Optimization on tensegrity structures

M. Sandnes, A. Skogøy, E. Winje, P. Vårheim

[a]Department of Mathematical Sciences, TMA 4180 - Optimization 1, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.

Abstract

A tensegrity structure is a configuration of nodes, bars and cables that stands upright based only on gravity and the stiffness of cables and bars. Initially, they were presented as art works, but have since been adapted to engineering, in particular as "smart structures," which can change their shape (by adjusting the length of the cables) depending on current requirements. Since the cables are of light materials, the idea of tensegrity has recently been applied to space based objects like satellites and space probes.[1]

Introduction

In this project we have studied an optimisation based model for the problem of form-finding of tensegrity structures. First we proved the existence of solutions for different constraints. Furthermore we proved convexity for cable nets and non-convexity from tensegrity domes. We later show that the optimisation problem is part of $\mathcal{C}^1$, but not in $\mathcal{C}^2$. We formulated the necessary and sufficient optimality conditions. Lastly we implemented a Quasi-Newton algorithm for solving the optimisation problem numerically.

1. Notation and definitions

We begin by collecting all node positions for the tensegrity structure in one matrix $X = (x^{(1)}, \ldots, x^{(N)}) \in \mathbb{R}^{3N}$. Where the position of node i is denoted as $x^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\right) \in \mathbb{R}^3$. To describe the organisation of a tensegrity structure, we model it as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, ..., N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The vertices represent the nodes in the structure, and an edge $e_{ij} = (i, j) \in \mathcal{E}$ indicates that the nodes i and j are connected through either a cable or bar.

External weight. For each node we have an external weight. The energy of all external weights are computed as such:

$$E_{ext}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)} \tag{1}$$

$m_i \geq 0$ is the mass of external node at position $x^{(i)}$ and $g \approx 9,81\text{m/s}^2$ is the acceleration of gravity near Earth's surface.

Individual cables. We now denote the elastic energy of a cable as $E_{elast}^{bar}(e_{ij})$ as a quadratic model:

$$E_{elast}^{cable}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2} \left(||x^{(i)} - x^{(j)}||_2 - \ell_{ij}\right)^2 & \text{if } \|x^{(i)} - x^{(j)}\|_2 > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\|_2 \leq \ell_{ij}, \end{cases} \tag{2}$$

where $k > 0$ is a material parameter and ℓ_{ij} is the resting length of the cable. We assume that each cable is light and the contribution from gravity is negligible.

Individual bars. We also have a contribution in elastic energy from the bars denoted as $E_{elast}^{bar}(e_{ij})$:

$$E_{elast}^{bar}(e_{ij}) = \frac{c}{2\ell_{ij}^2} \left(L(e_{ij}) - \ell_{ij}\right)^2 = \frac{c}{2\ell_{ij}^2} \left(||x^{(i)} - x^{(j)}||_2 - \ell_{ij}\right)^2, \tag{3}$$

Where $c > 0$ is a parameter depending on the material and the cross section of the bar, and ℓ_{ij} is the resting length of the bar.

In addition, each bar has gravitational energy denoted as $E^{bar}_{grav}(e_{ij})$:

$$E^{bar}_{grav}(e_{ij}) = \frac{\rho g \ell_{ij}}{2}\left(x_3^{(i)} + x_3^{(j)}\right) \tag{4}$$

ρ is the line density of the bar and $g \approx 9.81\text{m/s}^2$ is the gravitational acceleration near Earth's surface.

Total Energy. Denote $\mathcal{B}, \mathcal{C} \subset \mathcal{E}$ as the set of bars and cables. Then the total energy is given as

$$E(X) = \sum_{e_{i,j}\in\mathcal{B}} (E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})) + \sum_{e_{i,j}\in\mathcal{C}} E^{\text{cable}}_{\text{elast}}(e_{ij}) + E_{\text{ext}}(X) \tag{5}$$

Additional constraints. Observe that minimizing (5) will usually not admit a solution, since the function is unbounded from below. Thus we have to add some additional constraints. One idea is to add some fixed nodes $p^{(i)}$ such that

$$x^{(i)} = p^{(i)}, \qquad \text{for } i = 1, \dots M. \tag{6}$$

This means that we now solve a $3(N - M)$ lower dimensional, free optimization problem instead of using all $3N$ nodes.

The second type of constraints corresponds to a self-supported free standing structure, where the only constraint is that the structure must be above ground. We can now model the level of the ground at a given point $(x_1^{(i)}, x_2^{(i)})$ by the C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$.

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \qquad \text{for all } i = 1, \dots, N. \tag{7}$$

By using either one of these two constraints it is possible to show that the problem $\min\limits_X E(X)$ admits a solution.

2. Existence of solutions

In this section we want to prove existence of solutions for fixed nodes and height constraint optimization of equation (5).

