CAUCHY-COMPLETE ∞-CATEGORIES AND LAX ADDIVITIY

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Abstract. Additive categories can be characterized as those Ab-enriched categories that admit finite coproducts, which automatically coincide with the respective products. This is a particular instance of a paradigm that makes sense for any enrichment category $\mathcal{V}$: A (weighted) colimit is called absolute if it can be described by a dual limit diagram, and following Lawvere a $\mathcal{V}$-enriched category is called Cauchy-complete if it admits all absolute colimits. Generalizing to enriched ∞-categories, I explain how a category enriched over cocomplete ∞-categories is Cauchy-complete iff it is idempotent complete and admits lax colimits, i.e. it is a lax semiadditive $(\infty, 2)$-category. Up to idempotent completion, this lets me recover Angus' previous statement about the category of profunctors being the free lax semiadditive $(\infty, 2)$-category on a point, generalize it to enriched profunctors, and explain its relation to multifusion categories.

This is an informal addendum to a talk of the same name given in the *Higher Structures* seminar at the University of Hamburg. In particular parts of the results are based on ongoing work together with David Reutter, so use at your own risk. Comments and typos are very welcome!

1. Cocomplete categories

Fix universes $\text{ת} < \hat{\text{ת}} < \hat{\hat{\text{ת}}}$ of small, large and very large sets. Denote by $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ the very large (locally large) category of large categories admitting small colimits, and functors preserving small colimits. We will also refer to them as *cocomplete categories* and *cocontinuous functors*. A notable full subcategory is the large category Pr^{L} spanned by the presentable categories.

Lemma 1.1. For any collection of κ-small categories $\mathcal{K}$, the forgetful functor $\mathrm{Cat}^{\mathcal{K}} \to \mathrm{Cat}$ from the category of categories with $\mathcal{K}$-shaped colimits and functors preserving $\mathcal{K}$-shaped colimits, creates κ-filtered colimits.

Proof. Since the forgetful functor is conservative, it suffices to show that it preserves κ-filtered colimits. Similarly to [Lur09, Proposition 5.5.7.11], show that the inclusions into the colimit calculated in $\mathcal{C}at$ already preserve $\mathcal{K}$-shaped colimits. $\square$

Observation 1.2. In particular, the forgetful functor $\widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}$ creates ת-filtered colimits. Therefore its left adjoint free cocompletion functor preserves ת-compact objects, meaning that for any small category C the presheaf category $\mathcal{P}(\mathrm{C}) \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is ת-compact. Since the forgetful functor is conservative and small categories generate $\widehat{\mathcal{C}at}$ under colimits, we learn that $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ is ת-compactly generated by the presheaf categories. In fact by [Ste20, Proposition 5.1.4], a category $\mathcal{M} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is ת-compact iff it is presentable!

Proposition 1.3. Given $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, a module $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is ת-compact iff it is presentable, i.e. lies in the full subcategory $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}}) =: \mathrm{Pr}_{\mathcal{V}}$.

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Proof. The case $\mathcal{V} = \mathcal{S}$ follows from Observation 1.2. Further by [Ste20, Proposition 5.1.7], $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is ת-compactly generated by the free modules $\mathcal{P} \otimes \mathcal{V}$ for $\mathcal{P} \in \mathrm{Pr}^{\mathrm{L}}$ (even for $\mathcal{V} \in \mathrm{Alg}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$). In particular, this implies that its ת-compact objects are precisely the small colimits of such $\mathcal{P} \otimes \mathcal{V}$ by a $\mathrm{id}_{\mathcal{M}} \in \mathrm{Map}(\mathcal{M}, \mathcal{M}) = \mathrm{colim}_i \mathrm{Map}(\mathcal{M}, \mathcal{P}_i \otimes \mathcal{V})$ retract argument. Now $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ contains all of these free modules (as $\mathcal{V}$ is presentable), in fact it is generated by them under geometric realizations, and it is closed under small colimits so we are finished. □

elaborate?

Remark 1.4. In particular, any cocomplete $\mathcal{M}$ can be written as a large, ת-filtered colimit of presentable categories in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. For example $\mathrm{Pr}^{\mathrm{L}} = \mathrm{colim}_\kappa \mathrm{Pr}_\kappa$ is the colimit over all regular cardinals of the categories Pr_κ of κ-compactly generated categories and cocontinuous functors preserving κ-compact objects. Since $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is dense, there is even a canonical such colimit diagram indexed by $\mathrm{Pr}^{\mathrm{L}}_{/\mathcal{M}}$ for each $\mathcal{M}$.

Lemma 1.5. For C a small category, the functor $\mathrm{Fun}(\mathrm{C}, -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$ preserves ת-filtered colimits.

Proof. By Proposition 1.3 any cocomplete category is a ת-filtered colimit of presentable categories; also any presentable category is a small colimit of presheaf categories so the functors $\mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D}), -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \mathcal{S}$ for all $\mathrm{D} \in \mathcal{C}at$ are jointly conservative. Since $\mathcal{P}(\mathrm{D})$ is presentable, they also preserve ת-filtered colimits and hence jointly reflect them. Therefore it suffices to show that for any D the functor

$$\mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D}), \mathrm{Fun}(\mathrm{C}, -)) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{P}(\mathrm{D} \times \mathrm{C}), -) : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$$

preserves ת-filtered colimits, which follows from $\mathcal{P}(\mathrm{D} \times \mathrm{C})$ being presentable. □

Lemma 1.6. Let C be a small, and $\mathcal{M}$ a cocomplete category. Then $\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M})$.

