Cauchy-complete (∞, n)-categories and Higher Idempotents

Markus Zetto

University of Hamburg

Joint work in progress with David Reutter

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Content

Motivation

Additive Categories

Definition

An *additive category* is an Ab-enriched category that admits finite (including empty) coproducts.

This agrees with the usual notion, in particular the initial object is terminal and coproducts agree with products.

Proof Sketch.

There can only exist precisely one morphism from the initial object $\emptyset$ to itself, so $\mathrm{id}_\emptyset$ and the zero morphism must agree. Given any $f : c \to \emptyset$ from some $c \in \mathcal{C}$, this implies

$$f = \mathrm{id}_\emptyset \circ f = 0 \circ f = 0$$

so there is precisely one morphism from any c to $\emptyset$. (Co)Products work similarly. $\square$

Additive Categories

We say that initial objects are absolute colimits when enriching over pointed sets, and finite coproducts are absolute colimits over Ab:

- They can be written as limits,
- Any enriched functor preserves them,
- They are characterized by a diagrammatic property:

$$\begin{tikzcd}
X \arrow[dr, "i_1"'] & & X \\
& X \oplus Y \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
Y \arrow[ur, "i_2"] & & Y
\end{tikzcd}$$

Are those all absolute colimits over Ab? And what about other enrichment categories?

Motivation: Morita Theory

Can we recover:

- A ring R from its category of modules $\mathrm{LMod}_R(\mathrm{Ab})$?
- A ring R from its derived category $D(R) \simeq \mathrm{LMod}_{HR}(\mathrm{Sp})$?
- A category $\mathcal{C}$ from its presheaf category $\mathcal{P}(\mathcal{C})$?
- A $\mathcal{V}$-enriched category $\mathcal{C}$ from its enriched presheaf category $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) := \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}^{op}, \mathcal{V})$?

Generally not, only up to Morita equivalence. Note that the case of enriched ∞-categories subsumes all of the above.

Motivation: Morita Theory

- If we knew which module $M \in \mathrm{LMod}_R(\mathrm{Ab})$ was the trivial module ${}_R R$, we could recover $R \simeq \underline{\mathrm{End}}_R({}_R R)$.
- We know ${}_R R$ is always dualizable and generates $\mathrm{LMod}_R(\mathrm{Ab})$ under colimits.
- However by the main theorem of Morita theory, *any* dualizable module M that generates $\mathrm{LMod}_R(\mathrm{Ab})$ under colimits satisfies $\mathrm{LMod}_R(\mathrm{Ab}) \simeq \mathrm{LMod}_{\underline{\mathrm{End}}_R(M)}(\mathrm{Ab})$. Any ring Morita-equivalent to R arises this way.

Recovering a Category from its Presheaf Category

Similarly, we could recover $\mathcal{C}$ as a full subcategory of $\mathcal{P}(\mathcal{C})$ if we knew which presheaves are representable:

Observation

The Yoneda embedding $よ : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ embeds $\mathcal{C}$ as a full subcategory. Also, for $c \in \mathcal{C}$ and $F \in \mathcal{P}(\mathcal{C})$,

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(よ_c, F) \simeq F(c) = \mathrm{ev}_c(F)$$

implying that $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(よ_c, -) : \mathcal{P}(\mathcal{C}) \to \mathrm{Set}$ preserves colimits. Because of this, we call $よ_c \in \mathcal{P}(\mathcal{C})$ *tiny*.

Recovering a Category from its Presheaf Category

Proposition

The tiny objects in $\mathcal{P}(\mathcal{C})$ are precisely the retracts of representable presheaves. The full subcategory of $\mathcal{P}(\mathcal{C})$ on the tiny presheaves is called the *idempotent completion* of $\mathcal{C}$.

$$\begin{tikzcd}
Y \arrow[d, bend left, "i"] \\
X \arrow[u, bend left, "r"] \arrow[loop below, "e=ir"]
\end{tikzcd}$$

Enriched ∞-categories

Tiny Objects

Fix a presentably monoidal ∞-category $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, and a presentably $\mathcal{V}$-tensored ∞-category $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$.

Definition

An object $m \in \mathcal{M}$ is called *tiny* if the internal Hom

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(m, -) : \mathcal{M} \to \mathcal{V}$$

preserves colimits and $\mathcal{V}$-tensoring.

Valent enriched ∞-categories

Definition

A *valent $\mathcal{V}$-enriched ∞-category* $\mathcal{C}$ is specified by

- An underlying space X of objects,
- An enriched presheaf category $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) \in \mathrm{Pr}_{\mathcal{V}}$,
- A Yoneda functor $よ^{\mathcal{V}} : X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$,

satisfying the following conditions:

1. The full image $\mathrm{Im}(よ^{\mathcal{V}}) \subseteq \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ consists of tiny objects,
2. The full image $\mathrm{Im}(よ^{\mathcal{V}})$ generates $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ under colimits and $\mathcal{V}$-tensoring.

In other words, valent $\mathcal{V}$-categories form a full subcategory $\mathrm{vCat}_X(\mathcal{V})$ of $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})_{X/}$. We say that $よ^{\mathcal{V}}$ *marks* the representable presheaves.

