Corollary 2.26. For $\mathcal{V} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Pr}^{\mathrm{L}})$ and a an $\mathbb{E}_2$-algebra in it, a $\mathrm{LMod}_a(\mathcal{V})$-enriched category $\mathcal{C}$ is Cauchy-complete iff its underlying $\mathcal{V}$-category $U_!\mathcal{C}$ is Cauchy-complete.

Proof. An enriched category is Cauchy-complete iff any atomic presheaf over it is representable, so combine Lemma 2.24 and Lemma 2.25. □

2.3. **Characterization using the norm map.**

Notation 2.27. Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in v\mathcal{C}at(\mathcal{V})$ be valent $\mathcal{V}$-categories. We refer to

$$\mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{P}_{\mathcal{V}}(\mathcal{D}))$$

as the category of $\mathcal{V}$-*enriched profunctors* $\mathcal{C} \twoheadrightarrow \mathcal{D}$. Given profunctors $P : \mathcal{C} \twoheadrightarrow \mathcal{D}$ and $Q : \mathcal{D} \twoheadrightarrow \mathcal{E}$, we write $P \otimes_{\mathcal{D}} Q : \mathcal{C} \twoheadrightarrow \mathcal{E}$ for the composition $Q \circ P : \mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{P}_{\mathcal{V}}(\mathcal{E})$.

Observation 2.28. There is ample reason for this notation: By Eilenberg-Watts **??** a profunctor $P : \mathcal{C} \twoheadrightarrow \mathcal{D}$ is the same thing as a bimodule in ${}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{D}}(\mathrm{Fun}(ob\mathcal{C} \times ob\mathcal{D}, \mathcal{V}))$, and by [Lur17, Rem. 4.8.4.9] in this picture the composition

$$\otimes_{\mathcal{D}} : {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{D}}(\mathrm{Fun}(ob\mathcal{C}\times ob\mathcal{D}, \mathcal{V}))\times {}_{\mathcal{D}}\mathrm{Bimod}_{\mathcal{E}}(\mathrm{Fun}(ob\mathcal{D}\times ob\mathcal{E}, \mathcal{V})) \to {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{E}}(\mathrm{Fun}(ob\mathcal{C}\times ob\mathcal{E}, \mathcal{V}))$$

is given by the relative tensor product of bimodules. This can be written out as

$$\begin{aligned} P \otimes Q(c,e) &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} P \odot (\mathrm{Hom}_{\mathcal{D}})^{\odot n} \odot Q(c,e) \simeq \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname*{colim}_{(d_0,\dots,d_n)\in(ob\mathcal{D})^{\times n}} P(c,d_0) \otimes \mathrm{Hom}_{\mathcal{D}}(d_0,d_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{D}}(d_{n-1},d_n) \otimes Q(d_n,e) \end{aligned}$$

using [Lur17, Thm. 4.4.2.8] as well as our expression [**?**, Cor. 2.29] for the matrix product.

Example 2.29. A profunctor $B1_{\mathcal{V}} \twoheadrightarrow \mathcal{C}$ is the same thing as a module functor $\mathcal{V} \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, i.e. an enriched presheaf on $\mathcal{C}$. Similarly a profunctor $\mathcal{C} \twoheadrightarrow B1_{\mathcal{V}}$ can by identified as an enriched copresheaf on $\mathcal{C}$ using **??**. We obtain a canonical pairing sending an enriched presheaf $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ and an enriched copresheaf $V \in \mathcal{P}^{\vee}_{\mathcal{V}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{V})$ to the composition $W \otimes_{\mathcal{C}} V := V \circ W \in \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{V}, \mathcal{V}) \simeq \mathcal{V}$, which may as in Observation 2.28 be expanded as

$$W \otimes_{\mathcal{C}} V \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname*{colim}_{(c_0,\dots,c_n)\in(ob\mathcal{C})^{\times n}} W(c_0) \otimes \mathrm{Hom}_{\mathcal{C}}(c_0,c_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(c_{n-1},c_n) \otimes V(c_n) \,.$$

This is also referred to as the W-*weighted colimit* of V (imagined as an enriched functor $\mathcal{C} \to \mathcal{V}$).

