MAT02023 - Inferência A

Lista 2 - Continuação de Conceitos de Probabilidade

Amostra Aleatória

Exercício 1 **(a)** Descreva a σ álgebra e suas probabilidades.

(b) Uniforme discreta

(c) Calcule os momentos de uma uniforme discreta, exemplo $\mathbb{E}X = \sum_{k=1}^{n} kP(X = k)$.

(d)

(e)

(f) Binomial

Exercício 2 **(a)** $P(\boldsymbol{X} = \boldsymbol{x}) = \left(\frac{2}{3}\right)^{\sum_{i=1}^{9} x_i} \left(\frac{1}{3}\right)^{9-\sum_{i=1}^{9} x_i}$.

(b) Defina $Y = \sum_{i=1}^{9} X_i$, então $Y \sim \mathit{Binomial}\,(9, 2/3)$.

(c) $E(\bar{X}) = E(\frac{Y}{9}) = 9 \times \frac{2}{3} \times \frac{1}{9} = \frac{2}{3}$.

(d) $E(S^2) = \frac{n}{n-1}\left[E(X_i^2) - E(\bar{X}^2)\right] = \ldots = Var(X_i)$. Então $\mathrm{E(S^2)} = \frac{2}{9}$..

Exercício 3 **(a)** $P(\boldsymbol{X} = \boldsymbol{x}; \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$.

(b) $e^{-2\lambda n}$

Exercício 4 **(a)**

(b)

(c)

(d)

Exercício 5 Dica: $X_i \sim \text{Bernoulli}(p) \quad \Longrightarrow \quad \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$.

$$P(\bar{X} = x) = \binom{n}{nx} p^{nx}(1-p)^{nx}.$$

(a) Dica: Soma de variáveis aleatórias com distribuição Poisson é ainda Poisson.

$$P(\bar{X} = k/n) = \frac{e^{-n\lambda}(n\lambda)^k}{k!}.$$

(b) Dica: $\text{Exp}(\lambda) = \Gamma(1, 1/\lambda)$ e soma de n variáveis $\Gamma(1, 1/\lambda)$ é $\Gamma(n, 1/\lambda)$.

Exercício 6 **(a)** $E(\overline{X}) = \overline{X}$ *e* $Var(\overline{X}) = \sigma^2/n$.

(b) $E(S^2) = \sigma^2$.

Função Geradora de Momentos

Exercício 7 $M_X(t) = e^{\lambda(e^t - 1)}$.

Exercício 8 $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)$.

Exercício 9 $M_X(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$.

Exercício 10 usar $M_X(t)$.

Exercício 11 $M_X(t) = e^{\frac{t_1^2 + t_2^2}{2}}$.

Teoremas Limite

Exercício 12 $l = 0,328$.

Exercício 13 $n \approx 62$.

Exercício 14 $P(0.45 < \overline{X} < 0.55) \approx 0,866$.

Exercício 15 **(a)** $Y(n, p)$.

(b) $P(47,5 < Y < 52,5) \approx 0,3829$.

Exercício 16 $P(Y/n > 0.25) \approx 0,0062$.

Exercício 17 $0,3085$.

Exercício 18

a) $\lim_{m\to\infty} P(|\bar{X}_m - \mu| < \varepsilon) = 1$

Pela desigualdade de Chebychev: $P(g(x) \geqslant r) \leqslant \frac{E[g(x)]}{r}$

$P(|\bar{X}_m - \mu| < \varepsilon) = 1$, então $P(|\bar{X}_m - \mu| \geqslant \varepsilon) = 0$

$\lim_{m\to\infty} P(|\bar{X}_m - \mu| \geqslant \varepsilon^2) \leqslant \frac{E(\bar{X}_m - \mu)^2}{\varepsilon^2} = \lim_{m\to\infty} P(|\bar{X}_m - \mu| \geqslant \varepsilon^2) \leqslant \frac{\sigma^2}{m\varepsilon^2}$

