A CONSTRUCTIVE APPROACH TO ZAUNER'S CONJECTURE VIA THE STARK CONJECTURES

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ABSTRACT. We propose a construction of d^2 complex equiangular lines in $\mathbb{C}^d$, also known as SICs or SIC-POVMs, which were conjectured by Zauner to exist for all d. The construction gives a putatively complete list of SICs with Weyl-Heisenberg symmetry in all dimensions $d > 3$. Specifically, we give an explicit expression for an object that we call a ghost SIC which is constructed from the real multiplication values of a special function and which is Galois conjugate to a SIC. The special function, which we call the Shintani-Faddeev modular cocycle, is a family of meromorphic functions parameterized by congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and may be of independent interest. We prove that our construction gives a valid SIC assuming two conjectures: first, we conjecture that the ghost SIC is idempotent, and second, we require Tate's refinement of the rank-1 abelian Stark conjecture for real quadratic fields. The latter condition allows us to prove that the ghost and the SIC are Galois conjugate over an extension of $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = (d+1)(d-3)$. We provide computational tests of our SIC construction by cross validating it with known exact solutions, with the numerical work of Scott and Grassl, and by constructing four numerical examples of SICs in $d = 100$, three of which are new. We further consider rank-r generalizations of SICs given by equiangular configurations of r-dimensional complex subspaces, known also as MEFFs (maximal equichordal tight fusion frames). We give similar conditional constructions for MEFFs for all r, d such that $r(d-r)$ divides (d^2-1). Finally, we study the structure of the field extensions conjecturally generated by the SICs and MEFFs. If K is any real quadratic field, then either every ray class field over K, or else every ray class field for which 2 is unramified, is generated by our construction.

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1. Introduction

- History and motivation. (1/2 page)
- Introduce the main definitions (Shintani-Faddeev modular cocycle, SIC, MEFF).
- State main conjecture.
- Point reader to further material that helps provide background (e.g., the WH group). (literature review)

Currently subsection 1 is Marcus's "statement in words of what the paper is doing", subsections 2, 3, 4 are Gene's "bare-bones introduction without any discussion, just the definitions and theorems that are going to go in the introduction". These bits will need to be worked together more coherently.

[Here will be a statement in words of what the paper is doing, < 1 page?]

1.1. History and motivation.

The purpose of this paper is to establish some connections between Hilbert's 12^{th} problem [1] and the SIC (symmetric informationally complete measurement) problem [2, 3]. We will be particularly concerned with the Stark conjectures [4–8] (an instantiation of the 12^{th} problem).

Hilbert's 12^{th} problem is of central importance to algebraic number theory. Recently, a p-adic formulation has been proved for totally real fields [9]. However, its original formulation, in terms of the complex number field, remains open. On the other side, the SIC problem is of major interest in quantum information. It is, for instance, one of the eponymous "five open problems in quantum information theory" identified in ref. [10] as having "huge breakthrough potential".

We will review Hilbert's twelfth problem and the SIC problem below. But first we will summarize our results.

Our main result is that the Stark conjectures for a real quadratic field together with a conjectural special function identity imply SIC existence in every finite dimension. Besides existence, these conjectures (Stark conjectures + special function identity) also imply numerous special properties of SICs. SICs have been calculated [**?**, 2, 3, 11–17] numerically (and in many cases analytically) in every dimension up to 193, and in numerous other dimensions up to a current maximum of $39\,604$. In the course of these investigations a large body of observations has accumulated, concerning the symmetry groups of SICs, the number of unitarily inequivalent SICs in each dimension, relations between SICs in different dimensions, and so on. This mass of material—SIC phenomenology as it might be called—is rich, and complicated, and until now no one has managed to discern any simple, overall pattern to it. However, it turns out that it is all directly implied by the above mentioned conjectures. Our results are also relevant on a straightforward numerical level, in that we can use them to actually calculate SICs. Specifically, we were able to calculate a SIC in dimension 100 in less than 2 days on a 2018 Macbook Pro. This is comparable with existing methods. It is possible that with further development number-theoretic methods will come to be the optimal way of calculating SICs. Finally, we show that, the conjectures also imply the existence of a generalization of SICs, which we refer to as MEFFs ("maximal equichordal tight fusion frames"). **DMA:** add references

The connection goes the other way too: SICs are relevant to number theory. In the first place, our conjectured special function identity is also a relation between Stark units, and if true it has a direct bearing on number theory. In the second place, our conjectures imply that a subgroup of the Galois group acts as orthogonal transformations on quantum state space. In the third place the work on ray class fields of an order in ref. [18], and the work on the square roots of Stark units in ref. [19] was directly inspired by work on the SIC problem. We believe that this may be just the beginning of a long and fruitful interaction between quantum information theory and algebraic number theory.

In *A Mathematician's Apology* [20] G.H. Hardy defines mathematical "significance" as follows:

> We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in mathematics itself and even in other sciences. No chess problem has ever affected the general development of scientific thought; Pythagoras, Newton, Einstein have in their times changed its whole direction.

We would apply this to the connections between Hilbert's 12^{th} problem and the SIC problem, which illustrate the significance of both. As we discuss below, the existence of a SIC in a given dimension is an important geometrical feature of quantum state space. The various properties which we lump together under the heading "SIC phenomenology" are likewise highly intricate, dimension-dependent features of the geometry. It is remarkable that these geometrical facts should all be direct consequences of results in what, to a practically-minded physicist, might seem the highly arcane subject of algebraic number theory. Although it is not the focus of the present paper, let us note that there may also be some interesting connections with a different area of physics, in that the functions we use to calculate SICs and Stark units (what, following ref. [21], we call the double sine function [22, 23], and what, following ref. [19], we call the Shintani-Faddeev Jacobi cocycle [19]) also play a role in conformal field theory [24–26].

The purpose of this paper is to build a bridge between two subjects which have hitherto had little contact. The problem we face is that many of the number theorists to whom we wish to appeal do not know much about quantum information theory, while many of the physicists to whom we also wish to appeal know correspondingly little about algebraic number theory. Although there is only so much that can be done in this regard, we have done our best to make this paper intelligible to both communities. In particular, in the remainder of the introduction we describe Hilbert's twelfth problem and the SIC problem in terms that require the minimum of background knowledge.

We wish to formulate the SIC problem in a way that makes clear its relation to the geometry of quantum state space. We therefore begin by saying something about state space. Let $\mathcal{H}(d)$ be the set of $d \times d$ Hermitian complex matrices. Although the matrices are complex the fact that they are Hermitian means $\mathcal{M}_d$ is a d^2-dimensional *real* vector space. We equip it with the Hilbert-Schmidt inner product

$$\langle A, B\rangle = \mathrm{Tr}(AB), \tag{1.1}$$

and define $\|A\| = \sqrt{\langle A, A\rangle}$ to be the corresponding norm. A quantum state in dimension d is a matrix $\rho \in \mathcal{M}_d$ such that the eigenvalues are all non-negative and $\mathrm{Tr}(\rho) = 1$. A pure quantum state is a state with one eigenvalue equal to 1, and the others all zero (in other words it is a rank 1 projector). The trace condition means quantum states all lie in the $d^2 - 1$ dimensional hyperplane $\mathcal{T}(d) = \{M \in \mathcal{H}(d)\colon\ \mathrm{Tr}(M) = 1\}$. Quantum state space $\mathfrak{S}$ is the set of all quantum states. It is easily seen that it spans $\mathcal{T}(d)$.

Let $\mathfrak{S}_b$ be the boundary of $\mathfrak{S}$, and $\mathfrak{P}$ the set of pure states. Then it is easily seen that $\mathfrak{P} = \mathfrak{S}_b \cap S$, where S is the sphere of radius $\sqrt{(d-1)/d}$ centred on the state $(1/d)I$. The point to notice here is that $\mathfrak{P}$ is diffeomorphic to $\mathbb{CP}^{d-1}$, and is therefore a $2(d-1)$ dimensional real manifold, whereas $S \cap \mathcal{T}(d)$ is a $d^2 - 2$ dimensional real manifold. So if $d > 2$ then $\mathfrak{P}$ is a lower dimensional submanifold of S (*much* lower dimensional if d is large).

These preliminaries completed we may now define a SIC to be a $d^2 - 1$-dimensional regular simplex in $\mathcal{T}(d)$ centred on $(1/d)I$ and with vertices in $\mathfrak{P}$. In other words it consists of d^2 rank 1 projectors $\Pi_1, \dots, \Pi_{d^2}$ such that**DMA:** I am defining a SIC to be a family of projectors, rather than projectors scaled by $1/d$. To my mind that is the most natural definition: the scale factor is only needed to make them a POVM, and my instinct is to de-emphasise the measurement aspect. That is besides the approach taken in the frame theory literature. But perhaps you guys don't agree?

$$\mathrm{Tr}(\Pi_j \Pi_k) = \begin{cases} 1 & j = k \\ \frac{1}{d+1} & j \neq k \end{cases} \tag{1.2}$$

It is of course easy to construct a regular simplex with vertices in S. The problem is to rotate it, so that the vertices all lie in the much lower dimensional sub-manifold $\mathfrak{P} \cap S$. When the problem was first posed [2, 3] it seemed to many people (including one of the present authors) that it should not be too difficult to solve one way or the other: the thought being that the solution depends on the way that $\mathbb{CP}^{d-1}$ embeds in $\mathfrak{S}$, and $\mathbb{CP}^{d-1}$ is a very well-studied object. However, in the course of many years concentrated effort it has become clear that the problem is hard (meaning, in particular, that we do not understand the embedding of $\mathbb{CP}^{d-1}$ in $\mathfrak{S}$ as well as one might think). The connection with Hilbert's twelfth problem indicates why it is hard. Of course, the fact that the Stark conjectures plus our conjectured special function identity imply SIC existence does not exclude the possibility that existence might be proved some other way. However, any alternative approach would still have to explain the fact that the squares of SIC overlaps are conjugates of

Stark units [add reference to formula], and the fact that SIC phenomenology is all explicable in terms of the underlying number theory. For instance, SIC simplices can be grouped into equivalence classes depending on whether they can be continuously rotated into one another without leaving $\mathfrak{S}$. The number of such equivalence classes is a geometrical feature of quantum state space, and the embedding of $\mathbb{CP}^{d-1}$, which varies with dimension in a way that directly follows from the underlying number theory. Although it might conceivably be deduced independently of the Stark conjectures, it is hard to see how it could be proved in a way that did not involve number theory at all.

The term "symmetric informationally complete measurement" comes from the fact that if $\Pi_1, \dots, \Pi_{d^2}$ is a SIC, then the operators $(1/d)\Pi_j$ are the effects for a quantum measurement which is optimal for purposes of statistically inferring the quantum state [27] (the optimality being due to the fact that the operators are spread out evenly over S). SICs have various other applications in quantum information [14, 28] and play an important role in the QBist approach to quantum foundations [29, 30]. They also have applications to Lie and Jordan algebras [31] and Classical signal processing [32].

A MEFF is a generalization of a SIC consisting of d^2 rank-r projectors Π_j, where r is not assumed to equal 1. Then the rank r projectors Π_j are a MEFF if the states $(1/r)\Pi_j$ are a d^2-1 dimensional regular simplex in $\mathcal{T}(d)$ centred on $(1/d)I$. In other words they satisfy,

$$\mathrm{Tr}(\Pi_j \Pi_k) = \begin{cases} r & j = k \\ \frac{r(rd-1)}{d^2-1} & j \neq k \end{cases} \tag{1.3}$$

Let $\mathfrak{P}_r$ be the subspace of $\mathfrak{S}$ consisting of states of the form $(1/r)\Pi$ with Π a rank r projector. Then the dimension of $\mathfrak{P}_r$ is $2r(d-r)$. So a naive dimension-counting argument suggests that MEFFs might be easier to find as the rank increases. Indeed d is odd and $r = (d \pm 1)/2$ then it is trivial to construct an example (Wigner POVM in ref. [33]). For d odd and $r = (d \pm 1)/2$ one can construct further examples from a SIC if one exists (STFFs [as they are there called) in ref. [34]). In this paper we construct MEFFs for other values of r, and without the restriction to d odd. Although the MEFF existence problem is not (so far was we know) on anyone's list of major open problems in quantum information, MEFFs are nonetheless important if one wants to understand the interplay between number theory and geometry.

Hilbert's twelfth problem asks for the characterization of extensions of an algebraic number fields having an Abelian Galois group. Galois theory has not hitherto played a prominent role in theoretical physics, so we will begin by saying a little about that. There is one Galois automorphism that is familiar to every physicist, namely, complex conjugation. Complex conjugation is a bijection of $\mathbb{C}$ onto itself with the property that it respects addition and multiplication, so that $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 z_2)^* = z_1^* z_2^*$ for all $z_1, z_2 \in \mathbb{C}$. A Galois automorphism is any bijection of a number field onto itself which has these two properties. However, unlike complex conjugation Galois automorphisms are in general discontinuous. For instance, consider the field $\mathbb{Q}(\sqrt{2})$, consisting of all combinations of the form $a + b\sqrt{2}$ with a, $b \in \mathbb{Q}$. The map $\sigma\colon a + b\sqrt{2} \to a - b\sqrt{2}$ is a Galois automorphism of $\mathbb{Q}(\sqrt{2})$, which is superficially analogous to complex conjugation. However, unlike complex conjugation it is discontinuous at every point of its domain of definition. It is also unbounded on every finite interval, no matter how small. At first glance it might seem that this feature of Galois theory means it is physically irrelevant. Physics is concerned with quantities we can empirically measure, and empirical measurements have finite accuracy. For this reason one might think that physics is really only concerned with rational numbers. It is true that physics makes use of symbols like $\sqrt{2}$. However, it might be argued that such symbols are really just a convenient way of talking about the rational numbers that approximate them, and that their ability to do this depends on the fact that a function like $x \to \sqrt{x}$ is continuous. By contrast, Galois automorphisms are discontinuous everywhere, and so it may be thought that Galois theory cannot be relevant to physics. Notwithstanding its superficial plausibility, this conclusion is incorrect, as appears from the fact that in this paper, and also in refs. [**?**, 35], Galois automorphisms play a central role in the calculation of physically significant quantities. It raises the question, what other physical applications Galois theory may have.

Hilbert's twelfth problem asks for the characterization of algebraic field extensions having an Abelian Galois group. The simplest situation is Abelian extensions of the rationals: i.e. fields of the form $\mathbb{Q}(a)$ where a is a rational number. The Kronecker-Weber theorem states that $\mathbb{Q}(a)$ has an Abelian Galois group if and only if it is a subfield of $\mathbb{Q}(e^{\frac{2\pi i}{n}})$, for some integer n. This result is important for two reasons. In the first place it provides an explicit expression for the generators of such fields, in terms of special values of the analytic function e^x. In the second place it provides a simple geometrical interpretation for the action of the Galois group. In their action on $\mathbb{Q}(e^{\frac{2\pi i}{n}})$ as a whole the automorphisms are highly discontinuous. But their action on the field generator takes the extremely simple form $e^{\frac{2\pi i}{n}} \to e^{\frac{2k\pi i}{n}}$, for some integer k coprime to n. An analogous result has been proved for Abelian extensions of imaginary quadratic fields: i.e. fields of the form $K(a)$, where $K = \mathbb{Q}(i\sqrt{m})$ for some positive square-free integer m, and such that any pair of Galois automorphisms fixing $i\sqrt{m}$ commute. Hilbert asked for the generalization of these results to Abelian extensions of other number fields. The obvious place to start is Abelian extensions of a real quadratic field, $K = \mathbb{Q}(\sqrt{m})$, for m. Although a p-adic version has recently been proved [9], the original complex number version remains open.

Although still open, there have been a number of significant advances. In the first place, it has been shown that every Abelian extension of the field $\mathbb{Q}(\sqrt{m})$ is contained in a *ray class field*, characterized by a pair $(d, \mathfrak{i})$, called its *modulus*, where d is a positive integer (more generally an ideal in the ring of algebraic integers) and $\mathfrak{i}$ is a subset of $\{\infty_1, \infty_2\}$. For the purposes of this introduction it is not i In the second place, Stark conjectured [4–8] that the ray class fields are generated by special values of a transcendental function associated to the field, called an L-function, called *Stark units*.

Before explaining the connection between SICs and Hilbert's twelfth problem we should say that every known SIC in dimensions 2 and 3, and certain of the known SICs in dimension 8, have some special properties, and are accordingly known as sporadic SICs [36]. In this paper we will exclusively be concerned with non-sporadic SICs. We will accordingly drop the qualifier and simply refer to them as SICs. In particular, the dimensions of the spaces we consider will always be greater than 3.

The first hint that there may be a connection between SICs and Hilbert's twelfth problem came in ref. [37], where it was shown that, in the appropriate basis, and for every exact SIC then known, the matrix elements of a set of SIC projectors in dimension d generate an Abelian extension of the real quadratic field $\mathbb{Q}(\sqrt{(d-3)(d+1)})$. In ref. [38] it was further shown, that in each of the twenty four dimensions there considered the lowest degree field generated by a SIC is the ray-class field with modulus $(\bar{d}, \{\infty_1, \infty_2\})$, where $\bar{d} = d$ (respectively $\bar{d} = 2d$) if d is odd (respectively even). It follows that Preliminary calculations were also reported which suggested that the SIC overlap phases (defined below) might be related to the Stark units for the ray class field they generate. In ref. [35] this speculation was investigated for the ray class SICs in prime dimensions equal to 5 mod 6. A specific procedure for calculating a SIC using the Stark units in these dimensions was proposed, and verified for $d = 5$, 11, 17, 23. In ref. [**?**] a different procedure was proposed for calculating

SICs from Stark units in prime dimensions of the form n^2+3, and verified (exactly or numerically) for $d = 7, 19, 67, 103, 199, 487, 787, 1\,447, 2\,707, 4\,099, 5\,779, 19\,603$. In ref. [39] the results in refs. [35,37,38] are studied from the point of view of projective geometry, and developed in various ways.

The purpose of this paper is to take these investigations further forward.

(1) The results obtained in refs. [?,35,38] only apply to SICs of a special kind: namely, the ones which generate a ray class field. Moreover, the results in refs. [?,35] only apply to a special set of dimensions: namely prime dimensions equal to 5 mod 6, or to n^2+3 for some integer n. In this paper we remove both those restrictions: our results and conjectures apply to *every* SIC in *every* dimension. In particular we rely on the results in refs. [18,40], where the ray class concept is generalized to the ray class field of an arbitrary order, and where numerical evidence is presented that the field generated by any SIC is of this kind. We also rely on the Stark conjectures as they apply to these more general fields.

(2) As mentioned above, we show that the Stark conjectures together with the special function identity in Conjecture 1.7 below imply SIC existence in every dimension. This gives us a proof strategy.

(3) The Stark units are typically**DMA:** can I omit "typically"? not fundamental units. For this reason in refs. [?,35] one had to solve the non-trivial problem of taking the square roots with the appropriate signs. In this paper we use the results in ref. [19] to avoid that problem. Instead of calculating the L-function we calculate the Shintani-Fadeev modular cocycle, which gives us the square roots directly, complete with the appropriate signs.

(4) Until now there has been no explanation of the following features of the tables in refs. [11,13]:
   (a) the way in which the number of unitarily equivalent orbits of SICs depends on the dimension,
   (b) what determines the orders of the SIC symmetry groups in each dimension,
   (c) what determines the number of F_a orbits in dimensions equal to 3 mod 9,
   (d) why there is no analogue of the F_a orbits in dimensions equal to 6 mod 9 (see ref. [41]).

   . There has also been no explanation of the phenomenon of SIC alignment [42,43]. We show that these phenomena (what we referred to above as "SIC phenomenology") are all consequences of the Stark conjectures together with Conjecture 1.7.

(5) SICs and MEFFs do not generate every ray class field over every real quadratic field. However, they do generate a large subset. Specifically ...

1.2. SICs and MEFFs.

Definition 1.1. *A* ***symmetric informationally-complete positive operator-valued measure (SIC-POVM or SIC)*** *in dimension d is a set $\left\{\frac{1}{d}\Pi_1, \dots, \frac{1}{d}\Pi_{d^2}\right\}$, where the Π_j are $d\times d$ complex matrices satisfying the following conditions, for some fixed α.*

(1) $\mathrm{Tr}(\Pi_i\Pi_j) = \alpha$ *whenever* $i \neq j$.
(2) *For all* j, $\Pi_j^2 = \Pi_j$.
(3) *For all* j, $\mathrm{rk}\,\Pi_j = 1$.
(4) *For all* j, $\Pi_j^\dagger = \Pi_j$.

A SIC is equivalent to a set of d^2 equiangular lines in $\mathbb{C}^d$. The lines are taken to be the images of the projections Π_j. Moreover, for any SIC, $\alpha = \frac{1}{d+1}$.

A SIC is also equivalent to a maximal equiangular tight frame (ETF) and to a minimal 2-design.

Definition 1.2. *A **maximal equichordal fusion frame (MEFF)** of rank r in dimension d is a set $\{\Pi_1, \ldots, \Pi_{d^2}\}$, where the Π_j are $d \times d$ complex matrices satisfying the following conditions, for some fixed α.*

(1) $\mathrm{Tr}(\Pi_i \Pi_j) = \alpha$ *whenever* $i \neq j$.
(2) *For all* j, $\Pi_j^2 = \Pi_j$.
(3') *For all* j, $\mathrm{rk}\, \Pi_j = r$.
(4) *For all* j, $\Pi_j^\dagger = \Pi_j$.

...follows that $\alpha = \ldots$

Definition 1.3. *Let $\zeta = e^{2\pi i/d}$ and $\xi = -e^{\pi i/d}$. Consider the $d \times d$ unitary matrices*

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{d-1} \end{pmatrix} \quad \textit{and} \quad Z = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{1.4}$$

*with X being a diagonal "modulation" operator and Z a cyclic permutation or "shift" operator. The **Weyl-Heisenberg group** is the group*

$$\mathbf{WH}(d) = \{\xi^c X^{p_1} Z^{p_2} : c \in \mathbb{Z}/d'\mathbb{Z} \text{ and } p_1, p_2 \in \mathbb{Z}/d\mathbb{Z}\}, \tag{1.5}$$

*where $d' = d$ is d is odd, and $d' = 2d$ if d is even. The center of $\mathbf{WH}(d)$ is $\langle \xi I \rangle$, and a set of coset representatives for the center is given by the **displacement operators***

$$D_{\mathbf{p}} = \xi^{p_1 p_2} X^{p_1} Z^{p_2} \tag{1.6}$$

for $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/d\mathbb{Z})^2$.

Displacement operators satisfy the relations $D_{\mathbf{p}}^\dagger = D_{-\mathbf{p}}$ and $D_{\mathbf{p}} D_{\mathbf{q}} = \xi^{\langle p,q \rangle} D_{\mathbf{p}+\mathbf{q}}$, where $\langle p, q \rangle = p_1 q_2 - p_2 q_1$.

All known SICs satisfy a group convariance property—they are the orbit of a single projector under the action of a finite subgroup of the unitary group. With the exception of the Hoggar lines in dimension 8, all are (up to equivalence) covariant for the same group.

Definition 1.4. *A **Weyl-Heisenberg covariant SIC** is a SIC of the form $\{\frac{1}{d} D_{\mathbf{p}}^\dagger \Pi D_{\mathbf{p}}\}$. A **Weyl-Heisenberg covariant MEFF** is a MEFF of the form $\{D_{\mathbf{p}}^\dagger \Pi D_{\mathbf{p}}\}$.*

*A **SIC fiducial** (respectively, **MEFF fiducial**) is a projector Π generating a Weyl-Heisenberg covariant SIC (respectively, MEFF) in this fashion.*

Definition 1.5. *A **quasi-MEFF fiducial** in dimension d is a complex matrix Π satisfying the following conditions, for some fixed α:*

(1) *For all* $\mathbf{p} \in (\mathbb{Z}/d\mathbb{Z})^2$, $\mathrm{Tr}(\Pi D_{\mathbf{p}}) \,\mathrm{Tr}(\Pi D_{-\mathbf{p}}) = \alpha$.
(2) $\Pi^2 = \Pi$.
(3) $\mathrm{rk}\, \Pi = r$.

*If $r = 1$, Π is called a **quasi-SIC fiducial**.*

It follows from this definition that $\alpha = \frac{r(d-r)}{d^2-1}$. If a quasi-MEFF fiducial is Hermitian, then it is a MEFF fiducial. We also define another special type of quasi-MEFF fiducial.

Definition 1.6. *A **ghost MEFF fiducial** in dimension d with rank r is a quasi-MEFF fiducial satisfying the additional parity-Hermitian property*

(4') $\Pi^\dagger = U_{-I}\Pi U_{-I}$,

*where U_{-I} is the **parity operator***

$$U_{-I} = \sum_{\mathbf{p}\in(\mathbb{Z}/d\mathbb{Z})^2} D_{\mathbf{p}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}. \tag{1.7}$$

If $\operatorname{rk}\Pi = 1$, Π *is called a **ghost SIC fiducial**.*

1.3. The Shintani-Faddeev cocycle.

1.4. Main theorem and conjectures. Our main theorem is a conditional proof of Zauner's conjecture under the assumption of a particular Stark conjecture and a conjectured identity for the Shintani-Faddeev modular cocycle. We will state the identity now and state the required Stark conjecture in Section 2.2.

The following identity is our fundamental conjecture about the Shintani-Faddeev modular cocycle. It expresses the vanishing of a certain "twisted convolution" of the Shintani-Faddeev modular cocycle and its inverse evaluated at a fixed point.

Conjecture 1.7 (Twisted Convolution Vanishing $\mathbf{TCV}(d,r,B,\beta)$)**.** *Let d be a positive integer, and let r be an integer with $0 < r < d$. Suppose that $r(d-r)|(d^2-1)$, and let n be the ratio, so $n\in\mathbb{Z}$ and $nr(d-r) = d^2-1$. Fix some $B = \left(\begin{smallmatrix} s & t \\ u & v\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{Tr}(B) = n-2$, and let $A = B^k$ be the smallest positive power of B in $\Gamma(d)$. Let β be a fixed point of B under the fractional linear transformation action. Define the matrix*

$$C := rBS = \begin{pmatrix} rt & -rs \\ rv & -ru \end{pmatrix}. \tag{1.8}$$

Then,

$$\sum_{\mathbf{q}\in\mathbb{Z}^2/d\mathbb{Z}^2} \mathrm{e}\left(\frac{\mathbf{q}C\mathbf{p}^\top}{d}\right) \varpi_A^{d^{-1}\mathbf{q}}(\beta)^{-1}\, \varpi_A^{d^{-1}(\mathbf{p}+\mathbf{q})}(\beta) = 0 \tag{1.9}$$

for all $\mathbf{p}\in\mathbb{Z}^2\setminus d\mathbb{Z}^2$.

DMA: It seems to me that the TCV isn't stated correctly. Let

$$M = \frac{1}{d}\sum_{\mathbf{p}} \nu'_{\mathbf{p}} D_{\mathbf{p}} \tag{1.10}$$

be a ghost projector. Then $M^2 = M$ if and only if (see "idempotentConditionEvenDimension.tex")

$$\mathrm{Tr}(MD^\dagger_{\mathbf{r}}) = \frac{1}{d}\sum_{p_1,p_2=0}^{d-1} e^{\frac{\pi i(d+1)}{d}\langle\mathbf{p},\mathbf{r}\rangle}\, \mathrm{Tr}(MD^\dagger_{\mathbf{p}})\, \mathrm{Tr}(MD^\dagger_{\mathbf{r}-\mathbf{p}}) \tag{1.11}$$

for all $\mathbf{r}$, implying

$$\nu'_{\mathbf{r}} = \frac{1}{d}\sum_{p_1,p_2=0}^{d-1} e^{\frac{\pi i(d+1)}{d}\langle\mathbf{p},\mathbf{r}\rangle}\nu'_{\mathbf{p}}\nu'_{\mathbf{r}-\mathbf{p}} \tag{1.12}$$

for all $\mathbf{r}$. Using Eq. (1.14) (I think there may be a mistake in Eq. (1.14) also, but that is not important here) and specializing to the case $r = 1$ for simplicity (so that $\nu'_{\mathbf{p}} = \nu_{\mathbf{p}}$), this becomes

$$\varpi_A^{d^{-1}\mathbf{r}}(\beta) = -\frac{n^{-\frac{1}{2}}\zeta_{24}^{\mu(A)}}{d} \sum_{p_1,p_2=0}^{d-1} e^{\frac{\pi i(d+1)}{d}\langle \mathbf{p},\mathbf{r}\rangle} e^{\frac{\pi i(d^2-1)}{d}\left(Q^S(\mathbf{p})+Q^S(\mathbf{r}-\mathbf{p})-Q^S(\mathbf{r})\right)} \varpi_A^{d^{-1}\mathbf{p}}(\beta)\, \varpi_A^{d^{-1}(\mathbf{r}-\mathbf{p})}(\beta)$$

$$= -\frac{n^{-\frac{1}{2}}\zeta_{24}^{\mu(A)}}{d} \sum_{p_1,p_2=0}^{d-1} e^{\frac{\pi i(d+1)}{d}\langle \mathbf{p},\mathbf{r}\rangle} e^{\frac{2\pi i(d^2-1)}{d}\left(\mathbf{p}^{\mathrm{T}}Q^S(\mathbf{p}-\mathbf{r})\right)} \varpi_A^{d^{-1}\mathbf{p}}(\beta)\, \varpi_A^{d^{-1}(\mathbf{r}-\mathbf{p})}(\beta) \tag{1.13}$$

for all $\mathbf{r}$ (where in the last expression Q^S denotes the matrix associated to the quadratic form Q^S—i.e. if $Q^S(x,y) = ax^2 + bxy + cy^2$ then the matrix Q^S is $\left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$). Perhaps I am missing something, but I do not see how this can be massaged into Eq. (1.9).

We may now state our main theorem. We construct SICs directly from real multiplication (RM) values of the Shintani-Faddeev cocycle, under the assumption of the two conjectures. More precisely, the RM values define overlaps of a ghost SIC up to an explicit root of unity, and the SIC is obtained from the ghost SIC by Galois conjugation. The Twisted Convolution Vanishing Conjecture is needed for the construction of the ghost SIC, whereas the Stark Conjecture is needed for the Galois conjugation. Remarkably, the construction generalizes in a straightforward manner to a construction of a family of MEFFs. We present it in that generality.

Theorem 1.8. *Let d be a positive integer, and let r be an integer with $0 < r < d$. Suppose that $r(d-r)|(d^2-1)$, and let n be the ratio, so $n \in \mathbb{Z}$ and $nr(d-r) = d^2 - 1$. Let $\Delta = n(n-4)$. Let Q be a (not necessarily primitive) binary quadratic form of discriminant Δ. Let β be a root of $Q\left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right) = 0$. Fix $B \in \mathrm{SL}_2(\mathbb{Z})$ with the property that $Q_B = Q$, and let $A = B^k$ be the smallest positive power of B in $\Gamma(d)$. For $\mathbf{p} \in \mathbb{Z}^2/d\mathbb{Z}^2$, set*

$$\nu_{\mathbf{p}} = \begin{cases} r, & \text{if } p_1 = p_2 = 0; \\ -n^{-1/2}\zeta_{24}^{\mu(A)} \exp\!\left(\frac{\pi i r(d^2-1)}{d} Q^S(\mathbf{p})\right) \varpi_A^{d^{-1}\mathbf{p}}(\beta), & \text{otherwise}. \end{cases} \tag{1.14}$$

***DMA:** Isn't there a missing sign here for the even dimensional case? —see $(-1)^{s_{\mathbf{q}}(\mathbf{p})}$ in Eq. (5.51) (my expression for the ghost overlaps). Also am I right to assume $Q^s(\mathbf{p})$ means $Q(S\mathbf{p})$? Let $\nu'_{\mathbf{p}} = \nu_{\mathbf{p}\left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right)}$.*

Assume the case $\mathbf{TCV}(d,r,B,\beta)$ of Conjecture 1.7. The $\nu'_{\mathbf{p}}$ are ghost overlaps of a rank r ghost MEFF in dimension d. In particular, the $\nu'_{\mathbf{p}}$ are real, and the matrix

$$M = \frac{1}{d}\sum_{\mathbf{p}} \nu'_{\mathbf{p}} D_{\mathbf{p}} \tag{1.15}$$

is a rank r idempotent.

