ENRICHED ∞-OPERADS AS MARKED ALGEBRAS

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Abstract. The following is a preliminary and partial draft (last modified 03.11.25); an up-to-date version is maintained at `www.markus-zetto.com/enriched-operads.pdf`.

We prove that an enriched ∞-operad is completely determined by its category of right modules together with a 'marking' by the representable modules. This description allows for a direct comparison of (colored) $\mathcal{S}$-enriched ∞-operads, defined as algebras in symmetric sequences, and Lurie's model of ∞-operads. Additionally, we show that the categories of algebras and right modules defined in both models agree.

We develop the theory of enriched ∞-operads by defining a Boardman-Vogt product, operadic weighted colimits and free algebras; and consider the question how much about a $\mathcal{V}$-enriched ∞-operad $\mathcal{O}$ can be recovered from its categories of algebras or from its category of right modules, leading to an operatic version of Cauchy-completion. This resolves questions about the Morita-theory of ∞-operads posed in [KM01].

Finally, we provide a new proof that the envelope functor from ∞-operads to symmetric monoidal ∞-categories is a subcategory inclusion, and investigate when this is the case for general enrichment.

Contents

1. Introduction

The notion of a *(colored) operad*, or multicategory, generalizes that of a category by allowing for *k-ary multimorphisms* with k sources and a single target, for any $k \geq 0$. Operads were originally introduced in algebraic topology to encode the algebraic structure on iterated loop spaces [BV06], [May06]. The *little cubes operads* $\mathbb{E}_n$ introduced for this purpose already indicated two fundamental features that a general theory of operads should accommodate:

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- The sets of n-ary multimorphisms often carry additional structure, in this case they are chain complexes. One is thus naturally let to enriched operads, which generalize enriched categories.
- For a morphism of operads to be an equivalence it should suffice for it to induce quasi-isomorphisms between the multimorphism chain complexes, i.e. in modern language we want to homotopy coherently enrich in the derived category of chain complexes $\mathrm{D}(k)$.

Lurie's approach in [Lur17] incorporates the second approach while circumventing the first: He defines a fully homotopy-theoretic (and non-linear) version of operads, called *∞-operads*, of which the $\mathbb{E}_n$-operads are central examples. At first sight this may suggest that the k-linear structure inherent in the approach via dg-operads is superfluous — but there is more to this story.

The duality between grouplike $\mathbb{E}_n$-spaces and pointed n-connected spaces (i.e. n-fold loop spaces) is a specific instance of *Koszul duality* between augmented $\mathbb{E}_n$-algebras and augmented $\mathbb{E}_n$-coalgebras, see 5.2.6. Conceptually, this relies on the fact that the $\mathbb{E}_n$-operad is Koszul self-dual — a statement that only makes sense if we regard $\mathbb{E}_n$ as an enriched ∞-operad. Koszul duality is typically formulated for operads enriched in a semiadditive ∞-category, since this allows us to view non-unital cooperads as coalgebras in symmetric sequences [Hei24a, Corollary 2.27]. It is particularly well-behaved for enrichment in a stable ∞-category, where Koszul duality enhances to an equivalence between reduced non-unital operads and cooperads [Chi12], [Heu24, Thm. 3.4], [Hei24a, Prop. 3.6]. For instance, the $\mathbb{E}_\infty$-operad enriched in spectra is Koszul dual to the *spectral Lie operad*, which has no incarnation as an ordinary ∞-operad — already the ordinary or dg Lie operads require linear enrichment.

Further applications of enriched operads, particularly in vector spaces or chain complexes, arise in mathematical physics (related to factorization algebras and chiral algebras), geometric representation theory, deformation theory and algebraic geometry; see the introduction of [Hau22] for further references.

1.1. **Enriched ∞-operads.** Unfortunately, Lurie's definition of ∞-operads as certain fibrations over the category Fin_* of finite pointed sets does not immediately extend to the enriched setting. We therefore adopt a different approach, originating with [BD98] for classical coloured operads and extended to single-coloured ∞-operads in [Bra17]; see §1.2 of [Hau22] for a more detailed historical account.

Fix a presentably symmetric monoidal ∞-category $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr})$ as our enrichment category. Any space X freely generates a presentably symmetric monoidal $\mathcal{V}$-module ∞-category $\mathcal{P}\,\mathrm{Sym}(X) \otimes \mathcal{V} \simeq \prod_{k \geq 0} \mathrm{Fun}(B\Sigma_k, \mathcal{V})$, where Σ_k denotes the symmetric group on k elements. This means that there is an equivalence

$$\mathrm{End}_{\mathcal{V}}^{\mathrm{L},\otimes}(\mathcal{P}\,\mathrm{Sym}(X) \otimes \mathcal{V}) \simeq \mathrm{Fun}(X, \mathcal{P}\,\mathrm{Sym}(X) \otimes \mathcal{V}) \simeq \mathrm{Fun}(X \times \mathrm{Sym}\,X, \mathcal{V})\ .$$

Transporting the composition of endofunctors along this equivalence induces a monoidal structure on the right, (the reverse of) which is often referred to as the *composition product* or *plethysm product* $\circledast$ on the category of *X-colored symmetric sequences.* We describe it in Corollary 2.5.

A functor $\mathrm{Mul}_{\mathcal{O}} \in \mathrm{Fun}(X \times \mathrm{Sym}\,X, \mathcal{V})$ assigns to a colour in X and a finite symmetric tuple of colours an object of $\mathcal{V}$ — exactly the data of multimorphisms in a $\mathcal{V}$-enriched operad. Giving $\mathrm{Mul}_{\mathcal{O}}$ an algebra structure with respect to $\circledast$ encodes homotopy-coherent composition and identities. This motivates the following definition:

Definition 1.1. A $\mathcal{V}$-*enriched operad* with space of colors X is a monad on $\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V}$ in the $(\infty,2)$-category $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$. In other words, it is an algebra for the composition product $\circledast$ on $\mathrm{Fun}(X\times\operatorname{Sym}X,\mathcal{V})$.

1.2. **Marked algebras.** We follow the main idea of [RZ25] to rewrite this definition. n any $(\infty,2)$-category admitting Eilenberg–Moore objects, every monad determines a canonical monadic adjunction, and conversely the monad can be recovered from such an adjunction. Applied to a $\mathcal{V}$-enriched operad $\mathcal{O}\in\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V})$ we obtain and adjunction

$$\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V}\leftrightarrows\mathrm{LMod}_{\mathcal{O}}(\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V})=:\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$$

in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$, and we refer to the Eilenberg-Moore object $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})\in\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ as the *operadic presheaf category* of $\mathcal{O}$[1]. Conversely, this monadic adjunction lets us recover the original monad $\mathcal{O}$. Characterizing the monadic 1-morphisms in $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$, which we do in Proposition 4.1, gives an alternative description of enriched operads.

Theorem A (Corollary 4.6)**.** Assign to a $\mathcal{V}$-enriched operad $\mathcal{O}$ the composition $X\subseteq\operatorname{Sym}(X)\subseteq\mathcal{P}\operatorname{Sym}(X)\to\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V}\to\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$, which maps x to the free module on $よ_{(x)}\otimes 1_{\mathcal{V}}$. This induces a fully faithful embedding

$$v\mathcal{O}p_X(\mathcal{V})=\mathrm{Alg}\left(\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V})\right)\hookrightarrow\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{X/}$$

whose image consists those functors $よ : X\to\mathcal{M}$ with $\mathcal{M}\in\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ such that:

- The unique extension $Y:\mathcal{P}\operatorname{Sym}(X)\otimes\mathcal{V}\to\mathcal{M}$ is an internally left adjoint 1-morphism in $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$,
- The image of $よ$ generates $\mathcal{M}$ under colimits, symmetric monoidal structure and $\mathcal{V}$-tensoring.

We refer to functors $X\to\mathcal{M}$ in the essential image of this embedding as *marked* $\mathcal{V}$*-algebras*.

We use this result to define the category of *valent* $\mathcal{V}$*-enriched operads* as the fully subcategory of the pullback $\mathrm{Arr}(\widehat{\mathcal{C}at})\times_{\widehat{\mathcal{C}at}}\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ on the marked $\mathcal{V}$-algebras. The terminology *valent* as opposed to *univalent* indicates that the space of objects X that we begin with need not coincide with the actual underlying space of our operad. To recover the correct underlying space, by the Yoneda lemma, one must use the Yoneda lemma and require that X embeds as a subcategory of $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$.

Definition 1.2. A $\mathcal{V}$-enriched operad $\mathcal{O}$ is called *univalent* if its associated marked algebra $よ : col\mathcal{O}\to\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ is a subcategory inclusion, in other words $col\mathcal{O}\simeq\mathrm{Im}(よ)^{\simeq}$. Denote by $\mathcal{O}p(\mathcal{V})\subseteq v\mathcal{O}p(\mathcal{V})$ the full subcategory on the univalent $\mathcal{V}$-operads.

Theorem B (Theorem 5.9,Corollary 5.12)**.** The category $v\mathcal{O}p(\mathcal{S})$ of valent $\mathcal{S}$-enriched operads is equivalent to the category of *flagged* ∞*-operads*, i.e. pairs consisting of an ∞-operad O in the sense of Lurie together with a surjective functor $X\to col\mathrm{O}$ from some space X to its colors.

Moreover, the full subcategory $\mathcal{O}p(\mathcal{S})$ of univalent $\mathcal{S}$-enriched operads is equivalent to Lurie's category $\mathcal{O}p$ of ∞-operads.

Let us sketch how this comparison works. We first explain how to associate a monad in $\mathrm{CAlg}(\mathrm{Pr})$ to a flagged ∞-operad $X\to\mathrm{O}$. We regard X as an operad with only unary

[1] This is usually called the category of *right* $\mathcal{O}$*-modules*.

operations, so that $\operatorname{Env} X \simeq \operatorname{Sym} X \in \mathrm{CAlg}(\mathcal{C}at)$. Applying the envelope functor Env and then the presheaf functor $\mathcal{P}$ to the map $i : X \to \mathrm{O}$ yields an adjunction

$$\begin{tikzcd} \mathcal{P}\operatorname{Sym}(col\mathcal{O}) \arrow[r, bend left, "\operatorname{Env}(i)_!", "{\otimes,\, \mathrm{L}}"'] & \mathcal{P}\operatorname{Env}\mathcal{O} \arrow[l, bend left, "\operatorname{Env}(i)^*", "{\otimes,\, \mathrm{L}}"'] \end{tikzcd}$$

in which the left adjoint is symmetric monoidal and the right adjoint preserves colimits. In Theorem 3.42 we show, using an argument due to Jan Steinebrunner, that the induced lax symmetric monoidal structure on the right adjoint is in fact strong. Hence the above is an adjunction in CAlg(Pr), and we obtain a symmetric monoidal colimit-preserving monad

$$(\operatorname{Env} f)^*(\operatorname{Env} f)_! \in \operatorname{Alg}\left(\operatorname{End}^{\mathrm{L},\otimes}(\mathcal{P}\operatorname{Sym}(col\mathcal{O}))\right) \simeq \operatorname{Alg}(\operatorname{Fun}(X \times \operatorname{Sym} X, \mathcal{V}))\ .$$

The latter equivalence precomposes with the Yoneda embedding $o \mapsto \text{よ}_{(o)} \otimes 1_{\mathcal{V}}$, so we obtain

$$(\operatorname{Env} f)^*(\operatorname{Env} f)_!(\text{よ}_{(o)}) = (\operatorname{Env} f)^* \text{よ}_{(o)} = \operatorname{Map}_{\operatorname{Env}(\mathcal{O})}(-,(o)) \circ \operatorname{Env}(f)$$

which sends a symmetric tuple $(o_1, \ldots, o_n) \in \operatorname{Sym}(col\mathcal{O})$ to

$$\operatorname{Map}_{\operatorname{Env}(\mathcal{O})}((o_1, \ldots, o_n), (o)) \simeq \operatorname{Mul}_{\mathcal{O}}(o_1, \ldots, o_n; o)\ .$$

Thus, as expected, the multimorphism object $\operatorname{Mul}_{\mathcal{O}}$ is promoted to an algebra in $\operatorname{Fun}(X \times \operatorname{Sym} X, \mathcal{V})$. By construction, the marked algebra associated to this operad is $X \to \mathcal{P}\operatorname{Env}\mathrm{O}$, because the induced functor $\mathcal{P}\operatorname{Sym} X \to \mathcal{P}\operatorname{Env}\mathrm{O}$ is a monadic 1-morphism in $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$. Hence the argument identifies $\mathcal{P}\operatorname{Env}\mathrm{O}$ with the operadic presheaf category of O, from which we deduce that the categories of $\mathcal{O}$-algebras that can be defined in both settings agree.

Conversely, to any marked algebra $\text{よ} : X \to \mathcal{M}$, we associate the flagged operad $X \to \operatorname{Im}(\text{よ})$: Since the essential image $\operatorname{Im}(\text{よ})$ is a full subcategory of a symmetric monoidal category, it inherits an operad structure, and $X \to \operatorname{Im}(\text{よ})$ is automatically essentially surjective.

Proving that these two constructions are inverse requires understanding when a presentably symmetric monoidal $\mathcal{V}$-module $\mathcal{M} \in \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ arises as the operadic presheaf category of an enriched operad. This is analogous to the characterization in [RZ25, Rem. 5.10], which states that $\mathcal{N} \in \mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr})$ is an enriched presheaf category precisely when it is atomically generated. A priori, there are several possible definitions of what it means for $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ to be $\otimes$-*atomically generated*,listed below in increasing order of strength:

(1) $\mathcal{M}$ is generated under colimits, $\mathcal{V}$-tensoring and symmetric monoidal structure by its full subcategory (or equivalently, any full subcategory) of $\otimes$-*atomic objects*, i.e. those $m \in \mathcal{M}$ for which the unique morphism $\mathcal{P}\operatorname{Sym}(*) \otimes \mathcal{V} \to \mathcal{M}$ sending $* \mapsto m$ is internally left adjoint in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$,

(2) There exists a small full subcategory $\mathcal{M}_0 \subseteq \mathcal{M}$ generating $\mathcal{M}$ which generates $\mathcal{M}$ under these operations, and such that the induced morphism $\mathcal{P}\operatorname{Sym}(\mathcal{M}_0) \otimes \mathcal{V} \to \mathcal{M}$ is internally left adjoint,

(3) $\mathcal{M}$ is generated under these operations by its full subcategory $\mathcal{M}^{\otimes\mathrm{at}}$ of $\otimes$-atomic objects, and the induced morphism $\mathcal{P}\operatorname{Sym}(\mathcal{M}^{\otimes\mathrm{at}}) \otimes \mathcal{V} \to \mathcal{M}$ is internally left adjoint.

The corresponding notions in $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ are immediately seen to be equivalent [RZ25, Prop. 3.32]. However, in the present context, condition (1) is too weak, as shown in Warning 3.16. In Corollary 7.23 we prove that conditions (2) and (3) are equivalent, but the proof requires the machinery of Cauchy-complete operads developed later in the paper. In fact, that equivalence is precisely what ensures the existence of a Cauchy-completion functor for operads.

1.3. **Applications.** We use the marked-algebra picture to develop the higher algebra of enriched ∞-operads. For example, we construct:

- An envelope functor $\mathrm{Env}_{\mathcal{V}} : \mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))$ from enriched operads to symmetric monoidal enriched categories in Construction 2.7 and Definition 9.3,
- A Boardman-Vogt tensor product $\otimes_{\mathrm{BV}}$ on $\mathcal{V}$-operads that is left adjoint to the construction of $\mathcal{V}$-categories of algebras, see Proposition 6.2,
- Operadic weighted colimits in Definition 6.6, with factorization homology as a specific example Example 6.9,
- Free algebras over $\mathcal{V}$-operads in Proposition 6.8,

all generalizing the corresponding constructions for ordinary ∞-operads in [Lur17].

