A 1d Stationary Convection Diffusion Problem with Finite Element Methods

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Introduction

In this project we will develop and implement a finite element scheme for a 1D stationary convection diffusion problem. We will prove the existence of a solution in a weak formulation of the PDE, and do an analytical error analysis. We will then solve the problem numerically to confirm the analytical results, and explore the convergence of different right hand sides. Finally we will study the effectiveness of different grids.

1. Defining the problem

We want to look at a 1d convective diffusion problem, i.e. the transport/mixing/decay of a liquid in a tube. Let u be the concentration of the substance and $\Omega = (0, 1)$ be the domain we are working in. Using that the problem is stationary ($\partial_t u = 0$) we can model the problem as follows

$$-\partial_x(\alpha(x)\partial_x u) + \partial_x(b(x)u) + c(x)u = f(x) \quad \text{in} \quad \Omega = (0, 1), \tag{1}$$

where $\alpha(x) > 0$ is a diffusion coeffisient, $b(x)$ is the fluid velocity, $c(x) \geq 0$ is the decay rate of the substance and $f(x)$ is a source term. This equation can be obtained by conservation of mass.

2. Theory

We consider the case when $\alpha(x) = \cos(\frac{\pi}{3}x)$, $c = 5$, $||b||_{L^\infty} + ||f||_{L^2} < \infty$, with Dirichlet boundary conditions $u(0) = 0 = u(1)$. Denote this problem as P.

Theorem 2.1. *Any classical solution u of P satisfies*

$$a(u, v) = F(v) \quad \forall v \in H_0^1(0, 1), \tag{2}$$

where $a(u, v) = \int_0^1 (\alpha(x)u_x v_x - b(x)uv_x + c(x)uv)\, dx$, $F(v) = \int_0^1 f(x)vdx$ and $H_0^1(0, 1)$ denotes the space of continuously weak differentiable functions $v(x)$ on the interval $(0, 1)$, with $v(0) = v(1) = 0$.

Proof. By multiplying (1) by v and integrating we obtain that

$$\int_0^1 (-\partial_x(\alpha(x)u_x) + \partial_x(b(x)u) + c(x)u)\, vdx = \int_0^1 f(x)vdx = F(v),$$

where $u_x = \partial_x u$. Now we simplify the left hand side using integration by parts

$$\begin{aligned}
&\int_0^1 (-\partial_x(\alpha(x)u_x)v + \partial_x(b(x)u)v + c(x)uv)\, dx \\
&= \underbrace{[-(\alpha(x)u_x)v]_0^1}_{=0} + \int_0^1 \alpha(x)u_x v_x dx + \underbrace{[(b(x)u)v]_0^1}_{=0} - \int_0^1 b(x)uv_x dx + \int_0^1 c(x)uvdx \\
&= \int_0^1 (\alpha(x)u_x v_x - b(x)uv_x + c(x)uv)\, dx,
\end{aligned}$$

where we have used that $v(0) = v(1) = 0$. Thus we get

$$a(u,v) = \int_0^1 \left(\alpha(x)u_x v_x - b(x)uv_x + c(x)uv\right) dx = \int_0^1 f(x)v dx = F(v) \quad \forall v \in H_0^1(0,1),$$

which proves Theorem 2.1. $\square$

Equation (2) is the weak formulation of the problem and is true for every $v \in H_0^1(0,1)$. Thus we have to check if these functions actually exists and converges in $H_0^1(0,1) \subset H^1(0,1)$.

2.1. *Uniqueness of a solution u*

To show that the problem in equation (2) has a unique solution, we use Lax-Milgrams theorem. The theorem states that if $a(u,v)$ is bilinear, bounded and coercive, and $F(v)$ is linear and bounded then we have a unique solution u, which we will prove in this section.

Lemma 2.1. *$a(u,v)$ is a bilinear and continuous form (function) on $H^1 \times H^1$, and $F(v)$ is a linear continuous (bounded) functional on H^1.*