Furthermore, assume the Stark Conjecture 2.9. Fix a Galois automorphism $\sigma \in \mathrm{Gal}\big(\mathbb{Q}(\Delta, \zeta_d, \nu_{(0,1)})/\mathbb{Q}\big)$ with the property that $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta}$. Then, the matrix $\sigma(M)$ is the fiducial projector of a rank r Weyl-Heisenberg covariant MEFF in dimension d.

The headline result from our main theorem may be summarized as the following corollary.

Corollary 1.9. *Assume the Stark Conjecture 2.9, and assume those cases of Conjecture 1.7 for which $r = 1$. Then, SICs exist in every dimension $d \geq 1$.*

A more refined corollary of the main theorem gives a lower bound (which we expect to be tight) for the number of Weyl-Heisenberg covariant SICs in dimension d. This[1] count is consistent with the prediction of Kopp and Lagarias [40] and with the numerical data of Scott and Grassl [11, 13]. The[2] proof of the asymptotic relies on [40].

Corollary 1.10. *Assume the Stark Conjecture 2.9, and assume those cases of Conjecture 1.7 for which $r = 1$. Let $\mathcal{C}_{d-1}$ be the set of $\mathrm{GL}_2(\mathbb{Z})$-conjugacy class of elements of $\mathrm{SL}_2(\mathbb{Z})$ of trace $d-1$. For $d \neq 3$, there are at least*

$$|\mathcal{C}_{d-1}| = d^{1+o(1)} \tag{1.16}$$

$\mathrm{PEC}(d)$*-equivalence classes of Weyl-Heisenberg covariant SICs in dimension d.*

1.5. Future work.

2. The Shintani-Faddeev modular cocycle (Gene)

- Introduce the relevant background from class field theory (breezy intro with some references to textbooks)
- Theorem (to reference from Gene's other paper): Connect the zeta function to the special value of the cocycle.
- Two key properties of $\nu_{\mathbf{p}}$: $\nu_{\mathbf{p}}\nu_{-\mathbf{p}} = 1$ (follows from modularity of Jacobi theta, and more specifically a theorem of [19]) and $\nu_{\mathbf{p}}$ is real (follows from connection to zeta functions established in [19]).
- Theorem: SF modular cocycle transformation properties match those of a WH SIC
- Double sine formula

GSK: Mtg with Marcus Sept 21: May split this into two sections, one on "Class field theory and zeta functions" and one on "The Shintani-Faddeev modular cocycle"

Fix a positive integer d. The **Shintani-Faddeev modular cocycle with characteristics** $\mathbf{r} \in \frac{1}{d}\mathbb{Z}^2$ is a family of meromorphic functions $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ taking values in the complex plane. Here, A is a 2×2 matrix in a level d congruence subgroup $\Gamma_{\mathbf{r}}$ of $\mathrm{SL}_2(\mathbb{Z})$, and τ is a complex number for which $c\tau + d$ is not a nonpositive real number. Specifically,

$$\Gamma_{\mathbf{r}} = \{A \in \mathrm{SL}_2(\mathbb{Z}) : \mathbf{r}A \equiv \mathbf{r} \pmod{1}\}, \tag{2.1}$$

although we will sometimes restrict to A in the smaller principal congruence subgroup

$$\Gamma(d) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{d}\}. \tag{2.2}$$

Thinking of $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ as a function of A gives a multiplicative 1-cocycle; that is, the identity

$$\mathfrak{w}^{\mathbf{r}}_{AB}(\tau) = \mathfrak{w}^{\mathbf{r}}_A(B \cdot \tau)\, \mathfrak{w}^{\mathbf{r}}_B(\tau) \tag{2.3}$$

holds, where [$\Gamma_{\mathbf{r}}$ acts on $\mathbb{H}$ by fractional linear transformations].

Our construction of ghost SICs and ghost MEFFs uses **real multiplication values** of $\mathfrak{w}^{\mathbf{r}}_A(\tau)$. These are special values $\mathfrak{w}^{\mathbf{r}}_A(\beta)$ taken at the points $\tau = \beta$ where $A \cdot \beta = \beta$. We will need three properties of the Shintani-Faddeev modular cocycle for our construction of ghost SICs and ghost MEFFs. These are:

- $\mathfrak{w}^{\mathbf{r}}_A(\tau)\, \mathfrak{w}^{-\mathbf{r}}_A(\tau) = 1$ (or whatever is true).
- The "real multiplication" values of $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ are real numbers times explicit roots of unity.

[1] **DMA:** In this sentence I have replaced "KoppLagarias2" with "Kopp2020c"

[2] **DMA:** In this sentence I have replaced "KoppLagarias2" with "Kopp2020c"

- Under the assumption of Tate's refinement of Stark's conjecture, the "real multiplication" values of $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ are in a particular number field, which is an abelian extension of a real quadratic field.

2.1. Definition and properties of the Shintani-Faddeev modular cocycle.

Definition 2.1. *Suppose that* $\mathbf{r} \in \frac{1}{d}\mathbb{Z}^2$ *and* $A = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right) \in \Gamma(d)$*. For* $\tau \in \mathbb{H}$*, define*

$$\mathfrak{w}^{\mathbf{r}}_A(\tau) = \frac{\varpi_{\mathbf{r}}(A \cdot \tau)}{\varpi_{\mathbf{r}}(\tau)}, \tag{2.4}$$

where $\varpi_{\mathbf{r}}(\tau) = \prod_{k=1}^{\infty}(1 - \mathrm{e}((k + r_1)\tau + r_2))$*. We use the same notation for the analytic continuation of this function to* $\tau \in \mathbb{C} \setminus \{\omega \in \mathbb{R} : \ell\omega + m \leq 0\}$*.*

We also define

$$\widetilde{\mathfrak{w}}^{\mathbf{r}}_A(\tau) = \frac{\widetilde{\varpi}_{\mathbf{r}}(A \cdot \tau)}{\widetilde{\varpi}_{\mathbf{r}}(\tau)}, \tag{2.5}$$

where $\widetilde{\varpi}_{\mathbf{r}}(\tau) = \prod_{k=1}^{\infty}(1 - \mathrm{e}((k + r_2)\tau - r_1))$*.*

Theorem 2.2. *Suppose that* $\mathbf{r} \in \frac{1}{d}\mathbb{Z}^2$ *and* $A = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right) \in \Gamma(d)$*. Consider a fixed point* β *of* A*, that is, some* $\beta \in \mathbb{C}$ *such that* $A \cdot \beta = \beta$*. Then,*

$$\mathfrak{w}^{\mathbf{r}}_A(\beta)\, \mathfrak{w}^{-\mathbf{r}}_A(\beta) = -i\epsilon(A)^2 \mathrm{e}\!\left(\frac{d+1}{2}(kr_1^2 - (j-m)r_1r_2 - \ell r_2^2)\right). \tag{2.6}$$

Proof. This is [Proposition 3.25] of [19]. □

2.2. Class field theory (for orders).

Class field theory gives an abstract description of the abelian Galois extensions of a number field. If K is a number field, class field theory defines abelian **ray class groups** in terms of data intrinsic to K. It then proves the existence of **ray class fields** having those groups as Galois groups, and it shows that every abelian extension of K is contained in some ray class field.

A classical treatment of class field theory may be found in [44]; a modern treatment that also explains the classical viewpoint may be found in [45].

To describe the fields that we expect general SICs to be defined over, we require not only traditional ray class groups, but also a generalization due to Kopp and Lagarias [18]. The generalization interpolates between ray class groups and "ring class groups" by incorporating an order in the number field K. It relies on a few basic definitions from commutative algebra and algebraic number theory:

- An **order** $\mathcal{O}$ in a number field K is a subring of K containing 1 for which $\mathbb{Q}\mathcal{O} = K$.
- A **fractional ideal** of $\mathcal{O}$ is an $\mathcal{O}$-submodule of K. (An **ideal** of $\mathcal{O}$, or **integral ideal** for emphasis, is an $\mathcal{O}$-submodule of $\mathcal{O}$).
- A fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is **invertible** if there is another fractional ideal $\mathfrak{b}$ of $\mathcal{O}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}$, where the ideal product is defined by

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{j=1}^{n} a_j b_j : a_j \in \mathfrak{a}, b_j \in \mathfrak{b}, \text{ and } n \in \mathbb{N} \right\}. \tag{2.7}$$

- The order denoted $\mathcal{O}_K$ is the unique maximal order of $\mathcal{O}$ (so $\mathcal{O}_K$ contains all other orders $\mathcal{O}$ of K); $\mathcal{O}_K$ is also the integral closure of $\mathbb{Z}$ in K.

Definition 2.3. *Let K be a number field, $\mathcal{O}$ an order in K (that is, a subring containing 1 for which $\mathbb{Q}\mathcal{O} = K$), $\mathfrak{m}$ an ideal of $\mathcal{O}$, and Σ a set of embeddings of K into $\mathbb{R}$. The **ray class group** of $\mathcal{O}$ modulo $(\mathfrak{m}, \Sigma)$ is*

$$\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}) = \frac{\{\text{invertible fractional ideals of } \mathcal{O} \text{ coprime to } \mathfrak{m}\}}{\{\alpha\mathcal{O} \text{ such that } \alpha \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(\alpha) > 0 \text{ for all } \rho \in \Sigma\}}. \tag{2.8}$$

*The group $\mathrm{Cl}_{\mathcal{O},\{\}}(\mathcal{O})$ is called the **ring class group** of $\mathcal{O}$, whereas the group $\mathrm{Cl}_{\mathcal{O}_K,\{\}}(\mathcal{O}_K)$ is simply the **class group** of K. The group $\mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O}_K)$ is traditionally called the **ray class group** of K modulo $(\mathfrak{m}, \Sigma)$.*

2.3. **Partial zeta functions.** Let K be a number field, and let $\mathcal{O}$ be an order in K.

Definition 2.4 (Ray class partial zeta function)**.** *Let $\mathscr{A} \in \widetilde{\mathrm{Cl}}_{\mathfrak{m},\Sigma}(\mathcal{O})$. For $\mathrm{Re}(s) > 1$, define*

$$\zeta_{\mathfrak{m},\Sigma}(s, \mathscr{A}) = \sum_{\mathfrak{a} \in \mathscr{A}} \mathrm{Nm}(\mathfrak{a})^{-s}. \tag{2.9}$$

Definition 2.5 (Differenced ray class partial zeta function)**.** *Let $\mathscr{A} \in \widetilde{\mathrm{Cl}}_{\mathfrak{m},\Sigma}(\mathcal{O})$, and let $\mathscr{R}$ be the element of $\widetilde{\mathrm{Cl}}_{\mathfrak{m},\Sigma}(\mathcal{O})$ defined by*

$$\mathscr{R} := \{\alpha\mathcal{O} : \alpha \equiv -1 \pmod{\mathfrak{m}} \text{ and } \rho(\alpha) > 0 \text{ for all } \rho \in \Sigma\}. \tag{2.10}$$

For $\mathrm{Re}(s) > 1$, define

$$Z_{\mathfrak{m},\Sigma}(s, \mathscr{A}) = \zeta_{\mathfrak{m},\Sigma}(s, \mathscr{A}) - \zeta_{\mathfrak{m},\Sigma}(s, \mathscr{R}\mathscr{A}). \tag{2.11}$$

Definition 2.6 (Galois-theoretic partial zeta function)**.** *Let L/K be an abelian Galois extension. Let S be a finite set of places of K containing all the places that ramify in L, and let $S = S_{\mathrm{fin}} \sqcup S_\infty$ for a set of finite places S_{fin} and a set of infinite places S_∞. For $\mathrm{Re}(s) > 1$, define*

$$\zeta_S^{\mathrm{Gal}}(\sigma, s) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ (\forall \mathfrak{p} \in S_{\mathrm{fin}}) \mathfrak{a} + \mathfrak{p} = \mathcal{O}_K \\ \mathrm{Art}([\mathfrak{a}]) = \sigma}} \mathrm{Nm}(\mathfrak{a})^{-s}. \tag{2.12}$$

GSK: Some of the details of the following statement are wrong. WIP.

Theorem 2.7. *Let $\mathscr{A} \in \mathrm{Cl}_{\mathfrak{m},\Sigma}(\mathcal{O})$. Let $\sigma = \mathrm{Art}(\mathscr{A}) \in \mathrm{Gal}(H^{\mathcal{O}}_{\mathfrak{m},\Sigma}/K)$. Let S_{fin} be the set of primes of $\mathcal{O}_K$ dividing $\mathfrak{m}\mathcal{O}_K$, let $S_\infty = \Sigma$, and let $S = S_{\mathrm{fin}} \sqcup S_\infty$. Then,*

$$\zeta_S^{\mathrm{Gal}}(\sigma, s) = \zeta_{\mathfrak{m},\Sigma}(s, \mathscr{A}). \tag{2.13}$$

Probably reference one of my other papers here. [Or maybe proof is easy in which case we just include it.] □

2.4. **The Stark conjectures.** We will need a special case of Tate's refinement [8] of Stark's conjectures [4–7, 46]. We first state Tate's refinement in general. The following statement is part (II)(a) of Conjecture 4.2 of [8], which is equivalent to the full statement of that conjecture.

Conjecture 2.8 (Stark's rank 1 conjecture with Tate's refinement; $\mathrm{St}(S, L/K)$)**.** *Let L/K be an abelian extension of number fields. Let S be a finite set of places of K (that is, absolute values on K; or equivalently, prime ideals of $\mathcal{O}_K$, real embeddings of K, or pairs of nonreal complex embeddings of K). Suppose that S contains a place $\mathfrak{p}$ that splits completely in K. Let $T = S \setminus \{\mathfrak{p}\}$, and suppose that T is nonempty and contains the places of K that ramify in L. Let $\tilde{S}$ be the set of places of L*

lying over the places in S (that is, extending the absolute values), and let $\tilde{T}$ be the set of places of L lying over the places in T. Define the following subgroups of $K^\times$:

$$U = \{\alpha \in K^\times : |\alpha|_{\mathfrak{P}} = 1 \text{ for all } \mathfrak{P} \notin \tilde{S}\}; \tag{2.14}$$

$$U^T = \begin{cases} \{\alpha \in U : |\alpha|_{\mathfrak{Q}} = 1 \text{ for } \mathfrak{Q} \in \tilde{T}\} & \text{if } |T| \geq 2, \\ \{\alpha \in U : |\alpha|_{\mathfrak{Q}} \text{ is constant for } \mathfrak{Q} \in \tilde{T}\} & \text{if } T = \{\mathfrak{q}\}. \end{cases} \tag{2.15}$$

Let W be the number of roots of unity in L. Let $\mathfrak{P}$ be a place of L lying over $\mathfrak{p}$. Then, there is an element $\varepsilon \in \mathcal{O}_L^\times$ such that

$$\log|\sigma(\varepsilon)|_{\mathfrak{P}} = -W\zeta_S'(\sigma, 0) \qquad \text{for each } \sigma \in \mathrm{Gal}(L/K), \tag{2.16}$$

and such that $L(\varepsilon^{1/W})$ is abelian over K.

We now state the special case of interest to this paper.

Conjecture 2.9 (Stark's real quadratic Archimedean rank 1 conjecture with Tate's refinement)**.** *Let L/K be an abelian extension of number fields, with K real quadratic, such that the infinite place ∞_1 (corresponding to real embedding ρ_1) splits in L and the infinite place ∞_2 (corresponding to real embedding ρ_2) is ramified in L. [Also, suppose that at least one finite place of K ramifies in L...is this really necessary? NOT NECESSARY...IMPLIED BECAUSE NARROW HILBERT CF IS DEGREE 2 EXTENSION OF HILBERT CF] Let ρ be a real embedding of L extending ρ_1. Let S be the set of places of K consisting of ∞_1, ∞_2, and the finite places of K that ramify in L. There is an element $\varepsilon \in \mathcal{O}_L^\times$ such that*

$$\rho(\sigma(\varepsilon)) = \exp(-2\zeta_S'(\sigma, 0)) \qquad \text{for each } \sigma \in \mathrm{Gal}(L/K), \tag{2.17}$$

$$|\rho(\sigma(\varepsilon)| = 1 \qquad \text{for each } \sigma \in \mathrm{Gal}(L/\mathbb{Q}) \setminus \mathrm{Gal}(L/\mathbb{Q}), \tag{2.18}$$

and $L(\varepsilon^{1/2})$ is abelian over K.

Proof of Conjecture 2.9 from Conjecture 2.8. □

Conjecture 2.10 (Stark conjecture for differenced ray class zeta values)**.** *Let K be a real quadratic field and $\{\rho_1, \rho_2\}$ the real embeddings of K. Let $\mathcal{O}$ be an order of K and $\mathfrak{m}$ an ideal of $\mathcal{O}$, and let $\mathscr{A} \in \mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O})$. If $\mathscr{R}$ is not the identity of $\mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O})$, then $Z_{\mathfrak{m}\infty_2}'(0, \mathscr{A}) = \log(\rho_1(\varepsilon_{\mathscr{A}}))$ for an algebraic unit $\varepsilon_{\mathscr{A}}$ generating the ray class field $H_{\mathfrak{m}\infty_2}^{\mathcal{O}}$ corresponding to $\mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O})$. The units are compatible with the Artin map: $\varepsilon_{\mathrm{id}}^{\mathrm{Art}(\mathscr{A})} = \varepsilon_{\mathscr{A}}$. Moreover, $H_{\mathfrak{m}\infty_2}^{\mathcal{O}}(\varepsilon_{\mathscr{A}}^{1/2})$ is abelian over K.*

Proof of Conjecture 2.10 from Conjecture 2.9. □

2.5. Real multiplication values of the Shintani-Faddeev modular cocycle.

We will now relate the **real multiplication values** of the Shintani-Faddeev modular cocycle to special values of partial zeta functions and to presumptive Stark units. These are complex values $\mathfrak{w}_A^{N^{-1}\mathbf{p}}(\beta)$ associated to pairs $(A, \beta) \in \Gamma(N) \times \mathbb{R}$ where $A \cdot \beta = \beta$.

Theorem 2.11. *Let $\mathcal{O}$ be an order in a real quadratic field K, and let $d \in \mathbb{N}$. Let $\mathscr{A} \in \mathrm{Cl}(\mathcal{O})$, choose some $\mathfrak{b} \in \mathscr{A}^{-1}$ coprime to $d\mathcal{O}$, and write $\mathfrak{b} = \mathbb{Z} + \beta\mathbb{Z}$ for some $\beta \in K$ [such that $0 < \beta' < 1 < \beta$ **GSK:** remove this?]. Write*

$$\{B \in \Gamma(d) : B \cdot \beta = \beta\} = \langle A \rangle \tag{2.19}$$

such that $A\left(\begin{smallmatrix}\beta\\1\end{smallmatrix}\right) = \varepsilon\left(\begin{smallmatrix}\beta\\1\end{smallmatrix}\right)$ *for* $\varepsilon > 1$*, and write* $A = \left(\begin{smallmatrix}j & k\\ \ell & m\end{smallmatrix}\right)$ *with* $\ell > 0$*. Let* $\beta_0 \in \mathcal{O}$ *such that* $\beta \equiv \beta_0 \pmod{d}$*. For* $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/d\mathbb{Z})^2$*, let*

$$\mathscr{A}_{\mathbf{p}} = \{\alpha\mathcal{O} : \alpha \equiv p_1\beta + p_2 \pmod{d} \text{ and } \rho_1(\alpha) > 0\}. \tag{2.20}$$

Then

$$\begin{aligned}&\left[\mathrm{U}_{(d\mathcal{O}:d\mathcal{O}+\beta_0\mathcal{O}),\{\infty_1\}}(\mathcal{O}) : \mathrm{U}_{d\mathcal{O},\{\infty_1\}}(\mathcal{O})\right] \exp(Z'_{d\infty_2}(0, \mathfrak{b}^{-1}\mathscr{A}_{\mathbf{p}}))\\ &\qquad = \left(\zeta_8^{-1}\epsilon(A)\mathrm{e}\left(\frac{d+1}{4}\left(kp_1^2 - (j-m)p_1p_2 - \ell p_2^2\right)\right)\mathfrak{w}_A^{N^{-1}\mathbf{p}}(\beta)\right)^2,\end{aligned} \tag{2.21}$$

where $\mathfrak{w}_A^{N^{-1}\mathbf{p}}$ *denotes the Shintani-Faddeev modular cocycle.*

Proof. This is [Theorem 1.5] of [19]. □

Corollary 2.12. ...

$$\zeta_8^{-1}\epsilon(A)\mathrm{e}\left(\frac{d+1}{4}\left(kp_1^2 - (j-m)p_1p_2 - \ell p_2^2\right)\right)\mathfrak{w}_A^{N^{-1}\mathbf{p}}(\beta) \in \mathbb{R}^\times \tag{2.22}$$

Proof. The left-hand side of eq. (2.21) is a nonnegative real number because the partial zeta function $Z_{d\infty_2}(s, \mathfrak{b}^{-1}\mathscr{A}_{\mathbf{p}})$ takes real values when $s \in \mathbb{R}$. Thus, the factor under the square is a nonzero real number. □

Corollary 2.13. *Assume Conjecture 2.9. .... Then,* $\mathfrak{w}_A^{N^{-1}\mathbf{p}}(\beta)$ *is in an abelian extension of* $\mathbb{Q}(\beta)$*.*

Proof. ... □

3. SICs, MEFFs, and their ghosts

3.1. WH and Clifford Groups.

3.2. SICs.

- Explain that SICs for us will always mean WH-SIc in dimension > 4.
- Define $\xi = -e^{\frac{\pi i}{d}}$ (i.e. fix macro).
- include F_z and F_a symmetries

3.3. Fields.

Not sure if this subsection should be included at all. If it is

- Summarize what is known about SIC fields from AYZ [37], AFMY [38], Gene + Jeff [18,40].
- Explain refs. [**?**, 38] only talk about minimal SICs

3.4. Units and dimension towers. Let $K = \mathbb{Q}(\sqrt{\Delta_0})$ be a real quadratic field with discriminant Δ_0. Let $\mathcal{O}_{\Delta_0}$ be its ring of integers and $\mathcal{U}_{\Delta_0}$ its unit group.

As described in AFMY [38] associated to K is an infinite sequence, or *tower* of dimensions $d_1, d_2, \ldots$ such that the SICs in those dimensions give rise to an Abelian extension of K. The tower is constructed as follows. Let u be the unique fundamental unit of $\mathcal{U}$ which is greater than 1, and define

$$v = \begin{cases} u & N(u) = 1\\ u^2 & N(u) = -1\end{cases} \tag{3.1}$$

where $N(u)$ is the norm. The tower is then given by

$$d_r = 1 + v^r + v^{-r}, \qquad r = 1, 2, \ldots \tag{3.2}$$

Define

$$\Delta_r = (d_r - 1)^2 - 4 \tag{3.3}$$

Then

$$\Delta_r = f_r^2 \Delta_0 \qquad\qquad v^r = \frac{d_r - 1 + \sqrt{\Delta_r}}{2} \tag{3.4}$$

where f_r is the conductor of Δ_r. One then has

$$d_{kr} = T_k^*(d_r), \qquad\qquad T_k^*(x) = 1 + 2T_k\left(\frac{x-1}{2}\right), \tag{3.5}$$

$$f_{kr} = f_r U_k^*(d_r) \qquad\qquad U_k^*(x) = U_{k-1}\left(\frac{x-1}{2}\right) \tag{3.6}$$

where the T_k and U_k are respectively Chebyshev polynomials of the first and second kind.

Lemma 3.1. *A. $T_k^*(x)$, $U_k^*(x)$ are polynomials satisfying the recursion relations*

$$T_1^*(x) = x, \qquad T_2^*(x) = x(x-2), \qquad T_k^*(x) = 3 - x + (x-1)T_{k-1}^*(x) - T_{k-2}^*(x), \tag{3.7}$$

$$U_1^*(x) = 1, \qquad U_2^*(x) = x - 1, \qquad U_k^*(x) = (x-1)U_{k-1}^*(x) - U_{k-2}^*(x). \tag{3.8}$$

B. For all $k \in \mathbb{N}$

$$T_k^*(x) = \begin{cases} 3 + x\left(-\frac{k^2}{3}x + \mathrm{O}(x^2)\right) & k = 0 \textit{ mod } 3 \\ x\left(k + \frac{k(k-1)}{6}x + \mathrm{O}(x^2)\right) & k = 1 \textit{ mod } 3 \\ x\left(-k + \frac{k(k+1)}{6}x + \mathrm{O}(x^2)\right) & k = 2 \textit{ mod } 3 \end{cases} \tag{3.9}$$

$$U_k^*(x) = \begin{cases} x\left(-\frac{2k}{3} + \frac{k}{3}x + \mathrm{O}(x^2)\right) & k = 0 \textit{ mod } 3 \\ 1 + x\left(\frac{k-1}{3} - \frac{(k-1)(k+2)}{6}x + \mathrm{O}(x^2)\right) & k = 1 \textit{ mod } 3 \\ -1 + x\left(\frac{k+1}{3} + \frac{(k+1)(k-2)}{6}x + \mathrm{O}(x^2)\right) & k = 2 \textit{ mod } 3 \end{cases} \tag{3.10}$$

C. d_r is a factor of d_{kr} if and only if $k \neq 0$ mod 3.
D. If d_r is odd then d_{kr} is odd for all k.
E. If d_r is even then, for all $k \geq 1$,

(1) *d_{kr} is even if and only if $k \neq 0$ mod 3,*
(2) *f_{kr} is odd if and only if $k \neq 0$ mod 3,*
(3) *$\Delta_{kr} = \Delta_0 = 1$ (mod 4) if and only if $k \neq 0$ mod 3.*

Proof. A is a straightforward consequence of the recursion relations for the Chebyshev polynomials. B is a consequence of A. To prove C, it follows from Eq. (3.9) that $d_{kr} = 0 \bmod d_r$ (respectively $d_{kr} = 3 \bmod d_r$) if $k \neq 0 \bmod 3$ (respectively $k = 0 \bmod 3$). Since $d_r \geq 4$, this means $d_{kr} = 0 \bmod d_r$ if and only if $k \neq 0 \bmod 3$. To prove D, if d_r is odd it follows from Eq. (3.7) that $T_k^*(d_r) = T_{k-2}^*(d_r) \bmod 2$ for all k, and $T_1^*(d_r) = T_2^*(d_r) = 1 \bmod 2$. To prove E, if d_r is even it follows from Eq. (3.9) that $T_k{*}(d_r)$ is odd (respectively even) if $k = 0 \bmod 3$ (respectively $k \neq 0 \bmod 3$). The fact that $\Delta_r = f_r^2\Delta_0 = (d_r - 3)(d_r + 1)$ means Δ_r, Δ_0, f_r are all odd. It follows from this and Eq. (3.10) that f_{kr}, and consequently Δ_{kr} is odd if and only if $k \neq 0 \bmod 3$. The last statement is a consequence of this and the fact that Δ_{kr}, Δ_0 are discriminants (so that they must $= 0$ or $1 \bmod 4$). ∎

The following theorem, proved in ref. [**?**], tells us the dimension towers (equivalently, the fields) for which $N(u) = -1$.

Theorem 3.2. *The following statements are equivalent*

(1) $N(u) = -1$,
(2) $d_r - 3$ *is a perfect square for all odd values of* r,
(3) $d_r - 3$ *is a perfect square for one odd value of* r.

In that case $\frac{d_r+1}{\Delta_0}$ *is a perfect square for all odd values of* r *and*

$$u^r = \frac{\sqrt{d_r - 3} + \sqrt{d_r + 1}}{2} \tag{3.11}$$

for all r*, odd or even.*

Irrespective of the value of $N(u)$*, if* r *is even then* $d_r - 3$ *is not a perfect square, and* $d_r + 1$ *is a perfect square.*

We also need to consider the unit group of an arbitrary order, and its associated dimension sub-tower.

Let f be a positive integer,

$$\mathcal{O}_{\Delta_0,f} = \mathbb{Z}\left[f\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right)\right] \tag{3.12}$$

the corresponding order, and $\mathcal{U}_{\Delta_0,f}$ its unit group. We refer to f as the conductor of the order. Since Dirichlet's unit theorem generalizes to an arbitrary order (see, for example, ref. [47])

$$\mathcal{U}_{\Delta_0,f} = \{\pm u_f^n \colon n \in \mathbb{Z}\} \tag{3.13}$$

where u_f is the unique fundamental unit of $\mathcal{U}_{\Delta_0,f}$ greater than 1. Define

$$v_f = \begin{cases} u_f & N(u_f) = 1 \\ u_f^2 & N(u_f) = -1 \end{cases} \tag{3.14}$$

Since $\mathcal{U}_{\Delta_0,f} \subseteq \mathcal{U}$ we must have $v_f = v^{r_f}$ for some positive integer r_f. If $N(u_f) = -1$ then we also have $u_f = u^{r_f}$.

Proposition 3.3. *For all* $f \in \mathbb{N}$

$$r_f = \min\bigl(\{r \in \mathbb{N} \colon f \mid f_r\}\bigr). \tag{3.15}$$

For all f*,* $r \in \mathbb{N}$

$$f \mid f_r \quad \iff \quad r_f \mid r \tag{3.16}$$

Proof. Let $r \in \mathbb{N}$ be arbitrary. Then

$$v^r = \frac{d_r - 1 - f_r\Delta_0}{2} + f_r\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{3.17}$$

So $v^r \in \mathcal{U}_{\Delta_0,f}$ if and only if $f_r \mid r$. On the other hand $v^r \in \mathcal{U}_{\Delta_0,f}$ if and only if $r_f \mid r$. This establishes the second statement. The first is then immediate. □

Corollary 3.4. *Every positive integer is a divisor of* f_r *for some* r.

Corollary 3.5. *For all* r*,* $s \in \mathbb{N}$*,* $r \mid s$ *if and only if* $f_r \mid f_s$

We now define the dimension sub-tower associated to the order $\mathcal{O}_{\Delta_0,f}$ to be the sequence $d_{r_f}, d_{2r_f}, d_{3r_f}, \dots$. Finally we have the following generalization of Theorem 3.2. As we will see, it tells us which SICs have an anti-unitary symmetry.

Theorem 3.6. *f The following statements are equivalent*

(1) $N(u_f) = -1$,
(2) $d_{kr_f} - 3$ *and* $(d_{kr_f} + 1)/(f^2\Delta_0)$ *are perfect squares for some odd positive integer* k.
(3) $d_{kr_f} - 3$ *and* $(d_{kr_f} + 1)/(f^2\Delta_0)$ *are perfect squares for every odd positive integer* k.

Proof. 1 $\implies$ 3. Suppose $N(u_f) = -1$. Then r_f is odd. Theorem 3.2 implies $d_{kr_f} - 3 = m^2$, $d_{kr_f} + 1 = l^2\Delta_0$ and

$$u_f = \frac{m - l\Delta_0}{2} + l\left(\frac{\Delta_0 + \sqrt{\Delta_0}}{2}\right) \tag{3.18}$$

for some $m, l \in \mathbb{N}$. The fact that $u_f \in \mathcal{U}_{\Delta_0,f}$ means $f \mid l$.

3 $\implies$ 2. Immediate.

2 $\implies$ 1. If $d_{kr_f} - 3$ is a perfect square then Theorem 3.2 means $N(u) = -1$ and kr_f is odd, implying $N(u_f) = -1$.

□

3.5. **Quadratic forms and Unit Group Representations.** As described in Section 5, according to our conjecture SICs are associated to quadratic forms over $\mathbb{Z}$. Each such form determines a representation of the unit group by matrices in $\mathrm{GL}(2,\mathbb{Z})$, which in turn determines the Clifford symmetries of the corresponding SIC. In this subsection we summarize the necessary background material. For more details see refs. [48, 49].

