A CONSTRUCTIVE APPROACH TO ZAUNER'S CONJECTURE VIA THE STARK CONJECTURES

MARCUS APPLEBY, STEVEN T. FLAMMIA, AND GENE S. KOPP

ABSTRACT. We propose a construction of d^2 complex equiangular lines in $\mathbb{C}^d$, also known as SICs or SIC-POVMs, which were conjectured by Zauner to exist for all d. The construction gives a putatively complete list of SICs with Weyl-Heisenberg symmetry in all dimensions $d > 3$. Specifically, we give an explicit expression for an object that we call a ghost SIC which is constructed from the real multiplication values of a special function and which is Galois conjugate to a SIC. The special function, which we call the Shintani-Faddeev modular cocycle, is a family of meromorphic functions parameterized by congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and may be of independent interest. We prove that our construction gives a valid SIC assuming two conjectures: first, we conjecture that the ghost SIC is idempotent, and second, we require Tate's refinement of the rank-1 abelian Stark conjecture for real quadratic fields. The latter condition allows us to prove that the ghost and the SIC are Galois conjugate over an extension of $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = (d+1)(d-3)$. We provide computational tests of our SIC construction by cross validating it with known exact solutions, with the numerical work of Scott and Grassl, and by constructing four numerical examples of SICs in $d = 100$, three of which are new. We further consider rank-r generalizations called r-SICs given by equiangular configurations of r-dimensional complex subspaces. We give similar conditional constructions for r-SICs for all r, d such that $r(d-r)$ divides (d^2-1). Finally, we study the structure of the field extensions conjecturally generated by the r-SICs. If K is any real quadratic field, then either every ray class field over K, or else every ray class field for which 2 is unramified, is generated by our construction.

CONTENTS

Date: March 8, 2023.
2010 *Mathematics Subject Classification.* 11R37, 11R42, 81P15, 81R05.

1. Introduction

1.1. History and Motivation.

1.2. Main Result of this paper.

In the following H_d is a d-dimensional complex Hilbert space, and $\mathcal{L}(H_d)$ the space of linear operators on H_d. We say $\Pi \in \mathcal{L}(H_d)$ is a H-projectorif $\Pi^2 = \Pi$. If Π is also Hermitian we say it is an H-projector.

A SIC is constructed from rank-1 H-projectors. We need to consider a generalization, in which the H-projectors are rank-r. We refer to this object as an r-SIC.

Definition 1.1. *For $1 \leq r \leq d-1$, an r-SIC is a family of d^2 rank-r H-projectors $\Pi_1, \dots, \Pi_{d^2}$ in $\mathcal{L}(H_d)$ such that*

$$\mathrm{Tr}(\Pi_j \Pi_k) = \alpha \tag{1.1}$$

for all $j \neq k$ and some fixed constant $\alpha \neq r$.

Definition 1.2. *A SIC is the same thing as a 1-SIC.*

A few words about terminology. Some authors scale the H-projectors by a factor $1/d$. What we are calling r-SICs are also called maximal equichordal tight fusion frames [1–3] or maximal symmetric tight fusion frames [4]. They are instances of structures which have been variously described as SI-POVMs [5], general SIC-POVMs [6] and SIMs [7]. They are also instances of conical designs [7]. SICs (or 1-SICs) can alternatively be described as maximal equiangular tight frames [8], or minimal complex projective 2-designs [9, 10]. The 1-dimensional subspaces associated to the H-projectors form a maximal set of complex equiangular lines [11–13].

Hitherto SICs have received much more attention than r-SICs for general r. However, it turns out that the results proved in this paper for SICs naturally extend to a much larger class of r-SICs. Moreover, the r-SICs have major significance from a number-theoretic point of view. We therefore raise r-SICs to a starring role.

SICs have been constructed exactly in every dimension ≤ 53, and in many further dimensions up to a maximum of $5\,799$. High precision numerical solutions have been calculated in every dimension ≤ 193, and in many further dimensions up to a maximum of $39\,604$. These results are the work of many people obtained over a period of 25 years, starting with the original work of Hoggar [13] and Zauner [9]. For the current state of knowledge, and a review of the history see refs. [14–17]. For high dimensions the calculations are computationally intensive. The calculations reported in ref. [17] used two supercomputers, both on the TOP500 list [18], and each having $> 10^5$ cores.

The large number of examples that have been constructed encourages belief in Zauner's conjecture [9], that SICs exist in every finite dimension. However, there is still no proof of this. There is not even a proof that SICs exist in an infinite sequence of dimensions.

Although r-SICs with $r > 1$ have received much less attention, there actually is a proof in their case: namely, it can be shown [5] that in every odd dimension d there exists an r-SIC with $r = (d-1)/2$. It can also be shown [4] that, to every SIC in odd-dimension d of the kind described in Definition 1.5 below, there is a corresponding r-SIC with $r = (d-1)/2$ (different from the one constructed in ref. [5]). However, as we will see in the following, these two examples are only the tip of the iceberg.

The r-SICs we consider all carry a transitive action of the Weyl-Heisenberg group, which we now define. Let $|0\rangle, \dots, |d-1\rangle$ be an orthonormal basis for H_d, and let X, Z be the unitary operators which act on the basis according to

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \omega|j\rangle, \qquad \omega = e^{\frac{2\pi i}{d}}, \tag{1.2}$$

where addition of indices in the first equation is modulo d. Define

$$\xi = -e^{\frac{\pi i}{d}}, \qquad \bar{d} = \begin{cases} d & d \text{ odd}, \\ 2d & d \text{ even}. \end{cases} \tag{1.3}$$

STF: Possible alternative: Define a new symbol $\gamma = \frac{1}{2}\left(3 + (-1)^d\right)$ so that $\bar{d} = \gamma d$. Doesn't have to be γ, of course, but that's the idea.