Proof. A function is bilinear if $f(k_1u_1 + k_2u_2, v) = k_1 f(u_1, v) + k_2 f(u_2, v)$ and $f(u, k_1v + k_2v) = k_1 f(u, v_1) + k_2 f(u, v_2)$, where k_1 and k_2 are constants. For $a(u,v)$ we get that

$$\begin{aligned}
a(k_1u_1 + k_2u_2, v) &= \int_0^1 \left(\alpha(x)\partial_x(k_1u_1 + k_2u_2)v_x - b(x)(k_1u_1 + k_2u_2)v_x + c(x)(k_1u_1 + k_2u_2)v\right) dx \\
&= k_1 \int_0^1 \left(\alpha(x)\partial_x u_1 v_x - b(x)u_1v_x + c(x)u_1v\right) dx + k_2 \int_0^1 \left(\alpha(x)\partial_x u_2 v_x - b(x)u_2v_x + c(x)u_2v\right) dx \\
&= k_1 a(u_1, v) + k_2 a(u_2, v).
\end{aligned}$$

A similar calculation shows that $a(u, k_1v + k_2v) = k_1 a(u, v_1) + k_2 a(u, v_2)$, which shows that $a(u,v)$ is bilinear.
To show continuity of $a(u,v)$ on $H^1 \times H^1$ we have to show that it satisfies Cauchy-Schwarz inequality:

$$|a(u,v)| \leq M||u||_{H^1}||v||_{H^1}, \tag{C-S}$$

where $M > 0$ is some constant. Recall that we are working with the problem where $\alpha(x) = \cos(\frac{\pi}{3}x)$, $c = 5$ and $||b||_{L^\infty} + ||f||_{L^2} < \infty$. Thus we obtain that $||\alpha||_\infty = 1$, $||c||_\infty = 5$ and $||b||_\infty < \infty$. Then we have

$$\begin{aligned}
|a(u,v)|^2 &= \left| \int_0^1 \left(\alpha(x)u_x v_x - b(x)uv_x + c(x)uv\right) dx \right|^2 \\
&\leq \int_0^1 |\alpha(x)|^2 |u_x v_x|^2 dx + \int_0^1 |b(x)|^2 |uv_x|^2 dx + \int_0^1 |c(x)|^2 |uv|^2 dx \\
&\leq ||\alpha||_\infty^2 ||u_x||_{L^2}^2 ||v_x||_{L^2}^2 + ||b||_\infty^2 ||u||_{L^2}^2 ||v_x||_{L^2}^2 + ||c||_\infty^2 ||u||_{L^2}^2 ||v||_{L^2}^2 \\
&\leq \max\left\{||\alpha||_\infty^2, ||b||_\infty^2, ||c||_\infty^2\right\} \left(||u||_{L^2}^2 + ||u_x||_{L^2}^2\right) \left(||v||_{L^2}^2 + ||v_x||_{L^2}^2\right) \\
&= \max\left\{||\alpha||_\infty^2, ||b||_\infty^2, ||c||_\infty^2\right\} ||u||_{H^1}^2 ||v||_{H^1}^2 \\
\Rightarrow |a(u,v)| &\leq M||u||_{H^1}||v||_{H^1},
\end{aligned}$$

where $M = \max\{||\alpha||_\infty, ||b||_\infty, ||c||_\infty\} < \infty$. This proves that $a(u,v)$ is a continuous form on $H^1 \times H^1$.
To show that $F(v)$ is a linear continuous (bounded) functional on H^1 we can show linearity of F and that the dual norm of F is bounded. Thus we get that

Linearity:

$$F(k_1v_1 + k_2v_2) = \int_0^1 f(x)(k_1v_1 + k_2v_2)dx = k_1 \int_0^1 f(x)v_1 dx + k_2 \int_0^1 f(x)v_2 dx = k_1 F(v_1) + k_2 F(v_2)$$

Boundedness:

$$||F(v)||_{H^1} = \sup_{0 \neq v \in H_0^1} \frac{|F(v)|}{||v||_{H^1}} = \sup_{0 \neq v \in H_0^1} \frac{\left|\int_0^1 f(x)v dx\right|}{||v||_{H^1}} \leq \sup_{0 \neq v \in H_0^1} \frac{||f||_{L^2}||v||_{L^2}}{||v||_{H^1}} \leq ||f||_{L^2} < \infty,$$

and we can conclude that $F(v)$ is a linear continuous (bounded) functional on H^1. $\square$

Now it remains to show coercivity for the form $a(u,v)$. To be able to do this we need the following Gårding inequality.

Lemma 2.2. *$a(u,v)$ satisfies the Gårding inequality*

$$a(u,u) \geq (\alpha_0 - \frac{\epsilon}{2}||b||_{L^\infty}) \int_0^1 u_x^2 dx + (c_0 - \frac{1}{2\epsilon}||b||_{L^\infty}) \int_0^1 u^2 dx \quad \text{for all } \epsilon > 0,$$

where $\alpha_0 = \min_{x\in[0,1]} \alpha(x)$ *and* $c_0 = \min_{x\in[0,1]} c(x)$.

