O_f contains a negative norm unit it follows that $N(v_f) = -1$.

For the final statement see Eq. (2.73) above. □

For each $f \in \mathbb{N}$ define

$$\mathcal{R}_f = \{r \in \mathbb{N} \colon f \mid f_r\} \tag{2.77}$$

Proposition 2.9. *For all* $f \in \mathbb{N}$

$$r_{\min}(f) = \min(\mathcal{R}_f), \tag{2.78}$$
$$\mathcal{R}_f = r_{\min}(f)\mathbb{N}. \tag{2.79}$$

Remark. This is the reason for the notation $r_{\min}(f)$.

Proof. Let $r \in \mathbb{N}$ be arbitrary. Then

$$u_1^r = \frac{d_r - 1 + f_r\sqrt{\Delta_0}}{2} = \frac{d_r - 1 - \Delta_r}{2} + f_r\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.80}$$

So $u_1^r \in \mathcal{U}_f$ if and only if $r \in \mathcal{R}_f$. On the other hand we know $u_1^r \in \mathcal{U}_f$ if and only if it is a power of $u_1^{r_{\min}(f)}$. So $\mathcal{R}_f = r_{\min}(f)\mathbb{N}$, which establishes the second statement. The first is then immediate. □

It follows that each order O_f, and each discriminant $f^2\Delta_0$, is associated the sub-tower $d_{r_{\min}(f)}$, $d_{2r_{\min}(f)}, \dots$. Since

$$d_{kr_{\min}(f)} = 1 + 2T_k\left(\frac{d_{r_{\min}(f)} - 1}{2}\right) \tag{2.81}$$

the sub-tower contains infinitely many sequences of dimensions of the form

$$d_{k_1 r_{\min}(f)} \mid d_{k_2 r_{\min}(f)} \mid d_{k_3 r_{\min}(f)} \mid \dots \tag{2.82}$$

in which each term is a divisor of the succeeding one.

As we discussed back in January, in a given dimension d_r the Galois multiplets of SICs are in bijective correspondence to the elements of the set

$$\mathcal{F}_r = \{f \in \mathbb{N} \colon f \mid f_r\} \tag{2.83}$$

(the number of SICs corresponding to a given f being determined by the class number of the discriminant $f^2\Delta_0$). The present discussion adds two things.

In the first place it reveals that

$$\bigcup_{r\in\mathbb{N}} \mathcal{F}_r = \mathbb{N}. \tag{2.84}$$

In other words, every $f \in \mathbb{N}$ is a divisor of some f_r. This wasn't obvious at the outset. We know from Proposition 2.4 in ref. [9] that there exist primes which do not divide d_r for any r. Indeed, there is a well-defined sense in which that is true for the majority of primes. However, it turns out that it is quite different with the numbers f_r: for, not only every *prime*, but every *positive integer* is a divisor of f_r for some sufficiently large r.

In the second place our discussion reveals (on the assumption that our conjectures are correct) that one can find SICs corresponding to a given f in each of an infinite sequence of dimensions. It would be interesting to see if there are relations between these SICs for arbitrary f similar to the ones observed for the case $f = 1$.

2.3. **Stability group.** In this subsection we describe a natural isomorphism between the unit group of the order O_f and the stability group of a quadratic form with discriminant $f^2\Delta_0$.

Let $\mathcal{M}(R)$ be the ring of 2×2 matrices over the ring R, and let $\mathcal{M}_0(R)$ be the subspace consisting of those matrices in $\mathcal{M}(R)$ with trace zero.

Lemma 2.10. *(1) For every ring R and $A \in \mathcal{M}_0(R)$*

$$A^2 = -\operatorname{Det}(A)I \tag{2.85}$$

(2) For every field F and non-zero matrices A_1, $A_2 \in \mathcal{M}_0(F)$, A_1 commutes with A_2 if and only if $A_2 = \lambda A_1$ for some non-zero $\lambda \in F$.

Proof. Let

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} \tag{2.86}$$

be a pair of matrices in $\mathcal{M}_0$. Then

$$A_1A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 - b_1a_2 \\ c_1a_2 - a_1c_2 & c_1b_2 + a_1a_2 \end{pmatrix} \tag{2.87}$$

To prove the first statement specialize to the case $A_2 = A_1$. To prove the second observe

$$A_1A_2 - A_2A_1 = \begin{pmatrix} (\mathbf{v}_1 \times \mathbf{v}_2)_1 & (\mathbf{v}_1 \times \mathbf{v}_2)_3 \\ (\mathbf{v}_1 \times \mathbf{v}_2)_2 & -(\mathbf{v}_1 \times \mathbf{v}_2)_1 \end{pmatrix} \tag{2.88}$$

where

$$\mathbf{v}_j = \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} \tag{2.89}$$

So $A_1A_2 - A_2A_1 = 0$ if and only if $\mathbf{v}_1 \times \mathbf{v}_2 = 0$, which in turn is true if and only if $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. □

Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.90}$$

be the two canonical generators of $\mathrm{SL}(2, \mathbb{Z})$.

Lemma 2.11. *The map*

$$Q \mapsto SQ \tag{2.91}$$

is a linear isomorphism of the space of quadratic forms over $\mathbb{R}$ *onto* $\mathcal{M}_0(\mathbb{R})$*. We have*

$$\mathrm{Det}(SQ) = \mathrm{Det}(Q). \tag{2.92}$$

Proof. Let

$$Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \tag{2.93}$$

be a quadratic form over $\mathbb{R}$. Then

$$SQ = \begin{pmatrix} -\frac{b}{2} & -c \\ a & \frac{b}{2} \end{pmatrix} \tag{2.94}$$

which is trace-zero. The fact that $\mathrm{Det}(SQ) = \mathrm{Det}(Q)$, and the fact that the map is bijective are immediate. □

Lemma 2.12. *Let* $f \in \mathbb{N}$*, let* $Q \in \mathcal{Q}_f$*, and let* $\langle I, SQ \rangle$ *be the subalgebra of* $\mathcal{M}(\mathbb{Q})$ *generated by* I *and* SQ*. Then*

$$\langle I, SQ \rangle = \{xI + ySQ \colon x, y \in \mathbb{Q}\} \tag{2.95}$$

Proof. It is immediate that $\{xI + ySQ \colon x, y \in \mathbb{Q}\}$ is contained in $\langle I, A \rangle$, and that it is closed under addition and scalar multiplication. To see that it is closed under matrix multiplication use Lemmas 2.10, 2.11 to deduce

$$(xI + ySQ)(x'I + y'SQ) = \frac{1}{4}(4xx' + yy'f^2\Delta)I + (xy' + x'y)SQ \tag{2.96}$$

□

Proposition 2.13. *Let $f \in \mathbb{N}$ and $Q \in \mathcal{Q}_f$, and let $\eta_Q \colon K \to \langle I, SQ\rangle$ be the map defined by*

$$\eta_Q \colon x + yf\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \mapsto \left(x + \frac{yf\Delta_0}{2}\right) I + ySQ \tag{2.97}$$

Then

(1) *η_Q is a $\mathbb{Q}$-algebra isomorphism of K onto $\langle I, SQ\rangle$. It has the property that*
$$\operatorname{Det}(\eta_Q(\kappa)) = N(\kappa). \tag{2.98}$$
for all $\kappa \in K$ (where N denotes the norm).
(2) *η_Q restricts to a ring isomorphism of O_f onto $\langle I, SQ\rangle \cap \mathcal{M}(\mathbb{Z})$.*
(3) *η_Q restricts again to a group isomomorphism of $O_f^\times$ onto $\langle I, SQ\rangle \cap \mathrm{GL}(2,\mathbb{Z})$.*
(4) *For all $\kappa \in O_f$ and $d \in \mathbb{N}$, $\kappa = 1$ (mod d) if and only if $\eta_Q(\kappa) = I$ (mod d).*

Remark. In the last statement we are, of course, primarily interested in the case $d = d_{kr_{\min}(f)}$.

Proof. (1). The fact that η_Q is a $\mathbb{Q}$-linear isomorphism is immediate. To see that it is an algebra isomorphism let

$$\kappa = x + yf\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \qquad \kappa' = x' + y'f\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{2.99}$$

Then

$$\begin{aligned}\eta_Q(\kappa\kappa') &= \left(\left(x + \frac{yf\Delta_0}{2}\right)\left(x' + \frac{y'f\Delta_0}{2}\right) + \frac{yy'f^2\Delta_0}{4}\right) I \\ &\qquad + \left(xy' + x'y + yy'f\Delta_0\right) SQ \\ &= \eta_Q(\kappa)\eta_Q(\kappa') \end{aligned}\tag{2.100}$$

Also

$$\begin{aligned} N(\kappa) &= \left(x + \frac{yf\Delta_0}{2}\right)^2 - \frac{y^2 f^2\Delta_0}{4} &(2.101)\\ &= \left(x + \frac{yf\Delta_0}{2}\right)^2 - \frac{y^2(b^2 - 4ac)}{4} &(2.102)\\ &= \operatorname{Det}\begin{pmatrix} x + \frac{y(f\Delta_0 - b)}{2} & -yc \\ ya & x + \frac{y(f\Delta_0+b)}{2}\end{pmatrix} \\ &= \operatorname{Det}(\eta_Q(\kappa)). &(2.103)\end{aligned}$$

(2). Let κ be as above, and let

$$M = \eta_Q(\kappa) = \begin{pmatrix} x + \frac{y(f\Delta_0 - b)}{2} & -yc \\ ya & x + \frac{y(f\Delta_0+b)}{2}\end{pmatrix}. \tag{2.104}$$

The fact that $f^2\Delta_0 = b^2$ (mod 4) means that $f\Delta_0 = b$ (mod 2). It follows that $(f\Delta_0 \pm b)/2 \in \mathbb{Z}$. So if $x, y \in \mathbb{Z}$ then $M \in \mathcal{M}(\mathbf{Z})$. Suppose, on the other hand, that $M \in \mathcal{M}(\mathbb{Z})$. Then ya, $yc \in \mathbb{Z}$. Also

$$yb = \left(x + \frac{y(f\Delta_0 + b)}{2}\right) - \left(x + \frac{y(f\Delta_0 - b)}{2}\right) \in \mathbb{Z}. \tag{2.105}$$

Since $\gcd(a,b,c) = 1$ it follows that $y \in \mathbb{Z}$. Since $x + y(f\Delta_0 + b)/2$ and $(f\Delta_0 + b)/2$ are also in $\mathbb{Z}$ this means $x \in \mathbb{Z}$. We conclude that $M \in \mathcal{M}(\mathbb{Z})$ if and only if $\kappa \in O_f$.

(3). Let $u \in O_f$ be arbitrary. In view of the foregoing $\eta_Q(u) \in \mathcal{M}(\mathbb{Z})$. Hence $u \in O_f^\times \iff N(u) = \pm 1 \iff \mathrm{Det}(\eta_Q(u)) = \pm 1 \iff \eta_Q(u) \in \mathrm{GL}(2, \mathbb{Z})$.

(4) Let

$$\kappa = x + yf\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right), \qquad M = \eta_Q(\kappa) = \begin{pmatrix} x + \frac{y(f\Delta_0 - b)}{2} & -yc \\ ya & x + \frac{y(f\Delta_0+b)}{2} \end{pmatrix} \tag{2.106}$$

Assume $\kappa \in O_f$. Then x, $y \in \mathbb{Z}$ and $M \in \mathcal{M}(\mathbb{Z})$.Taking account of the fact proved above, that $(f\Delta_0 \pm b)/2 \in \mathbb{Z}$, it can be seen that if $x = 1$, $y = 0$ (mod d) then $M = I$ (mod d). Conversely, suppose $M = I$ (mod d). Then $yc = ya = 0$ (mod d). Also

$$yb = \left(x + \frac{y(f\Delta_0 + b)}{2}\right) - \left(x + \frac{y(f\Delta_0 - b)}{2}\right) = 0 \text{ (mod } d). \tag{2.107}$$

Since $\gcd(a, b, c) = 1$ it follows that $y = 0$ (mod d). In view of this and the fact that $x + y(f\Delta_0 + b) = 1$ (mod d) we must also have $x = 1$ (mod d). □

Define the stability group $\mathcal{S}(Q)$ of a quadratic form Q to be the set of matrices $L \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$L^{\mathrm{T}} Q L = \mathrm{Det}(L) Q. \tag{2.108}$$

Also define

$$\mathcal{S}^{(1)}(Q) = \mathcal{S}(Q) \cap \mathrm{SL}(2, \mathbb{Z}) \tag{2.109}$$

$$S_d^{(1)}(Q) = \mathcal{S}(Q) \cap \Gamma(d) \tag{2.110}$$

where

$$\Gamma(d) = \{L \in \mathrm{SL}(2, \mathbb{Z})\colon L = I \text{ (mod } d)\} \tag{2.111}$$

Proposition 2.14. *For every $Q \in \mathcal{Q}_f$ the map η_Q defined in Proposition 2.13 is a group isomorphism of $O_f^\times$ onto $\mathcal{S}(Q)$.*

Proof. Let

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.112}$$

be any element of $\mathrm{GL}(2, \mathbb{Z})$. It is in $\mathcal{S}(Q)$ if and only if

$$QL = \mathrm{Det}\, L \left(L^{\mathrm{T}}\right)^{-1} Q. \tag{2.113}$$

Observe that

$$\mathrm{Det}\, L \left(L^{\mathrm{T}}\right)^{-1} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} = S^{-1} L S. \tag{2.114}$$

So the condition can be written

$$SQL = LSQ. \tag{2.115}$$

Define $t = \mathrm{Tr}(L)$, $\bar{L} = L - tI$. Then the condition becomes

$$SQ\bar{L} = \bar{L}SQ \tag{2.116}$$

Since SQ and $\bar{L}$ are both non-zero elements of $\mathcal{M}_0(\mathbb{Q})$ we can use Lemma 2.10 to deduce $\bar{L} = \lambda SQ$ for some $\lambda \in \mathbb{Q}$. It follows that $L \in \langle I, SQ \rangle \cap \mathrm{GL}(2, \mathbb{Z})$. The converse is immediate. □

Propositions 2.13 and 2.14 allow us to calculate can calculate many properties of the stability group of Q just from a knowledge of the unit group $O_f^{\times}$, without needing to consider specific properties of the matrix Q. For instance, we can use them to calculate the order of the stability group mod $d_{kr_{\min}(f)}$, as we now show.

Lemma 2.15. *For all f, $k \in \mathbb{Z}$*

$$u_f^{2k} = (d_{kr_{\min}(f)} - 1)u_f^k - 1, \tag{2.117}$$
$$u_f^{3k} = d_{kr_{\min}(f)} \big((d_{kr_{\min}(f)} - 2)u_f^k - 1\big) + 1. \tag{2.118}$$

Proof. Using $u_f = u_1^{r_{\min}(f)}$ and

$$d_{kr_{\min}(f)} = 1 + u_1^{kr_{\min}(f)} + u_1^{-kr_{\min}(f)} \tag{2.119}$$

we deduce

$$u_f^{2k} = u_1^{2kr_{\min}(f)} = (d_{kr_{\min}(f)} - 1)u_f^k - 1. \tag{2.120}$$

Hence

$$\begin{aligned} u_f^{3k} &= (d_{kr_{\min}(f)} - 1)u_f^{2k} - u_f^k \\ &= (d_{kr_{\min}(f)} - 1)\big((d_{kr_{\min}(f)} - 1)u_f^k - 1\big) - u_f^k \\ &= d_{kr_{\min}(f)}\big((d_{kr_{\min}(f)} - 2)u_f^k - 1\big) + 1. \end{aligned} \tag{2.121}$$

□

Lemma 2.16. *Let x, $x' \in \mathbb{Z}_K$ be such that $x = x'$ (mod n) for some $n \in \mathbb{N}$. Then $N(x) = N(x')$ (mod n).*

Proof. Write

$$x = x_1 + x_2\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right), \tag{2.122}$$

$$x' = x_1' + x_2'\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right). \tag{2.123}$$

Then

$$\begin{aligned} N(x) &= \left(x_1 + \frac{x_2\Delta_0}{2}\right)^2 - \frac{x_2^2\Delta_0}{4} \\ &= x_1^2 + x_1x_2\Delta_0 + x_2^2\left(\frac{\Delta_0(\Delta_0 - 1)}{4}\right). \end{aligned} \tag{2.124}$$

Similarly

$$N(x') = x_1'^2 + x_1'x_2'\Delta_0 + x_2'^2\left(\frac{\Delta_0(\Delta_0 - 1)}{4}\right). \tag{2.125}$$

The fact that $\Delta_0 = 0$ or 1 (mod 4) means $\Delta_0(\Delta_0 - 1)/4 \in \mathbb{Z}$. Since $x_j' = x_j$ (mod n it follows that $N(x') = N(x)$ (mod n). □

Proposition 2.17. *For all f, $k \in \mathbb{N}$*

(1) *u_f is order $3k$ mod $d_{kr_{\min}(f)}$,*

(2) *v_f is order $3k$ (respectively $6k$) mod $d_{kr_{\min}(f)}$ if $N(v_f)=1$ (respectively $N(v_f)=-1$).*

Proof. Let t be the order of u_f mod $d_{kr_{\min}(f)}$. It follows from Lemma 2.15 that

$$u_f^{3k} = 1 \pmod{d_{kr_{\min}(f)}} \tag{2.126}$$

So $t \mid 3k$. Also the fact that

$$u_f^t = 1 + d_{kr_{\min}(f)}\left(m + n\left(\frac{\Delta_0+\sqrt{\Delta_0}}{2}\right)\right) \tag{2.127}$$

for some $m, n \in \mathbb{Z}$, combined with the fact

$$u_f^t = u_1^{tr_{\min}(f)} = \frac{(d_{tr_{\min}(f)}-1) + f_{tr_{\min}(f)}\sqrt{\Delta_0}}{2} \tag{2.128}$$

means

$$nd_{kr_{\min}(f)} = f_{tr_{\min}(f)}. \tag{2.129}$$

Hence

$$d^2_{kr_{\min}(f)} \mid f^2_{tr_{\min}(f)}\Delta_0 = (d_{tr_{\min}(f)}-1)^2 - 4 \tag{2.130}$$

$$\implies \quad d_{kr_{\min}(f)} < d_{tr_{\min}(f)} \tag{2.131}$$

In view of Proposition 2.2 it follows that $k < t$. Since $t \mid 3k$ this means $3k = \lambda t$, with $\lambda = 1$ or 2. If $\lambda = 1$ we are through. Suppose $\lambda = 2$. Then $k = 2\xi$, $t = 3\xi$ for some $\xi \in \mathbb{N}$, and

$$u_f^{3\xi} = 1 \pmod{d_{2\xi r_{\min}(f)}}. \tag{2.132}$$

On the other hand it follows from Lemma 2.15 that

$$u_f^{3\xi} = d_{\xi r_{\min}(f)}\big((d_{\xi r_{\min}(f)}-2)u_f^{\xi} - 1\big) + 1. \tag{2.133}$$

Putting these facts together gives

$$d_{\xi r_{\min}(f)}\big((d_{\xi r_{\min}(f)}-2)u_f^{\xi} - 1\big) = \eta d_{2\xi r_{\min}(f)}. \tag{2.134}$$

for some $\eta \in \mathbb{N}$. But that would mean $u_f^{\xi} \in \mathbb{Q}$, which is impossible.

The second statement is immediate if $N(v_f) = 1$. Suppose $N(v_f) = -1$, and let t be the order of v_f mod $d_{kr_{\min}(f)}$. The fact that $u_f = v_f^2$ means $t \mid 6k$. On the other hand it follows from Lemma 2.16 that

$$(-1)^t = \big(N(v_f)\big)^t = N(v_f^t) = N(1) = 1. \tag{2.135}$$

So t is even and $u_f^{\frac{t}{2}} = 1 \pmod{d_{kr_{\min}(f)}}$, implying $6k \mid t$. Consequently $t = 6k$. □

Let $Q \in \mathcal{Q}_f$, let $\eta_Q \colon O_f^\times \to \mathcal{S}(Q)$ be the isomorphism described in Propositions 2.13 and 2.14, and let

$$P_Q = \eta_Q(v_f) \tag{2.136}$$

Then

$$\mathcal{S}(Q) = \langle -I, P_Q \rangle \tag{2.137}$$

If $N(v_f) = 1$ then

$$v_f = \frac{d_{r_{\min}(f)} - 1 + f_{r_{\min}(f)}\sqrt{\Delta_0}}{2} \tag{2.138}$$

$$\implies \qquad P_Q = \frac{d_{r_{\min}(f)} - 1}{2} + \frac{f_{r_{\min}(f)}}{f} SQ \tag{2.139}$$

while if $N(v_f) = -1$ then

$$v_f = \frac{\sqrt{d_{r_{\min}(f)} - 3} + \sqrt{d_{r_{\min}(f)} + 1}}{2} \tag{2.140}$$

$$\begin{aligned} \implies \qquad P_Q &= \frac{\sqrt{d_{r_{\min}(f)} - 3}}{2} + \frac{1}{f}\sqrt{\frac{d_{r_{\min}(f)} + 1}{\Delta_0}} SQ \\ &= \frac{\sqrt{d_{r_{\min}(f)} - 3}}{2} I + \frac{f_{r_{\min}(f)}}{f\sqrt{d_{r_{\min}(f)} - 3}} SQ \end{aligned} \tag{2.141}$$

(recall that it was shown in Proposition 2.8 that if $N(v_f) = -1$ then f divides $f_{r_{\min}(f)}/(d_{r_{\min}(f)} - 3)$.

Corollary 2.18. *Let f, $k \in \mathbb{N}$ be arbitrary, and let Q be any element of $\mathcal{Q}_f$. Then $S^{(1)}_{d_{kr_{\min}(f)}}(Q)$ is the infinite cyclic group generated by $\eta_Q(u_f^{3k})$.*

3. Reduced Forms and Continued Fractions

3.1. **Basic facts about Continued Fractions.** According to the usual definition [5, 7] a continued fraction is an expansion, finite or infinite, of the form

$$[f_0, f_1, f_2, f_3, \dots]^+ = f_0 + \cfrac{1}{f_1 + \cfrac{1}{f_2 + \cfrac{1}{f_3 + \cdots}}} \tag{3.1}$$

where f_j is any sequence of integers such that $f_j \geq 1$ for $j \geq 1$. I will refer to this as an ordinary continued fraction. We are more interested in Hirzebruch-Jung (HJ continued fractions [12], which are finite or infinite expansions of the form[1]

$$[f_0, f_1, f_2, f_3, \dots] = f_0 - \cfrac{1}{f_1 - \cfrac{1}{f_2 - \cfrac{1}{f_3 - \cdots}}} \tag{3.2}$$

where f_j is any sequence of integers such that $f_j \geq 2$ for $j \geq 1$ and which, if infinite, is not ultimately constant equal to 2. Every real number f has an HJ continued fraction expansion. The expansion is finite if and only if f is rational. We say the expansion is periodic if it is recurring past a certain point (i.e. is of the form $[f_0, \dots, f_n, \overline{f_{n+1}, \dots, f_{n+m}}]$, where we indicate the recurring part of the expansion using an overline). A number f has a periodic HJ expansion if and only if it is of the form $a + b\sqrt{n}$ for some $n \in \mathbb{N}$, $a, b \in \mathbb{Q}$. In the sequel we will be particularly interested in numbers having purely periodic HJ expansions, of the form $[\overline{f_0, f_1, \dots, f_n}]$. Note that whereas in general f_0 can take any integral value, if the expansion is purely periodic we must have $f_0 \geq 2$ (because $f_0 = f_{n+1}$ where n is the periodic).

[1] Pompescu-Pampu [12] uses the notation $[f_0, f_1, f_2, f_3, \dots]^-$. However, since we are almost exclusively concerned with HJ-continued fractions I will omit the superscript.

The HJ expansion of a number f can be calculated by taking successive ceilings. For example

$$\begin{aligned}
\sqrt{2} &= \lceil\sqrt{2}\rceil - \frac{1}{(\lceil\sqrt{2}\rceil - \sqrt{2})^{-1}} \\
&= 2 - \frac{1}{1+2^{-\frac{1}{2}}} \\
&= 2 - \cfrac{1}{\left\lceil 1+2^{-\frac{1}{2}}\right\rceil - \cfrac{1}{\left(\left\lceil 1+2^{-\frac{1}{2}}\right\rceil - \left(1+2^{-\frac{1}{2}}\right)\right)^{-1}}} \\
&= 2 - \cfrac{1}{2 - \cfrac{1}{2+2^{\frac{1}{2}}}} \\
&= 2 - \cfrac{1}{2 - \cfrac{1}{\left\lceil 2+2^{\frac{1}{2}}\right\rceil - \cfrac{1}{\left(\left\lceil 2+2^{\frac{1}{2}}\right\rceil - \left(2+2^{\frac{1}{2}}\right)\right)^{-1}}}} \\
&= 2 - \cfrac{1}{2 - \cfrac{1}{4 - \cfrac{1}{1+2^{-\frac{1}{2}}}}} \\
&= [2, \overline{2,4}].
\end{aligned} \tag{3.3}$$

In particular if $[f_0, \dots]$ is the HJ expansion of arbitrary $f \in \mathbb{R}$ then $f_0 = \lceil f \rceil$.

One can express an HJ continued fraction as an ordinary fraction (if it is rational) or as the limit of a sequence of ordinary fractions (if it is irrational) using the following construction. Define

$$\begin{aligned}
(f_0) &= f_0, \\
(f_0, f_1) &= f_0 f_1 - 1, \\
(f_0, \dots, f_n) &= (f_0, \dots, f_{n-1}) f_n - (f_0, \dots, f_{n-2}), \qquad (n \geq 2).
\end{aligned} \tag{3.4}$$

Then

$$[f_0, \dots, f_n] = \frac{(f_0, \dots, f_n)}{(f_1, \dots, f_n)}, \qquad (n \geq 1). \tag{3.5}$$

Proposition 3.1. *The quantities* $(f_0, \dots, f_n)$ *are symmetric under reversal:*

$$(f_n, f_{n-1}, \dots, f_1, f_0) = (f_0, f_1, \dots, f_{n-1}, f_n). \tag{3.6}$$

In particular they satisfy the alternative recursion relation

$$(f_0, \dots, f_n) = f_0(f_1, f_2, \dots, f_n) - (f_2, \dots, f_n), \qquad (n \geq 2). \tag{3.7}$$

Proof. The same, *mutatis mutandi*, as the proof of the corresponding statements for ordinary continued fractions (see, e.g., Davenport [5]). ☐

Proposition 3.2. *For all $n \geq 2$*

$$(f_0,\dots,f_n)(f_1,\dots,f_{n-1}) - (f_0,\dots,f_{n-1})(f_1,\dots,f_n) = -1 \tag{3.8}$$

In particular $(f_0,\dots,f_n)$ is relatively prime $(f_1,\dots,f_n)$ for all n (so the fraction on the RHS of Eq. (3.5) is in its lowest terms).

