

CAUCHY-COMPLETE ∞ -CATEGORIES AND LAX ADDITIVITY

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ABSTRACT. Additive categories can be characterized as those Ab-enriched categories that admit finite coproducts, which automatically coincide with the respective products. This is a particular instance of a paradigm that makes sense for any enrichment category \mathcal{V} : A (weighted) colimit is called absolute if it can be described by a dual limit diagram, and following Lawvere a \mathcal{V} -enriched category is called Cauchy-complete if it admits all absolute colimits. Generalizing to enriched ∞ -categories, I explain how a category enriched over cocomplete ∞ -categories is Cauchy-complete iff it is idempotent complete and admits lax colimits, i.e. it is a lax semiadditive $(\infty, 2)$ -category. Up to idempotent completion, this lets me recover Angus' previous statement about the category of profunctors being the free lax semiadditive $(\infty, 2)$ -category on a point, generalize it to enriched profunctors, and explain its relation to multifusion categories.

1. NOTES ON PROFUNCTORS

1.1. Cocomplete categories. Denote by $\widehat{\mathcal{C}at}^{\text{colim}}$ the very large (locally large) category of large categories admitting small colimits, and functors preserving small colimits. We will also refer to them as *cocomplete categories* and *cocontinuous functors*. A notable full subcategory is the large category Pr^{L} spanned by the presentable categories.

Proposition 1.1. Given $\mathcal{V} \in \text{Alg}(\text{Pr}^{\text{L}})$, a module $\mathcal{M} \in \text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\text{colim}})$ is \mathcal{U} -compact iff it is presentable, i.e. lies in the full subcategory $\text{RMod}_{\mathcal{V}}(\text{Pr}^{\text{L}}) =: \text{Pr}_{\mathcal{V}}$.

Proof. For $\mathcal{V} = \mathcal{S}$, a category $\mathcal{M} \in \widehat{\mathcal{C}at}^{\text{colim}}$ is \mathcal{U} -compact iff it is presentable by [?, Proposition 5.1.4]. Further by [?, Proposition 5.1.7], $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\text{colim}})$ is \mathcal{U} -compactly generated by the free modules $\mathcal{P} \otimes \mathcal{V}$ for $\mathcal{P} \in \text{Pr}^{\text{L}}$ (even for $\mathcal{V} \in \text{Alg}(\widehat{\mathcal{C}at}^{\text{colim}})$). In particular, this implies that its \mathcal{U} -compact objects are precisely the small colimits of such $\mathcal{P} \otimes \mathcal{V}$ by a $\text{id}_{\mathcal{M}} \in \text{Map}(\mathcal{M}, \mathcal{M}) = \text{colim}_i \text{Map}(\mathcal{M}, \mathcal{P}_i \otimes \mathcal{V})$ retract argument. Now $\text{RMod}_{\mathcal{V}}(\text{Pr}^{\text{L}})$ contains all of these free modules (as \mathcal{V} is presentable), in fact it is generated by them under geometric realizations, and it is closed under small colimits so we are finished. \square

elaborate?

Remark 1.2. In particular, any cocomplete \mathcal{M} can be written as a large, \mathcal{U} -filtered colimit of presentable categories in $\widehat{\mathcal{C}at}^{\text{colim}}$. For example $\text{Pr}^{\text{L}} = \text{colim}_{\kappa \text{ regular cardinal}} \text{Pr}_{\kappa}$ is the colimit over all regular cardinals of the categories Pr_{κ} of κ -compactly generated categories and cocontinuous functors preserving κ -compact objects. Since $\text{Pr}^{\text{L}} \subseteq \widehat{\mathcal{C}at}^{\text{colim}}$ is dense, there is even a canonical such colimit diagram indexed by $\text{Pr}_{\mathcal{M}}^{\text{L}}$ for each \mathcal{M} .

Recall that the tensor product \otimes of cocomplete categories induces a symmetric monoidal structure on $\widehat{\mathcal{C}at}^{\text{colim}}$ that preserves large colimits in both variables separately, and restricts to a symmetric monoidal structure on Pr^{L} .

Date: May 29, 2024, Hamburg.

Proposition 1.3. The category of spectra $\mathcal{S}p$ is an idempotent algebra in the symmetric monoidal category $\widehat{\mathcal{C}at}^{\text{colim}}$ of cocomplete categories (equipped with the tensor product of cocomplete categories). Its category of modules $\text{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\text{colim}}) \subseteq \widehat{\mathcal{C}at}^{\text{colim}}$ consists precisely of the cocomplete stable categories.

Proof. In the case where $\mathcal{M} \in \text{Pr}^{\text{L}}$, we know from [Lur17, Example 4.8.1.23] that $\mathcal{S}p \otimes \mathcal{M} \simeq \mathcal{S}p(\mathcal{M})$ agrees with the stabilization of \mathcal{M} , in particular $\mathcal{S}p \otimes \mathcal{M} \simeq \mathcal{M}$ iff \mathcal{M} is stable. As $\mathcal{S}p$ is an idempotent algebra in Pr^{L} , this is once again equivalent to \mathcal{M} being a module over $\mathcal{S}p$. Now since $\text{Pr}^{\text{L}} \subseteq \widehat{\mathcal{C}at}^{\text{colim}}$ is a monoidal subcategory, $\mathcal{S}p$ is once again an idempotent algebra in $\widehat{\mathcal{C}at}^{\text{colim}}$, so it suffices to show that $\mathcal{S}p \otimes \mathcal{M}$ agrees with the stabilization $\lim_{\mathbb{N}}(\cdots \rightarrow \mathcal{M} \xrightarrow{\Omega} \mathcal{M})$ even if $\mathcal{M} \in \widehat{\mathcal{C}at}^{\text{colim}}$. Once again we expand $\mathcal{M} \simeq \text{colim}_i \mathcal{M}_i$ as a \mathcal{U} -filtered colimit with $\mathcal{M}_i \in \text{Pr}^{\text{L}}$, then

$$\mathcal{S}p \otimes \mathcal{M} \simeq \text{colim}_i \mathcal{S}p \otimes \mathcal{M}_i \simeq \text{colim}_i \lim_{\mathbb{N}}(\cdots \rightarrow \mathcal{M}_i \xrightarrow{\Omega} \mathcal{M}_i) \simeq \lim_{\mathbb{N}}(\cdots \rightarrow \mathcal{M} \xrightarrow{\Omega} \mathcal{M})$$

since \mathcal{U} -filtered colimit commute with small limits, in particular limits over \mathbb{N} and loop functors. \square

Proposition 1.4. Let C be a small, and \mathcal{M} a cocomplete category. Then $\mathcal{P}(C) \otimes \mathcal{M} \simeq \text{Fun}(C^{\text{op}}, \mathcal{M})$.

Proof. The statement is true if \mathcal{M} is presentable, since then $\mathcal{P}(C) \otimes \mathcal{M} \simeq \text{Fun}^{\text{lim}}(\mathcal{P}(C)^{\text{op}}, \mathcal{M}) \simeq \text{Fun}^{\text{L}}(\mathcal{P}(C), \mathcal{M}^{\text{op}}) \simeq \text{Fun}(C, \mathcal{M}^{\text{op}})$. Using Proposition 1.1, let us write \mathcal{M} as an \mathcal{U} -filtered (large) colimit $\mathcal{M} \simeq \text{colim}_i \mathcal{M}_i$ with $\mathcal{M}_i \in \text{Pr}^{\text{L}}$ and fix a small category D . Then using that \otimes preserves large colimits in both arguments separately,

$$\begin{aligned} \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D), \mathcal{P}(C) \otimes \mathcal{M}) &\simeq \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D), \text{colim}_i \mathcal{P}(C) \otimes \mathcal{M}_i) \simeq \text{colim}_i \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D), \text{Fun}(C^{\text{op}}, \mathcal{M}_i)) \simeq \\ &\simeq \text{colim}_i \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D) \otimes \mathcal{P}(C^{\text{op}}), \mathcal{M}_i) \simeq \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D \times C^{\text{op}}), \text{colim}_i \mathcal{M}_i) \simeq \text{Map}_{\text{Pr}^{\text{L}}}(\mathcal{P}(D), \text{Fun}(C^{\text{op}}, \mathcal{M})) \end{aligned}$$

making use of the fact that $\mathcal{P}(D), \mathcal{P}(D \times C^{\text{op}})$ are \mathcal{U} -compact and thus commute with \mathcal{U} -filtered colimits. Now, note that the functors $\text{Map}_{\widehat{\mathcal{C}at}^{\text{colim}}}(\mathcal{P}(D), -)$ for all $D \in \mathcal{C}at$ are jointly conservative since any presentable category is a small colimit of presheaf categories, and any cocomplete category is a \mathcal{U} -filtered colimit of presentable categories. \square

Proposition 1.5. Let $\mathcal{O}^{\otimes} \rightarrow \text{Fin}_*$ be a small operad and \mathcal{M} a cocomplete category. Then, the categories of monoid objects satisfy $\text{Mon}_{\mathcal{O}}(\mathcal{S}) \otimes \mathcal{M} \simeq \text{Mon}_{\mathcal{O}}(\mathcal{M})$, where $\text{Mon}_{\mathcal{O}}(\mathcal{M}) \subseteq \text{Fun}(\mathcal{O}^{\otimes}, \mathcal{M})$ denotes the full subcategory on those functors $M : \mathcal{O}^{\otimes} \rightarrow \mathcal{M}$ exhibiting $M(o_1, \dots, o_n) \simeq M(o_1) \times \cdots \times M(o_n)$, for any collection of colors o_n .

Proof. Note that $\text{Mon}_{\mathcal{O}}(\mathcal{S}) \simeq \mathcal{P}(\mathcal{O}^{\otimes, \text{op}})$, so we can use Proposition 1.4 and show that the respective subcategories agree. This is evident if \mathcal{M} is presentable by the explicit formula for \otimes , so going through the above calculation it suffices to show that $\text{Mon}_{\mathcal{O}}(-) : \widehat{\mathcal{C}at}^{\text{colim}} \rightarrow \widehat{\mathcal{C}at}^{\text{colim}}$ commutes with \mathcal{U} -filtered colimits. \square

Proposition 1.6. The category of spectra $\mathcal{S}p^{\text{cn}}$ is an idempotent algebra in the symmetric monoidal category $\widehat{\mathcal{C}at}^{\text{colim}}$ of cocomplete categories. Its category of modules $\text{Mod}_{\mathcal{S}p^{\text{cn}}}(\widehat{\mathcal{C}at}^{\text{colim}}) \subseteq \widehat{\mathcal{C}at}^{\text{colim}}$ consists precisely of the cocomplete additive categories.

Proof. \square

what is this functor? and why is this true?

confusion

1.2. Lax \mathcal{V} -additive categories.

Reminder 1.7. content...

Lemma 1.8. Let $\mathcal{V} \in \text{Alg}_{\mathbb{E}_2}(\text{Pr}^{\text{L}})$, fix an \mathbb{E}_2 -algebra $a \in \text{Alg}_{\mathbb{E}_2}(\text{Pr}^{\text{L}})$, and denote by $F : \mathcal{V} \rightarrow \text{LMod}_a(\mathcal{V}) : U$ the free-forgetful adjunction. We can regard any $\text{LMod}_a(\mathcal{V})$ -tensored category $\mathcal{M} \in \text{Pr}_{\text{LMod}_a(\mathcal{V})}$ as a \mathcal{V} -tensored category $\mathcal{M}_{\mathcal{V}}$ with the same underlying category, by restricting scalars along F . An object $m \in \mathcal{M}$ is tiny with respect to the $\text{LMod}_a(\mathcal{V})$ -tensoring iff it is tiny with respect to the \mathcal{V} -tensoring on $\mathcal{M}_{\mathcal{V}}$.

Proof. By definition, $m \otimes - := m \otimes F(-) : \mathcal{V} \rightarrow \mathcal{M}$ in $\mathcal{M}_{\mathcal{V}}$, so passing to adjoints

$$\underline{\text{Hom}}_{\mathcal{M}_{\mathcal{V}}}(m, -) \simeq U \circ \underline{\text{Hom}}_{\mathcal{M}}(m, -).$$

Since U is conservative, and preserves colimits by [Lur17, arg1 arg2] as \mathcal{V} is presentably monoidal, it creates colimits. Similarly it creates \mathcal{V} -tensorings since it preserves them, so we are finished. \square

Definition 1.9. For $\mathcal{V} \in \text{Alg}_{\mathbb{E}_2}(\widehat{\text{Cat}}^{\text{colim}})$, we define the category of *lax \mathcal{V} -additive $(\infty, 2)$ -categories* as $\text{CauchyCat}(\text{RMod}_{\mathcal{V}}(\widehat{\text{Cat}}^{\text{colim}}))$. Explicitly by Lemma 1.8, it consists of $(\infty, 2)$ -categories that are locally cocomplete, locally tensored over \mathcal{V} in a way that is compatible with composition and local colimits, and admits lax colimits and idempotent splittings.

Example 1.10. Let us note several cases of interest:

- A lax \mathcal{S} -additive $(\infty, 2)$ -category is a locally cocomplete $(\infty, 2)$ -category with lax colimits and idempotent splittings, also known as an i.c. *lax semiadditive $(\infty, 2)$ -category*.
- A lax Set-additive $(\infty, 2)$ -category is an i.c. locally cocomplete $(2, 2)$ -category with lax colimits, so we call it an i.c. *lax semiadditive $(2, 2)$ -category*.
- A lax $\mathcal{S}p$ -additive $(\infty, 2)$ -category is an i.c. lax semiadditive $(\infty, 2)$ -category that is locally tensored over $\mathcal{S}p$, which by Proposition 1.3 means that it is locally stable. Hence, we recover *lax additive $(\infty, 2)$ -categories*.
- A lax Ab-additive $(\infty, 2)$ -category using Proposition 1.6 is a lax semiadditive $(2, 2)$ -category that is locally additive.
- Similarly for $\mathcal{S}_{\leq m}, \mathcal{S}_{\leq m, *}, \mathcal{S}p_{\leq m}$ we obtain locally semiadditive $(m + 2, 2)$ -categories (that are locally pointed/ additive).
- One should consider lax $\text{Pr}_{\text{st}}^{\text{L}}$ -additive $(\infty, 2)$ -categories as *2-lax additive $(\infty, 3)$ -categories*. This is because they are enriched over $\text{Mod}_{\text{Pr}_{\text{st}}^{\text{L}}}(\widehat{\text{Cat}}^{\text{colim}})$ which is the $\text{Ind}_{\mathcal{U}}$ -completion of the category of presentable stable 2-categories introduced in [?] (just like $\text{Mod}_{\mathcal{S}p}(\widehat{\text{Cat}}^{\text{colim}})$ is the $\text{Ind}_{\mathcal{U}}$ -completion of $\text{Pr}_{\text{st}}^{\text{L}}$ in the lax additive case).

Observation 1.11. Any lax \mathcal{V} -additive $(\infty, 2)$ -category is automatically 2-idempotent complete (which is a priori a stronger condition). This is because any cocomplete category is idempotent complete, so the forgetful functor $\text{LMod}_{\mathcal{V}}(\widehat{\text{Cat}}^{\text{colim}}) \rightarrow \widehat{\text{Cat}}^{\text{colim}} \rightarrow \widehat{\text{Cat}}$ factors through $\widehat{\text{Cat}}^{\text{idem}}$. Since all of these functors are right adjoints of monoidal functors, change-of-enrichment along them preserves Cauchy-completeness, in particular if $\mathbb{C} \in \text{CauchyCat}(\text{LMod}_{\mathcal{V}}(\widehat{\text{Cat}}^{\text{colim}}))$ then the underlying $\widehat{\text{Cat}}^{\text{idem}}$ -enriched category is Cauchy-complete, i.e. 2-idempotent complete.

1.3. Universal Property of Profunctors.

Definition 1.12. Denote by $\mathbb{P}\text{rof}_{\mathcal{V}}$ the full sub-2-category of $\mathbb{P}\text{r}_{\mathcal{V}}$ spanned by the tiny-generated categories, i.e. those of the form $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ for \mathcal{C} a small \mathcal{V} -enriched category. We call it the 2-category of \mathcal{V} -enriched profunctors.

Example 1.13. For $\mathcal{V} = \mathcal{S}$, this agrees with Haugseng's Morita 2-category $\mathbb{P}\text{rof}_{\mathcal{S}}^H$ of profunctors in [?]: Consider the corepresentable 2-presheaf $\text{Hom}_{\mathbb{P}\text{rof}_{\mathcal{S}}^H}(*, -) : \mathbb{P}\text{rof}_{\mathcal{S}}^H \rightarrow \mathcal{C}\text{at}$. It sends a small category C to $\mathcal{P}(C)$, and a profunctor $P : C \times D^{op} \rightarrow \mathcal{S}$ to the postcomposition $P \circ - : \mathcal{P}(C) \rightarrow \mathcal{P}(D)$. It is immediate to see that this construction factors through $\mathbb{P}\text{r}^L$, where it is fully faithful as it induces the equivalence $\text{Fun}(C \times D^{op}, \mathcal{S}) \simeq \text{Fun}^L(\mathcal{P}(C), \mathcal{P}(D))$ on morphism categories. Also, its essential image consists of precisely the presheaf categories, as claimed.

more general proof?