Proof. In the proof we need Young's inequality: $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ for any $\epsilon > 0$. We rewrite $a(u,u)$,

$$a(u,u) = \int_0^1 \alpha(x)u_x^2 dx - \int_0^1 b(x)uu_x dx + \int_0^1 c(x)u^2 dx \geq \alpha_0 \int_0^1 u_x^2 dx - ||b||_{L^\infty} \int_0^1 uu_x dx + c_0 \int_0^1 u^2 dx$$

Then use Young's inequality for the middle term we get that

$$\begin{aligned} a(u,u) &\geq \alpha_0 \int_0^1 u_x^2 dx - ||b||_{L^\infty} \left(\frac{\epsilon}{2} \int_0^1 u_x^2 dx + \frac{1}{2\epsilon} \int_0^1 u^2 dx \right) + c_0 \int_0^1 u^2 dx \\ &= (\alpha_0 - \frac{\epsilon}{2}||b||_{L^\infty}) \int_0^1 u_x^2 dx + (c_0 - \frac{1}{2\epsilon}||b||_{L^\infty}) \int_0^1 u^2 dx \qquad \text{for all } \epsilon > 0, \end{aligned}$$

which completes the proof. □

Theorem 2.2. *If $||b||_{L^\infty} < \sqrt{2\alpha_0 c_0} = \sqrt{5}$, then $a(u,v)$ are coercive.*

Proof. The function is coercive if

$$a(u,u) \geq C||u||_{H^1}^2,$$

for some constant $C > 0$. Thus using Lemma 2.2 we get that

$$a(u,u) > (\frac{1}{2} - \frac{\epsilon\sqrt{5}}{2})||u_x||_{L^2}^2 + (5 - \frac{\sqrt{5}}{2\epsilon})||u||_{L^2}^2$$

We can solve the system

$$\begin{aligned} &(\frac{1}{2} - \frac{\epsilon\sqrt{5}}{2}) = (5 - \frac{\sqrt{5}}{2\epsilon}) \Rightarrow \epsilon^2 - \frac{9\sqrt{5}}{5}\epsilon - 1 = 0 \Rightarrow \epsilon = \frac{\sqrt{101} - 9}{2\sqrt{5}} \\ &\Rightarrow a(u,u) > \frac{1}{4}(11 - \sqrt{101}) \left(||u_x||_{L^2}^2 + ||u||_{L^2}^2 \right) \approx 0.2375||u||_{H^1}^2, \end{aligned}$$

which shows that $a(u,v)$ are coercive. □

Another approach to show coercivity is by integration by parts and we obtain that

$$a(u,u) \geq \alpha_0||u_x||_{L^2}^2 + (c_0 - \frac{1}{2}||b_x||_{L^\infty})||u||_{L^2}^2,$$

so if b is constant then $c_0 \geq 0$ is sufficient for coercivity.

We have proven that $a(u,v)$ is continuous, bilinear and coercive, and that $F(v)$ is linear and bounded in the space H^1. Since $H_0^1 \subset H^1$ these characteristics are also true in H_0^1. Thus from Lax-Milgrams theorem there exists a unique solution $u \in H_0^1$ which solves (2).

2.2. Error estimate

To get an error estimate for the problem in (1) we firstly have to show that the Galerkin orthogonality holds for the biliniear form $a(u,v)$.

Theorem 2.3. *Galerkin orthogonality given by*

$$a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h, \tag{3}$$

holds for $a(u,v)$.

Proof. Recall that $V_h \subset H_0^1$. Thus we get that

$$a(u,v) = F(v) \quad \forall v \in H_0^1 \Rightarrow a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Using that $a(u,v)$ is bilinear form we get that

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0,$$

which completes the proof. $\square$

Since we have Galerkin orthogonality we can use Cea's lemma to find an error bound for the method

Lemma 2.3. *Cea's lemma given by*

$$||u - u_h||_{H^1} \leq \frac{M}{\gamma} \inf_{v_h \in V_h} ||u - v_h||_{H^1}, \tag{4}$$

holds when we have Galerkin orthogonality, and $|F(v)| \leq C||v||_{H^1}$, $|a(u,v)| \leq M||u||_{H^1}||v||_{H^1}$ *and* $a(u,v) \geq \gamma ||u||_{H^1}^2$ *for all* $u, v \in H^1$.