In the following, unless the contrary is explicitly stated, forms will always be assumed to be binary, quadratic, integral (coefficients in F), primitive (coefficients have GCD equal to 1), irreducible (roots are not in $\mathbb{Q}$) and indefinite (roots are in $\mathbb{R}$). If there is no risk of confusion we use the same symbol $Q = \langle a, b, c\rangle$ to denote the bivariate polynomial $Q(x, y) = ax^2 + bxy + cy^2$, the univariate polynomial $Q(x) = ax^2 + bx + c$ and the matrix

$$Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}. \tag{3.19}$$

Let $\mathcal{Q}_{\Delta_0,f}$ be the set of all forms with fundamental discriminant Δ_0 and conductor f. Also define $\mathcal{Q}_{\Delta_0} = \bigcup_f \mathcal{Q}_{\Delta_0,f}$, and $\mathcal{Q} = \bigcup_{\Delta_0} \mathcal{Q}_{\Delta_0}$. Two forms Q, $Q' \in \mathcal{Q}_{\Delta_0,f}$ are properly equivalent (respectively equivalent), denoted $Q \sim_P Q'$ (respectively $Q \sim Q'$), if their matrices are related by

$$Q' = (\mathrm{Det}\, L)L^{\mathrm{T}}QL \tag{3.20}$$

for some $L \in \mathrm{SL}(2,\mathbb{Z})$ (respectively $L \in \mathrm{GL}(2,\mathbb{Z})$). The class number $c(\Delta_0, f)$ is the number of equivalence classes of $\mathcal{Q}_{\Delta_0,f}$ relative to the relation $\sim$.

Given a commutative ring with identity R let $\mathcal{M}(R)$ be the R-module of 2×2 matrices over R, $\mathcal{M}_0$ (respectively $\mathcal{M}_{\mathrm{s}}(R)$) the submodule consisting of all trace zero (respectively symmetric) matrices in $\mathcal{M}(R)$, and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.21}$$

Lemma 3.7.

(1) *For all* $M_0 \in \mathcal{M}_0(R)$

$$M_0^2 = -\operatorname{Det}(M_0)I \tag{3.22}$$

(2) *The map*

$$M_{\mathrm{S}} \to SM_{\mathrm{S}} \tag{3.23}$$

is a bijection of $\mathcal{M}_{\mathrm{s}}(R)$ *onto* $\mathcal{M}_0(R)$ *such that* $\operatorname{Det}(M_{\mathrm{s}}) = \operatorname{Det}(SM_{\mathrm{s}})$.

(3) *If* R *is a field and* M_0, $M_0' \in \mathcal{M}_0(R)$ *are non-zero then* $M_0M_0' = M_0'M_0$ *if and only if* $M_0' = \lambda M_0$ *for some non-zero* $\lambda \in R$.

If F *is a field and* M_0, $M_0' \in \mathcal{M}_0(F)$ *are both non-zero then* $M_0M_0' = M_0'M_0$ *if and only if* $M_0' = \lambda M_0$ *for some non-zero* $\lambda \in F$.

Proof. Straightforward consequences of the definitions. □

If R is a field then $\mathcal{M}(R)$ is an algebra. Given arbitrary $A \in \mathcal{M}(R)$ define $\langle I, A\rangle$ to be the subalgebra generated by I and A. If $A \in \mathcal{M}_0(R)$ then it follows from Eq. (3.22) that

$$\langle I, A\rangle = \{xI + yA \colon x, y \in R\}. \tag{3.24}$$

We say a map $\eta\colon K = \mathbb{Q}(\sqrt{\Delta_0}) \to \mathcal{M}(\mathbb{Q})$ is a canonical representation of K if it is a $\mathbb{Q}$-algebra isomorphism of K onto a sub-algebra of $\mathcal{M}(\mathbb{Q})$ such that

$$N(\kappa) = \operatorname{Det}(\eta(\kappa)) \tag{3.25}$$

$$\operatorname{Tr}(\kappa) = \operatorname{Tr}(\eta(\kappa)) \tag{3.26}$$

for all $\kappa \in K$ (where $\operatorname{Tr}(\kappa)$ is the Galois trace, and $\operatorname{Tr}(\eta(\kappa))$ is the matrix trace).

There is a natural bijective correspondence between the canonical representations of K and the forms in $\mathcal{Q}_{\Delta_0}$. Define, for each $Q \in \mathcal{Q}_{\Delta_0,f}$, a map $\eta_Q\colon K \to \langle I, SQ\rangle$ by

$$\eta_Q\colon x + y\sqrt{\Delta_0} \mapsto xI + \frac{2y}{f}SQ. \tag{3.27}$$

It is easily seen that $\eta_Q = \eta_{Q'}$ if and only if $Q = Q'$.

Theorem 3.8. *A map* $\eta\colon K \to \mathcal{M}(\mathbb{Q})$ *is a canonical representation of* K *if and only if* $\eta = \eta_Q$ *for some* $Q \in \mathcal{Q}_{\Delta_0}$.

Proof. The fact that η_Q is a canonical representation for all $Q \in \mathcal{Q}_{\Delta_0}$ is a straightforward consequence of the definitions and Lemma 3.7.

Let η be an arbitrary canonical representation and define

$$L_1 = \eta(1), \qquad \bar{L}_1 = L_1 - I. \tag{3.28}$$

The fact that $L_1^2 = L_1$ and $\operatorname{Tr}(\bar{L}_1) = 0$ together with Eq. (3.22) implies $L_1 = I$.

Also define

$$L_2 = \eta(\sqrt{\Delta_{K,0}}). \tag{3.29}$$

Since $\operatorname{Tr}(L_2) = 0$ it follows from Lemma 3.7 that $L_2 = \lambda SQ$ for some positive $\lambda \in \mathbb{Q}$ and primitive integral form Q. Let Δ be the discriminant of Q. Then

$$\Delta = -\frac{4}{\lambda^2}\operatorname{Det}(L_2) = \frac{4}{\lambda^2}\Delta_0. \tag{3.30}$$

implying $Q \in \mathcal{Q}_{\Delta_0}$ and $\eta = \eta_Q$. □

Theorem 3.9. *For all $Q \in \mathcal{Q}_{\Delta_0,f}$*

(1) *η_Q restricts to a ring isomorphism of $\mathcal{O}_{\Delta_0,f}$ onto $\langle I, SQ\rangle \cap \mathcal{M}(\mathbb{Z})$.*
(2) *η_Q restricts again to a group isomorphism of $\mathcal{U}_{\Delta_0,f}$ onto $\langle I, SQ\rangle \cap \mathrm{GL}(2,\mathbb{Z})$.*
(3) *Let $n \in \mathbb{N}$ and $\kappa \in \mathcal{O}_{\Delta_0,f}$ be arbitrary. Then $\kappa \in n\mathcal{O}_{\Delta_0,f}$ if and only if $\eta_Q(\kappa) \in n\mathcal{M}(\mathbb{Z})$.*

Proof. Let $Q = \langle a, b, c\rangle \in \mathcal{Q}_{\Delta_0,f}$ and $\kappa = x + 2^{-1}yf\left(\Delta_0 + \sqrt{\Delta_0}\right)$. Then

$$\eta_Q(\kappa) = \begin{pmatrix} x + \frac{y(f\Delta_0 - b)}{2} & -yc \\ ya & x + \frac{y(f\Delta_0+b)}{2} \end{pmatrix} \tag{3.31}$$

The fact that $f^2\Delta_0 = b^2 \pmod 4$ means $f\Delta_0 \pm b$ is even, implying $\eta_Q(\kappa) \in \mathcal{M}(\mathbb{Z})$ for all $\kappa \in \mathcal{O}_{\Delta_0,f}$. Conversely, if $\eta_Q(\kappa) \in \mathcal{M}(\mathbb{Z})$ then ya, yb, $yc \in \mathbb{Z}$. Since Q is primitive this means y, and consequently $x \in \mathbb{Z}$. So $\kappa \in \mathcal{O}_Q$, which proves (1). (2) is a consequence of (1) and Eq. (3.25). (3) is proved in the same way as (1). □

The stability group of $Q \in \mathcal{Q}_{\Delta_0,f}$, denoted $\mathcal{S}(Q)$, is the set of all $L \in \mathrm{GL}(2,\mathbb{Z})$ such that $L^{\mathrm{T}}QL = \mathrm{Det}(L)Q$. The fact that $L^{\mathrm{T}} = -(\mathrm{Det}(L))SL^{-1}S$ means an element of $\mathrm{GL}(2,\mathbb{Z})$ is in $\mathcal{S}(Q)$ if and only if it commutes with SQ. It follows from this, Lemma 3.7 and Theorem 3.9 that $\mathcal{S}(Q) = \eta_Q(\mathcal{U}_{\Delta_0,f})$. In particular $\mathcal{S}(Q)$ is Abelian. Indeed it is maximal Abelian: for if L is in the centralizer of $\mathcal{S}(Q)$ then it commutes with SQ and is therefore itself in $\mathcal{S}(Q)$.

Let Q, Q' be any pair of forms. Then $\mathcal{S}(Q) \cap \mathcal{S}(Q')$ must contain $\{I, -I\}$, the centre of $\mathrm{GL}(2,\mathbb{Z})$. If $Q' \neq \pm Q$ then one finds in fact

$$\mathcal{S}(Q) \cap \mathcal{S}(Q') = \{I, -I\} \tag{3.32}$$

(so $\mathcal{S}(Q)$, $\mathcal{S}(Q')$ are, so to speak, "as disjoint as possible"). Indeed, if $L \in \mathcal{S}(Q) \cap \mathcal{S}(Q')$ then

$$L = \lambda I + \mu SQ = \lambda' I + \mu' SQ' \tag{3.33}$$

for some $\lambda, \mu, \lambda', \mu' \in \mathbb{Q}$. Taking the trace on both sides we deduce $\lambda = \lambda'$. So $\mu Q = \mu' Q'$. Given the assumption $Q' \neq \pm Q$ this is only possible if $\mu = \mu' = 0$, implying $L = \pm I$.

Since $\mathcal{S}(-Q) = \mathcal{S}(Q)$ it is natural to restrict attention to the set $\mathcal{Q}^+$ consisting of all forms $\langle a, b, c\rangle$ such that $a > 0$. We would like to characterize the set $\bigcup_{\mathcal{Q}_+} \mathcal{S}(Q)$. Define $\mathcal{H}$ to be the set of all $L \in \mathrm{GL}(2,\mathbb{Z})$ such that $(\mathrm{Tr}(L))^2 - 4\,\mathrm{Det}\, L > 4$. It is easily seen that $\mathcal{H}$ consists of those matrices in $\mathrm{GL}_2(\mathbb{Z})$ having two distinct fixed points, both in $\mathbb{R} \setminus \mathbb{Q}$. Alternatively, $\mathcal{H}$ may be characterized as the set of hyperbolic matrices with matrices such that $\mathrm{Det}\, L = -1$, $\mathrm{Tr}(L) = 0$ removed (these being the ones with two distinct fixed points in $\mathbb{Q}$).

Theorem 3.10.

$$\bigcup_{Q \in \mathcal{Q}^+} \mathcal{S}(Q) = \{-I, I\} \cup \mathcal{H}. \tag{3.34}$$

For each $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right) \in \mathcal{H}$ the unique $Q \in \mathcal{Q}^+$ and unit w such that $L = \eta_Q(w)$ are

$$Q = \langle n^{-1}\ell, n^{-1}(m-j), -n^{-1}k\rangle, \tag{3.35}$$

$$w = \frac{\mathrm{Tr}(L) + s\sqrt{(\mathrm{Tr}\, L)^2 - 4\,\mathrm{Det}\, L}}{2} \tag{3.36}$$

where

$$n = s\gcd(|k|, |\ell|, |j-m|), \qquad s = \mathrm{sign}(\ell). \tag{3.37}$$

Proof. Suppose $L \in \mathcal{S}(Q)$ for some form $Q = \langle a, b, c\rangle$. Then

$$L = xI + ySQ = \begin{pmatrix} x - \frac{yb}{2} & -yc \\ ya & x + \frac{yb}{2} \end{pmatrix} \tag{3.38}$$

for some x, y. The fact that $ya, yb, yc \in \mathbb{Z}$ means $y \in \mathbb{Z}$. If $y = 0$ then $L = \pm I$. Otherwise

$$\big(\operatorname{Tr}(L)\big)^2 - 4 \operatorname{Det} L = y^2(b^2 - 4ac) > 4. \tag{3.39}$$

Conversely, suppose that $L \in \mathcal{H}$. Let Q, w be as defined in Eqs. (3.35), (3.36), and let Δ_0 (repectively f) be the fundamental discriminant (respectively conductor) of Q. Then

$$w = \frac{\operatorname{Tr}(L) + nf\sqrt{\Delta_0}}{2} \qquad \Longrightarrow \qquad \eta_Q(w) = L. \tag{3.40}$$

□

The $\mathcal{S}(Q)$ are the only maximal Abelian subgroups of $\{-I, I\} \cup \mathcal{H}$. Indeed, let $G \subseteq \mathcal{H}$ be maximal Abelian, and let L be an element of G which is not a multiple of the identity. Then $L \in \mathcal{S}(Q)$ for some $Q \in \mathcal{O}^+$. If L' is any other element of G then L' commutes with SQ, and therefore belongs to $\mathcal{S}(Q)$.

We conclude with some technical results which will be needed in the sequel.

Definition 3.11. *Given $Q = \langle a, b, c\rangle \in \mathcal{Q}$ and $L \in \mathrm{GL}_2(\mathbb{Z})$ define*

$$Q_L = \operatorname{Det}(L) L^{\mathrm{T}} Q L, \qquad\qquad \beta_{Q,\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.41}$$

Lemma 3.12. *For all $Q \in \mathcal{Q}$ and $L \in \mathrm{GL}_2(\mathbb{Z})$*

$$L^{-1}\beta_{Q,\pm} = \beta_{Q_L,\pm}. \tag{3.42}$$

In particular, $L^{-1}\beta_{Q,\pm} = \beta_{Q,\pm}$ if and only if $L \in \mathcal{S}(Q)$.

Proof. Straightforward consequence of the definitions. □

Definition 3.13. *For $Q \in \mathcal{Q}_{\Delta_0,f}$ and $k \in \mathbb{N}$ define*

$$\mathcal{S}_k(Q) = \mathcal{S}(Q) \cap \Gamma(d_{kr_f}). \tag{3.43}$$

Theorem 3.14. [3] *Let $Q \in \mathcal{Q}_{\Delta_0,f}$, $k \in \mathbb{N}$, $L = \eta_Q(u_f)$, and*

$$A = \begin{cases} L^{3k} & N(u_f) = 1, \\ L^{6k} & N(u_f) = -1. \end{cases} \tag{3.44}$$

Then

$$\mathcal{S}(Q) = \langle -I, L\rangle, \tag{3.45}$$

$$\mathcal{S}_k(Q) = \langle A\rangle. \tag{3.46}$$

In particular every element of $\mathcal{S}_k(Q)$ has positive determinant and trace. Moreover if d_{kr_f} is even then

$$A = (d_{kr_f} + 1)I \qquad (\mathit{mod}\ 2d_{kr_f}) \tag{3.47}$$

[3] **DMA:** Can the proof be shortened?

Proof. Eq. (3.45) is immediate. To prove Eq. (3.46) let m be the multiplicative order of v_f mod d_{kr_f}. Since

$$v_f^m = \frac{d_{mr_f} - 1}{2} + \frac{f_{mr_f}}{2}\sqrt{\Delta_0} \tag{3.48}$$

we must have $d_{mr_f} = 3 \mod d_{kr_f}$, implying $m > k$. On the other hand it follows from $d_{kr_f} = 1 + v_f^k + v_f^{-k}$ that

$$v_f^{3k} = d_{kr_f}\big((d_{kr_f} - 2)v_f^k - 1\big) + 1 = 1 \mod d_{kr_f}. \tag{3.49}$$

So $m \mid 3k$, implying $m = 3k$. We also see that if d_{kr_f} is even then $v_f^{3k} = d_{kr_f} + 1 \pmod{2d_{kr_f}}$, from which Eq. (3.47) follows.

Now let m' be the multiplicative order of u_f mod d_{kr_f}. If $N(u_f) = 1$ then $m' = 3k$. Suppose $N(u_f) = -1$. The fact $u_f^{6k} = 1$ means $m' \mid 6k$. On the other hand $N(u_f^{m'}) = 1$ implying $m' = 2m''$ for some $m'' \in \mathbb{N}$. So $v_f^{m''} = 1$, implying $3k \mid m''$, and consequently $6k \mid m'$. So $m' = 6k$. It follows that $\langle L\rangle \cap \Gamma(d_{kr_f}) = \langle A\rangle$.

It remains to show $-L^\ell \neq I \mod d_{kr_f}$ for any integer ℓ. Suppose there were such an integer ℓ. Then there would be an integer ℓ' such that $v_f^{\ell'} = -1 \mod d_{kr_f}$. Indeed, this is immediate if $N(u_f) = 1$, while if $N(u_f) = -1$ the fact that $u_f^\ell = -1 \mod d_{kr_f}$ means $N(u_f^\ell) = 1$, implying that ℓ is even, $= 2\ell'$. Choose j such that $0 < \ell'' = \ell' + 3jk < 3k$. Then $-1 = v_f^{\ell''} = 2^{-1}\left(d_{\ell'' r_f} - 1 + f_{\ell'' r_f}\sqrt{\Delta_0}\right) = -1 \mod d_{kr_f}$ implies $d_{l'' r_f} = -1 \mod 2d_{kr_f}$ implying $k < \ell'' < 3k$. The fact that $v_f^{2\ell''} = 1 \mod d_{kr_f}$ means $3k \mid 2\ell''$. So $2\ell'' = 3k$, implying $k = 2k'$, $\ell'' = 3k'$ for some k'. The argument leading to Eq. (3.49) then implies

$$-1 = v_f^{3k'} = d_{k'r_f}\big((d_{k'r_f} - 2)v_f^{k'} - d_{k'r_f} + 1 \mod d_{k'r_f}(d_{k'r_f} - 2) \tag{3.50}$$

implying in turn $d_{k'r_f} \mid 2$, which is not possible. □

3.6. **MEFFs.**

3.7. **Ghost SICs and MEFFs.**

4. [Main conjecture and results]

- Conjecture: Twisted Convolution Vanishing conjecture (TCV). [will be stated in introduction]
- Theorem: TCV implies ghost SIC existence
- Definition: clearly state what we mean by "the class field hypothesis"
- Theorem: TCV + Stark(-Tate) $\implies$ Zauner
- Theorem: Choice of quadratic form doesn't change the orbit (could possibly go in next section)

5. SIC Properties

- Symmetries
- Going up a tower
- Tables of data.

5.1. **Requisites.** This subsection contains definitions and results which are needed in what follows. Most,if not all, probably belong in Section 4 (not yet written).

Differences from `SIC_construction_notes`:

(1) $\rho_A(\mathbf{p},\tau)$ instead of Gene's $\sigma_{A,\mathbf{p}}(\tau)$.
(2) I use the notation $\langle \mathbf{p},\tau\rangle$.
(3) Meyer invariant defined directly in terms of matrix elements as in [50].

I have reasons for all these changes (see below); they are, however, negotiable.

5.1.1. *Q-Pochhammer symbol.* For all $\tau\in\mathbb{H}$, $z\in\mathbb{C}$

$$\varpi(\tau,z)=\prod_{j=0}^{\infty}(1-e(z+j\tau)) \tag{5.1}$$

$$\varpi_m(\tau,z)=\begin{cases}\prod_{j=0}^{m-1}(1-e(z+j\tau)) & m\geq 1\\ 1 & m=0\\ \prod_{j=m}^{-1}(1-e(z+j\tau))^{-1} & m\leq -1\end{cases} \tag{5.2}$$

where $e(z)=e^{2\pi i z}$.

5.1.2. *SF modular cocycle.* For all $L=\left(\begin{smallmatrix} j & k\\ \ell & m\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, $\tau\in\mathbb{H}$, $z\in\mathbb{C}$ define

$$j_L(\tau)=\ell\tau+m \tag{5.3}$$

and

$$\sigma_L(z,\tau)=\frac{\varpi\left(\frac{z}{j_L(\tau)},L\tau\right)}{\varpi(z,\tau)}. \tag{5.4}$$

Then

$$j_{LL'}(\tau)=j_L(L'\tau)j_{L'}(\tau) \tag{5.5}$$

$$\sigma_{LL'}(z,\tau)=\sigma_L\left(\frac{z}{j_{L'}(\tau)},L'\tau\right)\sigma_{L'}(z,\tau) \tag{5.6}$$

$$\sigma_{L^{-1}}(z,\tau)=\frac{1}{\sigma_L(\frac{z}{j_{L^{-1}}(\tau)},L^{-1}\tau)} \tag{5.7}$$

for all $L,L'\in\mathrm{SL}_2(\mathbb{Z})$.

σ_L continues to a meromorphic function on $\mathbb{C}\times\mathcal{D}_L$, where

$$\mathcal{D}_L=\begin{cases}\mathbb{H} & \ell=0,m=-1,\\ \mathbb{C} & \ell=0,m=1,\\ \mathbb{C}\setminus[-m/\ell,\infty) & \ell<0,\\ \mathbb{C}\setminus(-\infty,-m/\ell] & \ell>0.\end{cases} \tag{5.8}$$

The singularity and zero sets are both subsets of

$$\bigcup_{m_1,m_2=-\infty}^{\infty}\{(z,\tau)\in\mathbb{C}\times\mathcal{D}_L\colon z+m_1\tau+m_2=0\}. \tag{5.9}$$

Although σ_L is only defined for $L \in \mathrm{SL}_2(\mathbb{Z})$ it is convenient[4] to define $\mathcal{D}_L$ for all $L \in \mathrm{GL}_2(\mathbb{Z})$ by setting $\mathcal{D}_L = \mathcal{D}_{JL} = J\mathcal{D}^*_{LJ}$ when $\mathrm{Det}(L) = -1$, where $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. It is easily seen that, for all $L \in \mathrm{GL}_2(\mathbb{Z})$,

$$\mathcal{D}_{L^{-1}} = \begin{cases} L\mathcal{D}_L & \mathrm{Det}(L) = 1, \\ L\mathcal{D}^*_L & \mathrm{Det}(L) = -1, \end{cases} \tag{5.10}$$

and

$$\mathcal{D}_L \cup \mathcal{D}_{-L} = \begin{cases} \mathbb{C} & \ell = 0, \\ \mathbb{C} \setminus \{-m/\ell\} & \ell \neq 0. \end{cases} \tag{5.11}$$

$$\mathcal{D}_L \cap \mathcal{D}_{-L} = \begin{cases} \mathbb{H} & \ell = 0, \\ \mathbb{C} \setminus \mathbb{R} & \ell \neq 0. \end{cases}. \tag{5.12}$$

Also define[5]

$$\langle\!\langle \mathbf{p}, \tau \rangle\!\rangle = p_2\tau - p_1. \tag{5.13}$$

Then

$$\langle\!\langle L\mathbf{p}, L\tau \rangle\!\rangle = \frac{\langle\!\langle \mathbf{p}, \tau \rangle\!\rangle}{j_L(\tau)} \tag{5.14}$$

for all $L \in \mathrm{SL}_2(\mathbb{Z})$.

Lemma 5.1. *Let $\beta \in \mathbb{R}$ be a fixed point of $L \in \mathrm{SL}_2(\mathbb{Z})$. Then $\beta \in \mathcal{D}_L$ if and only if* $\mathrm{Tr}(L) > 0$.

Proof. Let $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$. If $\ell = 0$ then the fact that L has a fixed point means $k = 0$, implying $L = \pm I$. The statement is then immediate. Suppose, on the other hand, that $\ell \neq 0$. Then

$$\beta = \frac{\mathrm{Tr}(L) \pm \sqrt{(\mathrm{Tr}(L))^2 - 4}}{2\ell} - \frac{m}{\ell}. \tag{5.15}$$

from which the statement again follows. □

5.1.3. *Rescaled cocycle, as it appears in overlaps.* For $A = \begin{pmatrix} 1+d\bar{j} & d\bar{k} \\ d\bar{\ell} & 1+d\bar{m} \end{pmatrix} \in \Gamma(d)$, $\mathbf{p} \in \mathbb{Z}^2$, $\tau \in \mathcal{D}_A$ define[6]

$$\rho_{A,d}(\mathbf{p}, \tau) = \frac{\sigma_A\left(\frac{\langle\!\langle A^{-1}\mathbf{p}, \tau \rangle\!\rangle}{d}, \tau\right)}{\varpi_{-\bar{\ell}p_1 + \bar{j}p_2}\left(\frac{\langle\!\langle \mathbf{p}, \tau \rangle\!\rangle}{d}, \tau\right)}. \tag{5.16}$$

[4]**DMA:** For instance the statement of Theorem 5.4 would be more complicated without this extended definition.

[5]**DMA:** If we defined $\langle \mathbf{p}, \tau \rangle = p_1\tau + p_2$ then Eq. (5.14) would take the less symmetric form $\langle\!\langle (L^{\mathrm{T}})^{-1}\mathbf{p}, L\tau \rangle\!\rangle = \frac{\langle\!\langle \mathbf{p}, \tau \rangle\!\rangle}{j_L(\tau)}$

[6]**DMA:** Since there is now an extra subscript d one notational option would be to call this function $\sigma_{A,d}$.

By construction $\rho_{A,d}(\mathbf{p},\tau)$ is meromorphic on $\mathbb{C}^2 \times \mathcal{D}_A$. In the special case when $\mathbf{p} \in \mathbb{Z}^2$ and $\tau \in \mathbb{H}$ one has[7]

$$\rho_{A,d}(\mathbf{p},\tau) = \frac{\varpi\left(\frac{\langle\langle \mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)}{\varpi\left(\frac{\langle\langle \mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)} \tag{5.18}$$

For arbitrary $\mathbf{p}, \mathbf{n} \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$, $A \in \Gamma(d)$, $\tau \in \mathcal{D}_A$

$$\rho_{A,d}(\mathbf{p}+d\mathbf{n},\tau) = \frac{\varpi_{n_2}\left(\frac{\langle\langle \mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)}{\varpi_{n_2}\left(\frac{\langle\langle \mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)}\rho_{A,d}(\mathbf{p},\tau). \tag{5.19}$$

In the special case when $A\tau = \tau$ this becomes:

$$\rho_{A,d}(\mathbf{p}+d\mathbf{n},\tau) = \rho_{A,d}(\mathbf{p},\tau). \tag{5.20}$$

Theorem 5.2. *For all $A \in \Gamma(d)$, $\tau \in \mathcal{D}_{A^{-1}}$, and $\mathbf{p} \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$,*

$$\rho_{A^{-1},d}(\mathbf{p},\tau) = \frac{1}{\rho_{A,d}(\mathbf{p},A^{-1}\tau)} \tag{5.21}$$

Proof. Suppose $\tau \in \mathbb{H}$. Then

$$\rho_{A^{-1},d}(\mathbf{p},\tau) = \frac{\varpi\left(\frac{\langle\langle \mathbf{p}, A^{-1}\tau\rangle\rangle}{d}, A^{-1}\tau\right)}{\varpi\left(\frac{\langle\langle \mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)} = \left(\frac{\varpi\left(\frac{\langle\langle \mathbf{p}, AA^{-1}\tau\rangle\rangle}{d}, AA^{-1}\tau\right)}{\varpi\left(\frac{\langle\langle \mathbf{p}, A^{-1}\tau\rangle\rangle}{d}, A^{-1}\tau\right)}\right)^{-1} = \frac{1}{\rho_{A,d}(\mathbf{p},A^{-1}\tau)} \tag{5.22}$$

The statement follows by analytically continuing to $\mathcal{D}_{A^{-1}}$. ☐

Theorem 5.3. *Let $A \in \Gamma(d)$, $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then for all $\tau \in \mathcal{D}_A$ and $\mathbf{p} \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$*

$$\rho_{JAJ,d}(\mathbf{p},J\tau^*) = \big(\rho_{A,d}(-J\mathbf{p},\tau)\big)^* \tag{5.23}$$

[7]**DMA:** It can be seen from this that Gene's function $\sigma_{A,\mathbf{p}}(\tau)$ is given by

$$\sigma_{A,\mathbf{p}}(\tau) = \rho_{A,d}(dS\mathbf{p},\tau)) \tag{5.17}$$

Defining it the way I have done makes it clear that $\sigma_{A,\mathbf{p}}(\tau)$ is a re-scaled Shintani-Fadeeev modular cocycle. In particular, it shows that $\sigma_{A,\mathbf{p}}(\tau)$ is meromorphic on $\mathcal{D}_A$. It also makes it clear how to express $\sigma_{A,\mathbf{p}}(\tau)$ in terms of the double-sine function. Finally, it is potentially of interest that $\mathbf{p}$ can be allowed to take arbitrary complex values, in which case $\sigma_{A,\mathbf{p}}(\tau)$ is meromorphic on $\mathbb{C}^2 \times \mathcal{D}_A$. To reflect this fact I write $\mathbf{p}$ an argument of the function, instead of a subscript.

Proof. Suppose $\tau \in \mathbb{H}$. Then it follows from Eq. (5.16) that

$$\begin{aligned}
\rho_{JAJ,d}\left(\mathbf{p}, J\tau^*\right) &= \frac{\varpi\left(\frac{\langle\langle \mathbf{p}, JA\tau^*\rangle\rangle}{d}, JA\tau^*\right)}{\varpi\left(\frac{\langle\langle \mathbf{p}, J\tau^*\rangle\rangle}{d}, J\tau^*\right)} \\
&= \frac{\varpi\left(\frac{\langle\langle J\mathbf{p}, A\tau^*\rangle\rangle}{d}, -A\tau^*\right)}{\varpi\left(\frac{\langle\langle J\mathbf{p}, \tau^*\rangle\rangle}{d}, -\tau^*\right)} \\
&= \left(\frac{\varpi\left(\frac{\langle\langle -J\mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)}{\varpi\left(\frac{\langle\langle -J\mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)}\right)^* \\
&= \big(\rho_{A,d}\left(-J\mathbf{p}, \tau\right)\big)^*.
\end{aligned} \tag{5.24}$$

Now analytically continue to $\mathcal{D}_A$. $\square$

Theorem 5.4. *Let $A \in \mathcal{H} \cap \Gamma(d)$ be such that $\mathrm{Tr}(A) > 0$, let β be a fixed point of A, and let $L \in \mathrm{GL}_2(\mathbb{Z})$ be such that $\beta \in \mathcal{D}_{L^{-1}}$. Then*

$$\rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) = \begin{cases} \rho_{A,d}\left(L\mathbf{p}, \beta\right) & \mathrm{Det}(L) = 1, \\ \big(\rho_{A,d}\left(-L\mathbf{p}, \beta\right)\big)^* & \mathrm{Det}(L) = -1. \end{cases} \tag{5.25}$$

for all $\mathbf{p} \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$.

Proof. Suppose $\mathrm{Det}(L) = 1$. Then it follows from Eq. (5.16) that for $\tau \in \mathbb{H}$,

$$\begin{aligned}
\rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\tau\right) &= \left(\frac{\varpi\left(\frac{\langle\langle \mathbf{p}, L^{-1}A\tau\rangle\rangle}{d}, L^{-1}A\tau\right)\varpi\left(\frac{\langle\langle L\mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)}{\varpi\left(\frac{\langle\langle \mathbf{p}, L^{-1}\tau\rangle\rangle}{d}, L^{-1}\tau\right)\varpi\left(\frac{\langle\langle L\mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)}\right)\rho_{A,d}\left(L\mathbf{p}, \tau\right) \\
&= \left(\frac{\varpi\left(\frac{\langle\langle L\mathbf{p}, A\tau\rangle\rangle}{dj_{L^{-1}}(A\tau)}, L^{-1}A\tau\right)\varpi\left(\frac{\langle\langle L\mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)}{\varpi\left(\frac{\langle\langle L\mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)\varpi\left(\frac{\langle\langle L\mathbf{p}, \tau\rangle\rangle}{dj_{L^{-1}}(\tau)}, L^{-1}\tau\right)}\right)\rho_{A,d}\left(L\mathbf{p}, \tau\right) \\
&= \left(\frac{\sigma_{L^{-1}}\left(\frac{\langle\langle L\mathbf{p}, A\tau\rangle\rangle}{d}, A\tau\right)}{\sigma_{L^{-1}}\left(\frac{\langle\langle L\mathbf{p}, \tau\rangle\rangle}{d}, \tau\right)}\right)\rho_{A,d}\left(L\mathbf{p}, \tau\right)
\end{aligned} \tag{5.26}$$

Taking the limit as $\tau \to \beta$ the statement follows for the case $\mathrm{Det}(L) = 1$.