Warning 1.14. The 2-category $\mathbb{P}\text{rof}_{\mathcal{V}}$ does *not* admit all (conical) colimits, in fact its underlying 1-category is not even idempotent complete since regarded as a full subcategory of $\mathbb{P}\text{r}_{\mathcal{V}}$, it is not closed under retracts: For $\mathcal{V} = \mathcal{S}$ a counterexample is given in [hh], for $\mathcal{V} = \mathcal{S}p$ there is a large supply of compactly assembled stable categories that are not compactly generated, e.g. in [?]. However, $\mathbb{P}\text{r}_{\mathcal{V}}$ is idempotent complete, so the idempotent completion $\widehat{\mathbb{P}\text{rof}_{\mathcal{V}}^{\text{ic}}}$ can be identified with the full subcategory of $\mathbb{P}\text{r}_{\mathcal{V}}$ spanned by the retracts of tiny-generated categories.

Proposition 1.15. Given $\mathcal{V} \in \text{Alg}(\mathbb{P}\text{r}^L)$, a module $\mathcal{M} \in \text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$ is dualizable iff it is the retract of a tiny-generated category.

Proof. This statement is known to hold in $\text{RMod}_{\mathcal{V}}(\mathbb{P}\text{r}^L)$ by [?], so it suffices to show that any dualizable \mathcal{M} is automatically presentable. By Proposition 1.1 it suffices to show that \mathcal{M} is \mathcal{U} -compact, which follows from

$$\text{Map}_{\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})}(\mathcal{M}, -) \simeq \text{Map}_{\widehat{\mathcal{C}\text{at}}^{\text{colim}}}(\mathcal{S}, \underline{\text{Hom}}_{\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})}(\mathcal{M}, -)) \simeq \text{Map}_{\widehat{\mathcal{C}\text{at}}^{\text{colim}}}(\mathcal{S}, \mathcal{M}^{\vee} \otimes_{\mathcal{V}} -)$$

since $\mathcal{M}^{\vee} \otimes_{\mathcal{V}} -$ preserves all colimits, and $\mathcal{S} \in \widehat{\mathcal{C}\text{at}}^{\text{colim}}$ is \mathcal{U} -compact since it is presentable. \square

Remark 1.16. By [?], any dualizable $\mathcal{M} \in \text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$ is hence even \aleph_1 -compactly generated.

Theorem 1.17. The $(\infty, 2)$ -category $\mathbb{P}\text{rof}_{\mathcal{V}}$ of \mathcal{V} -enriched profunctors is the free lax semiadditive category on the delooping $B\mathcal{V}$. Further, it is the free lax \mathcal{V} -additive category on the point.

$$\begin{aligned} \text{Fun}(\mathbb{P}\text{rof}_{\mathcal{V}}, \mathbb{D}) &\simeq \text{Fun}(B\mathcal{V}, \mathbb{D}) \\ \text{Fun}(\mathbb{P}\text{rof}_{\mathcal{V}}, \mathbb{E}) &\simeq \text{Fun}(B\mathcal{V}, \mathbb{E}) \simeq \text{Fun}(*, \mathbb{E}) \simeq \mathbb{E} \end{aligned}$$

Proof. Note that $B\mathcal{V}$ is the free $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$ -enriched category on the point, since \mathcal{V} is the image of $*$ under the left adjoint to the forgetful functor $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}}) \rightarrow \mathcal{C}\text{at}$. Hence, it suffices to show that $\mathbb{P}\text{rof}_{\mathcal{V}}$ is the Cauchy-completion of $B\mathcal{V}$ both regarded as a $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$ -enriched category and as a $\widehat{\mathcal{C}\text{at}}^{\text{colim}}$ -enriched category. However in both settings, its enriched presheaf category is given by $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$, and the notions of tiny objects agree, so it suffices to consider the first case. Tiny objects in $\text{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}\text{at}}^{\text{colim}})$ regarded as a category presentably tensored over itself are precisely the dualizable objects by ???. Now, we are finished after combining Proposition 1.15 with ???. \square

rewrite

Corollary 1.18. The $(\infty, 2)$ -category $\mathbb{P}\text{rof}$ of profunctors is the free lax semiadditive category on both $B\mathcal{S}$ and the point.

Proof. The $(\infty, 2)$ -category $\mathbb{P}\mathrm{rof}^{\mathrm{ex}}$ of stable categories and exact profunctors is the free lax additive category on the point, and the free lax semiadditive category on BSp . \square

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