Proof. The statement is true if $\mathcal{M}$ is presentable, since then $\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathcal{P}(\mathrm{C})^{\mathrm{op}}, \mathcal{M}) \simeq \mathrm{Fun}^{L}(\mathcal{P}(\mathrm{C}), \mathcal{M}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathrm{C}, \mathcal{M}^{\mathrm{op}})$. Using Proposition 1.3, let us write $\mathcal{M}$ as an ת-filtered (large) colimit $\mathcal{M} \simeq \mathrm{colim}_i \mathcal{M}_i$ with $\mathcal{M}_i \in \mathrm{Pr}^{\mathrm{L}}$. Then using that $\otimes$ preserves large colimits in both arguments separately,

$$\mathcal{P}(\mathrm{C}) \otimes \mathcal{M} \simeq \mathrm{colim}_i \mathcal{P}(\mathrm{C}) \otimes \mathcal{M}_i \simeq \mathrm{colim}_i \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M}_i) \simeq \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathcal{M})$$

where the last equivalence follows from Lemma 1.5. □

2. Modes

Reminder 2.1. For $\mathcal{K} \subseteq \mathcal{L}$ any small sets of small categories that we want to use as shapes of colimit diagrams, [Lur09, Example 5.3.6.4] defines a $\mathcal{L}$-$\mathcal{K}$-*cocompletion functor* $\mathcal{P}^{\mathcal{L}}_{\mathcal{K}} : \mathcal{C}at^{\mathcal{K}} \to \mathcal{C}at^{\mathcal{L}}$ that is left adjoint to the forgetful functor. Explicitly, $\mathcal{P}^{\mathcal{L}}_{\mathcal{K}}(\mathrm{C})$ can be constructed by taking the full subcategory of $\mathcal{P}(\mathrm{C})$ generated by $\mathcal{L}$-shaped colimits, and further restricting to those presheaves $F : \mathrm{C}^{\mathrm{op}} \to \mathcal{S}$ that send $\mathcal{K}$-colimits in C to limits. Denote by $\mathcal{P}_{\mathcal{K}} : \mathcal{C}at^{\mathcal{K}} \subseteq \widehat{\mathcal{C}at}^{\mathcal{K}} \to \mathrm{Pr}^{\mathrm{L}}$ the special case where $\mathcal{L}$ consists of *all* small categories, in other words $\mathcal{P}_{\mathcal{K}}(\mathcal{C}) := \mathrm{Fun}^{\mathcal{K}\text{-lim}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which is presentable as its is an accessible localization of $\mathcal{P}(\mathrm{C})$.

The functor $\mathcal{P}^{\mathcal{L}}_{\mathcal{K}} : \mathcal{C}at^{\mathcal{K}} \to \mathcal{C}at^{\mathcal{L}}$ is symmetric monoidal with respect to the *tensor products* $\otimes^{\mathcal{K}}, \otimes^{\mathcal{L}}$ of $\mathcal{K}$- and $\mathcal{L}$-cocomplete categories. Those are defined by the universal property

$$\mathrm{Fun}^{\mathcal{K}}(\mathrm{C} \otimes^{\mathcal{K}} \mathrm{D}, \mathrm{E}) \simeq \mathrm{Fun}^{\mathcal{K} \times \mathcal{K}}(\mathrm{C} \times \mathrm{D}, \mathrm{E})$$

for $\mathrm{C}, \mathrm{D}, \mathrm{E} \in \mathcal{C}at^{\mathcal{K}}$, where $\mathrm{Fun}^{\mathcal{K}\times\mathcal{K}}$ denotes those functors that preserve $\mathcal{K}$-shaped colimits in C and D separately. Explicitly (as for the tensor product of vector spaces), we can construct $\mathrm{C} \otimes^{\mathcal{K}} \mathrm{D} := \mathcal{P}^{\mathcal{K}}_{\mathcal{K}\times\mathcal{K}}(\mathrm{C} \times \mathrm{D})$. By [Lur17, Lemma 4.8.4.2], $\otimes^{\mathcal{K}}$ preserves small colimits in both arguments separately.

Definition 2.2. For $\mathcal{K}$ a collection of categories, a *$\mathcal{K}$-mode* is a pair (A, a) of a $\mathcal{K}$-cocomplete category $\mathrm{A} \in \mathcal{C}at^{\mathcal{K}}$ and $a \in \mathrm{A}$ such that the functor $\mathrm{A} \simeq \mathrm{A} \otimes^{\mathcal{K}} \mathcal{P}_{\mathcal{K}}(*) \to \mathrm{A} \otimes \mathrm{A}$ induced by a is an equivalence. In this case, the inverse to this equivalence equips A with the structure of a commutative algebra in $\mathcal{C}at^{\mathcal{K}}$ (algebras that arise this way, i.e. whose multiplication map is an isomorphism, are called *idempotent*), and the forgetful functor $\mathrm{Mod}_{\mathrm{A}}(\mathcal{C}at^{\mathcal{K}}) \to \mathcal{C}at^{\mathcal{K}}$ is fully faithful with essential image those $\mathcal{M} \in \mathrm{Pr}^{\mathrm{L}}$ where the map $\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{P}_{\mathcal{K}}(*) \to \mathcal{M} \otimes \mathrm{A}$ induced by a is an equivalence. Such $\mathcal{M}$ will be called A-*modal*.