Suppose $\mathrm{Det}(L) = -1$. Define $M = LJ$. Then $\beta \in \mathcal{D}_{M^{-1}} = \mathcal{D}_{L^{-1}}$, so it follows from Theorem 5.3 together with the result just proved

$$\begin{aligned}
\rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) &= \rho_{JM^{-1}AMJ,d}\left(\mathbf{p}, JM^{-1}\beta\right) \\
&= \big(\rho_{M^{-1}AM,d}\left(-J\mathbf{p}, M^{-1}\beta\right)\big)^* \\
&= \big(\rho_{A,d}\left(-MJ\mathbf{p}, \beta\right)\big)^* \\
&= \big(\rho_{A,d}\left(-L\mathbf{p}, \beta\right)\big)^*
\end{aligned} \tag{5.27}$$

$\square$

Corollary 5.5. *Let $A \in \mathcal{H} \cap \Gamma(d)$ be such that $\operatorname{Tr}(A) > 0$ and let β be a fixed point of A. Let n be any positive integer. Then for all L (respectively M) in $\mathrm{GL}_2(\mathbb{Z})$ there exists M (respectively L) in $\mathrm{GL}_2(\mathbb{Z})$ such that $L = \pm M$ (mod nd) and*

$$\rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) = \begin{cases} \rho_{A,d}\left(M\mathbf{p},\beta\right) & \operatorname{Det}(L)=1, \\ \left(\rho_{A,d}\left(M\mathbf{p},\beta\right)\right)^* & \operatorname{Det}(L)=-1. \end{cases} \tag{5.28}$$

for all $\mathbf{p} \in \mathbb{Z}^2 \setminus d\mathbb{Z}^2$.

Proof. Suppose we are given $L \in \mathrm{GL}_2(\mathbb{Z})$. If $\beta \in \mathcal{D}_{L^{-1}}$ then the statement follows from Theorem 5.4 with $M = \operatorname{Det}(L)L$. If $\beta \notin \mathcal{D}_{L^{-1}}$ then Eq. (5.11) implies $\beta \in \mathcal{D}_{-L^{-1}}$, and so Theorem 5.4 implies

$$\rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) = \begin{cases} \rho_{A,d}\left(-L\mathbf{p},\beta\right) & \operatorname{Det}(L)=1, \\ \left(\rho_{A,d}\left(L\mathbf{p},\beta\right)\right)^* & \operatorname{Det}(L)=-1. \end{cases} \tag{5.29}$$

from which the statement follows with $M = -\operatorname{Det}(L)L$.

Suppose, on the other hand, we are given $M \in \mathrm{GL}_2(\mathbb{Z})$. If $\beta \in \mathcal{D}_{\operatorname{Det}(M)M^{-1}}$ the statement follows from Theorem 5.4 with $L = \operatorname{Det}(M)M$. If $\beta \in \mathcal{D}_{-\operatorname{Det}(M)M^{-1}}$ define

$$H_{\pm} = \begin{pmatrix} -1-nd & \pm nd \\ \mp nd & -1+nd \end{pmatrix}. \tag{5.30}$$

The fact that $\mathbb{C} = \mathcal{D}_{H_+} \cup \mathcal{D}_{H_-}$ means we can choose s so that $M^{-1}\beta \in \mathcal{D}_{H_s}$. Define $L = MH_s^{-1}$. Then it follows from Theorem 5.4 and Eq. (5.20) that

$$\begin{aligned} \rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) &= \rho_{M^{-1}AM}\left(H_s^{-1}\mathbf{p}, M^{-1}\beta\right) \\ &= \begin{cases} \rho_{A,d}\left(-MH_s^{-1}\mathbf{p},\beta\right) & \operatorname{Det}(M)=1 \\ \left(\rho_{A,d}\left(-MH_s^{-1}\mathbf{p},\beta\right)\right)^* & \operatorname{Det}(M)=-1 \end{cases} \\ &= \begin{cases} \rho_{A,d}\left(M\mathbf{p},\beta\right) & \operatorname{Det}(M)=1 \\ \left(\rho_{A,d}\left(M\mathbf{p},\beta\right)\right)^* & \operatorname{Det}(M)=-1 \end{cases} \end{aligned} \tag{5.31}$$

□

5.1.4. *Meyer invariant.* The Meyer invariant is a class-function $\phi\colon\ \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$. Define

$$((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & x \notin \mathbb{Z} \end{cases} \tag{5.32}$$

and, for arbitrary $a, b \in \mathbb{Z}$ such that $b \neq 0$, the Dedekind sum

$$s(a,b) = \sum_{n=1}^{|b|-1} \left(\left(\frac{n}{b}\right)\right)\left(\left(\frac{na}{b}\right)\right). \tag{5.33}$$

Then [50] for all $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$ the Meyer invariant is given by

$$\phi(L) = \begin{cases} -\frac{\operatorname{Tr}(L)}{3\ell} + \operatorname{sgn}\left(\ell \operatorname{Tr}(L)(|\operatorname{Tr}(L)|-2)\right) + 4\operatorname{sgn}(\ell)s(j,\ell) & \ell \neq 0 \\ \operatorname{sgn}(\operatorname{Tr}(L))\left(\operatorname{sgn}(k) - \frac{k}{3}\right) & \ell = 0 \end{cases} \tag{5.34}$$

where $\operatorname{sgn}(0) = 0$.

Lemma 5.6. *For all $L \in \mathrm{SL}_2(\mathbb{Z})$, $M \in \mathrm{GL}_2(\mathbb{Z})$,*

$$\phi(I) = 0, \tag{5.35}$$

$$\phi(-L) = \phi(L), \tag{5.36}$$

$$\phi(L^{-1}) = -\phi(L), \tag{5.37}$$

$$\phi(MLM^{-1}) = (\mathrm{Det}\, M)\phi(L) \tag{5.38}$$

Proof. Eqs. (5.35), (5.36), (5.37) are straightforward consequences of Eq. (5.34). If $\det M = 1$ then Eq. (5.38) follows from the fact that ϕ is a class-function. It follows from Eq. (5.34) that $\phi(JLJ^{-1}) = -\phi(L)$. If M is any other element of $\mathrm{GL}_2(\mathbb{Z})$ such that $\mathrm{Det}(M) = -1$ let $M' = MJ$. Then $\phi(MLM^{-1}) = \phi(JLJ^{-1}) = -\phi(L)$. □

Corollary 5.7. *Suppose $L \in \mathrm{SL}_2(\mathbb{Z})$ commutes with a matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ having negative determinant. Then $\phi(L) = 0$.*

Theorem 5.8. *For all $L \in \mathcal{H} \cap \mathrm{SL}_2(\mathbb{Z})$ and $n \in \mathbb{Z}$*

$$e^{\frac{\pi i}{4}\phi(L^n)} = e^{\frac{n\pi i}{4}\phi(L)} \tag{5.39}$$

Proof. Let

$$L^n = \begin{pmatrix} j_n & k_n \\ \ell_n & m_n \end{pmatrix} \tag{5.40}$$

It follows from Theorem 3.10 that

$$L = \eta_Q(x + y\sqrt{\Delta_0}) \tag{5.41}$$

for some $Q \in \mathcal{Q}_+$ with fundamental discriminant Δ_0, and x, $y \in \mathbb{Q}$. Assume to begin with that $\mathrm{Tr}(L) > 0$, $\ell_1 > 0$. Then x, $y > 0$ which in turn implies $\mathrm{Tr}(L^n) > 0$, $\ell_n > 0$ for all $n \in \mathbb{N}$. Then [51]

$$e^{\frac{\pi i}{4}\phi(L^n)} = \sqrt{j_{L^n}(\tau)}\left(\frac{\eta(\tau)}{\eta(L^n\tau)}\right) \tag{5.42}$$

for all $n \in \mathbb{N}$ and $\tau \in \mathbb{H}$, where $\eta(\tau)$ is the Dedekind η-function and where the principal branch of the square root is taken. It is trivial that $\varepsilon^{\frac{\pi i}{4}\phi(L^n)} = e^{\frac{n\pi i}{4}\phi(L)}$ if $n = 0$. Make the inductive hypothesis that the statement holds for some arbitrary non-negative integer n. Then

$$\begin{aligned} e^{\frac{\pi i}{4}\phi(L^{n+1})} &= \sqrt{j_{L^{n+1}}(\tau)}\left(\frac{\eta(\tau)}{\eta(L\tau)}\right)\left(\frac{\eta(L\tau)}{\eta(L^{n+1}\tau)}\right) \\ &= \sqrt{j_{L^{n+1}}(\tau)}\left(\frac{e^{\frac{\pi i}{4}\phi(L)}}{\sqrt{j_L(\tau)}}\right)\left(\frac{e^{\frac{\pi i}{4}\phi(L^n)}}{\sqrt{j_{L^n}(L\tau)}}\right) \end{aligned} \tag{5.43}$$

for all $\tau \in \mathbb{H}$, The fact that ℓ_1, $\ell_n > 0$ means $j_L(\tau)$, $j_{L^n}(L\tau)$, $j_{L^{n+1}}(\tau) \in \mathbb{H}$. In view of Eq. (5.5) this implies $\sqrt{j_L(\tau)}\sqrt{j_{L^n}(L\tau)} = \sqrt{j_{L^{n+1}}(\tau)}$. Consequently $e^{\frac{\pi i}{4}\phi(L^{n+1})} = e^{\frac{(n+1)\pi i}{4}\phi(L)}$. This proves the statement for all $n \geq 0$. If $n < 0$ we use Eq. (5.37) to deduce

$$e^{\frac{\pi i}{4}\phi(L^n)} = e^{-\frac{\pi i}{4}\phi(L^{|n|})} = e^{-\frac{|n|\pi i}{4}\phi(L)} = e^{\frac{n\pi i}{4}\phi(L)} \tag{5.44}$$

The statement thus holds for all $n \in \mathbb{Z}$. Using Eqs. (5.36), (5.37) we also see that the statement continues to hold if either or both of ℓ_1, $\mathrm{Tr}(L)$ are negative. □

5.1.5. *Ghost Projector.* A ghost projector is an operator Π such that

$$\Pi^\dagger = U_P \Pi U_P \tag{5.45}$$

$$\Pi^2 = \Pi \tag{5.46}$$

$$\mathrm{Tr}(\Pi) = 1 \tag{5.47}$$

and, for all $\mathbf{p} \in \mathbb{Z}^2$,

$$\mathrm{Tr}(\Pi D_p^\dagger)\,\mathrm{Tr}(\Pi D_{-p}^\dagger) = \begin{cases} 1 & \mathbf{p} = \mathbf{0} \mod d \\ \frac{1}{d+1} & \text{otherwise} \end{cases} \tag{5.48}$$

If Π is a ghost projector then $D_{\mathbf{p}}^\dagger \Pi D_{\mathbf{p}}$ is also a ghost projector if and only if

$$\mathbf{p} = \begin{cases} \mathbf{0} \mod d & d \text{ odd}, \\ \mathbf{0} \mod \frac{d}{2} & d \text{ even}. \end{cases} \tag{5.49}$$

5.2. **SIC construction.** Suppose we want to calculate the SICs in dimension d. Let

(1) K be the field $\mathbb{Q}(\sqrt{(d-3)(d+1)})$,
(2) Δ_0 the discriminant of K,
(3) $d_1, d_2, \dots$ the associated dimension tower and $f_1, f_2, \dots$ the corresponding sequence of conductors,
(4) r the integer such that $d = d_r$,
(5) $\mathcal{F}$ the divisors of f_r.

Then the Stark conjectures together with the twisted convolution identity conjectured in Section 4 imply that for each $f \in \mathcal{F}$ there is a corresponding SIC multiplet, which we denote (d, f). The individual SICs in the multiplet (d, f) are constructed from the forms in $\mathcal{Q}_{\Delta_0, f}$ in the way described below. Equivalent forms give rise to SICs on the same $\mathrm{EC}(d)$ orbit. Numerical investigations further suggest (a) one gets the full set of SICs in this way and (b) distinct classes in $\mathcal{Q}_{\Delta_0,f}$ give rise to distinct $\mathrm{EC}(d)$ orbits.

To construct the SIC corresponding to a form $Q = \langle a, b, c\rangle \in \mathcal{Q}_{\Delta_0,f}$, let

$$A = \begin{pmatrix} j & k \\ \ell & m \end{pmatrix} \tag{5.50}$$

be either of the two generators of $\mathcal{S}_{r/r_f}(Q)$ (as defined in Eq. (3.43)), and let β be either of the two roots of Q. Define[8], for $\mathbf{q}$ one of $(0,0)^{\mathrm{T}}$, $(0,1)^{\mathrm{T}}$, $(1,0)^{\mathrm{T}}$, $(1,1)^{\mathrm{T}}$,

$$\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) = \begin{cases} 1 & \mathbf{p} = \mathbf{0} \mod d \\ \frac{1}{\sqrt{d+1}} (-1)^{s_{\mathbf{q}}(\mathbf{p})} e^{\frac{\pi i}{4}\phi(A)} \xi_d^{-\mathbf{p}^{\mathrm{T}}.Q_A.\mathbf{p}} \rho_{A,d}(\mathbf{p}, \beta) & \text{otherwise} \end{cases} \tag{5.51}$$

and

$$\Pi_{A,\beta,\mathbf{q},d} = \frac{1}{d} \sum_{p_1,p_2=0}^{d-1} \nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) D_{\mathbf{p}} \tag{5.52}$$

[8]**DMA:** I am trying to make the notation uniform for d odd or even. But it is a bit unsatisfactory, because there is no $\mathbf{q}$-dependence when d is odd. So maybe we should state the cases d odd or even separately.

where $\xi_d = -e^{\frac{\pi i}{d}}$,

$$s_{\mathbf{q}}(\mathbf{p}) = \begin{cases} 1 & d \text{ odd}, \\ (1+p_1)(1+p_2) + \langle \mathbf{q}, \mathbf{p} \rangle & d \text{ even}, \end{cases} \tag{5.53}$$

$$Q_A = \frac{1}{d(d-2)} \langle l, m-j, -k \rangle \tag{5.54}$$

Note that it follows from Theorem 3.14 and Lemma 5.1 that $\beta \in \mathcal{D}_A$, so that $\rho_{A,d}(\mathbf{p}, \beta)$ is well defined.

Proposition 5.9. *$A = \eta_Q(v^{3tr})$ and $Q_A = \frac{tf_r}{f} Q$ where $t = \pm 1$.*

Proof. The fact that $A = \eta_Q(v^{3tr})$ where $t = \pm 1$ is a consequence of Theorem 3.14. It follows that

$$A = \frac{d_{3r} - 1}{2} I + \frac{t f_{3r}}{f} SQ \tag{5.55}$$

implying

$$Q_A = \frac{1}{d_r(d_r - 2)} \langle \ell, m-j, -k \rangle = \frac{t f_{3r}}{d_r(d_r-2) f} \langle a, b, c \rangle = \frac{t f_r}{f} Q \tag{5.56}$$

where we used Lemma 3.1 in the last step. □

If the twisted convolution identity is correct then it follows[9] that Π is a ghost projector. If the Stark conjectures are correct then it can further be shown[10] that, if g is any $\sqrt{\Delta_0}$ sign-switching automorphism, then

$$\tilde{\Pi}_{A,\beta,\mathbf{q},d} = \frac{1}{d} \sum_{p_1,p_2=0}^{d-1} g\big(\nu_{A,\beta,\mathbf{q},d}(\mathbf{p})\big) D_{\mathbf{p}} \tag{5.57}$$

is a strongly centred SIC fiducial projector.

The fact that when d is even one gets 4 different ghost and SIC projectors corresponding to the 4 different values of $\mathbf{q}$ is a consequence of the fact adduced in the sentence containing Eq. (5.49).

It can be seen from Eq. (5.20) that $\rho_{A,d}(\mathbf{p}, \tau)$ has period d irrespective of whether d is odd or even. On the other hand $\nu_{A,\beta,\mathbf{q},d}(\mathbf{p})$ has period $2d$ when d is even. To see this we need

Lemma 5.10. *Suppose d is even. Let $f \in \mathcal{F}$ and $Q = \langle a, b, c \rangle \in \mathcal{Q}_{\Delta_0, f}$. Then Δ_0, f, f_r, a, b, c are all odd.*

Proof. $f_r^2 \Delta_0 = (d_r - 1)^2 - 4$ is odd, implying f_r, Δ_0 and f are odd. Hence $b^2 - 4ac = f^2 \Delta_0$ is odd, implying b is odd.

To show that a, c are also odd, assume the contrary. So ac is even. Define $\langle a', b', c' \rangle = \frac{f_r}{f} \langle a, b, c \rangle$. Then

$$b'^2 - 4a'c' = (d_r - 1)^2 - 4 \tag{5.58}$$

Since $a'c'$ is even this means $b'^2 = 5 \bmod 8$, which is impossible. □

[9]**DMA:** We should say a bit more about this. However, I am waiting until Section 4 has been written before inserting the details (which possibly belong in Section 4?)

[10]**DMA:** Add some references here

Theorem 5.11. *For all* $\mathbf{p} \in \mathbb{Z}^2$

$$\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) = \operatorname{Tr}\left(\Pi_{A,\beta,\mathbf{q},d} D^\dagger_{\mathbf{p}}\right) \tag{5.59}$$

Proof. Immediate consequence of the definitions when d is odd. Suppose d is even. Then the equation holds by construction when $0 \le p_j < d$. Let $\mathbf{p} \in \mathbb{Z}^2$ be arbitrary, and let $\mathbf{n} \in \mathbb{Z}^2$ be such that $0 \le \bar{p}_j = p_j - dn_j < d$. Then Lemma 5.10 implies

$$\xi_d^{-\frac{s' f_r}{f}\mathbf{p}^{\mathrm{T}}.Q.\mathbf{p}} = (-1)^{\langle \mathbf{n}, \bar{\mathbf{p}}\rangle} \xi_d^{-\frac{s' f_r}{f}\bar{\mathbf{p}}^{\mathrm{T}}.Q.\bar{\mathbf{p}}}. \tag{5.60}$$

Hence

$$\begin{aligned} \nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) &= (-1)^{\langle \mathbf{n}, \bar{\mathbf{p}}\rangle} \nu_{A,\beta,\mathbf{q},d}(\bar{\mathbf{p}}) \\ &= (-1)^{\langle \mathbf{n}, \bar{\mathbf{p}}\rangle} \operatorname{Tr}\left(\Pi_{A,\beta,\mathbf{q},d} D^\dagger_{\bar{\mathbf{p}}}\right) \\ &= \operatorname{Tr}\left(\Pi_{A,\beta,\mathbf{q},d} D^\dagger_{\mathbf{p}}\right). \end{aligned} \tag{5.61}$$

$\square$

Lemma 5.12. *For all* $L \in \mathrm{GL}_2(\mathbb{Z})$ *and* $\mathbf{q}, \mathbf{p} \in \mathbb{Z}^2$

$$(-1)^{(1+(Lp)_1)(1+(Lp)_2)} = (-1)^{(1+p_1)(1+p_2)}, \tag{5.62}$$

$$(-1)^{s_{L^{-1}\mathbf{q}}(\mathbf{p})} = (-1)^{s_{\mathbf{q}}(L\mathbf{p})}. \tag{5.63}$$

Proof. Let $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$. Then

$$(-1)^{(1+(Lp)_1)(1+(Lp)_2)} = (-1)^{(1+p_1)(1+p_2)}. \tag{5.64}$$

Since j, ℓ and k, m are coprime $(1+j)(1+\ell)$ and $(1+k)(1+m)$ must both be even, from which Eq. (5.62) follows. Eq. (5.63) follows from this and the fact that

$$\langle \mathbf{q}, L\mathbf{p}\rangle = \operatorname{Det}(L)\langle L^{-1}\mathbf{q}, \mathbf{p}\rangle \tag{5.65}$$

$\square$

Theorem 5.13. *Let* d*,* A*,* β*,* $\mathbf{q}$ *be as above. Then for all* $L \in \mathrm{GL}_2(\mathbb{Z})$ *(respectively* $F \in \mathrm{ESL}(\mathbb{Z}/\bar{d}\mathbb{Z})$*) there exists* $F \in \mathrm{ESL}(\mathbb{Z}/\bar{d}\mathbb{Z})$ *(respectively* $L \in \mathrm{GL}_2(\mathbb{Z})$*) such that* $L = \pm F$ *(mod* $\bar{d}$*) and*

$$\Pi_{L^{-1}AL, L^{-1}\beta, L^{-1}\mathbf{q}, d} = U_F^\dagger \Pi_{A,\beta,\mathbf{q},d} U_F. \tag{5.66}$$

Proof. Suppose $\mathbf{p} \neq \mathbf{0}$ (mod d). Then corollary 5.5, Lemma 5.12, Eq. (5.38), and the fact that

$$Q_{L^{-1}AL} = \operatorname{Det}(L) L^{\mathrm{T}} Q_A L \tag{5.67}$$

mean that for all L (respectively M) in $\mathrm{GL}_2(\mathbb{Z})$ there exists M (respectively L) in $\mathrm{GL}_2(\mathbb{Z})$ such that $L = \pm M$ (mod $\bar{d}$) and

$$\begin{aligned} \nu_{L^{-1}AL, L^{-1}\beta, L^{-1}\mathbf{q}, d}(\mathbf{p}) &= \frac{1}{\sqrt{d+1}} (-1)^{s_{\mathbf{q}}(L\mathbf{p})} e^{\frac{\pi i}{4}\det(L)\phi(A)} \xi_d^{-\operatorname{Det}(L)(L\mathbf{p})^{\mathrm{T}}.Q_A.L\mathbf{p}} \rho_{L^{-1}AL,d}\left(\mathbf{p}, L^{-1}\beta\right) \\ &= \begin{cases} \nu_{A,\beta,\mathbf{q},d}(M\mathbf{p}) & \operatorname{Det}(L) = 1 \\ \nu_{A,\beta,\mathbf{q},d}(M\mathbf{p})^* & \operatorname{Det}(L) = -1 \end{cases} \end{aligned} \tag{5.68}$$

Since the ghost overlaps are real, it follows that

$$\nu_{L^{-1}AL, L^{-1}\beta, L^{-1}\mathbf{q}, d}(\mathbf{p}) = \nu_{A,\beta,\mathbf{q},d}(M\mathbf{p}) \tag{5.69}$$

irrespective of the sign of $\mathrm{Det}(M)$. The statement follows from this, and the fact that the canonical homomorphism $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{ESL}(\mathbb{Z}/\bar{d}\mathbb{Z})$ is surjective. □

There is some redundancy in the parameterisation of the ghost projectors $\Pi_{A,\beta,\mathbf{q},d}$, in that for given Q, d there are two generators of $\mathcal{S}_{r/r_f}(Q)$, two roots of Q, and when d is even 4 possible values of the vector $\mathbf{q}$. The following theorem shows that replacing A with A^{-1}, or $\mathbf{q}$ with $\mathbf{q}+\mathbf{r}$ is equivalent to conjugating with a symplectic unitary.

Theorem 5.14. *Let d, A, β, $\mathbf{q}$ be as above. Let $P = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $M_{\mathbf{r}} = \left(\begin{smallmatrix} 1 & r_1 d \\ r_2 d & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$. Then*

$$\Pi_{A^{-1},\beta,\mathbf{q},d} = U_P^\dagger \Pi_{A,\beta,\mathbf{q},d} U_P \tag{5.70}$$

$$\Pi_{A,\beta,\mathbf{q}+\mathbf{r},d} = U_{M_{\mathbf{r}}}^\dagger \Pi_{A,\beta,\mathbf{q},d} U_{M_{\mathbf{r}}} \tag{5.71}$$

Proof. Theorem 5.2 and Eq. (5.48) imply

$$\nu_{A^{-1},\beta,\mathbf{q},d}(\mathbf{p}) = \nu_{A,\beta,\mathbf{q},d}(-\mathbf{p}) \tag{5.72}$$

from which Eq. (5.70) follows. The second relation is trivial if d is odd. Suppose that d is even. Then $f_r^2\Delta_0$ is odd, implying $m-j$ is odd. Hence

$$\xi_d^{-\mathbf{p}^{\mathrm{T}}.M_{\mathbf{r}}^{\mathrm{T}}.Q_A.M_{\mathbf{r}}\cdot\mathbf{p}} = (-1)^{\langle \mathbf{r},\mathbf{p}\rangle} \xi_d^{-\mathbf{p}^{\mathrm{T}}.Q_A\cdot\mathbf{p}} \tag{5.73}$$

Since $M_{\mathbf{r}} = I \pmod d$ it follows that $\rho_{A,d}\left(M_{\mathbf{r}}\mathbf{p},\beta\right) = \rho_{A,d}\left(\mathbf{p},\beta\right)$. Hence

$$\nu_{A,\beta,\mathbf{q},d}(M_{\mathbf{r}}\mathbf{p}) = \nu_{A,\beta,\mathbf{q}+\mathbf{r},d}(\mathbf{p}) \tag{5.74}$$

from which Eq. (5.71) follows. □

It remains to consider the effect of replacing one root of Q with the other. The next result gives a partial answer to this question.

Theorem 5.15. *Let d, A, β, $\mathbf{q}$ be as above, and suppose $-Q$ is equivalent to Q. Then there exists $F \in \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ such that*

$$\Pi_{A,\beta_{Q,-},\mathbf{q},d} = U_F^\dagger \Pi_{A,\beta_{Q,+},\mathbf{q},d} U_F \tag{5.75}$$

(with notations as in Definition 3.11).

Proof. By assumption $Q_L = -Q$ for some $L \in \mathrm{GL}_2(\mathbb{Z})$. It follows from Lemma 3.12 that $L^{-1}\beta_{Q,+} = \beta_{-Q,+} = \beta_{Q,-}$. Also, the fact that $A = \eta_Q(v^{\pm 3r})$ (see Theorem 3.14) implies $L^{-1}AL = A^{-1}$. In view of Theorem 5.13 this means there exists $G \in \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ such that

$$U_G^\dagger \Pi_{A,\beta_{Q,+},\mathbf{q},d} U_G = \Pi_{A^{-1},\beta_{Q,-},L^{-1}\mathbf{q},d}. \tag{5.76}$$

The result follows from this and Theorem 5.14. □

This leaves open the question, of what happens when $-Q$ and Q are inequivalent. One finds in practice, in every case examined, that $\Pi_{A,\beta_{Q,+},\mathbf{q},d}$ and $\Pi_{A,\beta_{Q,-},\mathbf{q},d}$ are on different $\mathrm{ESL}_2(\mathbb{Z}/\bar{d})$ orbits. However, we have not been able to prove that is always the case.

We next describe the symmetries of the ghost projector.

Definition 5.16. *For all d define the Zauner matrix by*

$$F_z = \begin{pmatrix} 0 & d-1 \\ d+1 & d-1 \end{pmatrix} \tag{5.77}$$

When $d = 3$ mod 9 also define

$$F_a = \begin{pmatrix} 1 & d+3 \\ \frac{4d-3}{3} & d-2 \end{pmatrix}. \tag{5.78}$$

Theorem 5.17. *Let Q, d, A, β, $\mathbf{q}$ be as above. Let G be the image of $\eta_Q(u_f)$ under the canonical projection of $\mathrm{GL}_2(\mathbb{Z})$ onto $\mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ and define*

$$G_{\mathbf{q}} = \begin{cases} M_{\mathbf{q}} G M_{\mathbf{q}}^{-1} & N(u_f) = 1 \\ -M_{\mathbf{q}} G M_{\mathbf{q}}^{-1} & N(u_f) = -1 \end{cases} \tag{5.79}$$

where $M_{\mathbf{q}} = \left(\begin{smallmatrix} 1 & q_1 d \\ q_2 d & 1 \end{smallmatrix}\right)$. Let

$$m = \begin{cases} \frac{r}{r_f} & N(u_f) = 1, \\ \frac{2r}{r_f} & N(u_f) = -1. \end{cases} \tag{5.80}$$

Then

(1) *$U_{G_{\mathbf{q}}}$ stabilizes $\Pi_{A,\beta,\mathbf{q},d}$*

$$\Pi_{A,\beta,\mathbf{q},d} = U^{\dagger}_{G_{\mathbf{q}}} \Pi_{A,\beta,\mathbf{q},d} U_{G_{\mathbf{q}}}, \tag{5.81}$$

(2) *$G_{\mathbf{q}}$ is order $3m$ (respectively $6m$) if d is odd (respectively even),*
(3) *$G_{\mathbf{q}}^{3m} = (d+1)I$,*
(4) *$U_{G_{\mathbf{q}}}$ is order $3m$ as an element of $\mathrm{PU}(d)$,*
(5) *$\det G_{\mathbf{q}} = 1$ (respectively $\det G_{\mathbf{q}} = -1$) and $U_{G_{\mathbf{q}}}$ is unitary (respectively anti-unitary) if $N(u_f) = 1$ (respectively $N(u_f) = -1$).*
(6) *If $d \neq 3$ (mod 9) then $G_{\mathbf{q}}^m$ is conjugate to F_z.*
(7) *If $d = 3$ (mod 9) then $G_{\mathbf{q}}^m$ is conjugate to F_a if f_r/f is divisible by 3, and conjugate to F_z otherwise. In particular f_r is divisible by 3, so there is at least one orbit for which $G_{\mathbf{q}}^m$ is conjugate to F_a (namely the $f = 1$ orbit) and at least one orbit for which $G_{\mathbf{q}}^m$ is conjugate to F_z (namely the $f = f_r$ orbit).*

Remark 5.18. *Although we have not been able to prove that there are no other symmetries, in every known case $\langle G_{\mathbf{q}} \rangle$ is the full symmetry group.*

Proof. Let $L = \eta_Q(u_f)$. It follows from Theorem 3.8, Lemma 3.12 and Theorem 3.14 that $\mathrm{Tr}(L) = \mathrm{Tr}(u_f) > 0$, $L\beta = \beta$ and $A = L^{\pm 3m}$. It then follows from Lemma 5.1 that $\beta \in \mathcal{D}_L$ which in view of Theorem 5.4 means

$$\rho_{A,d}(\mathbf{p}, \beta) = \rho_{LAL^{-1},d}(\mathbf{p}, L\beta) = \begin{cases} \rho_{A,d}(L^{-1}\mathbf{p}, \beta) & N(u_f) = 1 \\ \left(\rho_{A,d}(-L^{-1}\mathbf{p}, \beta)\right)^* & N(u_f) = -1 \end{cases} \tag{5.82}$$

for all $\mathbf{p} \in \mathbb{Z}^2/d\mathbb{Z}^2$. It follows from Lemma 5.12, Corollary 5.7, the fact that

$$Q_A = Q_{L^{-1}AL} = \mathrm{Det}(L) L^{\mathrm{T}} Q_A L \tag{5.83}$$

that

$$\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) = \begin{cases} \nu_{A,\beta,L^{-1}\mathbf{q},d}(L^{-1}\mathbf{p}) & N(u_f) = 1 \\ \left(\nu_{A,\beta,L^{-1}\mathbf{q},d}(-L^{-1}\mathbf{p})\right)^* & N(u_f) = -1 \end{cases} \tag{5.84}$$

for $\mathbf{p}$. Since the ghost overlaps are real this implies

$$\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) = \nu_{A,\beta,L^{-1}\mathbf{q},d}\left(\mathrm{Det}(L)L^{-1}\mathbf{p}\right) \tag{5.85}$$

for all $\mathbf{p} \in \mathbb{Z}^2/d\mathbb{Z}^2$. The statement is trivial when $\mathbf{p} \in d\mathbb{Z}^2$. It follows that

$$\Pi_{A,\beta,\mathbf{0},d} = U^\dagger_{\mathrm{Det}(L)G}\Pi_{A,\beta,\mathbf{0},d}U_{\mathrm{Det}(L)G} \tag{5.86}$$

In view of Theorem 5.14 it follows that

$$\begin{aligned}\Pi_{A,\beta,\mathbf{q},d} &= U^\dagger_{M_\mathbf{q}}\Pi_{A,\beta,\mathbf{0},d}U_{M_\mathbf{q}} \\ &= U^\dagger_{M_\mathbf{q}}U^\dagger_{\mathrm{Det}(L)G}U_{M_\mathbf{q}}\Pi_{A,\beta,\mathbf{q},d}U^\dagger_{M_\mathbf{q}}U_{\mathrm{Det}(L)G}U_{M_\mathbf{q}} \\ &= U^\dagger_{G_\mathbf{q}}\Pi_{A,\beta,\mathbf{q},d}U_{G_\mathbf{q}}\end{aligned} \tag{5.87}$$

Statements (2) to (5) are straightforward consequences of Theorem 3.14. To prove (6) and (7), we know from (3) that $G^m_\mathbf{q}$ is order 3 (respectively 6) if d is odd (respectively even). Also

$$\begin{aligned}G^m_\mathbf{q} &= \begin{cases} M_\mathbf{q}G^mM^{-1}_\mathbf{q} & N(u_f)=1 \\ (-1)^m M_\mathbf{q}G^mM^{-1}_\mathbf{q} & N(u_f)=-1\end{cases} \\ &= M_\mathbf{q}\pi\big(\eta_Q(v^r)\big)M^{-1}_\mathbf{q}\end{aligned} \tag{5.88}$$

(where $\pi\colon\ \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$), implying $\mathrm{Tr}(G_\mathbf{q}) = \mathrm{Tr}(v^r) = d-1$. It follows from Theorem A.3 that if $d \neq 3$ or 6 mod 9 then $G^m_\mathbf{q}$ is conjugate to F_z, while it follows from Theorems A.3 and 3.9 that if $d=3$ mod 9 (respectively $d=6$ mod 9 then $G^m_\mathbf{q}$ is conjugate to F_a (respectively F'_a) if and only if $v^r \in 3\mathcal{O}_{\Delta_0,f}$m, and is otherwise conjugate to F_z. Suppose $d=3$ or 6 mod 9. Since

$$v^r = 1 + \frac{d_r - 3 - f_r\Delta_0}{2} + \frac{f_r}{f}\left(\frac{f\left(\Delta_0+\sqrt{\Delta_0}\right)}{2}\right) \tag{5.89}$$

$v^r \in 3\mathcal{O}_{\Delta_0,f}$ if and only if $f_r\Delta_0 = f_r/f = 0$ mod 4. If $d=6$ mod 9 then

$$f_r^2\Delta_0 = (d-3)(d+1) = 3 \mod 9 \tag{5.90}$$

implying f_r is coprime to 3. Consequently $v^r \notin 3\mathcal{O}_{\Delta_0,f}$ for any f. If, on the other hand, $d=3$ mod 9 then

$$f_r^2\Delta_0 = (d-3)(d+1) = 0 \mod 9 \tag{5.91}$$

implying $f_r = 0$ mod 3. Consequently $v^r \notin 3\mathcal{O}_{\Delta_0,f}$ if and only if f_r/f is divisible by 3. □

We now come to the phenomenon of *SIC alignment* [42, 43]. It is empirically observed, in many cases, that, up to a sign, the squares of the SIC overlap phases at position d_r in a dimension tower reappear among the SIC overlap phases at position d_{2r}. This phenomenon is a provable consequence of our conjecture. Moreover, it generalizes to a relation between the phases at positions d_r and d_{nr} in the tower, for any integer n coprime to 3.