Construction 2.30. For $\mathcal{C}, \mathcal{D}, \mathcal{E} \in v\mathcal{C}at(\mathcal{V})$ and $P : \mathcal{C} \twoheadrightarrow \mathcal{D}$ an enriched profunctor, the composition maps

$$\begin{aligned} - \otimes_{\mathcal{C}} P = P \circ - &: \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{E}), \mathcal{P}_{\mathcal{V}}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{E}), \mathcal{P}_{\mathcal{V}}(\mathcal{D})) \\ P \otimes_{\mathcal{D}} - = - \circ P &: \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{D}), \mathcal{P}_{\mathcal{V}}(\mathcal{E})) \to \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{P}_{\mathcal{V}}(\mathcal{E})) \end{aligned}$$

preserve colimits and hence admit rights adjoints, which we denote by $\underline{\mathrm{Nat}}_{\mathcal{D}}(P, -)$ and ${}_{\mathcal{C}}\underline{\mathrm{Nat}}(P, -)$ respectively. elaborate?

Example 2.31. We have seen in [**?**] that under Eilenberg-Watts, the identity functor $\mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ corresponds to the Yoneda bimodule $\text{よ}^{\mathcal{V}}_{\mathcal{C}} \in {}_{\mathcal{C}}\mathrm{Bimod}_{\mathcal{C}}(\mathrm{Fun}(X \times X, \mathcal{V}))$. In particular, for any $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ regarded as a profunctor $B1_{\mathcal{V}} \twoheadrightarrow \mathcal{C}$, the Yoneda-weighted colimit $W \otimes_{\mathcal{C}} \text{よ}^{\mathcal{V}}_{\mathcal{C}} = \mathrm{id}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})} \circ W \simeq W$ agrees with W. This is precisely the *coYoneda Lemma*: Any enriched presheaf is a weighted colimit of representable presheaves.

Observation 2.32. For $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, the functor $\underline{\mathrm{Nat}}_{\mathcal{C}}(W, -) : \mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{V}$ is right adjoint to $- \otimes_{B1_{\mathcal{V}}} W \simeq W \otimes - : \mathcal{V} \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, so it coincides with the internal Hom $\underline{\mathrm{Hom}}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})}(W, -)$ in $\mathcal{P}_{\mathcal{V}}(\mathcal{C})$.

Notation 2.33. For $P : \mathcal{C} \to\!\!\!\!\!\to \mathcal{D}$, denote by $\mathrm{id}_P : \mathrm{id}_{\mathcal{C}} \to \underline{\mathrm{Nat}}_{\mathcal{D}}(W, W)$ the map induced by the isomorphism $\mathrm{id}_{\mathcal{C}} \otimes_{\mathcal{C}} W \to W$, and dually by $\mathrm{id}_P : \mathrm{id}_{\mathcal{D}} \to {}_{\mathcal{C}}\underline{\mathrm{Nat}}(W, W)$ the map induced by the isomorphism $W \otimes_{\mathcal{D}} \mathrm{id}_{\mathcal{D}} \to W$. Further, note that adding $Q : \mathcal{C}' \to \mathcal{D}, R : \mathcal{C}'' \to\!\!\!\!\!\to \mathcal{D}$ as well as $Q' : \mathcal{C} \to \mathcal{D}', R' : \mathcal{C} \to\!\!\!\!\!\to \mathcal{D}''$ into the mix, there are canonical composition maps

$$\underline{\circ} : \underline{\mathrm{Nat}}_{\mathcal{D}'}(Q, R) \otimes_{\mathcal{C}'} \underline{\mathrm{Nat}}_{\mathcal{D}}(P, Q) \to \underline{\mathrm{Nat}}_{\mathcal{D}}(P, R)\ ,$$
$$\underline{\circ} : {}_{\mathcal{C}}\underline{\mathrm{Nat}}(P, Q') \otimes_{\mathcal{D}'} {}_{\mathcal{C}}\underline{\mathrm{Nat}}(Q', R') \to {}_{\mathcal{C}}\underline{\mathrm{Nat}}(P, R')\ .$$