If $\mathcal{K}$ is the collection of κ-small categories (together with, for $\kappa = \aleph_0$, the idempotent splitting diagram), we call A a *κ-mode*. Since the functor $\mathcal{P}_\kappa : \mathcal{C}at^{\kappa\text{-rex,ic}} \simeq \mathrm{Pr}^L_\kappa$ is an equivalence of symmetric monoidal categories by [Lur17, Lemma 5.3.2.11], this is equivalently an idempotent algebra of Pr^L_κ. A presentable category $\mathcal{A}$ together with $a \in \mathcal{A}$ inducing $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{A}$ will just be called a *mode*. Note that $\mathcal{A}$ is always κ-compactly generated for some κ, making $\mathcal{A}$ into a κ-mode as well. Conversely if A is a $\mathcal{K}$-mode, then $\mathcal{P}_{\mathcal{K}}(\mathrm{A})$ is a mode since $\mathcal{P}_{\mathcal{K}} : \mathcal{C}at^{\mathcal{K}} \to \mathrm{Pr}^{\mathrm{L}}$ is symmetric monoidal.

Proposition 2.3. For (A, a) a $\mathcal{K}$-mode and $\mathcal{M}$ a cocomplete category, the following are equivalent:

(1) $\mathcal{M}$ regarded as a $\mathcal{K}$-complete category is A-modal,
(2) $\mathcal{M}$ is $\mathcal{P}_{\mathcal{K}}(\mathrm{A})$-modal,
(3) $\mathcal{P}_{\mathcal{K}}(\mathcal{M})$ regarded as a $\mathcal{K}$-complete category is A-modal,
(4) $\mathcal{P}_{\mathcal{K}}(\mathcal{M})$ is $\mathcal{P}_{\mathcal{K}}(\mathrm{A})$-modal.

Proof. Since $\mathcal{P}_{\mathcal{K}} : \widehat{\mathcal{C}at}^{\mathcal{K}} \to \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is symmetric monoidal, (1) implies (4). Also if we knew that (1) was equivalent to (2), then (3) $\Leftrightarrow$ (4) would follow by replacing $\mathcal{M}$ with $\mathcal{P}_{\mathcal{K}}(\mathcal{M})$.

For (2) $\Rightarrow$ (1), note that if $\mathcal{M}$ admits a $\mathcal{P}_{\mathcal{K}}(\mathrm{A})$-module structure classified by a colimit-preserving monoidal functor $\mathcal{P}_{\mathcal{K}}(\mathrm{A}) \to \mathrm{End}^{\mathrm{colim}}(\mathcal{M})$, then the restriction $\mathrm{A} \to \mathrm{End}^{\mathrm{colim}}(\mathcal{M}) \subseteq \mathrm{End}^{\mathcal{K}\text{-colim}}$ exhibits $\mathcal{M}$ as an A-module in $\mathcal{C}at^{\mathcal{K}}$.

We finish by proving (4) $\Rightarrow$ (2). Note that $\mathcal{P}_{\mathcal{K}}(\mathrm{A}) \otimes \mathcal{M}$ is a full subcategory of $\mathcal{P}_{\mathcal{K}}(\mathrm{A}) \otimes \mathcal{P}_{\mathcal{K}}(\mathcal{M})$, namely the former consists of those presheaves in $\mathcal{P}(\mathrm{A} \times \mathcal{M})$ that send $\mathcal{K}$-shaped colimits in A as well as colimits in $\mathcal{M}$ to limits in $\mathcal{S}$, while for the latter this only needs to hold for $\mathcal{K}$-shaped colimits in A and $\mathcal{M}$ respectively. But by assumption, the map $\mathcal{P}_{\mathcal{K}}(\mathcal{M}) \simeq \mathcal{S} \otimes \mathcal{P}_{\mathcal{K}}(\mathcal{M}) \to \mathcal{P}_{\mathcal{K}}(\mathrm{A}) \otimes \mathcal{P}_{\mathcal{K}}(\mathcal{M})$ induced by $a \in \mathrm{A}$ is an isomorphism, so the latter is generated under colimits by elements of the form $\text{よ}_a \otimes \text{よ}_m$ with $m \in \mathcal{M}$. Since these are contained in $\mathcal{P}_{\mathcal{K}}(\mathrm{A}) \otimes \mathcal{M}$, the full inclusion must be an equivalence exhibiting $\mathcal{M}$ as $\mathcal{P}_{\mathcal{K}}(\mathrm{A})$-modal. $\square$

Proposition 2.4. The category of spectra $\mathcal{S}p$ is an idempotent algebra in the symmetric monoidal category $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ of cocomplete categories (equipped with the tensor product of cocomplete categories). Its category of modules $\mathrm{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete stable categories.

Proof. Recall e.g. from [CDH+20, Construction 5.1.1] that the category $\mathcal{S}p^{\mathrm{fin}}$ of finite spectra can do without this reference
(which can be defined as the compact objects in $\mathcal{S}p$) is an $\aleph_0$-mode, and a category C admitting finite colimits is stable iff it is $\mathcal{S}p^{\mathrm{fin}}$-modal. By Proposition 2.3 a cocomplete category $\mathcal{M}$ is hence modal for $\mathcal{S}p = \mathcal{P}_{\aleph_0}(\mathcal{S}p^{\mathrm{fin}}) = \mathrm{Ind}(\mathcal{S}p^{\mathrm{fin}})$ iff it is stable. $\square$

Remark 2.5. This does *not* immediately follow from [Lur17, Example 4.8.1.23] which tells us that $\mathcal{S}p$ is the mode classifying presentable stable categories: While this does imply that $\mathcal{S}p$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$, it is tricky to classify the $\mathcal{S}p$-modal objects.