2.1. Fixed nodes

Theorem 2.1. *The problem of minimizing $E(X)$ admits a global solution, given that the graph $\mathcal{G}$ is connected and follows the fixed nodes constraint.*

Proof. In the proof we will have use for Remark 1.18 and Theorem 1.33 from the lecture notes [2].
Remark 1.18. A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is coercive, if we have for every sequence $\{x_k\}_{k\in\mathcal{N}}$ with $|||x_k|| \to \infty$ then $f(x_k) \to \infty$ [2, s.4].
Theorem 1.33. If a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is coercive and lower semi-continuous, then the optimization problem $\min\limits_{x\in\mathbb{R}^d} f(x)$ has a global solution [2, s.8].
Thus we need to show that the energy function is coercive and lower-semi-continuous. We start by proving coercivity. For $||x^{(i)}||_2$ to approach infinity, we know that either $x_1^{(i)}$, $x_2^{(i)}$ or $x_3^{(i)}$ have to approach infinity for $i = 1, 2, \dots, N$. Since we know that the graph is connected with a finite amount of vertices, we know that the distance between at least one node pair j and i approaches infinity, which implies that $||x^{(i)} - x^{(j)}||_2 \to \infty$ as $||X||_2 \to \infty$. Thus E^{bar}_{elast} and E^{cable}_{elast} tends to infinity as $||X||_2 \to \infty$. We might have the case where $x_3^{(i)} \to -\infty$, and then E^{bar}_{grav} and E_{ext} tends to negative infinity. Thus we cannot directly state that $E(X) \to \infty$ as $||X||_2 \to \infty$. However when examining E^{bar}_{elast} and E^{cable}_{elast} we observe that they grow with a quadratic rate, while the E^{bar}_{grav} and E_{ext} grows linearly. We therefore state that the $E(X) \to \infty$ as $||X||_2 \to \infty$, and hence $E(X)$ is coercive.
Now we prove that $E(X)$ is lower-semi-continuous. This is true if all terms of $E(X)$ are continuous. Using continuity of the norm and polynomials, it is clear that $E^{bar}_{elast}, E^{bar}_{grav}$ and E_{ext} are continuous. To prove continuity of E^{cable}_{elast} we must examine the limit of $E^{cable}_{elast}(e_{i,j})$ as $||x^{(i)} - x^{(j)}|| \to \ell^+_{ij}$,

$$\lim_{\|x^{(i)}-x^{(j)}\|\to\ell_{ij}^+} E_{elast}^{cable}(e_{i,j}) = \lim_{\|x^{(i)}-x^{(j)}\|\to\ell_{ij}^+} \frac{k}{2\ell_{ij}^2}\left(||x^{(i)}-x^{(j)}||-\ell_{ij}\right)^2 = \frac{k}{2\ell_{ij}^2}\cdot 0 = 0.$$

Therefore E_{elast}^{cable} is continuous and we can state that $E(X)$ is continuous. Thus by Theorem 1.33, $E(X)$ admits a global solution if the graph $\mathcal{G}$ is connected and follows the fixed nodes constraint.

□

2.2. Level constraint

We now switch from the fixed point constraint, to a level constraint. We claim that the energy function is coercive if the level function is coercive. Before we prove the claim, we need to the following lemma.

Lemma 2.1. *If $E(X)$ follows the level constraint, the level function $f(x_1, x_2) \in \mathcal{C}^1(\mathbb{R}^2)$ is coercive, the mass of the system is greater than zero, and $\mathcal{G}$ is connected, then $E(X)$ is coercive as well.*

Proof. We define a sequence $\{X^k\}_{k\in\mathcal{N}}$ with X^k being a $3N$ vector, and $x_3^{i,k} \geq f(x_1^{i,k}, x_2^{i,k})$. We want to show that if $\lim_{k\to\infty} \|X^k\| = \infty$, then $\lim_{k\to\infty} E(X^k) = \infty$. If $\|X^k\| \to \infty$, then at least one coordinate for the i-th node must tend to infinity, $x_1^{i,k} \vee x_2^{i,k} \vee x_3^{i,k} \to \pm\infty$. Because $f(x_1, x_2)$ is coercive, we know that if $x_1^{i,k} \vee x_2^{i,k} \to \pm\infty$, then $x_3^{i,k} \geq f(x_1^{i,k}, x_2^{i,k}) \to \infty$. We also note that by Theorem 1.33, since $f(x_1, x_2)$ is continuous differentiable and coercive then it must admit a global minima $X^* \in \mathbb{R}^{3N}$. Thus we have

$$x_3^i \geq \min_{x_1^i, x_2^i \in \mathbb{R}^2} f(x_1^i, x_2^i) > -\infty.$$

By the arguments above we can state $\|X^k\| \to \infty \implies x_3^{i,k} \to \infty$. If either the i-th node or a connected bar has mass, then E_{grav}^{bar} or E_{ext} approaches infinity. If the i-th node does not contain a mass or a connected bar, then the elastic energy between the i-th node, and the rest of the system must approach infinity. Either way, one of the energy-terms must approach infinity. We also note that none of the terms in the total energy can approach negative infinity. $E_{elast}^{bar}, E_{elast}^{cable} > 0$, and since $x_3^{i,k} > -\infty$ then $E_{grav}^{bar}, E_{ext} > -\infty$. By the same reasoning as for fixed node constraint, we state that if $\lim_{k\to\infty} \|X^k\| = \infty$, then $\lim_{k\to\infty} E(X^k) = \infty$. □

We now return to our original goal of proving that the problem of minimizing $E(X)$ admits a solution if $f(x_1, x_2)$ is coercive.

Theorem 2.2. *Minimizing E(X) admits a solution with level constraint f, if $f(x_1, x_2) \in \mathcal{C}^1(\mathbb{R}^2)$ is coercive, the total mass of the system is greater then zero and the graph $\mathcal{G}$ is connected.*

Proof. To show that $E(X)$ admits a solution we again use Theorem 1.33. As we have already shown, $E(X)$ is continuous. Combining this with Lemma 2.1, we can directly apply Theorem 1.33, and thereby conclude that the problem admits a global solution. □

Remark 2.1. *It is not possible to prove the existence of a unique solution for the case of a flat ground profile $f(x_1, x_2) = 0$, for $x_1, x_2 \in \mathbb{R}$.*

Proof. Since the height function now becomes a constant, we can freely move the structure around the xy-plane without changing the energy of the structure. This prohibits the possibility for a unique global solution for the problem. However, we can not exclude the possibility for a global solution that's non-unique. □

3. Cable net structure

Instead of evaluating the full structure with both bars and cables, we look at the simpler case when all the nodes are connected by cables. As before, we assume that the mass of these cables are negligible. Thus we have to solve the following problem

$$\min_X E(X) = \sum_{e_{ij}\in\varepsilon} E_{elast}^{cable}(e_{ij}) + E_{ext}(X) \quad \text{s.t. } x^{(i)} = p^{(i)},\ i = 1, \ldots, M. \tag{8}$$

Theorem 3.1. *Problem* (8) *is C^1, but typically not C^2.*

Proof. To show that (8) is a C^1 function we have to show that both of the terms $E^{cable}_{elast}(e_{ij})$ and $E_{ext}(X)$ are at least C^1 continuous functions. First we evaluate the term $E_{ext}(X)$ and observe that

$$\frac{\partial}{\partial X}E_{ext}(X) = (0,0,0,\dots,0,0,gm_{M+1},\dots,0,0,gm_N), \text{ and}$$
$$\frac{\partial^k}{\partial X^k}E_{ext}(X) = \mathbf{0}, \text{ for all } k = 2,3,\dots$$

which shows that $E_{ext}(X)$ is a smooth function or in other words C^∞. Now evaluate the second term $E^{cable}_{elast}(e_{ij})$. Using that norms are smooth functions we observe from equation (2) that $E^{cable}_{elast}(e_{ij})$ is smooth in all points except when $||x^{(i)} - x^{(j)}|| \to \ell^{\pm}_{ij}$. Trivially we get that

$$\lim_{||x^{(i)}-x^{(j)}||_2 \to \ell^-_{ij}} \left(\frac{\partial}{\partial x^{(i)}_d}\right)^n E^{cable}_{elast}(e_{ij}) = 0, \text{ for all } n = 0,1,2,\dots,$$

where $x^{(i)}_d$, $d = 1,2,3$, denotes one of coordinates for the i^{th} node. For this term to be C^1 we have to show that the limit as $\|x^{(i)} - x^{(j)}\|_2 \to \ell^+_{ij}$ of $\frac{\partial}{\partial x^{(i)}_d}E^{cable}_{elast}$ also tends to zero. Taking the first and second order derivatives we get that

$$\frac{\partial}{\partial x^{(i)}_d}E^{cable}_{elast}(X) = \frac{k}{\ell^2_{ij}}\left(\|x^{(i)} - x^{(j)}\|_2 - \ell_{ij}\right)\cdot\frac{(x^{(i)}_d - x^{(j)}_d)}{\|x^{(i)} - x^{(j)}\|_2}$$
$$\frac{\partial}{\partial x^{(i)}_2}\frac{\partial}{\partial x^{(i)}_1}E^{cable}_{elast}(X) = \frac{\partial}{\partial x^{(i)}_2}\frac{k}{\ell^2_{ij}}\left(1 - \frac{\ell_{ij}}{||x^{(i)} - x^{(j)}||_2}\right)\cdot(x^{(i)}_1 - x^{(j)}_1) = \frac{k}{\ell_{ij}}\frac{(x^{(i)}_2 - x^{(j)}_2)(x^{(i)}_1 - x^{(j)}_1)}{||x^{(i)} - x^{(j)}||^3_2}.$$

When taking the limit $\|x^{(i)} - x^{(j)}\|_2 \to \ell^+_{ij}$ we observe that the first order derivative tends to zero. Observe that all of the second order mixed derivatives are only zero if all nodes i and j have the same coordinates, or are equal to zero. This is typically not the case. Thus E^{cable}_{elast} is C^1, but typically not C^2. Since both terms in problem (8) are C^1, but typically not C^2, problem (8) is typically not C^2. □

Since $E(X)$ in (8) is typically not C^2, the hessian, H_E is not defined. A consequence of this is that some numerical methods which use the hessian cannot be used to solve the problem. For example Newton's method cannot be used. Some good options for the problem are Quasi-Newton methods or gradient descent methods with a fitting linesearch..

Lemma 3.1. *If $h(x)$ is convex, $f(x)$ is convex and non-decreasing then the composition $f \circ h(x)$ is also convex.*

Proof. Since $h(x)$ is convex and $f(x)$ is non-decreasing we know that

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y), \quad f(x_0) \le f(x_1), \quad x_0 \le x_1$$
$$\Downarrow$$
$$f(h(\lambda x + (1-\lambda)y)) \le f(\lambda h(x) + (1-\lambda)h(y)) \overset{f(x)\ convex}{\le} \lambda f(h(x)) + (1-\lambda)f(h(y))$$
$$\Downarrow$$
$$f \circ h(\lambda x + (1-\lambda)y) \le \lambda f \circ h(x) + (1-\lambda) f \circ h(y)$$

□

The same holds when $f(x)$ is strictly convex and can be proven in a similar way. We will use this result to show convexity of problem (8).

Theorem 3.2. *Problem* (8) *is convex*

Proof. It is known that the sum of convex functions are also convex. Thus if both terms in (8) are convex the problem is also convex. Since $E_{ext}(X)$ is a linear function of X this is convex, but not strictly convex. For the other term, $E^{cable}_{elast}(e_{ij})$, it's slightly more complicated. Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(t) = \begin{cases} K(t-\ell_{ij})^2 & t > \ell_{ij} \\ 0 & t \le \ell_{ij}, \end{cases}$$

where $K = \frac{k}{2\ell_{ij}^2} > 0$ is a constant. Note that f is convex when $t \leq \ell_{ij}$. Thus if $f(t)$ is convex for every $t > \ell_{ij}$ the problem is also convex. By differentiating f we obtain

$$f'(t) = \begin{cases} 2K(t - \ell_{ij}) & t > \ell_{ij} \\ 0 & t \leq \ell_{ij}, \end{cases}$$

since $K > 0$, the derivative is increasing for increasing values of t, this means that the function $f(t)$ is non-decreasing. When $t > \ell_{ij}$, $f(t)$ is strictly convex since $K(t - \ell_{ij})^2$ is a polynomial of second degree. When $t \leq \ell_{ij}$, then $f(t)$ is constant and thus convex. Since $\|x^{(i)} - x^{(j)}\|_2$ is convex, $f(t)$ is convex and non-decreasing, the composition $f(\|x^{(i)} - x^{(j)}\|_2)$ is convex. Note that $f(\|x^{(i)} - x^{(j)}\|_2)$ is equal to $E_{elast}^{cable}(e_{ij})$. □

With weights on the free nodes, we will always have a combination of $x^{(i)}$ and $x^{(j)}$ such that $\|x^{(i)} - x^{(j)}\|_2 > l_{ij}$. Which means at least one of the $E_{elast}^{cable}(e_{ij})$ are strictly convex. This implies that problem (8), with weighted nodes, is strictly convex. Since the sum of convex and strictly convex functions are strictly convex. Thus we have a global and unique solution for problem (8), for weighted nodes. Without weights, the problem is convex and we can have multiple global solutions.

Since problem (8) is convex and C^1, we only need first order optimality conditions to find a solution X^*. Thus the necessary and sufficient optimality conditions are simply

$$\nabla E(X^*) = 0,$$

but as discussed earlier we cannot guarantee that X^* is unique.

4. Tensegrity-Domes

Now we look at a set with both cables and bars, often called tensegrity-domes. As denoted earlier; $\mathcal{E}$ is the full set of both bars and cables, and $\mathcal{B}$ is the set of bars and $\mathcal{C}$ is the set of cables.

4.1. Tensegrity-Domes with fixed nodes

As earlier, we fix some nodes in the system such that we solve a smaller optimization problem. Thus we can model the the problem as

$$\begin{aligned} \min_X E(X) = \sum_{e_{ij}\in\mathcal{B}} \left(E_{elast}^{bar}(e_{ij}) + E_{grav}^{bar}(e_{ij})\right) + \sum_{e_{ij}\in\mathcal{C}} E_{elast}^{cable}(e_{ij}) + E_{ext}(X) \\ \text{s.t. } x^{(i)} = p^{(i)},\ i = 1, \ldots, M. \end{aligned} \tag{9}$$

Let $x_d^{(i)}$, $d = 1, 2, 3$, denote one of the coordinates for the node. Then by differentiating the first term in (9) by $x_d^{(i)}$ we obtain that

$$\frac{\partial}{\partial x_d^{(i)}} E_{elast}^{bar}(e_{ij}) = \frac{c}{\ell_{ij}^2}\left(1 - \frac{\ell_{ij}}{||x^{(i)} - x^{(j)}||_2}\right)(x_d^{(i)} - x_d^{(j)}),$$

which is not defined when the norm $||x^{(i)} - x^{(j)}||_2$ tends to zero. Thus, problem (9) is not typically differentiable. In practice, this poses no problem since the nodes i and j will never actually coincide. Thus $||x^{(i)} - x^{(j)}||_2$ will never be equal to zero. By studying the expression, we also observe that

$$||x^{(i)} - x^{(j)}||_2 = \sqrt{\sum_{l=1}^{3}(x_l^{(i)} - x_l^{(j)})^2} \geq \sqrt{(x_d^{(i)} - x_d^{(j)})^2} = |x_d^{(i)} - x_d^{(j)}| \Leftrightarrow \frac{|x_d^{(i)} - x_d^{(j)}|}{||x^{(i)} - x^{(j)}||_2} \leq 1.$$

Thus we observe that (9) never diverges, at least for all resting lengths of the cables $\ell_{ij} > 0$, which we have assumed to be true.

Theorem 4.1. *Problem* (9) *is non convex if* $\mathcal{B} \neq \emptyset$.

Proof. Assume that (9) is convex, then it satisfies

$$E(\lambda X + (1-\lambda)Y) \leq \lambda E(X) + (1-\lambda)E(Y),$$

for all $\lambda \in (0,1)$. Consider the case when we have one fixed node, p, free nodes, $x^{(i)}$, and with the following properties:

$$\begin{cases} ||x^{(i)} - x^{(j)}||_2 = \ell_{ij} \quad \forall \quad e_{ij} \in \mathcal{C} \cup \mathcal{B} \quad \Rightarrow \quad E_{bar}^{elast}(X) = 0,\ E_{cable}^{elast}(X) = 0 \\ \lambda = \frac{1}{2} \\ Y = -X, \\ p = (0,0,0), \quad \text{this ensures that we solve the same system for } E(X) \text{ and } E(Y) \end{cases}$$

Assume that there is at least one bar in the system such that $\mathcal{B} \neq \emptyset$. Then we obtain the left hand side

$$E(\lambda X + (1-\lambda)Y) = E(\frac{1}{2}X - \frac{1}{2}X) = E(\mathbf{0}) = \sum_{e_{ij} \in \mathcal{B}} \left(\frac{c}{2\ell_{ij}^2} (0 - \ell_{ij})^2 + 0 \right) + \sum_{e_{ij} \in \mathcal{C}} 0 + 0$$
$$= \sum_{e_{ij} \in \mathcal{B}} \frac{c}{2} \geq \frac{c}{2} \overset{(c>0)}{>} 0,$$

Now for the right hand side, we note that $E_{bar}^{elast}(-X) = E_{bar}^{elast}(X)$ and $E_{cable}^{elast}(-X) = E_{cable}^{elast}(X)$ by the homogeneity of the norm. $E_{ext}(-X) = -E_{ext}(X)$ and $E_{bar}^{grav}(-X) = -E_{bar}^{grav}(X)$ by linearity. Thus all terms considering gravity cancels

$$\begin{aligned} \lambda E(X) + (1-\lambda)E(Y) &= \frac{1}{2}E(X) + \frac{1}{2}E(-X) \\ &\Downarrow \text{ homogeneity and linearity remark} \\ &= \frac{1}{2}(E_{bar}^{elast}(X) + E_{cable}^{elast}(X)) + \frac{1}{2}(E_{bar}^{elast}(X) + E_{cable}^{elast}(X)) \\ &= \frac{1}{2} \sum_{e_{ij} \in \mathcal{B}} \frac{c}{\ell_{ij}^2} \left(||x^{(i)} - x^{(j)}||_2 - \ell_{ij} \right)^2 + \frac{1}{2} \sum_{e_{ij} \in \mathcal{C}} \frac{k}{\ell_{ij}^2} \left(||x^{(i)} - x^{(j)}||_2 - \ell_{ij} \right)^2 \\ &= \sum_{e_{ij} \in \mathcal{B}} \frac{c}{2\ell_{ij}^2} (\ell_{ij} - \ell_{ij})^2 + \sum_{e_{ij} \in \mathcal{C}} \frac{k}{2\ell_{ij}^2} (\ell_{ij} - \ell_{ij})^2 = 0 \end{aligned}$$

Thus we have

$$E(\lambda X + (1-\lambda)Y) \nleq \lambda E(X) + (1-\lambda)E(Y),$$

which contradicts our assumption of convexity. Hence problem (9) is non-convex. □

The first order necessary condition for problem (9) is

$$\nabla E(X^*) = 0.$$

Since we are not working with a C^2 function, we cannot use the second order optimality conditions, and the necessary conditions are thus not sufficient. This means that we can have multiple solutions $X_1^*, \ldots, X_N^*$ which satisfy $\nabla E(X^*) = 0$. When implementing this numerically, we will converge to one of these points depending on the initial value X^0.

As a consequence of not having sufficient optimlaity conditions, problem (9) may admit local minimisers which are not global. Figure 3 in section 5 give a numeircal examples of this.

4.2. Tensegrity-Domes with constraints

Instead of fixed nodes as before we now solve a full optimisation problem but with some constraint. The only constraint we want to introduce is that the nodes remains above ground. Let $f(x_1^{(i)}, x_2^{(i)})$ be the height profile of the ground and thus the constraints are simply

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \quad \forall \quad i = 1, \dots, N.$$

Thus we get the following equation for the problem

$$\begin{aligned} \min_X E(X) = \sum_{e_{ij} \in \mathcal{B}} \left(E_{elast}^{bar}(e_{ij}) + E_{grav}^{bar}(e_{ij}) \right) + \sum_{e_{ij} \in \mathcal{C}} E_{elast}^{cable}(e_{ij}) + E_{ext}(X) \\ \text{s.t. } x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \; i = 1, \dots, N. \end{aligned} \tag{10}$$

We now have to solve a constraint optimisation problem with only inequality constraints

$$c_i(X) = x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \geq 0, \quad \forall \quad i = 1, \dots, N.$$

Denote the set of indices i as $\mathcal{I}$. Then X^* is a KKT point if there exists $\lambda_i^* \in \mathbb{R}$, $i \in \mathcal{I}$, such that

$$\begin{cases} \nabla E(X^*) = \sum_{i \in \mathcal{I}} \lambda_i^* \nabla c_i(X^*) & \text{for all } i \in \mathcal{I}, \\ c_i(X^*) \geq 0 & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* \geq 0 & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* c_i(X^*) = 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

The first condition ($\nabla E(X^*) = \sum_{i \in \mathcal{I}} \lambda_i^*$) can by difficult to find, but it can be shown that the critical points of the Lagrangian

$$\mathcal{L}(X, \lambda) = E(X) - \sum_{i \in \mathcal{I}} \lambda_i c_i(X)$$

satisfies this condition. Thus we can search for critical points of this function instead, and we get the following equations

$$\begin{cases} \nabla_X \mathcal{L}(X^*, \lambda^*) = 0 & \\ c_i(X^*) \geq 0 & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* \geq 0 & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* c_i(X^*) = 0 & \text{for all } i \in \mathcal{I}. \end{cases}$$

Theorem 4.2. *LICQ holds for problem* (10).

Proof. Observe that each constraint is only dependent on the i^{th} node $x^{(i)}$. Thus we have that

$$\nabla c_i(X) = \left((0,0,0), \dots, \left(-\frac{\partial f(x_1^{(i)}, x_2^{(i)})}{\partial x_1^{(i)}}, -\frac{\partial f(x_1^{(i)}, x_2^{(i)})}{\partial x_2^{(i)}}, 1 \right), \dots, (0,0,0) \right),$$

where the i^{th} component is the only component different from zero. Then we have that the inner product $\langle \nabla c_i(X), \nabla c_j(X) \rangle = 0$, for all $i \neq j$. Thus every constraint component are independent and thereby LICQ holds. □

Since LICQ holds the KKT-conditions are necessary, but it is not sufficient since we are working with non-convex set.

5. Numerical optimisation

Furthermore we did numerical experiments on cable nets and tensegrity domes. Since we have functions without computable Hessian's we choose to implemented gradient descent with weak Wolfe condition, and the Quasi-Newton method BFGS with strong Wolfe conditions. We will only present the results from BFGS since they are significantly better than gradient descent. The implementation of these methods can be found in `project_definitions.py` and the numerical results can be found in `project_optimization.ipynb`.

5.1. Unconstrained optimization for cable nets

We started by looking at cable structures with 8 nodes where p_1, p_2, p_3, p_4 are fixed and x_5, x_6, x_7, x_8 are free.

$$\begin{gathered} k = 3, \quad l_{ij} = 3 \; \forall \; e_{ij}, \quad m_i g = 1/6 \; \forall \; i \\ p_1 = (5,5,0), \quad p_2 = (-5,5,0), \quad p_3 = (-5,-5,0), \quad p_4 = (5,-5,0) \\ x_5 = (2,2,1), \quad x_6 = (-2,2,1), \quad x_7 = (-2,-2,1), \quad x_8 = (2,-2,1) \end{gathered}$$

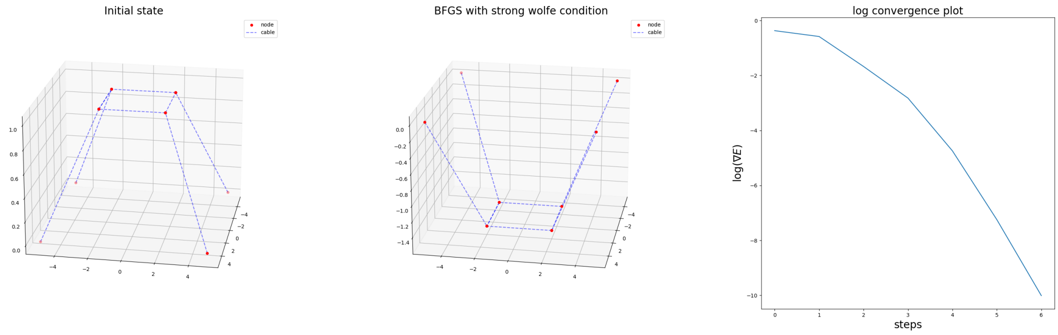

Figure 1: Plot of initial state of the cable structure, optimization with BFGS and log convergence plot.

From figure 1 we we see that the BFGS alogrithm find the unique global solution $x_5 = (2,2,-\frac{3}{2})$, $x_6 = (-2,2,-\frac{3}{2})$, $x_7 = (-2,-2,-\frac{3}{2})$ and $x_8 = (2,-2,-\frac{3}{2})$, with energy $E \approx 1.167$. We also observe that the BFGS method has a super linear convergence with only 7 steps.

5.2. Unconstrained optimisation for tensegrity domes

Now we look at tensegrity domes, a structure with nodes, cables and bars. We again use gradient descent and BFGS with strong wolfe conditions since we do not have a Hessain for the energy function. We use the following parameters:

$$\begin{gathered} l_{1,5} = l_{2,6} = l_{3,7} = l_{4,8} = 10 \quad \text{(bars)} \\ l_{1,8} = l_{2,5} = l_{3,6} = l_{4,7} = 8 \quad l_{5,6} = l_{6,7} = l_{7,8} = l_{5,8} = 1 \quad \text{(cables)} \\ c = 1 \quad k = 0.1 \quad \rho g = 0 \quad m_i g = 0 \; \forall \; i \\ p_1 = (1,1,0), p_2 = (-1,1,0), p_3 = (-1,-1,0), p_4 = (1,-1,0) \\ x_5 = (0.5,0.5,1) \quad x_6 = (-0.5,0.5,1), \quad x_7 = (-0.5,-0.5,1), \quad x_8 = (0.5,-0.5,1) \end{gathered} \tag{P}$$

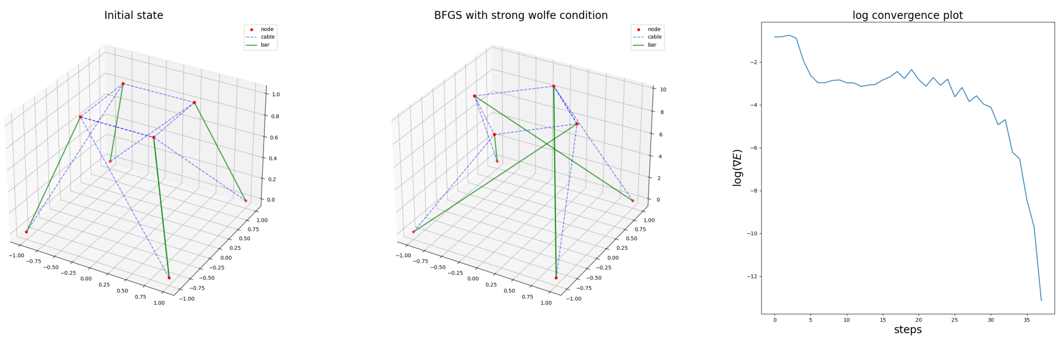

Figure 2: Initial state of the tensegrity dome, optimised structure from BFGS algorithm and log convergence plot.

From figure 2 we observe that the structure is rotating anticlockwise to find the lowest energy configuration. The BFGS algorithm finds the local minima where $x_5 = (-s, 0, t)$, $x_6 = (0, -s, t)$, $x_7 = (s, 0, t)$ and $x_8 = (0, s, t)$. With $s \approx 0.70970$ and $t \approx 9.54287$. The total energy is $E \approx 0.0661$. As in figure 1 BFGS has a super linear convergence, now with 38 steps.

We now provide a numerical example that shows that problem (9) has both local and global solutions.

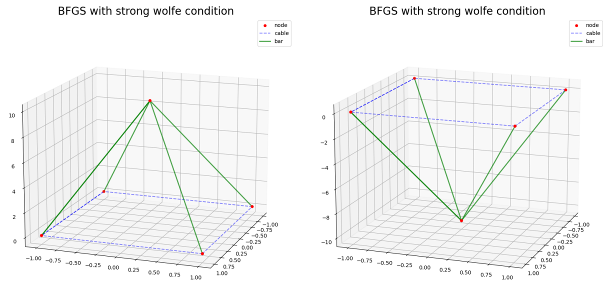

Figure 3: Plots of structures with one free node $x^{(0)}$. On the left we initialize with $x^{(0)} = (0, 0, 4)$ and $x^{(0)} = (0, 0, -4)$ on the right. Fixed nodes: $p^{(1)} = (1, 1, 0)$, $p^{(2)} = (-1, 1, 0)$, $p^{(3)} = (-1, -1, 0)$, $p^{(4)} = (1, -1, 0)$, $k = 0.1$, $c = 1$, $l_{ij} = 10 \; \forall \; e_{ij}$, $m_i g = 10^{-4}$ and $\rho g = 10^{-4}$ for both plots.

In figure 3 the left solution has energy $E(X^*_{left}) \approx 0.0207$ and gradient $\nabla E(X^*_{left}) \approx 5.33 \cdot 10^{-16}$. In the plot to the right the energy is $E(X^*_{right}) \approx -0.0209$ and gradient $\nabla E(X^*_{right}) \approx 7.066 \cdot 10^{-16}$. Since the gradient is close to the floating point error we can conclude that these points are minimums. In both plots we have used the same parameters, but different initial value $x^{(0)}$. Since $E(X_{left}) \geq E(X_{right})$ we have one local and one global minima.

5.3. Constrained optimisation for tensegrity domes

For constrained optimisation we choose $c_i(x^{(i)}) = x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \geq 0$ with $f(x_1, x_2) = \frac{1}{20}(x_1^2 + x_2^2)$ as the ground function. We used the quadratic penalty method as our function to optimize

$$Q(X, \mu) = E(X) + \frac{\mu}{2} \sum_{i \in \mathcal{I}} c_i^2(x)$$

We used BFGS with strong Wolfe conditions to find the node positions. Implementation can be found in the jupyter notebook. The stopping criteria is when $\sum_i c_i = 0$, but for our implementation in python we use a tolerance, 10^{-14}, to account for floating point errors. We use the same parameters as in (P) with some additional parameters

$$l_{1,2} = l_{2,3} = l_{3,4} = l_{1,4} = 2$$
$$m_i g = 10^{-4}, \;\; \rho g = 10^{-4}$$

As for unconstrained optimization we see that the structure rotate anticlockwise to find the local minimizer. The blue shaded area in figure 4 is $f(x_1, x_2)$. We see that all the nodes fulfill the constraint.

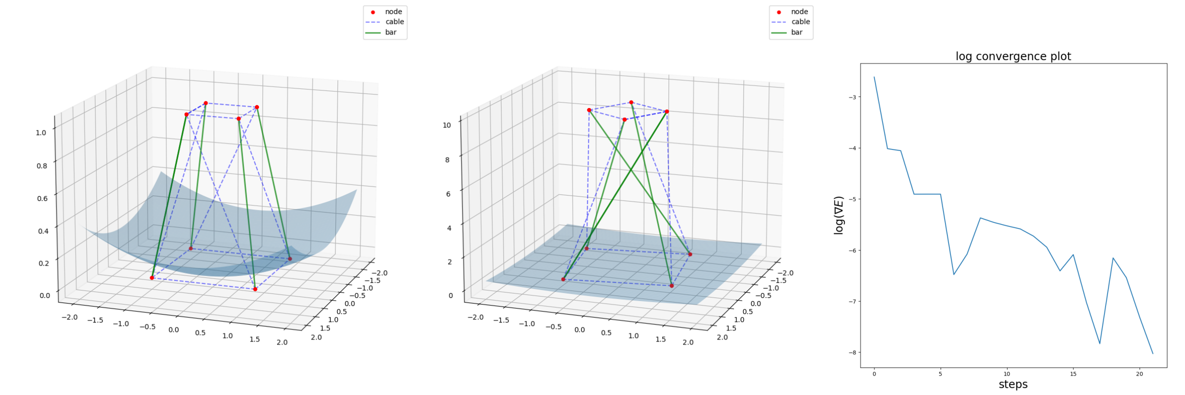

Figure 4: Initial state state of structure, optimised structure from BFGS, log convergence plot for $\mu = 10^4$.

5.4. Two level tensegrity dome

We simulated a solution for a more complicated setting. Here we put two identical tensegrity domes on top of each other by setting the top nodes of the bottom structure equal to the bottom nodes of the structure on top. We again used the quadratic penalty method for constrained optimization. We used BFGS with strong Wolfe conditions and $c_i(X) = x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \geq 0$ as our inequality constraint.

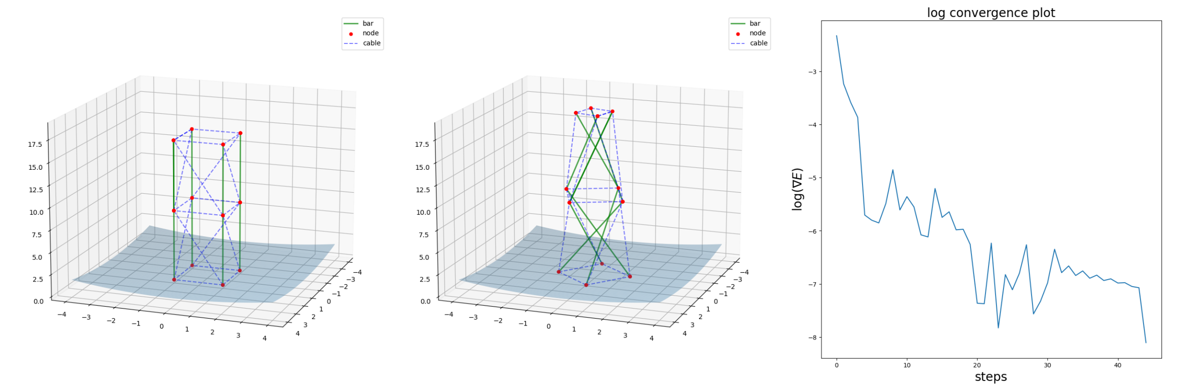

Figure 5: Initial state state of structure, optimised structure from BFGS, log convergence plot for $\mu = 10^4$

For the two level structure in figure 5 we again observe that the structure twists to find its lowest energy configuration. The structure is symmetric and seems like a plausible real world approximation.

6. Conclusion

In this project we have examined energy minimization problems for tensegrity structures. We have analytically proven the existence of solutions for both fixed nodes- and level constraints. Then we derived analytical results to check convergence of both constrained and unconstrained optimization problems. In addition we implemented an algorithm to solve the problem numerically, and confirmed our analytical results. We found that the most efficient solver was the BFGS method.

References

[1] Grasmair, M. *Project description* (16th February 2024). TMA4180-Optimization 1, NTNU Trondheim.
[2] Grasmair, M. *THEORY OF UNCONSTRAINED OPTIMISATION* (29th February 2024). TMA4180-Optimization 1, NTNU Trondheim.