Proof. Recall that

$$|F(v)| \leq C||v||_{H^1}, \quad |a(u,v)| \leq M||u||_{H^1}||v||_{H^1} \quad \text{and} \quad a(u,v) \geq \gamma ||u||_{H^1}^2,$$

where $C = ||f||_{L^2}$, $M = \max\{||\alpha||_\infty, ||b||_\infty, ||c||_\infty\}$ and $\gamma = \min\{\alpha_0, c_0\}$. From Theorem 2.3 we have already shown Galerkin orthogonality and thus (4) holds for the scheme. $\square$

Using interpolation of u we can derive an upper bound for the error

$$||u - u_h||_{H^1} \leq \frac{M}{\gamma} \inf_{v_h \in V_h} ||u - v_h||_{H^1} \leq \frac{M}{\gamma} \inf ||u - I_h u||_{H^1} \leq \frac{M}{\gamma} \sqrt{2} h ||u_{xx}||_{L^2}, \tag{E}$$

where $h = \max_i h_i$, $M = \max\{||\alpha||_\infty, ||b||_\infty, ||c||_\infty\}$ and $\gamma = \min\{\alpha_0, c_0\}$ when b is constant. Thus we have a linear convergence in the H^1 norm.

Remark 2.1. *Proof for* (E) *can be found in Week 14 of the Lecture notes and holds only if* $u \in H^2(0,1)$. *This is not the case if the solutions are not differentiable in some points in* $(0,1)$*, as we will show later.*

2.3. *Non differentiable functions*

Non-smooth manufactured solutions can be defined on $(0,1)$ by

$$w_1(x) = \begin{cases} \sqrt{2}x & \text{if } x \in \left[0, \frac{\sqrt{2}}{2}\right], \\ \frac{1-x}{1-\sqrt{2}/2} & \text{if } x \in \left(\frac{\sqrt{2}}{2}, 1\right], \end{cases} \quad \text{and} \quad w_2(x) = x - x^{\frac{3}{4}}. \tag{5}$$

Observe that both ω_1 and ω_2 are continuous, but not differentiable at $x = \frac{\sqrt{2}}{2}$ and $x = 0$ respectively, since

$$\lim_{x^- \to \frac{\sqrt{2}}{2}} \partial_x \omega_1(x) = \sqrt{2} \neq -\frac{1}{1 - \sqrt{2}/2} = \lim_{x^+ \to \frac{\sqrt{2}}{2}} \partial_x \omega_1(x)$$
$$\lim_{x^+ \to 0} \partial_x \omega_2(x) = \lim_{x^+ \to 0} (1 - \frac{3}{4} x^{-\frac{1}{4}}) = -\infty.$$

Even though w_1 and w_2 are not differentiable at these points we can show that both of them belong to $H^1(0,1)$. We have that

$$\int_0^1 |\omega_1(x)|^2 dx = \int_0^{\frac{\sqrt{2}}{2}} |\omega_1(x)|^2 dx + \int_{\frac{\sqrt{2}}{2}}^1 |\omega_1(x)|^2 dx = \frac{1}{3\sqrt{2}} + \frac{1}{6}(2 - \sqrt{2}) < \infty$$
$$\int_0^1 |\partial_x \omega_1(x)|^2 dx = \int_0^{\frac{\sqrt{2}}{2}} |\partial_x \omega_1(x)|^2 dx + \int_{\frac{\sqrt{2}}{2}}^1 |\partial_x \omega_1(x)|^2 dx = \sqrt{2} + (2 + \sqrt{2}) < \infty$$
$$\Rightarrow ||w_1||_{H^1} = ||w_1||_{L^2} + ||\partial_x w_1||_{L^2} < \infty$$

and

$$\int_0^1 |\omega_2(x)|^2 dx = \lim_{r\to 0}\int_r^1 |\omega_2(x)|^2 dx = \frac{1}{165} < \infty$$
$$\int_0^1 |\partial_x\omega_2(x)|^2 dx = \lim_{r\to 0}\int_r^1 |\partial_x\omega_2(x)|^2 dx = \frac{1}{8} < \infty$$
$$\Rightarrow ||w_2||_{H^1} = ||w_2||_{L^2} + ||\partial_x w_2||_{L^2} < \infty,$$

thus both w_1 and w_2 belong to $H^1(0,1)$. Observe that $\partial_x^2 w_1 = 0$ in all other points than in $x = \frac{\sqrt{2}}{2}$. In this point the second order derivative will be a Dirac delta function, which is known to not belong to any L^P spaces. Then $||\partial_x^2 w_1||^2_{L^2(0,1)}$ diverges and hence w_1 is not in $H^2(0,1)$. For w_2 we have that

$$\lim_{r\to 0}\int_r^1 |\partial_x^2\omega_2(x)|^2 dx = \lim_{r\to 0}\frac{3}{128}\left(\frac{1}{r^{\frac{3}{2}}} - 1\right) = \infty.$$