Corollary 3.3. *Note the difference with the case of ordinary continued fractions, where one has $+1$ on the RHS.*

Proof. The same, *mutatis mutandi*, as the proof of the corresponding statements for ordinary continued fractions (see, e.g., Davenport [5]). ☐

3.2. **More on Quadratic Forms and their Stability Groups.** Let $Q = \langle a,b,c\rangle$ be a primitive quadratic form with discriminant $\Delta = f^2\Delta_0$, let $r = r_{\min}(f)$, and let

$$\omega_\pm = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{3.9}$$

be the two roots of $ax^2 + bx + c$. If we restrict ourselves to primitive forms then, up to a sign, the roots $\omega_\pm$ determine the form uniquely[2].

Given any $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, let

$$M(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{3.12}$$

be the corresponding fractional linear transformation.

Proposition 3.4. *Let $Q = \langle a,b,c\rangle$ be a quadratic form, and let $\omega_\pm$ be the associated roots. Let $Q' = \langle a',b',c'\rangle$ be the quadratic form $M^{\mathrm{T}}QM$, for some $M \in \mathrm{GL}(2,\mathbb{Z})$, and let $\omega'_\pm$ be the roots associated to Q'. Then*

$$\omega'_\pm = M^{-1}(\omega_\pm) \text{ or } M^{-1}(\omega_\mp). \tag{3.13}$$

Proof. Write $M^{-1} = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Then

$$0 = \begin{pmatrix}\omega_\pm & 1\end{pmatrix} (M^{-1})^{\mathrm{T}} M^{\mathrm{T}} Q M M^{-1} \begin{pmatrix}\omega_\pm \\ 1\end{pmatrix} \tag{3.14}$$

$$= \begin{pmatrix}\alpha\omega_\pm + \beta & \gamma\omega_\pm + \delta\end{pmatrix} Q' \begin{pmatrix}\alpha\omega_\pm + \beta \\ \gamma\omega_\pm + \delta\end{pmatrix} \tag{3.15}$$

$$= a'\left(\alpha\omega_\pm + \beta\right)^2 + b'\left(\alpha\omega_\pm + \beta\right)\left(\gamma\omega_\pm + \delta\right) + c'\left(\gamma\omega_\pm + \delta\right)^2 \tag{3.16}$$

[2] Indeed, let $Q' = \langle a',b',c'\rangle$ be another primitive form for which $a'x^2 + b'x + c'$ has the same roots $\omega_\pm$. Then

$$\frac{b'}{a'} = -\left(\omega_+ + \omega_-\right) = \frac{b}{a}, \qquad \frac{c'}{a'} = \omega_+\omega_- = \frac{c}{a}. \tag{3.10}$$

Define $g = \gcd(a,a')$, $\bar{a} = a/g$, $\bar{a}' = a'/g$. Then

$$b' = \frac{\bar{a}'b}{\bar{a}}, \qquad c' = \frac{\bar{a}'c}{\bar{a}}. \tag{3.11}$$

Since b, $b' \in \mathbb{Z}$ and $\bar{a}$, $\bar{a}'$ this means $b = \bar{b}\bar{a}$, $c = \bar{c}\bar{a}$ for some $\bar{b}$, $\bar{c} \in \mathbb{Z}$. So $\bar{a}$ divides all three of a, b, c. Since Q is primitive it follows that $\bar{a} = \pm 1$. That in turn implies $\bar{a}'$ divides all three of a', b', c' which, given the assumption that Q' is primitive, means $\bar{a}' \in \{-1,+1\}$. Consequently $Q' = (\bar{a}'/\bar{a})Q = \pm Q$, as claimed.

$$= (\gamma\omega_\pm + \delta)^2 \left(a'\big(M^{-1}(\omega_\pm)\big)^2 + b' M^{-1}(\omega_\pm) + c' \right) \tag{3.17}$$

□

Recall that

$$u_f = u_r = \frac{d_r - 1 + f_r\sqrt{\Delta_0}}{2}. \tag{3.18}$$

Then it follows from Propositions 2.13, 2.14 that $\mathcal{S}^{(1)}(Q)$ (the intersection of the stability group with $\mathrm{SL}(2,\mathbb{Z})$) is generated by $-I$ and L_Q, where

$$L_Q = \eta_Q(u_f) = \frac{d_r - 1}{2} I + \frac{f_r}{f} SQ \tag{3.19}$$

or, more explicitly,

$$L_Q = \begin{pmatrix} \frac{d_r-1}{2} - \frac{f_r b}{2f} & -\frac{f_r c}{f} \\ \frac{f_r a}{f} & \frac{d_r-1}{2} + \frac{f_r b}{2f} \end{pmatrix}. \tag{3.20}$$

Proposition 3.5.

$$\eta_Q(u_f^{\pm 1}) = \eta_{-Q}(u_f^{\mp 1}), \tag{3.21}$$

$$L_{-Q} = L_Q^{-1}. \tag{3.22}$$

Proof.

$$\begin{aligned} \eta_Q(u_f^{\pm 1}) &= \eta_Q\left(\frac{d_r - 1 \pm f_r\sqrt{\Delta_0}}{2}\right) \\ &= \left(\frac{d_r-1}{2}\right) \pm \frac{f_r}{f} SQ \\ &= \eta_{-Q}\left(\frac{d_r - 1 \mp f_r\sqrt{\Delta_0}}{2}\right) \\ &= \eta_Q(u_f^{\mp 1}). \end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned} L_{-Q}L_Q &= \eta_{-Q}(u_f)\eta_Q(u_f) \\ &= \eta_Q(u_f^{-1})\eta_Q(u_f) \\ &= \eta_Q(u_f^{-1}u_f) \\ &= I \end{aligned} \tag{3.24}$$

□

Proposition 3.6. *The eigenvalues of L_Q are u_f, u_f^{-1}. Specifically*

$$L_Q \begin{pmatrix} \omega_\pm \\ 1 \end{pmatrix} = u_f^{\pm 1} \begin{pmatrix} \omega_\pm \\ 1 \end{pmatrix}. \tag{3.25}$$

Also

$$L_Q(\omega_\pm) = \omega_\pm. \tag{3.26}$$

Proof.

$$
\begin{aligned}
L_Q\begin{pmatrix}\omega_\pm\\ 1\end{pmatrix} &= \begin{pmatrix}\left(\frac{d_r-1}{2}-\frac{f_rb}{2f}\right)\omega_\pm-\frac{f_rc}{f}\\ \frac{f_ra\omega_\pm}{f}+\left(\frac{d_r-1}{2}+\frac{f_rb}{2f}\right)\end{pmatrix}\\
&= \begin{pmatrix}\left(\frac{d_r-1}{2}\right)\omega_\pm-\frac{f_r}{4fa}\left(b\left(-b\pm\sqrt{\Delta}\right)+4ac\right)\\ \frac{d_r-1}{2}+\frac{f_r}{2f}\left(2a\omega_\pm+b\right)\end{pmatrix}\\
&= \begin{pmatrix}\left(\frac{d_r-1}{2}\right)\omega_\pm-\frac{f_r}{4fa}\left(-\Delta\pm b\sqrt{\Delta}\right)\\ \frac{d_r-1}{2}\pm\frac{f_r\sqrt{\Delta}}{2f}\end{pmatrix}\\
&= \begin{pmatrix}\left(\frac{d_r-1}{2}\right)\omega_\pm\pm\frac{f_r\sqrt{\Delta}}{4fa}\left(-b\pm\sqrt{\Delta}\right)\\ \frac{d_r-1\pm f_r\sqrt{\Delta_0}}{2}\end{pmatrix}\\
&= \begin{pmatrix}\left(\frac{d_r-1\pm f_r\sqrt{\Delta_0}}{2}\right)\omega_\pm\\ \frac{d_r-1\pm f_r\sqrt{\Delta_0}}{2}\end{pmatrix}\\
&= u_f^{\pm1}\begin{pmatrix}\omega_\pm\\ 1\end{pmatrix}.
\end{aligned}
\tag{3.27}
$$

Writing $L_Q=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$ it follows

$$
\begin{pmatrix}\alpha\omega_\pm+\beta\\ \gamma\omega_\pm+\delta\end{pmatrix}=u_f^{\pm1}\begin{pmatrix}\omega_\pm\\ 1\end{pmatrix}
\tag{3.28}
$$

implying

$$
L_Q(\omega_\pm)=\frac{\alpha\omega_\pm+\beta}{\gamma\omega_\pm+\delta}=\omega_\pm
\tag{3.29}
$$

□

3.3. **Reducibility and pure periodicity in the ordinary case.** The form $Q=\langle a,b,c\rangle$ is reduced[4] if and only if

$$
b<\sqrt{\Delta},\qquad \sqrt{\Delta}-b<2|a|<\sqrt{\Delta}+b,
\tag{3.30}
$$

(note that it follows from this that $b>0$).

It can be shown [5, 7] that a number $\omega\in K=\mathbb{Q}(\sqrt{\Delta})$ has a purely periodic continued fraction expansion if and only if

$$
-1<\sigma(\omega)<0,\qquad 1<\omega
\tag{3.31}
$$

where σ is the non-trivial automorphism of K. These two notions are connected:

Proposition 3.7. (1) *If $a>0$ then Q is reduced if and only if $-\omega_-$ has a purely periodic continued fraction expansion.*

(2) *If $a<0$ then Q is reduced if and only if ω_- has a purely periodic continued fraction expansion.*

Proof. It is easily seen that if $a > 0$

$$Q \text{ is reduced} \iff -1 < -\omega_+ < 0, \quad 1 < -\omega_-. \tag{3.32}$$

while if $a < 0$

$$Q \text{ is reduced} \iff -1 < \omega_+ < 0, \quad 1 < \omega_-. \tag{3.33}$$

□

STF: I really don't like the notation here where σ is a number since it is used in really close proximity to places where σ is a Galois automorphism.

Corollary 3.8. *Let σ be any number having purely periodic HJ continued fraction expansion. Then there exists an HJ reduced form $Q = \langle a, b, c\rangle$ with $a > 0$ such that $\sigma = \omega_+$, where $\omega_\pm$ are the roots of Q.*

Proof. σ is the solution of an equation of the form $ax^2 + bx + c = 0$ with $\gcd(a, b, c) = 1$ and $a > 0$. So $\sigma = \omega_+$, where $\omega_\pm$ are the roots of Q. In view of Proposition 3.7 Q is HJ reduced. □

3.4. **Reducibility and pure periodicity in the HJ case.** An analogue of Proposition 3.7 holds in the HJ case. We say a form $Q = \langle a, b, c\rangle$ is HJ-reduced[3] if

$$b < -\sqrt{\Delta}, \qquad -\sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} - b. \tag{3.34}$$

It can be shown [13] that a number $\omega \in K$ has a purely periodic HJ continued fraction expansion if and only if

$$0 < \sigma(\omega) < 1 < \omega. \tag{3.35}$$

Proposition 3.9. (1) *If $a > 0$ then Q is HJ reduced if and only if ω_+ has a purely periodic HJ continued fraction expansion.*
(2) *If $a < 0$ then Q is HJ reduced if and only if $-\omega_+$ has a purely periodic HJ continued fraction expansion.*

Proof. It is easily seen that if $a > 0$

$$Q \text{ is HJ reduced} \iff 0 < \omega_- < 1 < \omega_+, \tag{3.36}$$

while if $a < 0$

$$Q \text{ is HJ reduced} \iff 0 < -\omega_- < 1 < -\omega_+. \tag{3.37}$$

□

In the case of ordinary reducibility it is known that for a given equivalence class there are only finitely many reduced forms. There is also an algorithm for calculating them. We now address the question, what can be said for the HJ case. Let N_Q (respectively N_Q^{HJ}) be the number of ordinary (respectively HJ) reduced forms properly equivalent to Q. We will show that it is always the case that $N_Q \le N_Q^{\mathrm{HJ}} < \infty$, and that there exist Q for which N_Q is strictly smaller than N_Q^{HJ}. We will also describe a method for calculating the complete set of HJ reduced forms.

Let Ω be the set of all pairs (ω_-, ω_+) corresponding to forms properly equivalent to Q. Let $W_+ \subseteq \mathbb{R}^2$ (respectively $W_- \subseteq \mathbb{R}^2$) be the set of pairs $(\omega_-, \omega_+) \in \Omega$ for which the corresponding form with $a > 0$ (respectively $a < 0$) is ordinary reduced. Let $W_+^{\mathrm{HJ}} \subseteq \mathbb{R}^2$ (respectively $W_-^{\mathrm{HJ}} \subseteq \mathbb{R}^2$)

[3] Note that the notion of HJ-reduction is something I am introducing in these notes.

be the set of pairs $(\omega_-,\omega_+)\in\Omega$ for which the corresponding form with $a>0$ (respectively $a<0$) is HJ reduced. Then

$$W_+ = \big((-\infty,-1)\times(0,1)\big)\cap\Omega, \tag{3.38}$$
$$W_- = \big((1,\infty)\times(-1,0)\big)\cap\Omega, \tag{3.39}$$
$$W_+^{\mathrm{HJ}} = \big((0,1)\times(1,\infty)\big)\cap\Omega, \tag{3.40}$$
$$W_-^{\mathrm{HJ}} = \big((-1,0)\times(-\infty,-1)\big)\cap\Omega. \tag{3.41}$$

More generally define, for any non-negative integer n,

$$W_{n,+} = \left((-\infty,-1)\times\left(0,\frac{1}{n+1}\right)\right)\cap\Omega, \tag{3.42}$$
$$W_{n,-} = \left((1,\infty)\times\left(-\frac{1}{n+1},0\right)\right)\cap\Omega, \tag{3.43}$$
$$W_{n,+}^{\mathrm{HJ}} = \left(\left(\frac{n}{n+1},\frac{n+1}{n+2}\right)\times(1,\infty)\right)\cap\Omega, \tag{3.44}$$
$$W_{n,-}^{\mathrm{HJ}} = \left(\left(-\frac{n+1}{n+2},-\frac{n}{n+1}\right)\times(-\infty,-1)\right)\cap\Omega. \tag{3.45}$$

So

$$W_+ = W_{0,+}\supseteq W_{1,+}\supseteq W_{2,+},\supseteq\dots, \tag{3.46}$$
$$W_- = W_{0,-}\supseteq W_{1,-}\supseteq W_{2,-},\supseteq\dots, \tag{3.47}$$

$$\bigcap_{n=0}^{\infty} W_{n,+} = \emptyset, \tag{3.48}$$
$$\bigcap_{n=0}^{\infty} W_{n,-} = \emptyset, \tag{3.49}$$

$$W_{n,+}^{\mathrm{HJ}}\cap W_{m,+}^{\mathrm{HJ}} = \emptyset, \tag{3.50}$$
$$W_{n,-}^{\mathrm{HJ}}\cap W_{m,-}^{\mathrm{HJ}} = \emptyset, \tag{3.51}$$

for all $n\neq m$ and (keeping in mind that the $\omega_\pm$ are irrational)

$$W_+^{\mathrm{HJ}} = \bigcup_{n=0}^{\infty} W_{n,+}^{\mathrm{HJ}}, \tag{3.52}$$
$$W_-^{\mathrm{HJ}} = \bigcup_{n=0}^{\infty} W_{n,-}^{\mathrm{HJ}}. \tag{3.53}$$

The fact that $\#(W_\pm)<\infty$ means that $\#(W_{n,\pm})=0$ for all $n\geq M_\pm$ for some pair of non-negative integers $M_\pm$. Define, for all non-negative $n\in\mathbb{Z}$,

$$F_{n,+} = TST^{n+1}S = \begin{pmatrix} n & -1 \\ n+1 & -1 \end{pmatrix} \tag{3.54}$$
$$F_{n,-} = T^{-1}ST^{-(n+1)}S = \begin{pmatrix} n & 1 \\ -(n+1) & -1 \end{pmatrix} \tag{3.55}$$

Then it is easily seen that $F_{n,\pm}$ maps $W_{n,\pm}$ bijectively onto $W^{\mathrm{HJ}}_{n,\pm}$. Hence

$$W^{\mathrm{HJ}}_{\pm} = \bigcup_{n=0}^{M_\pm} F_{n,\pm}\big(W_{n,\pm}\big) \tag{3.56}$$

It follows that $\#\big(W^{\mathrm{HJ}}_{\pm}\big) < \infty$ and that

$$\begin{aligned} N^{\mathrm{HJ}}_Q &= \#\big(W^{\mathrm{HJ}}_{+}\big) + \#\big(W^{\mathrm{HJ}}_{-}\big) \\ &\geq \#\Big(F_{0,+}\big(W_{0,+}\big)\Big) + \#\Big(F_{0,-}\big(W_{0,-}\big)\Big) \\ &= \#\big(W_{0,+}\big) + \#\big(W_{0,-}\big) \\ &= N_Q \end{aligned} \tag{3.57}$$

To see $N^{\mathrm{HJ}}_Q = N_Q$ and $N^{\mathrm{HJ}}_Q > N_Q$ are both possible consider the following examples.

Example 1. $\Delta = 5$. There is a single $\mathrm{SL}(2,\mathbb{Z})$ orbit which is also a $\mathrm{GL}(2,\mathbb{Z})$ orbit. It contains the two reduced forms $\langle 1, 1, -1\rangle$, $\langle -1, 1, 1\rangle$ with roots $\frac{1}{2}(-1 \pm \sqrt{5})$, $-\frac{1}{2}(-1 \pm \sqrt{5})$ respectively. In particular $N_Q = 2$, and the orbit gives rise to the single purely periodic number $\frac{1}{2}(1+\sqrt{5}) = [\overline{1}]_+$. Since $\frac{1}{2}(-1+\sqrt{5}) > \frac{1}{2}$ the sets $W_{n,\pm}$ are empty for $n > 0$. So there are two HJ reduced forms on the orbit, $\langle 1, -3, 1\rangle$, $\langle -1, -3, -1\rangle$, with roots $\frac{1}{2}(3 \pm \sqrt{5})$, $-\frac{1}{2}(3 \pm \sqrt{5})$ respectively. In particular, $N^{\mathrm{HJ}}_Q = 2 = N_Q$ and the orbit gives rise to the single HJ purely periodic number $\frac{1}{2}(3+\sqrt{5}) = [\overline{3}]$.

Example 2. $\Delta = 8$. There is a single $\mathrm{SL}(2,\mathbb{Z})$ orbit which is also a $\mathrm{GL}(2,\mathbb{Z})$ orbit. It contains the two reduced forms $\langle 1, 2, -1\rangle$, $\langle -1, 2, 1\rangle$ with roots $-1 \pm \sqrt{2}$, $-(-1 \pm \sqrt{2})$ respectively. In particular $N_Q = 2$, and the orbit gives rise to the single purely periodic number $1 + \sqrt{2} = [\overline{2}]$. Since $\frac{1}{3} < -1 + \sqrt{2} < \frac{1}{2}$ the sets $W_{n,\pm}$ are empty for $n > 1$, but non-empty for $n = 0,\ 1$. So there are four HJ reduced forms on the orbit $\langle 2, -4, 1\rangle$, $\langle 1, -4, 2\rangle$, $\langle -2, -4, -1\rangle$, $\langle -1, -4, -2\rangle$ with roots $\frac{1}{2}(2 \pm \sqrt{2})$, $2 \pm \sqrt{2}$, $-\frac{1}{2}(2 \pm \sqrt{2})$, $-(2 \pm \sqrt{2})$ respectively. In particular, $N^{\mathrm{HJ}}_Q = 4 > N_Q$ and the orbit gives rise to the two HJ purely periodic numbers $\frac{1}{2}(2+\sqrt{2}) = [\overline{2,4}]$, $2 + \sqrt{2} = [\overline{4,2}]$.

3.4.1. *Calculating the full set of HJ reduced forms.* So far we have been looking at how to transform the roots of reduced forms into the roots of HJ reduced forms. Let us now look at how this works for the forms themselves. Suppose $Q = \langle a, b, c\rangle$ is a reduced form with $a > 0$ and roots $\omega_\pm$. Let n be a non-negative integer such that

$$\omega_+ < \frac{1}{n+1} \tag{3.58}$$

Then it follows from Prop. 3.4 and Eq. (3.54) that

$$Q' = \big(F^{-1}_{n,+}\big)^{\mathrm{T}} Q F^{-1}_{n,+} \tag{3.59}$$

is an HJ-reduced form. Performing the algebra one finds $Q' = \langle a', b', c'\rangle$ where

$$a' = a + (n+1)b + (n+1)^2 c \tag{3.60}$$

$$b' = -2a - (2n+1)b - 2n(n+1)c \tag{3.61}$$

$$c' = a + nb + n^2 c. \tag{3.62}$$

Moreover

$$a' = a(n+1)^2 \left(\frac{1}{n+1} - \omega_+\right)\left(\frac{1}{n+1} - \omega_-\right) > 0. \tag{3.63}$$

Suppose, on the other hand,$Q = \langle a, b, c\rangle$ is a reduced from with $a < 0$ and roots $\omega_\pm$. Let n be a non-negative integer such that

$$-\frac{1}{n+1} < \omega_+ \tag{3.64}$$

Then it follows from Prop. 3.4 and Eq. (3.55) that

$$Q' = \left(F_{n,-}^{-1}\right)^{\mathrm{T}} Q F_{n,-}^{-1} \tag{3.65}$$

is an HJ-reduced form. Performing the algebra one finds $Q' = \langle a', b', c'\rangle$ where

$$a' = a - (n+1)b + (n+1)^2 c, \tag{3.66}$$

$$b' = 2a - (2n+1)b + 2n(n+1)c, \tag{3.67}$$

$$c' = a - nb + n^2 c. \tag{3.68}$$

Moreover

$$a' = a(n+1)^2 \left(\frac{1}{n+1} + \omega_+\right)\left(\frac{1}{n+1} + \omega_-\right) < 0. \tag{3.69}$$

3.4.2. *Principal HJ reduced form.* The principal reduced form for discriminant Δ is (see, e.g., ref. [4])

$$Q = \left\langle 1, b, (b^2 - \Delta)/4 \right\rangle, \qquad b = \begin{cases} 2\left\lfloor \frac{\sqrt{\Delta}}{2} \right\rfloor & \Delta \text{ even} \\ 2\left\lfloor \frac{\sqrt{\Delta}-1}{2} \right\rfloor + 1 & \Delta \text{ odd} \end{cases} \tag{3.70}$$

Setting $n = 0$ in Eqs. (3.60), (3.61), (3.62) we deduce that an equivalent HJ reduced form is

$$Q' = \langle (b'^2 - \Delta)/4, b', 1\rangle \qquad b' = \begin{cases} -2\left\lceil \frac{\sqrt{\Delta}}{2} \right\rceil & \Delta \text{ even} \\ -2\left\lceil \frac{\sqrt{\Delta}-1}{2} \right\rceil - 1 & \Delta \text{ odd} \end{cases} \tag{3.71}$$

We refer to this as the principal HJ-reduced form for discriminant Δ.

3.5. **Decomposition of Stability Group Generator.** The purpose of this subsection is to discuss the relationship between the HJ expansion of ω_+ and the expansion of L_Q in terms of T and S.

In the definition of $[k_0, \dots, k_n]$ it is required that $k_j \geq 2$ for all $j \geq 1$. It is to be observed, however, that the quantities $(k_0, \dots, k_n)$ are well-defined for arbitrary integers k_j (indeed, they are defined for arbitrary complex values). It is true that if one allows k_j outside the stated range then the equation

$$k_0 - \cfrac{1}{k_1 - \cfrac{1}{k_2 - \dots}} = \frac{(k_0, k_1, k_2, \dots, k_n)}{(k_1, k_2, \dots, k_n)} \tag{3.72}$$

may no longer be valid. This is for two reasons. In the first place, the denominator of the fraction on the right hand side may be zero. This is the case, for instance, if $k_0 = k_1 = k_2 = 1$, in which case neither of the fractions

$$k_0 - \cfrac{1}{k_1 - \cfrac{1}{k_2}}, \qquad \frac{(k_0, k_1, k_2)}{(k_1, k_2)} \tag{3.73}$$

is well-defined. In the second place, it may happen that the second fraction is well-defined, but the first is not. This is the case, for instance, if $k_0 = k_1 = 1$ and $k_2 = 0$. Nevertheless, the quantities $(k_0, k_1, \ldots, k_n)$ are still useful.

Continue to define the $(k_0, \ldots, k_n)$ recursively by

$$(k_r, \ldots, k_s) = \begin{cases} 1 & r > s, \\ k_r & r = s, \end{cases} \tag{3.74}$$

and

$$(k_0, \ldots, k_n) = (k_0, \ldots, k_{n-1})k_n - (k_0, \ldots, k_{n-2}), \qquad n \geq 1 \tag{3.75}$$

for arbitrary integral k_r. Then one has the following generalization of Proposition 3.2:

Proposition 3.10. *For all $n \geq 1$ and all $k_r \in \mathbb{Z}$*

$$(k_0, \ldots, k_n)(k_1, \ldots, k_{n-1}) - (k_0, \ldots, k_{n-1})(k_1, \ldots, k_n) = -1. \tag{3.76}$$

Proof. Straightforward proof by induction. □

Proposition 3.11. *For all $n \geq 1$ and all $k_r \in \mathbb{Z}$*

$$T^{k_0}ST^{k_1}S \ldots T^{k_n}S = \begin{cases} \begin{pmatrix} k_0 & -1 \\ 1 & 0 \end{pmatrix} & n = 0, \\ \begin{pmatrix} (k_0, \ldots, k_n) & -(k_0, \ldots, k_{n-1}) \\ (k_1, \ldots, k_n) & -(k_1, \ldots, k_{n-1}) \end{pmatrix} & n \geq 1. \end{cases} \tag{3.77}$$

Proof. Straightfoward proof by induction. □

Proposition 3.12. *Let $k_0, \ldots, k_n \in \mathbb{Z}$ be such that $k_r \geq 2$ for all r, and let ω be any complex number. Then $\big(T^{k_0}S \ldots T^{k_n}S\big)(\omega) = \omega$ if and only if $\omega = [\overline{k_0, \ldots, k_n}]$.*

Proof. If $n > 0$ it follows from Proposition 3.11 that

$$\omega = \big(T^{k_0}ST^{k_1}S \ldots T^{k_n}S\big)(\omega) \tag{3.78}$$

$$\iff \qquad \omega = \frac{(k_0, \ldots, k_n)\omega - (k_0, \ldots, k_{n-1})}{(k_1, \ldots, k_n)\omega - (k_1, \ldots, k_{n-1})} \tag{3.79}$$

$$\iff \qquad \omega = [k_0, \ldots, k_n, \omega] \tag{3.80}$$

$$\iff \qquad \omega = [\overline{k_0, \ldots, k_n}]. \tag{3.81}$$

It is straightfoward to show that the same is true when $n = 0$. □

Lemma 3.13. *If a sequence $k_0, k_1, \ldots$ is such that $k_j \geq 2$ for all j then for all $n \geq 1$*

$$(k_0, \ldots, k_n) \geq (k_0, \ldots, k_{n-1}) + 1, \tag{3.82}$$

and

$$(k_0, \ldots, k_n) \geq (k_1, \ldots, k_n) + 1. \tag{3.83}$$

In particular

$$(k_0, \ldots, k_n) \geq n + 2 \tag{3.84}$$

for all $n \geq 0$.