Valent enriched ∞-categories

Similarly, define $\mathrm{vCat}(\mathcal{V})$ as a full subcategory of the pullback

$$\mathcal{S} \times_{\mathsf{Fun}(\{0\},\mathsf{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}))} \mathsf{Fun}([1], \mathsf{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}))$$

and vEnr using $\mathsf{Fun}([1], \mathsf{RMod}(\mathrm{Pr}^{\mathrm{L}}))$.

Theorem (Reutter, Z.)

They are equivalent to Gepner-Haugseng's categorical algebras:

$$\mathrm{vCat}_X(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{Ass}_X}(\mathcal{V}) \quad \mathrm{vCat}(\mathcal{V}) \simeq \int^{X\in\mathcal{S}} \mathrm{Alg}_{\mathrm{Ass}_X}(\mathcal{V})$$

This equivalence is functorial in $\mathcal{V}$, and compatible with the respective monoidal structures.

Univalence

Among all markings $X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ specifying the same representable presheaves, one is distinguished:

Definition

A valent $\mathcal{V}$-category $\mathcal{C}$ is called *univalent* if the Yoneda-functor $よ^{\mathcal{V}} : X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is a monomorphism, i.e. exhibits $X \simeq \mathrm{Im}(よ^{\mathcal{V}})^{\simeq}$.
Denote by $\mathrm{Cat}(\mathcal{V}) \subseteq \mathrm{vCat}(\mathcal{V})$ their full subcategory.
The *univalization* of $よ^{\mathcal{V}} : X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is defined as $\mathrm{Im}(よ^{\mathcal{V}})^{\simeq} \hookrightarrow \mathcal{P}_{\mathcal{V}}(\mathcal{C})$.

This corresponds to Gepner-Haugseng's notion of completion, in particular Rezk-completeness of Segal spaces in case $\mathcal{V} = \mathcal{S}$.

Cauchy-complete $\mathcal{V}$-categories

Among *all* markings $X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, one is distinguished:

Definition

A univalent $\mathcal{V}$-category $\mathcal{C}$ is called *Cauchy-complete* if any tiny presheaf is representable, i.e. $\text{ よ}^{\mathcal{V}} : X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ exhibits $X \simeq \mathcal{M}^{\text{tiny}, \simeq}$. Denote by $\mathrm{CauchyCat}(\mathcal{V}) \subseteq \mathrm{Cat}(\mathcal{V})$ their full subcategory.
Given a valent $\mathcal{V}$-category $\mathcal{C} = (X \to \mathcal{P}_{\mathcal{V}}(\mathcal{C}))$, we can just replace X by $\mathcal{M}^{\text{tiny}, \simeq}$ to obtain its *Cauchy-completion* $\hat{\mathcal{C}}$.

By definition, every Morita-equivalence class contains an essentially unique Cauchy-complete $\mathcal{V}$-category.

Remark

A $\mathcal{V}$-category $\mathcal{C}$ is Cauchy-complete iff it "admits all absolute weighted colimits".

Examples

- CauchyCat $(\mathbb{R}_{\geq 0}, \geq, +)$ is the category of generalized metric spaces where all Cauchy-sequences converge.
- CauchyCat(Set) consists of idempotent complete categories, and CauchyCat($\mathcal{S}$) of idemp. compl. ∞-categories.
- CauchyCat($\mathrm{Cat}_{(n-1,m-1)}$) consists of idemp. compl. (n, m)-categories for $1 \leq m \leq n \leq \infty$.
- CauchyCat(Set_*) consists of idemp. compl. categories with zero object, similarly for $\mathcal{S}_*$.
- CauchyCat(Ab) consists of idemp. compl. additive categories, and CauchyCat($\mathrm{Sp}^{\mathrm{cn}}$) of idemp. compl. additive ∞-categories.
- CauchyCat(Sp) consists of idemp. compl. stable ∞-categories.
- Unfortunately, only idempotent splittings are absolute over CauchyCat(Cond($\mathcal{S}$)), just as over any local ∞-topos.

Cauchy-complete (∞, n)-categories

Iterative Cauchy-completion

If $\mathcal{V}$ is symmetric monoidal, so is $\mathrm{vCat}(\mathcal{V})$: The tensor product $\mathcal{C} \otimes \mathcal{D}$ has space of objects $X \times Y$ and morphism objects

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((x,y),(x',y')) := \mathrm{Hom}_{\mathcal{C}}(x,y) \otimes \mathrm{Hom}_{\mathcal{D}}(x',y') \in \mathcal{V}\,.$$

This localizes to a symmetric monoidal structure $\hat{\otimes}$ on $\mathrm{CauchyCat}(\mathcal{V})$ with $\mathcal{C}\hat{\otimes}\mathcal{D} := \widehat{\mathcal{C}\otimes\mathcal{D}}$ and unit $\widehat{B1_{\mathcal{V}}}$.