1.4. **Recovering on operad from its algebras.** We will study the question how much information about an operad $\mathcal{O}$ can be recovered from its categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ where $\mathcal{M} \in \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$. The following is more or less immediate from the above marked algebra description:

Proposition (**??**)**.** Let $f : \mathcal{O} \to \mathcal{P}$ be a map of valent $\mathcal{V}$-enriched operads whose underlying map on spaces of colors $col\mathcal{O} \to col\mathcal{P}$ is an equivalence. If the induced precomposition map $f^* : \mathrm{Alg}_{\mathcal{P}}(\mathcal{M}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ is an equivalence for any $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, then f is already an equivalence.

However, we might now always be given a comparison map f, but only the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ itself. Certainly, we can not fully recover $\mathcal{O}$ in this case:

- Let C be a category and $\widehat{\mathrm{C}}^{\mathrm{ic}}$ its idempotent completion. Then
$$\mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\mathrm{C})}(\mathcal{M}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{M}) \simeq \mathrm{Fun}(\hat{\mathrm{C}}^{\mathrm{ic}}, \mathcal{M}) \simeq \mathrm{Alg}_{\mathrm{Triv}_{\mathcal{V}}(\widehat{\mathrm{C}}^{\mathrm{ic}})}(\mathcal{M})\ .$$
- Given a spectral operad $\mathcal{O}$, we can define its *r-fold shift* $\mathcal{O}[r]$ whose multimorphism objects are given by the shifts
$$\mathrm{Mul}_{\mathcal{O}[r]}(o_1, \ldots, o_n; o) \simeq \mathrm{Mul}_{\mathcal{O}}(o_1, \ldots, o_n; o)[r \cdot (1-n)]\ ,$$
consider [Hei24a, Constr. 2.34] for a single-colored version of this construction. The auto-equivalence $[-r] : \mathcal{V} \xrightarrow{\simeq} \mathcal{V}$ induces, for any $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$, an equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \simeq \mathrm{Alg}_{\mathcal{O}[r]}(\mathcal{M})$.
- Let $\mathcal{V}$ be semiadditive. Then to any single-colored $\mathcal{V}$-operad $\mathcal{O}$, we can associate its r-matrix operad $\mathrm{Mat}_r(\mathcal{O})$, whose multimorphisms are given by
$$\mathrm{Mat}_r(\mathcal{O})(n) = \mathcal{O}(n)^{\oplus r^{n+1}}$$
regarded as matrices with n ingoing and one outgoing indices, and composition given by matrix products and composition in $\mathcal{O}$. In the case of dg-operads, [KM01] proves that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \simeq \mathrm{Alg}_{\mathrm{Mat}_r(\mathcal{O})}(\mathcal{M})$. We will see that the analogy of this statement with the fact that every ring is Morita-equivalent to its matrix ring is not accidental.

These examples indicate that how much about $\mathcal{O}$ we can recover is highly dependent on $\mathcal{V}$, and more specifically *absolute weighted colimits* in $\mathcal{V}$-enriched category theory. A weight $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ on some $\mathcal{V}$-category $\mathcal{C}$ is called *absolute* if W-weighted colimits are preserved by any $\mathcal{V}$-enriched functor, and $\mathcal{C}$ is called *Cauchy-complete* if any absolute weight on it (which are precisely the atomic objects in the enriched presheaf category $\mathcal{P}_{\mathcal{V}}\mathcal{C}$) is representable, c.f. [Hei24b, Prop. 5.51], [RZ26]. For instance, idempotent splittings are absolute colimits over

$\mathcal{S}$, shifts are over $\mathcal{S}p$, and direct sums are over $\mathcal{S}p^{\mathrm{cn}}$, which is closely related to the above constructions.

Analogously, we call an enriched operad Cauchy-complete if every $\otimes$-atomic object in $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ is representable. These form a full subcategory $\mathcal{O}p_{+}(\mathcal{V}) \subseteq v\mathcal{O}p(\mathcal{V})$ whose inclusion admits a left adjoint $\widehat{(-)}^{\mathcal{V}}$. We prove the following characterization:

Proposition 1.3. Given $\mathcal{V}$-operads $\mathcal{O}, \mathcal{P} \in v\mathcal{O}p(\mathcal{V})$, the following are equivalent:

(1) The functors $\mathrm{Alg}_{\mathcal{O}}(-), \mathrm{Alg}_{\mathcal{P}}(-) : \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) \to \widehat{\mathcal{C}at}$ are equivalent,
(2) The categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(\coprod_{k\geq 0} B\Sigma_k, \mathcal{V})) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Fun}(\coprod_{k\geq 0} B\Sigma_k, \mathcal{V}))$ are equivalent,
(3) The categories of operadic presheaves $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) \simeq \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})$ are equivalent,
(4) The Cauchy-completions $\widehat{\mathcal{O}}^{\mathcal{V}} \simeq \widehat{\mathcal{P}}^{\mathcal{V}}$ are equivalent.

In this case, we call $\mathcal{O}$ and $\mathcal{P}$ *Morita-equivalent.*

Further, we prove in Theorem 7.20 that a $\mathcal{V}$-operad $\mathcal{O}$ is Cauchy-complete iff its underlying $\mathcal{V}$-category $\mathrm{Col}_{\mathcal{V}}(\mathcal{O})$ is. This shows, for instance:

- Given two ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S})$. For instance, this is true for the $\mathbb{E}_k$-operad and its variants with tangential structures, and for the $\mathbb{E}_X$-operad classifying constructible factorization algebras on a stratified manifold X (since exit-path categories are always idempotent complete).
- Given spectral ∞-operads $\mathcal{O}, \mathcal{P}$ whose underlying ∞-categories are stable and idempotent complete, then $\mathcal{O} \simeq \mathcal{P}$ iff $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}p) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}p)$. Similarly for operads enriched over the derived category of a ring (e.g. dg-operads).

In fact, studying the idempotent completion lets us classify all single-colored operads Morita-equivalent to a given single-colored $\mathcal{O} \in v\mathcal{O}p(\mathcal{V})$. Namely, they are completely specified by an algebra $\mathcal{P}(1) \in \mathrm{Alg}(\mathcal{V})$ that is Morita-equivalent to the algebra $\mathcal{O}(1)$ of 1-ary operations in $\mathcal{O}$; or equivalently a dualizable generator in $\mathrm{Mod}_{\mathcal{O}(1)}(\mathcal{V})$. For instance:

- If $\mathcal{V} = \mathrm{Vec}_k$ over a field k and given a single-colored Vec_k-enriched operad $\mathcal{O} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathcal{O}(1) \simeq k$, then any single-colored operad $\mathcal{P} \in v\mathcal{O}p(\mathrm{Vec}_k)$ such that $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Vec}_k) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathrm{Vec}_k)$ is equivalent to $\mathrm{Mat}_r(\mathcal{O})$ for some $r \geq 0$.
- If $\mathcal{V} = \mathrm{D}(k)$ and $\mathcal{O} \in v\mathcal{O}p(\mathrm{D}(k))$ a single-colored dg-operad with $\mathcal{O}(1) \simeq k[0]$, then for any nonzero finite-dimensional graded vector space N we construct a generalized matrix operad $\mathrm{Mat}_N(\mathcal{O})$ that is Morita-equivalent to $\mathcal{O}$. We show in Example 8.13 that any operad Morita-equivalent to $\mathcal{O}$ arises like this; for instance the shifts $\mathcal{O}[r]$ correspond to $N = k[r]$, while $\mathrm{Mat}_r(\mathcal{O})$ corresponds to $k^{\oplus r}[0]$.

1.5. **Recovering an operad from its envelope.** We have seen that while we generally can not recover an enriched operad $\mathcal{O}$ from its operadic presheaf category, knowing which presheaves are representable is (by our definition of univalence) enough to recover the univalization of our operad. What if we are instead given the envelope of $\mathcal{O}$, i.e. the full subcategory of operadic presheaves that are generated by the representables under the symmetric monoidal structure?

It is a well-established result in the non-enriched setting [HK24], [BS22] that the envelope functor $\mathrm{Env} : \mathcal{O}p \to \mathrm{CAlg}(\mathcal{C}at)$ is a subcategory inclusion, in particular if $\mathrm{Env}(\mathrm{O}) \simeq \mathrm{Env}(\mathrm{P})$ then $\mathrm{O} \simeq \mathrm{P}$. Its image can be characterized in several ways; for instance as consisting of $\otimes$-disjunctive symmetric monoidal categories and $\otimes$-disjunctive symmetric monoidal functors, compare Definition 3.31. We generalize the latter notion to symmetric monoidal $\mathcal{V}$-enriched functors and prove the following statement about the enriched envelope functor:

Theorem 1.4. Let $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$ be the wide subcategory of $\mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$ on $\otimes$-disjunctive enriched functors, and similarly $\mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$. Then, the envelope functor

$$\mathrm{Env}_{\mathcal{V}} : \mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(\mathcal{C}at(\mathcal{V}))^{\otimes\text{-disj}}$$

admits a right adjoint, which we call the singlett functor $\mathrm{sing}_{\mathcal{V}}$.

- The counit $\mathrm{Env}_{\mathcal{V}}\,\mathrm{sing}_{\mathcal{V}}\mathcal{C} \to \mathcal{C}$ of this adjunction is always fully faithful, and an isomorphism iff $\mathcal{C}$ is generated under its symmetric monoidal structure by its singletts, i.e. the $\otimes$-atomic objects in $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$.
- The unit $\mathcal{O} \to \mathrm{sing}_{\mathcal{V}}\,\mathrm{Env}_{\mathcal{V}}\,\mathcal{O}$ is always fully faithful, and an isomorphism iff the full sub-$\mathcal{V}$-category $\mathrm{Col}_{\mathcal{V}}(\mathcal{O}) \subseteq \mathrm{Env}_{\mathcal{V}}(\mathcal{O})$ is closed under absolute colimits for $\mathcal{V}$-enrichment[2].

This yields an alternative proof that the envelope functor $\mathrm{Env}_{\mathcal{S}} : \mathcal{O}p(\mathcal{S}) \to \mathrm{CAlg}(\mathcal{C}at(\mathcal{S}))^{\otimes\text{-disj}} \hookrightarrow \mathrm{CAlg}(\mathcal{C}at(\mathcal{S}))$ is a subcategory inclusion for ordinary operads Example 9.17. Also we deduce that it is rarely a subcategory inclusion if $\mathcal{V}$ is semiadditive Example 9.18, in particular for neither $\mathrm{Vec}_k, \mathcal{S}p^{\mathrm{cn}}$ or $\mathcal{S}p$.

The best we can recover from $\mathrm{Env}_{\mathcal{V}}(\mathcal{O})$ is $\mathrm{sing}_{\mathcal{V}}\,\mathrm{Env}_{\mathcal{V}}(\mathcal{O}) =: \mathcal{O}^{\mathrm{env}}$, which we call the *envelope completion* of $\mathcal{O}$. Explicitly, this is the intersection of the Cauchy-completion and the envelope

$$\mathcal{O}^{\mathrm{env}} := \widehat{\mathcal{O}}^{\mathcal{V}} \cap \mathrm{Env}_{\mathcal{V}}(\mathcal{O}) \subseteq \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})\ .$$

We have $\mathrm{Env}_{\mathcal{V}}(\mathcal{O}) \simeq \mathrm{Env}_{\mathcal{V}}(\mathcal{P})$ iff $\mathcal{O}^{\mathrm{env}} \simeq \mathcal{P}^{\mathrm{env}}$; and $\mathrm{Env}_{\mathcal{V}}$ restricts to a subcategory inclusion on envelope-complete $\mathcal{V}$-operads.

Remark 1.5. It is crucial in this discussion that we work with univalent $\mathcal{V}$-operads and symmetric monoidal univalent $\mathcal{V}$-categories. The valent version of the envelope functor in Definition 9.3 is evidently a subcategory inclusion since one can recover a space X from the $\mathbb{E}_\infty$-space $\mathrm{Sym}(X)$, c.f. Theorem 9.13.

1.6. **Relation to other work.** In [Hau20], Haugseng proves that ∞-operads in the sense of Lurie are equivalent to algebras in symmetric sequences. However, his construction of the composition product differs from the one obtained by identifying symmetric sequences with an endomorphism object in $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$, and he does not supply a comparison between the two. As he notes in §1.2, this left open the question of whether this notion of enriched ∞-operads agrees with this more widely used one, which Theorem 5.9 resolves (the issue of comparing the monoidal structures still remains).

There is unpublished work proving this comparison for single-colored ∞-operads predating our result: Gijs Heuts and Lukas Brantner [BH] are preparing a proof using the theory of analytic monads. Also Thomas Blom, Connor Malin and Niall Taggart [BMT] give a proof that is closely to ours and describe Koszul duality in their formalism. Finally, Jan Steinebrunner and Shaul Barkan were considering a similar framework that extends to enriched ∞-properads, see the outlook of [BS22].

1.7. **Plan of the paper.** In Section 2 we introduce enriched ∞-operads as monads in the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$, which is constructed and studied in Appendix A. The subsequent Section 3 is concerned with understanding internally left adjoint morphisms in it, in particular $\otimes$-atomic objects and markings. This is applied in Section 4 to obtain our marked-algebra description, and compare it with Lurie's ∞-operads in Section 5.

[2]Absolute colimits are those weighted colimits that are preserved by any $\mathcal{V}$-enriched functor.

Section 6 considers further applications of our approach, defining a Boardman-Vogt product and operadic weighted colimits. In Section 7 we introduce Cauchy-completions of enriched ∞-operads, which is maybe the technically must subtle part of this paper. It is applied in Section 8 to study the Morita-theory of operads, and in Section 9 to study enriched envelopes.

1.8. **Acknowledgements.**

1.9. **Notation and Conventions.**

2. Enriched Operads

For $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr})$ a presentably symmetric monoidal category, in order to define $\mathcal{V}$-enriched operads we make use of the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ whose

- underlying 1-category is the category $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) := \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ of presentably symmetric monoidal $\mathcal{V}$-module categories,
- morphism categories $\mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{M}, \mathcal{N})$ consist of symmetric monoidal $\mathcal{V}$-linear colimit-preserving functors.

Constructing and studying this 2-category relies on a symmetric monoidal structure $\circledast_{\mathcal{V}}$ on it that we introduce in Appendix A; here we summarize some of its properties:

- By Observation A.28 the forgetful 2-functors
$$\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathrm{CAlg}(\mathbb{P}\mathrm{r}) \to \mathrm{CAlg}(\widehat{\mathbb{C}\mathrm{at}}) \to \widehat{\mathbb{C}\mathrm{at}}$$
admit (partially) left adjoint 2-functors
$$\mathbb{C}\mathrm{at} \xrightarrow{\mathrm{Sym}} \mathrm{CAlg}(\mathbb{C}\mathrm{at}) \xrightarrow{\mathcal{P}} \mathrm{CAlg}(\mathbb{P}\mathrm{r}) \xrightarrow{-\otimes\mathcal{V}} \mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$$
by which we mean in particular that for a given $\mathrm{C} \in \mathcal{C}at$ and $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, precomposing with the canonical functor $\mathrm{C} \subseteq \mathrm{Sym}(\mathrm{C}) \subseteq \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \to \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}$ induces an equivalence
$$\mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}\,\mathrm{C}) \otimes \mathcal{V}, \mathcal{M}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{M}) .$$
Since $\mathcal{P}, \mathrm{Sym}$ and $- \otimes \mathcal{V}$ merely enhance the respective 1-functors, we calculate
$$\mathcal{P}(\mathrm{Sym}\,\mathrm{C}) \otimes \mathcal{V} \simeq \mathrm{Fun}(\mathrm{Sym}\,\mathrm{C}^{\mathrm{op}}, \mathcal{V}) \simeq \prod_{n \geq 0} \mathrm{Fun}\left((\mathrm{C}^{\mathrm{op}})^{\times n}_{h\Sigma_n}, \mathcal{V}\right) .$$
- The 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ admits all partially (op)lax colimits. In particular it admits Eilenberg-Moore-objects, which are created by all of the above forgetful 2-functors.