Thus w_2 do not belong to $H^2(0,1)$. As a consequence of w_1 and w_2 not being $H^2(0,1)$ functions the error bound in (E) does not hold. Thus we expect a lower convergence rate for such functions.

2.4. ***Sharp gradients and refined grids***

We define the two functions

$$f_1(x) := x^{-\frac{2}{5}} \quad \text{and} \quad f_2(x) := x^{-\frac{7}{5}}. \tag{6}$$

Remark 2.2. *$f_1 \in L^2(0,1)$, $f_2 \notin L^2(0,1)$ and $f_2 = -\frac{5}{2}(f_1)_x$.*

Proof. We have that

$$||f_1||^2_{L^2(0,1)} = \int_0^1 \left(x^{-\frac{2}{5}}\right)^2 dx = \frac{5}{3}\left[x^{\frac{5}{3}}\right]_0^1 = \frac{5}{3} < \infty,$$
$$||f_2||^2_{L^2(0,1)} = \int_0^1 \left(x^{-\frac{7}{5}}\right)^2 dx = -\frac{5}{9}\left[x^{-\frac{9}{5}}\right]_0^1 = \infty,$$

which proves that $f_1 \in L^2(0,1)$ and $f_2 \notin L^2(0,1)$. Observe also that $-\frac{5}{2}(f_1)_x = -\frac{5}{2}\cdot -\frac{2}{5}x^{-\frac{7}{5}} = f_2$. □

Note that $f_2 \in H^{-1}(0,1)$. Typically solutions of (1) with such right hand sides, will have sharp gradients near $x = 0$. Thus we want to space our grid such that the the first intervals near $x = 0$ are shorter than the rest. One way to implement this is to choose a graded grid such that $x_0 = 0$ and $x_i = r^{M-i}$, $i = 1, 2, \ldots, M$, for some $r \in (0,1)$.

3. Numerics

Now we let $\alpha \geq 0$, b, $c \geq 0$ be nonzero constants and we want to set up a $\mathbb{P}_1$ finite element method (FEM) for the system given in (2). In mathematical terms we want to solve the problem

$$\text{Find } (u_h, v_h) \in V_h \text{ such that} \quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h, \tag{V_h}$$

where $V_h = X_h^1(0,1) \cap H_0^1(0,1)$ is the space of continuous functions with zero boundary values and are piecewise linear on the triangulation of the grid. One way to find u_h is to use Galerkin's method which generally is given by

$$\begin{gathered} u_h = \sum_{i=0}^{m} U_i\varphi_i \text{ where } \vec{U} = (U_0, \ldots, U_m) \in \mathbb{R}^{m+1} \text{ solves} \quad A\vec{U} = \vec{F}, \\ \text{where } A_{ij} = a(\varphi_j, \varphi_i) \text{ and } F_j = F(\varphi_j). \end{gathered} \tag{7}$$

The φ_i's in (7) are basis functions for a given triangulation/discretization $\tau_h = \{K_i\}_{i=1}^m$, where K_i denotes the i^{th} partition of the space Ω. We are working in the space $\Omega = (0,1)$. Thus we can use partitions $K_i = (x_{i-1}, x_i)$ with $0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = 1$ such that $\tau_h = \Omega$. Let $h_i = (x_i - x_{i-1})$ be the length of each interval K_i. Then a set of basis functions are

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & \text{for } x \in [x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h_{i+1}}, & \text{for } x \in [x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Having found a suitable basis we return to our original problem

$$a(u, v) = \int_0^1 (\alpha(x)u_x v_x - b(x)uv_x + c(x)uv)\, dx = \sum_{i=1}^{M} a^{k_i}(\varphi_i, \varphi_j), \qquad j = 0, \dots, M.$$

