ENRICHED ∞-OPERADS AS MARKED ALGEBRAS

MARKUS ZETTO

Abstract. We prove that an enriched ∞-operad is completely determined by its category of right modules together with a 'marking' by the representable modules. This description allows for a very explicit comparison of (colored) $\mathcal{S}$-enriched ∞-operads, defined as algebras in symmetric sequences, and Lurie's model of ∞-operads. Additionally, we show that the categories of algebras and right modules defined in both models agree.

We develop the theory of enriched ∞-operads by defining a Boardman-Vogt product, operadic weighted colimits, and consider the question how much about a $\mathcal{V}$-enriched ∞-operad $\mathcal{O}$ can be recovered from $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ or from its category of right modules, leading to an operatic version of Cauchy-completion.

Contents

1. Introduction

history

1.1. **Marked algebras.** explain the comparison functors

Proposition 1.1. A presentably symmetric monoidal $\mathcal{V}$-module category $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is equivalent to the operadic presheaf category $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ of some $\mathcal{V}$-operad $\mathcal{O}$ iff it is generated under colimits, monoidal structure and $\mathcal{V}$-tensoring by its $\otimes$-atomic objects.

1.2. **Applications.** We use the marked-module picture to develop the higher algebra of enriched ∞-operads, for instance we introduce:

- An envelope functor $\mathrm{Env}_{\mathcal{V}} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$,
- A Boardman-Vogt product $\otimes_{\mathrm{BV}}$ on $\mathcal{V}$-operads that is adjoint to taking $\mathcal{V}$-categories of algebras,
- Operadic weighted colimits (for instance, factorization homology),
- Free algebras,

Date: August 27, 2025, Hamburg.

generalizing the respective constructions in [Lur17].

1.3. **Recovering on operad from its algebras.** We will study the question how much information about an operad $\mathcal{O}$ can be recovered from its category of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. The following is more or less immediate from the above marked algebra description:

Proposition (??). Let $f : \mathcal{O} \to \mathcal{P}$ be a map of valent $\mathcal{V}$-enriched operads whose underlying map on spaces of colors $col\mathcal{O} \to col\mathcal{P}$ is an equivalence. If the induced precomposition map $f^* : \mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is an equivalence, then f is already an equivalence.

However, we might now always be given a comparison map f, but only the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ itself. Certainly, we can not fully recover $\mathcal{O}$ in this case:

introduce notation?

- Let C be a category and $\widehat{\mathrm{C}}^{\mathrm{ic}}$ its idempotent completion. Then
$$\mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\mathrm{C})}(\mathcal{V}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{V}) \simeq \mathrm{Fun}(\hat{\mathrm{C}}^{\mathrm{ic}}, \mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\widehat{\mathrm{C}}^{\mathrm{ic}})}(\mathcal{V}) \ .$$
- Given a spectral operad $\mathcal{O}$, we can define its *r-fold shift* $\mathcal{O}[r]$ whose multimorphism objects are given by
$$\mathrm{Mul}_{\mathcal{O}[r]}(o_1, \ldots, o_n; o) \simeq \mathrm{Mul}_{\mathcal{O}}(o_1, \ldots, o_n; o)[r \cdot (1-n)] \ ,$$
consider [**?**, Constr. 2.34] for a single-colored version of this construction. The auto-equivalence $[-r] : \mathcal{V} \xrightarrow{\simeq} \mathcal{V}$ induces, for any $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$, an equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{O}[r]}(\mathcal{V})$.
- Let $\mathcal{V}$ be semiadditive. Then to any single-colored $\mathcal{V}$-operad $\mathcal{O}$, we can associate its r-matrix operad $\mathrm{Mat}_r(\mathcal{O})$, whose multimorphisms are given by
$$\mathrm{Mat}_r(\mathcal{O})(n) = \mathcal{O}(n)^{\oplus r^{n+1}}$$
regarded as matrices with n ingoing and one outgoing indices, and composition given by matrix products and composition in $\mathcal{O}$. It turns out that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Mat}_r(\mathcal{O})}(\mathcal{V})$.

just like for rings

These examples indicate that the answer to this question is highly dependent on $\mathcal{V}$, and more specifically *absolute weighted colimits* in $\mathcal{V}$-enriched category theory. A weight $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ on some $\mathcal{V}$-category $\mathcal{C}$ is called *absolute* if W-weighted colimits are preserved by any $\mathcal{V}$-enriched functor, and $\mathcal{C}$ is called *Cauchy-complete* if any absolute weight on it (which are precisely the atomic objects in the enriched presheaf category $\mathcal{P}_{\mathcal{V}}\mathcal{C}$) is representable, c.f. [**?**]. For instance, idempotent splittings are absolute colimits over $\mathcal{S}$, shifts are over $\mathcal{S}p$, and direct sums are over $\mathcal{S}p^{\mathrm{cn}}$, which is closely related to the above constructions.

introduce?