Definition 1.3. *The Weyl-Heisenberg (WH) group in dimension d is the set of $d^2\bar{d}$ operators $\{\xi^{p_0} X^{p_1} Z^{p_2} \colon 0 \leq p_0 < \bar{d}, 0 \leq p_1, p_2 < d\}$.*

Definition 1.4. *The WH displacement operators are the WH group elements*

$$D_{\mathbf{p}} = \xi^{p_1 p_2} X^{p_1} Z^{p_2}, \qquad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{Z}^2. \tag{1.4}$$

Definition 1.5. *An r-SIC is said to be WH covariant if it is of the form $\{\Pi_{\mathbf{p}} \colon 0 \leq p_1, p_2 < d\}$, where*

$$\Pi_{\mathbf{p}} = D_{\mathbf{p}} \Pi D_{\mathbf{p}}^{\dagger} \tag{1.5}$$

for some fixed H-projector Π, called the fiducial projector.

From now on, unless the contrary is explicitly indicated, the term "r-SIC" will always mean "WH covariant r-SIC". It will also be assumed without comment that the dimension is greater than 3. The SICs thus excluded are all in some ways exceptional. They are accordingly called the sporadic SICs. For a detailed description see ref. [19]. We also exclude from consideration the r-SICs of rank $(d-1)/2$ described in refs. [4,5].

If Π is a fiducial projector for an r-SIC then the number

$$e^{i\theta_{\mathbf{p}}} = \sqrt{\frac{d^2-1}{r(d-r)}}\, \mathrm{Tr}\left(\Pi D_{\mathbf{p}}^{\dagger}\right). \tag{1.6}$$

has absolute value 1 for all $\mathbf{p} \neq \mathbf{0}$ (mod d).

Definition 1.6. *The numbers $\eta_{\mathbf{p}} = \mathrm{Tr}\left(\Pi D_{\mathbf{p}}^{\dagger}\right)$ are called the overlaps. The numbers $e^{i\theta_{\mathbf{p}}}$ are called the overlap phases.*

A fiducial can be recovered from its overlap phases using

$$\Pi = \frac{r}{d} I + \sqrt{\frac{r(d-r)}{d^2(d^2-1)}} \sum_{\mathbf{p} \notin d\mathbb{Z}^2} e^{i\theta_{\mathbf{p}}} D_{\mathbf{p}} \tag{1.7}$$

where the sum is over any set of coset representatives of $\mathbb{Z}^2/d\mathbb{Z}^2$ with the representative of $d\mathbb{Z}^2$ excluded.

The overlap phases are number-theoretically remarkable, and they are the focus of this paper. Let us briefly summarize what was previously known about them.

The story begins with the observation [20] that the components of every known fiducial are algebraic numbers. In ref. [21] the Galois group was investigated in detail and it was found (among other things) that the field generated by the fiducial components together with ξ ("the SIC field") is an Abelian extension of the real quadratic field $\mathbb{Q}\left(\sqrt{(d-3)(d+1)}\right)$.

Definition 1.7. *A set of fiducials which, together with ξ, all generate the same SIC field are called a Galois multiplet. The multiplet generating the SIC field of lowest degree in a given dimension is called the minimal multiplet.*

Refs. [22, 23] studied the known minimal multiplets. They showed (among other things) that

(1) the SIC field is the ray class field over $\mathbb{Q}\left(\sqrt{(d-3)(d+1)}\right)$ with modulus $\bar{d}$ and ramification at both infinite places,
(2) the overlap phases $e^{i\theta_{\mathbf{p}}}$ are algebraic integers, and therefore units.

The fact that the $e^{i\theta_{\mathbf{p}}}$ are units in a ray class field prompts the question: are they simply related to the Stark units? Ref. [24] answered that question in the affirmative for the minimal multiplets in the four lowest lying prime dimensions equal to $5 \bmod 6$. Ref. [17] investigated minimal multiplets in dimensions of the form n^2+3, and showed that in this special case, for the examples examined, Stark units are simply related the components of the fiducial projector itself. Refs. [17, 24] also showed that the Stark conjectures are of immediate computational relevance in that they can be used to calculate new SICs (in the case of ref. [17] pushing up the highest dimension in which SICs have been calculated by one order of magnitude).

In this paper, in this we extend these results to every dimension ≥ 4, and to every multiplet within a dimension. This depends on ref. [25], where the concept of ray class field is generalized to arbitrary orders, and ref. [26], where evidence is adduced that the fiducials in non-minimal multiplets, together with ξ, generate ray class fields in this generalized sense. It also depends on the generalization of the Stark conjectures corresponding to this more general concept of ray class field (see below).

In the second place, instead of calculating the Stark units using an L function, as was done in refs. [17, 24], we do so using the Shintani-Faddeev Jacobi cocycle (SFJC), and the closely related Shintani-Faddeev modular cocycle (SFMC) [27]. This has some very significant advantages over the L-function approach, as we will see. The SFJC is a generalization of Shintani's double-sine function, introduced in the context of algebraic number theory by Shintani [28, 29] and rediscovered by Faddeev [30] in the context of high energy physics, where it is known as the non-compact quantum dilogarithm, and where it features in in Liouville field theory, quantum Teichmüller theory, three-dimensional supersymmetric gauge theory, complex Chern-Simons theory, quantum group theory, and quantum knot theory (see refs. [31–35], and references cited therein). The connections between SICs and number theory are the subject of refs. [17, 22–24] and the present paper. It would be interesting to see if there are also connections with high energy physics.

In the third place, we show that SIC existence in every finite dimension follows from a special function identity (the Twisted Convolution Identity involving the SFJM together with the Stark conjectures. In the following we refer to the twisted convolution identity and Stark conjectures**DMA:** Not sure how to word this. "Stark's real quadratic Archimedean rank 1 conjecture with Tate's refinement" is a bit of a mouthfull. Is there a more compact way of saying it for the purposes of the Introduction? For instance, could we just say a case, or instantiation of the Stark conjectures? jointly as the TCSC.