Definition 2.1. Given a space $X \in \mathcal{S}$, a $\mathcal{V}$*-enriched operad with colors* X is a monad on $\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V} \simeq \mathrm{Fun}(\mathrm{Sym}\,X^{\mathrm{op}}, \mathcal{V})$ in the 2-category $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$. Using the above adjunction and that $X^{\mathrm{op}} \simeq X$, this unwinds to an algebra in
$$\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V}) \simeq \mathrm{Fun}(X, \mathrm{Fun}(\mathrm{Sym}\,X^{\mathrm{op}}, \mathcal{V})) \simeq \mathrm{Fun}\left(X \times \mathrm{Sym}\,X, \mathcal{V}\right) .$$
We refer to $\mathrm{sSeq}_X(\mathcal{V}) := \mathrm{Fun}\left(X \times \mathrm{Sym}\,X, \mathcal{V}\right)$, equipped with the monoidal structure $\oslash$ it obtains as an endomorphism object via composition, as the category of X*-colored symmetric sequences* in $\mathcal{V}$.

Given a $\mathcal{V}$-enriched operad $\mathcal{O}$, we call its underlying functor $\mathrm{Mul}_{\mathcal{O}} : X \times \mathrm{Sym}\,X \to \mathcal{V}$ the *multigraph* of $\mathcal{O}$, and denote its evaluation on a tuple $(x, (x_1, \ldots, x_n))$ by $\mathrm{Mul}_{\mathcal{O}}(x_1, \ldots, x_n; x) \in \mathcal{V}$. Its points $1_{\mathcal{V}} \to \mathrm{Mul}_{\mathcal{O}}(x_1, \ldots, x_n; x)$ are called *n-ary morphisms* in $\mathcal{O}$.

Warning 2.2. From the explicit formula in Corollary 2.5 we will see that the monoidal structure $\varogreaterthan$ on $\mathrm{sSeq}_X(\mathcal{V})$ is reverse to what is commonly called the composition product on symmetric sequences, which we denote $\varolessthan$. Of course, their categories of algebras are equivalent.

Notation 2.3. We write $\underline{x}$ for a symmetric tuple $(x_1, \ldots, x_n) \in \operatorname{Sym} X$. If $X = *$ and $\mathcal{O} \in \mathrm{sSeq}_*(\mathcal{V})$, we write $\mathcal{O}(n)$ for $\mathcal{O}(*; *, \ldots, *) \in \mathcal{V}$ evaluating on the unique n-tuple.

Lemma 2.4. Let $X \in \mathcal{C}at$ and $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. Under the equivalence $\mathrm{Fun}(X, \mathcal{M}) \simeq \mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\operatorname{Sym} X \otimes \mathcal{V}, \mathcal{M})$ from Observation A.28, a functor $A : X \to \mathcal{M}$ is sent to the morphism $\bar{A} : \mathrm{Fun}(\operatorname{Sym} X^{\mathrm{op}}, \mathcal{V}) \to \mathcal{M}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ which maps

$$(W : \operatorname{Sym} X^{\mathrm{op}} \to \mathcal{V}) \mapsto \oint^{\underline{x} \in \operatorname{Sym} X} A(x_1) \otimes \cdots \otimes A(x_n) \otimes W(\underline{x}) \ .$$

Proof. As a first step, A extends to a unique symmetric monoidal functor $\operatorname{Sym} X \to \mathcal{M}$ sending $(x_1, \ldots, x_n) \mapsto A(x_1) \otimes \cdots \otimes A(x_n)$. From here, the formula for the extension to $\mathcal{P}(\operatorname{Sym} X) \otimes \mathcal{V}$ follows from [RZ25, Lem. 2.13]. $\square$

Corollary 2.5. The monoidal structure $\varogreaterthan$ on $\mathrm{sSeq}_X(\mathcal{V}) \simeq \mathrm{Fun}\,(X \times \operatorname{Sym} X, \mathcal{V})$ sends $A, B : X \times \operatorname{Sym} X \to \mathcal{V}$ to

$$A \varogreaterthan B(z, \underline{x}) = \oint^{\underline{y} \in \operatorname{Sym} X} \operatorname*{colim}_{\underline{x}^{(1)} \otimes \cdots \otimes \underline{x}^{(k)} \to \underline{x}} A(y_1; \underline{x}^{(1)}) \otimes \cdots \otimes A(y_k; \underline{x}^{(k)}) \otimes B(z; \underline{y}) \ .$$

If additionally X is a space, this agrees with

$$\coprod_{k \geq 0} \coprod_{I_1 \sqcup \cdots \sqcup I_k = \underline{n}} \operatorname*{colim}_{(y_1, \ldots, y_k) \in X^{\times k}} (A(y_1; \underline{x}|_{I_1}) \otimes \cdots \otimes A(y_k; \underline{x}|_{I_k})) \otimes_{\Sigma_k} B(z; y_1, \ldots, y_k)$$

where the Σ_k-action permutes the blocks $\underline{x}|_{I_1}, \ldots, \underline{x}|_{I_k}$ partitioning $\underline{x}$ and the entries of $\underline{y}$. Specifically for $X = *$ we obtain

$$\coprod_{k \geq 0} \coprod_{I_1 \sqcup \cdots \sqcup I_k = \underline{n}} (A(\# I_1) \otimes \cdots \otimes A(\# I_k)) \otimes_{\Sigma_k} B(k) \simeq \coprod_{k \geq 0} \coprod_{n_1 + \cdots + n_k = n} \mathrm{Ind}^{\Sigma_n}_{\Sigma_{n_1} \times \ldots \Sigma_{n_k}} (A(n_1) \otimes \cdots \otimes A(n_k)) \otimes_{\Sigma_k} B(k) \ .$$

Proof. If we denote by $\bar{A}, \bar{B}$ the endofunctors of $\mathcal{P}(\operatorname{Sym} X) \otimes \mathcal{V}$ associated to A and B, then by definition we can write $A \varogreaterthan B(z, (x_i)) = \bar{A} \circ \bar{B}(\text{よ}_z \otimes 1_{\mathcal{V}})(x_1, \ldots, x_n) = \bar{A}(B(z; -))(x_1, \ldots, x_n)$ so the first expression follows from Lemma 2.4 as well as the formula [Lur17, Ex. 2.2.6.17] for the Day convolution product.

By [Lur18a, Tag 048H] and [Lur18a, Tag 048L], for a space X we have $\mathrm{Tw}(X) \simeq X$ so coends agree with colimits over X. Further, the functor $B\Sigma_k \to \mathcal{S}$ encoding the Σ_k-action on $X^{\times k}$ unstraightens into a coCartesian fibration $X^{\times k}_{h\Sigma_k} \to B\Sigma_k$ with fiber $X^{\times k}$. Hence, $\operatorname{colim}_{X^{\times k}_{h\Sigma_k}} \simeq \operatorname{colim}_{B\Sigma_k} \operatorname{colim}_{X^{\times k}}$, and the colimit over $B\Sigma_k$ turns the tensor product into a relative tensor product $\otimes_{\Sigma_k}$.

It remains to unwind the Day convolution product. The forgetful functor $\operatorname{Sym} X \to \mathrm{Fin}^{\simeq}$ into the category $\mathrm{Fin}^{\simeq}$ of finite sets and bijections induces an equivalence

$$((\operatorname{Sym} X)^{\times n}) \times_{\operatorname{Sym} X} (\operatorname{Sym} X)_{/\underline{x}} \simeq (\mathrm{Fin}^{\simeq, \times n}) \times_{\mathrm{Fin}^{\simeq}} (\mathrm{Fin}^{\simeq})_{/\underline{x}}$$

on the indexing category of $\operatorname{colim}_{\underline{x}^{(1)} \otimes \cdots \otimes \underline{x}^{(k)} \to \underline{x}}$, TODO: Fill in argument. Further, the right side (which is a priori a $(1,1)$-category) can be checked to be discrete, i.e. the set of partitions of $\underline{n}$.

Finally, note that for $X = *$ the colimit over this slice category calculates, on each connected component i.e. for fixed $n_1 + \cdots + n_k = n$, the left Kan extension along $B\Sigma_1 \times \cdots \times B\Sigma_k \to B\Sigma_n$ which is how the induced representation is defined. $\square$

Construction 2.6. By Observation A.28 the symmetric algebra functor

$$\mathrm{Sym}^{\otimes_{\mathcal{V}}} : \mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}) \to \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}))$$

is a 2-functor, so induces a monoidal functor between endomorphism objects

$$\mathrm{Fun}(X \times X, \mathcal{V}) \simeq \mathrm{End}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}(X) \otimes \mathcal{V}) \to \mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V}) \simeq \mathrm{Fun}(X \times \mathrm{Sym}\,X, \mathcal{V})$$

explicitly acting by left Kan extension along the full inclusion $X \subseteq \mathrm{Sym}\,X$. In particular it is fully faithful and preserves colimits, so it admits a lax monoidal right adjoint. We obtain an adjunction on algebra objects

$$\mathrm{Triv}_{\mathcal{V}} : v\mathcal{C}at_X(\mathcal{V}) \rightleftarrows v\mathcal{O}p_X(\mathcal{V}) : \mathrm{Col}_{\mathcal{V}} \ ,$$

where the fully faithful left adjoint $\mathrm{Triv}_{\mathcal{V}}$ sends a $\mathcal{V}$-enriched category $\mathcal{C}$ to the *trivial* $\mathcal{V}$*-operad* with colors $\mathcal{C}$, while $\mathrm{Col}_{\mathcal{V}}$ sends a $\mathcal{V}$-operad to its $\mathcal{V}$*-category of colors*.

Construction 2.7. Similarly, the forgetful 2-functor $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathbb{P}\mathrm{r}_{\mathcal{V}}$ induces a monoidal functor

$$\mathrm{Fun}(X \times \mathrm{Sym}\,X, \mathcal{V}) \simeq \mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V}) \to \mathrm{End}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V}) \simeq \mathrm{Fun}(\mathrm{Sym}\,X \times \mathrm{Sym}\,X, \mathcal{V})$$

explicitly given by extending symmetric monoidally in the first argument. On algebra objects we obtain a functor

$$\mathrm{vEnv}_{\mathcal{V}} : v\mathcal{O}p_X(\mathcal{V}) \to v\mathcal{C}at_{\mathrm{Sym}\,X}(\mathcal{V})$$

which we call the *(valent) envelope*. We will see in Definition 9.3 that this enhances to a functor $\mathrm{vEnv} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(v\mathcal{C}at(\mathcal{V}))$.

Construction 2.8. A morphism $f : \mathcal{V} \to \mathcal{W}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ induces by Proposition A.31 a 2-functor $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{W}})$. As in Construction 2.6, we obtain a *change-of-enrichment* adjunction

$$f_! : v\mathcal{O}p_X(\mathcal{V}) \leftrightarrows v\mathcal{O}p_X(\mathcal{W}) : f_!^{\mathrm{R}}$$

that acts on multigraphs by postcomposing $\mathrm{Mul}_{\mathcal{O}} : X \times \mathrm{Sym}\,X \to \mathcal{V}$ with f and f^{R}.

Definition 2.9. Given a $\mathcal{V}$-operad $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$, we define its *operadic (enriched) presheaf category* as the Eilenberg-Moore object

$$\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) := \mathrm{LMod}_{\mathcal{O}}(\mathcal{P}\,\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V}) \in \mathrm{CAlg}(\mathbb{P}\mathrm{r}) \ .$$

It is part of an Eilenberg-Moore adjunction

$$\mathcal{P}\,\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V} \rightleftarrows \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$$

between free and forgetful functor. We define the *operadic Yoneda functor* $よ_{\mathcal{O}} : col\mathcal{O} \to \mathcal{P}\,\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ of $\mathcal{O}$ by composing the unit of the free-forgetful adjunction $\widehat{\mathcal{C}at} \leftrightarrows \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathcal{C}at^{\mathrm{colim}}))$ with the free module functor.

Remark 2.10. This is usually known as the category of *right modules* over $\mathcal{O}$, c.f. [Hei24a]. The unfortunate change in direction comes from the fact that our composition product is reverse to the usual convention.

Construction 2.11. Given any $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr})$, by Observation A.27 the category $\mathrm{Fun}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V}, \mathcal{M}) \simeq \mathrm{Fun}(X, \mathcal{M})$ admits a $\mathcal{V}$-linear sifted-colimit-preserving right tensoring by $\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}(\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V}) = \mathrm{sSeq}_X(\mathcal{V})$ via precomposition.

Definition 2.12. Given a $\mathcal{V}$-enriched operad $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$ and $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr})$, we define the presentably symmetric monoidal $\mathcal{V}$-module category of $\mathcal{O}$*-algebras* in $\mathcal{M}$ as

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) := \mathrm{RMod}_{\mathcal{O}}(\mathrm{Fun}(X, \mathcal{M})) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})\,.$$

Remark 2.13. More generally for $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$ and $\mathcal{P} \in v\mathcal{O}p_Y(\mathcal{V})$, we can define a category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ of $\mathcal{O}$-*algebras in* $\mathcal{P}$ as the pullback

$$\mathrm{Map}_{\mathcal{S}}(X, Y) \times_{\mathrm{Fun}(X, \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P}))} \mathrm{RMod}_{\mathcal{O}}(\mathrm{Fun}(X, \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P}))) \ .$$

We expect that one can enhance this to a $\mathcal{V}$-operad, obtaining a functor $\mathrm{Alg}_{\mathcal{O}}(-) : v\mathcal{O}p(\mathcal{V}) \to v\mathcal{O}p(\mathcal{V})$ that is right adjoint to the Boardman-Vogt product $- \otimes_{\mathrm{BV}} \mathcal{O}$ we define in Proposition 6.2.

3. $\otimes$-atomic objects

We now study adjoint 1-morphisms in the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$, allowing us to define $\otimes$-atomic objects and $\otimes$-atomically generated categories. While this nomenclature suggests a comparison to atomic objects and atomically generated categories in the world of $\mathbb{P}\mathrm{r}_{\mathcal{V}}$, as studied in [Ram24] or [RZ25, §3], the symmetric monoidal world is more subtle as many statements are no longer true or require further assumptions.

Proposition 3.1. A 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in the 2-category $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ is internally left adjoint if its right adjoint functor F^{R} preserves colimits, and the canonical lax symmetric monoidal lax $\mathcal{V}$-linear structure on F^{R} is strong symmetric monoidal and strong $\mathcal{V}$-linear.