Analogously, we call an enriched operad Cauchy-complete if every $\otimes$-atomic object in $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ is representable. These form a full subcategory $\mathcal{O}p_+(\mathcal{V}) \subseteq v\mathcal{O}p(\mathcal{V})$ whose inclusion admits a left adjoint $\widehat{(-)}^{\mathcal{V}}$. We prove the following characterization:

Proposition 1.2. Given $\mathcal{V}$-operads $\mathcal{O}, \mathcal{P} \in v\mathcal{O}p(\mathcal{V})$, the following are equivalent:

(1) The categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{V})$ are equivalent,
(2) The categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\bigsqcup_{k\geq 0} B\Sigma_k, \mathcal{V})) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Fun}(\bigsqcup_{k\geq 0} B\Sigma_k, \mathcal{V}))$ are equivalent,
(3) The categories of operadic presheaves $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) \simeq \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})$ are equivalent,
(4) The Cauchy-completions $\widehat{\mathcal{O}}^{\mathcal{V}} \simeq \widehat{\mathcal{P}}^{\mathcal{V}}$ are equivalent.

In this case, we call $\mathcal{O}$ and $\mathcal{P}$ *Morita-equivalent.*

Further, we prove in **??** that a $\mathcal{V}$-operad $\mathcal{O}$ is Cauchy-complete iff its underyling $\mathcal{V}$-category $\mathrm{Col}_{\mathcal{V}}(\mathcal{O})$ is[1]. This shows, for instance:

[1]We give an alternative proof **??** for non-unital operads, which is significantly easier.

- Given two ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$. For instance, this is true for the $\mathbb{E}_k$-operad and its variants with tangential structures, and for the $\mathbb{E}_X$-operad classifying constructible factorization algebras on a stratified manifold X (since exit-path categories are always idempotent complete).
- Given spectral ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are stable and idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}p) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}p)$. Similarly for operads enriched over the derived category of a ring (e.g. dg-operads).

In fact, studying the idempotent completion lets us classify all single-colored operads Morita-equivalent to a given single-colored $\mathcal{O} \in v\mathcal{O}p(\mathcal{V})$. Namely, they are completely specified by an algebra $\mathcal{P}(1) \in \mathrm{Alg}(\mathcal{V})$ that is Morita-equivalent to the algebra $\mathcal{O}(1)$ of 1-ary operations in $\mathcal{O}$; or equivalently a dualizable generator in $\mathrm{Mod}_{\mathcal{O}(1)}(\mathcal{V})$. For instance:

- If $\mathcal{V} = \mathrm{Vec}_k$ over a field k and given a single-colored Vec_k-enriched operad $\mathcal{O} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathcal{O}(1) \simeq k$, then any single-colored operad $\mathcal{P} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Vec}_k) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Vec}_k)$ is equivalent to $\mathrm{Mat}_r(\mathcal{O})$ for some $r \geq 0$.
- If $\mathcal{V} = D(k)$ and $\mathcal{O} \in v\mathcal{O}p(D(k))$ a single-colored dg operad with $\mathcal{O}(1) \simeq k[0]$, then operads $\mathcal{P}$ with $\mathrm{Alg}_{\mathcal{O}}(D(k)) \simeq \mathrm{Alg}_{\mathcal{P}}(D(k))$ are classified by functions $f : \mathbb{Z} \to \mathbb{N}$ that are zero almost everywhere but not everywhere. For instance the shifts $\mathcal{O}[r]$ correspond to the functions δ_r that are 1 at r and zero everywhere else, while $\mathrm{Mat}_r(\mathcal{O})$ corresponds to $r \cdot \delta_0$.