In the fourth place, study of the known SICs has suggested a large number of conjectures concerning their properties. Collectively, these conjectures are sometimes referred to as SIC phenomenology. We show that all of them are consequences of the TCSC. We further show that the TCSC imply some additional, previously un-noticed properties.

In the fifth place, we show that the TCSC imply the existence of a hitherto unknown three-parameter family of r-SICs with $r > 1$.

In the sixth place, we address the question, how many of the ray class fields over a given real quadratic field are generated by r-SICs. Let K be a real quadratic field, let v be the smallest totally positive unit of K, and assume the TCSC. Then if $\mathrm{Tr}(v)$ is odd (respectively even) the fields generated by r-SICs over K is cofinal for the set of all ray class fields over K (respectively, the set of all ray class fields over K having odd modulus). So although r-SICs do not provide a full solution to Hilbert's 12^{th} problem for real quadratic fields, there is a well-defined sense in which they come close to doing so if the TCSC are true.

Our procedure for calculating r-SICs is a little reminiscent of the way in which, to calculate a Green's function on Minkowski space, one first calculates a correlation function on Euclidean space and then applies a Wick rotation. The analogue of the Minkowsi-space Green's function is an r-SIC fiducial, while the analogue of the Euclidean-space correlation function is a ghost r-SIC fiducial [36]:

Definition 1.8. *A ghost r-SIC fiducial is a rank-r projector Π given by*

$$\Pi = \frac{r}{d}I + \sqrt{\frac{r(d-r)}{d^2(d^2-1)}} \sum_{\mathbf{p}\notin d\mathbb{Z}^2} \nu_{\mathbf{p}} D_{\mathbf{p}} \tag{1.8}$$

where the sum is over any set of coset representatives of $\mathbb{Z}^2/d\mathbb{Z}^2$ with the representative of $d\mathbb{Z}^2$ excluded, and where the $\nu_{\mathbf{p}}$ are real numbers satisfying

$$\nu_{\mathbf{p}}\nu_{-\mathbf{p}} = \begin{cases} 1 & d \text{ odd}, \\ (-1)^{p_1+p_2} & d \text{ even}. \end{cases} \tag{1.9}$$

for all $\mathbf{p}$.

The $\nu_{\mathbf{p}}$ are calculated using the SFMC. The role of the Wick rotation is played by a Galois automorphism transforming the numbers $\nu_{\mathbf{p}}$ on the RHS of Eq. (1.8) into the numbers $e^{i\theta_{\mathbf{p}}}$ on the RHS of Eq. (1.7). The TC conjecture is needed to ensure that Π is a projector. The Stark conjectures are needed to ensure that the $\nu_{\mathbf{p}}$ are algebraic numbers, and that there exists a Galois automorphism having the required property.

We now describe the SFJC and SFMC.**DMA:** I have thought quite a lot about this. It seems to me there is no alternative to defining the Jacobi form in the introduction. I'll explain my reasoning when we next meet. Let $\mathbb{H}$ be the upper half plane.

Definition 1.9. *The finite exponentiated Q-Pochammer symbol is defined by*

$$\varpi_n(z,\tau) = \begin{cases} \prod_{j=0}^{n-1}\left(1-e^{2\pi i(z+j\tau)}\right) & n>0 \\ 1 & n=0 \\ \prod_{j=n}^{-1}\left(1-e^{2\pi i(z+j\tau)}\right)^{-1} & n<0 \end{cases} \tag{1.10}$$

for $n\in\mathbb{N}$, $z,\tau\in\mathbb{C}$. The infinite symbol is

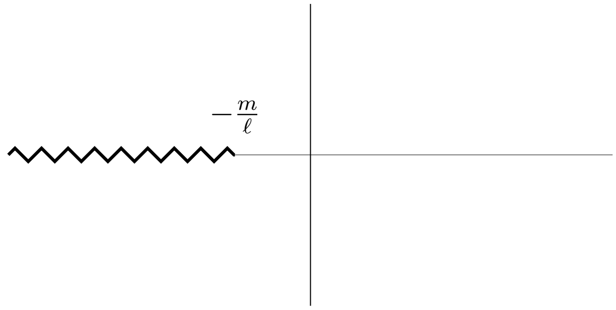
$$\varpi(z,\tau) = \prod_{j=0}^{\infty}\left(1-e^{2\pi i(z+j\tau)}\right) \tag{1.11}$$

for $z\in\mathbb{C}$, $\tau\in\mathbb{H}$

Definition 1.10. *For $\tau\in\mathbb{C}$, $L=\left(\begin{smallmatrix} j & k \\ \ell & m\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ define*

$$L\cdot\tau = \frac{j\tau+k}{\ell\tau+m}, \qquad\qquad j_L(\tau) = \ell\tau+m. \tag{1.12}$$

Definition 1.11. *For $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ define $\mathcal{D}_L$ to be the complex plane cut along $\{\tau \in \mathbb{R} \colon j_L(\tau) \le 0\}$, illustrated below for the case $\ell > 0$.*



Definition 1.12. *For $L \in \mathrm{SL}(2,\mathbb{Z})$, $z \in \mathbb{C}$, $\tau \in \mathbb{H}$ the Shintani-Faddeev Jacobi cocycle (SFJC) is defined by*

$$\sigma_L(z,\tau) = \frac{\varpi\left(\frac{z}{j_L(\tau)}, L\cdot\tau\right)}{\varpi(z,\tau)}. \tag{1.13}$$

It can be shown [27] that σ_L continues to a meromorphic function on $\mathbb{C} \times \mathcal{D}_L$.

Up to a scale factor $\sigma_S(z,\tau)$, with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, is the double sine function, or quantum dilogarithm.