Proof. In Observation A.30 we have exhibited $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ as a full sub-2-category of $\mathbb{F}\mathrm{un}(\mathrm{Fin}_*, \mathbb{P}\mathrm{r}_{\mathcal{V}})$; and adjunctions a functor 2-category are given by pointwise adjunctions satisfying a Beck-Chevalley condition, see for instance [Hei17, Cor. 5.15]: For any morphism $\alpha : \underline{n}_+ \to \underline{m}_+$ in Fin_* the diagram

$$\begin{tikzcd} \mathcal{M}^{\times n} \arrow[r, "F^{\times n}"] \arrow[d, "\alpha_!"] & \mathcal{N}^{\times n} \arrow[d, "\alpha_!"] \\ \mathcal{M}^{\times m} \arrow[r, "F^{\times n}"] & \mathcal{N}^{\times m} \end{tikzcd}$$

must be horizontally right adjointable, yielding the strong symmetric monoidality condition for F^{R}. It suffices to check the pointwise adjointness condition in $\mathrm{Pr}_{\mathcal{V}}$ on the component $F^{\times 1} : \mathcal{M}^{\times 1} \to \mathcal{N}^{\times 1}$, where it unwinds into strong $\mathcal{V}$-linearity and cocontinuity of F^{R} by a similar argument. Compare [BM24, § 4]. $\square$

Definition 3.2. For $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, an object $m \in \mathcal{M}$ is called $\otimes$-*atomic* if the unique map $\mathrm{Fun}(\bigsqcup_{k \geq 0} B\Sigma_k, \mathcal{V}) \simeq \mathcal{P}\,\mathrm{Sym}(*) \otimes \mathcal{V} \to \mathcal{M}$ sending $1_{\mathcal{V}} \mapsto m$ is internally left adjoint. In other words, the *internal multi-hom* functor

$$\underline{\mathrm{Hom}}^{\otimes}_{\mathcal{M}}(m, -) : \mathcal{M} \to \prod_{k \geq 0} \mathrm{Fun}\left(B\Sigma_k, \mathcal{V}\right)$$

right adjoint to the above map preserves colimits, and its canonical lax $\mathcal{O}$-monoidal lax $\mathcal{V}$-linear structure is strong. Denote by $\mathcal{M}^{\otimes \mathrm{at}} \subseteq \mathcal{M}$ the full subcategory on the $\otimes$-atomic objects.

More generally given a small category C, a functor $\mathrm{C} \to \mathcal{M}$ is called a $\otimes$-*atomic marking* if its unique extension $\mathcal{P}\,\mathrm{Sym}\,\mathrm{C} \to \mathcal{M}$ to $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is internally left adjoint, and specifically a sequence of objects $(m_1, \dots, m_n) \in \mathcal{M}$ is called $\otimes$-*atomic sequence* if the functor $\{1, \dots, n\} \to \mathcal{M}$ marking them is a $\otimes$-atomic marking.

Remark 3.3. This naming is chosen to resemble the case of $\mathcal{M} \in \mathbb{P}\mathrm{r}_{\mathcal{V}}$, where an object $m \in \mathcal{M}$ is called $\mathcal{V}$-*atomic* or simply *atomic* if the 1-morphism $\mathcal{P}(*) \otimes \mathcal{V} \simeq \mathcal{V} \to \mathcal{M}$ in $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ sending $* \mapsto m$ is internally left adjoint. Compare [BM24], [Ram24], [RZ25, Def. 3.20].

Observation 3.4. Internally left adjoint functors in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ are closed under composition; in particular they preserve $\otimes$-atomic markings and $\otimes$-atomic objects. We will prove partial converses in Proposition 3.26 and Corollary 7.24 respectively.

Observation 3.5. Let us work out explicitly what it means for an object $m \in \mathcal{M}$ to be $\otimes$-atomic in $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. The induced functor

$$\mathcal{P}\,\mathrm{Sym}(*) \otimes \mathcal{V} \simeq \prod_{k \geq 0} \mathrm{Fun}(B\Sigma_k, \mathcal{V}) \to \mathcal{M}$$

sends a tuple $(v_0, v_1, v_2, \dots)$ of objects $v_k \in \mathcal{V}$ with Σ_k-action to

$$\coprod_{k \geq 0} m^{\otimes k} \otimes_{\Sigma_k} v_k \in \mathcal{M}$$

where we set $m^{\otimes 0} := 1_{\mathcal{V}}$, and Σ_k acts on $m^{\otimes k}$ by permuting the components. So its right adjoint $\underline{\mathrm{Hom}}^{\otimes}_{\mathcal{M}}(m, -) : \mathcal{M} \to \prod_{k \geq 0} \mathrm{Fun}(B\Sigma_k, \mathcal{V})$ sends $m' \in \mathcal{M}$ to the tuple

$$\big(\underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes k}, m') \in \mathrm{Fun}(B\Sigma_k, \mathcal{V})\big)_{k \geq 0}$$

with Σ_k-action permuting the factors of m. This means that $m \in \mathcal{M}$ is $\otimes$-atomic iff for any $n \in \mathbb{N}_0$,

- the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes n}, -)$ preserves colimits and $\mathcal{V}$-tensoring; in other words $m^{\otimes n}$ is $\mathcal{V}$-atomic,
- $\underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes n}, 1_{\mathcal{M}}) \simeq \emptyset_{\mathcal{V}}$ unless $n = 0$, where $\underline{\mathrm{Hom}}_{\mathcal{M}}(1_{\mathcal{M}}, 1_{\mathcal{M}}) \simeq 1_{\mathcal{V}}$,
- for any $m_1, m_2 \in \mathcal{M}$, the multiplication map

  $$\coprod_{\underline{n} = S \sqcup T} \underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes \# S}, m_1) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes \# T}, m_2) \xrightarrow{\otimes} \underline{\mathrm{Hom}}_{\mathcal{M}}(m^{\otimes n}, m_1 \otimes m_2)$$

  is an isomorphism.

Observation 3.6. A similar discussion shows that a marking $y : \mathrm{C} \to \mathcal{M}$ is $\otimes$-atomic if for any finite sequence of objects $c_0, \dots, c_n \in \mathrm{C}$ where $n \geq 0$,

- The product $yc_1 \otimes \cdots \otimes yc_n$ is atomic,
- $\underline{\mathrm{Hom}}_{\mathcal{M}}(yc_1 \otimes \cdots \otimes yc_n, 1_{\mathcal{M}}) \simeq \emptyset_{\mathcal{V}}$ unless $n = 0$, in which case $\underline{\mathrm{Hom}}_{\mathcal{M}}(1_{\mathcal{M}}, 1_{\mathcal{M}}) \simeq 1_{\mathcal{V}}$,
- for any $m_1, m_2 \in \mathcal{M}$, the multiplication map

$$\coprod_{\underline{n} = S \sqcup T} \underline{\mathrm{Hom}}_{\mathcal{M}}\left(\bigotimes_{s \in S} yc_s, m_1\right) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}\left(\bigotimes_{t \in T} yc_t, m_2\right) \xrightarrow{\otimes} \underline{\mathrm{Hom}}_{\mathcal{M}}\left(\bigotimes_{k=1}^{n} yc_k, m_1 \otimes m_2\right)$$

  is an isomorphism. This is known as the *hereditary condition*, c.f. [HK24, Prop. 2.4.6, Rem. 2.4.7].

Observation 3.7. We learn that for any $\otimes$-atomic objects or $\otimes$-atomic marking to exist in $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, the unit $1_{\mathcal{M}} \in \mathcal{M}$ must be $\mathcal{V}$-atomic and the lax symmetric monoidal structure on the forgetful functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(1_{\mathcal{M}}, -) : \mathcal{M} \to \mathcal{V}$ must be strong.

Remark 3.8. Let $m_1, m_2 \in \mathcal{M}$. If the product $m_1 \otimes m_2$ is $\otimes$-atomic, then in particular

$$\mathrm{Map}_{\mathcal{M}}(m_1 \otimes m_2, m_1) \otimes \mathrm{Map}_{\mathcal{M}}(1, m_2) \sqcup \mathrm{Map}_{\mathcal{M}}(1, m_1) \otimes \mathrm{Map}_{\mathcal{M}}(m_1 \otimes m_2, m_2) \to \mathrm{Map}_{\mathcal{M}}(m_1 \otimes m_2, m_1 \otimes m_2)$$

so the identity $\mathrm{id}_{m_1 \otimes m_2}$ factors either through $\mathrm{Map}_{\mathcal{M}}(m_1 \otimes m_2, m_1) \otimes \mathrm{Map}_{\mathcal{M}}(1, m_2)$ or the other summand. This means there exist $i : m_1 \otimes m_2 \to m_1$ and $r : 1 \to m_2$ such that

$$\mathrm{id}_{m_1 \otimes m_2} \simeq r \otimes i \simeq (m_1 \otimes m_2 \xrightarrow{i} m_1 \simeq m_1 \otimes 1 \xrightarrow{m_1 \otimes r} m_1 \otimes m_2)$$

exhibiting $m_1 \otimes m_2$ as a retract of m_1.[3] Informally, it is rare for the tensor product of $\otimes$-atomic objects to remain $\otimes$-atomic; they should be imagined as singletts or indecomposables. We will formalize this in Section 9.

Lemma 3.9. Given any functor $\mathrm{C} \to \mathrm{D}$, the induced map $\mathcal{P}\operatorname{Sym}(\mathrm{C}) \otimes \mathcal{V} \to \mathcal{P}\operatorname{Sym}(\mathrm{D}) \otimes \mathcal{V}$ in $\operatorname{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ is internally left adjoint.

Proof. We apply $\operatorname{Sym}^{\otimes}$ and $-\otimes\mathcal{V}$, which are 2-functors by Observation A.28, to $\mathcal{P}(\mathrm{C}) \to \mathcal{P}(\mathrm{D})$ which is internally left adjoint in $\mathbb{P}\mathrm{r}$. ☐

Observation 3.10. If $y : \mathrm{C} \to \mathcal{M}$ is a $\otimes$-atomic marking, then for any $c \in \mathrm{C}$ the object $y(c) \in \mathcal{M}$ is $\otimes$-atomic, since the functor $\mathcal{P}\operatorname{Sym}(*) \otimes \mathcal{V} \to \mathcal{P}\operatorname{Sym}(\mathrm{C}) \otimes \mathcal{V} \to \mathcal{M}$ marking $y(c)$ is a composition of internally left adjoint functors by Lemma 3.9.

Warning 3.11. The converse is wrong: The Cartesian (which agrees with the coCartesian) symmetric monoidal structure on Pr induces a pointwise symmetric monoidal structure $\times$ on $\operatorname{CAlg}(\mathrm{Pr})$, in particular $\mathcal{P}\operatorname{Sym}(*)^{\times 2} \in \operatorname{CAlg}(\mathrm{Pr})$. Both inclusions into and projections out of it are maps in $\operatorname{CAlg}(\mathrm{Pr})$ by Lemma 3.9. The objects $(*, 1), (1, *) \in \mathcal{P}\operatorname{Sym}(*)^{\times 2}$ are both $\otimes$-atomic, e.g.

$$\underline{\operatorname{Hom}}^{\otimes}((*, 1), -) \simeq \mathrm{pr}_1 : \mathcal{P}\operatorname{Sym}(*)^{\times 2} \to \mathcal{P}\operatorname{Sym}(*)$$

preserves colimits and symmetric monoidal structure. However they do not form a $\otimes$-atomic pair, since otherwise their product $(*, *) \in \mathcal{P}\operatorname{Sym}(*)^{\times 2} \simeq \mathcal{P}(\operatorname{Sym}(*) \sqcup \operatorname{Sym}(*))$ would be $\mathcal{S}$-atomic and thereby representable as $\operatorname{Sym}(*) \sqcup \operatorname{Sym}(*)$ is a groupoid, which it evidently is not.

Remark 3.12. We will however see in Proposition 3.25 that to check that $y : \mathrm{C} \to \mathcal{M}$ is a $\otimes$-atomic marking, it is sufficient to verify that the restriction $\{c_1, \dots, c_n\} \to \mathcal{M}$ is a $\otimes$-marking for any finite set of objects $c_1, \dots, c_n$ in C. Also if we already suppose that $\mathcal{M}$ is $\otimes$-atomically generated, then we prove in Corollary 7.24 that the converse *is* in fact true, i.e. $y : \mathrm{C} \to \mathcal{M}$ is a $\otimes$-atomic marking iff it factors through the atomics.

Observation 3.13. By definition of the internal multihom, we have

$$\operatorname{Map}_{\operatorname{Fun}(B\Sigma, \mathcal{V})}(1_{\mathcal{V}}, \underline{\operatorname{Hom}}^{\otimes}_{\mathcal{M}}(m, -)) \simeq \operatorname{Map}_{\mathcal{M}}(m \otimes 1_{\mathcal{V}}, -) \simeq \operatorname{Map}_{\mathcal{M}}(m, -) \; .$$

This enhances to $\underline{\operatorname{Hom}}_{\operatorname{Fun}(B\Sigma, \mathcal{V})}(1_{\mathcal{V}}, \underline{\operatorname{Hom}}^{\otimes}_{\mathcal{M}}(m, -)) \simeq \underline{\operatorname{Hom}}_{\mathcal{M}}(m, -)$ since

$$\operatorname{Map}_{\mathcal{V}}\left(v, \underline{\operatorname{Hom}}_{\operatorname{Fun}(B\Sigma, \mathcal{V})}(1_{\mathcal{V}}, \underline{\operatorname{Hom}}^{\otimes}_{\mathcal{M}}(m, -))\right) \simeq \operatorname{Map}_{\operatorname{Fun}(B\Sigma, \mathcal{V})}\left(1_{\mathcal{V}} \otimes v, \underline{\operatorname{Hom}}^{\otimes}_{\mathcal{M}}(m, -)\right) \simeq \operatorname{Map}_{\mathcal{M}}(m \otimes v, -) \; .$$

Proposition 3.14. If κ is a regular cardinal such that the unit $1_{\mathcal{V}} \in \mathcal{V}$ is κ-compact, then every $\otimes$-atomic object in $\mathcal{M} \in \operatorname{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is also κ-compact. In particular, the full subcategory $\mathcal{M}^{\otimes\mathrm{at}} \subseteq \mathcal{M}$ on the $\otimes$-atomic objects is always small.

Proof. By Observation 3.5, $\otimes$-atomic objects are in particular $\mathcal{V}$-atomic, so this follows from [BM24, Prop. 5.9], [RZ25, Prop. 3.25]. ☐

Definition 3.15. For C a small category and $\mathcal{M} \in \operatorname{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, a functor $\mathrm{C} \to \mathcal{M}$ is called a *$\otimes$-atomically generating marking* if it is a $\otimes$-atomic marking, and the smallest full subcategory of $\mathcal{M}$ containing its image, that is closed under symmetric monoidal structure, $\mathcal{V}$-tensoring and colimits, is $\mathcal{M}$ itself. A full subcategory $\mathcal{M}_0 \subseteq \mathcal{M}$ is called a *system of $\otimes$-atomic*

[3] Conversely retracts of $\otimes$-atomic objects are always $\otimes$-atomic, since colimit-preserving $\mathcal{V}$-linear symmetric monoidal functors are closed under retracts in lax $\mathcal{V}$-linear lax symmetric monoidal functors (compare Example 7.21).

generators if its inclusion is a $\otimes$-atomically generating marking. We call $\mathcal{M}$ *$\otimes$-atomically generated* if it admits a $\otimes$-atomically generating marking, or equivalently (taking its image) a system of $\otimes$-atomic generators.

Warning 3.16. While a system of $\otimes$-atomic generators $\mathcal{M}_0 \subseteq \mathcal{M}$ always consists of $\otimes$-atomic objects by Observation 3.10, this is not enough to ensure that its inclusion is a $\otimes$-atomic marking. For instance, the category $\mathcal{P}\,\mathrm{Sym}(*)^{\times 2}$ from Warning 3.11 turns out not to be $\otimes$-atomically generated.

