ENRICHED ∞-OPERADS AS MARKED ALGEBRAS

MARKUS ZETTO

Abstract. We prove that an enriched ∞-operad is completely determined by its category of right modules together with a 'marking' by the representable modules. This description allows for a very explicit comparison of (colored) $\mathcal{S}$-enriched ∞-operads, defined as algebras in symmetric sequences, and Lurie's model of ∞-operads. Additionally, we show that the categories of algebras and right modules defined in both models agree.

We develop the theory of enriched ∞-operads by defining a Boardman-Vogt product, operadic weighted colimits, and consider the question how much about a $\mathcal{V}$-enriched ∞-operad $\mathcal{O}$ can be recovered from $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ or from its category of right modules, leading to an operatic version of Cauchy-completion.

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history

1.1. **Marked algebras.** explain the comparison functors

Proposition 1.1. A presentably symmetric monoidal $\mathcal{V}$-module category $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is equivalent to the operadic presheaf category $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ of some $\mathcal{V}$-operad $\mathcal{O}$ iff it is generated under colimits, monoidal structure and $\mathcal{V}$-tensoring by its $\otimes$-atomic objects.

1.2. **Applications.** We use the marked-module picture to develop the higher algebra of enriched ∞-operads, for instance we introduce:

- An envelope functor $\mathrm{Env}_{\mathcal{V}} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$,
- A Boardman-Vogt product $\otimes_{\mathrm{BV}}$ on $\mathcal{V}$-operads that is adjoint to taking $\mathcal{V}$-categories of algebras,
- Operadic weighted colimits (for instance, factorization homology),
- Free algebras,

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generalizing the respective constructions in [Lur17].

1.3. **Recovering on operad from its algebras.** We will study the question how much information about an operad $\mathcal{O}$ can be recovered from its category of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. The following is more or less immediate from the above marked algebra description:

Proposition (??). Let $f : \mathcal{O} \to \mathcal{P}$ be a map of valent $\mathcal{V}$-enriched operads whose underlying map on spaces of colors $col\mathcal{O} \to col\mathcal{P}$ is an equivalence. If the induced precomposition map $f^* : \mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is an equivalence, then f is already an equivalence.

However, we might now always be given a comparison map f, but only the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ itself. Certainly, we can not fully recover $\mathcal{O}$ in this case:

introduce notation?

- Let C be a category and $\widehat{\mathrm{C}}^{\mathrm{ic}}$ its idempotent completion. Then
$$\mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\mathrm{C})}(\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{V}) \simeq \mathrm{Fun}(\hat{\mathrm{C}}^{\mathrm{ic}}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\widehat{\mathrm{C}}^{\mathrm{ic}})}(\mathcal{V}) \ .$$
- Given a spectral operad $\mathcal{O}$, we can define its *r-fold shift* $\mathcal{O}[r]$ whose multimorphism objects are given by
$$\mathrm{Mul}_{\mathcal{O}[r]}(o_1, \ldots, o_n; o) \simeq \mathrm{Mul}_{\mathcal{O}}(o_1, \ldots, o_n; o)[r \cdot (1-n)] \ ,$$
consider [**?**, Constr. 2.34] for a single-colored version of this construction. The auto-equivalence $[-r] : \mathcal{V} \xrightarrow{\simeq} \mathcal{V}$ induces, for any $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$, an equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{O}[r]}(\mathcal{V})$.
- Let $\mathcal{V}$ be semiadditive. Then to any single-colored $\mathcal{V}$-operad $\mathcal{O}$, we can associate its r-matrix operad $\mathrm{Mat}_r(\mathcal{O})$, whose multimorphisms are given by
$$\mathrm{Mat}_r(\mathcal{O})(n) = \mathcal{O}(n)^{\oplus r^{n+1}}$$
regarded as matrices with n ingoing and one outgoing indices, and composition given by matrix products and composition in $\mathcal{O}$. It turns out that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Mat}_r(\mathcal{O})}(\mathcal{V})$.

just like for rings

These examples indicate that the answer to this question is highly dependent on $\mathcal{V}$, and more specifically *absolute weighted colimits* in $\mathcal{V}$-enriched category theory. A weight $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ on some $\mathcal{V}$-category $\mathcal{C}$ is called *absolute* if W-weighted colimits are preserved by any $\mathcal{V}$-enriched functor, and $\mathcal{C}$ is called *Cauchy-complete* if any absolute weight on it (which are precisely the atomic objects in the enriched presheaf category $\mathcal{P}_{\mathcal{V}}\mathcal{C})$ is representable, c.f. [**?**]. For instance, idempotent splittings are absolute colimits over $\mathcal{S}$, shifts are over $\mathcal{S}p$, and direct sums are over $\mathcal{S}p^{\mathrm{cn}}$, which is closely related to the above constructions.

introduce?