Proposition 2.6. The category $\mathrm{CMon}_m(\mathcal{S})$ of m-commutative monoids in $\mathcal{S}$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathrm{CMon}_m(\mathcal{S})}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete m-semiadditive categories.

Proof. Recall from [Har16, Proposition 5.6] that a category C admitting colimits over the small class $\mathcal{K}_m$ of m-finite spaces is m-semiadditive iff it is tensored over the category $\mathcal{S}^m$ of spans of m-finite spaces, which is a $\mathcal{K}_m$-mode. We obtain an idempotent algebra $\mathrm{CMon}_m(\mathcal{S}) := \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}^m) = \mathrm{Fun}^{\mathcal{K}_m\text{-lim}}(\mathcal{S}^{m,\mathrm{op}}, \mathcal{S})$ in Pr^{L} and consequently $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. By Proposition 2.3, modules over it are precisely the $\mathcal{S}^m$-modal, i.e. m-semiadditive cocomplete categories. □

Proposition 2.7. The category of spectra $\mathcal{S}p^{\mathrm{cn}}$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathcal{S}p^{\mathrm{cn}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete additive categories.

Proof. Let $Z := \mathrm{Sym}_{\mathbb{E}_\infty}(\{x, y\})$ be the free commutative algebra in $\mathcal{S}$ generated by two points, and define the *shearing map* $\sigma : Z \to Z$ as the unique algebra map extending $\{x, y\} \to \pi_0 Z$ mapping x to x and y to $x + y$. By the proof of [Lur18, Theorem C.4.1.1] a cocomplete semiadditive category $\mathcal{M}$ is additive if and only if for any cocontinuous map $H : \mathrm{CAlg}(\mathcal{S}) \to \mathcal{M}$, which is uniquely specified by an object $h \in \mathcal{M}$ using Proposition 2.6, the image of the shearing map $H(\sigma)$ is an isomorphism. But $\mathrm{CAlg}(\mathcal{S})$ is presentable, so if we write $\mathcal{M}$ as a $\daleth$-filtered colimit over presentable additive categories $\mathcal{M}_i$ (which we can do by Proposition 1.3) then H must factor through one of the $\mathcal{M}_i$, where it sends σ to an isomorphism. □

Remark 2.8. Alternatively, we could of course also prove this using our mode technology. Generally if $\mathcal{A}$ is any mode and κ-compactly generated, then $\mathcal{A}^{\kappa\text{-cpt}}$ is a κ-mode. From Proposition 2.3, we learn that a cocomplete category $\mathcal{M}$ is modal over $\mathcal{A} = \mathcal{P}_\kappa(\mathcal{A}^{\kappa\text{-cpt}})$ iff regarded as a κ-cocomplete category it is $\mathcal{A}^{\kappa\text{-cpt}}$-modal. A similar argument would also work if we replace the class of κ-small categories by any sound doctrine in the sense of [Rez21].

Proposition 2.9. The category of n-truncated spaces $\mathcal{S}_{\leq n}$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathcal{S}_{\leq n}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the cocomplete $(n, 1)$-categories.

Proof. It is an idempotent algebra in Pr^{L} classifying presentable $(n, 1)$-categories by [Lur17, Example 4.8.1.22]. Let us write a general cocomplete category $\mathcal{M}$ as a $\daleth$-filtered colimit $\mathrm{colim}_i\, \mathcal{M}_i$, then we may calculate

$$\mathcal{S}_{\leq n} \otimes \mathcal{M} \simeq \mathrm{colim}_i\, \mathcal{S}_{\leq n} \otimes \mathcal{M}_i \simeq \mathrm{colim}_i\, \tau_{\leq n}\mathcal{M}_i\,.$$

Now, note that the truncation functor $\tau_{\leq n} : \widehat{\mathcal{C}at} \to \widehat{\mathcal{C}at}_{(n,1)}$ is left adjoint to the inclusion and preserves κ-compact objects for any κ, in particular $\daleth$-compact objects (i.e. small categories). Hence, the composition $\tau_{\leq n} : \widehat{\mathcal{C}at} \to \widehat{\mathcal{C}at}$ with its right adjoint preserves τ-filtered colimits, so by Observation 1.2 the restriction $\tau_{\leq n} : \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}$ does so as well and we are finished since we recover $\tau_{\leq n}\mathcal{M}$. □

Proposition 2.10. The category of pointed spaces $\mathcal{S}_*$ is an idempotent algebra in $\widehat{\mathcal{C}at}^{\mathrm{colim}}$. Its category of modules $\mathrm{Mod}_{\mathcal{S}_*}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}^{\mathrm{colim}}$ consists precisely of the *pointed* cocomplete categories, i.e. those admitting a zero object.