Notation 2.34. Given any presheaf $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, we define the *norm map*

$$\mathrm{Nm} : W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \to \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W) \simeq \underline{\mathrm{Hom}}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})}(W, W)$$

induced by the counit ϵ of the adjunction $W \otimes - \dashv \underline{\mathrm{Nat}}_{\mathcal{C}}(W, -)$ as a mate to

$$W \otimes_{\mathcal{C}} \epsilon : W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \to W \otimes_{\mathcal{C}} \text{よ}^{\mathcal{V}} \simeq W\ .$$

Alternatively applying $W \simeq \underline{\mathrm{Nat}}_{\mathcal{C}}(\text{よ}^{\mathcal{V}}, W)$, it agrees with the composition map

$$- \otimes_{\mathcal{C}} - : \underline{\mathrm{Nat}}_{\mathcal{C}}(\text{よ}^{\mathcal{V}}, W) \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \to \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)\ .$$

Theorem 2.35. Let $\mathcal{C} \in \mathcal{C}at(\mathcal{V})$ and $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$, then the following are equivalent:

(1) W is atomic,
(2) The norm map $\mathrm{Nm} : W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \to \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)$ is an isomorphism,
(3) There exists a dashed lift in the following diagram:

$$\begin{tikzcd}
 & W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \arrow[d, "\mathrm{Nm}"] \\
1_{\mathcal{V}} \arrow[ur, dashed] \arrow[r, "\mathrm{id}_W"'] & \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)
\end{tikzcd}$$

Remark 2.36. Equivalently, the pullback $1_{\mathcal{V}} \times_{\underline{\mathrm{Nat}}_{\mathcal{C}}(W,W)} (W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}})) \to 1_{\mathcal{V}}$ must admit a section.

Remark 2.37. If we write $U_{\mathcal{V}} := \mathrm{Map}(1_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$, we may rephrase this as saying the full image of the induced map

$$U_{\mathcal{V}}(W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}})) \to \mathrm{Map}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})}(W, W)$$

contains the identity id_W.

Proof. For (1) $\Rightarrow$ (2) assume W is atomic, so the associated map $\mathcal{V} \to \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is internally left adjoint meaning there is some copresheaf $W^{\vee} : \mathcal{P}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{V}$ such that the composition functors

$$- \otimes W : \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{V}) \rightleftarrows \mathrm{Fun}^{\mathrm{L}}_{\mathcal{V}}(\mathcal{P}_{\mathcal{V}}(\mathcal{C}), \mathcal{P}_{\mathcal{V}}(\mathcal{C})) : - \otimes_{\mathcal{C}} W^{\vee}$$

Further the (co)units of both adjunctions are adjoint. By uniqueness of adjoints $- \otimes_{\mathcal{C}} W^{\vee} \simeq \underline{\mathrm{Nat}}_{\mathcal{C}}(W, -)$ and the (co)units of both adjunctions are isomorphic; in particular applying both functors to the identity $\text{よ}^{\mathcal{V}}_{\mathcal{C}}$ we learn $W^{\vee} \simeq \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}})$. By definition of the norm map we need to show that the counit $W \otimes_{\mathcal{C}} \epsilon : W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}) \otimes W \to W$ exhibits $W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}) \simeq W \otimes_{\mathcal{C}} W^{\vee}$ as pointwise right adjoint to $- \otimes W$ at W, but this follows from the adjunction data of $- \otimes W \dashv - \otimes_{\mathcal{C}} W^{\vee}$ and associativity of the relative tensor product.

Since (2) $\Rightarrow$ (3) is clear, we finish by proving (3) $\Rightarrow$ (1). Let $\eta : 1_{\mathcal{V}} \to W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}})$ be the assumed lift, and $\epsilon : \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \to \text{よ}^{\mathcal{V}}$ the unit of the adjunction $- \otimes W \dashv \underline{\mathrm{Nat}}_{\mathcal{C}}(W, -)$. In order to show that they exhibit $\underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}})$ as the adjoint profunctor to W, we must verify the triangle identities. On the one hand, the diagram