Definition 1.13. *For $\mathbf{p} = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \in \mathbb{Z}^2$ define*

$$\langle\!\langle \mathbf{p}, \tau \rangle\!\rangle = p_2\tau - p_1. \tag{1.14}$$

Definition 1.14. *For $d \in \mathbb{N}$ define $\Gamma(d)$ to be the subgroup consisting of matrices $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $A = I$ (mod d).*

Definition 1.15. ***DMA:** For the present I am giving my proposed definition, which differs from Gene's. For $d \in \mathbb{N}$, $A \in \Gamma(d)$, $\mathbf{p} \in \mathbb{Z}^2$, $\tau \in \mathcal{D}_A$ the Shintani-Faddeev modular cocycle (SFMC) is defined by*

$$\mathfrak{w}_A^{d,\mathbf{p}}(\tau) = \frac{\sigma_A\left(\frac{\langle\!\langle \mathbf{p},\tau\rangle\!\rangle}{d}, \tau\right)}{\varpi_{-(\bar{A}\mathbf{p})_2}\left(\frac{\langle\!\langle \mathbf{p},\tau\rangle\!\rangle}{d j_A(\tau)}, A\cdot\tau\right)} \tag{1.15}$$

where $\bar{A} = d^{-1}(A - I)$.

***DMA:** If $\mathbf{p} = \mathbf{0}$ (mod d) then it can happen that the RHS is infinite for all τ. We should exclude such $\mathbf{p}$ from the definition, but I am deferring that for now.*

If $A\tau = \tau$, $\mathbf{p} \neq \mathbf{0}$ and if the Stark conjectures **DMA:** Specify version of Stark conjectures? are true then $\mathfrak{w}_{\mathbf{p}}^{A,d}(\tau)$ is an algebraic integer and, in fact, a unit (though not a Stark unit). We are now ready to state the conjecture on which, along with the Stark conjectures, everything in this paper depends.

Conjecture 1.16 (Twisted Convolution Identity (TCI))**.** *Let n, r, $d \in \mathbb{N}$ be such that $nr(d-r) = d^2 - 1$ and let $L \in \mathrm{SL}(2,\mathbb{Z})$ be such that $\mathrm{Tr}(L) = n - 2$. Let*

(1) A be a generator of $\langle L \rangle \cap \Gamma(d)$ (where $\langle L \rangle$ is the cyclic subgroup generated by L),
(2) $\beta \in \mathcal{D}_A$ be such that $A\beta = \beta$,
(3) $s = \mathrm{sgn}(d - 2r)$ (with the convention $\mathrm{sgn}(0) = 1$),
(4) $\mathcal{I} = \{\mathbf{p} \in \mathbb{Z}^2 \colon 0 \le p_1, p_2 < d\}$.

Then there exists an integer t such that

$$\sum_{\mathbf{p}\in\mathcal{I}} \omega^{sr\langle \mathbf{p},(tI+L)\mathbf{q}\rangle} \mathfrak{w}_A^{d,\mathbf{p}}(\beta)\, \mathfrak{w}_{A^{-1}}^{d,\mathbf{p}-\mathbf{q}}(\beta) = d^2\delta_{\mathbf{q},\mathbf{0}} \tag{1.16}$$

for all $\mathbf{q}\in\mathcal{I}$.

We now show how the SFMC can be used to construct a ghost SIC.

Definition 1.17. *Let K be a real quadratic field, let Δ_0 be its discriminant, and let v be the smallest totally positive unit greater than 1. For all $a, m \in \mathbb{N}$ define*

$$f_a = \frac{v^a - v^{-a}}{\sqrt{\Delta_0}} \tag{1.17}$$

$$r_{a,m} = \frac{f_{am}}{f_a} \tag{1.18}$$

$$d_{a,m} = r_{a,m+1} + r_{a,m} \tag{1.19}$$

The numbers $d_{a,m}$ are the dimensions in which, according to our construction, assuming the TCSC, there exist r-SICs generating an Abelian extension of K, and the $r_{a,m}$ are their ranks. The situation is illustrated in Table 1.

(A) $d_{a,m}$

⋮	⋮	⋮	⋮	
48	2 255	105 937	4 976 784	…
19	341	6 119	109 801	…
8	55	377	2 584	…
4	11	29	76	…

(B) $r_{a,m}$

⋮	⋮	⋮	⋮	
1	47	2 208	103 729	…
1	18	323	5 796	…
1	7	48	329	…
1	3	8	21	…

TABLE 1. Dimensions and ranks associated to $\mathbb{Q}(\sqrt{5})$. In these grids a increases from top to bottom and m increases from left to right. The left-hand column of the first grid gives the sequence of dimensions in which one finds SICs.

Definition 1.18 (Meyer invariant)**.** *The Meyer invariant [37] is a class-function* $\mathrm{SL}(2,\mathbb{Z})$. *Define*

$$(\!(x)\!) = \begin{cases} 0 & x\in\mathbb{Z} \\ x - \lfloor x\rfloor - \frac{1}{2} & x\notin\mathbb{Z} \end{cases} \tag{1.20}$$

and, for arbitrary $a, b\in\mathbb{Z}$ such that $b\neq 0$, the Dedekind sum

$$s(a,b) = \sum_{n=1}^{|b|-1} \left(\!\!\left(\frac{n}{b}\right)\!\!\right)\left(\!\!\left(\frac{na}{b}\right)\!\!\right). \tag{1.21}$$

Then for all $L = \left(\begin{smallmatrix} j & k \\ \ell & m\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ the Meyer invariant is given by

$$\phi(L) = \begin{cases} -\frac{\mathrm{Tr}(L)}{3\ell} + \mathrm{sgn}\left(\ell\,\mathrm{Tr}(L)(|\,\mathrm{Tr}(L)| - 2)\right) + 4\,\mathrm{sgn}(\ell)s(j,\ell) & \ell\neq 0, \\ \mathrm{sgn}(\mathrm{Tr}(L))\left(\mathrm{sgn}(k) - \frac{k}{3}\right) & \ell = 0. \end{cases} \tag{1.22}$$

Theorem 1.19. *Assume the TCC is true. Let K be a real quadratic form, let Q be a primitive quadratic form*

2. Weyl-Heisenberg and Extended Clifford Groups

3. Material first attempt at intro: to be inserted in a subsequent section

3.1. **Preliminaries.**

3.1.1. *SICs.* Let H_d be a d-dimensional complex Hilbert space, and $\mathcal{L}(H_d)$ the space of linear operators on H_d. We say that $P \in \mathcal{L}(H_d)$ is a *projection operator* or *projector* if it is Hermitian ($P^\dagger = P$), and idempotent ($P^2 = P$).