Definition 5.19. *Define*

$$\tilde{\nu}_{A,\beta,\mathbf{q},d}(\mathbf{p}) = \begin{cases}\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) & \mathbf{p}=\mathbf{0} \textit{ mod } d, \\ \sqrt{d+1}\,\nu_{A,\beta,\mathbf{q},d}(\mathbf{p}) & \textit{otherwise}.\end{cases} \tag{5.92}$$

and

$$\mu_n(x) = \frac{1}{x} T_n^*(x) \tag{5.93}$$

It follows from Eq. (3.5) and Lemma 3.1 that $d_{nr} = \mu_n(d_r) d_r$, and that $\mu_n(d_r)$ is an integer if and only if n is coprime to 3.

Theorem 5.20. *Let $n \in \mathbb{N}$ be coprime to 3. Then for all $\mathbf{p} \neq \mathbf{0}$ mod d_r*

$$\tilde{\nu}_{A^n,\beta,\mathbf{0},d_{nr}}(\mu_n(d_r)\mathbf{p}) = (-1)^{n+1} \left(\tilde{\nu}_{A,\beta,\mathbf{0},d_r}(\mathbf{p})\right)^n \tag{5.94}$$

if d_r is odd, and

$$\tilde{\nu}_{A^n,\beta,\mathbf{0},d_{nr}}(\mu_n(d_r)\mathbf{p}) = \left(\tilde{\nu}_{A,\beta,\mathbf{0},d_r}(\mathbf{p})\right)^n \times \begin{cases} 1 & n = \pm 1 \text{ mod } 12 \\ (-1)^{(1+p_1)(1+p_2)} & n = \pm 2 \text{ mod } 12 \\ -1 & n = \pm 4 \text{ mod } 12 \\ -(-1)^{(1+p_1)(1+p_2)} & n = \pm 5 \text{ mod } 12 \end{cases} \tag{5.95}$$

if d_r is even.

Proof. Eq. (5.18) implies

$$\begin{aligned}
\rho_{A^n,d_{nr}}\left(\mu_n(d_r)\mathbf{p},\beta\right) &= \lim_{\tau\in\mathbb{H}\to\beta}\left(\frac{\varpi\left(\frac{\mu_n(d_r)\langle\langle\mathbf{p},A^n\tau\rangle\rangle}{d_{nr}}, A^n\tau\right)}{\varpi\left(\frac{\mu_n(d_r)\langle\langle\mathbf{p},\tau\rangle\rangle}{d_{nr}},\tau\right)}\right) \\
&= \lim_{\tau\in\mathbb{H}\to\beta}\left(\frac{\varpi\left(\frac{\langle\langle\mathbf{p},A^n\tau\rangle\rangle}{d_r}, A^n\tau\right)}{\varpi\left(\frac{\langle\langle\mathbf{p},A^{n-1}\tau\rangle\rangle}{d_r}, A^{n-1}\tau\right)} \times \frac{\varpi\left(\frac{\langle\langle\mathbf{p},A^{n-1}\tau\rangle\rangle}{d_r}, A^{n-1}\tau\right)}{\varpi\left(\frac{\langle\langle\mathbf{p},A^{n-2}\tau\rangle\rangle}{d_r}, A^{n-2}\tau\right)} \times \dots\right. \\
&\qquad\qquad \left.\dots \times \frac{\varpi\left(\frac{\langle\langle\mathbf{p},A\tau\rangle\rangle}{d_r}, A\tau\right)}{\varpi\left(\frac{\langle\langle\mathbf{p},\tau\rangle\rangle}{d_r},\tau\right)}\right) \\
&= \left(\rho_{A,d_r}\left(\mathbf{p},\beta\right)\right)^n.
\end{aligned} \tag{5.96}$$

Theorem 5.8 implies

$$e^{\frac{\pi i}{4}\phi(A^n)} = \left(e^{\frac{\pi i}{4}\phi(A)}\right)^n. \tag{5.97}$$

Proposition 5.9 and Eq. (3.6) imply

$$\xi_{d_{nr}}^{-\mu_n^2(d_r)\mathbf{p}^{\mathrm{T}}.Q_{A^n}.\mathbf{p}} = e^{-\left(\frac{\pi i}{d_r}\right)(d_{nr}+1)\mu_n(d_r)U_n^*(d_r)\left(\frac{tf_r}{f}\right)\mathbf{p}^{\mathrm{T}}.Q.\mathbf{p}} \tag{5.98}$$

where $t = \pm 1$. Suppose d_r is odd. Then it follows from Lemma 3.1 that d_{nr} and $\mu_n(d_r) = d_{nr}/d_r$ are both odd and

$$(d_{nr}+1)\mu_n(d_r)U_n^*(d_r) = n(d_{nr}+1) = n(d_r+1) \qquad \text{mod } 2d_r \tag{5.99}$$

implying

$$\xi_{d_{nr}}^{-\mu_n^2(d_r)\mathbf{p}^{\mathrm{T}}.Q_{A^n}.\mathbf{p}} = e^{-\left(\frac{\pi i}{d_r}\right)n(d_r+1)\left(\frac{tf_r}{f}\right)\mathbf{p}^{\mathrm{T}}.Q.\mathbf{p}} = \left(\xi_{d_r}^{-\mathbf{p}^{\mathrm{T}}.Q_A.\mathbf{p}}\right)^n. \tag{5.100}$$

Hence

$$\tilde{\nu}_{A^n,\beta,\mathbf{0},d_{nr}}(\mu_n(d_r)\mathbf{p}) = -\left(e^{\frac{\pi i}{4}\phi(A)}\xi_{d_r}^{-\mathbf{p}^{\mathrm{T}}.Q_A.\mathbf{p}}\rho_{A,d_r}\left(\mathbf{p},\beta\right)\right)^n = (-1)^{n+1}\left(\tilde{\nu}_{A,\beta,\mathbf{0},d_r}(\mathbf{p})\right)^n \tag{5.101}$$

Suppose, on the other hand, that d_r is even. Then it follows from Lemma 3.1 that

$$(d_{nr}+1)\mu_n(d_r)U_n^*(d_r) = \begin{cases} (1+nd_r)\left(n+\frac{n(n-1)}{6}d_r\right)\left(1+\frac{n-1}{3}d_r\right) \mod 2d_r, & n=1 \bmod 3, \\ (1-nd_r)\left(-n+\frac{n(n+1)}{6}d_r\right)\left(-1+\frac{n+1}{3}d_r\right) \mod 2d_r, & n=2 \bmod 3, \end{cases}$$
$$= \begin{cases} n(1+d_r)+\frac{3n(n-1)}{2}d_r \mod 2d_r, & n=1 \bmod 3, \\ n(1+d_r)-\frac{3n(n+1)}{2}d_r \mod 2d_r, & n=2 \bmod 3, \end{cases} \tag{5.102}$$

while it follows from Lemma 5.10 that

$$\frac{f_r}{f}Q = \langle 1,1,1\rangle \mod 2. \tag{5.103}$$

Consequently

$$\xi_{d_{nr}}^{-\mu_n^2(d_r)\mathbf{p}^{\mathrm{T}}.Q_{A^n}.\mathbf{p}} = \begin{cases} e^{-\frac{n\pi i}{d_r}(d_r+1)\left(\frac{tf_r}{f}\right)\mathbf{p}^{\mathrm{T}}.Q.\mathbf{p}}(-1)^{\frac{n(n-1)}{2}(p_1^2+p_1p_2+p_2^2)} & n=1 \bmod 3 \\ e^{-\frac{n\pi i}{d_r}(d_r+1)\left(\frac{tf_r}{f}\right)\mathbf{p}^{\mathrm{T}}.Q.\mathbf{p}}(-1)^{\frac{n(n+1)}{2}(p_1^2+p_1p_2+p_2^2)} & n=2 \bmod 3 \end{cases}$$
$$= \left(\xi_{d_r}^{-\mathbf{p}^{\mathrm{T}}.Q_A.\mathbf{p}}\right)^n \times \begin{cases} 1 & n=\pm 1 \text{ or } \pm 4 \bmod 12 \\ (-1)^{p_1^2+p_1p_1+p_2^2} & n=\pm 2 \text{ or } \pm 5 \bmod 12 \end{cases} \tag{5.104}$$

Also, it follows from Lemma 5.10 that $\mu_n(d_r)=n \bmod 2$. Hence

$$\tilde{\nu}_{A^n,\beta,\mathbf{0},d_{nr}}(\mu_n(d_r)\mathbf{p}) = (-1)^{(1+\mu_n(d_r)p_1)(1+\mu_n(d_r)p_2)}\left(e^{\frac{\pi i}{4}A}\xi_{d_r}^{-\mathbf{p}^{\mathrm{T}}.Q_A.\mathbf{p}}\rho_{A,d_r}(\mathbf{p},\beta)\right)^n$$
$$\times \begin{cases} 1 & n=\pm 1 \text{ or } \pm 4 \bmod 12 \\ (-1)^{p_1^2+p_1p_1+p_2^2} & n=\pm 2 \text{ or } \pm 5 \bmod 12 \end{cases}$$
$$= (-1)^{(1+np_1)(1+np_2)+ns_{\mathbf{0}}(\mathbf{p})}\left(\tilde{\nu}_{A,\beta,\mathbf{0},d_r}(\mathbf{p})\right)^n$$
$$\times \begin{cases} 1 & n=\pm 1 \text{ or } \pm 4 \bmod 12 \\ (-1)^{p_1^2+p_1p_1+p_2^2} & n=\pm 2 \text{ or } \pm 5 \bmod 12 \end{cases}$$
$$= \left(\tilde{\nu}_{A,\beta,\mathbf{0},d_r}(\mathbf{p})\right)^n \times \begin{cases} 1 & n=\pm 1 \bmod 12 \\ (-1)^{(1+p_1)(1+p_2)} & n=\pm 2 \bmod 12 \\ -1 & n=\pm 4 \bmod 12 \\ -(-1)^{(1+p_1)(1+p_2)} & n=\pm 5 \bmod 12 \end{cases} \tag{5.105}$$

□

6. Number theory implications (Gene)

- Class fields generated by MEFFs under our conjectures.
- Density of real quadratic fields with an odd trace unit.

7. Necromancy and numerical evidence

Suppose one wishes to use the preceding conjectures for constructing ghosts to compute an *explicit* SIC fiducial vector, either exactly or just a numerical approximation. We call any procedure for doing so *necromancy*, so-named because it "reanimates" the ghost fiducial as a SIC fiducial.

There is a straightforward brute-force algorithm to achieve necromancy. Using Eq. **??**STF: INCLUDE REF, we first compute a ghost fiducial to arbitrary precision. From this numerical approximation, powerful tools such as the Lenstra–Lenstra–Lovász (LLL) lattice basis reduction algorithm can round this into a candidate number field. A conjecture for this specific number field is provided by the existing conjectures [18, 38, 40]. Then a Galois automorphism that flips the sign of $\sqrt{\Delta_0}$ can be applied to compute the SIC fiducial. If the conjectures are correct, then with enough starting precision and patience for the lattice basis reduction algorithm, the result should be an exact expression for a SIC fiducial. This exact expression can then be evaluated to any precision that one likes for numerical approximation.

Unfortunately this approach is impractical for two reasons. The first difficulty is that the relevant number field typically has very high degree, and the precision required to round into this field would therefore be impractically large for even storing a SIC fiducial for modestly large d. Second, the convergence of LLL or similar algorithms is far too slow in practice with such high-degree number fields to allow a direct computation of a ghost in this fashion.

To circumvent these difficulties, we now describe an alternative heuristic approach to necromancy that avoids the complexity bottleneck by working entirely in the (typically much lower degree) Hilbert class field. One still requires very high precision if the dimension is at all large. For example, we required $\sim 10^5$ digit precision in dimension $d = 100$ to implement this approach in detail. However, this is not impractically large for a modern laptop. The price for this reduced complexity is that the method itself is more involved and it remains at this time merely a heuristic, even assuming our conjectures.

Let us sketch how our approach to necromancy works. We begin by computing a numerical estimate of a ghost fiducial. This can be done using the integral representation of the double sine function [19,22]. It is not practical to use this method to achieve the high numerical precision required by the subsequent steps, so we next amplify the precision of our initial numerical approximation using Newton's method.

STF: move this paragraph later... A second problem is that the method described in the last paragraph takes a given Galois orbit of ghost overlaps, and produces a corresponding set of candidate SIC overlaps. Deciding which of the candidates are the overlaps for an actual SIC requires further calculation. The problem is that the number of candidates that need to be checked grows rapidly with the number of distinct Galois orbits. The key to circumventing this problem is the fact already mentioned, that while the number of distinct overlaps is $\sim d^2$, it only needs $\sim 2d$ real numbers to specify a SIC fiducial vector. This means that the ghost overlaps on the maximal Galois orbit should provide enough information to reconstruct the SIC fiducial, using the method described below.

We give a detailed description of the procedure for calculating a SIC, and then describe the modifications needed for an MEFF. The main steps in the procedure are as follows:

- 7.1 Calculating the Shintani-Faddeev Modular Form
- 7.2 Precision bumping
- fix . . . (items to be added as writing progresses)
- **STF:** FIX Modifications for a MEFF

7.1. **Calculating the Shintani-Faddeev Modular Form.** Eq. (5.51) expresses a ghost overlap in terms of the function **DMA:** I am using my previous notation because currently there are some minor differences between my definition of $\rho_{A,d}\left(\mathbf{p},\beta\right)$ and Gene's definition of $\mathfrak{w}_{\mathbf{p}}^{A}(\beta)$ which need to be settled. $\rho_{A,d}\left(\mathbf{p},\tau\right)$. This in turn is expressed in terms of the function $\sigma_A(z,\tau)$ via Eq. (5.16). For $\tau \in \mathbb{H}$ the latter is given by a ratio of QPochhammer symbols, via Eq. (5.4). However, we are need its values for $\tau \in \mathbb{R}$. Although these can be obtained by taking a limit, it is numerically more efficient to calculate them directly, via an integral representation, using a procedure we now describe.

STF: I wonder if the subsequent discussion of the integral representation should be collected in a little lemma? Generally, I think we should have formal statements if we can rather than prose. (Happy to hear counterarguments, though.)

Let $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary. If $\ell = 0$ then either $L = T^k$, in which case $\sigma_L(z, \tau) = 1$ for all z, τ, or $L = -T^k$, in which case the function is singular for all $\tau \in \mathbb{R}$. If $\ell < 0$ we can use Eq. (5.7) to express σ_L in terms of $\sigma_{L^{-1}}$. It is therefore sufficient to give a procedure for calculating the function when $\ell > 0$. For this we use Theorem B.4 to deduce the existence of a sequence of integers $r_1, \dots, r_{n+1}$ such that

$$L = T^{r_1} S \dots S T^{r_{n+1}} \tag{7.1}$$

where $r_j \geq 2$ for $1 < j < n + 1$. The theorem also shows that if $L_j = T^j S \dots S T^{n+1}$ then $\mathcal{D}_L = \mathcal{D}_{L_1} \subset \mathcal{D}_{L_2} \dots \subset \mathcal{D}_{L_{n+1}}$. Consequently

$$\sigma_L(z, \tau) = \sigma_{T^{r_1} S}\left(\frac{z}{j_{L_2}(\tau)}, L_2\tau\right) \dots \sigma_{T^{r_n} S}\left(\frac{z}{j_{L_{n+1}}(\tau)}, L_{n+1}\tau\right) \sigma_{L_{n+1}}(z, \tau) \tag{7.2}$$

for all $\tau \in \mathcal{D}_L$. Using

$$\sigma_{L_{n+1}}(\tau, z) = \sigma_{T^{r_{n+1}}}(z, \tau) = 1 \tag{7.3}$$

and

$$\sigma_{T^m S}(z, \tau) = \sigma_{T^m}\left(\frac{z}{j_S(\tau)}, S\tau\right) \sigma_S(z, \tau) = \sigma_S(z, \tau) \tag{7.4}$$

this becomes

$$\sigma_L(z, \tau) = \sigma_S\left(\frac{z}{j_{L_2}(\tau)}, L_2\tau\right) \dots \sigma_S\left(\frac{z}{j_{L_{n+1}}(\tau)}, L_{n+1}\tau\right). \tag{7.5}$$

for all $\tau \in \mathcal{D}_L$. The problem thus reduces to calculating $\sigma_S(z, \tau)$. This can be done using the formula [19, 22]

$$\sigma_S(z, \tau) = e^{\frac{\pi i}{12\tau}\left(6z^2 + 6(1-\tau)z + \tau^2 - 3\tau + 1\right)} \operatorname{Sin}_2(z + 1, \tau) \tag{7.6}$$

where $\operatorname{Sin}_2(z + 1, \tau)$ is the double sine function. Note that the double sine function as usually defined has three arguments; we are employing the shorthand [19] $\operatorname{Sin}_2(z, \tau, 1) = \operatorname{Sin}_2(z, \tau)$. We can calculate $\operatorname{Sin}_2(z + 1, \tau)$ explicitly using the integral representation [19, 25],

$$\operatorname{Sin}_2(z + 1, \tau) = \exp\left(\int_0^\infty \left(\frac{\sinh\left(\frac{\tau - 1 - 2z}{2}\right) t}{2 \sinh\left(\frac{t}{2}\right) \sinh\left(\frac{\tau t}{2}\right)} - \frac{\tau - 1 - 2z}{\tau t}\right) \frac{dt}{t}\right) \tag{7.7}$$

valid for $\mathrm{Re}(\tau) > 0$ and $-1 < \mathrm{Re}(z) < \mathrm{Re}(\tau)$. To use this in Eq. (7.5) one needs, firstly, that $\mathrm{Re}(L_r\tau) > 0$ for $r = 2, \dots, n + 1$. A sufficient condition for that to be true is that $\mathrm{Re}(j_L(\tau)) > 0$. In particular, it is true for the case that interests us, $\tau \in \mathcal{D}_L \cap \mathbb{R}$. Indeed, let $\tau = x + iy$ with x, $y \in \mathbb{R}$. Then, in the notation of Theorem B.4, $\mathrm{Re}(j_L(\tau)) > 0$ implies $x + m_r/l_r > 0$ for $r = 2,,$

$n+1$ and, consequently,

$$\operatorname{Re}\left(L_r\tau\right) = \begin{cases} \frac{\ell_r}{\ell_{r+1}}\left(\frac{\left(x+\frac{m_r}{\ell_r}\right)\left(x+\frac{m_{r+1}}{\ell_{r+1}}\right)+y^2}{\left(x+\frac{m_{r+1}}{\ell_{r+1}}\right)^2+y^2}\right) & r=2,\dots,n \\ \ell_{n+1}\left(x+\frac{m_{n+1}}{\ell_{n+1}}\right) & r=n+1 \end{cases}$$
$$> 0 \tag{7.8}$$

To deal with the problem that $\operatorname{Re}\left(z/j_{L_r}(\tau)\right)$ may not be in the required interval we may use the fact

$$\sigma_S(z+m_1\tau+m_2,\tau)\,\frac{\varpi_{m_1}(z,\tau)}{\varpi_{-m_2}\left(\frac{z}{\tau},-\frac{1}{\tau}\right)}\sigma_Z((,z),\tau) \tag{7.9}$$

for all m_1, $m_2 \in \mathbb{Z}$ and all $\tau \in \mathcal{D}_S$.

If we can calculate one SIC on a given extended Clifford group orbit we can easily calculate all the others by applying the appropriate unitary or anti-unitary. To make the most efficient use of available resources one should choose the matrix A and associated quadratic form Q in such a way as to minimise the length of the expansion in Eq. (7.1). It follows from Theorem B.7 that to do this we need to choose Q to be HJ-reduced and such that the HJ-continued fraction expansion of $\beta_{Q,+}$ has minimal period. The HJ-reduced continued fraction then tells us the integers $r_1, \dots, r_{n+1}$ in the expansion in Eq. (7.1) (see Appendix B'for the definitions and a summary of the relevant properties of HJ-reduced forms and HJ-continued fractions). Appropriate choices of Q and the corresponding integers $r_1, \dots, r_{n+1}$ are tabulated in Appendix D, for $d=4$ to 50.

2. Precision Bumping. The method described below does not require us to calculate the full set of d^2 ghost overlaps. It does, however, require us to calculate an $\sim d^2$ subset. Moreover to convert this subset into a corresponding subset of SIC overlaps requires the ghost overlaps to be calculated to very high precision (10^5 digit precision in dimension 100). Doing this using Eq. (7.7) would be prohibitively slow. We therefore use the following alternative method, which relies on the fact that [52]

$$s|\psi\rangle\langle\psi|U_{\mathrm{P}} = \frac{1}{d}\sum_{p_1,p_2=0}^{d-1}\nu_{\mathbf{p}}D_{\mathbf{p}} \tag{7.10}$$

where $s=\pm1$, $|\psi\rangle$ is the ghost fiducial vector, U_{P} is the parity operator and $\nu_{\mathbf{p}}$ are the ghost overlaps. The simplest case is when none of the components of $|\psi\rangle$ is zero. In that case, in order to calculate the $2d$ real numbers determining the vector $|\psi\rangle$ one only needs to know $2d$ of the numbers $\nu_{\mathbf{p}}$. Specifically, let

$$\chi_{0,j} = \frac{1}{d}\sum_{k=0}^{d-1}\omega^{jk}\nu_{0,k}, \qquad\qquad \chi_{1,j} = \frac{1}{d}\sum_{k=0}^{d-1}\tilde{\omega}^k\omega^{jk}\nu_{1,k}. \tag{7.11}$$

The vector $|\psi\rangle$ is only determined up to an arbitrary phase. We may therefore assume, without loss of generality, that $\langle 0|\psi\rangle$ is positive real. We then have

$$\langle j|\psi\rangle = \begin{cases}\sqrt{|\chi_{0,0}|} & j=0, \\ \sqrt{|\chi_{0,0}|}\prod_{k=0}^{j-1}\left(\frac{\chi_{1,k}}{\chi_{0,k}}\right) & j>0.\end{cases} \tag{7.12}$$

Our strategy is therefore to calculate low precision approximations to the $2d$ numbers $\nu_{0,k}$, $\nu_{1,k}$ using the integral representation and then use these to calculate a low precision approximation to the vector

$|\psi\rangle$. We then apply Newton's method to the system of equations

$$\langle\psi|U_{\mathrm{P}} D_{\mathbf{p}}|\psi\rangle\langle\psi|U_{\mathrm{P}} D_{-\mathbf{p}}|\psi\rangle = \frac{d\delta_{\mathbf{p},\mathbf{0}}+1}{d+1} \tag{7.13}$$

to calculate a high precision approximation to $|\psi\rangle$ which in turn can be used to calculate high precision approximations to the numbers ν_{p}. If it should happen that one or more of the components of $|\psi\rangle$ is zero then it may be necessary to calculate more than $2d$ low precision ghost overlaps. However in no case is it necessary to calculate the numerical integral to high precision.

4. Constructing the Ghost Invariants. We next describe our method for constructing a set of numbers in the Hilbert class field which fully specify the ghost overlaps on a maximal Galois orbit.

The method relies on the fact that if E is the field generated by the ghost overlaps, and if H is the Hilbert class field, then there is an isomorphism of $\mathrm{Gal}(E/H)$ onto $\mathcal{M}/\mathcal{S}$, where $\mathcal{S}$ is the symmetry group (i.e. the set of $G \in \mathrm{GL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ such that $\nu_{G\mathbf{p}} = \nu_{\mathbf{p}}$ for all $\mathbf{p}$) and $\mathcal{M}$ is a maximal Abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ containing $\mathcal{S}$. If $g \in \mathrm{Gal}(E/H)$ and $F_g\mathcal{S}$ is the corresponding element of $\mathrm{GL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ then

$$g(\nu_{\mathbf{p}}) = \nu_{F_g\mathbf{p}}. \tag{7.14}$$

for all $\mathbf{p}$. This isomorphism was originally noted empirically, by studying the known examples [15, 37]. It can be shown to be a consequence of our main conjecture together with the Stark conjectures [**?**, 18, 19, 40, 53]. For a type z orbit there is only one maximal subgroup containing $\mathcal{S}$ (namely, the centralizer of $\mathcal{S}$). For the characterization of $\mathcal{M}$ in the case of a type a orbit see ref. [**?**]. Using this isomorphism one can calculate the action of $\mathrm{Gal}(E/H)$ on the ghost overlaps, without knowing them exactly.

Let $\nu_{\mathbf{p}_1}, \dots, \nu_{\mathbf{p}_n}$ be a maximal orbit of ghost overlaps under the action of $\mathrm{Gal}(E/H)$. It follows from the Stark Conjectures (see Subsection 2.4) that each element of the orbit generates the full field E, and therefore is not stabilized by a non-identity element of $\mathrm{Gal}(E/H)$. Choose $L_1, \dots, L_m \in \mathcal{M}$ such that

(1) Each $L_j\mathcal{S}$ is order $n_j = q_j^{r_j}$ in $\mathcal{M}/\mathcal{S}$ with q_j prime and r_j a positive integer greater than 1,
(2) $\mathcal{M}/\mathcal{S}$ is isomorphic to the direct product $\langle L_1\mathcal{S}\rangle \times \cdots \times \langle L_m\mathcal{S}\rangle$.

Let g_j be the element of $\mathrm{Gal}(E/H)$ such that $g_j(\nu_{\mathbf{p}}) = \nu_{L_j\mathbf{p}}$ for all $\mathbf{p}$, and let $E_j = \{\eta \in E\colon g_k(\eta) = \eta \text{ for all } k \neq j\}$. Finally choose some fixed element of the orbit, say $\nu_{\mathbf{p}_1}$, and define

$$\nu_{s_1,\dots,s_m} = \nu_{L_1^{s_1}\dots L_m^{s_m}\mathbf{p}_1}. \tag{7.15}$$

Then the map $(s_1, \dots, s_m) \to \nu_{s_1,\dots,s_m}$ is a bijective correspondence between $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_m\mathbb{Z}$ and the orbit. If follows that if we define

$$K_{j,t} = \{\nu_{s_1,\dots,s_m} \colon s_j = t\} \tag{7.16}$$

then

$$\bigcap_{j=1}^{m} K_{j,s_j} = \{\nu_{s_1,\dots,s_m}\}. \tag{7.17}$$

Let

$$P_{j,t}(x) = \prod_{\nu \in K_{j,t}} (x-\nu) = c_{j,t,0} + c_{j,t,1}x + \cdots + c_{j,t,l_j-1}x^{l_j-1} + x^{l_j} \tag{7.18}$$

where $l_j = n/n_j$. Then the coefficients of $c_{j,t,s}$ are all in E_j. So one approach to the problem of calculating the SIC overlaps would be to calculate exact versions of the coefficients of $c_{j,t,s}$ using an integer-relation algorithm, transform them using a $\sqrt{\Delta_0}$-sign switching automorphism, find the roots of the transformed polynomial, and then find the SIC overlaps using the transformed version of Eq. (7.17). Provided $m > 1$ this method would be more efficient than using an integer relation algorithm to calculate exact expressions for the $\nu_{\mathbf{p}}$ since E_j is then lower degree than E. However, we can do better than that.

Our method relies on the fact

Lemma 7.1. *For each j there exists at least one index l such that the numbers $\{c_{j,t,l} \colon t = 0, 1, \dots, n_j - 1\}$ are distinct non-zero. If l is such an index then $E_j = H(c_{j,t,l})$ for all t.*

Proof. Suppose there were no such index l. Since $\mathrm{Gal}(E_j/H) = \langle g_j \rangle$ is cyclic order n_j, and since $c_{j,t,l} = g_j^t(c_{j,0,l})$, it would follow that there existed a positive integer $r < n_j$ such that

$$g_j^r(c_{j,t,l}) = c_{j,t,l} \tag{7.19}$$

for all t, l. This in turn would mean $K_{j,t+r} = K_{j,t}$ for all t, contradicting the fact that these sets are disjoint.