Proof. For the case $n = 1$

$$(k_0, k_1) - (k_0) = k_0(k_1 - 1) - 1 \geq k_0 - 1 \geq 1. \tag{3.85}$$

Suppose Eq. (3.82) holds for arbitrary $n \geq 1$. Then

$$\begin{aligned}(k_0, \dots, k_{n+1}) - (k_0, \dots, k_n) &= (k_{n+1} - 1)(k_0, \dots, k_n) - (k_0, \dots k_{n-1}) \\ &\geq (k_0, \dots, k_n) - (k_0, \dots k_{n-1}) \\ &\geq 1.\end{aligned} \tag{3.86}$$

To prove Eq. (3.83) use the reversal symmetry and Eq. (3.82) to deduce

$$(k_0, \dots, k_n) = (k_n, \dots, k_0) \geq (k_n, \dots, k_1) + 1 = (k_1, \dots, k_n) + 1. \tag{3.87}$$

Eq. (3.84) is an immediate consequence of Eq. (3.83). □

Lemma 3.14. *Let n be a non-negative integer, and let $k_0, \dots, k_n$ be a set of integers greater than 1. Let*

$$T^{k_0} S \dots T^{k_n} S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.88}$$

Then

$$\alpha \geq n + 2, \qquad \beta \leq -(n + 1), \qquad \gamma \geq n + 1, \qquad \delta \leq -n. \tag{3.89}$$

Proof. Immediate consequence of Proposition 3.11 and Lemma 3.13. □

Proposition 3.15. *Let Q be a quadratic form, let $\omega_\pm$ be its roots, and let $M \in \mathrm{GL}(2, \mathbb{Z})$. Then $M(\omega_\pm) = \omega_\pm$ if and only if $M^{\mathrm{T}} Q M = (\det M) Q$.*

Remark. This result makes the usual assumptions, that Q is integral and irreducible, but does not assume that it is primitive.

Proof. Define

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}. \tag{3.90}$$

where the assumption that Q is irreducible means a, c are both non-zero. Let Δ be the discriminant of Q.

Now suppose $M^{\mathrm{T}} Q M = (\det M) Q$. Rewriting the condition as $QM = (\det M)(M^{\mathrm{T}})^{-1} Q$ and doing the algebra one finds that it is equivalent to

$$a\beta + c\gamma = 0, \qquad \text{and} \qquad a(\alpha - \delta) + b\gamma. = 0 \tag{3.91}$$

Hence

$$\begin{aligned}M(\omega_\pm) &= \frac{\alpha\omega_\pm + \beta}{\gamma\omega_\pm + \delta} \\ &= \frac{\alpha(-b \pm \sqrt{\Delta}) + 2a\beta}{\gamma(-b \pm \sqrt{\Delta}) + 2a\delta} \qquad (3.92) \\ &= \frac{(-b\alpha + 2a\beta) \pm \alpha\sqrt{\Delta}}{a(\alpha + \delta) \pm \gamma\sqrt{\Delta}}. \qquad (3.93)\end{aligned}$$

Since

$$\big(a(\alpha + \delta) \pm \gamma\sqrt{\Delta}\big)\big(a(\alpha + \delta) \mp \gamma\sqrt{\Delta}\big) = a^2(\alpha + \delta)^2 - \gamma^2(b^2 - 4ac)$$

$$
\begin{aligned}
&= a^2(\alpha+\delta)^2 - a^2(\alpha-\delta)^2 - 4a^2\beta\gamma \\
&= 4a^2 \det M \qquad (3.94)
\end{aligned}
$$

it follows

$$M(\omega_\pm) = \frac{\big((-b\alpha + 2a\beta) \pm \alpha\sqrt{\Delta}\big)\big(a(\alpha+\delta) \mp \gamma\sqrt{\Delta}\big)}{4a^2 \det M}. \qquad (3.95)$$

Since

$$
\begin{aligned}
&a(\alpha+\delta)(-b\alpha+2a\beta) - \alpha\gamma(b^2-4ac) \\
&\qquad = a(\alpha+\delta)(-b\alpha+2a\beta) + ab\alpha(\alpha-\delta) - 4a^2\alpha\beta \\
&\qquad = ab\alpha\big((\alpha-\delta)-(\alpha+\delta)\big) + 2a^2\beta\big((\alpha+\delta)-2\alpha\big) \\
&\qquad = -2ab\alpha\delta - 2a^2\beta(\alpha-\delta) \\
&\qquad = -2ab\alpha\delta + 2ab\beta\gamma - 2ab\alpha\delta \\
&\qquad = -2ab \det M \qquad (3.96)
\end{aligned}
$$

and

$$
\begin{aligned}
&a\alpha(\alpha+\delta) - \gamma(-b\alpha+2a\beta) \\
&\qquad = a\alpha(\alpha+\delta) - a\alpha(\alpha-\delta) - 2a\beta\gamma \\
&\qquad = 2a\alpha\delta - 2a\beta\gamma \\
&\qquad = 2a \det M \qquad (3.97)
\end{aligned}
$$

this means

$$M(\omega_\pm) = \frac{-2ab\det M \pm 2a\det M\sqrt{\Delta}}{4a^2\det M} = \omega_\pm \qquad (3.98)$$

which establishes sufficiency.

Suppose, on the other hand, that $M(\omega_\pm) = \omega_\pm$. Then

$$
\begin{aligned}
&& \frac{\alpha\omega_\pm + \beta}{\gamma\omega_\pm + \delta} &= \omega_\pm && (3.99) \\
\implies && \gamma\omega_\pm^2 + (\delta-\alpha)\omega_\pm - \beta &= 0 && (3.100) \\
\implies && \begin{pmatrix} \gamma & \frac{\delta-\alpha}{2} \\ \frac{\delta-\alpha}{2} & -\beta \end{pmatrix} &= \lambda Q && (3.101)
\end{aligned}
$$

for some $\lambda \in \mathbb{Q}$ (if Q is primitive then $\lambda \in \mathbb{Z}$, but we don't need this assumption). This implies, firstly, that

$$
\begin{aligned}
\lambda^2\Delta &= -4\lambda^2 \det Q \\
&= -4\left(-\beta\gamma - \frac{(\alpha-\delta)^2}{4}\right) \\
&= 4\beta\gamma + (\alpha+\delta)^2 - 4\alpha\delta \\
&= (\alpha+\delta)^2 - 4\det M. \qquad (3.102)
\end{aligned}
$$

Secondly, it implies

$$\lambda SQ = \begin{pmatrix} \frac{\alpha-\delta}{2} & \beta \\ \gamma & \frac{\delta-\alpha}{2} \end{pmatrix} \tag{3.103}$$

$$\implies \qquad M = \frac{\alpha+\delta}{2} I + \lambda SQ. \tag{3.104}$$

Consequently

$$\begin{aligned} M^{\mathrm{T}} Q M &= \left(\frac{\alpha+\delta}{2} I - \lambda QS\right) Q \left(\frac{\alpha+\delta}{2} I + \lambda SQ\right) \\ &= \left(\frac{\alpha+\delta}{2}\right)^2 Q - \lambda^2 QSQSQ. \end{aligned} \tag{3.105}$$

It follows from Lemmas 2.10 and 2.11

$$(SQ)^2 = \frac{\Delta}{4} I. \tag{3.106}$$

In view of Eq. (3.102) this means

$$M^{\mathrm{T}} Q M = \left(\frac{(\alpha+\delta)^2 - \lambda^2 \Delta}{4}\right) Q = (\det M) Q. \tag{3.107}$$

□

Let $R = TS$. Then $\mathrm{PSL}(2, \mathbb{Z})$ is a free-group generated by the order 2 element S, and order 3 element R (see, for example, ref. [1]). In particular, each element of $\mathrm{PSL}(2, \mathbb{Z})$ can uniquely be written as a product of of the form

$$x_1 x_2 \dots x_n \tag{3.108}$$

where each x_j is equal to S, R, or R^2, and where if x_j is a power of one generator then x_{j+1} is a power of the other. We will refer to such a product as alternating. or is powers of S and R, where the power of S is always 1 and the power of R is always 1 or 2. Define the length of an element of $\mathrm{PSL}(2, \mathbb{Z})$ to be the number of terms in its expansion (so I is length 0, R,R^2, S are length 1, etc). We say that that an element is type R–R (respectively type R^2–R^2) if its length is greater than 2 and its expansion begins and ends with the element R (respectively R^2). In other words it is type R–R if and only if it is of the form

$$\begin{array}{cc} RSR & \\ RSR^{n_1}SR, & n_1 = 1 \text{ or } 2 \\ RSR^{n_1}SR^{n_2}SR, & n_1, n_2 = 1 \text{ or } 2 \\ \vdots & \end{array}$$

and of type R^2–R^2 if and only if it is of the form

$$\begin{array}{cc} R^2SR^2 & \\ R^2SR^{n_1}SR^2, & n_1 = 1 \text{ or } 2 \\ R^2SR^{n_1}SR^{n_2}SR^2, & n_1, n_2 = 1 \text{ or } 2 \\ \vdots & \end{array}$$

Lemma 3.16. (1) *$M \in \mathrm{PSL}(2, \mathbb{Z})$ is type R–R (respectively R^2–R^2) if and only if M^{-1} is type R^2–R^2 (respectively R–R).*

(2) *Suppose M_1, $M_2 \in \mathrm{PSL}(2,\mathbb{Z})$ are both type R–R (respectively R^2–R^2). Then M_1M_2 is also type R–R (respectively R^2–R^2).*
(3) *Suppose $M, N \in \mathrm{PSL}(2,\mathbb{Z})$ are such that $M = N^k$ for some $k \in \mathbb{N}$. Then M is type R–R (respectively R^2–R^2) if and only N is type R–R (respectively R^2–R^2).*

Proof. (1) and (2) are immediate. Turning to (3), sufficiency follows from (2). To prove necessity write N as the alternating product

$$N = x_1 \dots x_n \tag{3.109}$$

(see above for the meaning of the term "alternating"). Then

$$M = \underbrace{x_1 \dots x_n}_{1} \underbrace{x_1 \dots x_n}_{2} \dots \underbrace{x_1 \dots x_n}_{k} \tag{3.110}$$

If x_1 and x_n are powers of different generators this expression is alternating. If, on the other hand, they are the powers of the same generator one of two things can happen. If $x_n x_1 = y \neq I$ then the length $nk - (k-1)$ expression

$$M = x_1 \underbrace{x_2 \dots x_{n-1}}_{1} y \underbrace{x_2 \dots x_{n-1}}_{2} y \dots y \underbrace{x_2 \dots x_{n-1}}_{k} x_n \tag{3.111}$$

is the unique alternating expression for M. If, on the other hand, $x_n x_1 = I$ then the expression for M reduces to the length $nk - 2(k-1)$ expression

$$M = x_1 \underbrace{x_2 \dots x_{n-1}}_{1} \underbrace{x_2 \dots x_{n-1}}_{2} \dots \underbrace{x_2 \dots x_{n-1}}_{k} x_n. \tag{3.112}$$

This expression is non-alternating because x_{n-1} and x_2 are powers of the same generator. There are again two possibilities: either $x_{n-1}x_2 \neq I$, in which case the right hand side of Eq. (3.112) reduces to an alternating expansion of length $nk - 3(k-1)$, or else $x_{n-1}x_2 = I$, in which case it reduces to a non-alternating expansion of length $nk - 4(k-1)$. Since M has length at least 3 this process of length reduction must stop at some point with an alternating expression of the form

$$M = x_1 \dots x_l P x_{n-l+1} \dots x_n \tag{3.113}$$

for some $l \leq n/2$ and group element P of length at least one. One sees from all this that, in every case, the unique alternating expression for M must begin with x_1 and end with x_n. It follows that if M is type R–R (respectively R^2–R^2) then $x_1 = x_n = R$ (respectively $x_1 = x_n = R^2$) and N is type R–R (respectively R^2–R^2). □

Lemma 3.17. (1) *Let $M \in \mathrm{PSL}(2,\mathbb{Z})$ be arbitrary. Then M is type R–R if and only if*

$$M = T^{k_0} S \dots T^{k_n} S \tag{3.114}$$

for some non-negative integer n, and sequence of integers $k_0, \dots, k_n$ all greater than one.
(2) *The expansion in Eq. (3.114) is unique: if n, n' are non-negative integers and $k_0, \dots, k_n$, $k'_0, \dots, k'_{n'}$ sequences of integers all greater than one such that*

$$T^{k_0} S \dots T^{k_n} S = T^{k'_0} S \dots T^{k'_{n'}} S \tag{3.115}$$

then $n' = n$ and $k'_j = k_j$ for all j.

Proof. Define P_k recursively by

$$P_1 = S, \tag{3.116}$$
$$P_k = P_{k-1}RS, \qquad (k \geq 2). \tag{3.117}$$

Then for all $k \geq 2$

$$T^k S = (RS)^k S = RP_{k-1}R \tag{3.118}$$

In particular $T^k S$ is type R-R. In view of Lemma 3.16 this proves sufficiency in (1). To prove necessity let M be type R–R. Then its alternating expansion is of the form

$$\begin{aligned} M &= \underbrace{\underbrace{RS\dots RS}_{l_0 \geq 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots RS}_{l_1 \geq 1 \text{ occurrences of } S} \quad \dots \quad \underbrace{SR\dots RS}_{l_{n-1} \geq 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots SR}_{l_n \geq 1 \text{ occurrences of } S}}_{n \geq 0 \text{ occurrences of } R^2} \\ &= (RP_{l_0}R)(RP_{l_1}R)\dots(RP_{l_{n-1}}R)(RP_{l_n}R) \\ &= T^{l_0+1}ST^{l_1+1}S\dots T^{l_{n-1}}ST^{l_n}S. \end{aligned} \tag{3.119}$$

The same argument also proves statement (2): for Eq. (3.115) implies

$$\begin{aligned} &\underbrace{\underbrace{RS\dots RS}_{k_0 - 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots RS}_{k_1 - 1 \text{ occurrences of } S} \quad \dots \quad \underbrace{SR\dots RS}_{k_{n-1} - 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots SR}_{k_n - 1 \text{ occurrences of } S}}_{n \text{ occurrences of } R^2} \\ &= \underbrace{\underbrace{RS\dots RS}_{k'_0 - 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots RS}_{k'_1 - 1 \text{ occurrences of } S} \quad \dots \quad \underbrace{SR\dots RS}_{k'_{n'-1} - 1 \text{ occurrences of } S} \quad R^2 \quad \underbrace{SR\dots SR}_{k'_{n'} - 1 \text{ occurrences of } S}}_{n' \text{ occurrences of } R^2}. \end{aligned} \tag{3.120}$$

In view of the uniqueness of the alternating expansion this means $n' = n$ and $k'_j = k_j$ for all j. □

We say that $[\overline{k_0, \dots, k_n}]$ is the minimal HJ expansion of ω if $k_0, \dots, k_n$ is the shortest sequence such that $\omega = [\overline{k_0, \dots, k_n}]$.

Proposition 3.18. *Let $Q = \langle a, b, c\rangle$ be an HJ reduced form such that $a > 0$, let $\omega_\pm$ be its roots and let $[\overline{k_0, \dots, k_n}]$ be the minimal HJ expansion of ω_+. Then*

$$L_Q = T^{k_0}S\dots T^{k_n}S. \tag{3.121}$$

Remark. The restriction to forms for which $a > 0$ is essential. Indeed, suppose $a < 0$, and let $M = T^{k_0}S\dots T^{k_n}S$. Then Proposition 3.12 implies $M(-\omega_+) = -\omega_+$, while Proposition 3.15 implies $L_Q(\omega_+) = \omega_+$. It follows that $M \neq L_Q$.

Let us note that every HJ reduced form with negative a is $\mathrm{GL}(2,\mathbb{Z})$-equivalent (though not necessarily $\mathrm{SL}(2,\mathbb{Z})$-equivalent) to a form with positive a. Indeed, let $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and let Q' be the form with matrix $(\det J)J^{\mathrm{T}}QJ$. Then $Q' = \langle -a, b, -c\rangle$. It is easy to see that Q' is also HJ-reduced.

Proof. Let $M = T^{k_0}S\dots T^{k_n}S$. Then it follows from Proposition 3.12 that $M(\omega_+) = \omega_+$ which in view of Proposition 3.15 means

$$M^{\mathrm{T}}QM = Q \tag{3.122}$$

implying that $M \in \mathcal{S}^{(1)}(Q)$ (the intersection of the stability group with $\mathrm{SL}(2,\mathbb{Z})$). Since $\mathcal{S}^{(1)}(Q)$ is generated by $-I$ and L_Q (where L_Q is as defined in Eqs. (3.19), (3.20)) it follows that

$$M = \pm N^m \tag{3.123}$$

where N is one of L_Q, L_Q^{-1} and m is a positive integer. We now proceed in two steps: we first show that $M = L_Q^m$ and we then show that $m = 1$.

It follows from Lemma 3.14 that if we write

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.124}$$

then

$$-\alpha < \beta < \delta \le 0 < \gamma < \alpha. \tag{3.125}$$

We now derive the corresponding inequalities satisfied by the matrix elements of L_Q^m. With notations as in Subsections 2.2, 2.3 and writing $r = r_{\min}(f)$ we find

$$\begin{aligned} L_Q^m &= \eta_Q(u_f^m) \\ &= \eta_Q\left(\frac{d_{mr}-1}{2} + \frac{f_{mr}}{2f}\sqrt{\Delta}\right) \\ &= \left(\frac{d_{mr}-1}{2}\right) I + \frac{f_{mr}}{f} SQ \\ &= \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \end{aligned} \tag{3.126}$$

where

$$\alpha' = \frac{d_{mr}-1}{2} - \frac{f_{mr}b}{2f}, \tag{3.127}$$

$$\beta' = -\frac{f_{mr}c}{f}, \tag{3.128}$$

$$\gamma' = \frac{f_{mr}a}{f}, \tag{3.129}$$

$$\delta' = \frac{d_{mr}-1}{2} + \frac{f_{mr}b}{2f}. \tag{3.130}$$

The fact that Q is HJ-reduced means

$$b < -\sqrt{\Delta} < 0. \tag{3.131}$$

Hence

$$b^2 > \Delta = b^2 - 4ac \tag{3.132}$$

$$\implies ac > 0. \tag{3.133}$$

Since a is assumed to be a positive integer this means c is a positive integer also. Next observe that it follows from Proposition 2.2 and the definition of $f_r = f_{r_{\min}(f)}$ that

$$f_{mr} \ge f_r \ge f. \tag{3.134}$$

Hence

$$\begin{aligned} (d_{mr}-1)^2 &= f_{mr}^2\Delta_0 + 4 \\ &= \frac{f_{mr}^2(b^2-4ac)}{f^2} + 4 \end{aligned}$$

$$= \frac{f_{mr}^2 b^2}{f^2} + \frac{4 f_{mr}^2 (1-ac)}{f^2}$$
$$\leq \frac{f_{mr}^2 b^2}{f^2} \tag{3.135}$$

where we used the fact that ac is a positive integer in the last step. Hence

$$-\alpha' < \beta' < \delta' \leq 0 < \gamma' < \alpha'. \tag{3.136}$$

So the matrix elements of L_Q^m satisfy the same inequalities as M. It also follows that the matrix elements of

$$-L_Q^m = \begin{pmatrix} -\alpha' & -\beta' \\ -\gamma' & -\delta' \end{pmatrix}, \qquad L_Q^{-m} = \begin{pmatrix} \delta' & -\beta' \\ -\gamma' & \alpha' \end{pmatrix}, \qquad -L_Q^{-m} = \begin{pmatrix} -\delta' & \beta' \\ \gamma' & -\alpha' \end{pmatrix}, \tag{3.137}$$

do not satisfy them. So

$$M = L_Q^m. \tag{3.138}$$

It remains to show that $m = 1$. It follows from Lemma 3.17 that M is type R–R. In view of Lemma 3.16 this means L_Q is also type R-R. By a further application of Lemma 3.17 it follows that

$$L_Q = T^{k'_0} S \dots T^{k'_{n'}} S \tag{3.139}$$

for some $n' \geq 0$ and $k_j \geq 2$. Hence

$$M = \left(T^{k'_0} S \dots T^{k'_{n'}} S\right)\Big)^m \tag{3.140}$$

It is shown in Lemma 3.17) that the expansion of a type R-R matrix as a product of operators $T^k S$ is unique. Hence $n+1 = m(n'+1)$ and the sequence $k_0, \dots, k_n$ is the concatenation of m copies of the sequence $k'_0, \dots, k'_{n'}$. Since $[\overline{k_0, \dots, k_n}]$ is the minimal HJ expansion of ω_+ it follows that $m = 1$, and the sequences are identical. □

Proposition 3.19. *Let Q be a primitive quadratic form, let ω either of its two roots, and let*

$$\omega = [l_0, l_1, \dots, l_m, \overline{k_0, \dots, k_n}] \tag{3.141}$$

be the HJ expansion of ω. Assume that the expansion is minimal in the sense that $k_0, \dots, k_n$ is not the concatenation of two or more identical subsequences. Define

$$M = \left(T^{l_0} S \dots T^{l_m} S\right)\left(T^{k_0} S \dots T^{k_n} S\right)\left(T^{l_0} S \dots T^{l_m} S\right)^{-1}. \tag{3.142}$$

Then $L_Q = M$ or M^{-1}.

Remark. Either way $\mathcal{S}^{(1)}(Q)$ (the intersection of the stability group with $\mathrm{SL}(2,\mathbb{Z})$) is generated by $-I$ and M.

Proof. Let σ be the non-trivial automorphism of K and define

$$\omega'_+ = [\overline{k_0, \dots, k_n}], \qquad \omega'_- = \sigma(\omega'_+). \tag{3.143}$$

Let $Q' = \langle a', b', c' \rangle$ be the unique primitive quadratic form with $a > 0$ and having $\omega'_\pm$ as roots. Then it follows from Proposition 3.18 that

$$L_{Q'} = T^{k_0} S \dots T^{k_n} S \tag{3.144}$$

Now define

$$N = T^{l_0} S \dots T^{l_m} S, \qquad Q'' = \left(N^{-1}\right)^{\mathrm{T}} Q' N^{-1} \tag{3.145}$$

Then it follows from Proposition 3.4 that $N(\omega'_\pm)$ are the roots of Q''. On the other hand

$$\begin{aligned} N(\omega'_+) &= T^{l_0}S\dots T^{l_{m-1}}S\left(l_m - \frac{1}{\omega'_+}\right) \\ &= T^{l_0}S\dots T^{l_{m-2}}S\left(l_{m-1} - \cfrac{1}{l_m - \cfrac{1}{\omega'_+}}\right) \\ &= [l_0,\dots,l_m,\omega'_+] \\ &= \omega \end{aligned} \tag{3.146}$$

and $N(\omega'_-) = \sigma(\omega)$. So $N(\omega'_\pm)$ are also the roots of Q. Since Q, Q'' are both primitive it follows that $Q = \pm Q''$. Referring to Eq. (3.19) we deduce

$$\begin{aligned} L_Q &= \left(\frac{d_r - 1}{2}\right) I + \frac{f_r}{f} SQ \\ &= \left(\frac{d_r - 1}{2}\right) I \pm S\left(N^{-1}\right)^{\mathrm{T}} Q' N^{-1} \\ &= \left(\frac{d_r - 1}{2}\right) I \pm NSQ'N^{-1} \\ &= NL_{\pm Q'}N^{-1} \end{aligned} \tag{3.147}$$

In view of Proposition 3.5 this means

$$\begin{aligned} L_Q &= N(L_{Q'})^{\pm 1}N^{-1} \\ &= M^{\pm 1}. \end{aligned} \tag{3.148}$$

□

3.6. **Special Cases.** It turns out that the purely periodic HJ number has period 1 if and only if it is the Zauner unit for some dimension:

Proposition 3.20. *For all $k \in \mathbb{Z}$ such that $k > 2$*

$$[\overline{k}] = \frac{k + \sqrt{k^2 - 4}}{2} \tag{3.149}$$

is the Zauner unit in dimension $k+1$.