A SIC is constructed from rank-1 projectors. We need to consider a generalization, in which the projectors are rank-r. We refer to this object as an r-SIC.

Definition 3.1. *For $1 \leq r \leq d-1$, an r-SIC is a family of d^2 rank-r projectors $\Pi_1, \dots, \Pi_{d^2}$ in $\mathcal{L}(H_d)$ such that*

$$\mathrm{Tr}(\Pi_j \Pi_k) = \alpha \tag{3.1}$$

for all $j \neq k$ and some fixed constant $\alpha \neq r$.

Remark 3.2. *The purpose of the restrictions $r \neq 0$, d and $\alpha \neq r$ is to exclude the degenerate case, $\Pi_1 = \cdots = \Pi_{d^2}$.*

Definition 3.3. *A SIC is the same thing as a 1-SIC.*

A few words about terminology. We are here adopting the same convention as, for example, refs. [8, 9], according to which a SIC is a family of projectors. Other authors, for example ref. [12], scale the projectors by a factor $1/d$. What we are calling r-SICs are also called maximal equichordal tight fusion frames [1–3] or maximal symmetric tight fusion frames [4]. They are instances of structures which have been variously described as SI-POVMs [5], general SIC-POVMs [6] and SIMs [7]. They are also instances of conical designs [7]. SICs (or 1-SICs) can alternatively be described as maximal equiangular tight frames [8], or minimal complex projective 2-designs [9, 10]. The 1-dimensional subspaces associated to the projectors form a maximal set of complex equiangular lines [11–13].

It is easily seen that if the projectors Π_j form an r-SIC, then the projectors $I - \Pi_j$ form a $(d-r)$-SIC. Moreover, the $I - \Pi_j$ have the same number theoretic properties as the Π_j. In the sequel, unless the contrary is explicitly stated, we will accordingly assume that $r \leq d/2$.

SICs have been constructed exactly in every dimension ≤ 53, and in many further dimensions up to a maximum of 5 799. High precision numerical solutions have been calculated in every dimension ≤ 193, and in many further dimensions up to a maximum of $39\,604$. These results are the work of many people obtained over a period of 25 years, starting with the original work of Hoggar [13] and Zauner [9]. For the current state of knowledge, and a review of the history see refs. [14–17]. For high dimensions the calculations are computationally intensive. The calculations reported in ref. [17] used two supercomputers, both on the TOP500 list [18], and each having $> 10^5$ cores. They also made essential use of the Stark conjectures which, besides their theoretical significance, have important applications to SIC-computation.

The large number of examples that have been constructed encourages belief in Zauner's conjecture [9], that SICs exist in every finite dimension. However, in spite of an immense amount of effort that has been expended on the problem during the last quarter century, there is still no proof of this. There is not even a proof that SICs exist in an infinite sequence of dimensions.

Hitherto, r-SICs with $r > 1$ have received much less attention in the literature. However, in their case there is an existence proof: namely, it can be shown [5] that in every odd dimension d there exists an r-SIC with $r = (d-1)/2$. It can also be shown [4] that, to every SIC in odd-dimension d, there is a corresponding r-SIC with $r = (d-1)/2$ (different from the one constructed in ref. [5]).

In this paper we will greatly enlarge the set of known r-SICs. We will show, in fact, that the class of SICs we consider is naturally embedded in a much larger class of r-SICs. Moreover, the embedding family of r-SICs has major significance from a number-theoretic point of view. We therefore raise r-SICs with general r to a starring role.

Theorem 3.4. *Let $\Pi_1, \dots, \Pi_{d^2}$ be an r-SIC. Then*

(1) For all j, k

$$\mathrm{Tr}(\Pi_j \Pi_k) = \left(\frac{rd(d-r)}{d^2-1}\right)\delta_{j,k} + \frac{r(rd-1)}{d^2-1}. \tag{3.2}$$

(2) Up to a scale factor the Π_j are a resolution of the identity:

$$\sum_{j=1}^{d^2} \Pi_j = rdI. \tag{3.3}$$

(3) The Π_j are a basis for $\mathcal{L}(H_d)$.

Remark 3.5. *These facts are, of course, well-known. However, existing discussions typically locate r-SICS in a larger context (symmetric POVMs, or fusion frames not assumed to be maximal), which tends to obscure the fact that everything follows from maximality plus Eq. (1.1). We therefore thought it appropriate to include a proof.*

Proof. Let $\mathcal{T}_0$ be the $d^2 - 1$ dimensional subspace of $\mathcal{L}(H_d)$ consisting of all operators A such that $\mathrm{Tr}(A) = 0$. Define operators $B_j \in \mathcal{T}_0$ by

$$B_j = \sqrt{\frac{d}{r(d-r)}}\Pi_j - \sqrt{\frac{r}{d(d-r)}}I. \tag{3.4}$$

There must exist at least one set of numbers c_j, not all 0, such that

$$\sum_{j=1}^{d^2} c_j B_j = 0. \tag{3.5}$$

The c_j must satisfy, for all k

$$0 = \mathrm{Tr}\left(\left(\sum_{j=1}^{d^2} c_j B_j\right) B_k\right) = -\left(\frac{d(\alpha - r)}{r(d-r)}\right) c_k + \frac{\alpha d - r^2}{r(d-r)} \sum_{j=1}^{d^2} c_j \tag{3.6}$$

implying $c_1 = \cdots = c_{d^2} = \mu$ for some fixed, non-zero constant μ. It follows that the orthogonal complement of the B_j is 1-dimensional, implying that the B_j are a spanning set for $\mathcal{T}_0$. Substituting $c_j = \mu$ into Eq. (3.6) we deduce

$$\alpha = \frac{r(rd-1)}{d^2-1}, \tag{3.7}$$

from which Eq. (3.2) follows. We have incidentally shown that

$$\sum_{j=1}^{d^2} B_j = 0 \tag{3.8}$$

from which Eq. (3.3) follows.