To prove the second statement suppose $H(c_{j,t,l})$ were a proper subfield of E_j. Then it would be fixed by g_j^r, for some positive integer $r < n_j$, which in turn would mean $c_{j,t+r,l} = c_{j,t,l}$ for all t, contradicting the fact that the these coefficients are distinct. □

In view of Lemma 7.1 we can choose, for each j, an index $l_{j,0}$ such that $E = H(c_{j,t,l_{j,0}}$ for all t. Suppose $l \neq l_{j,0}$. Then

$$c_{j,0,l} = \sum_{r=0}^{n_j-1} a_{j,r,l} c^r_{j,0,l_{j,0}} \tag{7.20}$$

for some $a_{j,r,l} \in H$. Repeatedly applying g_j to both sides it follows

$$\begin{pmatrix} c_{j,0,l} \\ \vdots \\ c_{j,n_j-1,l} \end{pmatrix} = V_j \begin{pmatrix} a_{j,0,l} \\ \vdots \\ a_{j,n_j-1,l} \end{pmatrix} \tag{7.21}$$

where V_j is the Vandermonde matrix

$$V_j = \begin{pmatrix} 1 & c_{j,0,l_{j,0}} & \dots & c^{n_j-1}_{j,0,l_{j,0}} \\ \vdots & \vdots & & \vdots \\ 1 & c_{j,n_j-1,l_{j,0}} & \dots & c^{n_j-1}_{j,n_j-1,l_{j,0}} \end{pmatrix} \tag{7.22}$$

In the case $l = l_{j,0}$ we have

$$c_{j,1,l_{j,0}} = \sum_{r=0}^{n_j-1} b_{j,r} c^r_{j,0,l_{j,0}} \tag{7.23}$$

for some $b_{j,r} \in H$, from which it follows that

$$\begin{pmatrix} c_{j,1,l_{j,0}} \\ \vdots \\ c_{j,n_j-1,l_{j,0}} \\ c_{j,0,l_{j,0}} \end{pmatrix} = V_j \begin{pmatrix} b_{j,0} \\ \vdots \\ b_{j,n_j-1} \end{pmatrix} \tag{7.24}$$

Another way of phrasing Eq. (7.24) is to say

$$g_j\left(c_{j,t,l_{j,0}}\right) = Q_j\left(c_{j,t,l_{j,0}}\right) \tag{7.25}$$

for all t, where

$$Q_j(x) = \sum_{u=0}^{n_j-1} b_{j,u} x^u \tag{7.26}$$

Following the terminology of ref. [**?**] we refer to the $Q_j(x)$ as *Galois polynomials*.

Numerical approximations to the numbers $a_{j,t,l}$, $b_{j,t}$ can be obtained from Eqs. (7.21), (7.24) by inverting the matrices V_j. In this connection it should be noted that standard algorithms for matrix inversion as applied to the Vandermonde matrix are numerically inefficient due to the problem typically being ill-conditioned [54, 55]. A variety of alternative algorithms have been described which surmount this problem and which are also faster than standard algorithms (complexity $O(n_j^2)$ as opposed to $O(n_j^3)$): see refs. [56, 57] and references cited therein.

Finally, define numbers $e_{j,t}$ by

$$e_{j,0} + e_{j,1}x + \dots + e_{j,n_j-1}x^{n_j-1} + x^{n_j} = \prod_{t=0}^{n_j-1} (x - c_{j,t,l_{j,0}}) \tag{7.27}$$

We refer to the numbers $a_{j,t,l}$, $b_{j,t}$, $e_{j,t}$ as *ghost invariants*. They are all in the Hilbert class field H, and exact versions can be calculated using an integer relation algorithm starting from high precision numerical approximations.

5. Constructing the SIC overlaps. Now let g be any automorphism which switches the sign of $\sqrt{\Delta_0}$. For the sake of simplicity assume $g(\omega) = \omega$ (although it is straightforward to construct a modified version of the argument which works when this condition is not satisfied). Define

$$\tilde{\Pi} = g(\Pi), \qquad \tilde{\nu}_{\mathbf{p}} = g(\nu_{\mathbf{p}}), \qquad \tilde{\nu}_{s_1,\dots,s_m} = g(\nu_{s_1,\dots,s_m}) \tag{7.28}$$

where Π is the ghost projector with which we started. Then $\tilde{\Pi}$ is a SIC projector, $\tilde{\nu}_{\mathbf{p}} = d^{-1}\operatorname{Tr}(\tilde{\Pi} D^\dagger_{\mathbf{p}})$ are the corresponding SIC overlaps, and $\tilde{\nu}_{s_1,\dots,s_m} = \tilde{\nu}_{L_1^{s_1}\dots L_m^{s_m}\mathbf{p}_1}$. Since the ghost invariants are all in H it is straightforward to calculate

$$\tilde{a}_{j,t,l} = g(a_{j,t,l}), \qquad \tilde{b}_{j,t} = g(b_{j,t}), \qquad \tilde{e}_{j,t} = g(e_{j,t}) \tag{7.29}$$

Also define

$$\tilde{c}_{j,t,l} = g(c_{j,t,l}) \tag{7.30}$$

Since we do not have exact expressions for the $c_{j,t,l}$ we cannot calculate the $\tilde{c}_{j,t,l}$ directly. However, we can calculate them indirectly using the quantities $\tilde{a}_{j,t,l}$, $\tilde{b}_{j,t}$, $\tilde{e}j,t,l$. Define

$$\tilde{g}_j = g g_j g^{-1}, \tag{7.31}$$

and,

$$\tilde{Q}_j(x) = \sum_{u=0}^{n_j-1} \tilde{b}_{j,u} x^u. \tag{7.32}$$

Then

$$\tilde{g}_j(\tilde{\nu}_{\mathbf{p}}) = \tilde{\nu}_{L_j \mathbf{p}}, \tag{7.33}$$

and

$$\tilde{g}_j\big(\tilde{c}_{j,t,l_{j,0}}\big) = \tilde{Q}_j\big(\tilde{c}_{j,t,l_{j,0}}\big) \tag{7.34}$$

It follows that, if we take $\tilde{c}_{j,0,l_{j,0}}$ to be any root of the equation

$$\sum_{t=0}^{n_j-1} \tilde{e}_{j,t} x^t = 0, \tag{7.35}$$

and if we define $\tilde{c}_{j,t,l_{j,0}}$ for $t > 0$ recursively by

$$\tilde{c}_{j,t+1,l_{j,0}} = \tilde{Q}_j\big(\tilde{c}_{j,t,l_{j,0}}\big) \tag{7.36}$$

then

$$\tilde{c}_{j,t,l_{j,0}} = \tilde{c}_{j,t+r_j,l_{j,0}} \tag{7.37}$$

for all t and some unknown constant r_j. It follows that if we define

$$\tilde{V}'_j = \begin{pmatrix} 1 & \tilde{c}_{j,0,l_{j,0}} & \dots & \tilde{c}^{n_j-1}_{j,0,l_{j,0}} \\ \vdots & \vdots & & \vdots \\ 1 & \tilde{c}_{j,n_j-1,l_{j,0}} & \dots & \tilde{c}^{n_j-1}_{j,n_j-1,l_{j,0}} \end{pmatrix} \tag{7.38}$$

and

$$\begin{pmatrix} \tilde{c}_{j,0,l} \\ \vdots \\ \tilde{c}_{j,n_j-1,l} \end{pmatrix} = \tilde{V}'_j \begin{pmatrix} a_{j,0,l} \\ \vdots \\ a_{j,n_j-1,l} \end{pmatrix} \tag{7.39}$$

for $l \neq l_{j,0}$, then

$$\tilde{c}_{j,t,l} = \tilde{c}_{j,t+r_j,l} \tag{7.40}$$

for all j, t, l. Now let $\tilde{K}'_{j,t}$ be the set of roots of the equation

$$\sum_{l=0}^{n_j} \tilde{c}_{j,t,l} x^l = 0, \tag{7.41}$$

and set $\tilde{\nu}'_{s_1,\dots,s_m}$ equal to the unique element in the set

$$\bigcap_{j=1}^{m} \tilde{K}'_{j,s_j}. \tag{7.42}$$

Finally, for each $\mathbf{p}_j$ define

$$\tilde{\nu}'_{\mathbf{p}_j} = \tilde{\nu}'_{s_1,\dots,s_m} \tag{7.43}$$

where $s_1, \dots, s_n$ is the unique set of integers in the range $0 \le s_j < n_j$ such that $\mathbf{p}_j = L_1^{s_1} \dots L_m^{s_m} \mathbf{p}_1 \mod \mathcal{S}\mathbf{p}_j$ (c.f. Eqs. (7.15) and (7.17)). Then

$$\tilde{\nu}_{\mathbf{p}} = \tilde{\nu}'_{L\mathbf{p}} \tag{7.44}$$

where $L = L^{-r_1} \dots L^{-r_m}$ is an unknown element of $\mathcal{M}/\mathcal{S}$, which can be found by trial-and-error.

There are two ways in which we can reduce the size of the search space. In the first place, if we only want *some* SIC on the same $\mathrm{EC}(d)$ orbit as Π, not necessarily Π itself, we can use the fact that if $L'L^{-1} \in \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ then $\{\tilde{\nu}'_{L'\mathbf{p}}\}$ is a set of SIC overlaps if and only if $\{\tilde{\nu}'_{L\mathbf{p}}\}$ is. It is consequently only necessary to test one element from each coset $\mathcal{M}/\big(\mathcal{M} \cap \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$.

The search space can be further reduced using the following condition. Let $\mathcal{P}$ be the set $\{M\mathbf{p}_j \colon j = 1, \dots, n \text{ and } M \in \mathcal{S}\}$ reduced mod d and with duplicates removed. Let

$$B = \frac{1}{d} \sum_{\mathbf{p} \in \mathcal{P}} \tilde{\nu}_{\mathbf{p}} D_{\mathbf{p}} \tag{7.45}$$

The fact that $-I \in \mathcal{M}$ means B is Hermitian. Let $\lambda_{\max}$ be its largest eigenvalue. Then

$$\lambda_{\max} \ge \mathrm{Tr}(B\Pi) = \mathrm{Tr}(B^2) = \frac{|\mathcal{P}|}{d(d+1)} \tag{7.46}$$

where $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$. We may therefore remove from consideration any set of candidate overlaps which do not satisfy this requirement.

6. Constructing the SIC fiducial.

(**STF:** Describe the method using low-rank matrix completion to find a rank-one vector compatible with the overlaps.)

8. Conclusion

- This is a new algorithm for (conjecturally) computing certain special values of zeta functions. Bootstrap from a Shintani method and then amplify using the geometry of the lines.
-

Appendix A. Canonical order 3 matrices

[11] For any $d \ge 2$, even or odd, let

$$\bar{F}_z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \bar{F}_a = \begin{pmatrix} 1 & 3 \\ \frac{d-3}{3} & -2 \end{pmatrix}, \qquad \bar{F}'_a = \begin{pmatrix} 1 & 3 \\ \frac{2d-3}{3} & -2 \end{pmatrix}. \tag{A.1}$$

Then Bos and Waldron [41] prove that the set of matrices of order 3 and trace -1 in $\mathrm{SL}_2(\mathbb{Z}/d\mathbb{Z})$ consist of

(1) The single conjugacy class $[\bar{F}_z]$ if $d \ne 0 \pmod 3$,
(2) The two disjoint conjugacy classes $[\bar{F}_z]$, $[\bar{F}_z^{-1}]$ if $d = 0 \pmod 9$ or $d = 3$,
(3) The three disjoint conjugacy classes $[\bar{F}_z]$, $[\bar{F}_z^{-1}]$, $[\bar{F}_a]$ if $d = 3 \pmod 9$ and $d \ne 3$,
(4) The three disjoint conjugacy classes $[\bar{F}_z]$, $[\bar{F}_z^{-1}]$, $[\bar{F}'_a]$ if $d = 6 \pmod 9$.

This result generalizes to $\mathrm{ESL}_2(\mathbb{Z}/d\mathbb{Z})$:

Lemma A.1. *The elements of order 3 and trace -1 in $\mathrm{ESL}_2(\mathbb{Z}/d\mathbb{Z})$ consist of*

(1) *The single conjugacy class $[\bar{F}_z]$ if $d \ne 0 \pmod 3$, or $d = 3$, or $d = 0 \pmod 9$.*

[11] **DMA:** Not sure if this should be included in final draft.

(2) *The two disjoint conjugacy classes* $[\bar{F}_z]$, $[\bar{F}_a]$ *if* $d = 3$ *(mod 9) and* $d \neq 3$,
(3) *The two disjoint conjugacy classes* $[\bar{F}_z]$, $[\bar{F}'_a]$ *if* $d = 6$ *(mod 9).*

Proof. The elements of $\mathrm{ESL}_2(\mathbb{Z}/d\mathbb{Z})$ which are order 3 and have trace -1 all have determinant $+1$. Indeed, suppose $G \in \mathrm{ESL}_2(\mathbb{Z}/d\mathbb{Z})$ is order 3 and such that $\mathrm{Det}(G) = \mathrm{Tr}(G) = -1$. Then $G^2 = -G+I$ implying $G^3 = 2G-I$. So $2G = 2I$. If d is odd it follows that $G = I$ which is a contradiction. If d is even then $G = I + \frac{d}{2}\left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$. The fact that $\mathrm{Det}\, G = -1$ means $-1 = 1 + \frac{d}{2}(j + m + jm - k\ell)$ implying $2 = 0 \bmod d/2$, while the fact that $\mathrm{Tr}\, G = -1$ means $-1 = 2 + \frac{d}{2}(j+m)$ implying $3 = 0 \bmod d/2$. This is only possible if $d = 2$, in which case $\mathrm{Det}(G) = -1 = +1 \bmod d$.

Also

$$M\bar{F}_z M^{-1} = \bar{F}_z^{-1}, \qquad M = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \tag{A.2}$$

where $M = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$. On the other hand, the fact that $\bar{F}_a = I \bmod 3$ (respectively $\bar{F}'_a = I \bmod 3$) if $d = 3 \bmod 9$ (respectively $d = 6 \bmod 9$) means $\bar{F}_z$ is not conjugate to $\bar{F}_a$ (respectively $\bar{F}'_a$) if $d = 3 \bmod 9$ (respectively $d = 6 \bmod 9$). The result now follows from the result proved by Bos and Waldron. □

Lemma A.2. *Let* $G \in \mathrm{ESL}_2(\mathbb{Z}/d\mathbb{Z})$ *be order 3 and trace* -1. *If* $d = 3 \bmod 9$ *(respectively* $d = 6 \bmod 9$*) then* G *is conjugate to* $\bar{F}_a$ *(respectively* $\bar{F}'_a$*) if and only if* $G = I \bmod 3$.

Proof. Consequence of the fact that if $d = 3 \bmod 9$ (respectively $d = 6 \bmod 9$) then $\bar{F}_z \neq I \bmod 3$ and $\bar{F}_a = I \bmod 3$ (respectively $\bar{F}'_a = I \bmod 3$). □

We are now ready to prove the main result of this appendix.

Let F_z and, when $d = 3$ (mod 9), F_a be as in Definition 5.16. When $d = 6$ (mod 9) also define

$$F'_a = \begin{pmatrix} 1 & d+3 \\ \frac{2d-3}{3} & d-2 \end{pmatrix}. \tag{A.3}$$

Theorem A.3. *(A). For all* d *the elements of order 3 and trace* -1 *in* $\mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ *consist of*

(1) *The single conjugacy class* $[F_z^2]$ *if* $d \neq 0$ *(mod 3), or* $d = 3$*, or* $d = 0$ *(mod 9).*
(2) *The two disjoint conjugacy classes* $[F_z^2]$, $[F_a^2]$ *if* $d = 3$ *(mod 9), and* $d \neq 3$,
(3) *The two disjoint conjugacy classes* $[F_z^2]$, $[F'^2_a]$ *if* $d = 6$ *(mod 9).*

If $G \in \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ *is order* 3 *and trace* -1 *and* $d = 3 \bmod 9$ *(respectively* $d = 6 \bmod 9$*) then* G *is conjugate to* F_a^2 *(respectively* F'^2_a*) if and only if* $G = I \bmod 3$.

(B) If d *is even the elements of order 6 and trace* $d-1$ *in* $\mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ *consist of*

(1) *The single conjugacy class* $[F_z]$ *if* $d \neq 0$ *(mod 3), or* $d = 3$*, or* $d = 0$ *(mod 9).*
(2) *The two disjoint conjugacy classes* $[F_z]$, $[F_a]$ *if* $d = 3$ *(mod 9), and* $d \neq 3$,
(3) *The two disjoint conjugacy classes* $[F_z]$, $[F_a]$ *if* $d = 6$ *(mod 9).*

If $G \in \mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ *is order* 6 *and trace* $d-1$ *and* $d = 3 \bmod 9$ *(respectively* $d = 6 \bmod 9$*) then* G *is conjugate to* F_a *(respectively* F'_a*) if and only if* $G = I \bmod 3$.

Proof. If d is odd then $F_z = \bar{F}_z$, $F_z^2 = \bar{F}_z^{-1}$, $F_a = \bar{F}_a$, $F_a^2 = \bar{F}_a^{-1}$, $F'_a = \bar{F}'_a$, $F'^2_a = \bar{F}_a^{-1}$. Also $F_a = I$ (mod 3) when $d = 3$ (mod 9), and $F'_a = I$ (mod 3) when $d = 6$ (mod 9). Statement (A) follows from this, and Lemmas A.1 and A.2.

Suppose d is even. Making the replacement $d \to 2d$ in Lemma A.1 one finds that the elements of order 3 and trace -1 in $\mathrm{ESL}_2(\mathbb{Z}/\bar{d}\mathbb{Z})$ consist of

(1) The single conjugacy class $[\bar{F}_z]$ if $2d \neq 0$ (mod 3), or $2d = 0$ (mod 9).

(2) The two disjoint conjugacy classes $[\bar{F}_z]$, $[\bar{F}'_a]$ if $2d = 3 \pmod 9$),
(3) The two disjoint conjugacy classes $[\bar{F}_z]$, $[\bar{F}''_a]$ if $2d = 6 \pmod 9$).

where

$$\bar{F}''_a = \begin{pmatrix} 1 & 3 \\ \frac{4d-3}{3} & -2 \end{pmatrix}. \tag{A.4}$$

(A) follows from this and Lemma A.2. To prove (B) observe that G is order 6 and $\mathrm{Tr}(G) = d-1$ if and only if $H = (d+1)G$ is order 3 and $\mathrm{Tr}(H) = -1$. (B) follows from this, (A) and the fact that $(d+1)H^2 = H^5 = MHM^{-1}$ where H is any of F_z, F_a, F'_a.

□

Appendix B. Hirzebruch-Jung Continued Fractions

‘ Expansions of the form

$$[k_1, k_2, k_3, k_4, \dots]_+ = k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cfrac{1}{k_4 + \cdots}}} \tag{B.1}$$

are extremely well-known, and are described in considerable detail in standard texts such as refs. [58, 59]. Following Popescu-Pampu [60] we refer to them as *Euclidean (E-) continued fractions*. In this paper we need a different kind of expansion, which Popescu-Pampu refers to as a *Hirzebruch-Jung (HJ-) continued fraction*, and which is defined by [60–70], of the form

$$[k_1, k_2, k_3, k_4, \dots]_- = k_1 - \cfrac{1}{k_2 - \cfrac{1}{k_3 - \cfrac{1}{k_4 - \cdots}}}. \tag{B.2}$$

In the literature such fractions are also described as backwards, negative-regular, or reduced regular continued fractions. For the convenience of the reader in this appendix we collect their essential properties. Since we are only concerned with HJ-continued fractions in this paper, we will drop the subscript, and simply denote them $[k_1, k_2, k_3, k_4, \dots]$. We also review the related concept, of a *Hirzebruch-Jung (HJ-) reduced form.*

For all $x \in \mathbb{Q}$ (respectively $x \in \mathbb{R} \setminus \mathbb{Q}$ there exists a unique finite (respectively infinite) sequence of integers k_j such that $x = [k_1, k_2 \dots]$, $k_j \geq 2$ for all $j \geq 2$ and (in case the sequence is infinite) there is no integer m such that $k_j = 2$ for all $j \geq m$.

Define = $(k_1, k_2, \dots, k_n)$ recursively by

$$(k_1, k_2, \dots, k_n) = \begin{cases} k_1 & n = 1, \\ k_1 k_2 - 1 & n = 2, \\ (k_1, k_2, \dots, k_{n-1})k_n - (k_1, k_2 \dots, k_{n-2}) & n > 2. \end{cases} \tag{B.3}$$

We refer to these quantities as HJ-convergents. They are symmetric under reversal:

$$(k_1, k_2, \dots, k_{n-1}, k_n) = (k_n, k_{n-1}, \dots, k_2, k_1). \tag{B.4}$$

One has

$$[k_1, k_2, \dots, k_n] = \begin{cases} (k_1) & n = 1, \\ \frac{(k_1,k_2,\dots,k_n)}{(k_2,\dots,k_n)} & n \geq 2, \end{cases} \tag{B.5}$$

and

$$T^{k_1} S T^{k_2} S \dots T^{k_n} S \begin{cases} \begin{pmatrix} (k_1) & -1 \\ 1 & 0 \end{pmatrix} & n = 1, \\ \begin{pmatrix} (k_1, k_2) & -(k_1) \\ (k_2) & -1 \end{pmatrix} & n = 2, \\ \begin{pmatrix} (k_1, \dots, k_n) & -(k_1, \dots, k_{n-1}) \\ (k_2, \dots, k_n) & -(k_2, \dots, k_{n-1}) \end{pmatrix} & n \geq 3, \end{cases} \tag{B.6}$$

(these relations being the reason HJ-continued fractions are relevant to this paper).

A continued fraction is said to be *periodic* if it is infinite and of the form

$$[l_1, l_2, \dots, l_m, \overline{k_1, k_2, \dots, k_n}] = [l_1, l_2, \dots, l_m, k_1, \dots, k_n, k_1, \dots, k_n, k_1, \dots, k_n, \dots] \tag{B.7}$$

It is said to be *purely periodic* if it is of the form $[\overline{k_1, \dots, k_n}]$. If $k_1, \dots, k_n$ doesn't break into two or more identical subsequences then we say that n is the *period* of $[\overline{k_1, \dots, k_n}]$.

The HJ-continued fraction expansion of a real number is periodic if and only if it is an irrational element of a real quadratic field. Let β be such a number. Then its HJ-continued fraction expansion is purely periodic if and only if $\beta > 1 > \beta' > 0$, where β' is its Galois conjugate.

There is a close connection between purely periodic HJ-continued fractions and a class of quadratic forms which we now define. A form $Q = \langle a, b, c \rangle$ with discriminant $\Delta = b^2 - 4ac$ is reduced in the ordinary sense [48, 49], or as we will say *Euclidean (E-) reduced*, if

$$0 < \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b. \tag{B.8}$$

Forms of this type have a connection with Euclidean continued fractions. Specifically, the number $\frac{b+\sqrt{\Delta}}{2|a|}$ has a purely periodic E-continued fraction expansion if and only if Q is E-reduced. Corresponding to this we say Q is *Hirzebruch-Jung (HJ-) reduced* if

$$0 < -\sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} - b. \tag{B.9}$$

The number $\frac{-b+\sqrt{\Delta}}{2|a|}$ has a purely periodic HJ-continued fraction expansion if and only if Q is HJ-reduced.

Let $Q = \langle a, b, c \rangle$ and $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Then $Q_J = \langle -a, b, -c \rangle$ (see Definition 3.11). It follows that Q_J is E-reduced (respectively HJ-reduced) if and only if Q is E-reduced (respectively HJ-reduced), in which case they define the same purely periodic E-continued (respectively HJ-continued) fraction. We may therefore, without loss of generality, confine ourselves to E-reduced (respectively HJ-reduced) forms $\langle a, b, c \rangle$ for which $a > 0$. In the following this restriction will be assumed without comment. The map taking Q to $-\beta_{Q,-}$ (respectively $\beta_{Q,+}$) is then a bijective correspondence of the set of E-reduced (respectively HJ-reduced) forms onto the set of purely periodic E-continued (respectively HJ-continued) fractions.

There is a well-known algorithm [48, 49] for calculating the complete set of E-reduced forms on a given $\mathrm{GL}(2, \mathbb{Z})$ orbit. We now show how this can be used to construct the complete set of

HJ-reduced forms on the same orbit. Let W be the forms $\langle a, b, c\rangle$ on a $GL(2,\mathbb{Z})$ orbit for which $a > 0$, and let

$$W_{\mathrm{E}} = \{Q \in W \colon \beta_{Q,-} < -1 < 0 < \beta_{Q,+} < 1\} \tag{B.10}$$

$$W_{\mathrm{HJ}} = \{Q \in W \colon 0 < \beta_{Q,-} < 1 < \beta_{Q,+}\} \tag{B.11}$$

(where $\beta_{Q,\pm}$ are given by Definition 3.11). Then W_{E} (respectively W_{HJ}) is precisely the set of E-reduced (respectively HJ-reduced) forms in W. Also define, for $n = 0, 1, \dots$

$$W_{\mathrm{E}}^{(n)} = \{Q \in W \colon \beta_{Q,-} < -1 < 0 < \beta_{Q,+} < \frac{1}{n+1}\}, \tag{B.12}$$

$$W_{\mathrm{HJ}}^{(n)} = \{Q \in W \colon \frac{n}{n+1} < \beta_{Q,-} < \frac{n+1}{n+2} < 1 < \beta_{Q,+}\}. \tag{B.13}$$

Then

$$W_{\mathrm{E}} = W_{\mathrm{E}}^{(0)} \supseteq W_{\mathrm{E}}^{(1)} \supseteq \dots, \qquad \bigcap_{n=0}^{\infty} W_{\mathrm{E}}^{(n)} = \emptyset. \tag{B.14}$$

$$W_{\mathrm{HJ}}^{(n)} \cap W_{\mathrm{HJ}}^{(n')} = \emptyset \qquad \text{if } n \neq n', \qquad \bigcup_{n=0}^{\infty} W_{\mathrm{HJ}}^{(n)} = W_{\mathrm{HJ}} \tag{B.15}$$

Since W_{E} is finite, non-empty there must exist $n_0 \in \mathbb{N}$ such that $W_{\mathrm{E}}^{(n)} = \emptyset$ if and only if $n \geq n_0$. Let

$$L_n = \begin{pmatrix} n & -1 \\ n+1 & -1 \end{pmatrix}. \tag{B.16}$$

Then $x < -1$ if and only if $n/(n+1) < L_n x < (n+1)/(n+2)$, and $0 < x < 1/(n+1)$ if and only if $1 < L_n x$. In view of Lemma 3.12 this means the map $Q \to Q_{L_n^{-1}}$ is a bijection of $W_{\mathrm{E}}^{(n)}$ onto $W_{\mathrm{HJ}}^{(n)}$. It follows that the cardinality of W_{HJ} is n_0 times the cardinality of W_{E}. It also provides an algorithm for calculating the set W_{HJ}, given the set W_{E}.

Lemma B.1. *Suppose $k_j \geq 2$ for all j. Then*

$$(k_1, \dots, k_n) > (k_1, \dots, k_{n-1}) > \dots > (k_1, k_2) > (k_1) > 1 \tag{B.17}$$

Proof. Straightforward consequence of the definition. □

Lemma B.2. *Suppose*

$$T^{k_1} S T^{k_2} S \dots T^{k_n} S = L^p \tag{B.18}$$

for $k_j \geq 2$ and $p \geq 1$. Then $n = pr$ for $r \in \mathbb{N}$, $k_{j+r} = k_j$ for $j = 1, \dots, n-r$ and

$$L = \begin{cases} T^{k_1} S \dots T^{k_r} S & p \text{ is odd} \\ \pm T^{k_1} S \dots T^{k_r} S & p \text{ is even} \end{cases}. \tag{B.19}$$

Proof. Let $\bar{T}$, $\bar{S}$ be the images of T, S under the canonical projection $h \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$, and let $\bar{R} = \bar{T}\bar{S}$. Then (see for example ref. [71]) $\bar{S}$ is order 2, $\bar{R}$ is order 3 and every element of $\mathrm{PSL}(2,\mathbb{Z})$ is either the identity or else has a unique alternating expansion of the form $\bar{M}_1 \dots \bar{M}_n$ where (1) each $\bar{M}_j$ is either $\bar{S}$ or $\bar{R}^k$ for $k = 1$ or 2 and (2) terms equal to $\bar{S}$ alternate with terms equal to a non-zero power of $\bar{R}$. Also define $\bar{S}_1 = \bar{S}$, $\bar{S}_2 = \bar{S}\bar{R}\bar{S}$, ... Now let $\bar{L} = h(L)$ and let

$\bar{L} = \bar{M}_1 ... \bar{M}_q$ be its expansion in terms of alternating powers of $\bar{S}$ and $\bar{R}$. It follows from Eq. (B.18) that

$$\bar{L}^p = \bar{R}\bar{S}_{k_1-1}\bar{R}^2\bar{S}_{k_2-1}\bar{R}^2 \dots \bar{R}^2\bar{S}_{k_n-1}\bar{R}. \tag{B.20}$$

So $\bar{M}_1 = \bar{M}_q = \bar{R}$ and

$$L = \bar{R}\bar{S}_{l_1-1}\bar{R}^2 \dots \bar{S}_{l_r-1}\bar{R} \tag{B.21}$$

for some sequence of integers $l_1, \dots, l_r$ all greater than one. Eq. (B.20) and the uniqueness of the alternating expansion then imply $n = pr$, $l_j = k_j$ for $j = 1, \dots, r$, and $k_{j+r} = k_j$ for $j = 1, \dots, n-r$. Eq. (B.19) then follows. □

Theorem B.3. *Let $Q = \langle a, b, c\rangle$ be a form with $a > 0$, let f be its conductor, and let*

$$\beta_{Q,+} = [l_1, \dots, l_q, \overline{k_1, \dots, k_p}] \tag{B.22}$$

(where we set $q = 0$ if $\beta_{Q,+}$ has a purely periodic expansion equal to $[\overline{k_1, \dots, k_p}]$). Assume the sequences $l_1, \dots, l_q$ and $k_1, \dots, k_p$ are as short as possible (i.e. $k_1, \dots, k_p$ is not the conjunction of 2 or more identical subsequences, and $l_q \neq k_p$). Then

$$\eta_Q(v_f) = (T^{l_1}S \dots T^{l_q}S)(T^{k_1}S \dots T^{k_p}S)(T^{l_1}S \dots T^{l_q}S)^{-1}. \tag{B.23}$$

Proof. Assume, to begin with, that $q = 0$ and $\beta_{Q,+} = [\overline{k_1, \dots, k_p}]$. Let

$$M = T^{k_1}S \dots T^{k_p}S. \tag{B.24}$$

Then it follows from Eq. (B.6) that

$$M\beta_{Q,+} = \beta_{Q,+}. \tag{B.25}$$

In view of Lemma 3.12 this means $M \in \mathcal{S}(Q)$. It then follows from Theorem 3.14 that

$$M = s_1\eta_Q(v^{s_2 tr_f}) = s_1\left(\frac{d_{tr_f} - 1}{2}I + \frac{s_2 f_{tr_f}}{f}SQ\right) \tag{B.26}$$

for some $t \in \mathbb{N}$ and signs s_1, s_2. Comparing this expression with Eq. (B.6) one sees

$$s_1(d_{tr_f} - 1) = \mathrm{Tr}(M) = \begin{cases} (k_1) & n = 1 \\ (k_1, k_2) - 1 & n = 2 \\ (k_1, \dots, k_p) - (k_2, \dots, k_{p-1}) & p \geq 3 \end{cases} \tag{B.27}$$

$$\frac{s_1 s_2 f_{tr_f} a}{f} = M_{21} = \begin{cases} 1 & p = 1, \\ (k_2, \dots, k_p) & p \geq 2. \end{cases} \tag{B.28}$$

In view of Lemma B.1 this means $s_1 = s_2 = +1$. So $M = (\eta_Q(v_f))^t$. It then follows from Lemma B.2 and the assumption that the sequence $k_1, \dots, k_p$ is not the conjunction of 2 or more identical subsequences, that $t = 1$ and $M = \eta_Q(v_f)$.

