2. Morita categories

The contents of this section are disjoint from our discussion of enriched operads, so the reader may freely skip it. Its purpose is to prove Theorem 2.22, a criterion for 2-functors to preserve Eilenberg-Moore objects which may be of independent interest. Our main tool for this is the Morita-category of monads in a given 2-category, which was introduced in **??**.

2.1. **Segal objects and oplax Segal categories.** For the purposes of this section, we model $(\infty, 2)$-categories as Segal objects valued in $\mathcal{C}at$:

Notation 2.1. Let Δ be the category of non-empty finite totally ordered sets $[n] := \{0 < \cdots < n\}$ and order preserving maps. A map $\alpha : [n] \to [m]$ in Δ is called

- *inert* if it is a subinterval inclusion, i.e. $\alpha(k+1) = \alpha(k) + 1$ for all $0 \le k < n$,
- *idle* if it has no jumps, i.e. $\alpha(k+1) \le \alpha(k) + 1$ for all $0 \le k < n$,
- *active* if $\alpha(0) = 0$ and $\alpha(n) = m$.

Behold for each $[n] \in \Delta$ the *standard inert maps* $\rho_i : [1] \to [n]$ for $1 \le k \le n$, sending $0 \mapsto k-1$ and $1 \mapsto k$.

Definition 2.2. Let C be a category with finite limits. A *Segal object* in C is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{C}$ such that for each $[n] \in \Delta^{\mathrm{op}}$, the n distinct inert maps $\rho_i : [n] \leftarrow [1]$ and the unique inert map $[n] \leftarrow [0]$ in Δ^{op} exhibit

$$X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \; .$$

A Segal object X is called *univalent* or *(Rezk-)complete* if the map $X_0 \to X_3 \times_{X_1 \times X_1} (X_0 \times X_0)$ along $d_1, d_3 : [3] \leftarrow [1]$ is an isomorphism[1]. Denote by $\mathrm{CSeg(C)} \subseteq \mathrm{Seg(C)} \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{C})$ the full subcategories on (univalent) Segal objects.

Example 2.3. A *(complete) Segal space* is a (univalent) Segal object in $\mathcal{S}$. By [**?**], $\mathrm{CSeg}(\mathcal{S})$ is equivalent to the category $\mathcal{C}at$ of categories.

Example 2.4. A *(univalent) double category* is a univalent Segal object in $\mathcal{C}at$. It is an *(univalent)* 2*-category* if X_0 lies in $\mathcal{S} \subseteq \mathcal{C}at$. Denote by $2\,\mathcal{C}at \subseteq \mathrm{CSeg}(\mathcal{C}at)$ the full subcategory on 2-categories.

Remark 2.5. Equivalently, a double category can be encoded as a coCartesian fibration $\int X \to \Delta^{\mathrm{op}}$ such that the coCartesian transports along the specified maps $[n] \leftarrow [1], [0]$ induce the above equivalence. In other words, $\mathrm{Seg}(\mathcal{C}at)$ can also be regarded as a full subcategory of $\mathrm{coCart}_{/\Delta^{\mathrm{op}}}$.

Example 2.6. Since the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}at$ preserves limits, any complete Segal space $X : \Delta^{\mathrm{op}} \to \mathcal{S}$ can be regarded as a 2-category X^{hor} by composing with it, we call this the *horizontal embedding* $(-)^{\mathrm{hor}} : \mathcal{C}at \simeq \mathrm{CSeg}(\mathcal{S}) \hookrightarrow 2\,\mathcal{C}at$. Explicitly, this sends a category C to the functor $[n] \mapsto \mathrm{Fun}([n], \mathrm{C})$. For instance:

- The *terminal* 2*-category* $* := *^{\mathrm{hor}}$ is encoded by the constant functor $\mathrm{const}_* : \Delta^{\mathrm{op}} \to \mathcal{C}at$, i.e. by the coCartesian fibration $\mathrm{id}_{\Delta^{\mathrm{op}}} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$.
- The *walking arrow* $[1]^{\mathrm{hor}}$ is encoded by the coCartesian fibration $\Delta^{\mathrm{op}}_{/[1]} := (\Delta^{\mathrm{op}})_{[1]/} \to \Delta^{\mathrm{op}}$.
- More generally, $[n]^{\mathrm{hor}}$ is encoded by the coCartesian fibration $\Delta^{\mathrm{op}}_{/[n]} \to \Delta^{\mathrm{op}}$.

[1] Informally, this expresses that any 1-morphism $f \in X_1$ that admits a right and a left inverse comes from the degeneracy map $s_0 : X_0 \to X_1$.