Finally, let $\mathcal{S}$ be the span of the Π_j. It follows from Eq. (3.3) that $I \in \mathcal{S}$, which in turn implies the B_j are all in $\mathcal{S}$, and consequently that $\mathcal{T}_0 \subseteq \mathcal{S}$. Since every element of $\mathcal{L}(H_d)$ can be written as a linear combination of I and an element of $\mathcal{T}_0$, it follows that the Π_j are a basis for $\mathcal{L}(H_d)$. $\square$

3.1.2. *Weyl-Heisenberg Group.* With the single exception[1] of the Hoggar lines described in ref. [13], every known r-SIC carries a transitive action of the Weyl-Heisenberg group, which we now describe.

Let $|0\rangle, \dots, |d-1\rangle$ be an orthonormal basis for H_d, and let X, Z be the unitary operators which act on the basis according to

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \omega|j\rangle, \qquad \omega = e^{\frac{2\pi i}{d}}, \tag{3.9}$$

where addition of indices in the first equation is modulo d. Define

$$\xi = -e^{\frac{\pi i}{d}}, \qquad \bar{d} = \begin{cases} d & d \text{ odd,} \\ 2d & d \text{ even.} \end{cases} \tag{3.10}$$

Then the set of operators $\{\xi^{n_1} X^{n_2} Z^{n_3} \colon 0 \le n_1 < \bar{d}, 0 \le n_2, n_3 < d\}$ form a group under matrix multiplication, which we refer to as the Weyl-Heisenberg group (WH group). Our focus will be on the subset of the WH group consisting of the displacement operators

$$D_{\mathbf{p}} = \xi^{p_1 p_2} X^{p_1} Z^{p_2}, \qquad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{Z}^2. \tag{3.11}$$

Also define the discrete symplectic form

$$\langle \mathbf{p}, \mathbf{q} \rangle = p_2 q_1 - p_1 q_2, \qquad \mathbf{p}, \mathbf{q} \in \mathbb{Z}^2. \tag{3.12}$$

Then the displacement operators satisfy

$$D^\dagger_{\mathbf{p}} = D_{-\mathbf{p}}, \qquad \forall \mathbf{p} \in \mathbb{Z}^2 \tag{3.13}$$

$$D_{\mathbf{p}} D_{\mathbf{q}} = \xi^{\langle \mathbf{p}, \mathbf{q} \rangle} D_{\mathbf{p}+\mathbf{q}}, \qquad \forall \mathbf{p}, \mathbf{q} \in \mathbb{Z}^2. \tag{3.14}$$

The fact $\xi^d = (-1)^{d+1}$ means

$$D_{\mathbf{p}+d\mathbf{q}} = (-1)^{(d+1)\langle \mathbf{p}, \mathbf{q} \rangle} D_{\mathbf{p}}. \tag{3.15}$$

for all $\mathbf{p}$, $\mathbf{q}$. So the displacement operators are d-periodic when d is odd, but not when d is even. It would be possible to define the displacement operators by $D_{\mathbf{p}} = X^{p_1} Z^{p_2}$, so that they were d-periodic for all values of d. Defining them the way we do introduces major simplifications later on, at the cost of some additional complexity at the outset. To get an idea of the relative merits of the two definitions see ref. [40].

One has

$$\operatorname{Tr}\left(D_{\mathbf{p}} D^\dagger_{\mathbf{q}}\right) = d(-1)^{\frac{d+1}{d}\langle \mathbf{p}, \mathbf{q} \rangle} \delta^{(d)}_{\mathbf{p},\mathbf{q}}, \qquad \delta^{(d)}_{\mathbf{p},\mathbf{q}} = \begin{cases} 1 & \text{if } \mathbf{p} = \mathbf{q} \bmod d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

It follows that an arbitrary operator $M \in \mathcal{L}(H_d)$ can be expanded in terms of the $D_{\mathbf{p}}$ using

$$M = \frac{1}{d} \sum_{\mathbf{p}} \operatorname{Tr}\left(M D^\dagger_{\mathbf{p}}\right) D_{\mathbf{p}} \tag{3.17}$$

[1] It was at one time suspected that there are other exceptions [12, 38]. However, it was shown in ref. [39] that these apparent exceptions are in fact all Weyl-Heisenberg SICs.

where the summation is over any transversal for the quotient group $\mathbb{Z}^2/(d\mathbb{Z}^2)$ (note that the product $\mathrm{Tr}\left(MD_{\mathbf{p}}^{\dagger}\right)D_{\mathbf{p}}$ is d-periodic, even though the two factors may not be).

3.1.3. *Weyl-Heisenberg SICs.* An r-SIC is said to be Weyl-Heisenberg covariant (WH covariant) if it is of the form $\{\Pi_{\mathbf{p}} \colon 0 \leq p_1, p_2 < d\}$, where

$$\Pi_{\mathbf{p}} = D_{\mathbf{p}}\Pi D_{\mathbf{p}}^{\dagger} \tag{3.18}$$

for some fixed projector $\Pi = \Pi_{\mathbf{0}}$. Notice that, unlike the $D_{\mathbf{p}}$, the $\Pi_{\mathbf{p}}$ are always d-periodic. We refer to Π as a fiducial projector. The transitivity of the group action means that we could choose any other of the $\Pi_{\mathbf{p}}$ in place of Π and still generate the SIC. However, we will see below that from a number-theoretic point of view some choices of fiducial are more natural than others.

WH covariance introduces a major simplification, for it means that, instead of having to study the full set of d^2 projectors constituting the r-SIC, we can focus on a single fiducial projector. This fact relies heavily on the following theorem, which enables us to replace the original definition of an r-SIC with one framed in terms of the fiducial projector.