Definition 3.17. We call $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ *Cauchy-$\otimes$-atomically generated* if its full subcategory $\mathcal{M}^{\otimes\mathrm{at}} \subseteq \mathcal{M}$ is a system of $\otimes$-atomic generators.

Observation 3.18. Clearly, a Cauchy-$\otimes$-atomically generated is $\otimes$-atomically generated. However the converse is not clear, since the extension $\mathcal{P}\,\mathrm{Sym}(\mathcal{M}^{\otimes\mathrm{at}}) \otimes \mathcal{V} \to \mathcal{M}$ might not be internally left adjoint, because its right adjoint might not be strong monoidal. We will prove this converse in Corollary 7.23, with quite some effort.

Notation 3.19. Given a functor $F : \mathrm{C} \to \mathrm{D}$ where D is cocomplete, its *colimit-closed image* $\mathrm{CIm}(F) \subseteq \mathrm{D}$ is the smallest full subcategory of D containing the image of F that is closed under colimits in D.

Lemma 3.20. Let $F : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. Then its colimit-closed image $\mathrm{CIm}(F) \subseteq \mathcal{N}$ is closed under $\mathcal{V}$-tensoring and the symmetric monoidal structure.

Proof. Let $\mathcal{M}_1$ denote the full subcategory of $\mathcal{M}$ on those m such that $m \otimes m_0 \in \mathrm{CIm}(F)$ for all m_0 in the actual image of F. This contains the image of F and is closed under colimits, hence it must agree with $\mathrm{CIm}(F)$. Now, let $\mathcal{M}_2$ be the full subcategory on those $m \in \mathcal{M}$ such that $m \otimes v \in \mathrm{CIm}(F)$ for all $v \in \mathcal{V}$, and $m \otimes m' \in \mathrm{CIm}(F)$ for all $m' \in \mathrm{CIm}(F)$. By the previous argument this contains the image of F, also it is closed under colimits, so $\mathcal{M}_2$ agrees with $\mathrm{CIm}(F)$ and we are done. ☐

Lemma 3.21. Let C be a small category, $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ and $y : \mathrm{C} \to \mathcal{M}$ a functor. Then the following are equivalent:

(1) The image of y generates $\mathcal{M}$ under colimits, $\mathcal{V}$-tensoring and the symmetric monoidal structure,
(2) The unique extension $Y : \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V} \to \mathcal{M}$ to a morphism in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is colimit-dominant,
(3) The right adjoint $Y^{\mathrm{R}} : \mathcal{M} \to \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}$, which sends m to $((c_1, \dots, c_k) \mapsto \underline{\mathrm{Hom}}_{\mathcal{M}}(c_1 \otimes \dots \otimes c_k, m)) \in \mathrm{Fun}(\mathrm{Sym}(\mathrm{C})^{\mathrm{op}}, \mathcal{V})$, is conservative.

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from [RZ25, Cor. 3.13]. Also (2) $\Rightarrow$ (1) since the image of $\mathrm{C} \to \mathcal{P}\,\mathrm{Sym}\,\mathrm{C} \otimes \mathcal{V}$ clearly generates under colimit, $\mathcal{V}$-tensoring and symmetric monoidal structure. Finally, (1) $\Rightarrow$ (2) follows since the colimit-closed image of Y contains the image of y and is closed under colimits, $\mathcal{V}$-tensoring and symmetric monoidal structure by Lemma 3.20 ☐

Lemma 3.22. Let $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ together with functors

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] & & \mathcal{P} \\ & \mathcal{N} \arrow[ul, "I", "\mathrm{iL,cdom}"'] \arrow[ur, "G"', "\mathrm{iL}"] & \end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ where I is internally left adjoint and colimit-dominant, and G is internally left adjoint. Then F is internally left adjoint.

Proof. Consider the full subcategory $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ on those $m \in \mathcal{M}$ such that for all diagrams $\mathrm{I} \to \mathcal{P}$, and all objects $p, p' \in \mathcal{P}$ and $v \in \mathcal{V}$ the morphisms

$$\mathrm{Map}_{\mathcal{M}}(m, \operatorname*{colim}_i F^{\mathrm{R}} p_i) \to \mathrm{Map}_{\mathcal{M}}(m, F^{\mathrm{R}} \operatorname*{colim}_i p_i)$$

$$\mathrm{Map}_{\mathcal{M}}(m, F^{\mathrm{R}} p \otimes v) \to \mathrm{Map}_{\mathcal{M}}(m, F^{\mathrm{R}}(p \otimes v))$$
$$\mathrm{Map}_{\mathcal{M}}(m, F^{\mathrm{R}} p \otimes F^{\mathrm{R}} p') \to \mathrm{Map}_{\mathcal{M}}(m, F^{\mathrm{R}}(p \otimes p'))$$

are isomorphisms. By assumption $\widetilde{\mathcal{M}}$ contains the full image of I, since $\mathrm{Map}_{\mathcal{M}}(I(n), -) \simeq \mathrm{Map}_{\mathcal{M}}(n, I^{\mathrm{R}} -)$ and both I^{R} and $I^{\mathrm{R}} F^{\mathrm{R}}$ belong to $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. Moreover, $\widetilde{\mathcal{M}}$ is closed under colimits since we can pull them out as limits, so by colimit-dominance we deduce $\widetilde{\mathcal{M}} = \mathcal{M}$. By Yoneda, imlies that F^{R} preserves colimits, symmetric monoidal structure and tensoring and thus F were internally left adjoint. $\square$

Lemma 3.23 (Adjoint Lifting Theorem in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$)**.** Let $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ together with functors

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "U", "\mathrm{iR}"'] & & \mathcal{P} \arrow[dl, "V"', "\mathrm{iR,cons}"] \\ & \mathcal{N} & \end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ where U is internally right adjoint, and V is internally right adjoint and conservative. Then F is internally right adjoint.

Proof. First, note that the full subcategory of $\mathcal{N}$ spanned by those n such that the copresheaf $\mathrm{Map}_{\mathcal{M}}(n, F-) : \mathcal{M} \to \mathcal{S}$ is representable contains the image of V^{L} by assumption that U is internally right adjoint, also it is closed under colimits since the coYoneda embedding preserves colimits. Hence we deduce that F admits a left adjoint, which automatically inherits an oplax $\mathcal{V}$-linear oplax symmetric monoidal structure. Once again using that U is internally right adjoint, we learn that F^{L} is strong $\mathcal{V}$-linear and symmetric monoidal on the image of V^{L}, so by colimit-dominance of V^{L} it is so everywhere. $\square$

Corollary 3.24. Let $y : \mathrm{C} \to \mathcal{M}$ be a functor with $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, and $f : \mathrm{C}' \to \mathrm{C}$ a surjective functor. Then y is a $\otimes$-atomically generating marking iff the composition $y \circ f : \mathrm{C}' \to \mathcal{M}$ is a $\otimes$-atomically generating marking.

In particular, $y : \mathrm{C} \to \mathcal{M}$ is a $\otimes$-atomically generating marking iff the fully faithful inclusion $\mathrm{Im}(y) \hookrightarrow \mathcal{M}$ is, iff the subcategory inclusion $\mathrm{Im}(y)^{\simeq} \hookrightarrow \mathcal{M}$ is.

Proof. Since $f : \mathrm{C} \to \mathrm{C}'$ is surjective, the induced $\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \to \mathcal{P}\,\mathrm{Sym}(\mathrm{C}')$ is colimit-dominant since it hits all representables, and internally left adjoint by Lemma 3.9. Now the result follows by applying Lemma 3.22, together with composition and cancellation properties of colimit-dominant and internally left adjoint functors. $\square$

Proposition 3.25. A functor $y : \mathrm{C} \to \mathcal{M}$ into $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is a $\otimes$-atomic marking iff for any tuple $(c_1, \ldots, c_n)$ of objects in C, the images $(y(c_1), \ldots, y(c_n))$ form a $\otimes$-atomic sequence in $\mathcal{M}$.

Proof. Let $Y : \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V} \to \mathcal{M}$ be the unique extension of y, and consider the full subcategory $\widetilde{\mathcal{C}} \subseteq \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}$ on those c such that for all diagrams $\mathrm{I} \to \mathcal{P}$, and all objects $p, p' \in \mathcal{P}$ and $v \in \mathcal{V}$ the morphisms

$$\mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}}(c, \operatorname*{colim}_i Y^{\mathrm{R}} p_i) \to \mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}}(c, Y^{\mathrm{R}} \operatorname*{colim}_i p_i)$$

$$\mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}}(c, Y^{\mathrm{R}} p \otimes v) \to \mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}}(c, Y^{\mathrm{R}}(p \otimes v))$$

$$\mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C})\otimes\mathcal{V}}(c, Y^{\mathrm{R}}p \otimes Y^{\mathrm{R}}p') \to \mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C})\otimes\mathcal{V}}(c, Y^{\mathrm{R}}(p \otimes p'))$$

are isomorphisms. We now show it contains the image of $\mathrm{Sym}\,\mathrm{C} \subseteq \mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \to \mathcal{P}\,\mathrm{Sym}(\mathrm{C})\otimes\mathcal{V}$. Then since it is closed under colimits and $\mathcal{V}$-tensoring, we are done. So let us choose $c_1, \ldots, c_n \in \mathrm{C}$ and denote $Z : \mathcal{P}\,\mathrm{Sym}\{1, \ldots, n\} \to \mathcal{P}\,\mathrm{Sym}(\mathrm{C})$ the induced functor; then for instance the last map agrees with

$$\mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}(\mathrm{C})\otimes\mathcal{V}}(Z(c_1\otimes\cdots\otimes c_n), Y^{\mathrm{R}}p\otimes Y^{\mathrm{R}}p') \simeq \mathrm{Map}_{\mathcal{P}\,\mathrm{Sym}\{1,\ldots,n\}\otimes\mathcal{V}}(c_1\otimes\cdots\otimes c_n, (Y\circ Z)^{\mathrm{R}}p\otimes(Y\circ Z)^{\mathrm{R}}p')$$

which is an isomorphism as $Y \circ Z$ is internally left adjoint by assumption. □

Proposition 3.26. Let $F : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, and $\mathrm{C} \to \mathcal{M}$ any $\otimes$-atomically generating marking. Then, F is internally left adjoint iff the composition $\mathrm{C} \to \mathcal{M} \to \mathcal{N}$ is a $\otimes$-atomic marking.

Proof. Consider the composition of the unique extension:

$$\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V} \to \mathcal{M} \to \mathcal{N}$$

Internally left adjoint functors compose, so the *if* direction is immediate. For the only if direction, apply Lemma 3.22 to this composition. □

Lemma 3.27. Let $y : \mathrm{C} \to \mathcal{M}$ be a $\otimes$-atomically generating marking. An internally left adjoint morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is fully faithful if and only if for all $c_1, \ldots, c_n, c' \in \mathrm{C}$, the induced morphism

$$F : \underline{\mathrm{Hom}}_{\mathcal{M}}(yc_1 \otimes \cdots \otimes yc_n, yc') \to \underline{\mathrm{Hom}}_{\mathcal{N}}(F(yc_1) \otimes \cdots \otimes F(yc_n), F(yc'_m))$$

is an isomorphism in $\mathcal{V}$.

Proof. Let $Y : \mathcal{P}\,\mathrm{Sym}(\mathrm{C})\otimes\mathcal{V} \to \mathcal{M}$ be the unique extension of y to a morphism in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, then the above comparison map is obtained by evaluating the transformation $Y^{\mathrm{R}}(yc') \to (F \circ Y)^{\mathrm{R}}(Fyc')$ induced by the unit $\mathrm{id} \to F^{\mathrm{R}}F$ at the tuple $(c_1, \ldots, c_n)$. Since both Y and F are internally left adjoint by assumption, the full subcategory of $m' \in \mathcal{M}$ where $Y^{\mathrm{R}}(m') \to Y^{\mathrm{R}}F^{\mathrm{R}}F(m')$ is an isomorphism is closed under colimits, $\mathcal{V}$-tensoring and symmetric monoidal structure, so since it contains the image of y it agrees with $\mathcal{M}$.

Conversely for fixed $m' \in \mathcal{M}$, the full subcategory on those $m \in \mathcal{M}$ such that $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, m') \to \underline{\mathrm{Hom}}_{\mathcal{M}}(Fm, Fm')$ is an isomorphism contains the image of the symmetric monoidal extension $\mathrm{Sym}\,\mathrm{C} \to \mathcal{M}$ by assumption, and is closed under colimits and $\mathcal{V}$-tensoring since we can pull them out. Hence it contains the colimit-closed image of $\mathcal{P}\,\mathrm{Sym}(\mathrm{C}) \otimes \mathcal{V}$ which is all of $\mathcal{M}$. □

Proposition 3.28. The subcategory $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iL}} \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is closed under sifted colimits.

Proof. Since the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) \to \mathrm{Pr}_{\mathcal{V}}$ preserves sifted colimits by [Lur17, Cor. 3.2.3.2], this is analogous to [Ram24, Corollary 1.32]. □

Warning 3.29. The subcategory $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iL}} \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is not closed under coproducts: By [Lur17, Prop. 3.2.4.7] the coproduct is the pointwise tensor product $\otimes_{\mathcal{V}}$ in $\mathrm{Pr}_{\mathcal{V}}$. Consider $\mathcal{P}\,\mathrm{Sym}(*) \otimes \mathcal{V}$, then internally left adjoint maps from it into $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ correspond to $\otimes$-objects, while internally left adjoint maps from $(\mathcal{P}\,\mathrm{Sym}(*) \otimes \mathcal{V}) \otimes_{\mathcal{V}} (\mathcal{P}\,\mathrm{Sym}(*) \otimes \mathcal{V}) \simeq \mathcal{P}\,\mathrm{Sym}(* \sqcup *) \otimes \mathcal{V}$ correspond to $\otimes$-atomic pairs, so compare Warning 3.11. We can fix this by restricting to $\otimes$-atomically generated categories, see Proposition 7.25.

The extensive list of conditions in Observation 3.5 indicates that checking explicitly whether a given object or pair is $\otimes$-atomic can be difficult. However the following discussion ensures a steady supply of internally left adjoint functors and $\otimes$-atomic objects:

Notation 3.30. Given a functor $F : \mathrm{C} \to \mathrm{D}$ and $d \in \mathrm{D}$, we write $\mathrm{C}_{/d}$ for the pullback $\mathrm{C} \times_{\mathrm{D}} \mathrm{D}_{/d}$.

Definition 3.31 (compare [BS22, Def. 3.2.14])**.** A symmetric monoidal functor $F : \mathrm{C} \to \mathrm{D}$ is *$\otimes$-disjunctive* if for any $a, b \in \mathrm{D}$ the induced functor

$$\otimes : \mathrm{C}_{/a} \times \mathrm{C}_{/b} \to \mathrm{C}_{/a \otimes b}$$

is an equivalence. More generally F is called *weakly $\otimes$-disjunctive* if this induced functor is colimit-cofinal for all $a, b \in \mathrm{D}$.

Observation 3.32. Inductively, note that F is (weakly) $\otimes$-disjunctive iff for any $n \in \mathbb{N}_0$ and $d_1, \ldots, d_n \in \mathrm{D}$ the functor

$$\otimes : \mathrm{C}_{/d_1} \times \cdots \times \mathrm{C}_{/d_n} \to \mathrm{C}_{/d_1 \otimes \cdots \otimes d_n}$$

induced by the symmetric monoidal structure is an equivalence (colimit-cofinal).