Proof. Recall from [Lur17, Example 4.8.1.21] that $\mathcal{S}_*$ is an idempotent algebra in Pr^{L} classifying pointed presentable categories. Also for any κ, a κ-cocomplete category C is pointed iff $\mathrm{Ind}_\kappa(\mathrm{C})$ is pointed, i.e. $\mathcal{S}_*$-tensored. Using the equivalence between κ-compactly generated and κ-cocomplete categories, we deduce that C is pointed iff it is tensored over the κ-mode $\mathcal{S}_*^{\kappa\text{-cpt}}$. Hence by Proposition 2.3, a cocomplete category is modal over $\mathrm{Ind}_\kappa(\mathcal{S}_*^{\kappa\text{-cpt}}) \simeq \mathcal{S}_*$ iff it is $\mathcal{S}_*^{\kappa\text{-cpt}}$-modal regarded as a κ-cocomplete category, i.e. pointed. ☐

3. Lax $\mathcal{V}$-additive categories

Lemma 3.1. Let $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$, fix an $\mathbb{E}_2$-algebra $a \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{V})$, and denote by $F : \mathcal{V} \to \mathrm{LMod}_a(\mathcal{V}) : U$ the free-forgetful adjunction. We can regard any $\mathrm{LMod}_a(\mathcal{V})$-tensored category $\mathcal{M} \in \mathrm{Pr}_{\mathrm{LMod}_a(\mathcal{V})}$ as a $\mathcal{V}$-tensored category $\mathcal{M}_{\mathcal{V}}$ with the same underlying category, by restricting scalars along F. An object $m \in \mathcal{M}$ is tiny with respect to the $\mathrm{LMod}_a(\mathcal{V})$-tensoring iff it is tiny with respect to the $\mathcal{V}$-tensoring on $\mathcal{M}_{\mathcal{V}}$.

Proof. By definition, $m \otimes - := m \otimes F(-) : \mathcal{V} \to \mathcal{M}$ in $\mathcal{M}_{\mathcal{V}}$, so passing to adjoints

$$\underline{\mathrm{Hom}}_{\mathcal{M}_{\mathcal{V}}}(m, -) \simeq U \circ \underline{\mathrm{Hom}}_{\mathcal{M}}(m, -)\,.$$

Since U is conservative, and preserves colimits by [Lur17, Corollary 4.2.3.5] as $\mathcal{V}$ is presentably monoidal, it creates colimits. Similarly it creates $\mathcal{V}$-tensorings since it preserves them, so we are finished. ☐

Lemma 3.2. For $F : \mathcal{V} \to \mathrm{LMod}_a(\mathcal{V}) : U$ as before, let $\mathcal{C}$ be a $\mathrm{LMod}_a(\mathcal{V})$-enriched category and $U_!\mathcal{C} \in \mathcal{C}at(\mathcal{V})$ its change-of-enrichment. Then $\mathcal{P}_{\mathcal{V}}(U_!\mathcal{C}) \in \mathrm{Pr}_{\mathcal{V}}$ is the restriction of scalars of $\mathcal{P}_{\mathrm{LMod}_a(\mathcal{V})}(\mathcal{C}) \in \mathrm{Pr}_{\mathrm{LMod}_a(\mathcal{V})}$ along F.

and the Yoneda functors agree

Proof. Since U is a $\mathcal{V}$-linear colimit-preserving functor, i.e. a map in $\mathrm{Pr}_{\mathcal{V}}$, we know by [RZ24b, ...] that $\mathcal{P}_{\mathcal{V}}(U_!\mathcal{C}) \simeq U_*\mathcal{P}_{\mathrm{LMod}_a(\mathcal{V})}(\mathcal{C})$ is the extension of scalars along U. But since U is right adjoint to F, we obtain the desired result. ☐

Corollary 3.3. For $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$ and a an $\mathbb{E}_2$-algebra in it, a $\mathrm{LMod}_a(\mathcal{V})$-enriched category $\mathcal{C}$ is Cauchy-complete iff its underlying $\mathcal{V}$-category $U_!\mathcal{C}$ is Cauchy-complete.

Proof. An enriched category is Cauchy-complete iff any tiny presheaf over it is representable, so combine Lemma 3.1 and Lemma 3.2. ☐

Remark 3.4. For $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$ presentably monoidal, a $\mathcal{V}$-enriched category $\mathcal{C}$ is called *Cauchy-complete* iff any tiny presheaf in $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is representable, i.e. the canonical inclusion $\mathcal{C} \subseteq \mathcal{P}_{\mathcal{V}}(\mathcal{C})^{\mathrm{tiny}}$ is an equivalence. Equivalently, $\mathcal{C}$ must admit all absolute weighted colimits.

Example 3.5. If a is an algebra in $\mathcal{V}$, then the tiny objects in $\mathrm{LMod}_a(\mathcal{V}) \in \mathrm{Pr}_{\mathcal{V}}$ are precisely the dualizable objects. This is because by definition, a left a-module m admits a dual if there is a right a-module $m^\vee$ such that $m \otimes - : \mathcal{V} \leftrightarrows \mathrm{LMod}_a(\mathcal{V}) : m^\vee \otimes_a -$ are adjoint. But then by definition $\underline{\mathrm{Hom}}_{\mathrm{LMod}_a(\mathcal{V})}(m, -) \simeq m^\vee \otimes_a -$, so this preserves colimits and tensoring. Conversely we can write any module n as a bar construction $a \otimes_a m$, so if m is tiny $\underline{\mathrm{Hom}}_{\mathrm{LMod}_a(\mathcal{V})}(m, n) \simeq \underline{\mathrm{Hom}}_{\mathrm{LMod}_a(\mathcal{V})}(m, a) \otimes_a n =: m^\vee \otimes_a n$.

Reminder 3.6. Let us call a category enriched over the large-presentable category $\widehat{\mathcal{C}at}^{\mathrm{colim}}$ a *locally cocomplete* 2*-category*, and a $\widehat{\mathcal{C}at}^{\mathrm{colim}}$-enriched functor *locally cocontinuous.* Explicitly, $\widehat{\mathcal{C}at}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \subseteq \widehat{\mathcal{C}at}_{(\infty,2)}$ is the subcategory spanned by those large 2-categories whose morphism categories admit small colimits and whose composition preserves them, together with 2-functors that on morphism-categories preserve small colimits.