Theorem 3.6. *Let Π be a rank r projector, with $0 < r < d$. Then the following statements are equivalent*

(1) Π is a fiducial projector for a WH covariant r-SIC
(2) For all $\mathbf{p} \neq \mathbf{0}$ (mod d)

$$\left|\mathrm{Tr}\left(\Pi D_{\mathbf{p}}^{\dagger}\right)\right|^2 = \frac{r(d-r)}{d^2-1} \tag{3.19}$$

(3) There exist phases $e^{i\theta_{\mathbf{p}}}$ such that $e^{i\theta_{\mathbf{p}+d\mathbf{q}}} = (-1)^{(d+1)\langle\mathbf{p},\mathbf{q}\rangle}e^{i\theta_{\mathbf{p}}}$ and

$$\Pi = \frac{r}{d}I + \sqrt{\frac{r(d-r)}{d^2(d^2-1)}}\sum_{\mathbf{p}\notin d\mathbb{Z}^2} e^{i\theta_{\mathbf{p}}}D_{\mathbf{p}} \tag{3.20}$$

where the sum is over any transversal for the quotient group $\mathbb{Z}^2/(d\mathbb{Z}^2)$ with the representative of $d\mathbb{Z}^2$ excluded.

The phases $e^{i\theta_{\mathbf{p}}}$ are given by

$$e^{i\theta_{\mathbf{p}}} = \sqrt{\frac{d^2-1}{r(d-r)}}\,\mathrm{Tr}\left(\Pi D_{\mathbf{p}}^{\dagger}\right). \tag{3.21}$$

for $\mathbf{p} \neq \mathbf{0}$ (mod d).

Proof. It follows from Eq. (3.17) that

$$\Pi_{\mathbf{p}} = \frac{1}{d}\sum_{\mathbf{l}}\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)D_{\mathbf{p}}D_{\mathbf{l}}D_{\mathbf{p}}^{\dagger} = \frac{1}{d}\sum_{\mathbf{l}}\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\omega^{\langle\mathbf{p},\mathbf{l}\rangle}D_{\mathbf{l}}. \tag{3.22}$$

Hence

$$\begin{aligned}\mathrm{Tr}\left(\Pi_{\mathbf{p}}\Pi_{\mathbf{q}}\right) &= \frac{1}{d}\sum_{\mathbf{l},\mathbf{l}'}\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\mathrm{Tr}\left(\Pi D_{\mathbf{l}'}^{\dagger}\right)\omega^{\langle\mathbf{p},\mathbf{l}\rangle+\langle\mathbf{q},\mathbf{l}\rangle}(-1)^{\frac{d+1}{d}\langle\mathbf{l},\mathbf{l}'\rangle}\delta^{(d)}_{\mathbf{l},-\mathbf{l}'} \\ &= \frac{1}{d}\sum_{\mathbf{l}}\left|\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\right|^2\omega^{\langle\mathbf{p}-\mathbf{q},\mathbf{l}\rangle} \end{aligned} \tag{3.23}$$

So Π is an r-SIC fiducial if and only if

$$\begin{aligned}
&\frac{1}{d}\sum_{\mathbf{l}}\left|\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\right|^2\omega^{\langle\mathbf{p},\mathbf{l}\rangle} = \left(\frac{rd(d-r)}{d^2-1}\right)\delta^{(d)}_{\mathbf{p},\mathbf{0}} + \frac{r(rd-1)}{d^2-1} && \forall\mathbf{p}\\
\iff\quad & \left|\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\right|^2 = \frac{r(d-r)}{d^2-1} + \left(\frac{rd(rd-1)}{d^2-1}\right)\delta^{(d)}_{\mathbf{l},\mathbf{0}} && \forall\mathbf{l}\\
\iff\quad & \left|\mathrm{Tr}\left(\Pi D_{\mathbf{l}}^{\dagger}\right)\right|^2 = \frac{r(d-r)}{d^2-1} && \forall\mathbf{l}\neq\mathbf{0}.
\end{aligned}\tag{3.24}$$

The remaining statements are now immediate. □

Definition 3.7. *The overlap phases for a WH covariant SIC fiducial Π are the quantities $e^{i\theta_{\mathbf{p}}}$ given by Eq.* (3.21).

3.1.4. *Number-theoretic properties of SICs: what is already known.*

3.2. **Main results of this paper.**

3.3. **Outline of the remainder of this paper.**

References

[1] P. G. Casazza, M. Fickus, D. G. Mixon, Y. Wang, and Z. Zhou, "Constructing tight fusion frames," *Applied and Computational Harmonic Analysis* **30** (2011) 175–187.

[2] M. Fickus, J. Jasper, D. G. Mixon, and C. E. Watson, "A brief introduction to equi-chordal and equi-isoclinic tight fusion frames," in *Wavelets and Sparsity XVII*, vol. 10394, pp. 186–194, SPIE. 2017.

[3] E. J. King, "Constructing subspace packings from other packings," *Linear Algebra and its Applications* **625** (2021) 68–80.

[4] M. Appleby, I. Bengtsson, S. Flammia, and D. Goyeneche, "Tight frames, Hadamard matrices, and Zauner's conjecture," *J. Phys. A* **52** (2019) 295301.

[5] D. M. Appleby, "Symmetric informationally complete measurements of arbitrary rank," *Opt. Spect.* **103** (2007) 416–428.

[6] G. Gour and A. Kalev, "Construction of all general symmetric informationally complete measurements," *J. Phys. A* **47** (2014) 335302.

[7] M. A. Graydon and D. M. Appleby, "Quantum conical designs," *J. Phys. A* **49** (2016) 085301.

[8] S. F. D. Waldron, *An Introduction to Finite Tight Frames*. Birkhäuser, 2018.