Observation 3.33. This generalizes the notion of $\otimes$-disjunctive symmetric monoidal categories from [BS22, Def. 3.2.14], in the sense that C is $\otimes$-disjunctive iff the identity functor id_{C} is. It follows from [BS22, Cor. 3.2.16] that the envelope $\mathrm{Env}(\mathrm{O})$ of any operad O is $\otimes$-disjunctive, i.e. for tuples $\underline{o}, \underline{o'} \in \mathrm{Env}(\mathrm{O})$ we have

$$\mathrm{Env}(\mathrm{O})_{/\underline{o}} \times \mathrm{Env}(\mathrm{O})_{/\underline{o'}} \xrightarrow{\simeq} \mathrm{Env}(\mathrm{O})_{/\underline{o} \otimes \underline{o'}}$$

where $\underline{o} \otimes \underline{o'}$ is the concatenation of tuples. This extends to map of operads, generalizing Lemma 3.9:

Definition 3.34 ([BS22, Def. 2.2.1])**.** A symmetric monoidal functor $F : \mathrm{C} \to \mathrm{D}$ is called *equifibered* if the following commutative square is a pullback:

$$\begin{tikzcd} \mathrm{C} \times \mathrm{C} \arrow[r, "F \times F"] \arrow[d, "\otimes"] & \mathrm{D} \times \mathrm{D} \arrow[d, "\otimes"] \\ \mathrm{C} \arrow[r, "F"] & \mathrm{D} \end{tikzcd}$$

Lemma 3.35. A symmetric monoidal functor $F : \mathrm{C} \to \mathrm{D}$ is $\otimes$-disjunctive iff the target projection $\mathrm{C} \times_{\mathrm{D}} \mathrm{Arr}(\mathrm{D}) \to \mathrm{D}$, regarded as a symmetric monoidal functor by equipping $\mathrm{Arr}(\mathrm{D})$ with the pointwise symmetric monoidal structure, is equifibered.

Proof. As (1) $\Leftrightarrow$ (2) in [BS22, Lem. 3.2.15]. $\square$

Lemma 3.36 ([BS22, above Cor. 2.2.28])**.** Equifibered symmetric monoidal functors are closed under composition and pullbacks in $\mathrm{CAlg}(\mathcal{C}at)$, since they form the right side of a factorization system.

Lemma 3.37. Let $\mathrm{C}, \mathrm{D}, \mathrm{E}$ be symmetric monoidal categories, $F : \mathrm{C} \to \mathrm{D}$ an equifibered symmetric monoidal functor and $G : \mathrm{D} \to \mathrm{E}$ a $\otimes$-disjunctive symmetric monoidal functor. Then the composite $G \circ F$ is $\otimes$-disjunctive.

Remark 3.38. Compare this statement to pasting for (lax) pullback squares: Pullback squares paste (just as equifibered functors compose), but also a pullback square can be pasted from the left to a lax pullback square to obtain another lax pullback.

Proof. By Lemma 3.35 we must show that the horizontal composition in the commutative diagram

$$\begin{tikzcd} \mathrm{C} \times_{\mathrm{E}} \mathrm{Arr}(\mathrm{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D} \times_{\mathrm{E}} \mathrm{Arr}(\mathrm{E}) \arrow[r, "\mathrm{trg}"] \arrow[d, "\mathrm{src}"] & \mathrm{E} \\ \mathrm{C} \arrow[r, "F"] & \mathrm{D} & \end{tikzcd}$$

is equifibered. But $\mathrm{D} \times_{\mathrm{E}} \mathrm{Arr}(\mathrm{E}) \to \mathrm{E}$ is equifibered by the same Lemma, so since the left square is a pullback this follows from Lemma 3.36. □

Proposition 3.39. Let $f : \mathrm{O} \to \mathrm{P}$ be a map of Lurie-operads. Then the induced symmetric monoidal functor $\mathrm{Env}(f) : \mathrm{Env}(\mathrm{O}) \to \mathrm{Env}(\mathrm{P})$ is $\otimes$-disjunctive.

Proof. By Lemma 3.37, writing $\mathrm{Env}(f) = \mathrm{id}_{\mathrm{Env}(\mathrm{P})} \circ \mathrm{Env}(f)$ it suffices to note that $\mathrm{Env}(f)$ is equifibered by [BS22, Thm. 3.2.13] and $\mathrm{id}_{\mathrm{Env}(\mathrm{P})}$ is $\otimes$-disjunctive by Observation 3.33. □

Remark 3.40. We will see in Corollary 9.11 that in fact, a symmetric monoidal functor $F : \mathrm{Env}(\mathrm{O}) \to \mathrm{Env}(\mathrm{P})$ is of the form $\mathrm{Env}(f)$ for some map of operads $f : \mathrm{O} \to \mathrm{P}$ iff it is $\otimes$-disjunctive iff it is weakly $\otimes$-disjunctive.

Observation 3.41. For $F : \mathrm{C} \to \mathrm{D}$ a symmetric monoidal functor, since $\mathcal{P} : \mathcal{C}at \to \mathrm{Pr}$ is symmetric monoidal by [Lur17, Rem. 4.8.1.8] the induced $F_! : \mathcal{P}(\mathrm{C}) \to \mathcal{P}(\mathrm{D})$ is symmetric monoidal with respect to the Day convolution product. Thus its right adjoint precomposition functor $F^* : \mathcal{P}(\mathrm{D}) \to \mathcal{P}(\mathrm{C})$ inherits a canonical lax symmetric monoidal structure.

We thank Jan Steinebrunner for communicating to us the following statement:

Theorem 3.42. Let $F : \mathrm{C} \to \mathrm{D}$ be a symmetric monoidal functor, then the lax symmetric monoidal structure on $F^* : \mathcal{P}(\mathrm{D}) \to \mathcal{P}(\mathrm{C})$ from Observation 3.41 is strong symmetric monoidal if and only if F is weakly $\otimes$-disjunctive.

Proof. To show that the lax symmetric monoidal functor F^* is monoidal, it suffices to prove that for any active morphism $\alpha : \underline{n}_+ \to \underline{m}_+$ and $w_1, \ldots, w_n \in \mathcal{P}(\mathrm{D})$ the commutative (by symmetric monoidality of $F_!$) diagram

$$\begin{tikzcd} \mathcal{P}(\mathrm{C}^{\times n}) \arrow[r, "F_!"] \arrow[d, "\otimes_\alpha"] & \mathcal{P}(\mathrm{D}^{\times n}) \arrow[d, "\otimes_\alpha"] \\ \mathcal{P}(\mathrm{C}^{\times m}) \arrow[r, "F_!"] & \mathcal{P}(\mathrm{D}^{\times m}) \end{tikzcd}$$

is horizontally right adjointable. But by the definition of a Lurie-operad, the operation $\otimes_\alpha : \mathcal{P}(\mathrm{D}^{\times n}) \to \mathcal{P}(\mathrm{D}^{\times m})$ splits into a product of m operations $\otimes_{\alpha_i} : \mathcal{P}(\mathrm{D}^{\times f^{-1}(\{i\})}) \to \mathcal{P}(\mathrm{D}^{\times\{i\}})$, so it suffices to consider the case where $m = 1$ and $\alpha : \underline{n}_+ \to \underline{1}_+$ is the unique active map. Horizontal right adjointability of the diagram

$$\begin{tikzcd} \mathcal{P}(\mathrm{C}^{\times n}) \arrow[r] \arrow[d] & \mathcal{P}(\mathrm{D}^{\times n}) \arrow[d] \\ \mathcal{P}(\mathrm{C}) \arrow[r] & \mathcal{P}(\mathrm{D}) \end{tikzcd}$$

can be checked on representable presheaves $よ_{(d_1,d_2)} \in \mathcal{P}(\mathrm{D} \times \mathrm{D})$, since all of the involved functors (including the horizontal right adjoints) preserve colimits. This presheaf is represented by the right fibration $(\mathrm{D}^{\times n})_{/(d_1,\ldots,d_n)} \to \mathrm{D}^{\times n}$, and its image $\otimes_! よ_{(d_1,d_2)} \simeq よ_{d_1 \otimes d_2}$ by the right fibration $\mathrm{D}_{/d_1 \otimes d_2} \to \mathrm{D}$. They fit into a commutative diagram

$$\begin{tikzcd}
& (\mathrm{C} \times \mathrm{C})_{/(d_1,d_2)} \arrow[rr, "\otimes"] \arrow[dl, "F \times F"'] \arrow[dd, "\otimes"] & & \mathrm{C}_{/d_1 \otimes d_2} \arrow[dl, "F"] \arrow[dd] \\
(\mathrm{D} \times \mathrm{D})_{/(d_1,d_2)} \arrow[rr, "\otimes"] \arrow[dd] & & \mathrm{D}_{/d_1 \otimes d_2} \arrow[dd] & \\
& \mathrm{C} \times \mathrm{C} \arrow[rr, "\otimes"] \arrow[dl, "F \times F"'] & & \mathrm{C} \arrow[dl, "F"] \\
\mathrm{D} \times \mathrm{D} \arrow[rr, "\otimes"] & & \mathrm{D} &
\end{tikzcd}$$

where the vertical morphisms are right fibrations, and the left and right squares are pullbacks. Recall that for a presheaf $W \in \mathcal{P}(\mathrm{B})$ represented by a right fibration $\mathrm{E} \to \mathrm{B}$ and a functor $g : \mathrm{B} \to \mathrm{B}'$, if we factor the composite map $\mathrm{E} \to \mathrm{B}'$ into a cofinal map $\mathrm{E} \to \mathrm{E}'$ and a right fibration $\mathrm{E}' \to \mathrm{B}'$ using that those classes form a factorization system, then $\mathrm{E}' \to \mathrm{B}'$ represents the left Kan extension $F_! W \in \mathcal{P}(\mathrm{B}')$: This follows from the composing the adjunction $g \circ - : \mathcal{C}at_{/\mathrm{B}} \to \mathcal{C}at_{/\mathrm{B}'} : g^*$ and the reflection of $\mathrm{RFib}_{/\mathrm{B}'} \subseteq \mathcal{C}at_{/\mathrm{B}'}$ by factoring through a right fibration. In other words, to show that the Beck-Chevalley map is an isomorphism it suffices to check that the horizontal map $(\mathrm{C} \times \mathrm{C})_{/(d_1,d_2)} \to \mathrm{C}_{/d_1 \otimes d_2}$ is colimit-cofinal, thereby exhibiting the presheaf $F^* \text{よ}_{/d_1 \otimes d_2}$ as $\otimes_! F^* \text{よ}_{(d_1,d_2)}$. But this is precisely the condition of weak $\otimes$-disjunctivity. $\square$

Corollary 3.43. If $f : \mathrm{O} \to \mathrm{P}$ is a map of Lurie-operads, then the induced functor $\mathcal{P}\,\mathrm{Env}(f) \otimes \mathcal{V} : \mathcal{P}\,\mathrm{Env}(\mathrm{O}) \otimes \mathcal{V} \to \mathcal{P}\,\mathrm{Env}(\mathrm{P}) \otimes \mathcal{V}$ is internally left adjoint in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$.

Proof. Since the free module functor $- \otimes \mathcal{V} : \mathrm{CAlg}(\mathrm{Pr}) \to \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is a 2-functor by Observation A.28, and thus preserves internal left adjoints, it suffices to consider the case $\mathcal{V} = \mathcal{S}$, which is part of Theorem 3.42. $\square$

Corollary 3.44. If O is a Lurie-operad, then $\mathcal{P}\,\mathrm{Env}(\mathrm{O}) \otimes \mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is $\otimes$-atomically generated via the marking given by the Yoneda functor $\underline{\mathrm{O}}^{\simeq} \hookrightarrow \mathrm{Env}(\mathrm{O}) \subseteq \mathcal{P}\,\mathrm{Env}(\mathrm{O}) \to \mathcal{P}\,\mathrm{Env}(\mathrm{O}) \otimes \mathcal{V}$. In particular for any color $o \in \mathrm{O}$, the representable presheaf $\text{よ}_{(o)} \otimes 1_{\mathcal{V}}$ on the singlett (o) is $\otimes$-atomic in $\mathcal{P}\,\mathrm{Env}\,\mathrm{O} \otimes \mathcal{V}$.

Corollary 3.45. For $\mathcal{O} \in v\mathcal{O}p_X(\mathcal{V})$ any $\mathcal{V}$-operad, its category $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) = \mathrm{LMod}_{\mathcal{O}}(\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V}) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ of operadic presheaves is $\otimes$-atomically generated by X via the Yoneda functor $\text{よ}_{\mathcal{O}} : X \to \mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$.

Proof. We know from Corollary 3.44 that the category $\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V} \simeq \mathcal{P}\,\mathrm{Env}\,\mathrm{Triv}_X \otimes \mathcal{V}$ is $\otimes$-atomically generated by X. The free functor $\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V} \to \mathrm{LMod}_{\mathcal{O}}(\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V})$ is internally left adjoint as well. We are done since any module is a geometric realization of free modules. $\square$

4. Marked algebras

We have defined enriched operads as certain monads in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$. Next, we give a description and characterization of the associated monadic 1-morphisms, leading to the notion of marked algebras.

Proposition 4.1. An adjunction $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ internal to $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ is monadic iff it satifies either of the following equivalent conditions:

- The underlying adjunction in $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ is monadic,

- The underlying adjunction in $\mathbb{P}\mathrm{r}$ is monadic,
- The underlying adjunction in $\widehat{\mathbb{C}\mathrm{at}}$ is monadic,
- F is colimit-dominant,
- G is conservative.

Proof. The forgetful 2-functors $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}}) \to \mathbb{P}\mathrm{r}_{\mathcal{V}} \to \mathbb{P}\mathrm{r} \to \widehat{\mathbb{C}\mathrm{at}}$ create Eilenberg-Moore-objects by Proposition A.35, in particular they create the associated colimit cones, which involve monadic 1-morphisms. In $\widehat{\mathbb{C}\mathrm{at}}$ we may apply Barr-Beck-Lurie [Lur17, Thm. 4.7.3.5], where the colimit conditions are automatically satisfied so only conservativity of G remains. This is equivalent to colimit-dominance of F by [RZ25, Cor. 3.15]. $\square$

Given a monad T on an object c in a 2-category $\mathbb{C}$, we should always be able to recover T from the associated monadic adjunction

$$\mathrm{free} : c \leftrightarrows \mathrm{LMod}_T(c) : \mathrm{fgt}$$

as the composition $\mathrm{fgt} \circ \mathrm{free} : c \to c$, assuming the Eilenberg-Moore object $\mathrm{LMod}_T(c)$ exists. The following statement makes this precise:

Proposition 4.2 ([Hei17, Hau20, Sto25])**.** Let $\mathbb{C}$ be a 2-category that admits Eilenberg-Moore objects, with underlying 1-category $\mathbb{C}^{\leq 1}$ and $c \in \mathbb{C}$. Then there exists a fully faithful functor

$$\mathcal{EM} : \mathrm{Alg}(\mathrm{End}_{\mathbb{C}}(c))^{\mathrm{op}} \hookrightarrow (\mathbb{C}^{\leq 1})_{/c}$$

sending a monad T on c to its Eilenberg-Moore object $\mathrm{LMod}_T(c) \in \mathbb{C}$, equipped with the monadic right adjoint $\mathrm{LMod}_T(c) \to c$ in the corresponding monadic adjunction.

Corollary 4.3. For any $\mathcal{M} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$, there exists a fully faithful functor

$$\mathcal{EM} : \mathrm{Alg}(\mathrm{End}^{\otimes,\mathrm{L}}_{\mathcal{V}}(\mathcal{M})) \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{\mathcal{M}/}$$

sending a colimit-preserving $\mathcal{V}$-linear symmetric monoidal monad T on $\mathcal{M}$ to its Eilenberg-Moore object $\mathrm{LMod}_T(\mathcal{M})$, equipped with the free module functor $\mathcal{M} \to \mathrm{LMod}_T(\mathcal{M})$.

