Cauchy Completions and Lax Additivity

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Motivation: A *lax semiadditive* $(\infty, 2)$-cat. is a locally cocomplete large $(\infty,2)$-cat $\mathbb{C}$ admitting lax colimits over small ∞-categories

2-cat. (= $(\infty,2)$)

Hom-categories admit small colims, composition preserves them (= locally cocomplete)

categories (= ∞-categories)

It is *lax additive* if it is locally stable.

Fact: In both cases, lax limit & colimit of any diagram coincide. Further they are absolute, i.e. any locally cocontinuous functor preserves them. } Lax matrix calculus

Thm [Angus, WIP] $\mathbb{P}\mathrm{rof}$ is the free lax semiadditive category generated by the point, in the sense that $\forall\, \mathbb{D}$ lax semiadd. we have

$$\mathrm{Fun}^{\text{loc. cocont}}(\mathbb{P}\mathrm{rof}, \mathbb{D}) \simeq \mathrm{Fun}^{\text{loc. cocont.}}(\mathbb{B}\mathcal{S}, \mathbb{D}) \simeq \mathbb{D}$$

Q:
- What is the free lax additive category? Maybe $\mathbb{P}\mathrm{rof}^{\mathrm{ex}}$?
- What about similar categories, e.g. enriched profunctors?
- Conceptual reason?

→ Will answer those, up to some caveats.

[$\mathcal{V}$ presentably monoidal ∞-category, i.e. closed monoidal, cocomplete, presentable (set-theoretic condition)]

Enriched ∞-categories & Cauchy-completions

Idea: Algebras / Monoids are 1-object categories!

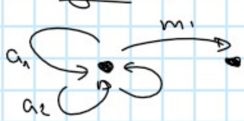
BA $a_1 a_2 a_3$

composition

algebra A $A^{\otimes n} \to A$

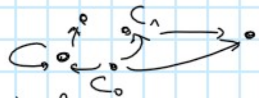
$\mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(c_{n-1}, c_n) \to \mathrm{Hom}_{\mathcal{C}}(c_0, c_n)$

category $\mathcal{C}$



$\rightsquigarrow a_1 a_2 m$

left module $A^{\otimes n} \otimes M \to M$



$\mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \otimes \cdots \otimes M(c_n) \to M(c_0)$

presheaf

Algebras $\xleftarrow{\text{regard } A \text{ as 1-obj. category } BA}$ **Enriched ∞-categories**

Algebras	Enriched ∞-categories
algebra $A, B \in \mathrm{Alg}(\mathcal{V})$	enriched ∞-category $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}(\mathcal{V})$
left module ${}_A M$	presheaf ${}_{\mathcal{C}}M \in \mathrm{PSh}_{\mathcal{V}}(\mathcal{C})$ " $\mathcal{C}^{op} \to \mathcal{V}$ "
right —"— N_A	copresheaf $N_{\mathcal{C}}$
bimodule ${}_A P_B$	profunctor ${}_{\mathcal{C}}P_{\mathcal{D}}$ (differentiate)
algebra homom. $f: A \to B$	enriched functor $f: \mathcal{C} \to \mathcal{D}$
$\mathrm{LMod}_A(\mathrm{Ab}) \circlearrowleft \mathrm{Ab}$	$\mathrm{PSh}_{\mathcal{V}}(\mathcal{C}) \circlearrowleft \mathcal{V}$
$\mathrm{Hom}_A({}_A M, {}_A M') \in \mathrm{Ab}$	$\mathrm{Hom}({}_{\mathcal{C}}M, -): \mathrm{PSh}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{V}$