From the definition of φ_i we recognize that the only contribution will be when $j \in i-1, i, i+1$. We can therefore solve it locally for each of the elemental matrices A^{k_i}, and then recombine each A^{k_i} into the stiffness matrix A. When solving the integral for the different cases we get the following solution to A^{k_i}.

$$A^{K_i} = \begin{bmatrix} a^{K_i}(\varphi_{i-1}, \varphi_{i-1}) & a^{K_i}(\varphi_{i-1}, \varphi_i) \\ a^{K_i}(\varphi_i, \varphi_{i-1}) & a^{K_i}(\varphi_i, \varphi_i) \end{bmatrix} = \frac{\alpha}{h_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{b}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{ch_i}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$F^{K_i} = \begin{bmatrix} F^{K_i}(\varphi_{i-1}) \\ F^{K_i}(\varphi_i) \end{bmatrix}$$

3.1. Convergence analysis for smooth functions

We chose to test our scheme on the function $u(x) = \sin(2\pi x)$, which admits the following right hand side $f(x) = 4\alpha\pi^2 \sin(2\pi x) + 2b\pi \cos(2\pi x) + c \sin(2\pi x)$. The error bound is given by equation (E) with $||u_{xx}||_{L^2} = 2\sqrt{2}\pi^2$.

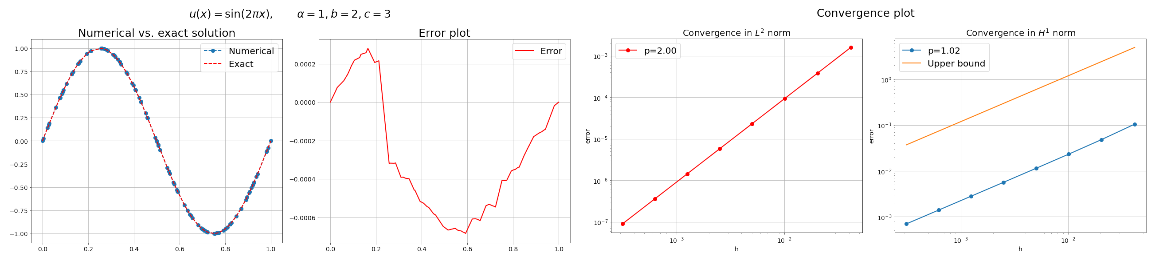

Figure 1: The figure shows the numerical solution with $M = 100$, analytical solution $u(x) = \sin(2\pi x)$, error $e = u - U$, and convergence rates in L^2 and H^1.

In figure (1) we observe that the numerical result seems to fit well with the exact solution, given a high enough number of random grid points. From the convergence plots we observe a convergence rate of $p = 2$ in L^2 norm, which was expected from the results found in week 14 of the lecture notes. We observe a convergence rate of $p = 1$ in H^1, which supports our claim of linear convergence in H^1. We also see that the error is well below the theoretical upper bound found in (E).

3.2. Convergence analysis of non differentiable functions

We now want to test our solver on non differentiable problems such as ω_1 and ω_2 in (5). We can no longer use our previous method, since the function no longer have a weak second derivative. However, since both functions have a weak derivative we can use integration by parts to find the load vector for both functions. We simply calculate the following integral for all values of ϕ^i, to find the value of F_i.

$$F(v) = -\alpha \int_0^1 u_{xx} v dx + b \int_0^1 u_x v dx + c \int_0^1 uv dx = \alpha \int_0^1 u_x v_x dx - b \int_0^1 uv_x dx + c \int_0^1 uv dx$$
$$F_i = F(\varphi^i) = \alpha \int_{x_{i-1}}^{x_{i+1}} u_x \varphi_x^i - b \int_{x_{i-1}}^{x_{i+1}} u\varphi_x^i dx + c \int_{x_{i-1}}^{x_{i+1}} u\varphi^i dx$$

From figure 3 and 4 we observe that our numerical solver approximates the analytical solution well. We observe that the L^2 convergence rate is less then $p = 2$, and a H^1 convergence rate less then $p = 1$ for both functions. This was expected since none of the functions belong to H^2-space. The convergence is slowed by the approximation close to the discontinuity- and singularity points. We note that the error bound found in (E) does not hold for the functions, since neither of them are in H^2.

