1 Documentation

This report contains the documentation of the model predictive controller (MPC) that was designed to direct the car around the test track of the udacity simulator. Provided as inputs from the simulator are:

- position (${}_E p_x$, ${}_E p_y$) and orientation (${}_E\psi$) of the vehicle in a global reference frame E
- current steering angle (δ) in radians
- current throttle (a; scaled between -1 and +1)
- current velocity in mph (is transformed to m/s in the program and denoted by v)
- waypoint arrays ${}_E pts_x$ and ${}_E pts_y$ that model the trajectory which the vehicle shall follow. The waypoints are given in the global coordinate system

1.1 Coordinate Transformation

To make things easier for the MPC computations the waypoint arrays are transformed into vehicle coordinates:

$$\begin{pmatrix} {}_V pts_x \\ {}_V pts_y \end{pmatrix} = \begin{pmatrix} cos(\psi^*) & sin(\psi^*) \\ -sin(\psi^*) & cos(\psi^*) \end{pmatrix} \left(\begin{pmatrix} {}_E pts_x \\ {}_E pts_y \end{pmatrix} - \begin{pmatrix} {}_E p_x^* \\ {}_E p_y^* \end{pmatrix} \right) \tag{1}$$

with ${}_V pts_{x,y}$ denoting the trajectory in vehicle coordinates. The transformed trajectory can also be displayed in the simulator. If the actual car position and orientation are used directly it turns out that the trajectory tends to „move“ in the simulator display. This can be improved by introducing a $t_m = 100ms$ measurement delay and to propagate the actual vehicle position that is used in the transformation by this delay as follows (this is signified by using the asterisk in eq. 1).

$$\begin{aligned} {}_E p_x^* &= {}_E p_x + v cos(\psi) t_m & (2) \\ {}_E p_y^* &= {}_E p_y + v sin(\psi) t_m & (3) \\ \psi^* &= \psi - \frac{v}{L_f} \delta t_m & (4) \end{aligned}$$

L_f is the vehicle length here.
In the following a (second order) polynomial of the form

$$f(x) = a_2 x^2 + a_1 x + a_0 \tag{5}$$

is fitted through the transformed trajectory points.

1.2 MPC

In this section the MPC algorithm is presented. As there exists a delay ($t_d = 100ms$) between control computation and actual output to the vehicle, the initialization values that will be sent to the MPC are predicted for this delay as follows:

$$
\begin{aligned}
{}_V p_x &= v * t_d && (6)\\
{}_V p_y &= 0 && (7)\\
{}_V \psi &= -\frac{v}{L_f} \delta t_d && (8)\\
v &= v + a t_d && (9)\\
{}_V cte &= -f(p_x) && (10)\\
{}_V epsi &= -atan\left(a_1 + 2a_2 \ {}_V p_x\right) && (11)
\end{aligned}
$$

${}_V cte$ and ${}_V cte$ in this context are the crosstrack and the orientation error respectively. The MPC problem can then finally be formulated as follows:

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \sum_{i=0}^{N-1} J &= \dots && (12)\\
s.t. \quad x_{i+1} &= x_i + v_i \cos(\psi_i) \Delta && (13)\\
y_{i+1} &= y_t + v_i \sin(\psi_i) \Delta && (14)\\
\psi_{i+1} &= \psi_i - \frac{v_i}{L_f} \delta_i \Delta && (15)\\
v_{i+1} &= v_i + a_i \Delta && (16)\\
cte_{i+1} &= (y_i - f(x_i)) + v_i \sin(\psi_i) \Delta && (17)\\
epsi_{i+1} &= (\psi_i - (a_1 + a_2 x_i)) - v_i \sin(\psi_i) \Delta && (18)\\
-0.44 &\le \delta_i \le 0.44 && (19)\\
-1 &\le a_i \le 1 && (20)
\end{aligned}
$$

with Δ for the MPC time step ($0.05s$ in this case). The prediction horizon is chosen as $N = 20$ steps. Together with the MPC time step this gives a lookahead time of $1s$ which was found to perform well for the tested vehicle speeds (less lookahead (e.g $N = 10$) resulted in jerky motion of the car; more lookahead (e.g $N = 30$) takes parts of the road into consideration that are currently still not relevant for the controller computations).
The J cost function that shall be minimized by the optimizer has the following goals:

- reduce crosstrack error
- reduce vehicle orientation error
- reduce deviation from vehicle reference speed (set to 60mph).
- penalize large control action (for steering angle and throttle)

- penalize rapid changes in control actions (for steering and throttle)

Mathematically this is formulated as follows:

$$
\begin{aligned}
J &= W_{cte} cte_i^2 + W_{epsi} epsi_i^2 + W_v \left(v_i - v_{ref}\right)^2 && (21)\\
&+ W_\delta \delta_i^2 + W_a a_i^2 && (22)\\
&+ W_{\dot{\delta}} \left(\delta_{i+1} - \delta_i\right)^2 + W_{\dot{a}} \left(a_{i+1} - a_i\right)^2 && (23)
\end{aligned}
$$

The coefficients that are used to „weigh“ the different aspects of the cost function are chosen by experiment as follows:

$$
\begin{aligned}
& W_{cte} = 2; \quad W_{epsi} = 2; \quad W_v = 1; && (24)\\
& W_\delta = 10; \quad W_a = 1; && (25)\\
& W_{\dot{\delta}} = 50000; \quad W_{\dot{a}} = 500 && (26)
\end{aligned}
$$