Proof. By Lemma 3.23, the fully faithful embedding in Proposition 4.2 sends any map of algebras to an internal right adjoint in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$. Hence, we can pass to left adjoints, which by [AGH24, Thm. 5.3.8] induces an equivalence $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iL}} \simeq (\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iR}})^{\mathrm{op}}$, so passing to slice categories and using Lemma 3.22 we obtain

$$\mathrm{Alg}(\mathrm{End}^{\otimes,\mathrm{L}}_{\mathcal{V}}(\mathcal{M})) \simeq (\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{/^{\mathrm{monra}}\mathcal{M}})^{\mathrm{op}} \simeq (\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iR}}_{/^{\mathrm{monra}}\mathcal{M}})^{\mathrm{op}} \simeq \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})^{\mathrm{iL}}_{\mathcal{M}/^{\mathrm{monra}}} \simeq \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{\mathcal{M}/^{\mathrm{monra}}}$$

where $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{\mathcal{M}/^{\mathrm{monra}}}$ denotes the full subcategory on the monadic right adjoints, and similarly for left adjoints. $\square$

Remark 4.4. We expect a version of Corollary 4.3 to hold in any 2-category that admits Eilenberg-Moore objects and locally admits geometric realizations. By Proposition A.35, the latter implies that Eilenberg-Moore objects agree with Kleisli objects, and the respective structure 1-morphisms are adjoint. Hence, one should be able to deduce this by applying Proposition 4.2 in the opposite 2-category.

Notation 4.5. For $X \in \mathcal{C}at$, let $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{X/}$ denote the pullback $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) \times_{\widehat{\mathcal{C}at}} \widehat{\mathcal{C}at}_{X/}$. By the same argument as in [RZ25, Lem. 5.4] this is equivalent to $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{\mathcal{P}\,\mathrm{Sym}\,X \otimes \mathcal{V}/}$.

Corollary 4.6. The assignment $\mathcal{O} \mapsto (X \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}))$ induces a fully faithful embedding

$$v\mathcal{O}p_X(\mathcal{V}) = \mathrm{Alg}(\mathrm{End}^{\mathrm{L},\otimes}_{\mathcal{V}}((\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V}))) \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{X/}$$

whose image consists those functors $よ : X \to \mathcal{M}$ with $\mathcal{M} \in \mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}))$ such that:

- The unique extension $Y : \mathcal{P}\,\mathrm{Sym}(X) \otimes \mathcal{V} \to \mathcal{M}$ is an internally left adjoint 1-morphism in $\mathrm{CAlg}(\mathrm{RMod}_{\mathcal{V}}(\mathbb{P}\mathrm{r}))$,
- The image of $よ$ generates $\mathcal{M}$ under colimits, symmetric monoidal structure and $\mathcal{V}$-tensoring.

Proof. Immediate from Corollary 4.3 if we can show the a morphism $F : \mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V} \to \mathcal{M}$ in $\mathrm{CAlg}(\mathbb{P}\mathrm{r}_{\mathcal{V}})$ is a monadic left adjoint iff the associated functor $よ : X \to \mathcal{M}$ by Notation 4.5 satisfies the given conditions. Using the characterization Proposition 4.1 it suffices to thow that the second is equivalent to colimit-dominance of F. Evidently $\mathcal{P}(\mathrm{Sym}\,X) \otimes \mathcal{V}$ is generated under colimits, $\mathcal{V}$-tensoring and symmetric monoidal structure by the image of X, so this follows from the cancellation property of colimit-dominant morphisms. $\square$

Definition 4.7. We refer to objects $(X \to \mathcal{M}) \in \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})_{X/}$ in the image of the Eilenberg-Moore functor in **??** as *marked $\mathcal{V}$-algebras.*

Definition 4.8. The *category of valent $\mathcal{V}$-enriched operads* is defined as the pullback

$$v\mathcal{O}p(\mathcal{V}) := \mathcal{S} \times_{\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})} \mathrm{Arr}^{\mathrm{iL,cdom}}(\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}))$$

of the source projection along the free functor $\mathcal{P}\,\mathrm{Sym}(-) \otimes \mathcal{V} : \mathcal{S} \to \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$. We refer to its morphisms as *maps of $\mathcal{V}$-enriched operads.*

Denote by $col : v\mathcal{O}p(\mathcal{V}) \to \mathcal{S}$ the projection to the first component, and call $col(\mathcal{O})$ the *space of colors* of $\mathcal{O}$. Similarly, denote the target projection by $\mathcal{P}^{\otimes}_{\mathcal{V}} : v\mathcal{O}p(\mathcal{V}) \to \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ as it sends $\mathcal{O}$ to its operadic enriched presheaf category.

Proposition 4.9. The space-of-colors functor $col : v\mathcal{O}p(\mathcal{V}) \to \mathcal{S}$ is a Cartesian and co-Cartesian fibration, where a map of $\mathcal{V}$-operads $F : \mathcal{O} \to \mathcal{P}$ encoded by a commutative square

$$\begin{tikzcd}
\mathcal{P}\,\mathrm{Sym}(col\mathcal{O}) \otimes \mathcal{V} \arrow[r] \arrow[d] & \mathcal{P}\,\mathrm{Sym}(col\mathcal{P}) \otimes \mathcal{V} \arrow[d] \\
\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) \arrow[r] & \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})
\end{tikzcd}$$

- is a col-Cartesian morphism iff $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})$ is fully faithful,
- is a col-coCartesian morphism iff this is a pushout square.

The category $v\mathcal{O}p(\mathcal{V})$ of valent $\mathcal{V}$-operads admits limits and colimits, and col preserves them. Its fiber over a space X is equivalent to the category $v\mathcal{O}p_X(\mathcal{V}) := \mathrm{Alg}(\mathrm{sSeq}_X(\mathcal{V}))$ defined in Definition 2.1.

Proof. Completely analogous to [RZ25, Cor. 6.6], [RZ25, Obs. 6.8] and [RZ25, Cor. 6.9]. $\square$

Construction 4.10. Given a morphism $f : \mathcal{V} \to \mathcal{W}$ in $\mathrm{CAlg}(\mathrm{Pr})$, we define the *change-of-enrichment* functor

$$f_! : v\mathcal{O}p(\mathcal{V}) \simeq \mathcal{S}\times_{\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})}\mathrm{Arr}^{\mathrm{iL,cdom}}(\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})) \to \mathcal{S}\times_{\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})}\mathrm{Arr}^{\mathrm{iL,cdom}}(\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})) \simeq v\mathcal{O}p(\mathcal{W})$$

using the identity on $\mathcal{S}$, and the extension-of-scalars functor $f_{\mathrm{ext}} : \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}}) \to \mathrm{CAlg}(\mathrm{Pr}_{\mathcal{W}})$. Note that it preserves colimit-dominant functors by [RZ25, Cor. 3.18][4], and internal left adjoints as it is a 2-functor by Proposition A.31.

As a left adjoint 2-functor f_{ext} even preserves Eilenberg-Moore objects, so we learn that fiberwise it agrees with the construction of Construction 2.8. Further, one can show analogously to [RZ25, Cor. 6.19] that $f_!$ preserves colimits, so it admits a right adjoint $f_!^{\mathrm{R}}$.

[4]Alternatively, as a 2-functor that locally preserves geometric realizations it preserves monadic morphisms by Proposition A.35.

5. Comparison and Univalence

Our next task will be to investigate how the above notion of $\mathcal{S}$-enriched operads compares to Lurie's category $\mathcal{O}p$ of ∞-operads, defined as a subcategory of $\mathcal{C}at_{/\,\mathrm{Fin}_*}$ (for distinction, we will refer to them as Lurie-operads). We prove that $v\mathcal{O}p(\mathcal{S})$ is equivalent to the category of *flagged operads*, and we will introduce a notion of univalence to obtain an equivalence with $\mathcal{O}p$.

Reminder 5.1. The functor $\underline{(-)} : \mathcal{O}p \to \mathcal{C}at$, sending a Lurie-operad P to the ∞-category $\mathrm{P}^{\otimes}_{\underline{1}_+}$ obtained by discarding all k-ary multimorphisms with $k \neq 1$, admits a left adjoint $\mathrm{Triv}_{(-)} : \mathcal{C}at \to \mathcal{O}p$ sending a category to the corresponding trivial operad admitting no k-ary multimorphisms for $k \neq 1$, see [Lur17, Prop. 2.1.4.11].

Also, the inclusion $\mathrm{CAlg}(\mathcal{C}at) \hookrightarrow \mathcal{O}p$ on those operads $\mathrm{O}^{\otimes} \to \mathrm{Fin}_*$ that are coCartesian fibrations admits a left adjoint by [Lur17, Prop. 2.2.4.9], called the symmetric monoidal envelope $\mathrm{Env} : \mathcal{O}p \to \mathrm{CAlg}(\mathcal{C}at)$. Composition of left adjoints implies that $\mathrm{Sym} \simeq \mathrm{Env} \circ \mathrm{Triv}_{(-)}$.

Definition 5.2. A map of Lurie-operads $f : \mathrm{O} \to \mathrm{P}$ is called *surjective* if the underlying functor $f : \mathrm{O}_{\underline{1}_+} \to \mathrm{P}_{\underline{1}_+}$ on categories of colors is surjective. We denote by $\mathrm{Arr}^{\mathrm{surj}}(\mathcal{O}p) \subseteq \mathrm{Arr}(\mathcal{O}p)$ the full subcategory on surjective maps of Lurie-operads.

A map of Lurie-operads $f : \mathrm{O} \to \mathrm{P}$ is called *fully faithful* if it induces isomorphisms on multimorphism spaces, i.e. for any $n \in \mathbb{N}$ and $o_1, \ldots, o_n, o \in \mathrm{O}$ the induced map $\mathrm{Mul}_{\mathrm{O}}(o_1, \ldots, o_n; o) \to \mathrm{Mul}_{\mathrm{P}}(f(o_1), \ldots, f(o_n); f(o))$ is an isomorphism.

Observation 5.3. The classes of surjective and fully faithful maps of operads form a factorization system on $\mathcal{O}p$.

Lemma 5.4. If $f : \mathrm{O} \to \mathrm{P}$ in $\mathcal{O}p$ is surjective, then the induced symmetric monoidal functor $\mathrm{Env}(f) : \mathrm{Env}(\mathrm{O}) \to \mathrm{Env}(\mathrm{P})$ is surjective, and $\mathcal{P}\,\mathrm{Env}(f) \otimes \mathcal{V} : \mathcal{P}\,\mathrm{Env}(\mathrm{O}) \otimes \mathcal{V} \to \mathcal{P}\,\mathrm{Env}(\mathrm{P}) \otimes \mathcal{V}$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is colimit-dominant.

Proof. The first claim is clear, since the image of $\mathrm{O} \hookrightarrow \mathrm{Env}\,\mathrm{O}$ generates $\mathrm{Env}\,\mathrm{O}$ under the symmetric monoidal structure, which is preserved by $\mathrm{Env}(f)$. Then $\mathcal{P}\,\mathrm{Env}(f)$ is colimit-dominant since it hits all representables, and so is $\mathcal{P}\,\mathrm{Env}(f) \otimes \mathcal{V}$ by [RZ25, Lem. 3.9]. $\square$

Definition 5.5. A *flagged operad* consists of a Lurie-operad O, a space X and a surjective map $X \to \mathrm{O}_{\underline{1}_+}$ into the colors of O. We will also shortly denote it as $X \to \mathrm{O}$.

The category $\mathcal{F}\mathcal{O}p$ of flagged operads is the full subcategory of the pullback $\mathrm{Arr}(\mathcal{C}at) \times_{\mathcal{C}at} \mathcal{O}p$, along the target projection and $\underline{(-)}$, on those pairs $(X \to \underline{\mathrm{O}}, \mathrm{O})$ where X is a space and the functor is surjective. The adjunction $\mathcal{C}at \leftrightarrows \mathcal{O}p$ exhibits this as equivalent to the pullback $\mathcal{S} \times_{\mathcal{O}p} \mathrm{Arr}^{\mathrm{surj}}(\mathcal{O}p)$, using [RZ25, Lem. A.7].

Construction 5.6. The *associated marked $\mathcal{S}$-algebra* of a flagged operad $X \to \mathrm{O}$ is the marked algebra $X \to \mathrm{O} \to \mathcal{P}(\mathrm{Env}\,\mathrm{O})$. Note that this is a $\otimes$-atomic marking by Corollary 3.44, and generates under colimits by Lemma 5.4. This construction assembles into a functor

$\mathcal{F}\mathcal{O}p \to v\mathcal{O}p(\mathcal{S})$ via the following map of pullback squares:

$$\begin{tikzcd}
 & v\mathcal{O}p(\mathcal{S}) \arrow[rr] \arrow[dd] & & \mathrm{Arr}^{\mathrm{iL,cdom}}(\mathrm{CAlg}(\mathrm{Pr})) \arrow[dd, "\mathrm{src}"] \\
\mathcal{F}\mathcal{O}p \arrow[ur, dashed] \arrow[rr] \arrow[dd] & & \mathrm{Arr}^{\mathrm{surj}}(\mathcal{O}p) \arrow[ur, "\mathcal{P}\,\mathrm{Env}"'] \arrow[dd, "\mathrm{src}"] & \\
 & \mathcal{S} \arrow[rr] & & \mathrm{CAlg}(\mathrm{Pr}) \\
\mathcal{S} \arrow[ur, equal] \arrow[rr] & & \mathcal{O}p \arrow[ur, "\mathcal{P}\,\mathrm{Env}"'] &
\end{tikzcd}$$

Definition 5.7. The *underlying flagged operad* of a marked $\mathcal{V}$-algebra $\left(col\mathcal{O} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})\right) \in v\mathcal{O}p(\mathcal{V})$ is the surjective functor $col\mathcal{O} \to \mathrm{Im}\left(col\mathcal{O} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})\right) \in \mathcal{F}\mathcal{O}p$, where we view the full image as an operad by virtue of it being a full subcategory of the symmetric monoidal category $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$.

Proposition 5.8. Let $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr})$ and let $\iota : \mathcal{S} \to \mathcal{V}$ in $\mathrm{Alg}(\mathrm{Pr})$ denote its unit. The composite

$$\mathcal{F}\mathcal{O}p \to v\mathcal{O}p(\mathcal{S}) \xrightarrow{\iota_!} v\mathcal{O}p(\mathcal{V})$$

admits a right adjoint, which sends a valent $\mathcal{V}$-enriched operad $\mathcal{O}$ to its underlying flagged operad $col\mathcal{O} \to \mathrm{Im}\left(\text{よ}_{\mathcal{O}} : col\mathcal{O} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})\right)$.