call this reduced

say what I mean by that, some of these might accidentally be the same

If we further equip the operadic presheaf category with information about which presheaves are generated by representables under the symmetric monoidal structure, we can recover a lot more: Introducing a notion of $\otimes$-*disjunctive enriched functors* between symmetric monoidal $\mathcal{V}$-categories we prove an enriched version of [**?**]:

Theorem 1.3. Let $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$ be the wide subcategory of $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$ on $\otimes$-disjunctive enriched functors, and similarly $\mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$. Then, the envelope functors

$$\mathrm{Env}_{\mathcal{V}} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$$
$$u \circ \mathrm{Env}_{\mathcal{V}} : \mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$$

are both fully faithful.

the latter is not proven yet, not sure if it is actually true for any $\mathcal{V}$.

The first statement is relatively easy, since using the flagging we can reduce to recovering a space X from $\mathrm{Sym}X$; however the second uses some of the Cauchy-completion machinery we develop.

1.4. Acknowledgements.

2. Enriched Operads

For $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr})$ a presentably symmetric monoidal category, in order to define $\mathcal{V}$-enriched operads we make use of the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ whose

- underlying 1-category is the category $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) := \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ of presentably symmetric monoidal $\mathcal{V}$-module categories,
- morphism categories $\mathrm{Fun}_{\mathcal{V}}^{\mathrm{L},\otimes}(\mathcal{M}, \mathcal{N})$ consist of symmetric monoidal $\mathcal{V}$-linear colimit-preserving functors.

Constructing this 2-category together with a symmetric monoidal structure $\bigotimes$ on it and verifying its relevant properties is slightly technical, so we list a few of them here referring to Appendix A for details:

say more about the symmetric monoidal structures? Or dont mention them at all here?

- By Observation A.8 there exist symmetric monoidal 2-functors
$$(\mathbb{C}\mathrm{at}, \times) \xrightarrow{\mathrm{Sym}} (\mathrm{CAlg}(\mathbb{C}\mathrm{at}), \bigotimes) \xrightarrow{\mathcal{P}} (\mathrm{CAlg}(\mathbb{P}\mathrm{r}), \bigotimes) \xrightarrow{-\otimes\mathcal{V}} (\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}), \bigotimes)$$
that are partially left adjoint to the forgetful 2-functors $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathrm{CAlg}(\mathbb{P}\mathrm{r}) \to \mathrm{CAlg}(\widehat{\mathbb{C}\mathrm{at}}) \to \widehat{\mathbb{C}\mathrm{at}}$. The symmetric monoidal structures $\bigotimes$ are described in **??**. Their composite sends a category C to
$$\mathcal{P}(\mathrm{Sym}\mathrm{C}) \otimes \mathcal{V} \simeq \mathrm{Fun}(\mathrm{Sym}\mathrm{C}^{\mathrm{op}}, \mathcal{V}) \simeq \mathrm{Fun}\big(\coprod_{n\geq 0} (\mathrm{C}^{\mathrm{op}})^{\times n}_{h\Sigma_n}, \mathcal{V}\big) \ .$$

self-enrichment? see commented out

- The 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ admits all partially (op)lax colimits. In particular it admits Eilenberg-Moore-objects, which are created by all of the above forgetful 2-functors.

Definition 2.1. Given a space $X \in \mathcal{S}$, a *$\mathcal{V}$-enriched operad with colors X* is a monad on $\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V} \simeq \mathrm{Fun}(\mathrm{Sym}X^{\mathrm{op}}, \mathcal{V})$ in the 2-category $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$. Using the above adjunction and that $X^{\mathrm{op}} \simeq X$, this unwinds to an algebra in
$$\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V}) \simeq \mathrm{Fun}(X, \mathrm{Fun}(\mathrm{Sym}X^{\mathrm{op}}, \mathcal{V})) \simeq \mathrm{Fun}\left(X \times \mathrm{Sym}X, \mathcal{V}\right) \ .$$
We refer to $\mathrm{sSeq}_X(\mathcal{V}) := \mathrm{Fun}\left(X \times \mathrm{Sym}X, \mathcal{V}\right)$, equipped with the monoidal structure $\circledcirc$ it obtains as an endomorphism object via composition, as the category of *X-colored symmetric sequences* in $\mathcal{V}$.