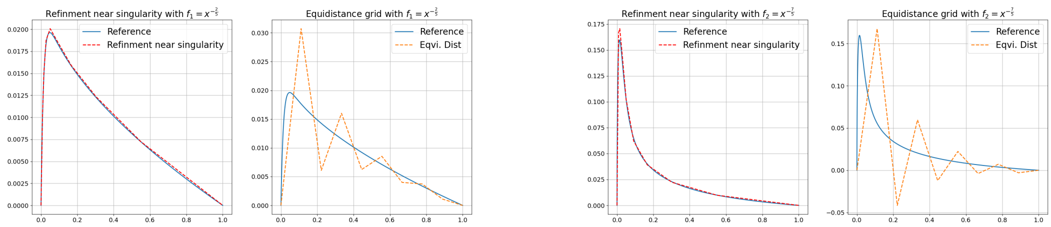

Figure 2: Comparison between grid approximation with refinement near singularity and equidistant grid for right hand side $f_1 = x^{-\frac{2}{5}}$ and $f_2 = x^{-\frac{7}{5}}$. Variables: $\alpha = 1$, b=-70, c=1,$r = 0.55$, $M = 10$ nodes.

3.3. ***Refinement- vs. equidistant grids***

We now want to solve problem (1) with f_1 and f_2 from (6) as the right hand side. When choosing r-values we computed numerical solutions and chose the r-value with the least L^2 and H^1 error. Since both equations have a sharp gradient near $x = 0$ we will compare a refinement $x_i = r^{M-i}$ with equidistant an grid.

From figure 2 we observe that the solution with a higher concentration of grid points closer to the sharper gradients gives a better approximation then with an equidistant grid. This was expected since we use less points on the linear part of the graph, and more points where the function is more flexible. From table (1) this is a general result, since both the L^2- and the H^1 error bound is consistently lower then with an equidistant grid. This can be clearly observed from figure 5 where we compare the equidistant error to several different r-values with a constant M value. From table 1 we also observe that as the value of M increases, the optimal r increases as well.

M	r	Refinement f_1		Equidistant f_1	
		L^2 error	H^1 error	L^2 error	H^1 error
10	0.51	8.79×10^{-5}	2.73×10^{-3}	5.00×10^{-3}	2.63×10^{-2}
20	0.73	2.21×10^{-5}	6.96×10^{-4}	1.97×10^{-3}	2.66×10^{-2}
30	0.81	1.00×10^{-5}	3.25×10^{-4}	9.80×10^{-4}	2.17×10^{-2}
40	0.85	5.84×10^{-6}	1.93×10^{-4}	5.60×10^{-4}	1.71×10^{-2}
50	0.88	3.80×10^{-6}	1.27×10^{-4}	3.52×10^{-4}	1.35×10^{-2}

Table 1: Optimal r-lengths for different M with errors in L^2 and H^1

Conclusion

In this paper we have implemented a finite element method for solving a 1d stationary convection diffusion problem. Before solving the problem numerically we have shown that the scheme has a unique solution u in H^1. We implemented a numerical solver to confirm the analytical results. We found that the convergence rate of H^2 functions were linear in H^1 and quadratic in L^2. Checking the convergence rate of non smooth functions not in H^2, we found them to have lower convergence rates. We then experimented with refined grids vs. equidistant grids on functions with singularities, and that refinement had a higher efficiency.

Appendix

M	r	Refinement f_2		Equidistant f_2	
		L^2 error	H^1 error	L^2 error	H^1 error
10	0.49	9.97×10^{-4}	4.11×10^{-2}	4.22×10^{-2}	2.17×10^{-1}
20	0.69	2.38×10^{-4}	2.54×10^{-2}	1.30×10^{-2}	3.41×10^{-1}
30	0.77	1.10×10^{-4}	1.69×10^{-2}	6.39×10^{-3}	3.71×10^{-1}
40	0.82	6.53×10^{-5}	1.19×10^{-2}	3.79×10^{-3}	3.62×10^{-1}
50	0.85	4.35×10^{-5}	8.75×10^{-3}	2.54×10^{-3}	3.38×10^{-1}

Table 2: Optimal r-lengths for different M with errors in L^2 and H^1

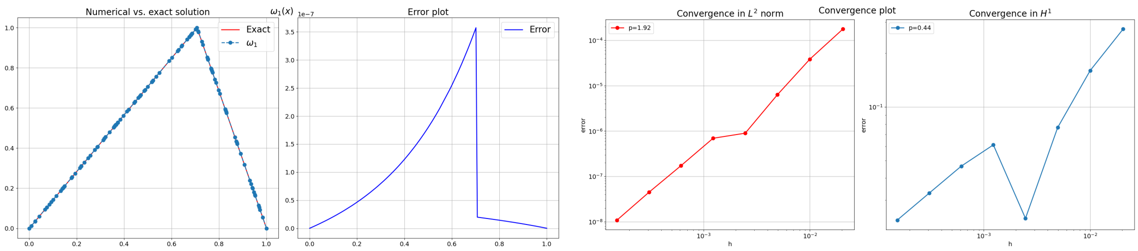

Figure 3: The figure shows the numerical solution, analytical solution, $\omega_1(x)$, error $e = u - U$, convergence in L^2 and H^1. Here $M = 111$, $\alpha = 1$, $b = 2$ and $c = 3$.

