6-Degrees of Freedom (6-DOF) Equations of Motion for BlueROV2 using Fossen Model

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Contents

1 Introduction

This document presents the complete equations of **BlueROV2** (6DOF) motion based on the **Fossen model**, along with detailed definitions of all involved matrices and symbols.

2 6-DOF Dynamics

Degree of Freedom	Pose / Angle	Velocity / Angular Velocity	Force / Moment
Surge	x_b	v_{xb}, u	X
Sway	y_b	v_{yb}, v	Y
Heave	z_b	v_{zb}, w	Z
Roll	ϕ	v_ϕ, p	K
Pitch	θ	v_θ, q	M
Yaw	ψ	v_ψ, r	N

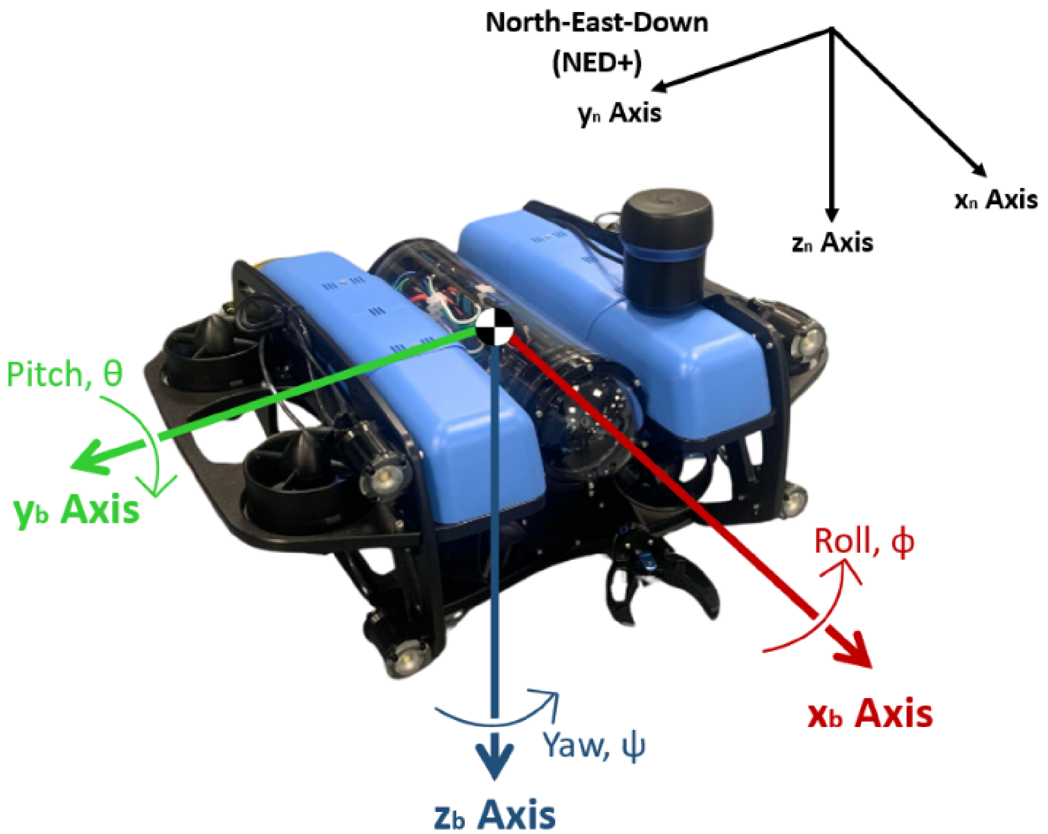

Figure 1: BlueROV.

2.1 Added Mass ($\mathbf{M}_A$)

Definition: Added Mass refers to the additional inertia that a vehicle experiences due to the acceleration of the surrounding water as it moves. When the ROV accelerates, it not only moves its own mass but also has to push some of the water out of the way, effectively increasing its total mass (pushing object through the water affects an extra "resistance").

Fossen: Hydrodynamic added mass can be seen as a virtual mass added to a system because an accelerating or decelerating body must move some volume of the surrounding fluid as it moves through it. Moreover, the object and fluid cannot occupy the same physical space simultaneously.

