Corollary 2.26. For $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$ and a an $\mathbb{E}_2$-algebra in it, a $\mathrm{LMod}_a(\mathcal{V})$-enriched category $\mathcal{C}$ is Cauchy-complete iff its underlying $\mathcal{V}$-category $U_!\mathcal{C}$ is Cauchy-complete.

Proof. An enriched category is Cauchy-complete iff any atomic presheaf over it is representable, so combine Lemma 2.24 and Lemma 2.25. □

2.3. **Characterization using the norm map.**

Notation 2.27. Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in v\mathcal{C}at(\mathcal{V})$ be valent $\mathcal{V}$-categories. We refer to

$$\mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{P}_{\mathcal{V}}(\mathcal{D}))$$

as the category of $\mathcal{V}$-*enriched profunctors* $\mathcal{C} \nrightarrow \mathcal{D}$. Given profunctors $P : \mathcal{C} \nrightarrow \mathcal{D}$ and $Q : \mathcal{D} \nrightarrow \mathcal{E}$, we write $P \otimes_{\mathcal{D}} Q : \mathcal{C} \nrightarrow \mathcal{E}$ for the composition $Q \circ P : \mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{P}_{\mathcal{V}}(\mathcal{E})$.

Observation 2.28. There is ample reason for this notation: By Eilenberg-Watts **??** a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is the same thing as a bimodule in ${}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{D}}(\mathrm{Fun}(ob\mathcal{C} \times ob\mathcal{D}, \mathcal{V}))$, and by [Lur17, Rem. 4.8.4.9] in this picture the composition

$$\otimes_{\mathcal{D}} : {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{D}}(\mathrm{Fun}(ob\mathcal{C}\times ob\mathcal{D}, \mathcal{V}))\times {}_{\mathcal{D}}\mathrm{Bimod}_{\mathcal{E}}(\mathrm{Fun}(ob\mathcal{D}\times ob\mathcal{E}, \mathcal{V})) \to {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{E}}(\mathrm{Fun}(ob\mathcal{C}\times ob\mathcal{E}, \mathcal{V}))$$

is given by the relative tensor product of bimodules. This can be written out as

$$P \otimes Q(c, e) \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} P \odot (\mathrm{Hom}_{\mathcal{D}})^{\odot n} \odot Q(c, e) \simeq$$

$$quad \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname*{colim}_{(d_0,\ldots,d_n)\in(ob\mathcal{D})^{\times n}} P(c, d_0) \otimes \mathrm{Hom}_{\mathcal{D}}(d_0, d_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{D}}(d_{n-1}, d_n) \otimes Q(d_n, e)$$

using [Lur17, Thm. 4.4.2.8] as well as our expression [**?**, Cor. 2.29] for the matrix product.

Example 2.29. A profunctor $B1_{\mathcal{V}} \nrightarrow \mathcal{C}$ is the same thing as a module functor $\mathcal{V} \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, i.e. an enriched presheaf on $\mathcal{C}$. Similarly a profunctor $\mathcal{C} \nrightarrow B1_{\mathcal{V}}$ can by identified as an enriched copresheaf on $\mathcal{C}$ using **??**. We obtain a canonical pairing sending an enriched presheaf $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ and an enriched copresheaf $V \in \mathcal{P}^{\vee}_{\mathcal{V}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{V})$ to the composition $W \otimes_{\mathcal{C}} V := V \circ W \in \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{V}, \mathcal{V}) \simeq \mathcal{V}$, which may as in Observation 2.28 be expanded as

$$W \otimes_{\mathcal{C}} V \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname*{colim}_{(c_0,\ldots,c_n)\in(ob\mathcal{C})^{\times n}} W(c_0) \otimes \mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(c_{n-1}, c_n) \otimes V(c_n)\ .$$

This is also referred to as the W-*weighted colimit* of V (imagined as an enriched functor $\mathcal{C} \to \mathcal{V}$).

Construction 2.30. For $\mathcal{C}, \mathcal{D}, \mathcal{E} \in v\mathcal{C}at(\mathcal{V})$ and $P : \mathcal{C} \nrightarrow \mathcal{D}$ an enriched profunctor, the composition maps

$$- \otimes_{\mathcal{C}} P = P \circ - : \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{E}), \mathcal{P}_{\mathcal{V}}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{E}), \mathcal{P}_{\mathcal{V}}(\mathcal{D}))$$
$$P \otimes_{\mathcal{D}} - = - \circ P : \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{D}), \mathcal{P}_{\mathcal{V}}(\mathcal{E})) \to \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{P}_{\mathcal{V}}(\mathcal{E}))$$

preserve colimits and hence admit rights adjoints, which we denote by $\underline{\mathrm{Nat}}_{\mathcal{D}}(P, -)$ and ${}_{\mathcal{C}}\underline{\mathrm{Nat}}(P, -)$ respectively. elaborate?

Example 2.31. We have seen in [**?**] that under Eilenberg-Watts, the identity functor $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ corresponds to the Yoneda bimodule $\text{よ}^{\mathcal{V}}_{\mathcal{C}} \in {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{C}}(\mathrm{Fun}(X \times X, \mathcal{V}))$. In particular, for any $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ regarded as a profunctor $B1_{\mathcal{V}} \nrightarrow \mathcal{C}$, the Yoneda-weighted colimit $W \otimes_{\mathcal{C}} \text{よ}^{\mathcal{V}}_{\mathcal{C}} = \mathrm{id}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})} \circ W \simeq W$ agrees with W. This is precisely the *coYoneda Lemma*: Any enriched presheaf is a weighted colimit of representable presheaves.