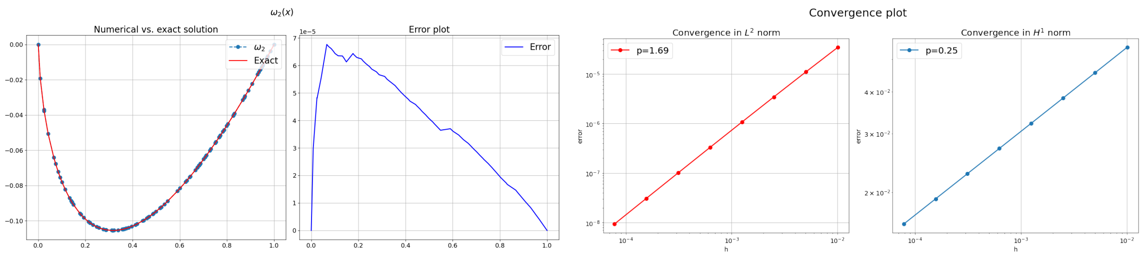

Figure 4: The figure shows the numerical solution, analytical solution, $\omega_2(x)$, error $e = u - U$, convergence in L^2 and H^1. Here $M = 111$, $\alpha = 1$, $b = 2$ and $c = 3$.

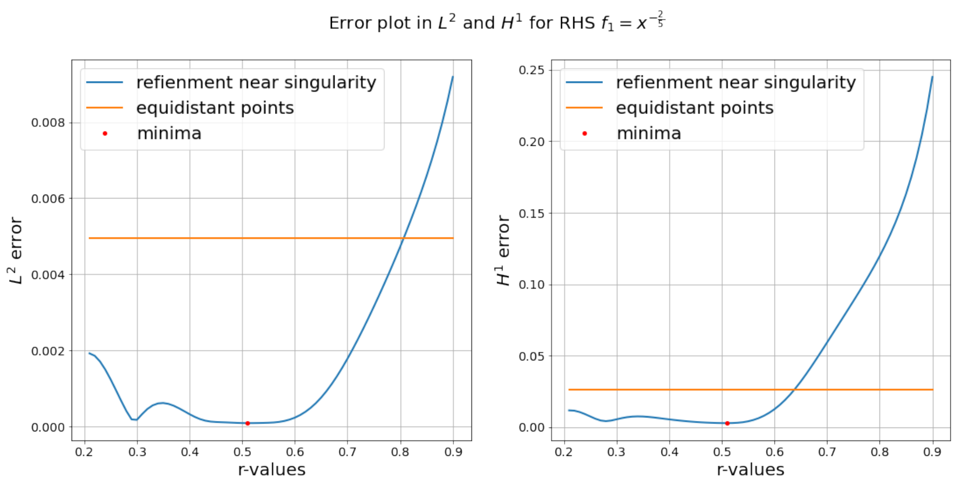

Figure 5: Comparison between grid approximation with refinement near singularity and equidistant grid for right hand side $f_1 = x^{-f\frac{2}{5}}$ and $f_2 = x^{-\frac{7}{5}}$. Here $M = 10$, $\alpha = 1$, $b = -100$ and $c = 1$.

```
[[14.28571429 -5.92857143  0.          0.          0.          0.        ]
 [-7.92857143 14.28571429 -5.92857143  0.          0.          0.        ]
 [ 0.         -7.92857143 14.28571429 -5.92857143  0.          0.        ]
 [ 0.          0.         -7.92857143 14.28571429 -5.92857143  0.        ]
 [ 0.          0.          0.         -7.92857143 14.28571429 -5.92857143]
 [ 0.          0.          0.          0.         -7.92857143 14.28571429]]
```

Figure 6: Stiffness matrix for $M = 8$, $\alpha = 1$, $b = 2$ and $c = 3$.