$\mathrm{Hom}_A({}_AM, {}_{A'}M) \in \mathrm{Ab}$	$\mathrm{Hom}({}_{\mathcal{C}}M, -) : \mathrm{PSh}_{\mathcal{V}}(\mathcal{C}) \to \mathcal{V}$
weighted limit	right adjoint to ${}_{\mathcal{C}}M \otimes - : \mathcal{V} \to \mathrm{PSh}_{\mathcal{V}}(\mathcal{C})$
$N_A \underset{A}{\otimes} {}_AM \in \mathrm{Ab}$	$N_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} {}_{\mathcal{C}}M \in \mathcal{V}$
$= \mathrm{coeq}(N \otimes A \otimes M \rightrightarrows M \otimes N)$	$= \underset{\Delta^{op}}{\mathrm{colim}}\left(\cdots \rightrightarrows \underset{c_0 \le c_1 \in \mathcal{C}^{\simeq}}{\mathrm{colim}} N(c_0) \otimes \mathrm{Hom}_{\mathcal{C}}(c_0,c_1) \otimes M(c_1) \downarrow\downarrow \underset{c_0 \in \mathcal{C}^{\simeq}}{\mathrm{colim}} N(c_0) \otimes M(c_0)\right)$
weighted colimit	
trivial bimodule ${}_AA_A$	Yoneda profunctor ${}_{\mathcal{C}}よ_{\mathcal{C}}$
$\mathrm{Nat}_A({}_AA_A, {}_AM) \simeq {}_AM$	Yoneda Lemma $\underset{\mathrm{PSh}_{\mathcal{V}}(\mathcal{C})}{\mathrm{Hom}}(よ_-, F) \simeq F$
${}_AA_A \otimes_A {}_AM \simeq {}_AM$	coYoneda Lemma ${}_{\mathcal{C}}よ_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} F \simeq F$
$f^*({}_BM')$ restriction of scalars,	$f^*_{\mathcal{D}}M' = $ "$M' \circ f$" precomposition
$f_*({}_AM) = B \underset{A}{\otimes} M$ extension –"–	$\mathrm{colim}^M(f) = M \underset{\mathcal{C}}{\otimes} f$ weighted colimit
${}_AM$ dualizable, i.e. $\exists M^\vee_A$ s.t.	${}_{\mathcal{C}}M \in \mathrm{PSh}_{\mathcal{V}}(\mathcal{C})$ tiny
$M \otimes - \dashv M^\vee \underset{A}{\otimes} -$	(Isbell dual $M^\vee_{\mathcal{C}}$)

Def $M \in \mathcal{P}^{\circlearrowleft \mathcal{V}}$ is called tiny $:\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{P}}(m, -)$ preserves $\underbrace{\text{colimits \& } \mathcal{V}\text{-tensoring}}_{\text{weighted colimits}}$

$\mathrm{Hom}_{\mathcal{P}}(m, m' \otimes v) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{M}}(m, m') \otimes v$

Ex/ For $\mathcal{P} = \mathcal{V}$ with $a \in \mathrm{Alg}(\mathcal{V})$, an element $m \in \mathcal{V}$ is tiny $\Leftrightarrow$ dualizable

Proof: Dualizable $:\Leftrightarrow \exists m^\vee \in \mathcal{V}$ st. $m^\vee \otimes - : \mathcal{V} \to \mathcal{V}$ r.a. to $m \otimes -$

But then $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, -) \simeq m^\vee \otimes -$ preserves colimits & tensoring.

Conversely $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, v) \simeq \underline{\mathrm{Hom}}_{\mathcal{M}}(m, 1 \otimes v) \simeq \underbrace{\underline{\mathrm{Hom}}_{\mathcal{M}}(m, 1)}_{m^\vee} \otimes v$ $\square$

Rem: Also true for $\mathcal{P} = \mathrm{LMod}_a(\mathcal{V}) = \mathrm{PSh}_{\mathcal{V}}(Ba)$ for $a \in \mathrm{Alg}(\mathcal{V})$

Def A $\mathcal{V}$-category $\mathcal{C}$ is called Cauchy-complete $\Leftrightarrow$ Any tiny presheaf $F \in \mathrm{PSh}_{\mathcal{V}}(\mathcal{C}) \circlearrowleft \mathcal{V}$ is representable. Otherwise, $\mathrm{PSh}_{\mathcal{V}}(\mathcal{C})^{\text{tiny}} =: \widehat{\mathcal{C}}^{\mathcal{V}}$ is called its Cauchy-completion. Denote $\mathrm{Cat}_+(\mathcal{V}) \subseteq \mathrm{Cat}(\mathcal{V})$ their full subcat.