Now suppose $q \geq 1$. Let

$$M = T^{k_1}S \dots T^{k_p}S, \tag{B.29}$$

$$N = T^{l_1}S \dots T^{l_q}S, \tag{B.30}$$

and let $Q' = \langle a', b', c' \rangle$ be the unique form such that $[\overline{k_1, \ldots, k_p}] = \beta_{Q',+}$ and $a' > 0$. It follows from the result just proved that $M = \eta_{Q'}(v_f)$, while it follows from Eq. (B.6) that

$$N\beta_{Q',+} = [l_1, \ldots, l_q, \beta_{Q',+}] = \beta_{Q,+}. \tag{B.31}$$

In view of Lemma 3.12 this means $\beta_{(N^{-1})^{\mathrm{T}} Q' N^{-1},+} = \beta_{Q,+}$, which in turn is easily seen to imply, $Q = (N^{-1})^{\mathrm{T}} Q' N^{-1}$. Hence

$$\begin{aligned} \eta_Q(v_f) &= \frac{d_{r_f} - 1}{2} I + \frac{f_{r_f}}{f} S (N^{-1})^{\mathrm{T}} Q' N^{-1} \\ &= N \left(\frac{d_{r_f} - 1}{2} I + \frac{f_{r_f}}{f} S Q' \right) N^{-1} \\ &= N M N^{-1} \end{aligned} \tag{B.32}$$

□

Theorem B.4. *Let $L = \left(\begin{smallmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{smallmatrix}\right)$ be any element of $\mathrm{SL}_2(\mathbb{Z})$. Then the following statements are equivalent:*

(1) $\ell_2 > 0$.
(2) *There exists an integer $n \geq 1$ and sequence of integers $r_1, r_2, \ldots, r_{n+1}$ for which $r_i \geq 2$ if $1 < i < n+1$ and such that*

$$L = T^{r_1} S T^{r_2} S \ldots S T^{r_n} S T^{r_{n+1}}. \tag{B.33}$$

If these conditions are satisfied the integer n and sequence $r_1, \ldots, r_{n+1}$ are unique. For $i = 1, \ldots, n+1$ let $L_i = T^{r_i} S \ldots S T^{r_{n+1}}$ and write

$$L_i = \begin{pmatrix} \ell_i & m_i \\ \ell_{i+1} & m_{i+1} \end{pmatrix}. \tag{B.34}$$

Then $\ell_{n+2} = 0$, $m_{n+2} = 1$, $\ell_{n+1} = 1$, $m_{n+1} = r_{n+1}$ and

$$\ell_2 > \ell_3 > \cdots > \ell_{n+1} \tag{B.35}$$

$$\frac{m_2}{\ell_2} < \frac{m_3}{\ell_3} < \cdots < \frac{m_{n+1}}{\ell_{n+1}} \tag{B.36}$$

$$\mathcal{D}_L = \mathcal{D}_{L_1} \subset \mathcal{D}_{L_2} \subset \cdots \subset \mathcal{D}_{L_{n+1}} = \mathbb{C} \tag{B.37}$$

Proof. Aside from uniqueness this is proved in ref. [19]. To prove uniqueness suppose

$$T^{r_1} S T^{r_2} S \ldots S T^{r_n} S T^{r_{n+1}} = T^{r'_1} S T^{r'_2} S \ldots S T^{r'_n} S T^{r'_{n'+1}}. \tag{B.38}$$

Assume to begin with that $r_i, r'_i \geq 2$ for all i, including $i = 1, n+1$. Let $h\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$ and $\bar{S}$, $\bar{T}$, $\bar{R}$ and $\bar{S}_i$ be as in the proof of Lemma B.1. Then

$$\bar{R}\bar{S}_{r_1 - 1} \bar{R}^2 \ldots \bar{R}^2 \bar{S}_{r_{n+1}-1} \bar{R}\bar{S}_1 = \bar{R}\bar{S}_{r'_1 - 1} \bar{R}^2 \ldots \bar{R}^2 \bar{S}_{r'_{n'+1}-1} \bar{R}\bar{S}_1 \tag{B.39}$$

which, in view of the uniqueness of the alternating expansion [71], means $n = n'$ and $r_i = r'_i$ for all i. In the general case choose integers m, l such that $r_1 + m$, $r'_1 + m$, $r_{n+1} + l$, $r'_{n'+1} + l \geq 2$. Then multiplying both sides of Eq. (B.38) by T^m on the left and T^l on the right gives

$$T^{r_1 + m} S T^{r_2} S \ldots S T^{r_n} S T^{r_{n+1} + l} = T^{r'_1 + m} S T^{r'_2} S \ldots S T^{r'_n} S T^{r'_{n'+1} + l}. \tag{B.40}$$

Applying the result just proved, the claim follows. □

Definition B.5. *We refer to the expression on the right hand side of Eq.* (B.33) *as the canonical expansion of* L*, and to the integer* n *as its length.*

Lemma B.6. *Let* $R = TS$*. Then*

$$T^m S = \begin{cases} -SR(T^2S)^{|m|-1}RS & m < 0 \\ S & m = 0 \\ (-1)^{m-1}(RS)^{m-1}R & m > 0 \end{cases} \tag{B.41}$$

and

$$STS = T^{-1}ST^{-1} \tag{B.42}$$

Proof. Straightforward consequences of the relations $R^3 = S^2 = -I$. □

Theorem B.7. *Let* $\mathcal{F}$ *be a* $\mathrm{GL}_2(\mathbb{Z})$ *orbit of forms with conductor* f *and discriminant* Δ*, let* $\mathcal{F}_+$ *be the subset consisting of* $\langle a, b, c\rangle \in \mathcal{F}$ *for which* $a > 0$*, and let* $\mathcal{F}_{\mathrm{HJ}}$ *be the set of HJ-reduced forms in* $\mathcal{F}_+$*. Let* $p_{\min}$ *be the minimum value of the period* p *of* $\beta_{Q,+} = [\overline{k_1, \dots, k_p}]$ *as* Q *ranges over the set* $\mathcal{F}_{\mathrm{HJ}}$*. Then*
A. Let $Q \in \mathcal{F}_{\mathrm{HJ}}$ *and let* $\beta_{Q,+} = [\overline{k_1, \dots, k_p}]$*. Then*

$$\eta_Q(v_f) = T^{k_1}S \dots T^{k_p}S \tag{B.43}$$

In particular, the length of $\eta_Q(v_f)$ *is the period of* $\beta_{Q,+}$*.*
B. For all $Q \in \mathcal{F}_+$*, the length of* $\eta_Q(v_f)$ *is greater than or equal to* $p_{\min}$*.*

Proof. A follows from Theorem B.3. To prove B, let $Q \in \mathcal{F}_+$ be arbitrary. The statement is immediate if Q is HJ-reduced, so assume not. Let $L = \eta_Q(v_f)$, and $\beta_{Q,+} = [l_1, \dots, l_q, \overline{k_1, \dots, k_p}]$ where the sequences $l_1, \dots, l_q$ and $k_1, \dots, k_p$ are chosen as short as possible. It follows from Theorem B.3 that

$$L = NMN^{-1} \tag{B.44}$$

where

$$N = T^{l_1}S \dots T^{l_q}S \tag{B.45}$$

$$M = T^{k_1}S \dots T^{k_p}S \tag{B.46}$$

and from Theorem B.4 that

$$L = T^{r_1}S \dots T^{r_n}ST^{r_{n+1}} \tag{B.47}$$

where $r_i \geq 2$ for $1 < i \leq n$. We have

$$LN = NM, \tag{B.48}$$

$$LN = T^{r_1}S \dots T^{r_n}ST^{m}ST^{l_2}S \dots T^{l_q}S \tag{B.49}$$

where $m = r_{n+1} + l_1$, and

$$NM = T^{l_1}S \dots T^{l_q}ST^{k_1}S \dots T^{k_p}S \tag{B.50}$$

The expression on the right hand side of Eq. B.50 is the canonical expansion of NM with length equal to $\mathrm{length}(N) + \mathrm{length}(M)$. If $m \geq 2$ the expression on the right hand side of Eq. (B.49) is the canonical expansion of $LN = NM$ which, in view of Theorem B.4, means $\mathrm{length}(L) + \mathrm{length}(N) =$

$\mathrm{length}(N) + \mathrm{length}(M)$ implying $\mathrm{length}(L) = \mathrm{length}(M) \geq p_{\min}$. Suppose $m < 2$. There are three cases to consider:

Case 1. $m = 1$. Then

$$LN = T^{r_1}S \dots T^{r_n}STST^{l_2}S \dots T^{l_q}S \tag{B.51}$$

Using Eq. (B.42) this becomes

$$LN = T^{r_1}S \dots T^{r_n-1}ST^{l_2-1}S \dots T^{l_q}S \tag{B.52}$$

One goes on in this way, making repeated applications of Eq. (B.42), until one obtains an expansion in canonical form. Each application of Eq. (B.42) reduces the number of S operators, so the length of the expansion which results will be less than $l + r$. It follows that $\mathrm{length}(L) + \mathrm{length}(N) > \mathrm{length}(LN) = \mathrm{length}(NM) = \mathrm{length}(N) + \mathrm{length}(M)$, implying $\mathrm{length}(L) > \mathrm{length}(M) \geq p_{\min}$.

Case 2. $m = 0$. Then

$$LN = -T^{r_1}S \dots T^{r_{n-1}}ST^{r_n+l_2}ST^{l_3}S \dots T^{l_q}S. \tag{B.53}$$

Writing $LN = MN = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$, Theorem B.4 implies $\ell < 0$. At the same time Theorem B.4, applied to the expansion on the right hand side of Eq. (B.49) implies $\ell > 0$. It follows that this case is not possible.

Case 3. $m < 0$. It follows from Lemma B.6 that

$$LN = -T^{r_1}S \dots T^{r_n+1}S(T^{r_2}S)^m T^{l_2+1}S \dots T^{l_q}S. \tag{B.54}$$

which is not possible for the same reason that case 2 is not possible. □

Appendix C. Shintani-Fadeev Jacobi cocycle

DMA: I am not sure whether this material should go in the paper at all, and if it does I am not sure where to put it. Maybe not an appendix. Anyway, we can decide that later.

Currently we are following Gene's definitions in his q-Pochhammer paper (see summary in Subsection 5.1.2). There is a problem with this, however. Consider Eq. (5.6):

$$\sigma_{LL'}(z,\tau) = \sigma_L\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau). \tag{5.6}$$

Using Eq. (5.4) one easily verifies that this is valid for all $\tau \in \mathbb{H}$. If the two sides of the equation are both defined on some non-empty interval on the real axis one can then continue into the lower half plane. However, that is not always possible. Suppose, for instance

$$L = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}, \qquad L' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{C.1}$$

and suppose $\tau \in \mathbb{R}$. Then for the right hand side of Eq. (5.6) to be defined one requires the two incompatible conditions $\tau \in \mathcal{D}_{L'} \implies \tau > 0$, and $L'\tau \in \mathcal{D}_L \implies -1 < \tau < 0$. So although (as we will see) the equation is actually valid for all $\tau \in \mathbb{C} \setminus \mathbb{R}$, this fact cannot be established by simply continuing from the upper-half plane.

I should stress that for the narrow purposes of this paper this problem doesn't matter (at least, I don't think it does). So one could deal with it by simply explaining that Eq. (5.6) isn't proved to hold for all values of τ in the intersection of the domains of definition of its two sides. But I don't like doing that because as a matter of fact it does hold and, moreover, we can prove it. So here is my suggested solution to the problem.

First a piece of notation. Looking at Eq. (5.8) it can be seen that matrices $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$ for which $\ell = 0$ require special treatment. These matrices constitute the group $\langle -I, T\rangle$, which we denote $\mathcal{G}$.

Next make a slight modification to the definitions. On the current definition

$$\sigma_{T^k}(z,\tau) = 1 \tag{C.2}$$

independently of k, for all $\tau \in \mathbb{C}$, and

$$\sigma_{-T^k}(z,\tau) = \sigma_S(z,\tau)\sigma_S\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) \tag{C.3}$$

independently of k, for all $\tau \in \mathbb{H}$. Observe that the right hand side of the second equation is well-defined for all $\tau \notin \mathbb{R}$. We therefore take it to be the definition of σ_{-T^k}, and accordingly redefine $\mathcal{D}_{T^k} = \mathbb{C}\setminus\mathbb{R}$. An immediate advantage of this is that the somewhat cumbersome, case-by-case definition of Eq. (5.8) can be replaced by the more compact expression

$$\mathcal{D}_L = (\mathbb{C}\setminus\mathbb{R}) \cup \{\tau \in \mathbb{R} \colon j_L(\tau) > 0\}, \tag{C.4}$$

valid for all L. It does, of course, mean that the domain $\mathcal{D}_{-T^k}$ is not connected. However, as we have seen, one is forced to consider disconnected sets in any case.

Now define

$$\mathcal{D}_{L,L'} = \mathcal{D}_{LL'} \cap \mathcal{D}_{L'} \cap (L')^{-1}\mathcal{D}_L \tag{C.5}$$

to be the set of all $\tau \in \mathbb{C}$ such that both sides of Eq. (5.6) are defined.

Lemma C.1.

$$\mathcal{D}_{L,L'} = \mathcal{D}_{LL'} \cap \mathcal{D}_{L'} = \mathcal{D}_{LL'} \cap (L')^{-1}\mathcal{D}_L = \mathcal{D}_{L'} \cap (L')^{-1}\mathcal{D}_L \tag{C.6}$$

Proof. Straightforward consequence of the definitions. □

We now prove that Eq. (5.6) is true for all $L, L' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{D}_{L,L'}$, irrespective of whether $\mathcal{D}_{L,L'}$ is connected. We need a few preliminary results.

Lemma C.2.

$$\sigma_{-I}(-z,-\tau) = -e^{-2\pi i z}\sigma_{-I}(z,\tau) \qquad \tau \in \mathcal{D}_{-I}, z \in \mathbb{C} \tag{C.7}$$

$$\sigma_{-I}(z,\tau+k) = \sigma_{-I}(z,\tau) \qquad \tau \in \mathcal{D}_{-I}, z \in \mathbb{C}, k \in \mathbb{Z}. \tag{C.8}$$

Proof. For all $\tau \in \mathcal{D}_S$, $z \in \mathbb{Z}$ (**DMA:** this formula, and definition of double sine not yet added)

$$\sigma_S(z,\tau) = 2\sin\left(\frac{\pi z}{\tau}\right) e^{\frac{\pi i}{12\tau}\left(6z^2+6(1-\tau)z+\tau^2-3\tau+1\right)} \operatorname{Sin}_2(z,\tau) \tag{C.9}$$

and $\operatorname{Sin}_2\left(\tau^{-1}z, \tau^{-1}\right) = \operatorname{Sin}_2(z,\tau)$. Hence

$$\sigma_S\left(\tau^{-1}z, \tau^{-1}\right) = \sin\pi z \csc\left(\frac{\pi z}{\tau}\right) e^{-\pi i(1-\tau)z}\sigma_S(z,\tau). \tag{C.10}$$

Making the replacements $z \to -z$, $\tau \to -\tau$ we deduce

$$\sigma_S\left(\tau^{-1}z, -\tau^{-1}\right) = -\sin\pi z \csc\left(\frac{\pi z}{\tau}\right) e^{\pi i(1+\tau)z}\sigma_S(-z,-\tau) \tag{C.11}$$

and, consequently,

$$\sigma_{-I}(z,\tau) = -\sin \pi z \csc\left(\frac{\pi z}{\tau}\right) e^{\pi i(1+\tau)z} \sigma_S(z,\tau)\sigma_S(-z,-\tau) \tag{C.12}$$

for all $\tau \in \mathcal{D}_{-I}$, $z \in \mathbb{C}$. Eq. (C.7) is now immediate. To prove Eq. (C.8), it follows from the definition that if $\tau \in \mathbb{H}$ then

$$\sigma_{-I}(z,\tau+k) = \frac{\varpi(-z,\tau+k)}{\varpi(z,\tau+k)} = \frac{\varpi(-z,\tau)}{\varpi(z,\tau)} = \sigma_{-I}(z,\tau). \tag{C.13}$$

We then use Eq. (C.7) to deduce that this relation continues to hold when $\tau \in -\mathbb{H}$. $\square$

Lemma C.3.

$$\sigma_{S^{-1}}(z,\tau) = \frac{1}{\sigma_S\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right)} \qquad \tau \in \mathcal{D}_{S^{-1}}, z \in \mathbb{C} \tag{C.14}$$

$$\sigma_{-I}(z,\tau) = \sigma_{S^{-1}}(z,\tau)\sigma_{S^{-1}}\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right) \qquad \tau \in \mathcal{D}_{-I}, z \in \mathbb{C} \tag{C.15}$$

$$\sigma_S(z,\tau)\sigma_S(-z,\tau) = e^{\frac{\pi i}{6\tau}(6z^2+\tau^2-3\tau+1)} \sin\left(\frac{\pi z}{\tau}\right) \csc \pi z \qquad \tau \in \mathcal{D}_S, z \in \mathbb{C} \tag{C.16}$$

$$\sigma_{-I}(z,\tau)\sigma_{-I}(-z,\tau) = 1, \qquad \tau \in \mathcal{D}_{-I}, z \in \mathbb{C} \tag{C.17}$$

Proof. Straightforward consequence of the definitions and the expression for $\sigma_S(z,\tau)$ in terms of the double sine function. $\square$

Theorem C.4. *Eq. (5.6) holds for all $L, L' \in \mathrm{SL}_2(\mathbb{Z})$ and all $\tau \in \mathcal{D}_{L,L'}$,*

Proof. Eq. (5.6) is an immediate consequence of the definitions if $\tau \in \mathbb{H}$. It follows that it holds for all $\tau \in \mathcal{D}_{L,L'}$ if the latter is connected. We need to consider the cases when $\mathcal{D}_{L,L'}$ is not connected. We group these as follows:

(1) All three of L, L', LL' are in $\langle -I, T\rangle$,
(2) Exactly one of L, L', LL' is in $\langle -I, T\rangle$,
(3) None of L, L', LL' are in $\langle -I, T\rangle$

(the fact that $\langle -I, T\rangle$ is a group means it is not possible for exactly two of L, L', LL' to be in $\langle -I, T\rangle$).

Case 1. If L, L' are both in $\langle T\rangle$ then $\mathcal{D}_{L,L'} = \mathbb{C}$ is connected. The other three possibilities are described in the following tabulation

L	L'	$\mathcal{D}_{L,L'}$	$\sigma_{LL'}(z,\tau)$	$\sigma_L\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau)$
T^k	$-T^{k'}$	$\mathbb{C}\setminus\mathbb{R}$	$\sigma_{-I}(z,\tau)$	$\sigma_{-I}(z,\tau)$
$-T^k$	$T^{k'}$	$\mathbb{C}\setminus\mathbb{R}$	$\sigma_{-I}(z,\tau)$	$\sigma_{-I}(z,\tau+k')$
$-T^k$	$-T^{k'}$	$\mathbb{C}\setminus\mathbb{R}$	1	$\sigma_{-I}(-z,\tau+k')\,\sigma_{-I}(z,\tau)$

The statement follows from this and Eqs. (C.8), (C.17).

Case 2. It is easily seen that $\mathcal{D}_{L,L'}$ is connected unless (a) $L = -T^n$, or (b) $L' = -T^n$, or (c) $LL' = -T^n$ for some integer n.

(a) $L = -T^n$ and $L' = \left(\begin{smallmatrix} j' & k' \\ \ell' & m' \end{smallmatrix}\right)$ with $\ell' \neq 0$. Assume to begin with that $n = 0$. It is easily seen that if $\ell' < 0$ then $\mathcal{D}_{S,SL'}$ and $\mathcal{D}_{S,L'}$ are both connected. So

$$\sigma_{-L'}(z,\tau) = \sigma_S\left(\frac{z}{j_{SL'}(\tau)}, SL'\tau\right)\sigma_{SL'}(z,\tau)\,, \tag{C.18}$$

and

$$\sigma_{SL'}(z,\tau) = \sigma_S\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau) \tag{C.19}$$

for all $\tau \in \mathcal{D}_{-I}$. Consequently

$$\begin{aligned}\sigma_{-L'}(z,\tau) &= \sigma_S\left(\frac{1}{L'\tau}\left(\frac{z}{j_{L'}(\tau)}\right), -\frac{1}{L'\tau}\right)\sigma_S\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau) \\ &= \sigma_{-I}\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau)\end{aligned} \tag{C.20}$$

for all $\tau \in \mathcal{D}_{-I,L'} = \mathcal{D}_{-I}$. If, on the other hand, $\ell' > 0$ then $\mathcal{D}_{S^{-1},S^{-1}L'}$ and $\mathcal{D}_{S^{-1},L'}$ are connected and so one can derive Eq. (C.20) by the same argument but with S replaced by S^{-1} and using Eq. (C.15).

Now suppose $n \neq 0$. Replacing L' by T^nL' in the result just proved gives, in view of Eq. (C.8),

$$\begin{aligned}\sigma_{-T^nL'}(z,\tau) &= \sigma_{-I}\left(\frac{z}{j_{T^nL'}(\tau)}, T^nL'\tau\right)\sigma_{T^nL'}(z,\tau) \\ &= \sigma_{-I}\left(\frac{z}{j_{L'}(\tau)}, L'\tau + n\right)\sigma_{L'}(z,\tau) \\ &= \sigma_{-T^n}\left(\frac{1}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau)\end{aligned} \tag{C.21}$$

for all $\tau \in \mathcal{D}_{-T^n,L'} = \mathcal{D}_{-I}$.

(b) $L' = -T^n$ and $L = \left(\begin{smallmatrix} j & k \\ \ell & m \end{smallmatrix}\right)$ with $\ell \neq 0$. Begin by assuming $n = 0$. It is easily seen that if $\ell < 0$ then $\mathcal{D}_{LS,S}$ and $\mathcal{D}_{L,S}$ are both connected. So

$$\sigma_{-L}(z,\tau) = \sigma_{LS}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)\sigma_S(z,\tau) \tag{C.22}$$

and

$$\sigma_{LS}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \sigma_L(-z,\tau)\,\sigma_S\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) \tag{C.23}$$

for all $\tau \in \mathcal{D}_{-I}$. Hence

$$\begin{aligned}\sigma_{-L}(z,\tau) &= \sigma_L(-z,\tau)\,\sigma_S\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)\sigma_S(z,\tau) \\ &= \sigma_L(-z,\tau)\,\sigma_{-I}(z,\tau)\end{aligned} \tag{C.24}$$

for all $\tau \in \mathcal{D}_{L,-I} = \mathcal{D}_{-I}$. If, on the other hand, $\ell > 0$ then $\mathcal{D}_{S^{-1},S^{-1}L'}$ and $\mathcal{D}_{S^{-1},L'}$ are connected and so one can derive Eq. (C.24) by the same argument but with S replaced by S^{-1} and using Eq. (C.15).

Now suppose $n \neq 0$. Replacing L by LT^n in the result just proved gives

$$\begin{aligned}\sigma_{-LT^n}(z,\tau) &= \sigma_{LT^n}(-z,\tau)\,\sigma_{-I}(z,\tau)\\ &= \sigma_L(-z,\tau+n)\,\sigma_{-T^n}(z,\tau)\end{aligned} \tag{C.25}$$

for all $\tau \in \mathcal{D}_{L,-T^n} = \mathcal{D}_{-I}$.

(c) $LL' = -T^n$, $L' = \left(\begin{smallmatrix} j' & k' \\ \ell' & m'\end{smallmatrix}\right)$, with $\ell' \neq 0$. $\mathcal{D}_{L,-L'} = \mathcal{D}_{-L'}$ is connected, implying

$$\sigma_{T^n}(z,\tau) = \sigma_L\left(-\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{-L'}(z,\tau)\,. \tag{C.26}$$

for all $\tau \in \mathcal{D}_{-L'}$ Using Eqs. (C.2) and (C.24) we deduce

$$1 = \sigma_L\left(-\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(-z,\tau)\,\sigma_{-I}(z,\tau)\,. \tag{C.27}$$

for all $\tau \in \mathcal{D}_{-I}$. In view of Eq. (C.17) this means

$$\sigma_{-I}(-z,\tau) = \sigma_L\left(-\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(-z,\tau)\,, \tag{C.28}$$

and, consequently,

$$\sigma_{LL'}(z,\tau) = \sigma_L\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau) \tag{C.29}$$

for all $\tau \in \mathcal{D}_{L,L'} = \mathcal{D}_{-I}$.

Case 3. By assumption $L' = \left(\begin{smallmatrix} j' & k' \\ \ell' & m'\end{smallmatrix}\right)$, $LL' = \left(\begin{smallmatrix} j'' & k'' \\ \ell'' & m''\end{smallmatrix}\right)$ with $\ell', \ell'' \neq 0$. It is easily seen that if ℓ' and ℓ'' have the same sign then $\mathcal{D}_{L,L'}$ is connected. So if $\mathcal{D}_{L,L'}$ is disconnected they have opposite signs, which means that $\mathcal{D}_{-L,L'}$ must be connected. We then have

$$\sigma_{-LL'}(z,\tau) = \sigma_{-L}\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau) \tag{C.30}$$

for all $\tau \in \mathcal{D}_{-L,L'}$. Using Eq. (C.20) we deduce

$$\begin{aligned}&\sigma_{-I}\left(\frac{z}{j_{LL'}(\tau)}, LL'\tau\right)\sigma_{LL'}(z,\tau)\\ &\qquad = \sigma_{-I}\left(\frac{z}{j_L(L'\tau)j_{L'}(\tau)}, LL'\tau\right)\sigma_L\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau)\end{aligned} \tag{C.31}$$

implying

$$\sigma_{LL'}(z,\tau) = \sigma_L\left(\frac{z}{j_{L'}(\tau)}, L'\tau\right)\sigma_{L'}(z,\tau) \tag{C.32}$$

for all $\tau \in \mathcal{D}_{L,L'}$. □

APPENDIX D. SIC DATA TABLES

Note that, at least in the purely periodic case, a Ghost is fully specified by its dimension and the corresponding HJ-continued fraction (I suspect this is true even if one allows HJ continued fractions which are not purely periodic, but I haven't checked in detail). So in the following tables I could have just listed the HJ expansions for each dimensions. But I feel an expanded data set might make the paper more digestible. So, in the following tables, the following items are specified:

(1) d is the dimension,
(2) r is position in tower,
(3) Δ_0 is the fundamental discriminant,
(4) f is the conductor,
(5) c is the class number,
(6) Q is a choice for the primitive HJ reduced form,
(7) τ is the larger of the two roots of Q,
(8) "HJ" is its HJ continued fraction expansion,
(9) L_Q is the generator of the stability group of Q,
(10) n is the order of L_Q in $\mathrm{SL}(2, \mathbb{Z})/\Gamma(d)$ (so $G = L_Q^n$).

For a given class we always include the principal HJ-reduced form. So if the class number is 1 the choice of form is unique. If the class number is 2 then for the remaining classes we always choose an HJ-reduced form for which the period of the continued fraction expansion is minimal. There is typically more than one such form, in which case the choice is not unique.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
4	1	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\left(\begin{smallmatrix}3&-1\\1&0\end{smallmatrix}\right)$	3
5	1	12	1	1	$\langle 1,-4,1\rangle$	$2+\sqrt{3}$	$[\overline{4}]$	$\left(\begin{smallmatrix}4&-1\\1&0\end{smallmatrix}\right)$	3
6	1	21	1	1	$\langle 1,-5,1\rangle$	$\frac{5+\sqrt{21}}{2}$	$[\overline{5}]$	$\left(\begin{smallmatrix}5&-1\\1&0\end{smallmatrix}\right)$	3
7	1	8	1	1	$\langle 2,-4,1\rangle$	$\frac{2+\sqrt{2}}{2}$	$[\overline{2,4}]$	$\left(\begin{smallmatrix}7&-2\\4&-1\end{smallmatrix}\right)$	3
			2	1	$\langle 1,-6,1\rangle$	$3+2\sqrt{2}$	$[\overline{6}]$	$\left(\begin{smallmatrix}6&-1\\1&0\end{smallmatrix}\right)$	3
8	2	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\left(\begin{smallmatrix}3&-1\\1&0\end{smallmatrix}\right)$	6
			3	1	$\langle 1,-7,1\rangle$	$\frac{7+3\sqrt{5}}{2}$	$[\overline{7}]$	$\left(\begin{smallmatrix}7&-1\\1&0\end{smallmatrix}\right)$	3
9	1	60	1	2	$\langle 1,-8,1\rangle$	$4+\sqrt{15}$	$[\overline{8}]$	$\left(\begin{smallmatrix}8&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 2,-10,5\rangle$	$\frac{5+\sqrt{15}}{2}$	$[\overline{5,2}]$	$\left(\begin{smallmatrix}9&-5\\2&-1\end{smallmatrix}\right)$	3
10	1	77	1	1	$\langle 1,-9,1\rangle$	$\frac{9+\sqrt{77}}{2}$	$[\overline{9}]$	$\left(\begin{smallmatrix}9&-1\\1&0\end{smallmatrix}\right)$	3
11	1	24	1	1	$\langle 3,-6,1\rangle$	$\frac{3+\sqrt{6}}{3}$	$[\overline{2,6}]$	$\left(\begin{smallmatrix}11&-2\\6&-1\end{smallmatrix}\right)$	3
			2	2	$\langle 1,-10,1\rangle$	$5+2\sqrt{6}$	$[\overline{10}]$	$\left(\begin{smallmatrix}10&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 3,-12,4\rangle$	$\frac{6+2\sqrt{6}}{3}$	$[\overline{4,3}]$	$\left(\begin{smallmatrix}11&-4\\3&-1\end{smallmatrix}\right)$	3
12	1	13	1	1	$\langle 3,-5,1\rangle$	$\frac{5+\sqrt{13}}{6}$	$[\overline{2,2,5}]$	$\left(\begin{smallmatrix}3&-2\\2&-1\end{smallmatrix}\right)$	6
			3	1	$\langle 1,-11,1\rangle$	$\frac{11+3\sqrt{13}}{2}$	$[\overline{11}]$	$\left(\begin{smallmatrix}11&-1\\1&0\end{smallmatrix}\right)$	3
13	1	140	1	2	$\langle 1,-12,1\rangle$	$6+\sqrt{35}$	$[\overline{12}]$	$\left(\begin{smallmatrix}12&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 2,-14,7\rangle$	$\frac{7+\sqrt{35}}{2}$	$[\overline{7,2}]$	$\left(\begin{smallmatrix}13&-7\\2&-1\end{smallmatrix}\right)$	3
14	1	165	1	2	$\langle 1,-13,1\rangle$	$\frac{13+\sqrt{165}}{2}$	$[\overline{13}]$	$\left(\begin{smallmatrix}13&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 3,-15,5\rangle$	$\frac{15+\sqrt{165}}{2}$	$[\overline{5,3}]$	$\left(\begin{smallmatrix}14&-5\\3&-1\end{smallmatrix}\right)$	3
15	2	12	1	1	$\langle 1,-4,1\rangle$	$2+\sqrt{3}$	$[\overline{4}]$	$\left(\begin{smallmatrix}4&-1\\1&0\end{smallmatrix}\right)$	6
			2	1	$\langle 4,-8,1\rangle$	$\frac{2+\sqrt{3}}{2}$	$[\overline{2,8}]$	$\left(\begin{smallmatrix}15&-2\\8&-1\end{smallmatrix}\right)$	3
			4	2	$\langle 1,-14,1\rangle$	$7+4\sqrt{3}$	$[\overline{14}]$	$\left(\begin{smallmatrix}14&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 3,-18,11\rangle$	$\frac{9+4\sqrt{3}}{3}$	$[\overline{6,2,2}]$	$\left(\begin{smallmatrix}16&-11\\3&-2\end{smallmatrix}\right)$	3
16	1	221	1	2	$\langle 1,-15,1\rangle$	$\frac{15+\sqrt{221}}{2}$	$[\overline{15}]$	$\left(\begin{smallmatrix}15&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 5,-21,11\rangle$	$\frac{21+\sqrt{221}}{10}$	$[\overline{4,3,2}]$	$\left(\begin{smallmatrix}18&-11\\5&-3\end{smallmatrix}\right)$	3