Analogously, we call an enriched operad Cauchy-complete if every $\otimes$-atomic object in $\mathcal{P}_{\mathcal{V}}^{\otimes}(\mathcal{O})$ is representable. These form a full subcategory $\mathcal{O}p_+(\mathcal{V}) \subseteq v\mathcal{O}p(\mathcal{V})$ whose inclusion admits a left adjoint $\widehat{(-)}^{\mathcal{V}}$. We prove the following characterization:

Proposition 1.2. Given $\mathcal{V}$-operads $\mathcal{O}, \mathcal{P} \in v\mathcal{O}p(\mathcal{V})$, the following are equivalent:

(1) The categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{V})$ are equivalent,
(2) The categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\bigsqcup_{k\geq 0} B\Sigma_k, \mathcal{V})) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Fun}(\bigsqcup_{k \geq 0} B\Sigma_k, \mathcal{V}))$ are equivalent,
(3) The categories of operadic presheaves $\mathcal{P}_{\mathcal{V}}^{\otimes}(\mathcal{O}) \simeq \mathcal{P}_{\mathcal{V}}^{\otimes}(\mathcal{P})$ are equivalent,
(4) The Cauchy-completions $\widehat{\mathcal{O}}^{\mathcal{V}} \simeq \widehat{\mathcal{P}}^{\mathcal{V}}$ are equivalent.

In this case, we call $\mathcal{O}$ and $\mathcal{P}$ *Morita-equivalent*.

Further, we prove in **??** that a $\mathcal{V}$-operad $\mathcal{O}$ is Cauchy-complete iff its underyling $\mathcal{V}$-category $\mathrm{Col}_{\mathcal{V}}(\mathcal{O})$ is[1]. This shows, for instance:

[1] We give an alternative proof **??** for non-unital operads, which is significantly easier.

- Given two ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$. For instance, this is true for the $\mathbb{E}_k$-operad and its variants with tangential structures, and for the $\mathbb{E}_X$-operad classifying constructible factorization algebras on a stratified manifold X (since exit-path categories are always idempotent complete).
- Given spectral ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are stable and idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}p) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}p)$. Similarly for operads enriched over the derived category of a ring (e.g. dg-operads).

In fact, studying the idempotent completion lets us classify all single-colored operads Morita-equivalent to a given single-colored $\mathcal{O} \in v\mathcal{O}p(\mathcal{V})$. Namely, they are completely specified by an algebra $\mathcal{P}(1) \in \mathrm{Alg}(\mathcal{V})$ that is Morita-equivalent to the algebra $\mathcal{O}(1)$ of 1-ary operations in $\mathcal{O}$; or equivalently a dualizable generator in $\mathrm{Mod}_{\mathcal{O}(1)}(\mathcal{V})$. For instance:

- If $\mathcal{V} = \mathrm{Vec}_k$ over a field k and given a single-colored Vec_k-enriched operad $\mathcal{O} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathcal{O}(1) \simeq k$, then any single-colored operad $\mathcal{P} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Vec}_k) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Vec}_k)$ is equivalent to $\mathrm{Mat}_r(\mathcal{O})$ for some $r \geq 0$.
- If $\mathcal{V} = D(k)$ and $\mathcal{O} \in v\mathcal{O}p(D(k))$ a single-colored dg operad with $\mathcal{O}(1) \simeq k[0]$, then (call this reduced) operads $\mathcal{P}$ with $\mathrm{Alg}_{\mathcal{O}}(D(k)) \simeq \mathrm{Alg}_{\mathcal{P}}(D(k))$ are classified by functions $f : \mathbb{Z} \to \mathbb{N}$ (say what I mean by that, some of these might accidentally be the same) that are zero almost everywhere but not everywhere. For instance the shifts $\mathcal{O}[r]$ correspond to the functions δ_r that are 1 at r and zero everywhere else, while $\mathrm{Mat}_r(\mathcal{O})$ corresponds to $r \cdot \delta_0$.

If we further equip the operadic presheaf category with information about which presheaves are generated by representables under the symmetric monoidal structure, we can recover a lot more: Introducing a notion of $\otimes$-*disjunctive enriched functors* between symmetric monoidal $\mathcal{V}$-categories we prove an enriched version of [**?**]:

Theorem 1.3. Let $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$ be the wide subcategory of $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$ on $\otimes$-disjunctive enriched functors, and similarly $\mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$. Then, the envelope functors

$$\mathrm{Env}_{\mathcal{V}} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$$
$$u \circ \mathrm{Env}_{\mathcal{V}} : \mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$$

are both fully faithful. (the latter is not proven yet, not sure if it is actually true for any $\mathcal{V}$.)

The first statement is relatively easy, since using the flagging we can reduce to recovering a space X from $\mathrm{Sym}X$; however the second uses some of the Cauchy-completion machinery we develop.

1.4. Acknowledgements.

2. Enriched Operads

For $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr})$ a presentably symmetric monoidal category, in order to define $\mathcal{V}$-enriched operads we make use of the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ whose

- underlying 1-category is the category $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) := \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ of presentably symmetric monoidal $\mathcal{V}$-module categories,
- morphism categories $\mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{M}, \mathcal{N})$ consist of symmetric monoidal $\mathcal{V}$-linear colimit-preserving functors.

Constructing this 2-category together with a symmetric monoidal structure $\bigotimes$ on it and verifying its relevant properties is slightly technical, so we list a few of them here referring to Appendix A for details: