Problem 1.

Consider two topologies on $\mathbb{R}$: the Euclidean (or usual) topology $\mathcal{T}_E$, and the topology $\mathcal{T}_{CF}$ in which the *closed* sets are $\mathbb{R}$ and all the finite subsets of $\mathbb{R}$.

(a) Suppose $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is continuous. Prove that it is also continuous as a map $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$.

(b) Give an example of a continuous map $g : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$ that is *not* continuous as a map $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$.

Suggestion: Try strictly increasing piecewise linear maps.

(c) Find all continuous maps $f : (\mathbb{R}, \mathcal{T}_{CF}) \to (\mathbb{R}, \mathcal{T}_E)$.

Solution

Part (a)

Let $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$. f is continuous iff $\forall U \subset \mathcal{T}_{CF}$ when U open in $\mathcal{T}_{CF} \implies f^{-1}(U)$ is open in $\mathcal{T}_E$. Let

$$U = \left(\bigcup_{i=1}^{N} x_i \right)^{\mathsf{c}}$$

be an open set in the topology $\mathcal{T}_{CF}$, then by De Morgan's Laws

$$U = \bigcap_{i=1}^{N} x_i^{\mathsf{c}}$$

where x_i^{c} is open in $\mathcal{T}_E$, and because finite intersections of open sets is open, U must be open in $\mathcal{T}_E$. The mapping $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is continuous $\implies f^{-1}(U)$ is open, thus f is continuous.

Part (b)

Part (c)

Problem 2.

(a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topologiclal spaces let $f, g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous maps, and suppose that $(Y, \mathcal{T}_Y)$ is Hausdorff. Prove that

$$E(f, g) = \{x \in X \mid f(x) = g(x)\}$$

is *closed* in $(X, \mathcal{T}_X)$.

(b) Give an example of two continuous maps f, g with $E(f, g)$ not closed. (If possible, try to find an example with $(X, \mathcal{T}_X)$ Hausdorff.)

Solution

Part (a)

Problem 3.

Let (X, d) be a metric space, let $x \in X$ and let $A \subset X$ be non-empty. Define the *distance between x and A*, denoted $d(x, A)$, by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

(a) Prove that $d(x, A)$ is a continuous function of x.

Suggestion: Prove more: it is a Lipschitz function, with Lipschitz constant 1.

(b) Prove that $d(x, A) = 0 \iff x \in \overline{A}$.

(c) Prove that, if A is closed, then there exists a continuous function $f : X \to \mathbb{R}$ so that $A = \{x \in X | f(x) = 0\}$.

(d) Suppose $A, B \subset X$ are *closed* sets, non-empty, and *disjoint*: $A \cap B = \emptyset$. Prove that there exists a contiuous function $g : X \to [0, 1]$ such that

$$g(x) = 0 \iff x \in A \text{ and } g(x) = 1 \iff x \in B.$$

Suggestion: Experiment with functions with $d(x, A) + d(x, B)$ as denominator.

(e) Prove that if A and B are disjoint, non-empty, closed sets as above, there exist open sets $U, V \subset X$ so that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Solution

Part (a)