Proof. One has

$$x = [\overline{k}] \tag{3.150}$$

$$\iff \qquad x = [k, x] \tag{3.151}$$

$$\iff \qquad x = k - \frac{1}{x} \tag{3.152}$$

$$\iff \qquad x^2 - kx + 1 = 0 \tag{3.153}$$

$$\iff \qquad x = \frac{k \pm \sqrt{k^2 - 4}}{2}. \tag{3.154}$$

The fact that $\lceil x \rceil = k \geq 3$ means

$$x = \frac{k + \sqrt{k^2 - 4}}{2}. \tag{3.155}$$

□

Proposition 3.21. *Let Δ be a discriminant with conductor f, Q the corresponding principal HJ reduced form, and $\omega_\pm$ the roots of Q. Let $r = r_{\min}(f)$. Then ω_+ is period 1 if and only if $f = f_r$. In that case*

$$\omega_+ = u_f = u_1^r = \frac{d_r - 1 + \sqrt{\Delta_r}}{2} \tag{3.156}$$

(the Zauner unit in dimension d_r),

$$\omega_+ = [\overline{(d_r - 1)}], \tag{3.157}$$

$$Q = \langle 1, -(d_r - 1), 1 \rangle \tag{3.158}$$

and

$$L_Q = T^{d_r - 1} S. \tag{3.159}$$

Proof. Suppose $f = f_r$. Then $\Delta = \Delta_r = (d_r - 1)^2 - 4$ implying

$$\frac{(d_r - 1)^2}{4} - 1 < \frac{\Delta}{4} < \frac{(d_r - 1)^2}{4} \tag{3.160}$$

Since $d_r > 3$

$$\frac{(d_r - 1)^2}{4} - 1 > \left(\frac{d_r - 1}{2} - 1 \right)^2 \tag{3.161}$$

Hence

$$\frac{d_r - 1}{2} - 1 < \frac{\sqrt{\Delta}}{2} < \frac{d_r - 1}{2}. \tag{3.162}$$

If Δ is even then d_r is odd implying $(d_r - 1)/2 \in \mathbb{N}$ and consequently

$$\left\lceil \frac{\sqrt{\Delta}}{2} \right\rceil = \frac{d_r - 1}{2}. \tag{3.163}$$

If Δ is odd then d_r is even implying $(d_r - 2)/2 \in \mathbb{N}$. Since

$$\frac{d_r - 2}{2} - 1 < \frac{\sqrt{\Delta} - 1}{2} < \frac{d_r - 2}{2}. \tag{3.164}$$

this means

$$\left\lceil \frac{\sqrt{\Delta} - 1}{2} \right\rceil = \frac{d_r - 2}{2}. \tag{3.165}$$

Either way the principal HJ reduced form is

$$Q = \begin{pmatrix} 1 & -\frac{d_r - 1}{2} \\ -\frac{d_r - 1}{2} & 1 \end{pmatrix}. \tag{3.166}$$

Its roots are

$$\omega_\pm = \frac{d_r - 1 \pm \sqrt{\Delta_r}}{2}. \tag{3.167}$$

In particular $\omega_+ = u_f = u_1^r$. In view of Proposition 3.20 this means

$$\omega_+ = [\overline{(d_r - 1)}]. \tag{3.168}$$

Finally, it follows from Eq. (3.19)

$$\begin{aligned} L_Q &= \frac{d_r - 1}{2} I + SQ \\ &= \begin{pmatrix} d_r - 1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= T^{d_r - 1} S. \end{aligned} \tag{3.169}$$

Suppose, on the other hand, that $\omega_+ = [\overline{k}]$ for some integer k greater than 1. Then it follows from Proposition 3.20 that ω_+ is the Zauner unit for dimension $d_s = k + 1$, for some s. Hence

$$\omega_\pm = \frac{(d_s - 1) \pm \sqrt{(d_s - 1)^2 - 4}}{2}. \tag{3.170}$$

It follows from the first part of the proof that $\omega_\pm$ are the roots of the form $\langle 1, -(d_s - 1), 1\rangle$. Since $\langle 1, -(d_s - 1), 1\rangle$ and Q are both primitive, with positive leading coefficient, they must be equal (see footnote 2). In particular, their conductors must be equal. So $f = f_s$. Hence $r_{\min}(f) = s$ and $f_{r_{\min}(f)} = f$. □

4. Shintani's Modularity Formula and its Implications

4.1. Initial results.

$$\mathcal{H} = \{\tau \in \mathbb{C} \colon \operatorname{Im}(\tau) > 0\} \tag{4.1}$$

be the upper half-plane, and for all $z \in \mathbb{C}$, $\tau \in \mathcal{H}$ let $\delta(z, \tau)$ be the exponentiated q-Pochhammer symbol:

$$\delta(z, \tau) = (e^{2\pi i z}, e^{2\pi i \tau})_\infty = \prod_{n=0}^{\infty} \left(1 - e^{2\pi i (z + n\tau)}\right). \tag{4.2}$$

Proposition 4.1. *For all $z \in \mathbb{C} \setminus (-\infty, 0]$, $\tau \in \mathcal{H}$*

$$\delta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = f(z, \tau)\delta(z, \tau). \tag{4.3}$$

where

$$f(z, \tau) = 2 \sin\left(\frac{\pi z}{\tau}\right) e^{\frac{\pi i}{12\tau}\left(6z^2 + 6(1-\tau)z + \tau^2 - 3\tau + 1\right)} \tilde{S}_2(z, \tau), \tag{4.4}$$

$\tilde{S}_2(z, \tau) = S_2(z, (1, \tau))$ being the reduced double-sine function defined in Eq. (A.25).

Remark. I will refer to this as Shintani's modularity formula. It is a modified version of Proposition 5 in ref. [14].

Proof. Setting $\omega_1 = 1, \omega_2 = \tau$, $q = e^{2\pi i\tau}$, $q' = e^{-\frac{2\pi i}{\tau}}$ in Proposition 5 of ref. [14] gives

$$\begin{aligned}
\frac{1}{\tilde{S}_2(z,\tau)} &= e^{\frac{\pi i}{12}\left(3+\tau+\frac{1}{\tau}\right)+\frac{\pi i}{2}\left(\frac{z^2}{\tau}-\left(1+\frac{1}{\tau}\right)z\right)}\left(\frac{\prod_{n=0}^{\infty}(1-q^n e^{2\pi i z})}{\prod_{n=1}^{\infty}(1-q'^n e^{\frac{2\pi i z}{\tau}})}\right) \\
&= e^{\frac{\pi i}{12\tau}\left(6z^2-6(1+\tau)z+\tau^2+3\tau+1\right)}\left(\frac{\prod_{n=0}^{\infty}(1-q^n e^{2\pi i z})}{\prod_{n=0}^{\infty}\left(1-q'^n e^{\frac{2\pi i(z-1)}{\tau}}\right)}\right) \\
&= e^{\frac{\pi i}{12\tau}\left(6z^2-6(1+\tau)z+\tau^2+3\tau+1\right)}\left(\frac{\delta(z,\tau)}{\delta\left(\frac{z-1}{\tau},-\frac{1}{\tau}\right)}\right).
\end{aligned} \tag{4.5}$$

Hence

$$\delta\left(\frac{z-1}{\tau},-\frac{1}{\tau}\right) = e^{\frac{\pi i}{12\tau}\left(6z^2-6(1+\tau)z+\tau^2+3\tau+1\right)}\tilde{S}_2(z,\tau)\delta(z,\tau). \tag{4.6}$$

Replacing $z \to z+1$ and using $\delta(z+1,\tau) = \delta(z,\tau)$, $\tilde{S}_2(z+1,\tau) = \sin\left(\frac{\pi z}{\tau}\right)\tilde{S}_2(z,\tau)$ the statement follows. $\square$

Let $\langle\cdot,\cdot\rangle$ be the symplectic form defined on $\mathbb{C}^2\times\mathbb{C}^2$ by

$$\langle \mathbf{p},\mathbf{q}\rangle = \left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle = \det\begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix} = p_2 q_1 - p_1 q_2. \tag{4.7}$$

By an abuse of notation define, for arbitrary $\mathbf{p}\in\mathbb{C}^2$, $\tau\in\mathbb{C}$,

$$\langle \mathbf{p},\tau\rangle = \left\langle \mathbf{p}, \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right\rangle = p_2\tau - p_1. \tag{4.8}$$

Given

$$L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \tag{4.9}$$

define

$$L\mathbf{p} = \begin{pmatrix} \alpha p_1 + \beta p_2 \\ \gamma p_1 + \delta p_2 \end{pmatrix} \tag{4.10}$$

for all $\mathbf{p}\in\mathbb{C}^2$, and

$$L\tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \tag{4.11}$$

for all $\tau\in\mathbb{C}$. Note that this notation is potentially misleading in that

$$-(L\tau) = -\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \tag{4.12}$$

while

$$(-L)\tau = \frac{-\alpha\tau-\beta}{-\gamma\tau-\delta} = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}. \tag{4.13}$$

Lemma 4.2. *For all* $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, $\mathbf{p}, \mathbf{q} \in \mathbb{C}^2$, $\tau \in \mathbb{C}$

$$\langle L\mathbf{p}, L\mathbf{q} \rangle = \langle \mathbf{p}, \mathbf{q} \rangle \tag{4.14}$$

$$\langle L\mathbf{p}, L\tau \rangle = \frac{\langle \mathbf{p}, \tau \rangle}{\gamma\tau + \delta} \tag{4.15}$$

Proof.

$$\begin{aligned} \langle L\mathbf{p}, L\mathbf{q} \rangle &= \left\langle \begin{pmatrix} (\alpha p_1 + \beta p_2) & (\gamma p_1 + \delta p_2) \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} (\alpha q_1 + \beta q_2) & (\gamma q_1 + \delta q_2) \end{pmatrix}^{\mathrm{T}} \right\rangle \\ &= (\gamma p_1 + \delta p_2)(\alpha q_1 + \beta q_2) - (\alpha p_1 + \beta p_2)(\gamma q_1 + \delta q_2) \\ &= p_2 q_1 - p_1 q_2 \\ &= \langle \mathbf{p}, \mathbf{q} \rangle. \end{aligned} \tag{4.16}$$

Using this result we then find

$$\begin{aligned} \langle L\mathbf{p}, L\tau \rangle &= \frac{1}{\gamma\tau + \delta} \left\langle L\mathbf{p}, L \begin{pmatrix} \tau & 1 \end{pmatrix}^{\mathrm{T}} \right\rangle \\ &= \frac{\langle \mathbf{p}, \tau \rangle}{\gamma\tau + \delta}. \end{aligned} \tag{4.17}$$

□

4.2. **Comparison of my notation with Gene's.** Gene defines the functions

$$\delta[p](\tau) = \delta(p_1\tau + p_2, \tau) \tag{4.18}$$

$$\sigma_L[p](\tau) = \frac{\delta[p](\tau)}{\delta[p](L\tau)} \tag{4.19}$$

(ouijaboard, Eqs. (5.2), (5.3)). Let us see how these look when expressed in terms of the functions introduced in the last sub-section. Observe

$$\langle S\mathbf{p}, \tau \rangle = \left\langle \begin{pmatrix} (-p_2 & p_1 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} \tau & 1 \end{pmatrix}^{\mathrm{T}} \right\rangle = p_1\tau + p_2. \tag{4.20}$$

So

$$\delta[p](\tau) = \delta\big(\langle S\mathbf{p}, \tau \rangle, \tau\big) \tag{4.21}$$

$$\sigma_L[p](\tau) = \frac{\delta\big(\langle S\mathbf{p}, \tau \rangle, \tau\big)}{\delta\big(\langle S\mathbf{p}, L\tau \rangle, L\tau\big)} \tag{4.22}$$

Recall that the Shintani-Faddeev modular cocycle (SFMC) is defined by (Ouijaboard Eq. (2.5))

$$\sigma_L(z, \tau) = \frac{\delta\left(\frac{z}{\gamma\tau+\delta}, L\tau\right)}{\delta(z, \tau)} \tag{4.23}$$

for all $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Note that it follows from Lemma 4.2 that

$$\sigma_L\left(\langle \mathbf{p}, \tau \rangle, \tau\right) = \frac{\delta\left(\frac{\langle \mathbf{p}, \tau \rangle}{\gamma\tau+\delta}, L\tau\right)}{\delta(\langle \mathbf{p}, \tau \rangle, \tau)} = \frac{\delta(\langle L\mathbf{p}, L\tau \rangle, L\tau)}{\delta(\langle \mathbf{p}, \tau \rangle, \tau)} \tag{4.24}$$

In the following, instead of $\delta[p]$, $\sigma_L[p]$ I will work with the two functions $\delta\big(\langle \mathbf{p}, \tau \rangle, \tau\big)$, $\sigma_L(\langle \mathbf{p}, \tau \rangle, \tau)$. This notation has the advantage that for "most" L (see below for a precise definition of "most") σ_L

continues to a meromorphic function defined on the cut complex τ-plane (Ouijaboard[4], Theorem 12.4), whereas it is not clear to me that the same is true of $\sigma_L[p]$.

It is convenient to introduce the shorthand

$$\tilde{\sigma}_L(\mathbf{p},\tau) = \sigma_L\big(\langle \mathbf{p},\tau\rangle,\tau\big). \tag{4.25}$$

Notice that σ_L can be expressed in terms of $\tilde{\sigma}_L$ using

$$\sigma_L(z,\tau) = \tilde{\sigma}_L\left(\begin{pmatrix}-z & 0\end{pmatrix}^{\mathrm{T}},\tau\right). \tag{4.26}$$

So formulae expressed in terms of $\tilde{\sigma}_L$ can easily be converted to ones expressed in terms σ_L

Let $L_1 = \left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{smallmatrix}\right)$, $L_2 = \left(\begin{smallmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Then in terms of σ_L the cocycle condition reads

$$\sigma_{L_1L_2}(z,\tau) = \sigma_{L_1}\left(\frac{z}{\gamma_2\tau+\delta_2}, L_2\tau\right)\sigma_{L_2}(z,\tau) \tag{4.27}$$

while in terms of $\tilde{\sigma}_L$ it is

$$\tilde{\sigma}_{L_1L_2}(\mathbf{p},\tau) = \tilde{\sigma}_{L_1}(L_2\mathbf{p}, L_2\tau)\tilde{\sigma}_{L_2}(\mathbf{p},\tau). \tag{4.28}$$

As noted above the two quantities σ_L and $\tilde{\sigma}_L$ are interchangeable. In the following I will mainly work with $\tilde{\sigma}_L$ since that is the function that is directly relevant to the problem of interest. However, there are certain results which are more easily stated in terms of σ_L. This is true, in particular, of statements regarding the domain of analyticity. It can be seen from the definition

$$\delta(z,\tau) = \prod_{m=0}^{\infty}\left(1-e^{2\pi i(z+m\tau)}\right) \tag{4.29}$$

that $\delta(z,\tau)$ is an analytic function on the set $\mathbb{C}\times\mathcal{H}$ whose zero-set is the variety $\mathcal{R}_\sigma = \{(z,\tau)\in\mathbb{C}\times\mathcal{H}\colon r+m\tau\colon r,m\in\mathbb{Z},\ m\leq 0\}$. So $\sigma_L(z,\tau)$ is a meromorphic function[5] on $\mathbb{C}\times\mathcal{H}$ whose singularity set is a subset of $\mathcal{R}_\sigma$ (the singularity may be a proper subset of $\mathcal{R}_\sigma$ since it can happen that zeros in the denominator are cancelled by zeros in the numerator—see below).

4.3. **Properties.**

Proposition 4.3. (1) *Let $L\in\mathrm{SL}(2,\mathbb{Z})$. Then*

$$\tilde{\sigma}_L(\mathbf{p},\tau) = \tilde{\sigma}_{T^kL}(\mathbf{p},\tau) \tag{4.30}$$

for all $k\in\mathbb{Z}$. In particular

$$\tilde{\sigma}_{T^k}(\mathbf{p},\tau) = 1 \tag{4.31}$$

for all $k\in\mathbb{Z}$.

(2) *For all $L\in\mathrm{SL}(2,\mathbb{Z})$*

$$\tilde{\sigma}_{L^{-1}}(\mathbf{p},\tau) = \frac{1}{\tilde{\sigma}_L(L^{-1}\mathbf{p}, L^{-1}\tau)} \tag{4.32}$$

(3)

$$\tilde{\sigma}_S(\mathbf{p},\tau) = \tilde{\sigma}_R(\mathbf{p},\tau) = f(\langle\mathbf{p},\tau\rangle,\tau) \tag{4.33}$$

(where $R = TS$, as usual).

---

[4] Note, however, that the Theorem as it stands is not quite correct—see below.

[5] See, for example, ref. [6] for the definition of a meromorphic function of several variables

Proof. It follows from the definition that

$$\tilde{\sigma}_{T^k}(\mathbf{p},\tau) = \frac{\delta\left(\langle \mathbf{p},\tau\rangle, \tau+k\right)}{\delta(\langle \mathbf{p},\tau\rangle,\tau)} = 1 \tag{4.34}$$

for all $\mathbf{p}$, τ. In view of the cocycle property this means

$$\tilde{\sigma}_{T^k L}(\mathbf{p},\tau) = \tilde{\sigma}_{T^k}(L\mathbf{p}, L\tau)\tilde{\sigma}_L(\mathbf{p},\tau) = \tilde{\sigma}_L(\mathbf{p},\tau) \tag{4.35}$$

To prove part 2 observe that it follows from part 1 together with the cocycle property that

$$1 = \tilde{\sigma}_{LL^{-1}}(\mathbf{p},\tau) = \tilde{\sigma}_L(L^{-1}\mathbf{p}, L^{-1}\tau)\tilde{\sigma}_{L^{-1}}(\mathbf{p},\tau). \tag{4.36}$$

Finally, it follows from the definition and Proposition 4.1

$$\tilde{\sigma}_S(\mathbf{p},\tau) = \frac{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{\tau}, -\frac{1}{\tau}\right)}{\delta(\langle \mathbf{p},\tau\rangle,\tau)} = f(\mathbf{p},\tau). \tag{4.37}$$

In view of Eq. (4.28) and part 1 this means

$$\tilde{\sigma}_R(\mathbf{p},\tau) = \tilde{\sigma}_T(S\mathbf{p}, S\tau)\tilde{\sigma}_S(\mathbf{p},\tau) = f(\mathbf{p},\tau). \tag{4.38}$$

□

Proposition 4.4. *For all* $\tau \in \mathbb{C} \setminus (-\infty, 0]$, $z \notin \mathbb{Z} + \tau\mathbb{Z}$

$$\tilde{S}_2(z,\tau)\tilde{S}_2(-z,\tau) = -\frac{1}{4}\csc(\pi z)\csc\left(\frac{\pi z}{\tau}\right) \tag{4.39}$$

Proof. Suppose z, τ are both real, $0 < z < 1$ and $0 < \tau$. Then $(z+\tau,\tau)$, $(1-z,\tau)$ are both in the domain of definition of the RHS of Eq. (A.62) and we can write

$$\begin{aligned}\tilde{S}_2(z+\tau,\tau)\tilde{S}_2(1-z,\tau) &= \exp\Bigg[\int_0^\infty \frac{1}{t}\Bigg(\frac{\sinh\left(\left(\frac{1+\tau}{2} - z - \tau\right)t\right) + \sinh\left(\left(\frac{1+\tau}{2} - 1 + z\right)t\right)}{2\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{\tau t}{2}\right)} \\ &\qquad - \frac{2\left(\frac{1+\tau}{2} - (z+\tau)\right) + 2\left(\frac{1+\tau}{2} - (1-z)\right)}{\tau t}\Bigg)dt\Bigg] \\ &= 1\end{aligned} \tag{4.40}$$

Using Eqs. (A.36), (A.38) we deduce

$$\tilde{S}_2(z,\tau)\tilde{S}_2(-z,\tau) = -\frac{1}{4}\csc(\pi z)\csc\left(\frac{\pi z}{\tau}\right) \tag{4.41}$$

for z, τ real and $0 < z < 1$, $0 < \tau$. Finally we use analyticity to conclude that the result holds for all $\tau \in \mathbb{C} \setminus (-\infty, 0]$, $z \notin \mathbb{Z} + \tau\mathbb{Z}$. □

Corollary 4.5. *The function*

$$g(z,\tau) = \tilde{S}_2(z,\tau)\tilde{S}_2(-z,\tau) \tag{4.42}$$

continues to a meromorphic function on $\mathbb{C} \times \mathbb{C}$ *with singularity set* $\mathbb{Z} \times \tau\mathbb{Z}$.

Remark. This is a striking result, if it is true[6] that $\tilde{S}_2$ is singular for all τ on the cut $(-\infty, 0]$. It is not impossible for the product of two singular functions to be non-singular. It does, however, say something interesting about the singularities that they should cancel so neatly for all values of τ. Another example of the cancellation of singularities is discussed in the remark to Corollary 4.8.

[6] I put it this way because I do not know of a proof that this is definitely case.

Proposition 4.6. *For $\tau \in \mathcal{H}$*

$$\sigma_{-I}(z,\tau) = 4ie^{-\pi iz}\sin(\pi z)\sin\left(\frac{\pi z}{\tau}\right)\tilde{S}_2(z,\tau)\tilde{S}_2(-z,-\tau) \tag{4.43}$$

and

$$\begin{aligned}\sigma_{-I}(z,\tau) = 8ie^{-\pi iz}\sin\left(\pi z\right)\sin\left(\frac{\pi z}{\tau}\right)\sin\left(\frac{\pi z}{\tau-1}\right)& \\ \times \tilde{S}_2(z,\tau)\tilde{S}_2\left(\frac{z}{\tau},\frac{\tau-1}{\tau}\right)\tilde{S}_2\left(\frac{z}{\tau-1},-\frac{1}{\tau-1}\right)&\end{aligned} \tag{4.44}$$

Remark. Eq. (4.43) implies[7] that $\sigma_{-I}(z,\tau)$ is singular everywhere on the real τ-axis, which means in turn that Theorem 12.4 in Ouijaboard is not correct as it stands. This is discussed in detail in Subsection 4.4.

Proof. It follows from Proposition 4.3, Eq. (A.29), and the fact that $S^2 = -I$ that

$$\begin{aligned}\tilde{\sigma}_{-I}(\mathbf{p},\tau) &= \tilde{\sigma}_S(S\mathbf{p},S\tau)\tilde{\sigma}_S(\mathbf{p},\tau) \\ &= f\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau},-\frac{1}{\tau}\right)f\big(\langle\mathbf{p},\tau\rangle,\tau\big) \\ &= -4\sin\big(\pi\langle\mathbf{p},\tau\rangle\big)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau}\right)e^{-\frac{\pi i\tau}{12}\left(6\frac{\langle\mathbf{p},\tau\rangle^2}{\tau^2}+6\left(1+\frac{1}{\tau}\right)\frac{\langle\mathbf{p},\tau\rangle}{\tau}+\frac{1}{\tau^2}+\frac{3}{\tau}+1\right)} \\ &\qquad\times e^{\frac{\pi i}{12\tau}\left(\langle\mathbf{p},\tau\rangle^2+6(1-\tau)\langle\mathbf{p},\tau\rangle+\tau^2-3\tau+1\right)}\tilde{S}_2\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau},-\frac{1}{\tau}\right))\tilde{S}_2\big(\langle\mathbf{p},\tau\rangle,\tau\big) \\ &= 4ie^{-\pi i\langle\mathbf{p},\tau\rangle}\sin\big(\pi\langle\mathbf{p},\tau\rangle\big)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau}\right)\tilde{S}_2\big(\langle\mathbf{p},\tau\rangle,\tau\big)\tilde{S}_2\big(-\langle\mathbf{p},\tau\rangle,-\tau\big)\end{aligned} \tag{4.45}$$

It follows from Proposition 4.3, and the fact that $R^3 = -I$ that

$$\begin{aligned}\tilde{\sigma}_{-I}(\mathbf{p},\tau) &= \tilde{\sigma}_R(R^2\mathbf{p},R^2\tau)\tilde{\sigma}_R(R\mathbf{p},R\tau)\tilde{\sigma}_R(\mathbf{p},\tau) \\ &= f\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau-1},-\frac{1}{\tau-1}\right)f\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau},\frac{\tau-1}{\tau}\right)f\big(\langle\mathbf{p},\tau\rangle,\tau\big) \\ &= -8\sin\big(\pi\langle\mathbf{p},\tau\rangle\big)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau-1}\right)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau}\right) \\ &\quad\times e^{-\frac{\pi i(\tau-1)}{12}\left(\frac{6\langle\mathbf{p},\tau\rangle^2}{(\tau-1)^2}+6\left(1+\frac{1}{\tau-1}\right)\frac{\langle\mathbf{p},\tau\rangle}{\tau-1}+\frac{1}{(\tau-1)^2}+\frac{3}{\tau-1}+1\right)} \\ &\quad\times e^{\frac{\pi i\tau}{12(\tau-1)}\left(\frac{6\langle\mathbf{p},\tau\rangle^2}{\tau^2}+6\left(1-\frac{\tau-1}{\tau}\right)\frac{\langle\mathbf{p},\tau\rangle}{\tau}+\frac{(\tau-1)^2}{\tau^2}-\frac{3(\tau-1)}{\tau}+1\right)} \\ &\quad\times e^{\frac{\pi i}{12\tau}\left(\langle\mathbf{p},\tau\rangle^2+6(1-\tau)\langle\mathbf{p},\tau\rangle+\tau^2-3\tau+1\right)} \\ &\quad\times\tilde{S}_2\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau-1},-\frac{1}{\tau-1}\right))\tilde{S}_2\left(\frac{\langle\mathbf{p},\tau\rangle}{\tau},\frac{\tau-1}{\tau}\right)\tilde{S}_2\big(\langle\mathbf{p},\tau\rangle,\tau\big) \\ &= 8ie^{-\pi i\langle\mathbf{p},\tau\rangle}\sin\big(\pi\langle\mathbf{p},\tau\rangle\big)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau}\right)\sin\left(\frac{\pi\langle\mathbf{p},\tau\rangle}{\tau-1}\right)\end{aligned}$$

[7]Assuming that the double sine function is singular for τ negative real.

$$\times \tilde{S}_2(\langle \mathbf{p}, \tau\rangle, \tau)\tilde{S}_2\left(\frac{\langle \mathbf{p}, \tau\rangle}{\tau}, \frac{\tau-1}{\tau}\right)\tilde{S}_2\left(\frac{\langle \mathbf{p}, \tau\rangle}{\tau-1}, -\frac{1}{\tau-1}\right)) \tag{4.46}$$

The result follows upon making the replacement $\mathbf{p} \to \begin{pmatrix}-z & 0\end{pmatrix}^{\mathrm{T}}$ in these expressions. ☐

Corollary 4.7. *The product $\sigma_{-I}(z,\tau)\sigma_{-I}(z,-\tau)$ continues to a holomorphic function on $\mathbb{C}\times\mathbb{C}$. Specifically*

$$\sigma_{-I}(z,\tau)\sigma_{-I}(z,-\tau) = -16e^{-2\pi i z}. \tag{4.47}$$

Remark. As noted in the remark to Proposition 4.6 the function $\sigma_{-I}(z,\tau)$ is singular for all $\tau\in\mathbb{R}$. So this is another instance of the cancellation of singularities noted in the remarks to Corollaries 4.5, 4.8.

Proof. It follows from Eq. (4.43) that

$$\sigma_{-I}(z,\tau) = 4ie^{-\pi i z}\tilde{S}_2(z+\tau,\tau)\tilde{S}_2(1-z,-\tau) \tag{4.48}$$

while it follows from Proposition 4.4 that

$$\tilde{S}_2(z+\tau,\tau)\tilde{S}_2(1-z,\tau) = 1. \tag{4.49}$$

Hence

$$\begin{aligned}\sigma_{-I}(z,\tau)\sigma_{-I}(z,-\tau) &= -16e^{-2\pi i z}\tilde{S}_2(z+\tau,\tau)\tilde{S}_2(1-z,-\tau) \qquad (4.50)\\ &\qquad\times \tilde{S}_2(z-\tau,-\tau)\tilde{S}_2(1-z,\tau)\\ &= -16e^{-2\pi i z}. \qquad (4.51)\end{aligned}$$

☐

Corollary 4.8. *The ratios*

$$\frac{\tilde{S}_2(z,\tau-1)}{\tilde{S}_2(z,\tau)}, \qquad \frac{\tilde{S}_2\left(z,\frac{1}{\tau-1}\right)}{\tilde{S}_2((\tau-1)z,\tau)} \tag{4.52}$$

analytically continue to meromorphic functions on $\mathbb{C}\times(\mathbb{C}\setminus[0,1])$. On this domain one has

$$\frac{\tilde{S}_2(z,\tau-1)}{\tilde{S}_2(z,\tau)} = \tilde{S}_2\left(\frac{z}{\tau-1}+1, \frac{\tau}{\tau-1}\right), \tag{4.53}$$

$$\frac{\tilde{S}_2\left(z,\frac{1}{\tau-1}\right)}{\tilde{S}_2((\tau-1)z,\tau)} = \tilde{S}_2\left(z+1, \frac{\tau}{\tau-1}\right). \tag{4.54}$$

Remark. We saw in the remark to Corollary 4.5 that the product of two double sine functions with second argument on the negative real line can be non-singular. We now see that the ratio of two such functions can also be non-singular. Indeed, if τ is negative real then the left-hand sides of Eqs. (4.53), (4.54) are the ratios of double sine functions with second arguments in the interval $(-\infty,0)$, whereas the right-hand sides are double sine functions with second arguments in the interval $(0,1)$.

Proof. Equating the right hand sides of Eqs. (4.43), (4.44) and using Eqs. (A.29) (A.36) gives, for $\tau\in\mathcal{H}$

$$\tilde{S}_2(-z,-\tau) = 2\sin\left(\frac{\pi z}{\tau-1}\right)\tilde{S}_2\left(\frac{z}{\tau}, \frac{\tau-1}{\tau}\right)\tilde{S}_2\left(\frac{z}{\tau-1}, -\frac{1}{\tau-1}\right)$$

$$= \tilde{S}_2\left(\frac{z}{\tau}+1, \frac{\tau-1}{\tau}\right)\tilde{S}_2\left(-z, 1-\tau\right). \tag{4.55}$$

The two sides of this equation are meromorphic on the set $\mathbb{C}\times\mathbb{C}\setminus[0,\infty)$. Making the replacements $z\to -z$, $\tau\to 1-\tau$ and rearranging we find that

$$\frac{\tilde{S}_2(z,\tau-1)}{\tilde{S}_2(z,\tau)} = \tilde{S}_2\left(\frac{z}{\tau-1}+1, \frac{\tau}{\tau-1}\right) \tag{4.56}$$

as meromorphic functions on the set $\mathbb{C}\times\mathbb{C}\setminus(-\infty,1]$. We now observe that the RHS is meromorphic on the set $\mathbb{C}\times\mathbb{C}\setminus[0,1]$. It follows that the LHS continues to a meromorphic function on this set.

Making the replacement $\tau\to\frac{\tau}{\tau-1}$ in Eq. (4.56) and making another application of Eq. (A.36) we find

$$\begin{aligned}\frac{\tilde{S}_2\left(z,\frac{1}{\tau-1}\right)}{\tilde{S}_2\left(z,\frac{\tau}{\tau-1}\right)} &= \tilde{S}_2\big((\tau-1)z+1,\tau\big)\\ &= 2\sin\left(\frac{\pi(\tau-1)z}{\tau}\right)\tilde{S}_2\big((\tau-1)z,\tau\big)\end{aligned} \tag{4.57}$$

as meromorphic functions on $\mathbb{C}\times\mathbb{C}\setminus(-\infty,1]$. Rearranging and making another application of Eq. (A.29) gives

$$\begin{aligned}\frac{\tilde{S}_2\left(z,\frac{1}{\tau-1}\right)}{\tilde{S}_2\big((\tau-1)z,\tau\big)} &= 2\sin\left(\frac{\pi(\tau-1)z}{\tau}\right)\tilde{S}_2\left(z,\frac{\tau}{\tau-1}\right)\\ &= \tilde{S}_2\left(z+1,\frac{\tau}{\tau-1}\right)\end{aligned} \tag{4.58}$$

as meromorphic functions on $\mathbb{C}\times\mathbb{C}\setminus(-\infty,1]$. As before the RHS is meromorphic on the set $\mathbb{C}\times\mathbb{C}\setminus[0,1]$, implying that the LHS continues to a meromorphic function on this set.

□

Corollary 4.9.

$$\delta(z,\tau)^2 = -4e^{\frac{\pi i}{12}}e^{2\pi i z}\frac{\tilde{S}_2(\tau-z,\tau)\tilde{S}_2(1-\tau+z,-\tau)\theta\left(z+\frac{1}{2}+\frac{\tau}{2},\tau\right)}{\eta(\tau)} \tag{4.59}$$

where θ is the Jacobi theta function and η is the Dedekind eta function.

Eq. (4.43) implies

$$\begin{aligned}\frac{\delta(z,\tau)}{\delta(-z,\tau)} &= 4ie^{\pi i z}\sin(\pi z)\sin\left(\frac{\pi z}{\tau}\right)\tilde{S}_2(-z,\tau)\tilde{S}_2(z,-\tau)\\ &= 4ie^{\pi i z}\tilde{S}_2(\tau-z,\tau)\tilde{S}_2(1+z,-\tau).\end{aligned} \tag{4.60}$$

On the other hand it follows from the Jacobi identity (Ouijaboard, Eq. (2.4)) that

$$\delta(z,\tau)\delta(-z,\tau) = e^{\frac{\pi i}{12}}(1-e^{2\pi i z})\frac{\theta\left(z+\frac{1}{2}+\frac{\tau}{2},\tau\right)}{\eta(\tau)}. \tag{4.61}$$

Hence

$$\delta(z,\tau)^2 = 8e^{\frac{\pi i}{12}}e^{2\pi i z}\sin(\pi z)\frac{\tilde{S}_2(\tau-z,\tau)\tilde{S}_2(1+z,-\tau)\theta\left(z+\frac{1}{2}+\frac{\tau}{2},\tau\right)}{\eta(\tau)}$$

$$= -4e^{\frac{\pi i}{12}} e^{2\pi i z} \frac{\tilde{S}_2(\tau - z, \tau)\tilde{S}_2(1 - \tau + z, -\tau)\theta\left(z + \frac{1}{2} + \frac{\tau}{2}, \tau\right)}{\eta(\tau)} \tag{4.62}$$

4.4. **Corrected version of Theorem 12.4 in Ouijaboard.**

If it is true that $\tilde{S}_2(z, \tau)$ is singular for all values of τ in the interval $(-\infty, 0]$ (see the remark to Corollary 4.5) then, failing some kind of miraculous cancellation (and I can't see how that would work), and with the possible exception of a discrete set of z values, the RHS of Eq. (4.43) is singular [8] for all real values of τ. If that is so then Theorem 12.4 in Ouijaboard is incorrect as stated. The question now arises, for how many other elements of $\mathrm{SL}(2, \mathbb{Z})$ does the theorem fail? At first sight one might think that the problem is going to occur for a large subset of $\mathrm{SL}(2, \mathbb{Z})$. Indeed, let H be any element of $\mathrm{SL}(2, \mathbb{Z})$. Then the cocycle condition means

$$\tilde{\sigma}_{-H}(\mathbf{p}, \tau) = \tilde{\sigma}_{-I}(H\mathbf{p}, H\tau)\tilde{\sigma}_H(\mathbf{p}, \tau) \tag{4.63}$$

If it is true that $\tilde{\sigma}_{-I}(H\mathbf{p}, H\tau)$ is singular for all real τ then one might naively think that the same will be true of $\tilde{\sigma}_{-H}(\mathbf{p}, \tau)$. However, the cancellation of singularities described in Corollary 4.5 (also see Corollary 4.8) means that this is not necessarily the case. For instance

$$\begin{aligned}
\sigma_{-S}(z, \tau) &= \sigma_{-I}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)\sigma_S(z, \tau) \\
&= ie^{\frac{\pi i}{12\tau}\left(6z^2 + 6(1-\tau)z + \tau^2 - 3\tau + 1\right)} e^{-\pi i z} \tilde{S}_2\left(\frac{z-1}{\tau}, -\frac{1}{\tau}\right)\tilde{S}_2\left(1 - \frac{z}{\tau}, \frac{1}{\tau}\right)\tilde{S}_2(1 + z, \tau) \\
&= ie^{\frac{\pi i}{12\tau}\left(6z^2 + 6(1-3\tau)z + \tau^2 - 3\tau + 1\right)}\tilde{S}_2(1 - z, -\tau)\tilde{S}_2(\tau - z, \tau)\tilde{S}_2(1 + z, \tau)
\end{aligned} \tag{4.64}$$

The function $\tilde{S}_2(1 - z, -\tau)$ is singular for $\tau \geq 0$, while $\tilde{S}_2(\tau - z, \tau)$ and $\tilde{S}_2(1 + z, \tau)$ are singular for $\tau \leq 0$. However, using Eq. (4.40) one sees that two of the singularities cancel, leaving us with

$$\sigma_{-S}(z, \tau) = ie^{\frac{\pi i}{12\tau}\left(6z^2 + 6(1-3\tau)z + \tau^2 - 3\tau + 1\right)}\tilde{S}_2(1 - z, -\tau) \tag{4.65}$$

which is non-singular for $\tau \in (-\infty, 0)$. The question we now address is, how often this happens. One case where it can easily be seen that the singularities do not cancel is σ_{-T^k}, $k \in \mathbb{Z}$. Indeed, it follows from Proposition 4.3 that $\sigma_{-T^k} = \sigma_{-I}$, for all k. We now prove a corrected version of Theorem 12.4 in Ouijaboard which says that these are in fact the only examples of cocycles which are singular everywhere on the real axis.

Proposition 4.10. *Let $L = \left(\begin{smallmatrix} r & s \\ u & v \end{smallmatrix}\right)$ be any element of $\mathrm{SL}(2, \mathbb{Z})$. The corresponding cocycle σ_L is meromorphic on $\mathbb{C} \times D_L$ where*

$$D_L = \begin{cases} \mathbb{C} & u = 0, v > 0 \\ \mathcal{H} & u = 0, v < 0 \\ \mathbb{C} \setminus (-\infty, -\frac{v}{u}] & u > 0 \\ \mathbb{C} \setminus [-\frac{v}{u}, \infty) & u < 0 \end{cases} \tag{4.66}$$

Remark. The proof which follows is closely modelled on Gene's proof of 12.4. The difference comes down to this: it makes explicit the fact that, in Ouijaboard Eq. (12.9), if c is positive, then so is a'. Consequently, the matrix one ends with at the conclusion of the recursive procedure is one of the

[8] I haven't attempted to carry out a systematic investigation. However a very cursory, unsystematic investigation in Mathematica is consistent with this. What seems to happen is that the function becomes violently oscillatory as τ approaches the real line

form T^k, for which the domain is $\mathbb{C}$. However, I decided to write out the modified proof in full so as to make sure I haven't missed anything.

Proof. If $u = 0$ then either $r = v = 1$ or $r = v = -1$. In the first case $L = T^k$ for some k, implying $\sigma_L(x, \tau) = \sigma_I(x, \tau) = 1$ for all x, τ. In the second case $L = -T^k$ for some k, implying $\sigma_L(x, \tau) = \sigma_{-I}(x, \tau)$ for all x, τ which as we saw is singular for τ on the real line, and theforefore can't be continued outside the set $\mathbb{C} \times \mathcal{H}$.

Suppose $u > 0$. Define a sequence of matrices

$$L_j = \begin{pmatrix} r_j & s_j \\ u_j & v_j \end{pmatrix} \tag{4.67}$$

recursively as follows. Begin by setting

$$L_1 = L. \tag{4.68}$$

Next, suppose j is such that $u_j > 0$. Then there exists a unique integer k_j such that $(k_j - 1)u_j < r_j \le k_j u_j$. Define

$$L_{j+1} = S^{-1}T^{-k_j}L_j = \begin{pmatrix} u_j & v_j \\ k_j u_j - r_j & k_j v_j - s_j \end{pmatrix} \tag{4.69}$$

In particular

$$0 \le u_{j+1} = k_j u_j - r_j < u_j \tag{4.70}$$

The u_j thus form a strictly decreasing sequence of non-negative numbers. There must consequently exist some integer $j_0 \ge 2$ such that $u_{j_0} = 0$. By construction

$$L_j = \begin{pmatrix} u_{j-1} & v_{j-1} \\ u_j & v_j \end{pmatrix} \tag{4.71}$$

for $j \ge 2$. In particular

$$L_{j_0} = \begin{pmatrix} u_{j_0-1} & v_{j_0-1} \\ 0 & v_{j_0} \end{pmatrix}. \tag{4.72}$$

Since $\det L_{j_0} = 1$ and $u_{j_0-1} > 0$ this means $u_{j_0-1} = v_{j_0} = 1$. Hence

$$L_j = \begin{cases} T^{k_j}SL_{j+1} & j = 1, \dots, j_0 - 1, \\ T^{v_{j_0-1}} & j = j_0. \end{cases} \tag{4.73}$$

Using Proposition 4.3 we deduce

$$\begin{aligned} \sigma_L(z, \tau) &= \tilde{\sigma}(\mathbf{z}, \tau) \\ &= \tilde{\sigma}_{T^{k_1}S}(L_2\mathbf{z}, L_2\tau) \dots \tilde{\sigma}_{T^{k_{j_0-1}}S}(L_{j_0}\mathbf{z}, L_{j_0}\tau)\tilde{\sigma}_{T^{v_{j_0-1}}}(\mathbf{z}, \tau) \\ &= \tilde{\sigma}_S(L_2\mathbf{z}, L_2\tau) \dots \tilde{\sigma}_S(L_{j_0}\mathbf{z}, L_{j_0}\tau) \end{aligned} \tag{4.74}$$

where $\mathbf{z} = \left(\begin{smallmatrix} -z \\ 0 \end{smallmatrix}\right)$. This expression is meromorphic as a function of z if τ is real and

$$0 < L_j\tau = \begin{cases} \frac{u_{j-1}\tau + v_{j-1}}{u_j\tau + v_j} & j = 2, \dots, j_0 - 1, \\ u_{j_0-1}\tau + v_{j_0-1} & j = j_0. \end{cases} \tag{4.75}$$

A sufficient condition for this to be true is that

$$\tau > -\frac{v_j}{u_j} \tag{4.76}$$

for $j = 1, \dots, j_0 - 1$. It follows from Eq. (4.71) that

$$\frac{v_j}{u_j} - \frac{v_{j-1}}{u_{j-1}} = \frac{\det L_j}{u_j u_{j-1}} = \frac{1}{u_j u_{j-1}} \tag{4.77}$$

for $j = 2, \dots, j_0 - 1$. Since $u_j u_{j-1} > 0$ for $j < j_0$ this means

$$-\frac{v_1}{u_1} > \cdots > -\frac{v_{j_0}}{u_{j_0}}. \tag{4.78}$$

We conclude that $\sigma_L(z,\tau)$ is meromorphic as a function of z for τ in the interval $\left(-\frac{v}{u}, \infty\right)$.

Suppose, on the other hand, that $u < 0$. Then it follows from Proposition 4.3 that

$$\sigma_L(z,\tau) = \tilde{\sigma}_L(\mathbf{z},\tau) = \frac{1}{\tilde{\sigma}_{L^{-1}}(L\mathbf{z}, L\tau)}. \tag{4.79}$$

where $\mathbf{z} = \left(\begin{smallmatrix} -z \\ 0 \end{smallmatrix}\right)$. We can apply the result just proved to L^{-1} to deduce that $\sigma_L(z,\tau)$ is meromorphic as a function of z if τ is real and $L\tau > \frac{r}{u}$. Suppose $\tau < -\frac{v}{u}$. Then $u\tau + v > 0$ and, consequently,

$$L\tau - \frac{r}{u} = \frac{us - rv}{u(u\tau + v)} = \frac{\det L}{|u|(u\tau + v)} > 0 \tag{4.80}$$

□

4.5. **Quasi-periodicities.** For $(z,\tau) \in \mathbb{C}\times\mathbb{H}$, $m \in \mathbb{Z}$ define the finite exponentiated Q-Pochhammer symbol by

$$\delta_m(z,\tau) = \frac{\delta(z,\tau)}{\delta(z + m\tau, \tau)} \tag{4.81}$$

Unlike the infinite symbol it continues to a meromorphic function on $\mathbb{C}\times\mathbb{C}$:

$$\delta_m(z,\tau) = \begin{cases} \prod_{j=0}^{m-1} \left(1 - e^{2\pi i(z + j\tau)}\right) & m \geq 1 \\ 1 & m = 0 \\ \prod_{j=m}^{-1} \left(1 - e^{2\pi i(z + j\tau)}\right)^{-1} & m \leq -1 \end{cases} \tag{4.82}$$

Lemma 4.11. *For all m_1, $m_2 \in \mathbb{Z}$ and all $(z,\tau) \in \mathbb{C}\times\mathbb{H}$*

$$\delta(z + m_1\tau + m_2, \tau) = \frac{\delta(z,\tau)}{\delta_{m_1}(z,\tau)}. \tag{4.83}$$

Proof. Immediate consequence of the definition. □

Lemma 4.12. *For all m, $n \in \mathbb{Z}$*

$$\delta_m(z + n\tau, \tau) = \frac{\delta_{m+n}(z,\tau)}{\delta_n(z,\tau)} \tag{4.84}$$

Proof. If $\tau \in \mathbb{H}$ then it follows from Eq. (4.81)

$$\begin{aligned} \delta_m(z + n\tau, \tau) &= \frac{\delta(z + n\tau, \tau)}{\delta(z + (n+m)\tau, \tau)} \\ &= \left(\frac{\delta(z + n\tau, \tau)}{\delta(z,\tau)}\right)\left(\frac{\delta(z,\tau)}{\delta(z + (n+m)\tau, \tau)}\right) \\ &= \frac{\delta_{m+n}(z,\tau)}{\delta_n(z,\tau)}. \end{aligned} \tag{4.85}$$

We then use analyticity to extend the result to arbitrary z, τ, in the domain of definition. □

Lemma 4.13. *For all* $m \in \mathbb{Z}$

$$\delta_{-m}(z,\tau) = \frac{1}{\delta_m(z-\tau,-\tau)} \tag{4.86}$$

Proof. It follows from Eq. (4.82)

$$\begin{aligned}
\delta_{-m}(z,\tau) &= \begin{cases} \prod_{j=0}^{-m-1}\left(1-e^{2\pi i(z+j\tau)}\right) & m \le -1, \\ 1 & m = 0, \\ \prod_{j=-m}^{-1}\left(1-e^{2\pi i(z+j\tau)}\right)^{-1} & m \ge 1, \end{cases} \\
&= \begin{cases} \prod_{j=m}^{-1}\left(1-e^{2\pi i(z-\tau-j\tau)}\right) & m \le -1, \\ 1 & m = 0, \\ \prod_{j=0}^{m-1}\left(1-e^{2\pi i(z-\tau-j\tau)}\right)^{-1} & m \ge 1, \end{cases} \\
&= \frac{1}{\delta_m(z-\tau,-\tau)}.
\end{aligned} \tag{4.87}$$

□

Proposition 4.14. *For all* $G = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$*, and* m_1*,* $m_2 \in \mathbb{Z}$

$$\sigma_G(z+m_1\tau+m_2,\tau) = \frac{\delta_{m_1}(z,\tau)}{\delta_{vm_1-um_2}\left(\frac{z}{u\tau+v}, G\tau\right)}\sigma_G(z,\tau) \tag{4.88}$$

Proof. Define

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = G\begin{pmatrix} \tau & 1 \end{pmatrix} = \begin{pmatrix} s\tau+t \\ u\tau+v \end{pmatrix} \tag{4.89}$$

Then $G\tau = \omega_1/\omega_2$ and

$$\begin{pmatrix} \tau \\ 1 \end{pmatrix} = G^{-1}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} v\omega_1 - t\omega_2 \\ -u\omega_1 + s\omega_2 \end{pmatrix} \tag{4.90}$$

Suppose $\tau \in \mathbb{H}$. Then

$$\sigma_G(z+m_1\tau+m_2,\tau) = \frac{\delta\left(\frac{z+m_1\tau+m_2}{\omega_2}, \frac{\omega_1}{\omega_2}\right)}{\delta(z+m_1\tau+m_2,\tau)} \tag{4.91}$$

Using Eq. (4.83)

$$\begin{aligned}
\delta\left(\frac{z+m_1\tau+m_2}{\omega_2}, \frac{\omega_1}{\omega_2}\right) &= \delta\left(\frac{z}{\omega_2} + (vm_1-um_2)\frac{\omega_1}{\omega_2} + (-tm_1+sm_2), \frac{\omega_1}{\omega_2}\right) \\
&= \frac{\delta\left(\frac{z}{\omega_2}, \frac{\omega_1}{\omega_2}\right)}{\delta_{vm_1-um_2}\left(\frac{z}{\omega_2}, \frac{\omega_1}{\omega_2}\right)},
\end{aligned} \tag{4.92}$$

$$\delta(z+m_1\tau+m_2,\tau) = \frac{\delta(z,\tau)}{\delta_{m_1}(z,\tau)}. \tag{4.93}$$

Hence

$$\begin{aligned}\sigma_G(z+m_1\tau+m_2,\tau) &= \frac{\delta\left(\frac{z}{\omega_2},\frac{\omega_1}{\omega_2}\right)\delta_{m_1}(z,\tau)}{\delta_{vm_1-um_2}\left(\frac{z}{\omega_2},\frac{\omega_1}{\omega_2}\right)\delta(z,\tau)}\\ &= \frac{\delta_{m_1}(z,\tau)}{\delta_{vm_1-um_2}\left(\frac{z}{\omega_2},\frac{\omega_1}{\omega_2}\right)}\sigma_G(z,\tau). \end{aligned}\tag{4.94}$$

Now use analyticity to extend to all values of z, τ in the domain of definition. $\square$

Corollary 4.15. *For all m_1, $m_2 \in \mathbb{Z}$ and all z, τ in the domain of definition*

$$\sigma_S(z+m_1\tau+m_2,\tau) = \frac{\delta_{m_1}(z,\tau)}{\delta_{-m_2}\left(\frac{z}{\tau},-\frac{1}{\tau}\right)}\sigma_S(z,\tau)\tag{4.95}$$

Comparing with Eq. (A.75) it can be seen that this formula is much simpler than the corresponding formula for the double sine function. In view of the relation

$$\sigma_S(z,\tau) = 2\sin\left(\frac{\pi z}{\tau}\right)e^{\frac{\pi i}{12\tau}\left(6z^2+6(1-\tau)z+\tau^2-3\tau+1\right)}\tilde{S}_2(z,\tau)\tag{4.96}$$

(c.f. Eqs. (4.4) and (4.33)) we can regard $\sigma_S(z,\tau)$ as a kind of modified version of the double Sine function, and work exclusively with it.

Analogous to Eq. (A.29):

Lemma 4.16. *For all z, τ in the domain of definition*

$$\sigma_S(\tau^{-1}z,\tau^{-1}) = \frac{1-e^{2\pi iz}}{1-e^{\frac{2\pi iz}{\tau}}}\sigma_S(z,\tau).\tag{4.97}$$

Proof. It follows from Eqs. (4.96), (A.29)

$$\begin{aligned}\sigma_S(\tau^{-1}z,\tau^{-1}) &= 2\sin(\pi z)e^{\frac{\pi i}{12}\left(\frac{6z^2}{\tau}+6(\tau-1)\frac{z}{\tau}+\tau-3+\frac{1}{\tau}\right)}\tilde{S}_2(\tau^{-1}z,\tau^{-1})\\ &= 2\sin(\pi z)e^{\frac{\pi i}{12\tau}\left(6z^2-6(1-\tau)z+\tau^2-3\tau+1\right)}\tilde{S}_2(z,\tau)\\ &= \frac{e^{\frac{\pi i(\tau-1)z}{\tau}}\sin(\pi z)}{\sin\left(\frac{\pi z}{\tau}\right)}\sigma_S(z,\tau)\\ &= \frac{1-e^{2\pi iz}}{1-e^{\frac{2\pi iz}{\tau}}}\sigma_S(z,\tau). \end{aligned}\tag{4.98}$$

$\square$

Definition 4.17. For $G = \left(\begin{smallmatrix}1+d\bar{s} & d\bar{t}\\ d\bar{u} & 1+d\bar{v}\end{smallmatrix}\right) \in \Gamma(d)$ define

$$\rho_G(\mathbf{p},\tau) = \frac{\sigma_G\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)}{\delta_{-\bar{u}p_1+\bar{s}p_2}\left(\frac{\langle\mathbf{p},\tau\rangle}{d},\tau\right)}\tag{4.99}$$

By construction ρ_G is meromorphic as a function of τ on the cut $\mathbb{C}$-plane.

Proposition 4.18. *Let $\tau \in \mathbb{H}$. Then*

$$\rho_G(\mathbf{p},\tau) = \frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)} \tag{4.100}$$

for all $G = \left(\begin{smallmatrix} 1+d\bar{s} & \bar{t} \\ \bar{u} & 1+d\bar{v}\end{smallmatrix}\right) \in \Gamma(d)$, $\mathbf{p} \in \mathbb{Z}^2$.

Proof. Suppose $\tau \in \mathbb{H}$. Then the ratio on the right-hand side of Eq. (4.100) is defined and

$$\begin{aligned} \frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)} &= \left(\frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}{\delta\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)}\right)\left(\frac{\delta\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}\right) \\ &= \sigma_G\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)\left(\frac{\delta\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}\right) \end{aligned} \tag{4.101}$$

It follows from Eq. (4.83)

$$\begin{aligned} \delta\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right) &= \delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d} + (-\bar{u}p_1+\bar{s}p_2)\tau - (\bar{v}p_1 - \bar{t}p_2),\tau\right) \\ &= \frac{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}{\delta_{-\bar{u}p_1+\bar{s}p_2}\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}. \end{aligned} \tag{4.102}$$

Hence

$$\frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)} = \frac{\sigma_G\left(\frac{\langle G^{-1}\mathbf{p},\tau\rangle}{d},\tau\right)}{\delta_{-\bar{u}p_1+\bar{s}p_2}\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)} = \rho_G(\mathbf{p},\tau). \tag{4.103}$$

$\square$

Proposition 4.19. *For all $\mathbf{p}$, $\mathbf{n} \in \mathbb{Z}^2$, all $G = \left(\begin{smallmatrix} 1+d\bar{s} & d\bar{t} \\ d\bar{u} & 1+d\bar{v}\end{smallmatrix}\right) \in \Gamma(d)$, and all τ in the cut plane*

$$\rho_G(\mathbf{p}+d\mathbf{n},\tau) = \frac{\delta_{n_2}\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}{\delta_{n_2}\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}\rho_G(\mathbf{p},\tau). \tag{4.104}$$

Proof. Suppose $\tau \in \mathbb{H}$. Then it follows from Eqs. (4.83), (4.100)

$$\begin{aligned} \rho_G(\mathbf{p}+d\mathbf{n},\tau) &= \frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d} + \langle \mathbf{n}, G\tau\rangle, G\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d} + \langle \mathbf{n},\tau\rangle,\tau\right)} \\ &= \frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d} + n_2 G\tau - n_1, G\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d} + n_2\tau - n_1,\tau\right)} \\ &= \left(\frac{\delta\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}{\delta_{n_2}\left(\frac{\langle \mathbf{p}, G\tau\rangle}{d}, G\tau\right)}\right)\left(\frac{\delta_{n_2}\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}{\delta\left(\frac{\langle \mathbf{p},\tau\rangle}{d},\tau\right)}\right) \end{aligned}$$

$$= \left(\frac{\delta_{n_2}\left(\frac{\langle \mathbf{p}, \tau \rangle}{d}, \tau\right)}{\delta_{n_2}\left(\frac{\langle \mathbf{p}, G\tau \rangle}{d}, G\tau\right)} \right) \rho_G(\mathbf{p} + d\mathbf{n}, \tau). \tag{4.105}$$

Then use analyticity to extend the result to the whole of the cut τ-plane. □

Corollary 4.20. *For all τ in the cut plane ρ_G is periodic in the first component of $\mathbf{p}$:*

$$\rho_G(\mathbf{p} + d(n_1, 0)^{\mathrm{T}}, \tau) = \rho_G(\mathbf{p}, \tau) \tag{4.106}$$

for all $\mathbf{p} \in \mathbb{Z}^2$, $n_1 \in \mathbb{Z}$.

Corollary 4.21. *If $G\tau = \tau$ then ρ_G is periodic in both components of $\mathbf{p}$:*

$$\rho_G(\mathbf{p} + d\mathbf{n}, \tau) = \rho_G(\mathbf{p}, \tau) \tag{4.107}$$

for all $\mathbf{p}$, $\mathbf{n} \in \mathbb{Z}^2$.

5. SIC CONSTRUCTION

Note that, at least in the purely periodic case, a Ghost is fully specified by its dimension and the corresponding HJ-continued fraction (I suspect this is true even if one allows HJ contineud fractions which are not purely periodic, but I haven't checked in detail). So in the following tables I could have just listed the HJ expansions for each dimensions. But I feel an expanded data set might make the paper more digestible. So, in the following tables, the following items are specified:

(1) d is the dimension,
(2) r is position in tower,
(3) Δ_0 is the fundamental discriminant,
(4) f is the conductor,
(5) c is the class number,
(6) Q is a choice for the primitive HJ reduced form,
(7) τ is the larger of the two roots of Q,
(8) "HJ" is its HJ continued fraction expansion,
(9) L_Q is the generator of the stability group of Q,
(10) n is the order of L_Q in $\mathrm{SL}(2, \mathbb{Z})/\Gamma(d)$ (so $G = L_Q^n$).

For a given class we always include the principal HJ-reduced form. So if the class number is 1 the choice of form is unique. If the class number is 2 then for the remaining classes we always choose an HJ-reduced form for which the period of the continued fraction expansion is minimal. There is typically more than one such form, in which case the choice is not unique.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
4	1	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$	3
5	1	12	1	1	$\langle 1,-4,1\rangle$	$2+\sqrt{3}$	$[\overline{4}]$	$\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}$	3
6	1	21	1	1	$\langle 1,-5,1\rangle$	$\frac{5+\sqrt{21}}{2}$	$[\overline{5}]$	$\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}$	3
7	1	8	1	1	$\langle 2,-4,1\rangle$	$\frac{2+\sqrt{2}}{2}$	$[\overline{2,4}]$	$\begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix}$	3
			2	1	$\langle 1,-6,1\rangle$	$3+2\sqrt{2}$	$[\overline{6}]$	$\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$	3
8	2	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$	6
			3	1	$\langle 1,-7,1\rangle$	$\frac{7+3\sqrt{5}}{2}$	$[\overline{7}]$	$\begin{pmatrix} 7 & -1 \\ 1 & 0 \end{pmatrix}$	3
9	1	60	1	2	$\langle 1,-8,1\rangle$	$4+\sqrt{15}$	$[\overline{8}]$	$\begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2,-10,5\rangle$	$\frac{5+\sqrt{15}}{2}$	$[\overline{5,2}]$	$\begin{pmatrix} 9 & -5 \\ 2 & -1 \end{pmatrix}$	3
10	1	77	1	1	$\langle 1,-9,1\rangle$	$\frac{9+\sqrt{77}}{2}$	$[\overline{9}]$	$\begin{pmatrix} 9 & -1 \\ 1 & 0 \end{pmatrix}$	3
11	1	24	1	1	$\langle 3,-6,1\rangle$	$\frac{3+\sqrt{6}}{3}$	$[\overline{2,6}]$	$\begin{pmatrix} 11 & -2 \\ 6 & -1 \end{pmatrix}$	3
			2	2	$\langle 1,-10,1\rangle$	$5+2\sqrt{6}$	$[\overline{10}]$	$\begin{pmatrix} 10 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-12,4\rangle$	$\frac{6+2\sqrt{6}}{3}$	$[\overline{4,3}]$	$\begin{pmatrix} 11 & -4 \\ 3 & -1 \end{pmatrix}$	3
12	1	13	1	1	$\langle 3,-5,1\rangle$	$\frac{5+\sqrt{13}}{6}$	$[\overline{2,2,5}]$	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$	6
			3	1	$\langle 1,-11,1\rangle$	$\frac{11+3\sqrt{13}}{2}$	$[\overline{11}]$	$\begin{pmatrix} 11 & -1 \\ 1 & 0 \end{pmatrix}$	3
13	1	140	1	2	$\langle 1,-12,1\rangle$	$6+\sqrt{35}$	$[\overline{12}]$	$\begin{pmatrix} 12 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2,-14,7\rangle$	$\frac{7+\sqrt{35}}{2}$	$[\overline{7,2}]$	$\begin{pmatrix} 13 & -7 \\ 2 & -1 \end{pmatrix}$	3
14	1	165	1	2	$\langle 1,-13,1\rangle$	$\frac{13+\sqrt{165}}{2}$	$[\overline{13}]$	$\begin{pmatrix} 13 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-15,5\rangle$	$\frac{15+\sqrt{165}}{2}$	$[\overline{5,3}]$	$\begin{pmatrix} 14 & -5 \\ 3 & -1 \end{pmatrix}$	3
15	2	12	1	1	$\langle 1,-4,1\rangle$	$2+\sqrt{3}$	$[\overline{4}]$	$\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}$	6
			2	1	$\langle 4,-8,1\rangle$	$\frac{2+\sqrt{3}}{2}$	$[\overline{2,8}]$	$\begin{pmatrix} 15 & -2 \\ 8 & -1 \end{pmatrix}$	3
			4	2	$\langle 1,-14,1\rangle$	$7+4\sqrt{3}$	$[\overline{14}]$	$\begin{pmatrix} 14 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-18,11\rangle$	$\frac{9+4\sqrt{3}}{3}$	$[\overline{6,2,2}]$	$\begin{pmatrix} 16 & -11 \\ 3 & -2 \end{pmatrix}$	3
16	1	221	1	2	$\langle 1,-15,1\rangle$	$\frac{15+\sqrt{221}}{2}$	$[\overline{15}]$	$\begin{pmatrix} 15 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 5,-21,11\rangle$	$\frac{21+\sqrt{221}}{10}$	$[\overline{4,3,2}]$	$\begin{pmatrix} 18 & -11 \\ 5 & -3 \end{pmatrix}$	3

TABLE 1. SICs in dimensions 4–16.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
17	1	28	1	1	$\langle 2,-6,1\rangle$	$\frac{3+\sqrt{7}}{2}$	$[\overline{3,6}]$	$\begin{pmatrix}17&-3\\6&-1\end{pmatrix}$	3
			3	2	$\langle 1,-16,1\rangle$	$8+3\sqrt{7}$	$[\overline{16}]$	$\begin{pmatrix}16&-1\\1&0\end{pmatrix}$	3
					$\langle 2,-18,9\rangle$	$\frac{9+3\sqrt{7}}{2}$	$[\overline{9,2}]$	$\begin{pmatrix}17&-9\\2&-1\end{pmatrix}$	3
18	1	285	1	2	$\langle 1,-17,1\rangle$	$\frac{17+\sqrt{285}}{2}$	$[\overline{17}]$	$\begin{pmatrix}17&-1\\1&0\end{pmatrix}$	3
					$\langle 3,-21,13\rangle$	$\frac{21+\sqrt{285}}{6}$	$[\overline{7,2,2}]$	$\begin{pmatrix}19&-13\\3&-2\end{pmatrix}$	3
19	3	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\begin{pmatrix}3&-1\\1&0\end{pmatrix}$	9
			2	1	$\langle 4,-6,1\rangle$	$\frac{3+\sqrt{5}}{4}$	$[\overline{2,2,2,6}]$	$\begin{pmatrix}21&-4\\16&-3\end{pmatrix}$	3
			4	1	$\langle 5,-10,1\rangle$	$\frac{5+2\sqrt{5}}{5}$	$[\overline{2,10}]$	$\begin{pmatrix}19&-2\\10&-1\end{pmatrix}$	3
			8	2	$\langle 1,-18,1\rangle$	$9+4\sqrt{5}$	$[\overline{18}]$	$\begin{pmatrix}18&-1\\1&0\end{pmatrix}$	3
					$\langle 4,-20,5\rangle$	$\frac{5+2\sqrt{5}}{2}$	$[\overline{5,4}]$	$\begin{pmatrix}19&-5\\4&-1\end{pmatrix}$	3
20	1	357	1	2	$\langle 1,-19,1\rangle$	$\frac{19+\sqrt{357}}{2}$	$[\overline{19}]$	$\begin{pmatrix}19&-1\\1&0\end{pmatrix}$	3
					$\langle 3,-21,7\rangle$	$\frac{21+\sqrt{357}}{6}$	$[\overline{7,3}]$	$\begin{pmatrix}20&-7\\3&-1\end{pmatrix}$	3
21	1	44	1	1	$\langle 5,-8,1\rangle$	$\frac{4+\sqrt{11}}{5}$	$[\overline{2,2,8}]$	$\begin{pmatrix}22&-3\\15&-2\end{pmatrix}$	3
			3	4	$\langle 1,-20,1\rangle$	$10+3\sqrt{11}$	$[\overline{20}]$	$\begin{pmatrix}20&-1\\1&0\end{pmatrix}$	3
					$\langle 2,-22,11\rangle$	$\frac{11+3\sqrt{11}}{2}$	$[\overline{11,2}]$	$\begin{pmatrix}21&-11\\2&-1\end{pmatrix}$	3
					$\langle 5,-26,14\rangle$	$\frac{13+3\sqrt{11}}{5}$	$[\overline{5,3,2}]$	$\begin{pmatrix}23&-14\\5&-3\end{pmatrix}$	3
					$\langle 5,-24,9\rangle$	$\frac{12+3\sqrt{11}}{5}$	$[\overline{5,2,3}]$	$\begin{pmatrix}22&-9\\5&-2\end{pmatrix}$	3
22	1	437	1	1	$\langle 1,-21,1\rangle$	$\frac{21+\sqrt{437}}{2}$	$[\overline{21}]$	$\begin{pmatrix}21&-1\\1&0\end{pmatrix}$	3
23	1	120	1	2	$\langle 6,-12,1\rangle$	$\frac{6+\sqrt{30}}{6}$	$[\overline{2,12}]$	$\begin{pmatrix}23&-2\\12&-1\end{pmatrix}$	3
					$\langle 2,-12,3\rangle$	$\frac{6+\sqrt{30}}{2}$	$[\overline{6,4}]$	$\begin{pmatrix}23&-6\\4&-1\end{pmatrix}$	3
			2	4	$\langle 1,-22,1\rangle$	$11+2\sqrt{30}$	$[\overline{22}]$	$\begin{pmatrix}22&-1\\1&0\end{pmatrix}$	3
					$\langle 4,-28,19\rangle$	$\frac{7+\sqrt{30}}{2}$	$[\overline{7,2,2,2}]$	$\begin{pmatrix}25&-19\\4&-3\end{pmatrix}$	3
					$\langle 3,-24,8\rangle$	$\frac{12+2\sqrt{30}}{3}$	$[\overline{8,3}]$	$\begin{pmatrix}23&-8\\3&-1\end{pmatrix}$	3
					$\langle 7,-30,15\rangle$	$\frac{15+2\sqrt{30}}{7}$	$[\overline{4,4,2}]$	$\begin{pmatrix}26&-15\\7&-4\end{pmatrix}$	3
24	2	21	1	1	$\langle 1,-5,1\rangle$	$\frac{5+\sqrt{21}}{2}$	$[\overline{5}]$	$\begin{pmatrix}5&-1\\1&0\end{pmatrix}$	6
			5	2	$\langle 1,-23,1\rangle$	$\frac{23+5\sqrt{21}}{2}$	$[\overline{23}]$	$\begin{pmatrix}23&-1\\1&0\end{pmatrix}$	3
					$\langle 3,-27,17\rangle$	$\frac{27+5\sqrt{21}}{6}$	$[\overline{9,2,2}]$	$\begin{pmatrix}25&-17\\3&-2\end{pmatrix}$	3

TABLE 2. SICs in dimensions 17–24.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
25	1	572	1	2	$\langle 1, -24, 1 \rangle$	$12 + \sqrt{143}$	$[\overline{24}]$	$\begin{pmatrix} 24 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2, -26, 13 \rangle$	$\frac{13+\sqrt{143}}{2}$	$[\overline{13, 2}]$	$\begin{pmatrix} 25 & -13 \\ 2 & -1 \end{pmatrix}$	3
26	1	69	1	1	$\langle 3, -9, 1 \rangle$	$\frac{9+\sqrt{69}}{6}$	$[\overline{3, 9}]$	$\begin{pmatrix} 26 & -3 \\ 9 & -1 \end{pmatrix}$	3
			3	3	$\langle 1, -25, 1 \rangle$	$\frac{25+3\sqrt{69}}{2}$	$[\overline{25}]$	$\begin{pmatrix} 25 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 5, -31, 17 \rangle$	$\frac{31+3\sqrt{69}}{10}$	$[\overline{6, 3, 2}]$	$\begin{pmatrix} 28 & -17 \\ 5 & -3 \end{pmatrix}$	3
					$\langle 5, -29, 11 \rangle$	$\frac{29+3\sqrt{69}}{10}$	$[\overline{6, 2, 3}]$	$\begin{pmatrix} 27 & -11 \\ 5 & -2 \end{pmatrix}$	3
27	1	168	1	2	$\langle 7, -14, 1 \rangle$	$\frac{7+\sqrt{42}}{7}$	$[\overline{2, 14}]$	$\begin{pmatrix} 27 & -2 \\ 14 & -1 \end{pmatrix}$	3
					$\langle 2, -16, 11 \rangle$	$\frac{8+\sqrt{42}}{2}$	$[\overline{8, 2, 2, 2}]$	$\begin{pmatrix} 29 & -22 \\ 4 & -3 \end{pmatrix}$	3
			2	4	$\langle 1, -26, 1 \rangle$	$13 + 2\sqrt{42}$	$[\overline{26}]$	$\begin{pmatrix} 26 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3, -30, 19 \rangle$	$\frac{15+2\sqrt{42}}{3}$	$[\overline{10, 2, 2}]$	$\begin{pmatrix} 28 & -19 \\ 3 & -2 \end{pmatrix}$	3
					$\langle 4, -28, 7 \rangle$	$\frac{7+\sqrt{42}}{2}$	$[\overline{7, 4}]$	$\begin{pmatrix} 27 & -7 \\ 4 & -1 \end{pmatrix}$	3
					$\langle 8, -32, 11 \rangle$	$\frac{8+\sqrt{42}}{4}$	$[\overline{4, 3, 3}]$	$\begin{pmatrix} 29 & -11 \\ 8 & -3 \end{pmatrix}$	3
28	1	29	1	1	$\langle 5, -7, 1 \rangle$	$\frac{7+\sqrt{29}}{10}$	$[\overline{2, 2, 2, 2, 7}]$	$\begin{pmatrix} 31 & -5 \\ 25 & -4 \end{pmatrix}$	3
			5	2	$\langle 1, -27, 1 \rangle$	$\frac{27+5\sqrt{29}}{2}$	$[\overline{27}]$	$\begin{pmatrix} 27 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 7, -37, 23 \rangle$	$\frac{37+5\sqrt{29}}{14}$	$[\overline{5, 3, 2, 2}]$	$\begin{pmatrix} 32 & -23 \\ 7 & -5 \end{pmatrix}$	3
29	1	780	1	4	$\langle 1, -28, 1 \rangle$	$14 + \sqrt{195}$	$[\overline{28}]$	$\begin{pmatrix} 28 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2, -30, 15 \rangle$	$\frac{15+\sqrt{195}}{2}$	$[\overline{15, 2}]$	$\begin{pmatrix} 29 & -15 \\ 2 & -1 \end{pmatrix}$	3
					$\langle 5, -30, 6 \rangle$	$\frac{15+\sqrt{195}}{5}$	$[\overline{6, 5}]$	$\begin{pmatrix} 29 & -6 \\ 5 & -1 \end{pmatrix}$	3
					$\langle 3, -30, 10 \rangle$	$\frac{15+\sqrt{195}}{3}$	$[\overline{10, 3}]$	$\begin{pmatrix} 29 & -10 \\ 3 & -1 \end{pmatrix}$	3
30	1	93	1	1	$\langle 7, -11, 1 \rangle$	$\frac{11+\sqrt{93}}{14}$	$[\overline{2, 2, 11}]$	$\begin{pmatrix} 31 & -3 \\ 21 & -2 \end{pmatrix}$	3
			3	3	$\langle 1, -29, 1 \rangle$	$\frac{29+3\sqrt{93}}{2}$	$[\overline{29}]$	$\begin{pmatrix} 29 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 7, -37, 19 \rangle$	$\frac{37+3\sqrt{93}}{14}$	$[\overline{5, 4, 2}]$	$\begin{pmatrix} 33 & -19 \\ 7 & -4 \end{pmatrix}$	3
					$\langle 7, -33, 9 \rangle$	$\frac{33+3\sqrt{93}}{14}$	$[\overline{5, 2, 4}]$	$\begin{pmatrix} 31 & -9 \\ 7 & -2 \end{pmatrix}$	3

TABLE 3. SICs in dimensions 25–30.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
31	1	56	1	1	$\langle 2,-8,1\rangle$	$\frac{4+\sqrt{14}}{2}$	$[\overline{4,8}]$	$\begin{pmatrix}31 & -4\\ 8 & -1\end{pmatrix}$	3
			2	2	$\langle 8,-16,1\rangle$	$\frac{4+\sqrt{14}}{4}$	$[\overline{2,16}]$	$\begin{pmatrix}31 & -2\\ 16 & -1\end{pmatrix}$	3
					$\langle 4,-20,11\rangle$	$\frac{5+\sqrt{14}}{2}$	$[\overline{5,2,3,2}]$	$\begin{pmatrix}35 & -22\\ 8 & -5\end{pmatrix}$	3
			4	4	$\langle 1,-30,1\rangle$	$15+4\sqrt{14}$	$[\overline{30}]$	$\begin{pmatrix}30 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 4,-36,25\rangle$	$\frac{9+2\sqrt{14}}{2}$	$[\overline{9,2,2,2}]$	$\begin{pmatrix}33 & -25\\ 4 & -3\end{pmatrix}$	3
					$\langle 5,-36,20\rangle$	$\frac{18+4\sqrt{14}}{5}$	$[\overline{7,3,2}]$	$\begin{pmatrix}33 & -20\\ 5 & -3\end{pmatrix}$	3
					$\langle 5,-34,13\rangle$	$\frac{17+4\sqrt{14}}{5}$	$[\overline{7,2,3}]$	$\begin{pmatrix}32 & -13\\ 5 & -2\end{pmatrix}$	3
32	1	957	1	2	$\langle 1,-31,1\rangle$	$\frac{31+\sqrt{957}}{2}$	$[\overline{31}]$	$\begin{pmatrix}31 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 3,-33,11\rangle$	$\frac{33+\sqrt{957}}{6}$	$[\overline{11,3}]$	$\begin{pmatrix}32 & -11\\ 3 & -1\end{pmatrix}$	3
33	1	1020	1	4	$\langle 1,-32,1\rangle$	$16+\sqrt{255}$	$[\overline{32}]$	$\begin{pmatrix}32 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 2,-34,17\rangle$	$\frac{17+\sqrt{255}}{2}$	$[\overline{17,2}]$	$\begin{pmatrix}33 & -17\\ 2 & -1\end{pmatrix}$	3
					$\langle 3,-36,23\rangle$	$\frac{18+\sqrt{255}}{3}$	$[\overline{12,2,2}]$	$\begin{pmatrix}34 & -23\\ 3 & -2\end{pmatrix}$	3
					$\langle 5,-40,29\rangle$	$\frac{20+\sqrt{255}}{5}$	$[\overline{8,2,2,2,2}]$	$\begin{pmatrix}36 & -29\\ 5 & -4\end{pmatrix}$	3
34	1	1085	1	2	$\langle 1,-33,1\rangle$	$\frac{33+\sqrt{1085}}{2}$	$[\overline{33}]$	$\begin{pmatrix}33 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 5,-35,7\rangle$	$\frac{35+\sqrt{1085}}{10}$	$[\overline{7,5}]$	$\begin{pmatrix}34 & -7\\ 5 & -1\end{pmatrix}$	3
35	2	8	1	1	$\langle 2,-4,1\rangle$	$\frac{2+\sqrt{2}}{2}$	$[\overline{2,4}]$	$\begin{pmatrix}7 & -2\\ 4 & -1\end{pmatrix}$	6
			2	1	$\langle 1,-6,1\rangle$	$3+2\sqrt{2}$	$[\overline{6}]$	$\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}$	6
			3	1	$\langle 7,-10,1\rangle$	$\frac{5+3\sqrt{2}}{7}$	$[\overline{2,2,2,10}]$	$\begin{pmatrix}37 & -4\\ 28 & -3\end{pmatrix}$	3
			4	1	$\langle 4,-12,1\rangle$	$\frac{3+2\sqrt{2}}{2}$	$[\overline{3,12}]$	$\begin{pmatrix}35 & -3\\ 12 & -1\end{pmatrix}$	3
			6	2	$\langle 9,-18,1\rangle$	$\frac{3+2\sqrt{2}}{3}$	$[\overline{2,18}]$	$\begin{pmatrix}35 & -2\\ 18 & -1\end{pmatrix}$	3
					$\langle 4,-20,7\rangle$	$\frac{5+3\sqrt{2}}{2}$	$[\overline{5,3,3}]$	$\begin{pmatrix}37 & -14\\ 8 & -3\end{pmatrix}$	3
			12	4	$\langle 1,-34,1\rangle$	$17+12\sqrt{2}$	$[\overline{34}]$	$\begin{pmatrix}34 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 4,-36,9\rangle$	$\frac{9+6\sqrt{2}}{2}$	$[\overline{9,4}]$	$\begin{pmatrix}35 & -9\\ 4 & -1\end{pmatrix}$	3
					$\langle 7,-44,28\rangle$	$\frac{22+12\sqrt{2}}{7}$	$[\overline{6,3,2,2}]$	$\begin{pmatrix}39 & -28\\ 7 & -5\end{pmatrix}$	3
					$\langle 7,-40,16\rangle$	$\frac{20+12\sqrt{2}}{7}$	$[\overline{6,2,2,3}]$	$\begin{pmatrix}37 & -16\\ 7 & -3\end{pmatrix}$	3
36	1	1221	1	4	$\langle 1,-35,1\rangle$	$\frac{35+\sqrt{1221}}{2}$	$[\overline{35}]$	$\begin{pmatrix}35 & -1\\ 1 & 0\end{pmatrix}$	3
					$\langle 3,-39,25\rangle$	$\frac{39+\sqrt{1221}}{6}$	$[\overline{13,2,2}]$	$\begin{pmatrix}37 & -25\\ 3 & -2\end{pmatrix}$	3
					$\langle 5,-41,23\rangle$	$\frac{41+\sqrt{1221}}{10}$	$[\overline{8,3,2}]$	$\begin{pmatrix}38 & -23\\ 5 & -3\end{pmatrix}$	3
					$\langle 5,-39,15\rangle$	$\frac{39+\sqrt{1221}}{10}$	$[\overline{8,2,3}]$	$\begin{pmatrix}37 & -15\\ 5 & -2\end{pmatrix}$	3

TABLE 4. SICs in dimensions 31–36.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
37	1	1292	1	4	$\langle 1, -36, 1\rangle$	$18+\sqrt{323}$	$[\overline{36}]$	$\left(\begin{smallmatrix} 36 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	3
					$\langle 2, -38, 19\rangle$	$\frac{19+\sqrt{323}}{2}$	$[\overline{19, 2}]$	$\left(\begin{smallmatrix} 37 & -19 \\ 2 & -1 \end{smallmatrix}\right)$	3
					$\langle 7, -44, 23\rangle$	$\frac{22+\sqrt{323}}{7}$	$[\overline{6, 4, 2}]$	$\left(\begin{smallmatrix} 40 & -23 \\ 7 & -4 \end{smallmatrix}\right)$	3
					$\langle 7, -40, 11\rangle$	$\frac{20+\sqrt{323}}{7}$	$[\overline{6, 2, 4}]$	$\left(\begin{smallmatrix} 38 & -11 \\ 7 & -2 \end{smallmatrix}\right)$	3
38	1	1365	1	4	$\langle 1, -37, 1\rangle$	$\frac{37+\sqrt{1365}}{2}$	$[\overline{37}]$	$\left(\begin{smallmatrix} 37 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	3
					$\langle 5, -45, 33\rangle$	$\frac{45+\sqrt{1365}}{10}$	$[\overline{9, 2, 2, 2, 2}]$	$\left(\begin{smallmatrix} 41 & -33 \\ 5 & -4 \end{smallmatrix}\right)$	3
					$\langle 3, -39, 13\rangle$	$\frac{39+\sqrt{1365}}{6}$	$[\overline{13, 3}]$	$\left(\begin{smallmatrix} 38 & -13 \\ 3 & -1 \end{smallmatrix}\right)$	3
					$\langle 11, -45, 15\rangle$	$\frac{45+\sqrt{1365}}{22}$	$[\overline{4, 4, 3}]$	$\left(\begin{smallmatrix} 41 & -15 \\ 11 & -4 \end{smallmatrix}\right)$	3

TABLE 5. SICs in dimensions 37–40.

6. ECTFF TABLES

(Added by GSK, 31 May 2021.)

In the following table, we list parameters for maximal ECTFFs predicted by our conjectures, of dimension $4 \le d \le 200$ and rank $2 \le r < \frac{d-1}{2}$. (Zauner predicts the existence of maximal ECTFFs of rank 1 (SICs) in every dimension. Maximal ECTFFs of rank $\frac{d-1}{2}$ are known (the Wigner ECTFF), and a one-parameter family is believed to exist. Our construction does not produce potential maximal ECTFFs of rank $\frac{d}{2}$, and ranks $r > \frac{d}{2}$ are redundant by rank complementation.)

(1) d is the dimension
(2) r is the rank
(3) $n = \frac{d^2-1}{r(d-r)}$
(4) $\Delta = n(n-4)$
(5) k is the order of B (or of $\frac{n-2+\sqrt{\Delta}}{2}$) modulo d
(6) Δ_0 is the fundamental part of Δ
(7) f is the conductor
(8) HJ is the HJ-expansion of β

d	r	n	Δ	k	Δ_0	f	HJ
11	3	5	5	5	5		
19	4	6	12	5	12		
29	5	7	21	5	21		
29	8	5	5	7	5		
41	6	8	32	5	8		
55	7	9	45	5	5		
71	8	10	60	5	60		
71	15	6	12	7	12		
76	21	5	5	9	5		
89	9	11	77	5	77		
109	10	12	96	5	24		
131	11	13	117	5	13		
139	24	7	21	7	21		
155	12	14	140	5	140		
181	13	15	165	5	165		
199	55	5	5	11	5		

APPENDIX A. THE DOUBLE GAMMA FUNCTION

A.1. **Definition.** I noticed that in the Wikipedia article on multiple gamma functions it says that the definition used differs by a constant from the one given by Barnes [3] himself. However, they don't say what the constant is. I think the Wikipedia article is following Spreafico [15], but Spreafico doesn't say what the constant is either. In any case it looks to me[9] as though Shintani [14], who is the author most relevant to us, is defining it the same way as Barnes. So I am taking Shintani's definition to be canonical in these notes. The purpose of this appendix is to fill in a few gaps in Shintani's discussion.

Define, for $\mathbf{b} = (b_1, b_2)$,

$$P(w, \boldsymbol{b}) = \prod_{\mathbf{m}\in N_+} \frac{e^{\frac{w}{\mathbf{m}.\mathbf{b}} - \frac{w^2}{2(\mathbf{m}.\mathbf{b})^2}}}{1 + \frac{w}{\mathbf{m}.\mathbf{b}}} \tag{A.1}$$

where

$$N_+ = \{\mathbf{m} \in \mathbb{Z}^2 \colon m_1, m_2 \geq 0, \text{ and } m_1 + m_2 > 0\}. \tag{A.2}$$

It can be shown [3] if b_1, b_2 are non-zero, and b_2/b_1 is not a negative real, then the product converges to a meromorphic function of w, with poles at the points $\{-\mathbf{m}.\mathbf{b} \colon \mathbf{m} \in N_+\}$. Also define

$$\gamma_{21}(\mathbf{b}) = \frac{1}{b_1 b_2} f_1\left(\frac{b_2}{b_1}\right) + \frac{\ln(b_1 b_2)}{2 b_1 b_2} \tag{A.3}$$

$$\gamma_{22}(\mathbf{b}) = \frac{1}{b_1} f_2\left(\frac{b_2}{b_1}\right) - \frac{1}{2}\left(\frac{\ln b_1}{b_1} + \frac{\ln b_2}{b_2}\right) \tag{A.4}$$

where γ is the Euler constant,

$$f_1(\beta) = \frac{3\ln\beta - \pi^2\beta - 6\gamma}{6} - \beta \sum_{n=1}^{\infty} \left(\psi'(n\beta) - \frac{1}{n\beta}\right) \tag{A.5}$$

$$f_2(\beta) = \frac{(2+\beta^{-1})\gamma + (2-\beta^{-1})\ln\beta - \ln(2\pi)}{2} - \sum_{n=1}^{\infty}\left(\psi(n\beta) - \ln n\beta + \frac{1}{2n\beta}\right) \tag{A.6}$$

and $\psi(\beta)$ is the logarithmic derivative of the gamma function. Shintani [14] then defines the double gamma function by (p. 172 of ref. [14])

$$\Gamma_2(w, \mathbf{b}) = \frac{1}{w} e^{-\gamma_{22}(\mathbf{b})w - \frac{1}{2}\gamma_{21}(\mathbf{b})w^2} P(w, \mathbf{b}). \tag{A.7}$$

Barnes (on p. 299 of ref. [3]) defines the function in the same way except that he gives different expressions for γ_{21}, γ_{22}.

The functions $f_1(\beta)$, $f_2(\beta)$ in Shintani's definition are analytic on the complex plane cut along the non-positive real axis. Indeed, suppose $|\arg(z)| < \pi/2 - \delta/2$ for positive real δ. Then [10]

$$\left|\psi'(z) - \frac{1}{z}\right| \leq \frac{1}{2|z|^2}\left(1 + \frac{1}{3|z|\sin^4\delta}\right), \qquad \left|\psi(z) - \ln(z) + \frac{1}{2z}\right| \leq \frac{1}{6|z|^2\sin^3\delta}. \tag{A.8}$$

[9] I say "looks" because Shintani's expression has a very different form to Barnes's. I suspect the expressions are actually the same, but I haven't proved this. Another fact contributing to my uncertainty is that, on the assumption that Shintani's version of the function is the same as Barnes's, Shintani's Lemma 1 is identical and his Proposition 1 is a special case of statements proved by Barnes. So why does Shintani go to all the trouble of re-proving them?

So the summations on the right hand sides of Eqs. (A.5), (A.6) converge absolutely and uniformly on every compact subset of $\mathbb{C} \setminus (-\infty, 0]$ and therefore define analytic functions.

A.2. **Reflection symmetry of double gamma.** One significant difference between Barnes's definition and Shintani's is that in Barnes's version the functions γ_{21}, γ_{22} are manifestly symmetric under the interchange $b_1 \leftrightarrow b_2$. In Shintani's version, by contrast, the symmetry is not obvious. It can, however, be proved. Specifically[10] in Lemma 1 of ref. [14] Shintani shows that for β positive real

$$f_1(\beta^{-1}) = f_1(\beta), \qquad f_2(\beta^{-1}) = \beta^{-1} f_2(\beta^{-1}). \tag{A.9}$$

It follows from the analyticity of f_1, f_2 that these relations continue to hold for all $\beta \in \mathbb{C} \setminus (-\infty, 0]$, and consequently that $\Gamma_2(w, \mathbf{b})$ is symmetric under the interchange $b_1 \leftrightarrow b_2$ for all b_1, b_2 such that $b_2/b_1 \in \mathbb{C} \setminus (-\infty, 0]$.

A.3. **Reduced double gamma.** It is convenient to decompose $\Gamma_2(w, \mathbf{b})$ into a function F of the three variables w, b_1, b_2, and a function $\tilde{\Gamma}_2$ of the two variables w/b_1, b_2/b_1:

$$\Gamma_2(w, \mathbf{b}) = F(w, \mathbf{b}) \tilde{\Gamma}_2 \left(\frac{w}{b_1}, \frac{b_2}{b_1} \right) \tag{A.10}$$

where

$$F(w, \mathbf{b}) = \frac{1}{b_1} e^{\frac{w}{2}\left(\frac{\ln b_1}{b_1} + \frac{\ln b_2}{b_2}\right) - \frac{w^2}{4}\left(\frac{\ln(b_1 b_2)}{b_1 b_2}\right)}, \tag{A.11}$$

$$\tilde{\Gamma}_2(\omega, \beta) = \frac{1}{\omega} e^{-\omega f_2(\beta) - \frac{\omega^2}{2\beta} f_1(\beta)} \tilde{P}(\omega, \beta), \tag{A.12}$$

$$\tilde{P}(\omega, \beta) = \prod_{\mathbf{m} \in N_+} \frac{e^{\frac{\omega}{m_1 + m_2 \beta} - \frac{\omega^2}{2(m_1 + m_2 \beta)^2}}}{1 + \frac{\omega}{m_1 + m_2 \beta}} \tag{A.13}$$

I will refer to $\tilde{\Gamma}_2$ as the reduced double gamma function.

As we will see the double sine function *only* depends on the two variables w/b_1, b_2/b_1.

A.4. **Domain of definition of double gamma and reduced double gamma.** We now want to specify the domain of definition of $\Gamma_2(w, \mathbf{b})$. Define the region $\mathcal{R} \subseteq \mathbb{C}^3$ to consist of all triples (w, b_1, b_2) such that $|b_1|$, $|b_2| > 0$,

$$|b_1| > 0, \qquad |b_2| > 0, \tag{A.14}$$

$$-\pi < \arg(b_2) + \arg(b_1) < \pi, \tag{A.15}$$

$$-\pi < \arg(b_2) - \arg(b_1) < \pi, \tag{A.16}$$

$$w \notin \{-\mathbf{m}.\mathbf{b} \colon m_1, m_2 \in \mathbb{Z} \text{ and } m_1, m_2 \geq 0\} \tag{A.17}$$

where by $\arg(z)$ we will always mean the principal argument, in the range $(-\pi, \pi]$. The set of poles of $P(w, \mathbf{b})$ is $\{-\mathbf{m}.\mathbf{b} \colon \mathbf{m} \in N_+\}$; due to the factor $1/w$ in Eq. (A.7) the function $\Gamma_2(w, \mathbf{b})$ has an additional pole at $w = 0$. The constraints on $\arg(b_1)$, $\arg(b_2)$ are illustrated in Fig. 1. Observe that

[10] Shintani's Lemma 1 is one of the things that makes me wonder if his version of the function is in fact the same as Barnes's. For if it were the same, wouldn't it be most natural for him to show that his definition is equivalent to Barnes's, and then appeal to the fact that Barnes's versions of γ_{21}, γ_{22} are manifestly symmetric? Of course this question should be easy to settle one way or another. At some point I will do that.

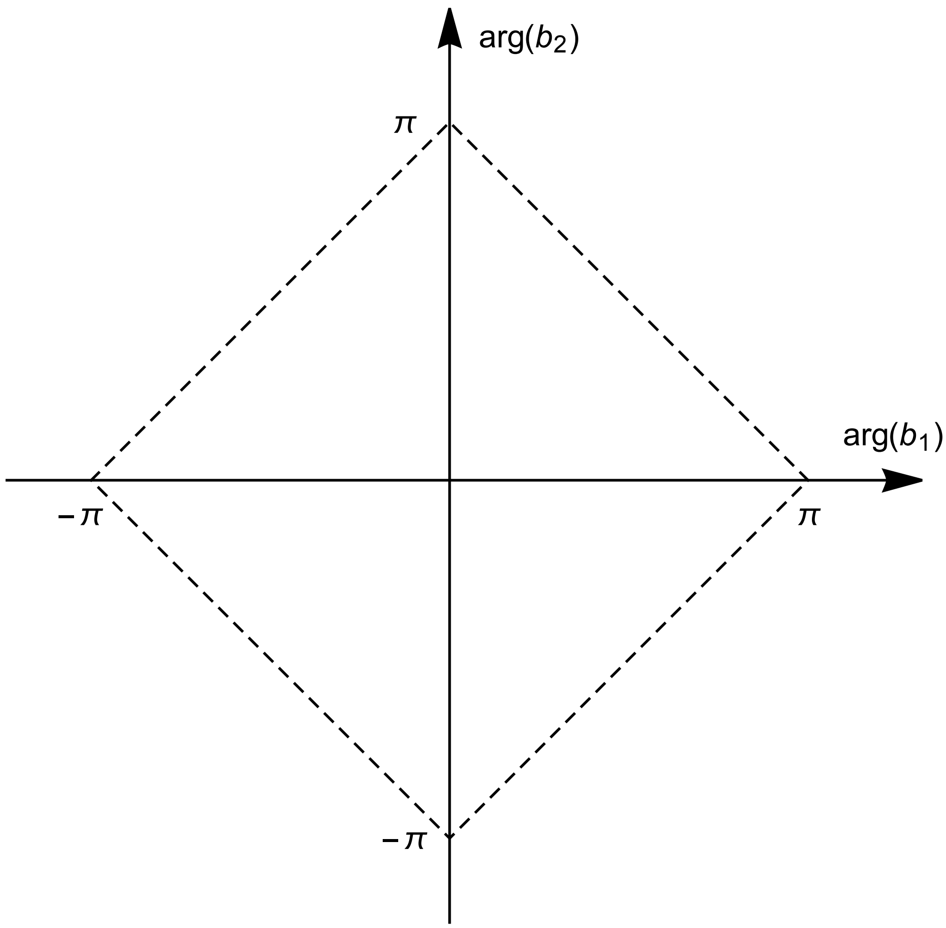

FIGURE 1. Allowed values of the arguments for $\mathbf{b} \in \mathcal{R}$

if b_1, b_2 are non-zero and satisfy Eqs. (A.15), (A.16), then b_2/b_1, $b_1 b_2$, b_1, b_2 are all in $\mathbb{C} \setminus (-\infty, 0]$. Consequently $\Gamma_2(w, \mathbf{b})$ is well-defined and single-valued on $\mathcal{R}$.

Note also that if b_1, b_2 are non-zero and satisfy Eqs. (A.15), (A.16) then

$$\ln(b_1 b_2) = \ln b_2 + \ln b_1, \tag{A.18}$$

$$\ln\left(\frac{b_2}{b_1}\right) = \ln b_2 - \ln b_1, \tag{A.19}$$

where in both cases the principal branches of the logarithm is taken.

$\tilde{\Gamma}_2$ is defined and single-valued on $\tilde{R}$, where $\tilde{R}$ is the region consisting of all $(\omega, \beta) \in \mathbb{C}^2$ such that

$$\begin{aligned} &\beta = \mathbb{C} \setminus (-\infty, 0], \\ &\omega \notin \{-m_1 - m_2\beta \colon m_1, m_2 \in \mathbb{Z} \text{ and } m_1, m_2 \geq 0\}. \end{aligned} \tag{A.20}$$

A.5. **Reflection symmetry of reduced double gamma.** Corresponding to the reflection symmetry of $\Gamma_2(w, \mathbf{b})$ (i.e. its invariance under the interchange $b_1 \leftrightarrow b_2$) one has

$$\tilde{\Gamma}_2(\omega, \beta) = \frac{\Gamma_2(\omega, (1, \beta))}{F(\omega, (1, \beta))}$$

$$
\begin{aligned}
&= \frac{\Gamma_2(\omega,(\beta,1))}{F(\omega,(1,\beta))} \\
&= \frac{F(w,(\beta,1))\tilde{\Gamma}_2(\beta^{-1}\omega,\beta^{-1})}{F(\omega,(1,\beta))}, \\
&= \beta^{-1}\tilde{\Gamma}_2(\beta^{-1}\omega,\beta^{-1}).
\end{aligned} \tag{A.21}
$$

A.6. **Definition of double sine and reduced double sine.** The double-sine function is defined

$$
S_2(w,\mathbf{b}) = \frac{\Gamma_2(w,\mathbf{b})}{\Gamma_2(b_1+b_2-w,\mathbf{b})} \tag{A.22}
$$

Since Γ_2 has no zeros (as can be seen from Eqs. (A.1), (A.7)) S_2 is well-defined and single-valued for all $(w,b_1,b_2)\in\mathcal{R}$, the same as Γ_2. Observe

$$
\begin{aligned}
\frac{F(w,\mathbf{b})}{F(b_1+b_2-w,\mathbf{b})} &= e^{\left(\frac{w}{2}-\frac{b_1+b_2-w}{2}\right)\left(\frac{\ln b_1}{b_1}+\frac{\ln b_2}{b_2}\right)-\left(\frac{w^2}{4}-\frac{(b_1+b_2-w)^2}{4}\right)\left(\frac{\ln(b_1b_2)}{b_1b_2}\right)} \\
&= e^{\frac{2w-b_1-b_2}{2}\left(\frac{\ln b_1}{b_1}+\frac{\ln b_2}{b_2}\right)-\frac{2w(b_1+b_2)-(b_1+b_2)^2}{4}\left(\frac{\ln(b_1b_2)}{b_1b_2}\right)} \\
&= e^{\frac{1}{4}\left(\left(\frac{b_2}{b_1}-\frac{b_1}{b_2}\right)+2w\left(\frac{1}{b_2}-\frac{1}{b_1}\right)\right)\ln\left(\frac{b_2}{b_1}\right)}.
\end{aligned} \tag{A.23}
$$

So

$$
S_2(w,\mathbf{b}) = \tilde{S}_2\left(\frac{w}{b_1},\frac{b_2}{b_1}\right) \tag{A.24}
$$

where

$$
\tilde{S}_2(\omega,\beta) = e^{-\frac{1}{4}(1-\beta)(1+\beta-2\omega)\frac{\ln\beta}{\beta}}\left(\frac{\tilde{\Gamma}_2(\omega,\beta)}{\tilde{\Gamma}_2(1+\beta-\omega,\beta)}\right). \tag{A.25}
$$

I will call $\tilde{S}_2$ the reduced double sine function.

A.7. **Domains of definition of double sine and reduced double sine.** By construction $\tilde{S}_2$ is defined and single-valued on $\tilde{R}$. Although Eq. (A.24) was derived on the assumption that $(w,b_1,b_2)\in\mathcal{R}$ we can use it to extend the domain of definition of S_2 to the set

$$
\mathcal{R}_e = \left\{(w,b_1,b_2)\in\mathbb{C}^3 : \left(\frac{w}{b_1},\frac{b_2}{b_1}\right)\in\tilde{\mathcal{R}}\right\}. \tag{A.26}
$$

A.8. **Symmetries of double sine and reduced double sine.** It follows from the above that S_2 has a dilation symmetry: for all non-zero $t\in\mathbb{C}$ and $(w,\mathbf{b})\in\mathcal{R}_e$, $(tw,t\mathbf{b})\in\mathcal{R}_e$ and

$$
S_2(tw,t\mathbf{b}) = S_2(w,\mathbf{b}). \tag{A.27}
$$

This fact explains the equivalence between the description in the Wikepedia article where S_2 is identified with the two-parameter function $S_2(w,(b,b^{-1})$, and the description in our `SIC_constructio-n_notes` where we work in terms of the two-parameter function $S_2(w,(1,\tau))$. Indeed, it follows from the foregoing that

$$
S_2(w,(b,b^{-1})) = \tilde{S}_2(wb^{-1},b^{-2}) = S_2(wb^{-1},(1,b^{-2})). \tag{A.28}
$$

It can also be seen that these two alternative parameterisations include every case.

It follows from the definition that S_2 is symmetric under the interchange $b_1 \leftrightarrow b_2$. In view of Eq. (A.24) this means

$$\begin{aligned}\tilde{S}_2(\omega, \beta) &= S_2(\omega, (1, \beta)) \\ &= S_2(\omega, (\beta, 1)) \\ &= \tilde{S}_2(\beta^{-1}\omega, \beta^{-1}) \end{aligned} \tag{A.29}$$

(compare Eq. (A.21)).

A.9. **Periodicities of double gamma and reduced double gamma.** Shintani shows (Proposition 1) that if b_1, b_2 are positive real then

$$\Gamma_2(w + b_1, \mathbf{b})^{-1} = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{w}{b_2} - \frac{1}{2}\right) \ln b_2} \Gamma\left(\frac{w}{b_2}\right) \Gamma_2(w, \mathbf{b})^{-1}, \tag{A.30}$$

$$\Gamma_2(w + b_2, \mathbf{b})^{-1} = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{w}{b_1} - \frac{1}{2}\right) \ln b_1} \Gamma\left(\frac{w}{b_1}\right) \Gamma_2(w, \mathbf{b})^{-1} \tag{A.31}$$

for all $w \in \mathbb{C}$. Note that the fact that Γ_2 has no zeros means that $\Gamma_2(w, \mathbf{b})^{-1}$ is holomorphic, so that this formula is valid even when w or $w + b_j$ is at one of the poles of Γ_2. We can then use analyticity to infer that the formulae continue to hold for all $\mathbf{b} \in \mathbb{C}^2$ satisfying Eqs. (A.14)–(A.16).

Using

$$\tilde{\Gamma}_2(\omega, \beta)^{-1} = F(\omega, (1, \beta))\Gamma_2(\omega, (1, \beta))^{-1} = e^{\frac{\omega(2-\omega)}{4}\frac{\ln \beta}{\beta}} \Gamma_2(\omega, (1, \beta))^{-1} \tag{A.32}$$

one finds that the corresponding formulae for the reduced function are

$$\begin{aligned}\tilde{\Gamma}_2(\omega + 1, \beta)^{-1} &= e^{\frac{(1-\omega^2)}{4}\frac{\ln \beta}{\beta}} \Gamma_2(\omega + 1, (1, \beta))^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{(1-\omega^2)}{4}\frac{\ln \beta}{\beta}} \Gamma\left(\frac{\omega}{\beta}\right) e^{\left(\frac{\omega}{\beta} - \frac{1}{2}\right) \ln \beta} \Gamma_2(\omega, (1, \beta))^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{1-\omega^2}{4} + \frac{2\omega - \beta}{2} - \frac{\omega(2-\omega)}{4}\right)\frac{\ln \beta}{\beta}} \Gamma\left(\frac{\omega}{\beta}\right) \tilde{\Gamma}_2(\omega, \beta)^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{1-2\beta+2\omega}{4}\right)\frac{\ln \beta}{\beta}} \Gamma\left(\frac{\omega}{\beta}\right) \tilde{\Gamma}_2(\omega, \beta)^{-1} \end{aligned} \tag{A.33}$$

$$\begin{aligned}\tilde{\Gamma}_2(\omega + \beta, \beta)^{-1} &= e^{\frac{(\omega+\beta)(2-\omega-\beta)}{4}\frac{\ln \beta}{\beta}} \Gamma_2(\omega + \beta, (1, \beta))^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{(\omega+\beta)(2-\omega-\beta)}{4}\frac{\ln \beta}{\beta}} \Gamma(\omega)\Gamma_2(\omega, (1, \beta))^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{(\omega+\beta)(2-\omega-\beta)}{4} - \frac{\omega(2-\omega)}{4}\right)\frac{\ln \beta}{\beta}} \Gamma(\omega)\tilde{\Gamma}_2(\omega, \beta)^{-1} \\ &= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{2-\beta-2\omega}{4}\right)\ln \beta} \Gamma(\omega)\tilde{\Gamma}_2(\omega, \beta)^{-1} \end{aligned} \tag{A.34}$$

for all $\omega \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus (-\infty, 0]$.

A.10. **Periodicities of double sine and reduced double sine.** Using Eqs. (A.25), (A.33), and (A.34) one finds

$$\tilde{S}_2(\omega + 1, \beta) = e^{-\frac{1}{4}(1-\beta)(\beta - 2\omega - 1)\frac{\ln \beta}{\beta}} \left(\frac{\tilde{\Gamma}_2(\omega + 1, \beta)}{\tilde{\Gamma}_2(\beta - \omega, \beta)}\right)$$

$$
\begin{aligned}
&= e^{-\frac{1}{4}(1-\beta)(\beta-2\omega-1)\frac{\ln\beta}{\beta}} \left(\frac{\sqrt{2\pi} e^{-\left(\frac{1-2\beta+2\omega}{4}\right)\frac{\ln\beta}{\beta}} \Gamma\left(\frac{\omega}{\beta}\right)^{-1} \tilde{\Gamma}_2(\omega,\beta)}{\frac{1}{\sqrt{2\pi}} e^{\left(\frac{1-2\omega}{4}\right)\frac{\ln\beta}{\beta}} \Gamma\left(1-\frac{\omega}{\beta}\right) \tilde{\Gamma}_2(1+\beta-\omega,\beta)} \right) \\
&= 2e^{-\frac{1}{4}(1-\beta)(1+\beta-2\omega)\frac{\ln\beta}{\beta}} \left(\frac{\pi}{\Gamma\left(\frac{\omega}{\beta}\right)\Gamma\left(1-\frac{\omega}{\beta}\right)} \right) \frac{\tilde{\Gamma}_2(\omega,\beta)}{\tilde{\Gamma}_2(1+\beta-\omega,\beta)} \\
&= 2\sin\left(\frac{\pi\omega}{\beta}\right) \tilde{S}_2(\omega,\beta) \qquad \text{(A.35)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{S}_2(\omega+\beta,\beta) &= e^{-\frac{1}{4}(1-\beta)(1-\beta-2\omega)\frac{\ln\beta}{\beta}} \left(\frac{\tilde{\Gamma}_2(\omega+\beta,\beta)}{\tilde{\Gamma}_2(1-\omega,\beta)} \right) \\
&= e^{-\frac{1}{4}(1-\beta)(1-\beta-2\omega)\frac{\ln\beta}{\beta}} \left(\frac{\sqrt{2\pi} e^{-\left(\frac{2-\beta-2\omega}{4}\right)\ln\beta} \Gamma(\omega)^{-1} \tilde{\Gamma}_2(\omega,\beta)}{\frac{1}{\sqrt{2\pi}} e^{\left(\frac{2\omega-\beta}{4}\right)\ln\beta} \Gamma(1-\omega) \tilde{\Gamma}_2(1+\beta-\omega)} \right) \\
&= 2e^{-\frac{1}{4}(1-\beta)(1+\beta-2\omega)\frac{\ln\beta}{\beta}} \left(\frac{\pi}{\Gamma(\omega)\Gamma(1-\omega)} \right) \frac{\tilde{\Gamma}_2(\omega,\beta)}{\tilde{\Gamma}_2(1+\beta-\omega,\beta)} \\
&= 2\sin(\pi\omega)\tilde{S}_2(\omega,\beta) \qquad \text{(A.36)}
\end{aligned}
$$

for all $\omega \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus (-\infty, 0]$.

Using Eq. (A.24) one infers

$$
\begin{aligned}
S_2(w+b_1,\mathbf{b}) &= \tilde{S}_2\left(\frac{w}{b_1}+1, \frac{b_2}{b_1}\right) \\
&= 2\sin\left(\frac{\pi w}{b_2}\right) \tilde{S}_2\left(\frac{w}{b_1}, \frac{b_2}{b_1}\right) \\
&= 2\sin\left(\frac{\pi w}{b_2}\right) S_2(w,\mathbf{b}), \qquad \text{(A.37)} \\
S_2(w+b_2,\mathbf{b}) &= \tilde{S}_2\left(\frac{w}{b_1}+\frac{b_2}{b_1}, \frac{b_2}{b_1}\right) \\
&= 2\sin\left(\frac{\pi w}{b_1}\right) \tilde{S}_2\left(\frac{w}{b_1}, \frac{b_2}{b_1}\right) \\
&= 2\sin\left(\frac{\pi w}{b_1}\right) S_2(w,\mathbf{b}) \qquad \text{(A.38)}
\end{aligned}
$$

for all $(w, b_1, b_2) \in \mathcal{R}_e$.

A.11. **Integral Representation of Double Gamma.** The Wikipedia article gives an integral representation of $\Gamma_2(w, \mathbf{b})$ which, though less general than the one proved by Shintani [14] (since it assumes $b_1 b_2 = 1$) is more convenient. In this subsection I will show how Shintani's method can be used to prove a fully general version of the representation in the Wikipedia article.

Consider the integral

$$
\mathcal{I}(w,\mathbf{b}) = \int_0^\infty I(t,w,\mathbf{b})dt, \qquad \text{(A.39)}
$$

$$I(t,w,\mathbf{b}) = \frac{1}{t}\left(\frac{e^{-wt}-e^{-\frac{(b_1+b_2)t}{2}}}{(1-e^{-b_1t})(1-e^{-b_2t})} - \frac{\left(\frac{b_1+b_2}{2}-w\right)^2 e^{-t}}{2b_1b_2} - \frac{\frac{b_1+b_2}{2}-w}{b_1b_2t}\right) \tag{A.40}$$

We begin by establishing the values of w, b_1, b_2 for which it is well-defined. For small t

$$\begin{aligned}
I(t,w,\mathbf{b}) = \frac{1}{t}\Bigg(&\frac{\left(\frac{b_1+b_2}{2}-w\right)t-\frac{1}{2}\left(\frac{(b_1+b_2)^2}{4}-w^2\right)t^2+O(t^2)}{\left(b_1t-\frac{1}{2}b_1^2t^2+O(t^3)\right)\left(b_2t-\frac{1}{2}b_2^2t^2+O(t^3)\right)}\\
&-\frac{\left(\frac{b_1+b_2}{2}-w\right)^2(1+O(t))}{2b_1b_2}-\frac{\frac{b_1+b_2}{2}-w}{b_1b_2t}\Bigg)\\
=&\frac{\left(\frac{b_1+b_2}{2}-w\right)\left(1-\frac{1}{2}\left(\frac{b_1+b_2}{2}+w\right)t+O(t^2)\right)}{b_1b_2t^2\left(1-\frac{1}{2}b_1t+O(t^2)\right)\left(1-\frac{1}{2}b_2t+O(t^2)\right)}\\
&-\frac{\left(\frac{b_1+b_2}{2}-w\right)\left(\left(\frac{b_1+b_2}{2}-w\right)(t+O(t^2))+2\right)}{2b_1b_2t^2}\\
=&\frac{\left(\frac{b_1+b_2}{2}-w\right)}{2b_1b_2t^2}\left(2\left(1+\frac{1}{2}\left(\frac{b_1+b_2}{2}-w\right)t+O(t^2)\right)\right.\\
&\left.-\left(2+\left(\frac{b_1+b_2}{2}-w\right)t+O(t^2)\right)\right)\\
=&O(t^0).
\end{aligned} \tag{A.41}$$

So if b_1, b_2 are both non-zero then $I(t,w,\mathbf{b})$ is integrable in the vicinity of $t=0$.

Turning to the behaviour at large t one has, for t large and positive,

$$\begin{aligned}
\left|\frac{e^{-\frac{(b_1+b_2)t}{2}}}{(1-e^{-b_1t})(1-e^{-b_2t})}\right| - = &\left|\frac{1}{\sinh\left(\frac{b_1t}{2}\right)\sinh\left(\frac{b_2t}{2}\right)}\right|\\
&\leq \frac{1}{\sinh\left(\frac{|t\operatorname{Re}(b_1)|}{2}\right)\sinh\left(\frac{|t\operatorname{Re}(b_2)|}{2}\right)}\\
&\sim e^{-\left(\frac{|\operatorname{Re}(b_1)|+|\operatorname{Re}(b_2)|}{2}\right)t}.
\end{aligned} \tag{A.42}$$

Also

$$\left(\frac{1}{1-e^{-b_jt}}\right) = \begin{cases}1 & \operatorname{Re}(b_j)>0,\\ 0 & \operatorname{Re}(b_j)<0.\end{cases} \tag{A.43}$$

So if $\operatorname{Re}(b_1)$, $\operatorname{Re}(b_2)$ are non-zero then $1/(1-e^{-b_1t})$, $1/(1-e^{-b_2t})$ are bounded in the limit of large t. We conclude that $I(t,w,\mathbf{b})$ is integrable on the interval $[0,\infty)$ if $\operatorname{Re}(b_1)$, $\operatorname{Re}(b_2)$ are non-zero and $\operatorname{Re}(w)$ is positive.

Now specialize to the case when $w-1$, b_1-1, b_2-1 are positive real and consider the difference

$$\begin{aligned}
\mathcal{I}(w+b_1,\mathbf{b})-\mathcal{I}(w,\mathbf{b}) &= \int_0^\infty \frac{1}{t}\left(-\frac{e^{-wt}}{1-e^{-b_2t}}+\left(\frac{1}{2}-\frac{w}{b_2}\right)e^{-t}+\frac{1}{b_2t}\right)dt\\
&= \int_0^\infty \frac{1}{t}\left(-\frac{e^{-\omega t}}{1-e^{-t}}+\left(\frac{1}{2}-\omega\right)e^{-\frac{t}{b_2}}+\frac{1}{t}\right)dt
\end{aligned} \tag{A.44}$$

where $\omega = w/b_2$. Using the fact [11] that the Hurwitz zeta function has the integral representation

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^{ta}}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) dt \tag{A.45}$$

valid for $\mathrm{Re}(s) > -1$, $s \neq 1$, $\mathrm{Re}(a) > 0$, together with the fact[11]

$$\ln\left(\frac{\Gamma(a)}{2\pi}\right) = \frac{\partial}{\partial s}\zeta(s,a)\bigg|_{s=0} \tag{A.46}$$

valid for w positive real, one finds

$$\ln\left(\frac{\Gamma(a)}{2\pi}\right) = \left(a - \frac{1}{2}\right)\ln a - a + \int_0^\infty \frac{1}{t}\left(\frac{e^{-(a+1)t}}{1-e^{-t}} + \left(\frac{1}{2} - \frac{1}{t}\right)e^{-at}\right) dt. \tag{A.47}$$

Comparing with Eq. (A.44) one sees

$$\mathcal{I}(w+b_1,\mathbf{b}) - \mathcal{I}(w,\mathbf{b}) = -\ln\left(\frac{\Gamma(\omega-1)}{\sqrt{2\pi}}\right) + \left(\omega - \frac{3}{2}\right)\ln(\omega-1) - (\omega-1) + J \tag{A.48}$$

where

$$\begin{aligned} J &= \int_0^\infty \frac{1}{t}\left(\left(\frac{1}{2}-\omega\right)e^{-\frac{t}{b_2}} + \frac{1}{t} + \left(\frac{1}{2} - \frac{1}{t}\right)e^{-(\omega-1)t}\right) dt \\ &= \int_0^\infty \frac{1}{2t^2}\left((1-2\omega)te^{-\frac{t}{b_2}} + (t-2)e^{-(\omega-1)t} + 2\right) dt. \end{aligned} \tag{A.49}$$

Integrating by parts gives

$$\begin{aligned} J &= \left[-\frac{1}{2t}\left((1-2\omega)te^{-\frac{t}{b_2}} + (t-2)e^{-(\omega-1)t} + 2\right)\right]_0^\infty \\ &\qquad + \int_0^\infty \frac{1}{2t}\left((1-2\omega)\left(1-\frac{t}{b_2}\right)e^{-\frac{t}{b_2}} + (1-(t-2)(\omega-1))\,e^{-(\omega-1)t}\right) dt \\ &= \int_0^\infty \left(\frac{(1-2\omega)}{2t}\left(e^{-\frac{t}{b_2}} - e^{-(\omega-1)t}\right) + \frac{2\omega-1}{2b_2}e^{-\frac{t}{b_2}} - \frac{\omega-1}{2}e^{-(\omega-1)t}\right) dt \\ &= (\omega-1) + \left(\frac{1}{2}-\omega\right)\lim_{p\to 0}\left(\int_0^\infty t^{p-1}(e^{-\frac{t}{b_2}} - e^{-(\omega-1)t})dt\right) \\ &= (\omega-1) + \left(\frac{1}{2}-\omega\right)\lim_{p\to\infty}\left(\left(b_2^p - \frac{1}{(\omega-1)^p}\right)\Gamma(p)\right) \\ &= (\omega-1) + \left(\frac{1}{2}-\omega\right)\ln(b_2(\omega-1)). \end{aligned} \tag{A.50}$$

Hence

$$\begin{aligned} \mathcal{I}(w+b_1,\mathbf{b}) - \mathcal{I}(w,\mathbf{b}) &= -\ln\left(\frac{\Gamma(\omega-1)}{\sqrt{2\pi}}\right) - \ln(\omega-1) + \left(\frac{1}{2}-\omega\right)\ln b_2 \\ &= \ln(\sqrt{2\pi}) - \ln(\Gamma(\omega)) + \left(\frac{1}{2}-\omega\right)\ln b_2 \\ &= \ln(\sqrt{2\pi}) - \ln\left(\Gamma\left(\frac{w}{b_2}\right)\right) + \left(\frac{1}{2} - \frac{w}{b_2}\right)b_2. \end{aligned} \tag{A.51}$$

The fact that $\mathcal{I}$ is symmetric under the interchange $b_1 \leftrightarrow b_2$ means one also has

$$\mathcal{I}(w+b_2,\mathbf{b})-\mathcal{I}(w,\mathbf{b})=\ln(\sqrt{2\pi})-\ln\left(\Gamma\left(\frac{w}{b_1}\right)\right)+\left(\frac{1}{2}-\frac{w}{b_1}\right)b_1. \tag{A.52}$$

Now define

$$f(w,\mathbf{b})=e^{-\mathcal{I}(w,\mathbf{b})}\Gamma_2(w,\mathbf{b}). \tag{A.53}$$

Comparing with (A.30), (A.31) one sees that

$$f(w+b_1,\mathbf{b})=f(w+b_2,\mathbf{b}) \qquad\qquad =f(w,\mathbf{b}). \tag{A.54}$$

Lemma A.1. *Suppose b_2/b_1 is irrational. Then the set*

$$S(\mathbf{b})=\{\mathbf{n}.\mathbf{b}\colon \mathbf{n}\in\mathbb{Z}^2 \textit{ and } \mathbf{n}.\mathbf{b}>0\} \tag{A.55}$$

is dense in the positive real line.

Proof. For all $\epsilon>0$ there exist positive integers n_1, n_2 such that (ref. [7], Theorem 36)

$$\left|\frac{n_1}{n_2}-\frac{b_2}{b_1}\right|\le\frac{\epsilon}{b_1 n_2} \tag{A.56}$$

$$\implies \qquad |n_1b_1-n_2b_2|\le\epsilon \tag{A.57}$$

Let $s=\operatorname{sgn}(n_1b_1-n_2b_2)$ and define $\mathbf{n}=s(n_1,-n_2)$. Then $\mathbf{n}.\mathbf{b}\in S$ and $0<\mathbf{n}.\mathbf{b}\le\epsilon$. Define $l_j=j\mathbf{n}.\mathbf{b}$ for $j=0,1,\dots$. For every positive real ξ there exists j such that $l_j<\xi\le l_{j+1}$ implying $|\xi-l_j|\le\epsilon$. □

Let $Q=(b_1+b_2)/2$ and suppose b_2/b_1 is irrational. It follows from Eq. (A.54)

$$f(Q+\mathbf{n}.\mathbf{b},\mathbf{b})=f(Q,\mathbf{b}) \tag{A.58}$$

for all $\mathbf{n}.\mathbf{b}\in S(\mathbf{b})$. In view of Lemma A.1 and the fact that $f(w,\mathbf{b})$ is a continuous function of w this means

$$f(w,\mathbf{b})=f(Q,\mathbf{b}) \tag{A.59}$$

for all real $w\ge Q$. Since $\mathcal{I}(Q,\mathbf{b})=0$ it follows that

$$\frac{\Gamma_2(w,\mathbf{b})}{\Gamma_2(Q,\mathbf{b})}=\exp\left[\int_0^\infty\frac{1}{t}\left(\frac{e^{-wt}-e^{-Qt}}{(1-e^{-b_1t})(1-e^{-b_2t})}-\frac{(Q-w)^2e^{-t}}{2b_1b_2}-\frac{Q-w}{b_1b_2t}\right)dt\right] \tag{A.60}$$

This equation was derived on the assumption that $w-1$, b_1-1, b_2-1 are positive real and b_2/b_1 is irrational. We can now use analyticity to infer that it actually holds for all w, b_1, b_2 such that $\operatorname{Re}(w)$, $\operatorname{Re}(b_1)$, $\operatorname{Re}(b_2)$ are all positive.

A.12. **Integral Representation of double sine.** Suppose $\operatorname{Re}(b_1)$, $\operatorname{Re}(b_2)$ are positive and $0<\operatorname{Re}(w)<2\operatorname{Re}(Q)$, where $Q=(b_1+b_2)/2$. Then $(w,b_1,b_2)\in\mathcal{R}_e$. So $S_2(w,\mathbf{b})$ is well-defined and

$$\begin{aligned}S_2(w,\mathbf{b})&=\frac{\Gamma_2(w,\mathbf{b})}{\Gamma_2(2Q-w,\mathbf{b})}\\&=\exp\left[\int_0^\infty\frac{1}{t}\left(\frac{e^{-wt}-e^{-(2Q-w)t}}{(1-e^{-b_1t})(1-e^{-b_2t}}-\frac{2(Q-w)}{b_1b_2t}\right)dt\right]\\&=\exp\left[\int_0^\infty\frac{1}{t}\left(\frac{\sinh(Q-w)t}{2\sinh\left(\frac{b_1t}{2}\right)\sinh\left(\frac{b_2t}{2}\right)}-\frac{2(Q-w)}{b_1b_2t}\right)dt\right].\end{aligned} \tag{A.61}$$

In particular

$$\tilde{S}_2(\omega,\beta) = \exp\left[\int_0^\infty \frac{1}{t}\left(\frac{\sinh(Q-\omega)t}{2\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{\beta t}{2}\right)} - \frac{2(Q-\omega)}{\beta t}\right)dt\right] \tag{A.62}$$

for $\operatorname{Re}(\beta) > 0$, $0 < \operatorname{Re}(\omega) < 2\operatorname{Re}(Q)$, where $Q = (1+\beta)/2$

A.13. **Quasi-periodicity of double sine: further discussion.** Define, for arbitrary $m \in \mathbb{Z}$, $z \in \mathbb{C}$,

$$P(m,z,\tau) = \begin{cases} \prod_{j=0}^{m-1} 2\sin\left(\pi(z+j\tau)\right) & m \geq 1 \\ 1 & m = 0 \\ \left(\prod_{j=m}^{-1} 2\sin\left(\pi(z+j\tau)\right)\right)^{-1} & m \leq -1 \end{cases} \tag{A.63}$$

Then

Lemma A.2. *For all m_1, $m_2 \in \mathbb{Z}$*

$$\tilde{S}_2(z+m_1\tau+m_2,\tau) = (-1)^{m_1m_2} P\left(m_1,z,\tau\right) P\left(m_2,\tau^{-1}z,\tau^{-1}\right)\tilde{S}_2(z,\tau). \tag{A.64}$$

Proof. By repeated application of Eq. (A.36) we have, for arbitrary $m \in \mathbb{N}$,

$$\begin{aligned} \tilde{S}_2(z+m\tau,\tau) &= 2\sin\pi\big(z+(m-1)\tau\big)\tilde{S}_2\big(z+(m-1)\tau,\tau\big) \\ &= P(m,z,\tau)\tilde{S}_2(z,\tau). \end{aligned} \tag{A.65}$$

Making the replacement $z \to z - m\tau$ gives

$$\tilde{S}_2(z,\tau) = P(m,z-m\tau,\tau)\tilde{S}_2(z-m\tau,\tau). \tag{A.66}$$

Hence

$$\begin{aligned} \tilde{S}_2(z-m\tau,\tau) &= \left(\prod_{j=0}^{m-1} 2\sin\left(\pi(z+(j-m)\tau\right)\right)^{-1}\tilde{S}_2(z,\tau) \\ &= P(-m,z,\tau)\tilde{S}_2(z,\tau). \end{aligned} \tag{A.67}$$

This proves the result for $m_2 = 0$ and m_1 arbitrary.

It follows that, for arbitrary $m \in \mathbb{Z}$,

$$\begin{aligned} \tilde{S}_2(z+m,\tau) &= \tilde{S}_2\left(\frac{z}{\tau}+\frac{m}{\tau},\frac{1}{\tau}\right) \\ &= P\left(m,\frac{z}{\tau},\frac{1}{\tau}\right)\tilde{S}_2\left(\frac{z}{\tau},\frac{1}{\tau}\right) \\ &= P(m,\tau^{-1}z,\tau^{-1})\tilde{S}_2(z,\tau) \end{aligned} \tag{A.68}$$

which establishes the result for $m_1 = 0$ and m_2 arbitrary.

Finally, consider the general case. It follows from the foregoing that

$$\begin{aligned} \tilde{S}_2(z+m_1\tau+m_2,\tau) &= P(m_1,z+m_2,\tau)\tilde{S}_2(z+m_2,\tau) \\ &= P(m_1,z+m_2,\tau)P(m_2,\tau^{-1}z,\tau^{-1})\tilde{S}_2(z,\tau). \end{aligned} \tag{A.69}$$

Observe

$$P(m_1, z+m_2, \tau) = \begin{cases} \prod_{j=0}^{m_1-1} 2\sin\left(\pi(z+m_2+j\tau)\right) & m_1 \geq 1 \\ 1 & m_1 = 0 \\ \left(\prod_{j=m_1}^{-1} 2\sin\left(\pi(z+m_2+j\tau)\right)\right)^{-1} & m_1 \leq -1 \end{cases}$$
$$= (-1)^{m_1 m_2} P(m_1, z, \tau). \tag{A.70}$$

The claim now follows. □

The function $P(m, z, \tau)$ can be expressed in terms of a finite q-Pochhammer symbol. Indeed, for $m \in \mathbb{N}$ define

$$\delta_m(z,\tau) = \prod_{j=0}^{m-1} \left(1 - e^{2\pi i(z+j\tau)}\right). \tag{A.71}$$

Then

$$P(m,z,\tau) = \begin{cases} \prod_{j=0}^{m-1} 2\sin\left(\pi(z+j\tau)\right) & m \geq 1 \\ 1 & m = 0 \\ \left(\prod_{j=m}^{-1} 2\sin\left(\pi(z+j\tau)\right)\right)^{-1} & m \leq -1 \end{cases}$$
$$= \begin{cases} i^m \prod_{j=0}^{m-1} e^{-\pi i(z+j\tau)}\left(1 - e^{2\pi i(z+j\tau)}\right) & m \geq 1 \\ 1 & m = 0 \\ i^m \prod_{j=m}^{-1} e^{\pi i(z+j\tau)}\left(1 - e^{2\pi i(z+j\tau)}\right)^{-1} & m \leq -1 \end{cases}$$
$$= \begin{cases} e^{-m\pi i\left(z-\frac{1}{2}+\frac{(m-1)\tau}{2}\right)}\delta_m(z,\tau) & m \geq 1 \\ 1 & m = 0 \\ e^{-m\pi i\left(z-\frac{1}{2}+\frac{(m-1)\tau}{2}\right)}\left(\delta_{-m}(z,\tau)\right)^{-1} & m \leq -1 \end{cases} \tag{A.72}$$

If we define $\delta_m(z,\tau)$ for non-positive values of m by

$$\delta_m(z,\tau) = \begin{cases} 1 & m = 0 \\ \left(\delta_{|m|}(z-\tau, -\tau)\right)^{-1} & m \leq -1 \end{cases} \tag{A.73}$$

then we can write this in the form

$$P(m,z,\tau) = e^{-m\pi i\left(z-\frac{1}{2}+\frac{(m-1)\tau}{2}\right)}\delta_m(z,\tau). \tag{A.74}$$

Expressed in this way Eq. (A.64) becomes

$$\tilde{S}_2\left(z+m_1\tau+m_2,\tau\right) = (-1)^{m_1 m_2} e^{-m_1\pi i\left(z-\frac{1}{2}+\frac{(m_1-1)\tau}{2}\right)} e^{-m_2\pi i\left(\frac{z}{\tau}-\frac{1}{2}+\frac{(m_2-1)}{2\tau}\right)}$$
$$\times\, \delta_{m_1}(z,\tau)\delta_{m_2}(\tau^{-1}z, \tau^{-1})\tilde{S}_2(z,\tau). \tag{A.75}$$

This means that we can express the double sine function on the real axis in terms of the limit of a product of finite q-Pochhammer symbols. Indeed, choose some fixed $z_0 \in \mathbb{R}$. Then, for any any other real number z we can choose a sequence $\mathbf{m}_j \in \mathbb{N}^2$ such that $m_{j,1}\tau + m_{j,2} \to z - z_0$. Then

$$\tilde{S}_2(z,\tau) = \lim_{j\to\infty}\left((-1)^{m_{j,1}m_{j,2}} E_{j,1}E_{j,2}\delta_{m_{j,1}}(z,\tau)\delta_{m_{j,2}}(z,\tau)\right)\tilde{S}_2(z_0,\tau) \tag{A.76}$$

where

$$E_{j,1} = e^{-m_{j,1}\pi i\left(z-\frac{1}{2}+\frac{(m_{j,1}-1)\tau}{2}\right)} \tag{A.77}$$

APPENDIX B. EQUIVALENT EXISTENCE CONDITIONS

STF: I want to write some notes here about all the equivalent conditions that would let us prove something.

Theorem B.1. *Let $M = \sum_{\mathbf{p}} \nu_{\mathbf{p}} D^{\dagger}_{\mathbf{p}}$ where $\nu_{\mathbf{p}}$ are the ghost overlaps. (Add some conditions...) Then the following are equivalent:*

- $\mathrm{Tr}(M^3) = 1$
- $M^2 = M$
- $\nu_{\mathbf{r}} = \frac{1}{d}\sum_{\mathbf{p}} \tau^{\langle \mathbf{p},\mathbf{r}\rangle} \nu_{\mathbf{p}}\nu_{\mathbf{r}-\mathbf{p}}$
- *for all j,k, $M_{jj}M_{kk} = M_{jk}M_{kj}$*

B.1. **The zero convolution endgame (GSK).** This section describes a variant of the twisted convolution identity, that is, of the third equivalent condition in Theorem B.1. It may be easier to prove because it involves showing that a natural quantity is equal to zero.

Theorem B.2. *Let A and β be as in the main conjecture. Let $\mathbf{p} = (p_1, p_2) \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$ with $p_1, p_2 \geq 0$. Define C [to be a 2 by 2 integer matrix defined in terms of A in a particular way]. Then the statement that*

$$\sum_{r_1=1}^{d}\sum_{r_2=0}^{d-1} \mathrm{e}\left(\frac{\mathbf{p}C\mathbf{r}^{\top}}{d}\right) \sigma_{A,d^{-1}\mathbf{r}}(\beta)^{-1}\sigma_{A,d^{-1}(\mathbf{p}+\mathbf{r})}(\beta) = 0 \tag{B.1}$$

for all $\mathbf{p}$ is equivalent to the main conjecture.

This is proved using multiplicative relations of the Shintani-Faddeev cocycle and one “easy” special value.

For an explicit example, for the SIC in $d = 5$, $(c_1, c_2) := \mathbf{p}C = (-p_1 + p_2, -p_2)$.

When β in the above sum is replaced with $\tau \in \mathcal{H}$, the absolute value of the sum seems to grow linearly in $|\tau - \beta|$, at least locally near β. Plots show a fairly uniform growth. Perhaps bounding the absolute value of the sum would be a viable proof strategy.

B.2. **Equations in a power basis.** Let $K(a)$ be a degree n separable field extension over a base field K. The power basis in terms of the generator a is given by $\mathcal{B} = \{1, a, \ldots, a^{n-1}\}$. Let $f(x)$ be the minimal polynomial of a over K. Then f factorizes as follows:

$$f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n = (x - a)(\tilde{\alpha}_0 + \tilde{\alpha}_1 x + \ldots + \tilde{\alpha}_{n-1} x^{n-1}) \,. \tag{B.2}$$

The "reduced" coefficients $\tilde{\alpha}_k$ are simply related to the coefficients α_k by

$$\alpha_k = \tilde{\alpha}_{k-1} - a\tilde{\alpha}_k \,, \tag{B.3}$$

with the convention that $\tilde{\alpha}_k = 0$ for any $k < 0$. This can be inverted to get $\tilde{\alpha}_k$ in terms of the α_k as

$$\tilde{\alpha}_k = -\sum_{j=0}^{k} a^{j-k-1} \alpha_j \,. \tag{B.4}$$

In order to compute an expansion of an element of $K(a)$ in the basis $\mathcal{B}$, we need the dual basis of the power basis with respect to the field trace. The dual basis $\mathcal{B}^*$ can be obtained explicitly in terms of the $\tilde{\alpha}_k$ as follows.

$$\mathcal{B}^* = \frac{1}{f'(a)} \{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}\} \,, \tag{B.5}$$

where

$$f'(a) = \tilde{\alpha}_0 + \tilde{\alpha}_1 a + \ldots + \tilde{\alpha}_{n-1} a^{n-1} \,. \tag{B.6}$$

If we write all the roots of $f(x)$ as $\{a = a_1, a_2, \ldots, a_n\}$, then from the previous equations, alternate expressions for $f'(a)$ are given by

$$f'(a) = \prod_{i>1} (a - a_i) = -\sum_{k=0}^{n-1} \sum_{j=0}^{k} a^{j-1} \alpha_j \,. \tag{B.7}$$

It follows that an explicit expression for the elements of the dual basis solely in terms of a and α_k is given by

$$\frac{\tilde{\alpha}_k}{f'(a)} = \frac{\sum_{j=0}^{k} a^{j-k} \alpha_j}{\sum_{\ell=0}^{n-1} \sum_{m=0}^{\ell} a^m \alpha_m} \,. \tag{B.8}$$

The dual basis tells us how to expand any element in the power basis, but we also need to know how to expand products of elements. For this we need to compute the structure constants in terms of the coefficients of the minimal polynomial. We must first define some relevant concepts from the theory of symmetric polynomials. The complete homogeneous symmetric polynomials on $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ are

$$h_k(\boldsymbol{x}) = \sum_{1 \le j_1 \le \ldots \le j_k \le n} x_{j_1} x_{j_1} \cdots x_{j_k} \,. \tag{B.9}$$

The closely related elementary symmetric polynomials are defined by

$$e_k(\boldsymbol{x}) = \sum_{1 \le j_1 < \ldots < j_k \le n} x_{j_1} x_{j_1} \cdots x_{j_k} \,. \tag{B.10}$$

Thus, the elementary form does not allow collisions of the same variable in a given monomial term. We let $e_0 = h_0 = 1$. (We will not need the power sums, which is another natural symmetric polynomial.) The structure constants are then computable from the following formula, valid for all $m \geq 0$.

$$a^{n+m} = \sum_{k=0}^{n-1} \left(- \sum_{j=0}^{m} \alpha_{k-m+j} h_j(\boldsymbol{a}) \right) a^k \,. \tag{B.11}$$

Here $\boldsymbol{a} = \{a = a_1, a_2, \ldots, a_n\}$ are the roots of the minimal polynomial f.

Note that

$$f(x) = \prod_{j=1}^{n} (x - a_j) = \sum_{k=0}^{n} (-1)^{n-k} e_{n-k}(\boldsymbol{a}) x^k \,, \tag{B.12}$$

so that $\alpha_k = (-1)^{n-k} e_{n-k}(\boldsymbol{a})$.

The complete homogeneous symmetric polynomials can be expressed in terms of the elementary symmetric polynomials via the following relations

$$\sum_{i=0}^{m} (-1)^i e_i(\boldsymbol{a}) h_{m-i}(\boldsymbol{a}) = 0 \,, \tag{B.13}$$

which in particular implies the recursive relation for $m > 0$

$$h_m(\boldsymbol{a}) = - \sum_{i=1}^{m} (-1)^i e_i(\boldsymbol{a}) h_{m-i}(\boldsymbol{a}) = - \sum_{j=0}^{m-1} \alpha_{n-m+j} h_j(\boldsymbol{a}) \,. \tag{B.14}$$

Sticking points: Where does the fact that a is a Stark unit come in? How does the Galios group act on roots of unity, or at least their real part? Does it help to assume that the minimal polynomial is palindromic?

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