CAUCHY-COMPLETE ∞-CATEGORIES AND LAX ADDIVITIY

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Abstract. Additive categories can be characterized as those Ab-enriched categories that admit finite coproducts, which automatically coincide with the respective products. This is a particular instance of a paradigm that makes sense for any enrichment category $\mathcal{V}$: A (weighted) colimit is called absolute if it can be described by a dual limit diagram, and following Lawvere a $\mathcal{V}$-enriched category is called Cauchy-complete if it admits all absolute colimits. Generalizing to enriched ∞-categories, I explain how a category enriched over cocomplete ∞-categories is Cauchy-complete iff it is idempotent complete and admits lax colimits, i.e. it is a lax semiadditive $(\infty, 2)$-category. Up to idempotent completion, this lets me recover Angus' previous statement about the category of profunctors being the free lax semiadditive $(\infty, 2)$-category on a point, generalize it to enriched profunctors, and explain its relation to multifusion categories.

1. Notes on Profunctors

1.1. **Cocomplete categories.** Fix universes $\beth < \hat{\beth} < \hat{\hat{\beth}}$ of small, large and very large sets. Denote by $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ the very large (locally large) category of large categories admitting small colimits, and functors preserving small colimits. We will also refer to them as *cocomplete categories* and *cocontinuous functors*. A notable full subcategory is the large category Pr^{L} spanned by the presentable categories.

Lemma 1.1. For any collection of κ-small categories $\mathcal{K}$, the forgetful functor $\mathrm{Cat}^{\mathcal{K}} \to \mathrm{Cat}$ from the category of categories with $\mathcal{K}$-shaped colimits and functors preserving $\mathcal{K}$-shaped colimits, creates κ-filtered colimits

Proof. Since the forgetful functor is conservative, it suffices to show that it preserves κ-filtered colimits. Similarly to [Lur09, Proposition 5.5.7.11], show that the inclusions into the colimit calculated in $\mathcal{C}at$ already preserve $\mathcal{K}$-shaped colimits. $\square$

Observation 1.2. In particular, the forgetful functor $\widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}$ creates $\beth$-filtered colimits. Therefore its left adjoint free cocompletion functor preserves $\beth$-compact objects, meaning that for any small category C the presheaf category $\mathcal{P}(\mathrm{C}) \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is $\beth$-compact. Since the forgetful functor is conservative and small categories generate $\widehat{\mathcal{C}at}$ under colimits, we learn that $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ is $\beth$-compactly generated by the presheaf categories. In fact by [**?**, Proposition 5.1.4], a category $\mathcal{M} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is $\beth$-compact iff it is presentable!

Proposition 1.3. Given $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, a module $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is $\beth$-compact iff it is presentable, i.e. lies in the full subcategory $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}) =: \mathrm{Pr}_{\mathcal{V}}$.

Proof. The case $\mathcal{V} = \mathcal{S}$ follows from Observation 1.2. Further by [**?**, Proposition 5.1.7], $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is $\beth$-compactly generated by the free modules $\mathcal{P} \otimes \mathcal{V}$ for $\mathcal{P} \in \mathrm{Pr}^{\mathrm{L}}$ (even for $\mathcal{V} \in \mathrm{Alg}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$). In particular, this implies that its $\beth$-compact objects are precisely

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the small colimits of such $\mathcal{P} \otimes \mathcal{V}$ by a $\mathrm{id}_{\mathcal{M}} \in \mathrm{Map}(\mathcal{M}, \mathcal{M}) = \mathrm{colim}_i \mathrm{Map}(\mathcal{M}, \mathcal{P}_i \otimes \mathcal{V})$ retract

elaborate?

argument. Now $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ contains all of these free modules (as $\mathcal{V}$ is presentable), in fact it is generated by them under geometric realizations, and it is closed under small colimits so we are finished. $\square$

Remark 1.4. In particular, any cocomplete $\mathcal{M}$ can be written as a large, ת-filtered colimit of presentable categories in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. For example $\mathrm{Pr}^{\mathrm{L}} = \mathrm{colim}_\kappa \mathrm{Pr}_\kappa$ is the colimit over all regular cardinals of the categories Pr_κ of κ-compactly generated categories and cocontinuous functors preserving κ-compact objects. Since $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is dense, there is even a canonical such colimit diagram indexed by $\mathrm{Pr}^{\mathrm{L}}_{/\mathcal{M}}$ for each $\mathcal{M}$.

Lemma 1.5. For C a small category, the functor $\mathrm{Fun}(\mathrm{C}, -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$ preserves ת-filtered colimits.

Proof. By Proposition 1.3 any cocomplete category is a ת-filtered colimit of presentable categories; also any presentable category is a small colimit of presheaf categories so the functors $\mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D}), -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \mathcal{S}$ for all $\mathrm{D} \in \mathcal{C}at$ are jointly conservative. Since $\mathcal{P}(\mathrm{D})$ is presentable, they also preserve ת-filtered colimits and hence jointly reflect them. Therefore it suffices to show that for any D the functor

$$\mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D}), \mathrm{Fun}(\mathrm{C}, -)) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D} \times \mathrm{C}), -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$$

preserves ת-filtered colimits, which follows from $\mathcal{P}(\mathrm{D} \times \mathrm{C})$ being presentable. $\square$

Recall that the tensor product $\otimes$ of cocomplete categories induces a symmetric monoidal structure on $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ that preserves large colimits in both variables separately, and restricts to a symmetric monoidal structure on Pr^{L}.

Proposition 1.6. The category of spectra $\mathcal{S}p$ is an idempotent algebra in the symmetric monoidal category $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ of cocomplete categories (equipped with the tensor product of cocomplete categories). Its category of modules $\mathrm{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete stable categories.

Proof. In the case where $\mathcal{M} \in \mathrm{Pr}^{\mathrm{L}}$, we know from [Lur17, Example 4.8.1.23] that $\mathcal{S}p \otimes \mathcal{M} \simeq \mathcal{S}p(\mathcal{M})$ agrees with the stabilization of $\mathcal{M}$, in particular $\mathcal{S}p \otimes \mathcal{M} \simeq \mathcal{M}$ iff $\mathcal{M}$ is stable. As $\mathcal{S}p$ is an idempotent algebra in Pr^{L}, this is once again equivalent to $\mathcal{M}$ being a module over $\mathcal{S}p$. Now since $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is a monoidal subcategory, $\mathcal{S}p$ is once again an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$, so it suffices to show that $\mathcal{S}p \otimes \mathcal{M}$ agrees with the stabilization $\lim_{\mathbb{N}}(\cdots \to \mathcal{M} \xrightarrow{\Omega} \mathcal{M})$ even if $\mathcal{M} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$. Once again we expand $\mathcal{M} \simeq \mathrm{colim}_i \mathcal{M}_i$ as a ת-filtered colimit with $\mathcal{M}_i \in \mathrm{Pr}^{\mathrm{L}}$, then

$$\mathcal{S}p \otimes \mathcal{M} \simeq \operatorname*{colim}_i \mathcal{S}p \otimes \mathcal{M}_i \simeq \operatorname*{colim}_i \lim_{\mathbb{N}}(\cdots \to \mathcal{M}_i \xrightarrow{\Omega} \mathcal{M}_i) \simeq \lim_{\mathbb{N}}(\cdots \to \mathcal{M} \xrightarrow{\Omega} \mathcal{M})$$

since ת-filtered colimit commute with small limits, in particular limits over $\mathcal{N}$ and loop functors. $\square$

Proposition 1.7. Let C be a small, and $\mathcal{M}$ a cocomplete category. Then $\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M})$.

Proof. The statement is true if $\mathcal{M}$ is presentable, since then $\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathcal{P}(\mathrm{C})^{\mathrm{op}}, \mathcal{M}) \simeq \mathrm{Fun}^{L}(\mathcal{P}(\mathrm{C}), \mathcal{M}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{M}^{\mathrm{op}})$. Using Proposition 1.3, let us write $\mathcal{M}$ as an ת-filtered (large) colimit $\mathcal{M} \simeq \operatorname{colim}_i \mathcal{M}_i$ with $\mathcal{M}_i \in \mathrm{Pr}^{\mathrm{L}}$. Then using that $\otimes$ preserves large colimits in both arguments separately,

$$\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \operatorname*{colim}_i \mathcal{P}(\mathrm{C}) \otimes \mathcal{M}_i \simeq \operatorname*{colim}_i \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M}_i) \simeq \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M})$$

where the last equivalence follows from Lemma 1.5. □

Reminder 1.8. A presentable category $\mathcal{A}$ together with a fixed object $a \in \mathcal{A}$ is called a *mode* if the functor $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{S} \to \mathcal{A} \otimes \mathbb{A}$ induced by a is an equivalence. In this case, the inverse to this equivalence equips $\mathcal{A}$ with the structure of a commutative algebra in Pr^{L}, and the forgetful functor $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ is fully faithful with essential image those $\mathcal{M} \in \mathrm{Pr}^{\mathrm{L}}$ where the map $\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{S} \to \mathcal{M} \otimes \mathcal{A}$ is an equivalence. Generally, we call a cocomplete category $\mathcal{M}$ where this map $\mathcal{M} \to \mathcal{M} \otimes \mathcal{A}$ is an equivalence *$\mathcal{A}$-modal*.

Note that there always exists a regular cardinal κ such that $\mathcal{A} \in \mathrm{Pr}^{\mathrm{L}}_{\kappa}$, which also means that $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\kappa})$. In this case we call $\mathcal{A}$ a *κ-mode*. Since the functor $\mathrm{Pr}^{\mathrm{L}}_{\kappa} \simeq \mathcal{C}at^{\kappa\text{-rex,ic}}$ taking the κ-compact objects is an equivalence of symmetric monoidal categories by [Lur17, Lemma 5.3.2.11] (using the idempotent-completed tensor product of categories with κ-small colimits on the right), we can view $\mathcal{A}^{\kappa\text{-cpt}}$ as an idempotent algebra in $\mathcal{C}at^{\kappa\text{-rex,ic}}$, and call its algebras *$\mathcal{A}^{\kappa\text{-cpt}}$-modal* categories.

Example 1.9. The presentable categories Set, Ab

On the other hand, the category $\mathrm{CMon}_m(\mathcal{S})$ of m-commutative monoids in $\mathcal{S}$ is $\aleph_1$-modal, but *not* $\aleph_0$-modal unless $m \leq 0$.

Proposition 1.10. For $\mathcal{A}$ a $\mathcal{K}$-mode and $\mathcal{M}$ a cocomplete category, the following are equivalent:

- $\mathcal{M}$ is $\mathcal{K}$-modal,
- $\mathcal{M}^{\kappa\text{-cpt}}$ is $\mathcal{A}^{\kappa\text{-cpt}}$-modal.

Also, $\mathcal{M}$ is $\mathcal{A}^{\kappa\text{-cpt}}$-modal iff $\mathrm{Ind}_{\mathcal{K}}(\mathcal{M})$ is $\mathcal{A}$-modal.

Remark 1.11. For many modes of interest, all four of the above conditions are equivalent. In fact, we do not know of any counterexamples.

Reminder 1.12. Let $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be a small operad and C any category. Denote by $\mathrm{Mon}_{\mathcal{O}}(\mathrm{C}) \subseteq \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{C})$ the full subcategory on those functors $M : \mathcal{O}^{\otimes} \to \mathrm{C}$ exhibiting $M(o_1, \ldots, o_n) \simeq M(o_1) \times \cdots \times M(o_n)$, for any collection of colors o_n. In particular, this product needs to exist in C.

Proposition 1.13. For $\mathcal{M}$ a cocomplete category, its categories of $\mathcal{O}$-monoid objects can be calculated as $\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \otimes \mathcal{M} \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{M})$.

Proof. Note that $\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \subseteq \mathcal{P}(\mathcal{O}^{\otimes,\mathrm{op}})$, so we can use Proposition 1.7 and show that the respective subcategories agree. If $\mathcal{M}$ is presentable, we can calculate that

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \otimes \mathcal{M} \subseteq \mathrm{Fun}^{\lim}(\mathcal{M}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}^{\lim}(\mathcal{M}^{\mathrm{op}}, \mathcal{S})) \supseteq \mathrm{Mon}_{\mathcal{O}}(\mathcal{M})$$

translates the subcategories into each other. Following the calculation in the proof of Proposition 1.7, it suffices to show that $\mathrm{Mon}_{\mathcal{O}}(-) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$ commutes with ת-filtered colimits, which by **??** are calculated in $\widehat{\mathcal{C}at}$. □ why is this the case?

Corollary 1.14. The category $\mathrm{CMon}(\mathcal{S})$ of commutative monoids, i.e. $\mathbb{E}_{\infty}$-algebras in $\mathcal{S}$, is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{cn}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete semiadditive categories.

Proof. By [Lur18, Corollary C.4.1.9] we know that $\mathrm{CMon}(\mathcal{S})$ is an idempotent algebra in Pr^{L}, and hence also in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its modules are precisely those cocomplete categories $\mathcal{M}$ such that the functor $\mathrm{CMon}(\mathcal{S}) \otimes \mathcal{M} \to \mathcal{S} \otimes \mathcal{M} \simeq \mathcal{M}$ is an equivalence, which by **??** is equivalent to the forgetful functor $\mathrm{CMon}(\mathcal{M}) \to \mathcal{M}$ being an equivalence. If $\mathcal{M}$ is semiadditive, i.e. both the Cartesian and coCartesian symmetric monoidal structure exist and agree, then by [Lur17, Proposition 2.4.3.8] this is the case. Conversely if $\mathcal{M} \simeq \mathrm{CMon}(\mathcal{M})$, replace $\mathcal{M}$ by a larger cocomplete category $\mathcal{M}'$ that admits products. By [Lur17, Proposition 3.2.4.10], the monoidal structure on $\mathrm{CMon}(\mathcal{M}')$ that is induced by the pointwise product is coCartesian, meaning that product and coproduct in $\mathcal{M}'$ must agree. □

how exactly must $\mathcal{M}'$ be chosen? can this work?

Proposition 1.15. The category $\mathrm{CMon}_m(\mathcal{S})$ of m-commutative monoids in $\mathcal{S}$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathrm{CMon}_m(\mathcal{S})}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete m-semiadditive categories.

Proof. Recall from [**?**,] that a category C admitting colimits over the class $\mathcal{K}_m$ of m-finite spaces is m-semiadditive iff it is tensored over the category $\mathcal{S}^m$ of spans of m-finite spaces, which is an idempotent algebra in $\mathcal{C}at^{\mathcal{K}_m}$. Also, $\mathrm{CMon}_m(\mathcal{S}) := \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}^m) = \mathrm{Fun}^{\mathcal{K}_m\text{-lim}}(\mathcal{S}^{m,\mathrm{op}}, \mathcal{S})$ where $\mathcal{P}_{\mathcal{K}_m} : \mathcal{C}at^{\mathcal{K}_m} \to \mathrm{Pr}^{\mathrm{L}}$ is the (symmetric monoidal) relative presheaf category functor. This means that if $\mathcal{M} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is m-semiadditive, then $\mathcal{M} \simeq \mathcal{M} \otimes^{\mathcal{K}_m} \mathcal{S}^m$ so $\mathcal{P}_{\mathcal{K}_m}(\mathcal{M}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{M}) \otimes \mathrm{CMon}_m(\mathcal{S})$. □

Proposition 1.16. The category of spectra $\mathcal{S}p^{\mathrm{cn}}$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{cn}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete additive categories.

Proof. Let $Z := \mathrm{Sym}_{\mathbb{E}_\infty}(\{x, y\})$ be the free commutative algebra in $\mathcal{S}$ generated by two points, and define the *shearing map* $\sigma : Z \to Z$ as the unique algebra map extending $\{x, y\} \to \pi_0 Z$ mapping x to x and y to $x + y$. By the proof of [Lur18, Theorem C.4.1.1] a cocomplete semiadditive categery $\mathcal{M}$ is additive if and only if for any cocontinuous map $H : \mathrm{CAlg}(\mathcal{S}) \to \mathcal{M}$, which is uniquely specified by an object $h \in \mathcal{M}$ using Corollary 1.14, the image of the shearing map $H(\sigma)$ is an isomorphism. But $\mathrm{CAlg}(\mathcal{S})$ is presentable, so if we write $\mathcal{M}$ as a $\sqcap$-filtered colimit over presentable additive categories $\mathcal{M}_i$ (which we can do by Proposition 1.3) then H must factor through one of the $\mathcal{M}_i$, where it sends σ to an isomorphism. □

1.2. Lax $\mathcal{V}$-additive categories.

Reminder 1.17. content...

Lemma 1.18. Let $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$, fix an $\mathbb{E}_2$-algebra $a \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$, and denote by $F : \mathcal{V} \to \mathrm{LMod}_a(\mathcal{V}) : U$ the free-forgetful adjunction. We can regard any $\mathrm{LMod}_a(\mathcal{V})$-tensored category $\mathcal{M} \in \mathrm{Pr}_{\mathrm{LMod}_a(\mathcal{V})}$ as a $\mathcal{V}$-tensored category $\mathcal{M}_{\mathcal{V}}$ with the same underlying category, by restricting scalars along F. An object $m \in \mathcal{M}$ is tiny with respect to the $\mathrm{LMod}_a(\mathcal{V})$-tensoring iff it is tiny with respect to the $\mathcal{V}$-tensoring on $\mathcal{M}_{\mathcal{V}}$.

Proof. By definition, $m \otimes - := m \otimes F(-) : \mathcal{V} \to \mathcal{M}$ in $\mathcal{M}_{\mathcal{V}}$, so passing to adjoints

$$\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathcal{V}}}(m, -) \simeq U \circ \underline{\mathrm{Hom}}_{\mathcal{M}}(m, -)\,.$$

Since U is conservative, and preserves colimits by [Lur17, arg1 arg2] as $\mathcal{V}$ is presentably monoidal, it creates colimits. Similarly it creates $\mathcal{V}$-tensorings since it preserves them, so we are finished. □

Definition 1.19. For $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$, we define the category of *lax $\mathcal{V}$-additive* $(\infty, 2)$-*categories* as $\mathcal{C}auchy\mathcal{C}at(\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}))$. Explicitly by Lemma 1.18, it consists of $(\infty, 2)$-categories that are locally cocomplete, locally tensored over $\mathcal{V}$ in a way that is compatible with composition and local colimits, and admits lax colimits and idempotent splittings.

Example 1.20. Let us note several cases of interest:

- A lax $\mathcal{S}$-additive $(\infty, 2)$-category is a locally cocomplete $(\infty, 2)$-category with lax colimits and idempotent splittings, also known as an i.c. *lax semiadditive* $(\infty, 2)$-*category*.
- A lax Set-additive $(\infty, 2)$-category is an i.c. locally cocomplete $(2, 2)$-category with lax colimits, so we call it an i.c. *lax semiadditive* $(2, 2)$-*category*.
- A lax $\mathcal{S}p$-additive $(\infty, 2)$-category is an i.c. lax semiadditive $(\infty, 2)$-category that is locally tensored over $\mathcal{S}p$, which by Proposition 1.6 means that it is locally stable. Hence, we recover *lax additive* $(\infty, 2)$-*categories*.
- A lax Ab-additive $(\infty, 2)$-category using Proposition 1.16 is a lax semiadditive $(2, 2)$-category that is locally additive.
- Similarly for $\mathcal{S}_{\leq m}, \mathcal{S}_{\leq m,*}, \mathcal{S}p_{\leq m}$ we obtain locally semiadditive $(m + 2, 2)$-categories (that are locally pointed/ additive).
- One should consider lax $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$-additive $(\infty, 2)$-categories as 2-*lax additive* $(\infty, 3)$-*categories*. This is because they are enriched over $\mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ which is the $\mathrm{Ind}_{\sqcap}$-completion of the category of presentable stable 2-categories introduced in [?] (just like $\mathrm{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is the $\mathrm{Ind}_{\sqcap}$-completion of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ in the lax additive case).

Observation 1.21. Any lax $\mathcal{V}$-additive $(\infty, 2)$-category is automatically 2-idempotent complete (which is a priori a stronger condition). This is because any cocomplete category is idempotent complete, so the forgetful functor $\mathrm{LMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \to \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}$ factors through $\widehat{\mathcal{C}at}^{\mathrm{idem}}$. Since all of these functors are right adjoints of monoidal functors, change-of-enrichment along them preserves Cauchy-completeness, in particular if $\mathbb{C} \in \mathcal{C}auchy\mathcal{C}at(\mathrm{LMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}))$ then the underlying $\widehat{\mathcal{C}at}^{\mathrm{idem}}$-enriched category is Cauchy-complete, i.e. 2-idempotent complete.

1.3. **Universal Property of Profunctors.**

Definition 1.22. Denote by $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ the full sub-2-category of $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ spanned by the tiny-generated categories, i.e. those of the form $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ for $\mathcal{C}$ a small $\mathcal{V}$-enriched category. We call it the 2-*category of $\mathcal{V}$-enriched profunctors*.

Example 1.23. For $\mathcal{V} = \mathcal{S}$, this agrees with Haugseng's Morita 2-category $\mathbb{P}\mathrm{rof}^{H}_{\mathcal{S}}$ of profunctors in [?]: Consider the corepresentable 2-presheaf $\mathrm{Hom}_{\mathbb{P}\mathrm{rof}^{H}_{\mathcal{S}}}(*, -) : \mathbb{P}\mathrm{rof}^{H}_{\mathcal{S}} \to \mathbb{C}\mathrm{at}$. It sends a small category C to $\mathcal{P}(C)$, and a profunctor $P : C \times D^{\mathrm{op}} \to \mathcal{S}$ to the postcomposition $P \circ - : \mathcal{P}(C) \to \mathcal{P}(D)$. It is immediate to see that this construction factors through $\mathbb{P}\mathrm{r}^{L}$, where it is fully faithful as it induces the equivalence $\mathrm{Fun}(C \times D^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{L}(\mathcal{P}(C), \mathcal{P}(D))$ on morphism categories. Also, its essential image consists of precisely the presheaf categories, as claimed.

more general proof?

Warning 1.24. The 2-category $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ does *not* admit all (conical) colimits, in fact its underlying 1-category is not even idempotent complete since regarded as a full subcategory of $\mathrm{Pr}_{\mathcal{V}}$, it is not closed under retracts: For $\mathcal{V} = \mathcal{S}$ a counterexample is given in [hh], for $\mathcal{V} = \mathcal{S}p$ there is a large supply of compactly assembled stable categories that are not compactly generated, e.g. in [?]. However, $\mathrm{Pr}_{\mathcal{V}}$ *is* idempotent complete, so the idempotent completion $\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}$ can be identified with the full subcategory of $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ spanned by the retracts of tiny-generated categories.

Proposition 1.25. Given $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, a module $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is dualizable iff it is the retract of a tiny-generated category.

Proof. This statement is known to hold in $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ by [?], so it suffices to show that any dualizable $\mathcal{M}$ is automatically presentable. By Proposition 1.3 is suffices to show that $\mathcal{M}$ is ת-compact, which follows from

$$\mathrm{Map}_{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(\mathcal{M}, -) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{S}, \underline{\mathrm{Hom}}_{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(\mathcal{M}, -)) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{S}, \mathcal{M}^{\vee} \otimes_{\mathcal{V}} -)$$

sine $\mathcal{M}^{\vee} \otimes_{\mathcal{V}} -$ preserves all colimits, and $\mathcal{S} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is ת-compact since it is presentable. □

Remark 1.26. By [?], any dualizable $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is hence even $\aleph_1$-compactly generated.

Theorem 1.27. The idempotent completion of the $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ of $\mathcal{V}$-enriched profunctors is the free i.c. lax semiadditive category on the delooping $B\mathcal{V}$. Further, it is the free lax $\mathcal{V}$-additive category on the point.

$$\mathrm{Fun}^{\mathrm{loc.coc.}}(\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}, \mathbb{D}) \simeq \mathrm{Fun}^{\mathrm{loc.coc.}}(B\mathcal{V}, \mathbb{D})$$
$$\mathrm{Fun}^{\mathrm{loc.coc.}}(\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}, \mathbb{E}) \simeq \mathrm{Fun}^{\mathrm{loc.coc.}}(B\mathcal{V}, \mathbb{E}) \simeq \mathrm{Fun}(*, \mathbb{E}) \simeq \mathbb{E}$$

Proof. Note that $B\mathcal{V}$ is the free $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$-enriched category on the point, since $\mathcal{V}$ is the image of $*$ under the left adjoint to the forgetful functor $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \to \mathcal{C}at$. Hence, it suffices to show that $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ is the Cauchy-completion of $B\mathcal{V}$ both regarded as a $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$-enriched category and as a $\widehat{\mathcal{C}at}^{\mathrm{colim}}$-enriched category. However in both settings, its enriched presheaf category is given by $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$, and the notions of tiny objects agree, so it suffices to consider the first case. Tiny objects in $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ regarded as a category presentably tensored over itself are precisely the dualizable objects by **??**. Now, we are finished after combining Proposition 1.25 with **??**. □

rewrite

Corollary 1.28. The $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}$ of profunctors is the free lax semiadditive category on $B\mathcal{S}$, or on the point.

Proof. Up to idempotent completion this is immediate from the above theorem. For the full statement, use Angus' results that $\mathbb{P}\mathrm{rof}$ admits all lax colimits, and generated under lax colimits by the point. □

Proposition 1.29. The idempotent completion of the $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}^{\mathrm{ex}}$ of stable categories and exact profunctors (in other words, the category of compactly assembled stable categories and colimit-preserving functors) is the free i.c. lax additive category on the point, and the free i.c. lax semiadditive category on $B\mathcal{S}p$.

Proof. Combine the above theorem with the observation that the category of $\mathcal{S}p$-enriched profunctors is equivalent to its full subcategory on (i.c.) stable categories since any $\mathcal{S}p$-enriched category is equivalent to its Cauchy-completion which lies in there. Also, $\mathcal{S}p$-enriched functors between stable categories are the same thing as exact functors. □

Remark 1.30. Once again, this statement is true without idempotent completing.

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