2.2 Damping (D)

Definition: Damping represents the **resistive forces** that oppose the ROV's motion through water. It includes both linear damping (proportional to velocity) and nonlinear (quadratic) damping (proportional to the square of velocity).

2.3 Coriolis and Centripetal Forces (C)

Definition: These are velocity-dependent inertial forces that arise from the ROV's motion and rotation. **Coriolis forces** result from the interaction between the ROV's linear and angular velocities, while **Centripetal forces** arise from the ROV's rotational motion.

2.4 Restoring Forces and Moments (G)

Definition: Restoring Forces and **Moments** are natural forces that act to bring the ROV back to its equilibrium position and orientation. These arise from the balance between **gravity**, **buoyancy**, and the relative positions of the **Center of Gravity (CoG)** and **Center of Buoyancy (CoB)**.

2.5 Hydrostatic Forces

Definition: Hydrostatic Forces = Buoyancy Force + Gravitational Force

2.6 Hydrodynamic Forces

Definition: Hydrodynamic Forces = Added Mass (intertia added) + Drag Force + Lift Force encompass all the forces exerted by water on the **moving ROV**.

2.7 Summary of Effects

- **Added Mass ($\mathbf{M}_A$):** Extra inertia from accelerating surrounding water.
- **Damping (D):** Resistive forces opposing motion, both linear and nonlinear.
- **Coriolis and Centripetal Forces (C):** Inertial forces from motion and rotation.
- **Restoring Forces and Moments (G):** Natural forces pushing the ROV back to equilibrium.

- **Hydrostatic Forces:** Forces from water displacement balancing buoyancy and gravity.
- **Hydrodynamic Forces:** Forces from water flow affecting movement and stability.
- **Hydrostatic Stability:** Ability to maintain desired orientation and position.
- **Hydrodynamic Stability:** Smooth and controlled response to dynamic hydrodynamic forces.

3 Complete 6-DOF Equations of Motion

The **6-DOF equations of motion** for the **BlueROV2** using the **Fossen model** are expressed as follows:

$$\underbrace{\mathbf{M}_{RB}\dot{\nu} + \mathbf{C}_{RB}(\nu)\nu}_{\text{rigid-body forces}} + \underbrace{\mathbf{M}_{A}\dot{\nu} + \mathbf{C}_{A}(\nu_r)\nu_r + \mathbf{D}(\nu_r)\nu_r + \mathbf{G}(\eta)}_{\text{hydrodynamical forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{ext}}$$

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c, \quad \text{where } \boldsymbol{\nu}_c \in \mathbb{R}^{6\times 1} \text{ is the underwater current decomposed in the body frame.} \tag{1}$$

4 Generalized Coordinates

The generalized position and velocity vectors are defined as follows:
Generalized Position and Velocity

$$\boldsymbol{\eta} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

4.1 Kinematic Equations

In addition to the dynamic equations, the **kinematic equations** relate the vehicle's **position** and **orientation** to its **velocities**:

$$\dot{\mathbf{p}} = \mathbf{R}(\boldsymbol{\eta}) \cdot \boldsymbol{\nu}_{\text{trans}} \tag{2}$$
$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta}) \cdot \boldsymbol{\nu}_{\text{rot}} \tag{3}$$

5 Detailed Definitions of Matrices and Symbols

5.1 Mass-Inertia Matrix (M)

5.1.1 Rigid-Body Mass and Inertia ($\mathbf{M}_{RB}$)

$$\mathbf{M}_{RB} = \begin{bmatrix} m & 0 & 0 & 0 & mz_R & -my_R \\ 0 & m & 0 & -mz_R & 0 & mx_R \\ 0 & 0 & m & my_R & -mx_R & 0 \\ 0 & -mz_R & my_R & I_{xx} & 0 & 0 \\ mz_R & 0 & -mx_R & 0 & I_{yy} & 0 \\ -my_R & mx_R & 0 & 0 & 0 & I_{zz} \end{bmatrix}$$

Symbol Definitions:

Table 1: Rigid-Body Mass and Inertia Matrix Symbols

Symbol	Description
m	Mass of the ROV (kg)
I_{xx}	Moment of inertia about the X-axis (kg·m^2)
I_{yy}	Moment of inertia about the Y-axis (kg·m^2)
I_{zz}	Moment of inertia about the Z-axis (kg·m^2)
x_R, y_R, z_R	Center of Mass coordinates relative to the body frame (m)

5.1.2 Added Mass ($\mathbf{M}_A$)

$$\mathbf{M}_A = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{\dot{v}} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{\dot{w}} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{\dot{p}} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{\dot{q}} & 0 \\ 0 & 0 & 0 & 0 & 0 & N_{\dot{r}} \end{bmatrix}$$

Symbol Definitions:

Table 2: Notation and Description of Parameters

Symbol	Description
K_p, M_q, N_r	Linear damping coefficients for rotation in water.
$K_{p\|p}, M_{q\|q}, N_{r\|r}$	Quadratic damping coefficients for rotation in water.
$K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}}$	Increased inertia about x, y, z-axis due to rotation in water.
X_u, Y_v, Z_w	Linear damping coefficients for translation in water.
$X_{u\|u}, Y_{v\|v}, Z_{w\|w}$	Quadratic damping coefficients for translation in water.
$X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}$	Added mass in x, y, z-direction due to translation in water.

5.2 Coriolis and Centripetal Matrices (C)

$$\mathbf{C}(\boldsymbol{\nu}) = \mathbf{C}_{RB} + \mathbf{C}_A$$

5.2.1 Coriolis for Rigid Body ($\mathbf{C}_{RB}$)

$$\mathbf{C}_{RB} = \begin{bmatrix} 0 & 0 & 0 & 0 & mz_R & -my_R \\ 0 & 0 & 0 & -mz_R & 0 & mx_R \\ 0 & 0 & 0 & my_R & -mx_R & 0 \\ 0 & -mz_R & my_R & 0 & 0 & 0 \\ mz_R & 0 & -mx_R & 0 & 0 & 0 \\ -my_R & mx_R & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

5.2.2 Coriolis for Added Mass ($\mathbf{C}_A$)

$$\mathbf{C}_A = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}}w & Y_{\dot{v}}v \\ 0 & 0 & 0 & Z_{\dot{w}}w & 0 & -X_{\dot{u}}u \\ 0 & 0 & 0 & -Y_{\dot{v}}v & X_{\dot{u}}u & 0 \\ Z_{\dot{w}}w & -Y_{\dot{v}}v & X_{\dot{u}}u & 0 & -N_{\dot{r}}r & M_{\dot{q}}q \\ -Y_{\dot{v}}v & X_{\dot{u}}u & 0 & N_{\dot{r}}r & 0 & -K_{\dot{p}}p \\ X_{\dot{u}}u & 0 & -Z_{\dot{w}}w & -M_{\dot{q}}q & K_{\dot{p}}p & 0 \end{bmatrix}$$

5.3 Damping Matrices (D)

$$\mathbf{D}(\boldsymbol{\nu}) = \mathbf{D}_R + \mathbf{D}_A$$

5.3.1 Linear Damping ($\mathbf{D}_R$)

$$\mathbf{D}_R = \begin{bmatrix} X_u & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_w & 0 & 0 & 0 \\ 0 & 0 & 0 & K_p & 0 & 0 \\ 0 & 0 & 0 & 0 & M_q & 0 \\ 0 & 0 & 0 & 0 & 0 & N_r \end{bmatrix}$$

Symbol Definitions:

Table 3: Linear Damping Matrix Symbols

Symbol	**Description**
X_u	Linear damping in Surge (kg/s)
Y_v	Linear damping in Sway (kg/s)
Z_w	Linear damping in Heave (kg/s)
K_p	Linear damping for Roll (kg·m^2/s)
M_q	Linear damping for Pitch (kg·m^2/s)
N_r	Linear damping for Yaw (kg·m^2/s)

