Calculating the double sine function

Poles and zeros

Just a few remarks about poles and zeros of the double sine function. In particular, I want to comment on the first expression in Kurokawa and Wakayama, which might be wrongly interpreted.

If one starts with the expression for $S_2(z, w_1, w_2)$ in terms of q-Pochammer symbols (see below) one easily sees that it has zeros at the points

$$z = n_1 w_1 + n_2 w_2, \qquad n_1, n_2 \in \mathbb{Z}, \quad n_1, n_2 > 0 \tag{1}$$

and poles at the points

$$z = n_1 w_1 + n_2 w_2, \qquad n_1, n_2 \in \mathbb{Z}, \quad n_1, n_2 \leq 0 \tag{2}$$

Naively, the first equation in Kurokawa and Wakayama may suggest it is the other way round: poles when n_1, $n_2 > 0$ and zero's when $n_1, n_2 \leq 0$. Note, however that the product in their equation is “zeta-regularized”. I am not sure what this means exactly, but I think zeta-regularization must have the effect of interchanging poles and zeros.

Just to check I am right about this, suppose the zero's and poles are indeed as given by (1), (2) and let

$$w_1 = 1 + i, \qquad w_2 = 1 - i, \qquad z = w_2 - 2w_1 \tag{3}$$

as in your email. Then

$$S_2(w_2 - 2w_1, w_1, w_2) = 2\sin\left(\frac{-2\pi w_1}{w_1}\right) S_2(-2w_1, w_1, w_2) = 0 \times \infty \tag{4}$$

which checks. On the other hand, if the poles and zeros are as suggested by a naive reading of Kurokawa and Wakayama then the RHS $= 0$, while the LHS is non-zero, which is a contradiction.

Alternatively

$$
\begin{aligned}
S_2(w_2-2w_1,w_1,w_2) &= \frac{S_2(w_2-w_1,w_1,w_2)}{2\sin\left(\frac{\pi(w_2-2w_1)}{w_2}\right)} \\
&= \frac{S_2(w_2-w_1,w_1,w_2)}{2\sin\left(\frac{2\pi w_1}{w_2}\right)} \\
&= \frac{S_2(w_2,w_1,w_2)}{4\sin\left(\frac{2\pi w_1}{w_2}\right)\sin\left(\frac{\pi(w_2-w_1)}{w_2}\right)} \\
&= \frac{S_2(w_2,w_1,w_2)}{4\sin\left(\frac{2\pi w_1}{w_2}\right)\sin\left(\frac{\pi w_1}{w_2}\right)} \\
&= \frac{\sin\left(\frac{\pi\times 0}{w_1}\right)S_2(0,w_1,w_2)}{2\sin^2\left(\frac{2\pi w_1}{w_2}\right)} \\
&= 0\times\infty
\end{aligned}
\tag{5}
$$

which again checks if the poles and zeros are as in (1) and (2). On the other hand, if they are as suggested by a naive reading of Kurokawa and Wakayama one again gets a contradiction.

Expression in terms of q-Pochammer

In your email you start by asking for the value of $S_2(z,1,\tau)$ when $z=2-i$ and $\tau=i$. For a case like this, when τ is in the upper half plane, and well away from the real axis, one can use the expression in terms of q-Pochammer symbols:

$$
S_2(z,1,\tau) = e^{-\frac{\pi i}{12\tau}\left(6z^2-6(1+\tau)z+\tau^2+3\tau+1\right)}\left(\frac{\varpi\left(\frac{z-1}{\tau},-\frac{1}{\tau}\right)}{\varpi(z-1,\tau)}\right)
\tag{6}
$$

Possibly this is faster than the numerical integration. It might be worth comparing.