

Problem 1.

Consider two topologies on \mathbb{R} : the Euclidean (or usual) topology \mathcal{T}_E , and the topology \mathcal{T}_{CF} in which the *closed* sets are \mathbb{R} and all the finite subsets of \mathbb{R} .

(a) Suppose $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ is continuous. Prove that it is also continuous as a map $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$.

(b) Give an example of a continuous map $g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$ that is *not* continuous as a map $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$.

Suggestion: Try strictly increasing piecewise linear maps.

(c) Find all continuous maps $f : (\mathbb{R}, \mathcal{T}_{CF}) \rightarrow (\mathbb{R}, \mathcal{T}_E)$.

Solution**Part (a)**

Let $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$. f is continuous iff $\forall U \subset \mathbb{R}$ when U closed in $\mathcal{T}_{CF} \implies f^{-1}(U)$ is closed in \mathcal{T}_E . Let

$$U = \bigcup_{i=1}^N x_i \quad (1)$$

be a closed set in the topology \mathcal{T}_{CF} , then this set is also closed in \mathcal{T}_E because a single point x_i is closed and the closed sets are closed under finite union. The mapping $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ is continuous $\implies f^{-1}(U)$ is closed, thus f is continuous.

Part (b)

A continuous map $g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$ that is *not* continuous as $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ is

$$g(x) = \begin{cases} x & x \leq 0 \\ x + 1 & x > 0 \end{cases} \quad (2)$$

This map is continuous $\mathcal{T}_E \rightarrow \mathcal{T}_{CF}$ as the preimage of any finite set of points (which is closed in \mathcal{T}_{CF}) is closed in \mathcal{T}_E :

$$\text{Let } U = \bigcup_{i=1}^N x_i \quad (3)$$

$$f^{-1}(U) = \left\{ \bigcup x_i \mid \forall x_i \leq 0 \right\} \cup \left\{ \bigcup x_i + 1 \mid \forall x_i > 0 \right\} \quad (4)$$

where $x_i \in \mathbb{R}$, thus the preimage in \mathcal{T}_E is a finite union of closed sets, which is also closed.

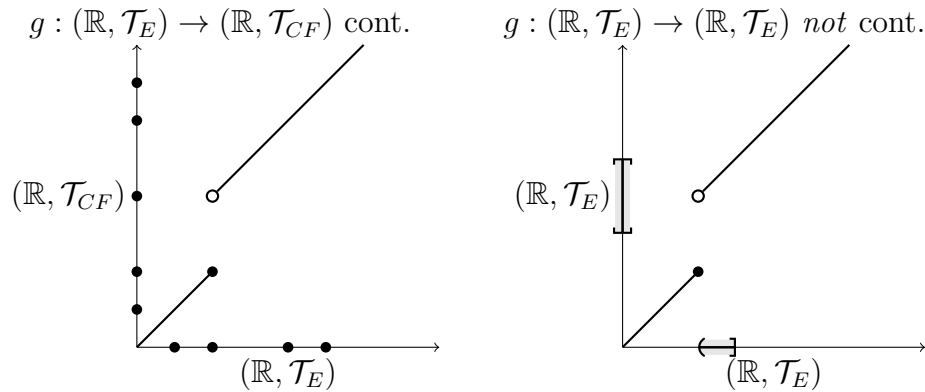
$g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ is *not* continuous because the preimage of a \mathcal{T}_E -closed set is not \mathcal{T}_E -closed:

$$\text{Let } U = \left[\frac{1}{2}, \frac{3}{2} \right] \quad (5)$$

$$f^{-1}(U) = \left(0, \frac{1}{2} \right] \quad (6)$$

This preimage is not \mathcal{T}_E -closed, so the map is not continuous, because a map is continuous iff the preimage of every closed set is closed.

These diagrams illustrate these cases:



Note that for this g , $g^{-1}((1, 2]) = \emptyset$, which is still closed.

Part (c)

Problem 2.

- (a) Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces let $f, g : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ be continuous maps, and suppose that (Y, \mathcal{T}_Y) is Hausdorff. Prove that

$$E(f, g) = \{x \in X \mid f(x) = g(x)\}$$

is *closed* in (X, \mathcal{T}_X) .

- (b) Give an example of two continuous maps f, g with $E(f, g)$ not closed. (If possible, try to find an example with (X, \mathcal{T}_X) Hausdorff.)

Solution**Part (a)**

Problem 3.

Let (X, d) be a metric space, let $x \in X$ and let $A \subset X$ be non-empty. Define the *distance between x and A* , denoted $d(x, A)$, by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

- (a) Prove that $d(x, A)$ is a continuous function of x .

Suggestion: Prove more: it is a Lipschitz function, with Lipschitz constant 1.

- (b) Prove that $d(x, A) = 0 \iff x \in \overline{A}$.

- (c) Prove that, if A is closed, then there exists a continuous function $f : X \rightarrow \mathbb{R}$ so that $A = \{x \in X \mid f(x) = 0\}$.

- (d) Suppose $A, B \subset X$ are *closed* sets, non-empty, and *disjoint*: $A \cap B = \emptyset$. Prove that there exists a continuous function $g : X \rightarrow [0, 1]$ such that

$$g(x) = 0 \iff x \in A \text{ and } g(x) = 1 \iff x \in B.$$

Suggestion: Experiment with functions with $d(x, A) + d(x, B)$ as denominator.

- (e) Prove that if A and B are disjoint, non-empty, closed sets as above, there exist open sets $U, V \subset X$ so that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Solution**Part (a)**