5.3.2 Nonlinear (Quadratic) Damping ($\mathbf{D}_A$)

$$\mathbf{D}_A = \begin{bmatrix} X_{uu} & X_{uv} & X_{uw} & X_{up} & X_{uq} & X_{ur} \\ Y_{vu} & Y_{vv} & Y_{vw} & Y_{vp} & Y_{vq} & Y_{vr} \\ Z_{wu} & Z_{wv} & Z_{ww} & Z_{wp} & Z_{wq} & Z_{wr} \\ K_{pu} & K_{pv} & K_{pw} & K_{pp} & K_{pq} & K_{pr} \\ M_{qu} & M_{qv} & M_{qw} & M_{qp} & M_{qq} & M_{qr} \\ N_{ru} & N_{rv} & N_{rw} & N_{rp} & N_{rq} & N_{rr} \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

Symbol Definitions:

Table 4: Nonlinear Damping Matrix Symbols

Symbol	Description
X_{uu}	Quadratic damping in Surge (kg/m)
Y_{vv}	Quadratic damping in Sway (kg/m)
Z_{ww}	Quadratic damping in Heave (kg/m)
K_{pp}	Quadratic damping for Roll (kg·m^2/m)
M_{qq}	Quadratic damping for Pitch (kg·m^2/m)
N_{rr}	Quadratic damping for Yaw (kg·m^2/m)

Note: Cross-damping terms ($X_{uv}, X_{uw}, \ldots$) are often negligible for symmetric ROVs and may be omitted for simplicity.

$$\mathbf{D} = \begin{bmatrix} X_{u|u} + X_u & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{v|v} + Y_v & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{w|w} + Z_w & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{p|p} + K_p & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{q|q} + M_q & 0 \\ 0 & 0 & 0 & 0 & 0 & N_{r|r} + N_r \end{bmatrix}$$

Symbol	Description
X_u	Linear damping in Surge (kg/s)
Y_v	Linear damping in Sway (kg/s)
Z_w	Linear damping in Heave (kg/s)
K_p	Linear damping for Roll (kg·m^2/s)
M_q	Linear damping for Pitch (kg·m^2/s)
N_r	Linear damping for Yaw (kg·m^2/s)
X_{uu}	Quadratic damping in Surge (kg/m)
Y_{vv}	Quadratic damping in Sway (kg/m)
Z_{ww}	Quadratic damping in Heave (kg/m)
K_{pp}	Quadratic damping for Roll (kg·m^2/m)
M_{qq}	Quadratic damping for Pitch (kg·m^2/m)
N_{rr}	Quadratic damping for Yaw (kg·m^2/m)

Table 5: Linear and quadratic damping coefficients.

5.4 Restoring Forces and Moments (G)

$$\mathbf{G}(\boldsymbol{\eta}) = \begin{bmatrix} G_x \\ G_y \\ G_z \\ G_\phi \\ G_\theta \\ G_\psi \end{bmatrix} = \begin{bmatrix} -(B-W)g \cdot \theta \\ (B-W)g \cdot \phi \\ Bg - Wg \\ B \cdot y_{CG} - W \cdot z_{CG} \\ -B \cdot x_{CG} + W \cdot z_{CG} \\ 0 \end{bmatrix}$$

Symbol Definitions:

Table 6: Restoring Forces and Moments Symbols

Symbol	**Description**
G_x	Restoring force in Surge (N)
G_y	Restoring force in Sway (N)
G_z	Restoring force in Heave (N)
G_ϕ	Restoring moment about Roll axis (N·m)
G_θ	Restoring moment about Pitch axis (N·m)
G_ψ	Restoring moment about Yaw axis (N·m)
B	Buoyant force (N)
W	Weight of the ROV (N)
g	Acceleration due to gravity ($\approx 9.81\,\mathrm{m/s^2}$)
x_{CG}, y_{CG}, z_{CG}	Center of Gravity coordinates relative to body frame (m)
ϕ, θ, ψ	Roll, Pitch, Yaw angles (rad)

5.5 Control Input Vector ($\boldsymbol{\tau}$)

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_u \\ \tau_v \\ \tau_w \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} F_{\text{surge}} \\ F_{\text{sway}} \\ F_{\text{heave}} \\ M_{\text{roll}} \\ M_{\text{pitch}} \\ M_{\text{yaw}} \end{bmatrix}$$