$$\begin{tikzcd} W \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}_W \otimes \mathrm{id}"'] & W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \arrow[d, "\mathrm{Nm}\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes_{\mathcal{C}} \epsilon"] & \\ & \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W) \otimes W \arrow[r] & W \end{tikzcd}$$

commutes by construction of Nm and assumption on η. The second identity is about maps into $\underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)$, so after currying it suffices to note that the composite map in the commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, equal] & \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \otimes_{\mathcal{C}} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \arrow[r, "\epsilon\otimes_{\mathcal{C}}\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\epsilon"] & \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \arrow[d, "\epsilon"] \\ & \underline{\mathrm{Nat}}_{\mathcal{C}}(W, \text{よ}^{\mathcal{V}}) \otimes W \arrow[r, "\epsilon"] & \text{よ}^{\mathcal{V}} \end{tikzcd}$$

is ϵ. The left triangle commutes by the first triangle identity. $\square$

Remark 2.38. This may be regarded as a specialization of the diagrammatic absoluteness criterion for profunctors in [?].

Corollary 2.39. In particular, to verify some given W is atomic it is sufficient (but not necessary) to specify a lift of $1 \to \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)$ through

$$\operatorname*{colim}_{c \in ob\mathcal{C}} \underline{\mathrm{Nat}}(W, \text{よ}^{\mathcal{V}}_c) \otimes W(c) \to \underline{\mathrm{Nat}}(W, \text{よ}^{\mathcal{V}}) \otimes_{\mathcal{C}} W \overset{\mathrm{Nm}}{\to} \underline{\mathrm{Nat}}_{\mathcal{C}}(W, W)$$

where the first functor is part of the geometric realization Example 2.29 calculating the subsequent weighted colimit.

The converse is true if we assume that $1_{\mathcal{V}} \in \mathcal{V}$ is projective, i.e. the forgetful functor $U_{\mathcal{V}} = \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$ preserves geometric realizations. Then $W \in \mathcal{P}_{\mathcal{V}}(\mathcal{C})$ is atomic *iff* such a lift exists: Since $\mathcal{S}$ is a topos the map

$$\mathrm{Map}_{\mathcal{V}}\left(1_{\mathcal{V}}, \operatorname*{colim}_{c \in \iota\mathcal{C}} \underline{\mathrm{Nat}}(W, \text{よ}^{\mathcal{V}}_c) \otimes W(c)\right) \to \mathrm{Map}_{\mathcal{V}}\left(1_{\mathcal{V}}, \underline{\mathrm{Nat}}(W, \text{よ}^{\mathcal{V}}) \otimes_{\mathcal{C}} W\right) \simeq$$

$$\simeq \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}}\left(1_{\mathcal{V}}, \operatorname*{colim}_{c_0, \ldots, c_n \in \iota\mathcal{C}} \underline{\mathrm{Nat}}(W, \text{よ}^{\mathcal{V}}_{c_0}) \otimes \mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \otimes \cdots \otimes W(c_n)\right)$$

is an effective epimorphism, i.e. surjective on connected components[2].

keep footnote? Cite the related [Lur09, Lem. Lemma 6.2.3.13]?

Corollary 2.40. If $1_{\mathcal{V}}$ is completely compact, i.e. $U_{\mathcal{V}} = \mathrm{Map}_{\mathcal{V}}(1_{\mathcal{V}}, -)$ preserves colimits, then W is atomic iff there exists some $c \in \mathcal{C}$ such that id_W lies in the full image of the composition map

$$U_{\mathcal{V}}(\underline{\mathrm{Nat}}(W, \text{よ}_c) \otimes \underline{\mathrm{Nat}}(\text{よ}_c, W)) \to \mathrm{Map}_{\mathcal{P}_{\mathcal{V}}(\mathcal{C})}(W, W)$$

[2]If $X_\bullet : \Delta^{op} \to \mathcal{S}$ and $x : * \to X := \mathrm{colim}_{\Delta^{\mathrm{op}}} X_\bullet$ is a point in its geometric realization, then since colimits are universal the pullback $* \times_X X_\bullet$ has colimit $*$. So it is impossible to have $* \times_X X_0 = \emptyset$ since then this colimit would also be empty.