Theorem (WIP)

The construction of associating to symmetric monoidal $\mathcal{V}$ its symmetric monoidal $\mathrm{CauchyCat}(\mathcal{V})$ *assembles into a functor*

$$\mathrm{CauchyCat}(-) : \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})\,.$$

In particular, we can iterate this endofunctor to define the category of *Cauchy-complete $\mathcal{V}$-enriched (∞, n)-categories* $\mathrm{CauchyCat}^n(\mathcal{V})$.

Higher Idempotents

Following [Gaiotto, Johnson-Freyd] a *2-retraction* in a 2-category $\mathcal{C}$ is a diagram of shape

$$\begin{tikzcd}
Y \arrow[d, bend left=60, "i_1"] \arrow[d, Rightarrow, bend left=20, "i_2"] \\
X \arrow[u, bend left=60, "r_1"] \arrow[u, Rightarrow, bend left=20, "r_2"]
\end{tikzcd}$$

where $r_2 : \mathrm{id}_X \to r_1 i_1, i_2 : r_1 i_1 \to \mathrm{id}_Y$ and $r_2 i_2 \simeq \mathrm{id}_{id_Y}$. Similarly define n-retractions for $n \leq \infty$.

Examples

- $\mathrm{CauchyCat}^n(\mathcal{S})$ consists of n-idemp. compl. (∞, n)-categories.
- $\mathrm{CauchyCat}^n(\mathrm{Ab})$ consists of n-idemp. compl. additive (n, n)-categories, similarly for $\mathcal{S}\mathrm{p}^{\mathrm{cn}}$.
- $\mathrm{CauchyCat}(\mathrm{LMod}_R(\mathrm{Ab}))$ consists of n-idemp. compl. R-linear (n, n)-categories, similarly for $\mathrm{LMod}_{HR}(\mathcal{S}\mathrm{p}^{\mathrm{cn}})$.
- $\widehat{\mathrm{CauchyCat}}(\widehat{\mathrm{Cat}}^{\mathrm{colim}})$ consists of 2-idemp. compl. locally cocomplete $(\infty, 2)$-categories admitting lax colimits, called *lax semiadditive* in [Christ, Dyckerhoff, Walde].
- $\mathrm{CauchyCat}^2(\mathcal{S}\mathrm{p})$ consists of 2-idemp. compl. locally stable $(\infty, 2)$-categories with recollements (lax colimits over Δ^1) [anticipated by Campion].
- $\mathrm{CauchyCat}^n(\mathrm{CMon}_m(\mathcal{S})) \subseteq \mathrm{CauchyCat}^{n-1}(\mathrm{Cat}^{m-\mathrm{sa}})$ consists of n-idemp. compl. m-semiadditive (∞, n)-categories, in the sense of [WIP by Scheimbauer, Walde].

Constructing Fully Dualizable Categories

Theorem (WIP, parts shown by Gaiotto, Johnson-Freyd)

The monoidal unit $\Sigma^n_{\mathcal{V}} 1 := \widehat{B \ldots \widehat{B 1_{\mathcal{V}}}} \in \mathrm{CauchyCat}^n(\mathcal{V})$ *is a fully dualizable* $(n-1)$*-category. In fact for* $\mathcal{V} = \mathrm{Vec}_k$*, it is the category of fully dualizable n-vector spaces.*

Example

- $\Sigma_{\mathcal{V}} 1_{\mathcal{V}}$ are the dualizable objects in $\mathcal{V}$.
- $\Sigma^2_{\mathrm{Vec}_k} k$ is the Morita category of separable k-algebras.
- $\Sigma^3_{\mathrm{Vec}_k} k$ is the Morita cat. of separable k-multifusion categories.
- $\Sigma^2_{D(k)} k[0]$ is the Morita cat. of smooth proper dg-algebras.

Much is still to study about these categories, e.g. higher Tannaka duality results, ∞-semiadditive structure...

Decategorification Functors

Given a lax monoidal functor $d : \mathrm{CauchyCat}(\mathcal{V}) \to \mathcal{V}$, change-of-enrichment along it followed by Cauchy-completion induces a diagram

$$\cdots \to \mathrm{CauchyCat}^3(\mathcal{V}) \longrightarrow \mathrm{CauchyCat}^2(\mathcal{V}) \xrightarrow{\hat{d}_!} \mathrm{CauchyCat}(\mathcal{V}) \to \mathcal{V}$$

so in particular functors $\mathrm{CauchyCat}^n(\mathcal{V}) \to \mathcal{V}$. We write $\mathrm{CauchyCat}^\infty(\mathcal{V}, d)$ for its limit.

Example

- For $\mathcal{V} = \mathcal{S}$, the realization $|-|$ and the maximal subgroupoid functor $(-)^{\simeq}$.
- For $\mathcal{V} = \mathrm{Ab}$, the Grothendieck group of K_0.
- For $\mathcal{V} = \mathcal{S}\mathrm{p}$, K-theory of stable categories K.
- For $\mathcal{V} = \mathcal{S}\mathrm{p}^{\mathrm{cn}}$, K-theory of additive categories $K \circ K^b$.

Thank you for listening!

Slides available at `www.markus-zetto.com/enriched`