$\frac{\sigma^2}{m\varepsilon^2} = \frac{\sigma^2}{\infty} = 0$ na medida em que m vai para infinito

Portanto, $\lim_{m\to\infty} P(|\bar{X}_m - \mu| < \varepsilon) = 1$

b) $Z_m = \frac{\bar{X}_m - E(\bar{X}_m)}{\sqrt{Var(\bar{X}_m)}} \sim N(0,1)$ na medida em que $m \to \infty$

$m_{Z_m}(t) = \frac{\bar{X} - E(\bar{X})}{\sqrt{Var(\bar{X})}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{m}} \xrightarrow[m\to\infty]{d} N(0,1)$

$m_{Z_m}(t) = E(e^{tZ_m}) = E\left(e^{t\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{m}}\right)}\right) = E\left(e^{t\left(\frac{\frac{\Sigma x_i}{m} - \frac{m\mu}{m}}{\sigma/\sqrt{m}}\right)}\right) = E\left(e^{\frac{t}{m}\left[\frac{\Sigma x_i - \mu m}{\sigma/\sqrt{m}}\right]}\right) =$

$= E\left(e^{\frac{t}{m}\Sigma\left(\frac{X_i - \mu}{\sigma/\sqrt{m}}\right)}\right) \overset{aa\ iid}{=} \prod_{i=1}^{m} E\left(e^{t/m\left(\frac{X_i-\mu}{\sigma/\sqrt{m}}\right)}\right)$ * Chamamos $Y_i = \frac{X_i - \mu}{\sigma/\sqrt{m}}$

. Utilizando Série de Taylor.

$m_Y\left(\frac{t}{m}\right)^m = \left[m_Y(0) \cdot \frac{(t/m - 0)^0}{0!} + m'_Y(0)\frac{(t/m - 0)^1}{1!} + m''_Y(0)\frac{\left(\frac{t}{m} - 0\right)}{2!} + R_Y(t/m)\right]^m$

$= \left[1 + 0 + m\frac{t^2/m^2}{2!} + R_Y(t/m)\right]^m = \left[1 + \frac{t^2}{2m} + R_Y(t/m)\right]^2 = \left[1 + \frac{1}{m}\left(\frac{t^2}{2} + mR_Y(t/m)\right)\right]^m$

. Aplicamos no limite de $m \to \infty$

$\lim_{m\to\infty}\left(1 + \frac{1}{m}\left(t^2/2 + mR_Y(t/m)\right)\right)^m = \exp\left\{\frac{t^2}{2} + mR_Y(t/m)\right\} = \underbrace{\exp\left\{\frac{t^2}{2}\right\}}_{\text{fgm da } N(0,1)}$

c) $E(\bar{x}) = \mu$ e $Var(\bar{x}) = \frac{1}{m}\sigma^2$

$$E(\bar{x}) = E\left(\frac{\sum x_i}{m}\right) = E\left[\frac{1}{m}\sum x_i\right] = \frac{1}{m}E\left(\sum x_i\right) = \frac{1}{m}\sum E(x_i) = \frac{1}{m}\cdot m\mu = \mu$$

$$Var(\bar{x}) = Var\left(\frac{1}{m}\sum x_i\right) = \frac{1}{m^2}Var\left(\sum x_i\right) = \frac{1}{m^2}\sum Var(x_i) = \frac{1}{m^2}\sum\sigma^2 = \frac{m\sigma^2}{m^2} = \frac{\sigma^2}{m}$$

d) $U = \sum_{i=1}^{k}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2 \sim \chi^2_k$ sabemos que $Z_i = \frac{x_i-\mu_i}{\sigma_i} \sim N(0,1)$

$$m_U(t) = E\left(e^{t\sum z_i^2}\right) = E\left[\prod e^{tz_i^2}\right] = \prod E\left(e^{tz_i^2}\right) = \prod\int_{-\infty}^{\infty} e^{tz_i^2}(2\pi)^{-1/2}e^{-1/2z^2}dz =$$