From lax matrix calculus, we know that any locally cocontinuous functor preserves lax colimits over small categories, i.e. those are *absolute.* Conversely, in WIP David and me show that a $\widehat{\mathcal{C}at}^{\mathrm{colim}}$-enriched category is Cauchy-complete, i.e. admits all absolute colimits, iff it is idempotent complete and admits all lax colimits over small categories.

Definition 3.7. For $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$, we define the category of *lax* $\mathcal{V}$*-additive* $(\infty,2)$*-categories* as $\mathcal{C}auchy\mathcal{C}at(\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}))$. Explicitly by Corollary 3.3, it consists of $(\infty,2)$-categories that are locally cocomplete, locally tensored over $\mathcal{V}$ in a way that is compatible with composition and local colimits, and admits lax colimits and idempotent splittings.

Example 3.8. Let us note several cases of interest:

- A lax $\mathcal{S}$-additive $(\infty,2)$-category is a locally cocomplete $(\infty,2)$-category with lax colimits and idempotent splittings, also known as an i.c. *lax semiadditive* $(\infty,2)$*-category.*
- A lax Set-additive $(\infty,2)$-category is an i.c. locally cocomplete $(2,2)$-category with lax colimits, so we call it an i.c. *lax semiadditive* $(2,2)$*-category.*
- A lax $\mathcal{S}p$-additive $(\infty,2)$-category is an i.c. lax semiadditive $(\infty,2)$-category that is locally tensored over $\mathcal{S}p$, which by Proposition 2.4 means that it is locally stable. Hence, we recover *lax additive* $(\infty,2)$*-categories.*
- A lax Ab-additive $(\infty,2)$-category using Proposition 2.7 is a lax semiadditive $(2,2)$-category that is locally additive.
- Similarly for $\mathcal{S}_{\leq m}, \mathcal{S}_{\leq m,*}, \mathcal{S}p_{\leq m}$ we obtain locally semiadditive $(m+2,2)$-categories (that are locally pointed/ additive).
- One should consider lax $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$-additive $(\infty,2)$-categories as *lax additive* $(\infty,3)$*-categories.* This is because they are enriched over $\mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ which is the $\mathrm{Ind}_{\sqcap}$-completion of the category $\mathrm{Pr}^2_{\mathrm{st}}$ of presentable stable 2-categories introduced in [Ste20]; just like $\mathrm{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is the $\mathrm{Ind}_{\sqcap}$-completion of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ in the lax additive case. This further suggests calling lax $\mathrm{Pr}^n_{\mathrm{st}}$-additive $(\infty,2)$-categories *lax additive* $(\infty,n+2)$*-categories.*

Observation 3.9. Any lax $\mathcal{V}$-additive $(\infty,2)$-category is automatically 2-idempotent complete (which is a priori a stronger condition). This is because any cocomplete category is idempotent complete, so the forgetful functor $\mathrm{LMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \to \widehat{\mathcal{C}at}^{\mathrm{colim}} \to \widehat{\mathcal{C}at}$ factors through $\widehat{\mathcal{C}at}^{\mathrm{idem}}$. Since all of these functors are right adjoints of monoidal functors, change-of-enrichment along them preserves Cauchy-completeness, in particular if $\mathbb{C} \in \mathcal{C}auchy\mathcal{C}at(\mathrm{LMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}))$ then the underlying $\widehat{\mathcal{C}at}^{\mathrm{idem}}$-enriched category is Cauchy-complete, i.e. 2-idempotent complete.

4. Universal Property of Profunctors

Definition 4.1. Denote by $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ the full sub-2-category of $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ spanned by the tiny-generated categories, i.e. those of the form $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$ for $\mathcal{C}$ a small $\mathcal{V}$-enriched category. We call it the 2*-category of* $\mathcal{V}$*-enriched profunctors.*

Example 4.2. For $\mathcal{V} = \mathcal{S}$, this agrees with Haugseng's Morita 2-category $\mathbb{P}\mathrm{rof}^H_{\mathcal{S}}$ of profunctors in [Hau15]: Consider the corepresentable 2-presheaf $\mathrm{Hom}_{\mathbb{P}\mathrm{rof}^H_{\mathcal{S}}}(*,-) : \mathbb{P}\mathrm{rof}^H_{\mathcal{S}} \to \mathbb{C}\mathrm{at}$. It sends a small category C to $\mathcal{P}(\mathrm{C})$, and a profunctor $P : \mathrm{C} \times \mathrm{D}^{\mathrm{op}} \to \mathcal{S}$ to the postcomposition $P \circ - : \mathcal{P}(\mathrm{C}) \to \mathcal{P}(\mathrm{D})$. It is immediate to see that this construction factors through $\mathbb{P}\mathrm{r}^L$, where it is fully faithful as it induces the equivalence $\mathrm{Fun}(\mathrm{C} \times \mathrm{D}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^L(\mathcal{P}(\mathrm{C}), \mathcal{P}(\mathrm{D}))$ on morphism categories. Also, its essential image consists of precisely the presheaf categories, as claimed.

more general proof?