Rem: Equivalently, $\mathcal{C}$ must admit absolute weighted colimits.

Ex/
- If $\mathcal{C} = B1 = \bullet \circlearrowleft 1$, then $\widehat{B1}^{\mathcal{V}} = \mathrm{Fun}^{\mathcal{V}}(B1, \mathcal{V})^{\text{tiny}} = \mathcal{V}^{\text{dual}}$
- Similarly $\widehat{Ba}^{\mathcal{V}} = \mathrm{Fun}^{\mathcal{V}}(Ba, \mathcal{V})^{\text{tiny}} = \mathrm{LMod}_a(\mathcal{V})^{\text{dual}}$
- If $\mathcal{V} = \mathrm{Set}$ or $\mathcal{S}$, then $F \in \mathrm{PSh}(C)$ is tiny $\Leftrightarrow$ retract of a repres. presheaf. Hence $\widehat{\mathcal{C}}^{\mathcal{S}} = \widehat{\mathcal{C}}^{\text{i.c.}}$, and Cauchy-complete $\Leftrightarrow$ i.c.
- If $\mathcal{V} = \mathrm{Ab}$ or Sp_{cn}, then $F \in \mathrm{PSh}_{\mathrm{Ab}}(C)$ tiny $\Leftrightarrow$ retract of a finite $\oplus$ of representables. In particular Cauchy-complete $\Leftrightarrow$ i.c. additive
- If $\mathcal{V} = \mathrm{Sp}$, then $\mathrm{Cat}_+(\mathrm{Sp}) = \{\text{i.c. stable cats}\}$
- For $\mathrm{Mod}_a(\mathcal{V})$ with $a \in \mathrm{CAlg}(\mathcal{V})$, can regard $\mathcal{P} \circlearrowleft \mathrm{Mod}_a(\mathcal{V}) \xleftarrow{F} \mathcal{V}$ as

- If $\mathcal{V} = Sp$, then $Cat_+(Sp) = \{$i.c. stable cats$\}$
- For $Mod_a(\mathcal{V})$ with $a \in CAlg(\mathcal{V})$, can regard $\mathcal{J} \circlearrowleft Mod_a(\mathcal{V}) \underset{u}{\overset{F}{\leftrightarrows}} \mathcal{V}$ as $\mathcal{V}$-tensored via $m \otimes - := m \otimes F(-) : \mathcal{V} \to \mathcal{J}$

Then $\underline{Hom}_{F^*\mathcal{J}}(m,-) \simeq U \circ \underline{Hom}_{\mathcal{J}}(m,-)$, with U conservative so it preserves & reflects colimits & tensoring $\leadsto$ $tiny_{Mod_a(\mathcal{V})} \Leftrightarrow tiny_{\mathcal{V}}$

turns out $\Rightarrow$ $Cat_+(Vec_k)$ = i.c. k-lin. additive cats, $Cat_+(D(R))$ = i.c. R-lin. stable cats

- Define $Cat_{(n,m)} := Cat(Cat_{(n-1,m-1)})$ where $1 \leq m \leq n \leq \infty$ and $Cat_{(n,0)} := \mathcal{S}_{\leq n}$. Then $Cat_+(Cat_{(n-1,m-1)}) = \{$i.c. (n,m)-categories$\}$
- $Cat(Cat^{ic})$ = "2-idemp. complete 2-categories"

What about Cat (categories with $\mathcal{K}$-colimits)? Or $Cat(Cat^{ex})$?

- If $\mathcal{K} \subseteq \{$groupoids$\}$, $\{$i.c. "locally $\mathcal{K}$-coc. 2-categories with $\mathcal{K}$-colimits"$\}$
- As it turns out, $Cat_+(Cat^{colim})$ = i.c. locally cocomplete 2-cats with lax colimits over small 1-cats!
- General $\mathcal{K}$ is difficult, e.g. $Cat(Cat^{ex}) = \left\{\begin{matrix}\text{i.c. locally stable 2-cats} \\ \text{with lax colimits over } \Delta^{\wedge}\end{matrix}\right\}$

Lax Semiadditivity

<u>Have learnt</u>: Lax colimits & idempotent splittings generate <u>all</u> absolute colimits over $Cat(Cat^{colim})$.