Symbol Definitions:

Table 7: Control Input Vector Symbols

Symbol	**Description**
F_{surge}	Control force in Surge (N)
F_{sway}	Control force in Sway (N)
F_{heave}	Control force in Heave (N)
M_{roll}	Control moment about Roll axis (N·m)
M_{pitch}	Control moment about Pitch axis (N·m)
M_{yaw}	Control moment about Yaw axis (N·m)

5.6 External Disturbance Vector ($\boldsymbol{\tau}_{\text{ext}}$)

$$\boldsymbol{\tau}_{\text{ext}} = \begin{bmatrix} \tau_{\text{ext},u} \\ \tau_{\text{ext},v} \\ \tau_{\text{ext},w} \\ \tau_{\text{ext},\phi} \\ \tau_{\text{ext},\theta} \\ \tau_{\text{ext},\psi} \end{bmatrix}$$

Symbol Definitions:

Table 8: External Disturbance Vector Symbols

Symbol	**Description**
$\tau_{\text{ext},u}$	External disturbance force in Surge (N)
$\tau_{\text{ext},v}$	External disturbance force in Sway (N)
$\tau_{\text{ext},w}$	External disturbance force in Heave (N)
$\tau_{\text{ext},\phi}$	External disturbance moment about Roll axis (N·m)
$\tau_{\text{ext},\theta}$	External disturbance moment about Pitch axis (N·m)
$\tau_{\text{ext},\psi}$	External disturbance moment about Yaw axis (N·m)

5.7 Kinematic Matrices

5.7.1 Rotation Matrix ($\mathbf{R}(\boldsymbol{\eta})$)

$$\mathbf{R}(\boldsymbol{\eta}) = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}$$

Where:

$$c_\alpha = \cos(\alpha), \quad s_\alpha = \sin(\alpha)$$

5.7.2 Jacobian Matrix ($\mathbf{J}(\boldsymbol{\eta})$)

$$\mathbf{J}(\boldsymbol{\eta}) = \begin{bmatrix} 1 & s_\phi \tan(\theta) & c_\phi \tan(\theta) \\ 0 & c_\phi & -s_\phi \\ 0 & \frac{s_\phi}{c_\theta} & \frac{c_\phi}{c_\theta} \end{bmatrix}$$

Note: This matrix becomes singular when $\theta = \pm 90^\circ$, leading to Euler angle singularities.

5.7.3 Jacobian Matrix ($\mathbf{J}(\boldsymbol{\eta})$)

$$\mathbf{J}(\boldsymbol{\eta}) = \begin{bmatrix} 1 & s_\phi \tan(\theta) & c_\phi \tan(\theta) \\ 0 & c_\phi & -s_\phi \\ 0 & \frac{s_\phi}{c_\theta} & \frac{c_\phi}{c_\theta} \end{bmatrix}$$

Note: This matrix becomes singular when $\theta = \pm 90^\circ$, leading to Euler angle singularities.