$$= \prod\int_{-\infty}^{+\infty}(2\pi)^{-1/2}e^{-\frac{1}{2}z^2(1-2t)}dz = \prod\int(2\pi)^{-1/2}\cdot\left(\frac{1}{1-2t}\right)^{1/2}\cdot\left(\frac{1}{1-2t}\right)^{1/2}\cdot\frac{1}{e^{2\left(\frac{1}{1-t^2}\right)}}z^2 =$$

$$= \prod\frac{1}{\sqrt{1-2t}}\underbrace{\int\left(2\pi\left(\frac{1}{1-2t}\right)\right)^{-\frac{1}{2}}e^{1/2\,(1/1-tz)}\,dz}_{1}$$

então, $m_U(t) = \prod_{i=1}^{k}\left(\frac{1}{\sqrt{1-2t}}\right) = \left[\frac{1}{1-2t}\right]^{k/2} \rightarrow$ FMG de χ^2_k.

e) Se $U = \sum_{i=1}^{k}\left(\frac{x_i-\mu_i}{\sigma}\right)^2 \sim \chi^2_k$

então $\sum_{i=1}^{m}\frac{(x_i-\mu)^2}{\sigma^2} \sim \chi^2_m$ pois $\sum_{i=1}^{k}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \sum_{i=1}^{m}\frac{(x_i-\mu)^2}{\sigma^2}$ se $k=m$

f) i. $\bar{Z} \sim N(0, 1/m)$

$\bar{Z} = \frac{1}{m}\sum Z_i$

$\sum z_i \sim N(\mu,\sigma^2)$ onde $\mu = \sum\mu_i = 0+0+\dots+0 = 0$

e $\sigma^2 = Var(\bar{Z}) = Var\left(\frac{\sum z_i}{m}\right) = \frac{1}{m^2}\cdot m\,\sigma_i = \frac{1}{m}$

$\therefore\ \bar{Z} \sim N(0, 1/m)$

ii) $\bar{Z}$ e $\sum_{i=1}^{m}(z_i-\bar{z})^2$ são independentes. $(m=2)$

$\bar{Z} = \frac{z_1-z_2}{2}$

$$\sum_{i=1}^{2}(z_1-z_2)^2 = \left[z_1-\frac{(z_1+z_2)}{2}\right]^2+\left[z_2-\frac{(z_1+z_2)}{2}\right]^2 = \frac{(z_1-z_2)^2}{4}+\frac{(z_2-z_1)^2}{4}$$

$\cdot\ \bar{Z} = \sum_{i=1}^{m} \frac{Z_i}{m} = \frac{Z_1 - Z_2}{m}$ e $\sum_{i=1}^{2} (Z_1 - \bar{Z})^2 = \frac{(Z_2 - Z_1)^2}{2}$

Temos que mostrar que $Z_1 + Z_2$ e $Z_2 - Z_1$ são independentes

$m_{Z_1+Z_2}(t)_1 = E\left(e^{t_1(Z_1+Z_2)}\right) = E\left(e^{t_1 Z_1 + t_1 Z_2}\right) = \exp\left\{\frac{t_1^2}{2}\right\} \exp\left\{\frac{t_1^2}{2}\right\} = \exp\{t_1^2\}$

$\cdot\ m_{Z_2-Z_1}(t_2) = \exp\{t_2^2\}$ (Usando resultado anterior)

Portanto, $m_{Z_1+Z_2,\, Z_2-Z_1}(t_1, t_2) = E\left(e^{t_1(Z_1+Z_2) + t_2(Z_2-Z_1)}\right) = E\left(e^{Z_1(t_1-t_2)} + e^{Z_2(t_1+t_2)}\right) =$

$= E\left(e^{Z_1(t_1-t_2)}\right) \cdot E\left(e^{Z_2(t_1+t_2)}\right) = \left[e^{\frac{t_1^2 - 2t_1t_2 + t_2^2}{2}}\right]\left[e^{\frac{t_1^2 + 2t_1t_2 + t_2^2}{2}}\right] = e^{t_1^2} \cdot e^{t_2^2} =$

