Stable ∞-categories

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Def A 1-category $\mathcal{A}$ is called <u>additive</u> $:\Leftrightarrow$ ← property!

- It has a zero object (i.e. both initial and terminal)
- It has products and coproducts
- $\forall X, Y \in \mathcal{A}$ the map $X \sqcup Y \xrightarrow{\begin{pmatrix} id_X & 0 \\ 0 & id_Y \end{pmatrix}} X \times Y$ is an isomorphism $=: X \oplus Y$ ← zero morphism $X \to 0 \to Y$
- For $f, g : X \to Y$, define their sum

$$f+g : X \xrightarrow{\Delta} \begin{matrix} X \\ \oplus \\ X \end{matrix} \begin{matrix} \xrightarrow{f} \\ \\ \xrightarrow{g} \end{matrix} \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{\nabla} Y$$

Then, for any f there should exist a morph. $-f : X \to Y$ s.t. $f + (-f) = 0$

Equivalently, an additive category is an Ab-enriched category with finite coproducts.

<u>Reason:</u> Let $\emptyset$ be the initial object, then $\mathrm{Hom}_{\mathcal{A}}(\emptyset, \emptyset) = \{id_\emptyset\} \in Ab$ so $id_\emptyset = 0$ is the zero object. Therefore, for all $X \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(X, \emptyset)$, we have

$f = id_\emptyset \circ f = 0 \circ f = 0$ by Ab-enrichment. → This proof up to now only uses enrichment in pointed sets

The argument for direct sums is similar:

$$\begin{tikzcd} X \arrow[dr, "i_1"] & & X \sqcup 0 \cong X \\ & X \sqcup Y \arrow[ur, "id_X \sqcup 0"] \arrow[dr, "0 \sqcup id_Y"'] & \\ Y \arrow[ur, "i_2"'] & & 0 \sqcup Y \cong Y \end{tikzcd}$$

□

Rem: Deeper reason is that finite coproducts (and idempotent splittings) are the <u>absolute colimits</u> over Ab, ie. those preserved by all Ab-enriched functors.

Def An additive category $\mathcal{A}$ is <u>abelian</u> $:\Leftrightarrow$

- It has kernels and cokernels
- $$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[dr, phantom, "\ulcorner"] & \mathrm{coker}(f) \\ 0 \arrow[r] & \mathrm{coim}(f) \arrow[r, "\cong"] & \mathrm{im}(f) \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Def An ∞-category $\mathcal{C}$ is called <u>stable</u> $:\Leftrightarrow$

- It admits a zero object 0
- It admits fibers and cofibers (analoga of kernels & cokernels)

$$\left(\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} , \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \end{tikzcd} \right)$$

- A sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ exhibits X' as fiber of g iff it exhibits X'' as the cofiber of f

Def For $X \in \mathcal{C}$, let

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

then $\Omega : \mathcal{C} \overset{\simeq}{\longleftrightarrow} \mathcal{C} : \Sigma$ are inverse equivalences if $\mathcal{C}$ stable.

→ Reason: $\Omega X \to 0 \to X$, $X \to 0 \to \Sigma X$ (co)fiber sequences

→ Explains the name. Also write $X[1] := \Sigma X$.

Note: $\pi_* \mathrm{Map}(X, Y) = \pi_* \mathrm{Map}(X, \Omega^2 \Sigma^2 Y) = \pi_* \Omega^2 \mathrm{Map}(X, \Sigma^2 Y) = \pi_* \mathrm{Map}(X, Y)$

$\to$ Explains the name. Also write $X[1] = \Sigma X$. $\to$ Reason: $\Omega X \to 0 \to X$, $X \to 0 \to \Sigma X$ (co)fiber sequences

Note: $\pi_0 \mathrm{Map}(X,Y) = \pi_0 \mathrm{Map}(X, \Omega^2 \Sigma^2 Y) = \pi_0 \Omega^2 \underbrace{\mathrm{Map}(X, \Sigma^2 Y)}_{\text{pointed by } X \to 0 \to Y} = \pi_2 \mathrm{Map}(X,Y)$

$\Rightarrow$ We can add & subtract morphisms, abelian group. $\to$ substract by reversing loops in $\Omega \mathrm{Map}(X, \Sigma Y)$

In fact, get a spectrum / infinite loop space.

Rem: A square
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\sim}{\Leftarrow}"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$
classifies a morphism $X \xrightarrow{f} \Omega \Sigma Y = Y$

The mirroring $\Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^1$ swapping the components acts on it by reversing the commutation homotopy
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\sim}{\Rightarrow}"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd},$$
thereby inversing a loop in $\mathrm{Map}(X,Y) = \Omega \mathrm{Map}(X, \Sigma Y)$, so the new square classifies $-f$.

Note:
$$\begin{tikzcd} \mathrm{fib}(X \xrightarrow{0} \Sigma Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \Omega \Sigma Y = Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$
shows $X \times Y = \mathrm{fib}(X \xrightarrow{0} \Sigma Y) \in \mathcal{C}$

[similarly $X \sqcup Y = \mathrm{cofib}(X[-1] \xrightarrow{0} Y)$]

$\Rightarrow h\mathcal{C}$ is Ab-enriched & has finite (co)products

Proposition: If $\mathcal{C}$ is stable, then $h\mathcal{C}$ is additive, in particular finite products & coproducts in $\mathcal{C}$ itself are isomorphic.

Fact:
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$
in $\mathcal{C}$ stable is a pullback iff it is a pushout square. Thus, we can apply the pasting Lemma in both directions.

Theorem If $\mathcal{C}$ is a stable ∞-category, then $h\mathcal{C}$ is triangulated, with shift functor $X[1] := \Sigma X$ and dist. triangles
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

Proof:
- $X \xrightarrow{f} Y$ can always be completed to dist. tr. $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$

(TR0)
- If $f = \mathrm{id}_X$, then $\mathrm{cofib}(f) = 0$
- Isomorphic to dist. tr. $\Rightarrow$ still one

(TR1) • Shift
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "-f[1]"] \\ & 0 \arrow[r] & Y[1] \end{tikzcd}$$
— since mirrored $\to$ "turns loop around"

$X \to Y$ / $\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd}$

(TR2) •
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
induce
$$\begin{tikzcd} \mathrm{cofib}(f) \arrow[d] \\ \mathrm{cofib}(f') \end{tikzcd}$$
fitting into dist. tr. $\Rightarrow$ functorial cone

(TR3) • Octaeder axiom
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ & 0 \arrow[r] & Z/Y \arrow[r] & Y[1] \arrow[r] & Y/X[1] \end{tikzcd}$$
This commut. diagram $\leadsto$ includes the octaeder diagram.

(Could go on to find "higher octaeder") $\square$

$$\begin{tikzcd}
0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & Z/Y \arrow[r] & Y[1] \arrow[r] & Y/X[1]
\end{tikzcd}$$

(Could go on to find "higher octaeder".) $\blacksquare$

Not every triangulated category arises this way, but "all the interesting ones", e.g.

- $Ch(\mathcal{A}) = \{$ chain cplxs, chain maps, chain homotopies, ... $\}$ is stable
- Any (pretriangulated) dg-(A_∞-)category has an associated (stable) ∞-category
- For $\mathcal{A}$ Grothendieck abelian (for simplicity),

  $D(\mathcal{A}) = \{$injective chain cplxes, chain maps, chain homot, ...$\}$ is stable

  & has all limits and colimits (presentable stable)
- $D(Sh(X;\mathcal{A})) = Sh_\infty^{hyp}(X; D(\mathcal{A}))$ is stable $\to$ Factorization algebras...
- $Sp = \lim (S_* \xleftarrow{\Omega} S_* \xleftarrow{\Omega} \cdots)$ is stable, in fact universal among them

Conceptional Remark:

We have seen that (idempotent complete) additive categories are the same thing as Ab-enriched categories that admit all absolute colimits for Ab-enrichement.

Say: $\{$i.c. additive categories$\} = \mathrm{CauchyCat}(\mathrm{Ab})$.

It turns out: $\{$i.c. stable ∞-categories$\} = \mathrm{CauchyCat}(\mathrm{Spectra})$

where the absolute colimits are finite colimits, shifts and idemp. splittings.

Also: $\{$i.c. additive ∞-categories$\} = \mathrm{CauchyCat}(\mathrm{Spectra}^{\text{connective}})$

Similarly i.c. additive (n,m)-categories.