Def Let $\mathcal{V} \in CAlg(Cat^{colim})$. We call $Cat_+(Mod_{\mathcal{V}}(Cat^{colim}))$ the category of i.c. lax $\mathcal{V}$-additive categories.

Ex/ Explicitly: Locally cocomplete & $\mathcal{V}$-tensored, i.e., admits lax colimits

- Lax $\mathcal{S}$-additive = lax semiadditive
- Lax Sp-additive = lax additive
- Lax Vec_k-additive = lax semiadditive, k-linear, locally additive
- (Lax Pr_{st}-additive =: lax additive $(\infty,3)$-cat.)

The universal property of Profunctors

Def $Cat^{colim} \hookrightarrow \widehat{Cat}$ sub-2-category on cocomplete categories & cocontinuous functors.

$Prof \subseteq Cat^{colim}$ full sub-2-category on presheaf categories $PSh(C)$ (small C)

<u>Notice</u>: $Hom_{Prof}(PSh(C), PSh(D)) = Fun^L(PSh(C), PSh(D)) =$
$= Fun(C, Fun(D^{op}, \mathcal{S})) = Fun(C \times D^{op}, \mathcal{S})$

Similarly for $\mathcal{V} \in Alg(Pr^L)$, let $Prof_{\mathcal{V}} \subseteq Mod_{\mathcal{V}}(Cat^{colim})$ on enriched

$= \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})) = \mathrm{Fun}(\mathcal{C}\times\mathcal{D}^{op}, \mathcal{S})$

Similarly for $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^L)$, let $\mathrm{Prof}_{\mathcal{V}} \subseteq \mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})$ on enriched presheaf categories $\mathrm{PSh}_{\mathcal{V}}(\mathcal{C})$.

⚠ Warning: Unlike $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})$, $\mathrm{Prof}_{\mathcal{V}}$ is not idemp. compl., so $\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}} \subseteq \mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})$ spanned by retracts of enriched presheaf cats.

Ex/ For $\mathcal{V}=\mathrm{Sp}$, $\mathrm{Prof}_{\mathrm{Sp}} \subseteq \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L) = \mathrm{Pr}_{st}$ consists of the cptly. generated stable cats while $\widehat{\mathrm{Prof}}^{ic}_{\mathrm{Sp}}$ consists of the cptly assembled ones. ($\to$ Efimov K-theory)

Thm [Ramzi + extra steps] $\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}}$ consists of precisely the dualizable obj. in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})$

Thm $\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}}$ is both the free i.c. lax semiadditive category on $B\mathcal{V}$, and the free i.c. lax $\mathcal{V}$-additive category on the point:

$\mathrm{Fun}^{loc.coc.}(\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}}, \mathbb{C}) \simeq \mathrm{Fun}^{loc.coc}(B\mathcal{V}, \mathbb{C}) =$ "$\mathcal{V}$-modules in $\mathbb{C}$" $\forall \mathbb{C}$ i.c. lax sa.

$\mathrm{Fun}^{\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})}(\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}}, \mathbb{D}) \simeq \mathrm{Fun}^{\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})}(B\mathcal{V}, \mathbb{D}) \simeq \mathbb{D}$ $\forall \mathbb{D}$ i.c. lax $\mathcal{V}$-additive.

Proof: Must show $\widehat{\mathrm{Prof}}^{ic}_{\mathcal{V}} \simeq \widehat{B\mathcal{V}}^{\mathrm{Cat}^{colim}} \simeq \widehat{B\mathcal{V}}^{\mathrm{Mod}_{\mathcal{V}}(\mathrm{Cat}^{colim})}$

$\mathrm{LMod}_{\mathcal{V}}(\mathrm{Cat}^{colim})^{dual}$ $\rightsquigarrow$ use above theorem □

Rem: For $\mathcal{V}=\mathcal{S}$, Angus' arguments show $\mathrm{Prof} \subseteq \widehat{\mathrm{Prof}}^{ic}$ is closed under lax colimits & generated by them from the point $\Rightarrow$ Prof is the free lax semiadditive cat.