Proof. Recall from Construction 4.10 that the change-of-enrichment functor $\iota_! : v\mathcal{O}p(\mathcal{S}) \to v\mathcal{O}p(\mathcal{V})$ sends a marked $\mathcal{V}$-algebra $X \to \mathcal{M}$ to $X \to \mathcal{M} \otimes_{\mathcal{S}} \mathcal{V} \simeq \mathcal{M} \otimes \mathcal{V}$. For $X \to \mathrm{O}$ a flagged operad and $\text{よ} : Y \to \mathcal{M}$ a marked $\mathcal{V}$-algebra, consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{F}\mathcal{O}p}\begin{pmatrix} X & Y \\ \downarrow, & \downarrow \\ \mathrm{O} & \mathrm{Im}(\text{よ}) \end{pmatrix} \arrow[r] \arrow[d] & \mathrm{Map}_{\widehat{\mathcal{O}p}}(\mathrm{O}, \mathrm{Im}(\text{よ})) \arrow[r] \arrow[d] & \mathrm{Map}_{\widehat{\mathcal{O}p}}(\mathrm{O}, \mathcal{M}) \arrow[d] \\
\mathrm{Map}_{\mathcal{S}}(X, Y) \arrow[r] & \mathrm{Map}_{\widehat{\mathcal{O}p}}(X, \mathrm{Im}(\text{よ})) \arrow[r] & \mathrm{Map}_{\widehat{\mathcal{O}p}}(X, \mathcal{M})
\end{tikzcd}$$

The left square is a pullback by definition of the mapping spaces of $\mathcal{F}\mathcal{C}at$. Orthogonality of surjective and fully faithful maps of operads shows that the right square is a pullback, since $X \to \mathrm{O}$ is surjective and $\mathrm{Im}(\text{よ}) \to \mathcal{M}$ fully faithful. Thus, the total square is a pullback and

$$\begin{aligned}
\mathrm{Map}_{\mathcal{F}\mathcal{O}p}\begin{pmatrix} X & Y \\ \downarrow, & \downarrow \\ \mathrm{O} & \mathrm{Im}(\text{よ}) \end{pmatrix} &\simeq \mathrm{Map}_{\mathcal{S}}(X, Y) \times_{\mathrm{Map}_{\widehat{\mathcal{O}p}}(X, \mathcal{M})} \mathrm{Map}_{\widehat{\mathcal{O}p}}(\mathrm{O}, \mathcal{M}) \simeq \\
&\simeq \mathrm{Map}_{\mathcal{S}}(X, Y) \times_{\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})}(\mathcal{P}\,\mathrm{Sym}\, X \otimes \mathcal{V}, \mathcal{M})} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})}(\mathcal{P}\,\mathrm{Env}\,\mathrm{O} \otimes \mathcal{V}, \mathcal{M})
\end{aligned}$$

which by definition agrees with the mapping space $\mathrm{Map}_{v\mathcal{O}p(\mathcal{V})}\begin{pmatrix} X & Y \\ \downarrow & , \downarrow \\ \mathcal{P}\,\mathrm{Env}\,\mathrm{O} \otimes \mathcal{V} & \mathcal{M} \end{pmatrix}$. ☐

Inspecting the unit and counit, we conclude that for $\mathcal{V} = \mathcal{S}$, this adjunction is in fact an equivalence:

Theorem 5.9. The associated-marked-$\mathcal{S}$-algebra functor $\mathcal{F}\mathcal{O}p \to v\mathcal{O}p(\mathcal{S})$ from Construction 5.6 is an equivalence.

Proof. Let $(X \to \mathrm{O}) \in \mathcal{F}\mathcal{O}p$; our goal is to prove that both unit and counit of the adjunction from Proposition 5.8 are isomorphisms. The unit is the map of flagged operads $(X \to \mathrm{O}) \to (X \to \mathrm{Im}(よ : \mathrm{O} \to \mathcal{P}\,\mathrm{Env}\,\mathrm{O}))$. Since $\mathrm{O} \to \mathrm{Env}(\mathrm{O}) \to \mathcal{P}\,\mathrm{Env}(\mathrm{O})$ is fully faithful, this is an isomorphism.

Conversely, for $(よ : Y \to \mathcal{M}) \in v\mathcal{O}p(\mathcal{S})$, the counit is the map of marked $\mathcal{S}$-algebras $(Y \to \mathcal{P}\,\mathrm{Env}(\mathrm{Im}(よ))) \to (Y \to \mathcal{M})$. Since $\mathrm{Im}(よ) \subseteq \mathcal{M}$ is a full suboperad, their multimorphism spaces agree, meaning the full inclusion $\mathrm{Env}(\mathrm{Im}(よ)) \to \mathcal{M}$ is fully faithful. We deduce from the fully-faithfulness criterion in Lemma 3.27, which is applicable since $\mathrm{Im}(よ) \subseteq \mathcal{P}\,\mathrm{Env}(\mathrm{Im}(よ))$ consists of $\otimes$-atomics by Corollary 3.44, that the functor $\mathcal{P}\,\mathrm{Env}(\mathrm{Im}(よ)) \to \mathcal{M}$ is fully faithful. Also its image contains all representable presheaves, i.e. a system of $\otimes$-atomic generators, so since it is closed under tensor products, $\mathcal{V}$-tensoring and colimits it must agree with $\mathcal{M}$. ☐

Definition 5.10. A $\mathcal{V}$-operad $\mathcal{O} \in v\mathcal{O}p(\mathcal{V})$, with associated marked $\mathcal{V}$-algebra $(よ : col\mathcal{O} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}))$, is called *univalent* if the functor $よ$ is a subcategory inclusion. In other words, it exhibits $col\mathcal{O} \simeq \mathrm{Im}(よ)^{\simeq}$. Denote by $\mathcal{O}p(\mathcal{V}) \subseteq v\mathcal{O}p(\mathcal{V})$ the full subcategory on univalent flagged categories.

Definition 5.11. The target projection $\mathrm{trg} : \mathcal{F}\mathcal{O}p \to \mathcal{O}p$ admits a fully faithful right adjoint $\mathrm{O} \mapsto (\underline{\mathrm{O}}^{\simeq} \to \mathrm{O})$. We call a flagged operad $(X \to \mathrm{O})$ *univalent* it lies in the image of this right adjoint, i.e. the induced map $X \to \underline{\mathcal{O}}^{\simeq}$ is an isomorphism, or equivalently $X \to \underline{\mathrm{O}}$ is a subcategory inclusion.

Corollary 5.12. The equivalence from Theorem 5.9 restricts to an equivalence $\mathcal{O}p(\mathcal{S}) \simeq \mathcal{O}p$, where we embed $\mathcal{O}p \subseteq \mathcal{F}\mathcal{O}p$ as in Definition 5.11.

Proof. Clearly if a marked algebra $(よ : X \to \mathcal{M})$ is univalent, then so is its associated flagged operad $(X \to \mathrm{Im}(よ))$. Conversely, given a flagged operad $(X \to \mathrm{O})$ such that $X \to \underline{\mathrm{O}}$ is a subcategory inclusion, then so is the composition $X \to \underline{\mathrm{O}} \subseteq \mathrm{Env}(\mathrm{O}) \subseteq \mathcal{P}\,\mathrm{Env}(\mathrm{O})$. ☐

Definition 5.13. To any $\mathcal{V}$-operad $\mathcal{O} \in v\mathcal{O}p(\mathcal{V})$ with associated marked algebra $col\mathcal{O} \hookrightarrow \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$, we associate a $\mathcal{V}$-operad $u\mathcal{O}$ described by the marked algebra $\mathrm{Im}(よ)^{\simeq} \hookrightarrow \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$, which we call the *univalization* of $\mathcal{O}$. Note that this is indeed a $\otimes$-atomically generating marking by Corollary 3.24.

Proposition 5.14. For a map of valent $\mathcal{V}$-operads $F : \mathcal{O} \to \mathcal{P}$, the following are equivalent:

(1) The underlying morphism $\mathcal{P}^{\otimes}_{\mathcal{V}}(F) : \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O}) \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})$ in $\mathrm{CAlg}(\mathrm{Pr}_{\mathcal{V}})$ is fully faithful,

(1) For any $o_1, \dots, o_n, o \in col\mathcal{O}$, the induced map on multigraphs

$$\mathrm{Mul}_{\mathcal{O}}(o_1, \dots, o_n; o) \simeq \underline{\mathrm{Hom}}_{\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})}(よ(o_1) \otimes \cdots \otimes よ(o_n), よ(o)) \to$$
$$\to \underline{\mathrm{Hom}}_{\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})}(よ(Fo_1) \otimes \cdots \otimes よ(Fo_n), よ(Fo)) \simeq \mathrm{Mul}_{\mathcal{O}}(Fo_1, \dots, Fo_n; Fo)$$

is an isomorphism in $\mathcal{V}$,

(1) F is col-Cartesian for the Cartesian fibration $col : v\mathcal{O}p(\mathcal{V}) \to \mathcal{S}$.

Proof. (1) $\Leftrightarrow$ (2) follows form Lemma 3.27 by definition of marked algebras. Further, (1) $\Leftrightarrow$ (3) is part of Proposition 4.9. ☐

Definition 5.15. A map of valent $\mathcal{V}$-operads $F : \mathcal{O} \to \mathcal{P}$ is called *fully faithful* if either of the equivalent conditions of Proposition 5.14 is satisfied.

Definition 5.16. A map of valent $\mathcal{V}$-operads $F : \mathcal{O} \to \mathcal{P}$ is called *surjective-on-colors* if the underlying map $col\mathcal{O} \to col\mathcal{P}$ in $\mathcal{S}$ is surjective on connected components.

On the other hand, $F : \mathcal{O} \to \mathcal{P}$ is called *surjective* if the induced map $\mathrm{Im}(\text{よ}_{\mathcal{O}}) \to \mathrm{Im}(\text{よ}_{\mathcal{P}})$ in $\mathcal{C}at$ is surjective, i.e. if the induced map on univalizations $u\mathcal{O} \to u\mathcal{P}$ is surjective-on-colors.

Theorem 5.17. The full inclusion $\mathcal{O}p(\mathcal{V}) \subseteq v\mathcal{O}p(\mathcal{V})$ admits a left adjoint u, called *univalization*, that sends a marked algebra $\text{よ} : col\mathcal{O} \to \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$ to the subcategory inclusion $\mathrm{Im}(\text{よ})^{\simeq} \hookrightarrow \mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{O})$. This left adjoint exhibits $\mathcal{O}p(\mathcal{V})$ as the localization of $v\mathcal{O}p(\mathcal{V})$ at the single morphism $(* \sqcup * \to \mathcal{P}\,\mathrm{Sym}(*)) \to (* \to \mathcal{P}\,\mathrm{Sym}(*))$. It inverts precisely those morphisms in $v\mathcal{O}p(\mathcal{V})$ that are both fully faithful and surjective.

Proof. The first claim is analogous to [RZ25, Thm. 7.27]. Further, note that for any marked algebra $X \to \mathcal{M}$, being local with respect to the above morphism is equivalent to any diagram

$$\begin{tikzcd} * \sqcup * \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] \arrow[ur, dashed] & \mathcal{M} \end{tikzcd}$$

admitting a unique filler, which is equivalent to the right map being a monomorphism in $\mathcal{C}at$, i.e. a subcategory inclusion.

A fully faithful and surjective morphism $\mathcal{O} \to \mathcal{P}$ induces an isomorphism on univalizations since by definition the induced map $\mathrm{Im}(\text{よ}_{\mathcal{O}}) \to \mathrm{Im}(\text{よ}_{\mathcal{P}})$ is an equivalence, which also means that a system of $\otimes$-atomic generators in $\mathcal{P}^{\otimes}_{\mathcal{V}}(\mathcal{P})$ is hit. Conversely if $uF : u\mathcal{O} \to u\mathcal{P}$ is an isomorphism, then it is an equivalence on operadic presheaf categories so F must have been fully faithful. Also the map $col(u\mathcal{O}) \simeq \mathrm{Im}(\text{よ}_{\mathcal{O}}) \to \mathrm{Im}(\text{よ}_{\mathcal{P}}) \simeq col(u\mathcal{P})$ is an equivalence implying that F is surjective. $\square$

Remark 5.18. We deduce that $\mathcal{O}p(\mathcal{V})$ is also obtained as the localization at the intermediate class of fully faithful and surjective-on-objects morphisms.

Remark 5.19. A straightforward proof shows that the univalization functor induces an equivalence between $v\mathcal{O}p(\mathcal{V})$ and the category of *flagged $\mathcal{V}$-operads*

$$\mathcal{S} \times_{\mathcal{O}p(\mathcal{V})} \mathrm{Arr}^{\mathrm{surj}}(\mathcal{O}p(\mathcal{V})) \simeq \mathrm{Arr}^{\mathrm{surj}}(\mathcal{S}) \times_{\mathcal{S}} \mathcal{O}p(\mathcal{V})$$

whose objects are pairs of a univalent $\mathcal{V}$-operad $\mathcal{O}$ and a surjective map $X \to col\mathcal{O}$.

References

[AGH24] Fernando Abellán, Andrea Gagna, and Rune Haugseng. Straightening for lax transformations and adjunctions of $(\infty, 2)$-categories. *arXiv preprint arXiv:2404.03971*, 2024.

[BD98] John C Baez and James Dolan. Higher-dimensional algebra III. n-categories and the algebra of opetopes. *Advances in Mathematics*, 135(2):145–206, 1998.

[BH] Lukas Brantner and Gijs Heuts. Symmetric sequences, ∞-operads, and the universal Lie algebra. *in preparation.*

[BM24] Shay Ben-Moshe. Naturality of the ∞-categorical enriched yoneda embedding. *Journal of Pure and Applied Algebra*, 228(6):107625, 2024.

[BMT] Thomas Blom, Connor Malin, and Niall Taggart. Envelopes of spectral operads and Koszul duality for right modules. *in preparation.*

[Bra17] David Lukas Benjamin Brantner. *The Lubin-Tate theory of spectral Lie algebras.* Harvard University, 2017.

[BS22] Shaul Barkan and Jan Steinebrunner. The equifibered approach to ∞-properads. *arXiv preprint arXiv:2211.02576*, 2022.

[BV06] John Michael Boardman and Rainer M Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347. Springer, 2006.

[Chi12] Michael Ching. Bar–cobar duality for operads in stable homotopy theory. *Journal of Topology*, 5(1):39–80, 2012.

[Hau20] Rune Haugseng. On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories. *arXiv preprint arXiv:2002.01037*, 2020.

[Hau22] Rune Haugseng. ∞-operads via symmetric sequences. *Mathematische Zeitschrift*, 301(1):115–171, 2022.

[Hei17] Hadrian Heine. A monadicity theorem for higher algebraic structures. *arXiv preprint arXiv:1712.00555*, 2017.

[Hei24a] Hadrian Heine. A derived milnor-moore theorem. *arXiv preprint arXiv:2408.06917*, 2024.

[Hei24b] Hadrian Heine. The higher algebra of weighted colimits. *arXiv preprint arXiv:2406.08925*, 2024.

[Heu24] Gijs Heuts. Koszul duality and a conjecture of Francis-Gaitsgory. *arXiv preprint arXiv:2408.06173*, 2024.

[HK24] Rune Haugseng and Joachim Kock. ∞-operads as symmetric monoidal ∞-categories. *Publicacions matemàtiques*, 68(1):111–137, 2024.

[KM01] Mikhail Kapranov and Yu Manin. Modules and Morita theorem for operads. *American Journal of Mathematics*, 123(5):811–838, 2001.

[Lur17] Jacob Lurie. Higher algebra. 2017.

[Lur18a] Jacob Lurie. Kerodon. `https://kerodon.net`, 2018.

[Lur18b] Jacob Lurie. Spectral algebraic geometry. *preprint*, 2018.

[May06] J. Peter May. *The geometry of iterated loop spaces*, volume 271. Springer, 2006.

[Ram24] Maxime Ramzi. Dualizable presentable ∞-categories. *arXiv preprint arXiv:2410.21537*, 2024.

[RZ25] David Reutter and Markus Zetto. Enriched ∞-categories as marked module categories. *arXiv preprint arXiv:2501.07697*, 2025.

[RZ26] David Reutter and Markus Zetto. Cauchy-complete (∞, n)-categories and higher idempotents. in preparation, 2026.

[Sto25] Devon Stockall. Large condensation in enriched ∞-categories. *arXiv preprint arXiv:2506.23632*, 2025.

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