[9] G. Zauner, *Quantendesigns – Grundzüge einer nichtkommutativen Designtheorie*. PhD thesis, University of Vienna, 1999. `http://www.mat.univie.ac.at/~neum/papers/physpapers.html`. Available in English translation as: G. Zauner, Quantum Designs: Foundations of a Noncommutative Design Theory, *Int. J. Quant. Info.*, 9(1):445–507, 2011.

[10] A. J. Scott, "Tight informationally complete quantum measurements," *J. Phys. A* **39** (2006) 13507–13530.

[11] P. Delsarte, J. M. Goethals, and J. J. Seidel, "Bounds for systems of lines, and Jacobi polynomials," *Philips Research Reports* **30** (1975) 91.

[12] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," *J. Math. Phys.* **45** no. 6, (June, 2004) 2171–2180, `quant-ph/0310075`.

[13] S. G. Hoggar, "64 lines from a quaternionic polytope," *Geometriae Dedicata* **69** no. 3, (1998) 287–289.

[14] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, "The SIC question: History and state of play," *Axioms* **6** (2017) 21.

[15] M. Grassl, "Computing numerical and exact SIC-POVMs." Talk at Jagiellonian University, Kraków, 29 March 2021. `https://www.youtube.com/watch?v=CGNxSRcqWts`.

[16] M. Grassl, "Computing SIC-POVMs using permutation symmetries and Stark units." Codex seminar, 26 October 2021. `https://www.youtube.com/watch?v=2vzS55SjaZI`.

[17] M. Appleby, I. Bengtsson, M. Grassl, M. Harrison, and G. McConnell, "SIC-POVMs from Stark units: Prime dimensions n^2+3," *Journal of Mathematical Physics* **63** (2022) 112205.

[18] TOP500 Project. Online listing of the world's most powerful supercomputers, as ranked by the Linpack benchmark. https://www.top500.org/project/.

[19] B. C. Stacey, *A First Course in the Sporadic SICs*. Springer Briefs in Mathematical Physics Vol. 41. Springer, 2021.

[20] A. J. Scott and M. Grassl, "Symmetric informationally complete positive-operator-valued measures: A new computer study," *J. Math. Phys.* **51** no. 4, (2010) 042203, arXiv:0910.5784.

[21] D. M. Appleby, H. Yadsan-Appleby, and G. Zauner, "Galois automorphisms of a symmetric measurement," *Quant. Info. Comput.* **13** no. 7&8, (2013) 672–720, arXiv:1209.1813.

[22] M. Appleby, S. Flammia, G. McConnell, and J. Yard, "Generating ray class fields of real quadratic fields via complex equiangular lines," *Acta Arithmetica* **192** no. 3, (2020) 211–233, arXiv:1604.06098.

[23] M. Appleby, S. Flammia, G. McConnell, and J. Yard, "SICs and algebraic number theory," *Found. Phys.* **47** (2017) 1042–1059.

[24] G. S. Kopp, "SIC-POVMs and the Stark conjectures," *International Mathematics Research Notices* (2019) rnz153, arXiv:1807.05877.

[25] G. S. Kopp and J. C. Lagarias, "Class field theory for orders of number fields," arXiv:2212.09177.

[26] G. S. Kopp and J. C. Lagarias, ""SICs and orders of real quadratic fields" (title tbd),". Forthcoming.

[27] G. S. Kopp, ""square roots of Stark units as ratios of q-Pochhammer symbols" (title tbd),". Forthcoming.

[28] T. Shintani, "On a Kronecker limit formula for real quadratic fields," *J. Fac. Sci. Univ. Tokyo* **24** (1977) 167–199.

[29] T. Shintani, "On certain ray class invariants of real quadratic fields," *J. Math. Soc. Japan* **30** (1977) 139–167.

[30] L. D. Faddeev, "Discrete heisenberg-weyl group and modular group," *Letters in Mathematical Physics* **34** (1995) 249–254.

[31] S. Woronowicz, "Quantum exponential function," *Reviews in Mathematical Physics* **12** (2000) 873–920.

[32] L. D. Faddeev, R. M. Kashaev, and A. Y. Volkov, "Strongly coupled quantum discrete liouville theory. i: Algebraic approach and duality," *Communications in Mathematical Physics* **219** (2001) 199–219.

[33] H. Murakami and Y. Yokota, *Volume Conjecture for Knots*. Springer Briefs in Mathematical Physics Vol. 30. Springer, 2018.

[34] C. Closset and H. Kim, "Three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and partition functions on Seifert manifolds: A review," *International Journal of Modern Physics A* **34** (2019) 1930011.

[35] S. Garoufalidis and R. Kashaev, "Resurgence of Faddeev's quantum dilogarithm," in *Topology and Geometry*, A. Papadopoulos, ed., pp. 257–272. European Mathematical Society, 2021. arXiv:2008.12465.

[36] M. Appleby, S. Flammia, and G. Kopp, "Ghost SICs and the Wigner function," 2022. Forthcoming.

[37] M. Atiyah, "The logarithm of the Dedekind η-function," *Math. Ann* **278** (1987) 335–380.

[38] M. Grassl, "Tomography of quantum states in small dimensions," *Electronic Notes in Discrete Mathematics* **20** (2005) 151–164.

[39] H. Zhu, *Quantum state estimation and symmetric informationally complete POMs*. PhD thesis, National University of Singapore, 2012. https://core.ac.uk/download/pdf/48656576.pdf.

[40] L. Bos and S. Waldron, "SICs and the elements of canonical order 3 in the Clifford group," *J. Phys. A* **52** (2019) 105301.

CENTRE FOR ENGINEERED QUANTUM SYSTEMS, SCHOOL OF PHYSICS, UNIVERSITY OF SYDNEY, SYDNEY AUSTRALIA
Email address: marcus.appleby@gmail.com

AWS CENTER FOR QUANTUM COMPUTING, PASADENA, CA 91125, USA

IQIM, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA
Email address: sflammia@gmail.com

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL, UK AND HEILBRONN INSTITUTE FOR MATHEMATICAL RESEARCH, BRISTOL, UK
Email address: gene.kopp@bristol.ac.uk