$= \exp\{t_1^2\} \cdot \exp\{t_2^2\}$

$\star\ m_{Z_1+Z_2}(t_1) \cdot m_{Z_2-Z_1}(t_2) = m_{Z_1+Z_2,\, Z_2-Z_1}(t_1, t_2)$, então provamos a independência de $\bar{Z}$ e $\sum_{i=1}^{m} (Z_i - \bar{Z})^2$

(iii) $\sum_{i=1}^{m} (Z_i - \bar{Z})^2 \sim \chi^2_{(m-1)}$

$$\sum Z_1^2 - \underbrace{2Z_1\bar{Z}}_{-2\bar{Z}^2} + \bar{Z}^2 + \dots + \sum Z_m^2 - \underbrace{2Z_m\bar{Z}}_{-2\bar{Z}^2} + \bar{Z}^2 = \sum (Z_1 - \bar{Z})^2 + m\bar{Z}^2$$

$$m_{\Sigma Z^2}(t) = m_{\Sigma(Z-\bar{Z})^2 + m\bar{Z}^2}(t) = m_{\Sigma(Z_1-\bar{Z})^2}(t) \cdot m_{m\bar{Z}^2}(t) = m_{\Sigma Z_i}(t)$$

$$m_{\Sigma(Z_i-\bar{Z})^2}(t) = \underbrace{\frac{m_{\Sigma Z_i^2}(t)}{m_{m\bar{Z}^2}(t)}}_{\frac{\left(\frac{1}{1-2t}\right)^{m/2}}{\left(\frac{1}{1-2t}\right)^{1/2}}} = m_{\Sigma(Z_i-\bar{Z})^2}(t) = \left(\frac{1}{1-2t}\right)^{\frac{m-1}{2}} \sim \chi^2_{(m-1)}$$

g) $\dfrac{Z}{\sqrt{U/K}} \sim t_K$ com $Z \sim N(0,1)$ e $U \sim \chi^2_K$

$$J = \begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial z}{\partial y} & \dfrac{\partial u}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{y/K} & 0 \\ & 1 \end{vmatrix} = \text{Det}|J| = \sqrt{y/K}$$

$$f_{xy}(x,y) = (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\,\frac{x^2 y}{K}\right\} \frac{1}{\Gamma(K/2)} \left(\frac{1}{2}\right)^{K/2} \cdot y^{K/2-1} \cdot \exp\left\{-\frac{1}{2}y\right\} \cdot \left\{\frac{y}{K}\right\}^{1/2}$$

$$f_x = \int_{-\infty}^{\infty} f_{xy}\, dy$$

$$\frac{1}{\sqrt{2\pi}\cdot\Gamma(K/2)} \cdot \left(\frac{1}{2}\right)^{K/2} \int_0^{\infty} y^{K/2+1/2-1} \exp\left\{-\frac{1}{2}y\left[1+\frac{x^2}{K}\right]\right\} dy \cdot A$$

$$A = \frac{\Gamma\left(\frac{K+1}{2}\right) \cdot \left(\frac{2}{1+x^2/K}\right)^{\frac{K+1}{2}}}{\Gamma\left(\frac{K+1}{2}\right) \cdot \left(\frac{2}{1+x^2/K}\right)^{\frac{K+1}{2}}} \sim K \cdot \text{Gama}\left(\frac{K+1}{2}, \frac{2}{[1+x^2/K]}\right)$$

$$f_x(x) = \frac{\Gamma\left(\frac{K+1}{2}\right) \cdot \left(\frac{2}{1+x^2/K}\right)^{\frac{K+1}{2}}}{\sqrt{2\pi K}\;\Gamma(K/2)} \cdot \left(\frac{1}{2}\right)^{K/2} \cdot \frac{\Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{K}{2}\right)} \frac{1}{\sqrt{\pi K}} \frac{1}{(1+x^2/K)^{\frac{K+1}{2}}} \sim T_K$$