Warning 4.3. The 2-category $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ does *not* admit all (conical) colimits, in fact its underlying 1-category is not even idempotent complete since regarded as a full subcategory of $\mathrm{Pr}_{\mathcal{V}}$, it is not closed under retracts: For $\mathcal{V} = \mathcal{S}$ a counterexample is given in [Har], for $\mathcal{V} = \mathcal{S}p$ there is a large supply of compactly assembled stable categories that are not compactly generated, e.g. consult [Efi24]. However, $\mathrm{Pr}_{\mathcal{V}}$ *is* idempotent complete, so the idempotent completion $\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}$ can be identified with the full subcategory of $\mathbb{P}\mathrm{r}_{\mathcal{V}}$ spanned by the retracts of tiny-generated categories.

Proposition 4.4. Given $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, a module $\mathcal{M} \in \mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is dualizable iff it is the retract of a tiny-generated category.

Proof. This statement is known to hold in $\mathrm{RMod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ by [Ram24, Theorem 1.47], so it suffices to show that any dualizable $\mathcal{M}$ is automatically presentable. By Proposition 1.3 is suffices to show that $\mathcal{M}$ is $\sqcap$-compact, which follows from

$$\mathrm{Map}_{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(\mathcal{M}, -) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{S}, \underline{\mathrm{Hom}}_{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(\mathcal{M}, -)) \simeq \mathrm{Map}_{\widehat{\mathcal{C}at}^{\mathrm{colim}}}(\mathcal{S}, \mathcal{M}^{\vee} \otimes_{\mathcal{V}} -)$$

sine $\mathcal{M}^{\vee} \otimes_{\mathcal{V}} -$ preserves all colimits, and $\mathcal{S} \in \widehat{\mathcal{C}at}^{\mathrm{colim}}$ is $\sqcap$-compact since it is presentable. ☐

Theorem 4.5. The idempotent completion of the $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ of $\mathcal{V}$-enriched profunctors is both the free i.c. lax semiadditive category on the delooping $B\mathcal{V}$, and it is the free lax $\mathcal{V}$-additive category on the point. By this we mean that for $\mathbb{C}$ i.c. lax semiadditive and $\mathbb{D}$ i.c. lax $\mathcal{V}$-additive:

$$\mathrm{Fun}^{\mathrm{loc.coc.}}(\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}, \mathbb{C}) \simeq \mathrm{Fun}^{\mathrm{loc.coc.}}(B\mathcal{V}, \mathbb{C})$$
$$\mathrm{Fun}^{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(\widehat{\mathbb{P}\mathrm{rof}}^{\mathrm{ic}}_{\mathcal{V}}, \mathbb{D}) \simeq \mathrm{Fun}^{\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})}(B\mathcal{V}, \mathbb{D}) \simeq \mathrm{Fun}(*, \mathbb{D}) \simeq \mathbb{D}$$

Proof. Note that $B\mathcal{V}$ is the free $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$-enriched category on the point, since $\mathcal{V}$ is the image of $*$ under the left adjoint to the forgetful functor $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}}) \to \mathcal{C}at$. Hence, it suffices to show that $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ is the Cauchy-completion of $B\mathcal{V}$ both regarded as a $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$-enriched category and as a $\widehat{\mathcal{C}at}^{\mathrm{colim}}$-enriched category. However in both settings, its enriched presheaf category is given by $\mathrm{RMod}_{\mathcal{V}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$, and the tiny objects agree with the dualizable objects by **??**. Hence we are finished after combining Proposition 4.4 with Example 3.5. ☐

Corollary 4.6. The $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}$ of profunctors is the free lax semiadditive category on $B\mathcal{S}$, or on the point.

Proof. Up to idempotent completion this is immediate from the above theorem. For the full statement, use Angus' results that $\mathbb{P}\mathrm{rof}$ admits all lax colimits, and generated under lax colimits by the point. ☐

Proposition 4.7. The idempotent completion of the $(\infty, 2)$-category $\mathbb{P}\mathrm{rof}^{\mathrm{ex}}$ of stable categories and exact profunctors (in other words, the category of compactly assembled stable categories and colimit-preserving functors) is the free i.c. lax additive category on the point, and the free i.c. lax semiadditive category on $B\mathcal{S}p$.

Proof. Combine Theorem 4.5 with the observation that the category of $\mathcal{S}p$-enriched profunctors is equivalent to its full subcategory on (i.c.) stable categories since any $\mathcal{S}p$-enriched category is equivalent to its Cauchy-completion, which lies in there. Also, $\mathcal{S}p$-enriched functors between stable categories are the same thing as exact functors. $\square$

Remark 4.8. A variation of Angus' proof suggests that this is still true without idempotent completing: Any stable category can be written as the lax colimit in $\mathbb{P}\mathrm{rof}^{\mathrm{ex}}$ over itself, and this full subcategory of $\mathbb{P}\mathrm{r}^L_{\mathrm{st}}$ is closed under lax colimits.

Warning 4.9. It is unclear whether $\mathbb{P}\mathrm{rof}_{\mathcal{V}}$ is the free lax $\mathcal{V}$-additive category on the point (or equivalently the free lax semiadditive category on $B\mathcal{V}$) for any $\mathcal{V}$. The issue is that using lax colimits over small categories, we can a priori only generate $\mathcal{V}$-enriched categories that are freely generated by categories. More general $\mathcal{V}$-categories could be generated using Eilenberg-Moore objects, but those can to our knowledge only be written as partially lax colimits/ lax colimits over lax functors.