Warning 2.2. We will see in Corollary 2.5 that the monoidal structure $\circledcirc$ defined on $\mathrm{sSeq}_X(\mathcal{V})$ by being an endomorphism object is reverse to the commonly used composition product on symmetric sequences, which we denote $\circledcirc$. Of course, their categories of algebras are equivalent.

possibly just use the usual to avoid confusion? Compare with literature?

Notation 2.3. We write $\underline{x}$ for a sequence $(x_1, \ldots, x_n) \in \mathrm{Sym}X$.

Lemma 2.4. Let $X \in \mathcal{C}at$ and $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. Under the equivalence $\mathrm{Fun}(X, \mathcal{M}) \simeq \mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\mathrm{Sym}X \otimes \mathcal{V}, \mathcal{M})$ from Observation A.8, a functor $A : X \to \mathcal{M}$ is sent to the morphism $\bar{A} : \mathrm{Fun}(\mathrm{Sym}X^{\mathrm{op}}, \mathcal{V}) \to \mathcal{M}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ which maps

to the coend

$$(W : \mathrm{Sym}X^{\mathrm{op}} \to \mathcal{V}) \mapsto \oint^{\underline{x} \in \mathrm{Sym}X} A(x_1) \otimes \cdots \otimes A(x_n) \otimes W(\underline{x}) \ .$$

Proof. As a first step, A extends to a unique symmetric monoidal functor $\mathrm{Sym}X \to \mathcal{M}$ sending $(x_1, \ldots, x_n) \mapsto A(x_1) \otimes \cdots \otimes A(x_n)$. From here, the formula for the extension to $\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V}$ follows from [**?**, arg1 arg2]. ☐

Corollary 2.5. The monoidal structure $\circledcirc$ on $\mathrm{sSeq}_X(\mathcal{V}) \simeq \mathrm{Fun}\left(X \times \mathrm{Sym}X, \mathcal{V}\right)$ sends $A, B : X \times \mathrm{Sym}X \to \mathcal{V}$ to
$$A \circledcirc B(z, \underline{x}) = \oint^{\underline{y} \in \mathrm{Sym}X} \operatorname*{colim}_{\underline{\tilde{x}}^{(1)} \otimes \cdots \otimes \underline{\tilde{x}}^{(k)} \to \underline{x}} A(y_1, \underline{\tilde{x}}^{(1)}) \otimes \cdots \otimes A(y_k; \underline{\tilde{x}}^{(k)}) \otimes B(z; \underline{y}) \ .$$
If additionally X is a space, this agrees with
$$\coprod_{k\geq 0} \coprod_{n_1+\cdots+n_k=n} \operatorname*{colim}_{(y_1,\ldots,y_k)\in X^{\times k}} \big(A(y_1; x_1, \ldots, x_{n_1}) \otimes \cdots \otimes A(y_k; x_{n_1+\cdots+n_{k-1}+1}, \ldots, x_n)\big) \otimes_{\Sigma_k} B(z; y_1, \ldots, y_k)$$
where the Σ_k-action permutes the blocks partitioning $\underline{x}$ and the entries of $\underline{y}$. Specifically for $X = *$ we obtain
$$\coprod_{k\geq 0} \coprod_{n_1+\cdots+n_k=n} (A(n_1) \otimes \cdots \otimes A(n_k)) \otimes_{\Sigma_n} B(n) \ .$$

Proof. If we denote by $\bar{A}, \bar{B}$ the endofunctors of $\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V}$ associated to A and B, then by definition we can write $A \circledast B(z, (x_i)) = \bar{A} \circ \bar{B}(\text{よ}_z \otimes 1_{\mathcal{V}})(x_1, \ldots, x_n) = \bar{A}(B(z; -))(x_1, \ldots, x_n)$ so the first expression follows from Lemma 2.4 as well as the formula [Lur17, arg1 arg2] for the Day convolution product.