6 Summary of Symbols

Table 9: Summary of Symbols

Symbol	Description
$\mathbf{M}$	Total Mass-Inertia Matrix ($\mathbf{M}_{RB} + \mathbf{M}_A$)
$\mathbf{M}_{RB}$	Rigid-Body Mass and Inertia Matrix
$\mathbf{M}_A$	Added Mass Matrix due to Fluid Acceleration
$\dot{\boldsymbol{\nu}}$	Acceleration Vector $[\dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}]^T$
$\mathbf{C}(\boldsymbol{\nu})$	Coriolis and Centripetal Matrix (6×6)
$\mathbf{C}_{RB}$	Coriolis and Centripetal Matrix for Rigid Body
$\mathbf{C}_A$	Coriolis and Centripetal Matrix for Added Mass
$\mathbf{D}(\boldsymbol{\nu})$	Damping (Hydrodynamic) Matrix (6×6)
$\mathbf{D}_R$	Linear Damping Matrix
$\mathbf{D}_A$	Nonlinear (Quadratic) Damping Matrix
$\mathbf{G}(\boldsymbol{\eta})$	Restoring Forces and Moments Vector (6×1)
$\boldsymbol{\tau}$	Control Input Vector $[F_{\text{surge}}, F_{\text{sway}}, F_{\text{heave}}, M_{\text{roll}}, M_{\text{pitch}}, M_{\text{yaw}}]^T$
$\boldsymbol{\tau}_{\text{ext}}$	External Disturbance Vector (6×1)
$\mathbf{p}$	Position Vector (x, y, z)
$\boldsymbol{\eta}$	Orientation Vector $[\phi, \theta, \psi]^T$
$\mathbf{R}(\boldsymbol{\eta})$	Rotation Matrix from body-fixed to inertial frame
$\mathbf{J}(\boldsymbol{\eta})$	Jacobian Matrix for Kinematics
u, v, w	Linear Velocities in Surge, Sway, Heave (m/s)
p, q, r	Angular Velocities in Roll, Pitch, Yaw (rad/s)
ϕ, θ, ψ	Roll, Pitch, Yaw Angles (rad)
X_u, Y_v, Z_w	Linear Damping Coefficients in Surge, Sway, Heave (kg/s)
K_p, M_q, N_r	Rotational Damping Coefficients in Roll, Pitch, Yaw (kg·m^2/s)
X_{uu}, Y_{vv}, Z_{ww}	Quadratic Damping Coefficients for Surge, Sway, Heave (kg/m)
K_{pp}, M_{qq}, N_{rr}	Quadratic Damping Coefficients for Roll, Pitch, Yaw (kg·m^2/m)
x_R, y_R, z_R	Center of Mass Coordinates relative to body frame (m)
I_{xx}, I_{yy}, I_{zz}	Moments of Inertia about Roll, Pitch, Yaw Axes (kg·m^2)
B	Buoyant Force (N)
W	Weight of the ROV (N)
g	Acceleration due to Gravity ($\approx 9.81\,\mathrm{m/s^2}$)

7 Thruster Allocation Matrix for BlueROV2 Heavy

The thruster allocation matrix T for the BlueROV2 Heavy is used to map the forces and torques generated by the vehicle's thrusters to the six degrees of freedom (DOF) in the vehicle's body-fixed frame. The matrix T is defined as follows:

$$T = \begin{bmatrix} -0.71 & -0.71 & 0.71 & 0.71 & 0 & 0 & 0 & 0 \\ 0.71 & -0.71 & 0.71 & -0.71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 1.0 & 1.0 & 1.0 \\ -0.06 & 0.06 & -0.06 & 0.06 & 0.22 & -0.22 & 0.22 & -0.22 \\ -0.06 & -0.06 & 0.06 & 0.06 & -0.12 & -0.12 & 0.12 & 0.12 \\ 0.99 & -0.99 & -0.99 & 0.99 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.1 Mapping of Rows to Directions

The rows of the thruster allocation matrix correspond to the six degrees of freedom (DOF) for the BlueROV2 Heavy:

- **Row 1 (Surge)**: Represents forward and backward movement along the x-axis in the body frame.
- **Row 2 (Sway)**: Represents left and right movement along the y-axis in the body frame.
- **Row 3 (Heave)**: Represents upward and downward movement along the z-axis in the body frame.
- **Row 4 (Roll)**: Represents rotational movement about the x-axis (rolling motion).
- **Row 5 (Pitch)**: Represents rotational movement about the y-axis (pitching motion).
- **Row 6 (Yaw)**: Represents rotational movement about the z-axis (yawing motion).

7.2 Explanation

The matrix T defines how the thrusters contribute to each direction or rotation. Each column corresponds to one of the eight thrusters on the vehicle. For example, the first column describes how Thruster 1 contributes to surge, sway, heave, roll, pitch, and yaw.

To calculate the forces and torques τ on the vehicle, the matrix is applied as:

$$\tau = Tm$$

where m is the vector of forces produced by the eight thrusters. Conversely, to determine the thruster forces required for a desired motion, the pseudo-inverse $T^{\dagger}$ is used:

$$m = T^{\dagger}\tau$$