5. Lax additivity and higher presentable categories

There are several notions of 2-categories that, in some aspects, appear similar to stable categories:

- The category of *finitely lax additive 2-categories* $\mathcal{C}auchy\mathcal{C}at^2(\mathcal{S}p)$, which explicitly consists of 2-idempotent complete locally stable 2-categories that admit lax colimits over Δ^1,
- The category $\mathcal{C}auchy\mathcal{C}at(\mathrm{Mod}_{\mathcal{S}p}(\widehat{\mathcal{C}at}^{\mathrm{colim}})) = \mathcal{C}auchy\mathcal{C}at(\widehat{\mathcal{C}at}^{\mathrm{colim,st}})$ of *lax additive categories*,
- The category $\mathrm{Pr}_{\mathcal{C}at^{\mathrm{st,ic}}}$ of *presentably* $\mathcal{C}at^{\mathrm{st,ic}}$*-tensored categories*,
- The category $\mathrm{Pr}^2_{\mathrm{st}} := \mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})^{\mathfrak{n}\text{-cpt}}$ of *presentable stable 2-categories*.

All of them can be regarded as 3-categories, as we will see below. It would be nice to have some functors translating between these theories. Let us develop them for general $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^{\mathrm{L}})$, where all of the above constructions make sense as well:

Construction 5.1. To any Cauchy-complete $\mathcal{V}$-enriched 2-category $\mathcal{C} \in \mathcal{C}auchy\mathcal{C}at^2(\mathcal{V})$, we can associate its enriched presheaf category $\mathcal{P}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}(\mathcal{C}) \in \mathrm{Pr}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}$. The functor $\mathcal{P}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})} : \mathcal{C}auchy\mathcal{C}at^2(\mathcal{V}) \to \mathrm{Pr}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}$ is symmetric monoidal by [RZ24b], so we can regard it as a $\mathcal{C}auchy\mathcal{C}at^2(\mathcal{V})$-linear so in particular $\mathcal{C}at_{(\infty,2)}$-linear functor. The $\mathcal{C}at_{(\infty,2)}$-tensoring on both sides is closed, with internal Hom given by 2-categories of $\mathcal{C}auchy\mathcal{C}at(\mathcal{V})$-enriched functors and $\mathcal{C}auchy\mathcal{C}at(\mathcal{V})$-linear cocontinuous functors respectively. Hence we can regard $\mathcal{P}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}$ as a 3-functor that is actually a sub-3-category inclusion inducing equivalences on categories of 2-morphisms, with essential image the tiny-generated categories and internally left adjoint functors. In particular, it factors through the dualizable objects and left adjoint 1-morphisms.

Construction 5.2. Given a presentably $\mathcal{C}auchy\mathcal{C}at(\mathcal{V})$-tensored category, we can extend scalars along the symmetric monoidal functor $\mathcal{P}_{\mathcal{V}} : \mathcal{C}auchy\mathcal{C}at(\mathcal{V}) \to \mathrm{Pr}_{\mathcal{V}}$ in $\mathrm{CAlg}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ and view it as a cocompletely $\mathrm{Pr}_{\mathcal{V}}$-tensored category. Note that the extension-of-scalars-functor $- \otimes_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})} \mathrm{Pr}_{\mathcal{V}} : \mathrm{Mod}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathrm{Pr}_{\mathcal{V}}}(\widehat{\mathcal{C}at}^{\mathrm{colim}})$ is symmetric monoidal, so it induces a functor of $\mathrm{Mod}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}(\mathrm{Pr}^{\mathrm{L}})$-tensored categories and can in particular be seen as a 3-functor (since both induced tensorings over $\mathcal{C}at_{(\infty,2)}$ admit internal Homs). Also the image of any $\mathcal{M} \in \mathrm{Pr}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})}$ is a geometric realization of objects

$\mathcal{M} \otimes \mathcal{C}auchy\mathcal{C}at(\mathcal{V}) \otimes \cdots \otimes \mathcal{C}auchy\mathcal{C}at \otimes \mathrm{Pr}_{\mathcal{V}}$, so since $\mathcal{M}$ and $\mathcal{C}auchy\mathcal{C}at(\mathcal{V})$ are presentable (using [RZ24a, ...]) using Proposition 1.3 we learn that our functor factors through the ת-compact objects, hence inducing a 3-functor $\mathrm{Pr}_{\mathcal{C}auchy\mathcal{C}at(\mathcal{V})} \to \mathrm{Pr}^2_{\mathcal{V}}$.

Construction 5.3.

TODO: How do lax V-additive categories fit in?

References

[CDH+20] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian k-theory for stable ∞-categories i: Foundations, 2020.

[Efi24] Alexander I Efimov. K-theory and lozalizing invariants of large categories. *arXiv preprint arXiv:2405.12169*, 2024.

[Har] Yonatan Harpaz. Dualizable object in the category of locally presentable categories. MathOverflow.

[Har16] Yonatan Harpaz. Higher semiadditivity, higher monoids, and theinfinity-category of finite spans. 2016.

[Hau15] Rune Haugseng. Bimodules and natural transformations for enriched ∞-categories. *arXiv preprint arXiv:1506.07341*, 2015.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. Higher algebra. 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. *preprint*, 2018.

[Ram24] Maxime Ramzi. The formal theory of dualizable presentable ∞-categories. Online draft, 2024.

[Rez21] Charles Rezk. Generalizing accessible ∞-categories. Online preprint, 2021.

[RZ24a] David Reutter and Markus Zetto. Cauchy-complete (∞, n)-categories and higher idempotents, 2024.

[RZ24b] David Reutter and Markus Zetto. Enriched ∞-categories as marked modules, 2024.

[Ste20] Germán Stefanich. Presentable (∞, n)-categories. *arXiv preprint arXiv:2011.03035*, 2020.