By [Lur18a, Tag arg1], for a space X we have $\mathrm{Tw}(X) \simeq X$ so coends agree with colimits over X. Further, the functor $B\Sigma_k \to \mathcal{S}$ encoding the Σ_k-action on $X^{\times k}$ unstraightens into a coCartesian fibration $X^{\times k}_{h\Sigma_k} \to B\Sigma_k$ with fiber $X^{\times k}$. Hence, $\mathrm{colim}_{X^{\times k}_{h\Sigma_k}} \simeq \mathrm{colim}_{B\Sigma_k} \mathrm{colim}_{X^{\times k}}$. Finally, since $X^{\times k}_{h\Sigma_k}$ is a space, any morphism in it is an equivalence meaning that $\mathrm{colim}_{\underline{\tilde{x}}^{(1)} \otimes \cdots \otimes \underline{\tilde{x}}^{(k)} \to \underline{x}}$ becomes a colimit over partitions of $\underline{x}$. $\square$

Construction 2.6. The symmetric algebra functor

$$\mathrm{Sym} : \mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}) \to \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}))$$

is symmetric monoidal and thereby enhances to a $\mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}})$-enriched functor. In particular it induces an adjunction between endomorphism objects possibly this enrichment does not exist

$$\mathrm{Fun}(X \times X, \mathcal{V}) \simeq \mathrm{End}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}(X) \otimes \mathcal{V}) \leftrightarrows \mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\mathrm{Sym}X \otimes \mathcal{V}) \simeq \mathrm{Fun}(X \times \mathrm{Sym}X, \mathcal{V})$$

whose left adjoint is monoidal for composition, agrees with the left-Kan-extension functor and is hence fully faithful. We obtain an adjunction on algebra objects

$$\mathrm{Triv}_{\mathcal{V}} : v\mathcal{C}at_X(\mathcal{V}) \leftrightarrows v\mathcal{O}p_X(\mathcal{V}) : \mathrm{Col}_{\mathcal{V}} \ ,$$

where the fully faithful left adjoint $\mathrm{Triv}_{\mathcal{V}}$ sends a $\mathcal{V}$-enriched category to the *trivial $\mathcal{V}$-operad* might be a bad name
with colors $\mathcal{C}$, while $\mathrm{Col}_{\mathcal{V}}$ sends a $\mathcal{V}$-operad to its *$\mathcal{V}$-category of colors*.

Construction 2.7. A functor $f : \mathcal{V} \to \mathcal{W}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ induces a symmetric monoidal functor $\mathrm{Pr}_{\mathcal{V}} \to \mathrm{Pr}_{\mathcal{W}}$ and hence a symmetric monoidal 2-functor $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{W}})$. As in Construction 2.6, we obtain a *change-of-enrichment* adjunction

$$f_! : v\mathcal{O}p_X(\mathcal{V}) \leftrightarrows v\mathcal{O}p_X(\mathcal{W}) : f_!^{\mathrm{R}}$$

that acts on multigraphs by postcomposing $\mathrm{Mul}_{\mathcal{O}} : X \times \mathrm{Sym}X \to \mathcal{V}$ with f (and postcom- notation
poting with f^{R} in the other direction).

Definition 2.8. Given a $\mathcal{V}$-operad $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$, we define its *operadic (enriched) presheaf category* as the Eilenberg-Moore object

$$\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) := \mathrm{LMod}_{\mathcal{O}}(\mathcal{P}\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V}) \in \mathrm{CAlg}(\mathbb{P}\mathrm{r}) \ .$$

Also, we define the *operadic Yoneda functor* $\text{よ} : col\mathcal{O} \to \mathcal{P}\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ by composing the unit of the free-forgetful adjunction $\widehat{\mathcal{C}at} \leftrightarrows \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}))$ with the free module functor.

Remark 2.9. This is classically known as the category of *right modules* over $\mathcal{O}$, c.f. [**?**]. The unfortunate change in direction comes from the fact that our composition product is reverse to the usual convention.

Construction 2.10. Given any $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr})$, the category $\mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V}, \mathcal{M}) \simeq \mathrm{Fun}(X, \mathcal{M})$ admits a $\mathcal{V}$-linear colimit-preserving right tensoring by $\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}X) \otimes \mathcal{V}) =$ not clear
$\mathrm{sSeq}_X(\mathcal{V})$ via precomposition.

Definition 2.11. Given a $\mathcal{V}$-enriched operad $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$ and $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr})$, we define the $\mathcal{V}$-enriched category of *$\mathcal{O}$-algebras* in $\mathcal{M}$ as

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) := \mathrm{RMod}_{\mathcal{O}}(\mathrm{Fun}(X, \mathcal{M})) \ .$$