TABLE 1. SICs in dimensions 4–16.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
17	1	28	1	1	$\langle 2,-6,1\rangle$	$\frac{3+\sqrt{7}}{2}$	$[\overline{3,6}]$	$\begin{pmatrix} 17 & -3 \\ 6 & -1 \end{pmatrix}$	3
			3	2	$\langle 1,-16,1\rangle$	$8+3\sqrt{7}$	$[\overline{16}]$	$\begin{pmatrix} 16 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2,-18,9\rangle$	$\frac{9+3\sqrt{7}}{2}$	$[\overline{9,2}]$	$\begin{pmatrix} 17 & -9 \\ 2 & -1 \end{pmatrix}$	3
18	1	285	1	2	$\langle 1,-17,1\rangle$	$\frac{17+\sqrt{285}}{2}$	$[\overline{17}]$	$\begin{pmatrix} 17 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-21,13\rangle$	$\frac{21+\sqrt{285}}{6}$	$[\overline{7,2,2}]$	$\begin{pmatrix} 19 & -13 \\ 3 & -2 \end{pmatrix}$	3
19	3	5	1	1	$\langle 1,-3,1\rangle$	$\frac{3+\sqrt{5}}{2}$	$[\overline{3}]$	$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$	9
			2	1	$\langle 4,-6,1\rangle$	$\frac{3+\sqrt{5}}{4}$	$[\overline{2,2,2,6}]$	$\begin{pmatrix} 21 & -4 \\ 16 & -3 \end{pmatrix}$	3
			4	1	$\langle 5,-10,1\rangle$	$\frac{5+2\sqrt{5}}{5}$	$[\overline{2,10}]$	$\begin{pmatrix} 19 & -2 \\ 10 & -1 \end{pmatrix}$	3
			8	2	$\langle 1,-18,1\rangle$	$9+4\sqrt{5}$	$[\overline{18}]$	$\begin{pmatrix} 18 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 4,-20,5\rangle$	$\frac{5+2\sqrt{5}}{2}$	$[\overline{5,4}]$	$\begin{pmatrix} 19 & -5 \\ 4 & -1 \end{pmatrix}$	3
20	1	357	1	2	$\langle 1,-19,1\rangle$	$\frac{19+\sqrt{357}}{2}$	$[\overline{19}]$	$\begin{pmatrix} 19 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-21,7\rangle$	$\frac{21+\sqrt{357}}{6}$	$[\overline{7,3}]$	$\begin{pmatrix} 20 & -7 \\ 3 & -1 \end{pmatrix}$	3
21	1	44	1	1	$\langle 5,-8,1\rangle$	$\frac{4+\sqrt{11}}{5}$	$[\overline{2,2,8}]$	$\begin{pmatrix} 22 & -3 \\ 15 & -2 \end{pmatrix}$	3
			3	4	$\langle 1,-20,1\rangle$	$10+3\sqrt{11}$	$[\overline{20}]$	$\begin{pmatrix} 20 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 2,-22,11\rangle$	$\frac{11+3\sqrt{11}}{2}$	$[\overline{11,2}]$	$\begin{pmatrix} 21 & -11 \\ 2 & -1 \end{pmatrix}$	3
					$\langle 5,-26,14\rangle$	$\frac{13+3\sqrt{11}}{5}$	$[\overline{5,3,2}]$	$\begin{pmatrix} 23 & -14 \\ 5 & -3 \end{pmatrix}$	3
					$\langle 5,-24,9\rangle$	$\frac{12+3\sqrt{11}}{5}$	$[\overline{5,2,3}]$	$\begin{pmatrix} 22 & -9 \\ 5 & -2 \end{pmatrix}$	3
22	1	437	1	1	$\langle 1,-21,1\rangle$	$\frac{21+\sqrt{437}}{2}$	$[\overline{21}]$	$\begin{pmatrix} 21 & -1 \\ 1 & 0 \end{pmatrix}$	3
23	1	120	1	2	$\langle 6,-12,1\rangle$	$\frac{6+\sqrt{30}}{6}$	$[\overline{2,12}]$	$\begin{pmatrix} 23 & -2 \\ 12 & -1 \end{pmatrix}$	3
					$\langle 2,-12,3\rangle$	$\frac{6+\sqrt{30}}{2}$	$[\overline{6,4}]$	$\begin{pmatrix} 23 & -6 \\ 4 & -1 \end{pmatrix}$	3
			2	4	$\langle 1,-22,1\rangle$	$11+2\sqrt{30}$	$[\overline{22}]$	$\begin{pmatrix} 22 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 4,-28,19\rangle$	$\frac{7+\sqrt{30}}{2}$	$[\overline{7,2,2,2}]$	$\begin{pmatrix} 25 & -19 \\ 4 & -3 \end{pmatrix}$	3
					$\langle 3,-24,8\rangle$	$\frac{12+2\sqrt{30}}{3}$	$[\overline{8,3}]$	$\begin{pmatrix} 23 & -8 \\ 3 & -1 \end{pmatrix}$	3
					$\langle 7,-30,15\rangle$	$\frac{15+2\sqrt{30}}{7}$	$[\overline{4,4,2}]$	$\begin{pmatrix} 26 & -15 \\ 7 & -4 \end{pmatrix}$	3
24	2	21	1	1	$\langle 1,-5,1\rangle$	$\frac{5+\sqrt{21}}{2}$	$[\overline{5}]$	$\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}$	6
			5	2	$\langle 1,-23,1\rangle$	$\frac{23+5\sqrt{21}}{2}$	$[\overline{23}]$	$\begin{pmatrix} 23 & -1 \\ 1 & 0 \end{pmatrix}$	3
					$\langle 3,-27,17\rangle$	$\frac{27+5\sqrt{21}}{6}$	$[\overline{9,2,2}]$	$\begin{pmatrix} 25 & -17 \\ 3 & -2 \end{pmatrix}$	3

TABLE 2. SICs in dimensions 17–24.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
25	1	572	1	2	$\langle 1,-24,1\rangle$	$12+\sqrt{143}$	$[\overline{24}]$	$\begin{pmatrix}24&-1\\1&0\end{pmatrix}$	3
					$\langle 2,-26,13\rangle$	$\frac{13+\sqrt{143}}{2}$	$[\overline{13,2}]$	$\begin{pmatrix}25&-13\\2&-1\end{pmatrix}$	3
26	1	69	1	1	$\langle 3,-9,1\rangle$	$\frac{9+\sqrt{69}}{6}$	$[\overline{3,9}]$	$\begin{pmatrix}26&-3\\9&-1\end{pmatrix}$	3
			3	3	$\langle 1,-25,1\rangle$	$\frac{25+3\sqrt{69}}{2}$	$[\overline{25}]$	$\begin{pmatrix}25&-1\\1&0\end{pmatrix}$	3
					$\langle 5,-31,17\rangle$	$\frac{31+3\sqrt{69}}{10}$	$[\overline{6,3,2}]$	$\begin{pmatrix}28&-17\\5&-3\end{pmatrix}$	3
					$\langle 5,-29,11\rangle$	$\frac{29+3\sqrt{69}}{10}$	$[\overline{6,2,3}]$	$\begin{pmatrix}27&-11\\5&-2\end{pmatrix}$	3
27	1	168	1	2	$\langle 7,-14,1\rangle$	$\frac{7+\sqrt{42}}{7}$	$[\overline{2,14}]$	$\begin{pmatrix}27&-2\\14&-1\end{pmatrix}$	3
					$\langle 2,-16,11\rangle$	$\frac{8+\sqrt{42}}{2}$	$[\overline{8,2,2,2}]$	$\begin{pmatrix}29&-22\\4&-3\end{pmatrix}$	3
			2	4	$\langle 1,-26,1\rangle$	$13+2\sqrt{42}$	$[\overline{26}]$	$\begin{pmatrix}26&-1\\1&0\end{pmatrix}$	3
					$\langle 3,-30,19\rangle$	$\frac{15+2\sqrt{42}}{3}$	$[\overline{10,2,2}]$	$\begin{pmatrix}28&-19\\3&-2\end{pmatrix}$	3
					$\langle 4,-28,7\rangle$	$\frac{7+\sqrt{42}}{2}$	$[\overline{7,4}]$	$\begin{pmatrix}27&-7\\4&-1\end{pmatrix}$	3
					$\langle 8,-32,11\rangle$	$\frac{8+\sqrt{42}}{4}$	$[\overline{4,3,3}]$	$\begin{pmatrix}29&-11\\8&-3\end{pmatrix}$	3
28	1	29	1	1	$\langle 5,-7,1\rangle$	$\frac{7+\sqrt{29}}{10}$	$[\overline{2,2,2,2,7}]$	$\begin{pmatrix}31&-5\\25&-4\end{pmatrix}$	3
			5	2	$\langle 1,-27,1\rangle$	$\frac{27+5\sqrt{29}}{2}$	$[\overline{27}]$	$\begin{pmatrix}27&-1\\1&0\end{pmatrix}$	3
					$\langle 7,-37,23\rangle$	$\frac{37+5\sqrt{29}}{14}$	$[\overline{5,3,2,2}]$	$\begin{pmatrix}32&-23\\7&-5\end{pmatrix}$	3
29	1	780	1	4	$\langle 1,-28,1\rangle$	$14+\sqrt{195}$	$[\overline{28}]$	$\begin{pmatrix}28&-1\\1&0\end{pmatrix}$	3
					$\langle 2,-30,15\rangle$	$\frac{15+\sqrt{195}}{2}$	$[\overline{15,2}]$	$\begin{pmatrix}29&-15\\2&-1\end{pmatrix}$	3
					$\langle 5,-30,6\rangle$	$\frac{15+\sqrt{195}}{5}$	$[\overline{6,5}]$	$\begin{pmatrix}29&-6\\5&-1\end{pmatrix}$	3
					$\langle 3,-30,10\rangle$	$\frac{15+\sqrt{195}}{3}$	$[\overline{10,3}]$	$\begin{pmatrix}29&-10\\3&-1\end{pmatrix}$	3
30	1	93	1	1	$\langle 7,-11,1\rangle$	$\frac{11+\sqrt{93}}{14}$	$[\overline{2,2,11}]$	$\begin{pmatrix}31&-3\\21&-2\end{pmatrix}$	3
			3	3	$\langle 1,-29,1\rangle$	$\frac{29+3\sqrt{93}}{2}$	$[\overline{29}]$	$\begin{pmatrix}29&-1\\1&0\end{pmatrix}$	3
					$\langle 7,-37,19\rangle$	$\frac{37+3\sqrt{93}}{14}$	$[\overline{5,4,2}]$	$\begin{pmatrix}33&-19\\7&-4\end{pmatrix}$	3
					$\langle 7,-33,9\rangle$	$\frac{33+3\sqrt{93}}{14}$	$[\overline{5,2,4}]$	$\begin{pmatrix}31&-9\\7&-2\end{pmatrix}$	3

TABLE 3. SICs in dimensions 25–30.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
31	1	56	1	1	$\langle 2,-8,1\rangle$	$\frac{4+\sqrt{14}}{2}$	$[\overline{4,8}]$	$\left(\begin{smallmatrix}31&-4\\8&-1\end{smallmatrix}\right)$	3
			2	2	$\langle 8,-16,1\rangle$	$\frac{4+\sqrt{14}}{4}$	$[\overline{2,16}]$	$\left(\begin{smallmatrix}31&-2\\16&-1\end{smallmatrix}\right)$	3
					$\langle 4,-20,11\rangle$	$\frac{5+\sqrt{14}}{2}$	$[\overline{5,2,3,2}]$	$\left(\begin{smallmatrix}35&-22\\8&-5\end{smallmatrix}\right)$	3
			4	4	$\langle 1,-30,1\rangle$	$15+4\sqrt{14}$	$[\overline{30}]$	$\left(\begin{smallmatrix}30&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 4,-36,25\rangle$	$\frac{9+2\sqrt{14}}{2}$	$[\overline{9,2,2,2}]$	$\left(\begin{smallmatrix}33&-25\\4&-3\end{smallmatrix}\right)$	3
					$\langle 5,-36,20\rangle$	$\frac{18+4\sqrt{14}}{5}$	$[\overline{7,3,2}]$	$\left(\begin{smallmatrix}33&-20\\5&-3\end{smallmatrix}\right)$	3
					$\langle 5,-34,13\rangle$	$\frac{17+4\sqrt{14}}{5}$	$[\overline{7,2,3}]$	$\left(\begin{smallmatrix}32&-13\\5&-2\end{smallmatrix}\right)$	3
32	1	957	1	2	$\langle 1,-31,1\rangle$	$\frac{31+\sqrt{957}}{2}$	$[\overline{31}]$	$\left(\begin{smallmatrix}31&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 3,-33,11\rangle$	$\frac{33+\sqrt{957}}{6}$	$[\overline{11,3}]$	$\left(\begin{smallmatrix}32&-11\\3&-1\end{smallmatrix}\right)$	3
33	1	1020	1	4	$\langle 1,-32,1\rangle$	$16+\sqrt{255}$	$[\overline{32}]$	$\left(\begin{smallmatrix}32&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 2,-34,17\rangle$	$\frac{17+\sqrt{255}}{2}$	$[\overline{17,2}]$	$\left(\begin{smallmatrix}33&-17\\2&-1\end{smallmatrix}\right)$	3
					$\langle 3,-36,23\rangle$	$\frac{18+\sqrt{255}}{3}$	$[\overline{12,2,2}]$	$\left(\begin{smallmatrix}34&-23\\3&-2\end{smallmatrix}\right)$	3
					$\langle 5,-40,29\rangle$	$\frac{20+\sqrt{255}}{5}$	$[\overline{8,2,2,2,2}]$	$\left(\begin{smallmatrix}36&-29\\5&-4\end{smallmatrix}\right)$	3
34	1	1085	1	2	$\langle 1,-33,1\rangle$	$\frac{33+\sqrt{1085}}{2}$	$[\overline{33}]$	$\left(\begin{smallmatrix}33&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 5,-35,7\rangle$	$\frac{35+\sqrt{1085}}{10}$	$[\overline{7,5}]$	$\left(\begin{smallmatrix}34&-7\\5&-1\end{smallmatrix}\right)$	3
35	2	8	1	1	$\langle 2,-4,1\rangle$	$\frac{2+\sqrt{2}}{2}$	$[\overline{2,4}]$	$\left(\begin{smallmatrix}7&-2\\4&-1\end{smallmatrix}\right)$	6
			2	1	$\langle 1,-6,1\rangle$	$3+2\sqrt{2}$	$[\overline{6}]$	$\left(\begin{smallmatrix}6&-1\\1&0\end{smallmatrix}\right)$	6
			3	1	$\langle 7,-10,1\rangle$	$\frac{5+3\sqrt{2}}{7}$	$[\overline{2,2,2,10}]$	$\left(\begin{smallmatrix}37&-4\\28&-3\end{smallmatrix}\right)$	3
			4	1	$\langle 4,-12,1\rangle$	$\frac{3+2\sqrt{2}}{2}$	$[\overline{3,12}]$	$\left(\begin{smallmatrix}35&-3\\12&-1\end{smallmatrix}\right)$	3
			6	2	$\langle 9,-18,1\rangle$	$\frac{3+2\sqrt{2}}{3}$	$[\overline{2,18}]$	$\left(\begin{smallmatrix}35&-2\\18&-1\end{smallmatrix}\right)$	3
					$\langle 4,-20,7\rangle$	$\frac{5+3\sqrt{2}}{2}$	$[\overline{5,3,3}]$	$\left(\begin{smallmatrix}37&-14\\8&-3\end{smallmatrix}\right)$	3
			12	4	$\langle 1,-34,1\rangle$	$17+12\sqrt{2}$	$[\overline{34}]$	$\left(\begin{smallmatrix}34&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 4,-36,9\rangle$	$\frac{9+6\sqrt{2}}{2}$	$[\overline{9,4}]$	$\left(\begin{smallmatrix}35&-9\\4&-1\end{smallmatrix}\right)$	3
					$\langle 7,-44,28\rangle$	$\frac{22+12\sqrt{2}}{7}$	$[\overline{6,3,2,2}]$	$\left(\begin{smallmatrix}39&-28\\7&-5\end{smallmatrix}\right)$	3
					$\langle 7,-40,16\rangle$	$\frac{20+12\sqrt{2}}{7}$	$[\overline{6,2,2,3}]$	$\left(\begin{smallmatrix}37&-16\\7&-3\end{smallmatrix}\right)$	3
36	1	1221	1	4	$\langle 1,-35,1\rangle$	$\frac{35+\sqrt{1221}}{2}$	$[\overline{35}]$	$\left(\begin{smallmatrix}35&-1\\1&0\end{smallmatrix}\right)$	3
					$\langle 3,-39,25\rangle$	$\frac{39+\sqrt{1221}}{6}$	$[\overline{13,2,2}]$	$\left(\begin{smallmatrix}37&-25\\3&-2\end{smallmatrix}\right)$	3
					$\langle 5,-41,23\rangle$	$\frac{41+\sqrt{1221}}{10}$	$[\overline{8,3,2}]$	$\left(\begin{smallmatrix}38&-23\\5&-3\end{smallmatrix}\right)$	3
					$\langle 5,-39,15\rangle$	$\frac{39+\sqrt{1221}}{10}$	$[\overline{8,2,3}]$	$\left(\begin{smallmatrix}37&-15\\5&-2\end{smallmatrix}\right)$	3

TABLE 4. SICs in dimensions 31–36.

d	r	Δ_0	f	c	Q	τ	HJ	L_Q	n
37	1	1292	1	4	$\langle 1,-36,1\rangle$	$18+\sqrt{323}$	$[\overline{36}]$	$\left(\begin{smallmatrix} 36 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	3
					$\langle 2,-38,19\rangle$	$\frac{19+\sqrt{323}}{2}$	$[\overline{19,2}]$	$\left(\begin{smallmatrix} 37 & -19 \\ 2 & -1 \end{smallmatrix}\right)$	3
					$\langle 7,-44,23\rangle$	$\frac{22+\sqrt{323}}{7}$	$[\overline{6,4,2}]$	$\left(\begin{smallmatrix} 40 & -23 \\ 7 & -4 \end{smallmatrix}\right)$	3
					$\langle 7,-40,11\rangle$	$\frac{20+\sqrt{323}}{7}$	$[\overline{6,2,4}]$	$\left(\begin{smallmatrix} 38 & -11 \\ 7 & -2 \end{smallmatrix}\right)$	3
38	1	1365	1	4	$\langle 1,-37,1\rangle$	$\frac{37+\sqrt{1365}}{2}$	$[\overline{37}]$	$\left(\begin{smallmatrix} 37 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	3
					$\langle 5,-45,33\rangle$	$\frac{45+\sqrt{1365}}{10}$	$[\overline{9,2,2,2,2}]$	$\left(\begin{smallmatrix} 41 & -33 \\ 5 & -4 \end{smallmatrix}\right)$	3
					$\langle 3,-39,13\rangle$	$\frac{39+\sqrt{1365}}{6}$	$[\overline{13,3}]$	$\left(\begin{smallmatrix} 38 & -13 \\ 3 & -1 \end{smallmatrix}\right)$	3
					$\langle 11,-45,15\rangle$	$\frac{45+\sqrt{1365}}{22}$	$[\overline{4,4,3}]$	$\left(\begin{smallmatrix} 41 & -15 \\ 11 & -4 \end{smallmatrix}\right)$	3

TABLE 5. SICs in dimensions 37–40.

APPENDIX E. MEFF DATA TABLES

In the following table, we list parameters for maximal ECTFFs predicted by our conjectures, of dimension $4 \leq d \leq 200$ and rank $2 \leq r < \frac{d-1}{2}$. (Zauner predicts the existence of maximal ECTFFs of rank 1 (SICs) in every dimension. Maximal ECTFFs of rank $\frac{d-1}{2}$ are known (the Wigner ECTFF), and a one-parameter family is believed to exist. Our construction does not produce potential maximal ECTFFs of rank $\frac{d}{2}$, and ranks $r > \frac{d}{2}$ are redundant by rank complementation.)

(1) d is the dimension
(2) r is the rank
(3) $n = \frac{d^2-1}{r(d-r)}$
(4) $\Delta = n(n-4)$
(5) k is the order of B (or of $\frac{n-2+\sqrt{\Delta}}{2}$) modulo d
(6) Δ_0 is the fundamental part of Δ
(7) f is the conductor
(8) HJ is the HJ-expansion of β

d	r	n	Δ	k	Δ_0	f	HJ
11	3	5	5	5	5		
19	4	6	12	5	12		
29	5	7	21	5	21		
29	8	5	5	7	5		
41	6	8	32	5	8		
55	7	9	45	5	5		
71	8	10	60	5	60		
71	15	6	12	7	12		
76	21	5	5	9	5		
89	9	11	77	5	77		
109	10	12	96	5	24		
131	11	13	117	5	13		
139	24	7	21	7	21		
155	12	14	140	5	140		
181	13	15	165	5	165		
199	55	5	5	11	5		

REFERENCES

[1] D. Hilbert, "Mathematische probleme," *Götttinger Nachrichten* (1900) 253–297. Reprinted in *Archiv der Mathematik und Physik* **3**, 44–63; 213–237 (1901). English translation in *Bulletin of the American Mathematical Society* **8**, 437–479 (1902).

[2] G. Zauner, *Quantendesigns – Grundzüge einer nichtkommutativen Designtheorie*. PhD thesis, University of Vienna, 1999. `http://www.mat.univie.ac.at/~neum/papers/physpapers.html`. Available in English translation as: G. Zauner, Quantum Designs: Foundations of a Noncommutative Design Theory, *Int. J. Quant. Info.*, 9(1):445–507, 2011.

[3] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," *J. Math. Phys.* **45** no. 6, (June, 2004) 2171–2180, `quant-ph/0310075`.

[4] H. M. Stark, "Values of L-functions at $s = 1$. I. L-functions for quadratic forms," *Advances in Mathematics* **7** no. 3, (1971) 301–343.

[5] H. M. Stark, "L-functions at $s = 1$. II. Artin L-functions with rational characters," *Advances in Mathematics* **17** no. 1, (1975) 60–92.

[6] H. M. Stark, "L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem," *Advances in Mathematics* **22** no. 1, (1976) 64–84.

[7] H. M. Stark, "L-functions at $s = 1$. IV. First derivatives at $s = 0$," *Advances in Mathematics* **35** no. 3, (1980) 197–235.

[8] J. Tate, "On stark's conjectures on the behavior of $l(s, \chi)$ at $s = 0$," *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** no. 3, (1981) 963–978.

[9] S. Dasgupta and M. Kakde, "Brumer-Stark units and Hilbert's 12th problem," `arXiv:2103.02516`.

[10] P. Horodecki, Ł. Rudnicki, and K. Życzkowski, "Five open problems in quantum information," *PRX Quantum* (2022) 010101.

[11] A. J. Scott and M. Grassl, "Symmetric informationally complete positive-operator-valued measures: A new computer study," *J. Math. Phys.* **51** no. 4, (2010) 042203, `arXiv:0910.5784`.

[12] M. Grassl and A. J. Scott, "Fibonacci-Lucas SIC-POVMs," *J. Math. Phys.* **58** (2017) 122201.

[13] A. J. Scott, "SICs: Extending the list of solutions," `arXiv:1703.03993`.

[14] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, "The SIC question: History and state of play," *Axioms* **6** (2017) 21.

[15] M. Appleby, T.-Y. Chien, S. T. Flammia, and S. Waldron, "Constructing exact symmetric informationally complete measurements from numerical solutions," *J. Phys. A* **51** (2018) 165302.

[16] M. Appleby and I. Bengtsson, "Simplified exact SICs," *Journal of Mathematical Physics* **60** (2019) 062203.

[17] M. Grassl, "Computing numerical and exact SIC-POVMs." Talk at Jagiellonian University, Kraków, 29 March 2021. `https://www.youtube.com/watch?v=CGNxSRcqWts`.

[18] G. S. Kopp and J. C. Lagarias, "Class field theory for orders of number fields," `arXiv:2212.09177`.

[19] G. S. Kopp, ""square roots of Stark units as ratios of q-Pochhammer symbols" (title tbd),". Forthcoming.

[20] G. H. Hardy, *A mathematician's apology*. Cambridge University Press, 1940.

[21] N. Kurokawa, "Multiple sine functions and Selberg zeta functions," *Proceedings of the Japan Academy, Series A, Mathematical Sciences* **67** (1991) 61–64.

[22] T. Shintani, "On a Kronecker limit formula for real quadratic fields," *J. Fac. Sci. Univ. Tokyo* **24** (1977) 167–199.

[23] T. Shintani, "On certain ray class invariants of real quadratic fields," *J. Math. Soc. Japan* **30** (1977) 139–167.

[24] L. D. Faddeev and R. M. Kashaev, "Quantum dilogarithm," *Modern Physics Letters A* **9** (1994) 427–434.

[25] B. Ponsot, "Recent progress in liouville field theory," *International Journal of Modern Physics A* **19** no. supp02, (2004) 311–335.

[26] G. A. Sarkissian and V. P. Spiridonov, "General modular quantum dilogarithm and beta integrals," *Proceedings of the Steklov Institute of Mathematics* **309** no. 1, (2020) 251–270.

[27] A. J. Scott, "Tight informationally complete quantum measurements," *J. Phys. A* **39** (2006) 13507–13530.

[28] S. K. Pandey, V. I. Paulsen, J. Prakash, and M. Rahaman, "Entanglement breaking rank and the existence of sic povms," *Journal of Mathematical Physics* **61** (2020) 042203.

[29] C. A. Fuchs and R. Schack, "Quantum-Bayesian coherence," *Rev. Mod. Phys.* **85** (2013) 1693–1715.

[30] M. Appleby, C. A. Fuchs, B. C. Stacey, and H. Zhu, "Introducing the qplex: A novel arena for quantum theory," *Eur. Phys. J. D* **71** (2017) 197.

[31] D. M. Appleby, C. A. Fuchs, and H. Zhu, "Group theoretic, Lie algebraic and Jordan algebraic formulations of the SIC existence problem," *Quantum Inf. Comput.* **15** (2015) 61–94.
[32] A. Fannjiang and T. Strohmer, "The numerics of phase retrieval," *Acta Numerica* **29** (2020) 125–228.
[33] D. M. Appleby, "Symmetric informationally complete measurements of arbitrary rank," *Opt. Spect.* **103** (2007) 416–428.
[34] M. Appleby, I. Bengtsson, S. Flammia, and D. Goyeneche, "Tight frames, Hadamard matrices, and Zauner's conjecture," *J. Phys. A* **52** (2019) 295301.
[35] G. S. Kopp, "SIC-POVMs and the Stark conjectures," *International Mathematics Research Notices* (2019) rnz153, arXiv:1807.05877.
[36] B. C. Stacey, *A First Course in the Sporadic SICs*. Springer Briefs in Mathematical Physics Vol. 41. Springer, 2021.
[37] D. M. Appleby, H. Yadsan-Appleby, and G. Zauner, "Galois automorphisms of a symmetric measurement," *Quant. Info. Comput.* **13** no. 7&8, (2013) 672–720, arXiv:1209.1813.
[38] M. Appleby, S. Flammia, G. McConnell, and J. Yard, "Generating ray class fields of real quadratic fields via complex equiangular lines," *Acta Arithmetica* **192** no. 3, (2020) 211–233, arXiv:1604.06098.
[39] K. Dixon and S. Salamon, "Moment maps and Galois orbits for SIC-POVMs," arXiv:1912.03209.
[40] G. S. Kopp and J. C. Lagarias, ""SICs and orders of real quadratic fields" (title tbd),". Forthcoming.
[41] L. Bos and S. Waldron, "SICs and the elements of canonical order 3 in the Clifford group," *J. Phys. A* **52** (2019) 105301.
[42] M. Appleby, I. Bengtsson, I. Dumitru, and S. Flammia, "Dimension towers of SICs. I. Aligned SICs and embedded tight frames," *Journal of Mathematical Physics* **58** no. 11, (2017) 112201.
[43] O. Andersson and I. Dumitru, "Aligned SICs and embedded tight frames in even dimensions," *Journal of Physics A: Mathematical and Theoretical* **52** (2019) 425302.
[44] H. Cohn, *Introduction to the Construction of Class Fields*. Dover Publications: New York, 1994. Corrected reprint of 1985 original.
[45] J. Neukirch, *Algebraic Number Theory*. Springer Berlin Heidelberg, 1999.
[46] H. M. Stark, "Class fields for real quadratic fields and L-series at 1," in *Algebraic Number Fields*, pp. 355–374, New York: Academic Press. 1977.
[47] H. Koch, *Number Theory. Algebraic Numbers and Functions*. Graduate Studies in Mathematics, vol. 24. American Mathematical Society, 2000.
[48] D. A. Buell, *Binary Quadratic Forms: Classical Theory and Modern Computations*. Springer-Verlag, 1989.
[49] J. Buchmann and U. Vollmer, *Binary Quadratic Forms: An Algorithmic Approach*. Algorithms and Computation in Mathematics, no. 20. Springer, 2007.
[50] M. Atiyah, "The logarithm of the Dedekind η-function," *Math. Ann* **278** (1987) 335–380.
[51] H. Rademacher, *Topics in Analytical Number Theory*. Die Grundlehren der mathematischen Wissenschaften. Springer, 1973.
[52] M. Appleby, I. Bengtsson, M. Grassl, M. Harrison, and G. McConnell, "SIC-POVMs from Stark units: Prime dimensions $n^2 + 3$," *Journal of Mathematical Physics* **63** (2022) 112205.
[53] G. S. Kopp, "A Kronecker limit formula for indefinite zeta functions," *Forthcoming* (2020) .
[54] W. Gautschi and G. Inglese, "Lower bounds for the condition number of Vandermonde matrices," *Numerische Mathematik* **52** (1987) 241–250.
[55] E. E. Tyrtyshnikov, "How bad are Hankel matrices?," *Numerische Mathematik* **67** (1994) 261–269.
[56] A. Eisinberg and G. Fedele, "On the inversion of the Vandermonde matrix," *Applied mathematics and computation* **174** (2006) 1384–1397.
[57] Y. A. Ghassabeh, "A recursive algorithm for computing the inverse of the Vandermonde matrix," *Cogent Engineering* **3** (2016) 1175061.
[58] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. Oxford University Press, sixth ed., 2009. Revised by D. R. Heath-Brown and J. H. Silverman.
[59] J. H. Davenport, *The Higher Arithmetic: An Introduction to the Theory of Numbers*. Cambridge University Press, eighth ed., 2008.
[60] P. Popescu-Pampu, "The geometry of continued fractions and the topology of surface singularities," in *Singularities in geometry and topology: : Proceedings of the third Franco-Japanese Symposium on Singularities, September 2004*, J.-P. Brasselet and T. Suwa, eds., Advanced Studies in Pure Mathematics, Volume 46, pp. 119–195, Mathematical Society of Japan. 2007. available online at https://arxiv.org/abs/math/0506432.

[61] H. W. E. Jung, "Darstellung der funktionen eines algebraischen körpers zweier unabhängigen veränderlichen x, y in der umgebung einer stelle $x = a$, $y = b$.," *Journal für die reine und angewandte Mathematik (Crelles Journal)* **1908** (1908) 289–314.
[62] F. Hirzebruch, "Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen," *Mathematische Annalen* **126** (1953) 1–22.
[63] F. Hirzebruch, W. Neumann, and S. Koh, *Differentiable Manifolds and Quadratic Forms*. Marcel Dekker, Inc. New York, 1971.
[64] F. Hirzebruch, "Hilbert modular surfaces," *L'Enseignement Mathématique* **19** (1973) 183–282.
[65] T. Shintani, "On evaluation of zeta functions of totally real algebraic number fields at non-positive integers," *J. Fac. Sci., Univ. Tokyo, Sect. IA* **23** (1976) 393–417.
[66] R. L. Adler and L. Flatto, "The backward continued fraction map and geodesic flow," *Ergodic Theory and Dynamical Systems* **4** (1984) 487–492.
[67] G. Myerson, "On semi-regular finite continued fractions," *Archiv der Mathematik* **48** (1987) 420–425.
[68] Y. Y. Finkel'shtein, "Klein polygons and reduced regular continued fractions," *Russian Mathematical Surveys* **48** (1993) 198–200.
[69] F. I. B. López, V. N. Efremov, and A. M. H. Magdaleno, "Algorithm for fast calculation of Hirzebruch-Jung continued fraction expansions to coding of graph manifolds," *Applied Mathematics* **6** (2015) 1676.
[70] C. Bjorklund and M. Litman, "Error approximation for backwards and simple continued fractions,". https://www.math.ucdavis.edu/~mclitman/img/portfolio/BCF-Paper.pdf.
[71] R. C. Alperin, "$\mathrm{PSL}_2(\mathrm{Z}) = \mathrm{Z}_2 * \mathrm{Z}_3$," *The American Mathematical Monthly* **100** (1993) 385–386.

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