For general $\mathcal{V}$, this might be wrong...

Corollary: $\widehat{\mathrm{Prof}}^{ic}_{\mathrm{Sp}} \simeq \widehat{\mathrm{Prof}}^{ic}_{ex} = \left\{ \begin{array}{l} \text{small stable categories} \\ \text{exact profunctors} \end{array} \right\}$

is the free i.c. lax additive category.

Proof: A priori $\mathrm{Prof}_{ex} \subseteq \mathrm{Prof}_{\mathrm{Sp}}$, but ess. surj. since any Sp-enriched category is Morita-equivalent to its Cauchy-completion which is stable. □

Rem: Probably Prof_{ex} is the free lax additive category.

Free Cauchy-complete categories & Morita categories

$\mathrm{Cat}^{colim, ex}$ is a bit large. What if we use Cat^{ex} instead?

Def A 2-i.c. finitely lax additive category is a $\mathrm{Cat}_+(\underbrace{\mathrm{Cat}^{ex,ic}}_{\mathrm{Cat}_+(\mathrm{Sp})})$, i.e. a 2-i.c. locally stable 2-cat with lax colimits over Δ^1.

Similarly in the R-linear case, for $R \in \mathrm{CAlg}(\mathrm{Sp})$.

Thm The free $\mathrm{Cat}_+(\mathrm{Cat}^{R\text{-lin},ex,ic}) = \mathrm{Cat}^2_+(\mathrm{Mod}_R(\mathrm{Sp}))$ is

$\widehat{B\,\widehat{BR}^{\mathrm{Mod}_R(\mathrm{Sp})}}^{\mathrm{Cat}^{R\text{-lin},ex,ic}} = \widehat{B\,\mathrm{Mod}_R^{perf}(\mathrm{Sp})}^{\mathrm{Cat}_+(\mathrm{Mod}_R(\mathrm{Sp}))} \simeq \left\{ \begin{array}{l} \text{Morita category of} \\ \text{smooth proper R-algebras} \end{array} \right\}$

Thm/ The free $\mathrm{Cat}_+(\mathrm{Cat} \quad) = \mathrm{Cat}_+(\mathrm{Mod}_R(\mathrm{Sp}))$ is

$$\widehat{\mathcal{B}\,\widehat{\mathcal{B}R}^{\mathrm{Mod}_R(\mathrm{Sp})}}^{\mathrm{Cat}^{\mathrm{Rlin,ex,ic}}} = \widehat{\mathcal{B}\,\mathrm{Mod}_R^{\mathrm{perf}}(\mathrm{Sp})}^{\mathrm{Cat}_+(\mathrm{Mod}_R(\mathrm{Sp}))} \simeq \left\{ \begin{matrix} \text{Morita category of} \\ \text{smooth proper } R\text{-algebras} \end{matrix} \right\}$$

Ex/ For $R = Hk$, char $k = 0$ get smooth proper dga's

$\Rightarrow$ Explains their appearance in 2D TFTs, e.g. Landau-Ginzburg models.

For general $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$, can define a symm. mon. str. on $\mathrm{Cat}_+(\mathcal{V})$ s.t. $\widehat{\mathcal{B}1}$ is its unit. Hence, $\mathrm{Cat}_+ : \mathrm{CAlg}(\mathrm{Pr}^L) \to \mathrm{CAlg}(\mathrm{Pr}^L)$ [Ramzi + Reutter - Z.]

Can iterate this to obtain $\mathrm{Cat}_+^n(\mathcal{V})$ with unit $\widehat{\mathcal{B}\,\widehat{\mathcal{B}\cdots 1}} =: \Sigma_{\mathcal{V}}^n 1$

Thm/ $\Sigma_{\mathcal{V}}^n 1$ is (the universal) fully dualizable $\mathcal{V}$-enriched n-category.

Ex/ $\Sigma_{\mathrm{Vect}_k}^3 1 = \{$Morita 3-cat separable multifusion categories$\}$