Problem 1.

Consider two topologies on $\mathbb{R}$: the Euclidean (or usual) topology $\mathcal{T}_E$, and the topology $\mathcal{T}_{CF}$ in which the *closed* sets are $\mathbb{R}$ and all the finite subsets of $\mathbb{R}$.

(a) Suppose $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is continuous. Prove that it is also continuous as a map $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$.

(b) Give an example of a continuous map $g : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$ that is *not* continuous as a map $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$.

Suggestion: Try strictly increasing piecewise linear maps.

(c) Find all continuous maps $f : (\mathbb{R}, \mathcal{T}_{CF}) \to (\mathbb{R}, \mathcal{T}_E)$.

Solution

Part (a)

Let $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$. f is continuous iff $\forall U \subset \mathbb{R}$ when U closed in $\mathcal{T}_{CF} \implies f^{-1}(U)$ is closed in $\mathcal{T}_E$. Let

$$U = \bigcup_{i=1}^{N} x_i \tag{1}$$

be a closed set in the topology $\mathcal{T}_{CF}$, then this set is also closed in $\mathcal{T}_E$ because a single point x_i is closed and the closed sets are closed under finite union. The mapping $f : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is continuous $\implies f^{-1}(U)$ is closed, thus f is continuous.

Part (b)

A continuous map $g : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_{CF})$ that is *not* continuous as $(\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is

$$g(x) = \begin{cases} x & x \leq 0 \\ x+1 & x > 0 \end{cases} \tag{2}$$

This map is continuous $\mathcal{T}_E \to \mathcal{T}_{CF}$ as the preimage of any finite set of points (which is closed in $\mathcal{T}_{CF}$) is closed in $\mathcal{T}_E$:

$$\text{Let } U = \bigcup_{i=1}^{N} x_i \tag{3}$$

$$f^{-1}(U) = \left\{ \bigcup x_i \ \forall x_i \leq 0 \right\} \cup \left\{ \bigcup x_i + 1 \ \forall x_i > 0 \right\} \tag{4}$$

where $x_i \in \mathbb{R}$, thus the preimage in $\mathcal{T}_E$ is a finite union of closed sets, which is also closed.

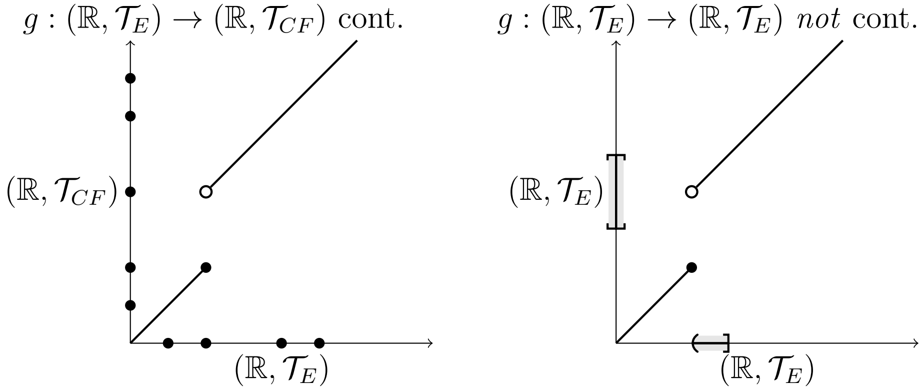
$g : (\mathbb{R}, \mathcal{T}_E) \to (\mathbb{R}, \mathcal{T}_E)$ is *not* continuous because the preimage of a $\mathcal{T}_E$-closed set is not $\mathcal{T}_E$-closed:

$$\text{Let } U = \left[\frac{1}{2}, \frac{3}{2}\right] \tag{5}$$

$$f^{-1}(U) = \left(0, \frac{1}{2}\right] \tag{6}$$

This preimage is not $\mathcal{T}_E$-closed, so the map is not continuous, because a map is continuous iff the preimage of every closed set is closed.

These diagrams illustrate these cases:



Note that for this g, $g^{-1}\big((1,2]\big) = \emptyset$, which is still closed.

Part (c)

Constant mappings.

Problem 2.

(a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces let $f, g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous maps, and suppose that $(Y, \mathcal{T}_Y)$ is Hausdorff. Prove that

$$E(f, g) = \{x \in X \mid f(x) = g(x)\}$$

is *closed* in $(X, \mathcal{T}_X)$.

(b) Give an example of two continuous maps f, g with $E(f, g)$ not closed. (If possible, try to find an example with $(X, \mathcal{T}_X)$ Hausdorff.)

Solution

Part (a)

Suppose $f(x) \neq g(x)$, because Y is Hausdorff $\exists U =$ nbd of $f(x)$, $V =$ nbd of $g(x)$ where $U \cap V = \emptyset$. The preimages of these open sets remain open as f, g continuous, and since they both contain x: $f^{-1}(U)$, $g^{-1}(V) \subset X$ are open nbds of x. Let $W = f^{-1}(U) \cap g^{-1}(V) \implies x \in W$. W is still open as $\mathcal{T}_X$ is closed under finite intersection. These x are such that $f(x) \neq g(x)$, so they are in the complement of E: $x \in W \subset E^{\mathsf{c}} \implies E^{\mathsf{c}}$ open $\implies E$ closed.

Part (b)

Problem 3.

Let (X, d) be a metric space, let $x \in X$ and let $A \subset X$ be non-empty. Define the *distance between x and A*, denoted $d(x, A)$, by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

(a) Prove that $d(x, A)$ is a continuous function of x.

Suggestion: Prove more: it is a Lipschitz function, with Lipschitz constant 1.

(b) Prove that $d(x, A) = 0 \iff x \in \overline{A}$.

(c) Prove that, if A is closed, then there exists a continuous function $f : X \to \mathbb{R}$ so that $A = \{x \in X | f(x) = 0\}$.

(d) Suppose $A, B \subset X$ are *closed* sets, non-empty, and *disjoint*: $A \cap B = \emptyset$. Prove that there exists a contiuous function $g : X \to [0, 1]$ such that

$$g(x) = 0 \iff x \in A \text{ and } g(x) = 1 \iff x \in B.$$

Suggestion: Experiment with functions with $d(x, A) + d(x, B)$ as denominator.

(e) Prove that if A and B are disjoint, non-empty, closed sets as above, there exist open sets $U, V \subset X$ so that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Solution

Part (a)

To show this, let x and y be points in X and p be a point in A:

$$d(x, p) \leq d(x, y) + d(y, p) \qquad \text{(triangle inequality)}$$

$$d(x, A) \leq d(x, y) + d(y, p)$$

as $d(x, A)$ is the infimum. Then:

$$d(x, A) - d(x, y) \leq d(y, p)$$

$$d(x, A) - d(x, y) \leq d(y, A)$$

as $d(y, A)$ is the infimum. Thus:

$$d(x, A) - d(y, A) \leq d(x, y) \tag{7}$$

If we look at the definition of Lipschitz, f is Lipshitz iff $d'(f(x), f(y)) \leq Cd(x, y)$. In this case f is $d(x, A)$, thus:

$$d'(f(x), f(y)) = d'(d(x, A), d(y, A)) \leq Cd(x, y)$$

$$|d(x, A) - d(y, A)| \leq Cd(x, y) \tag{8}$$

In (7), the definition of Lipschitz, (8), was proven with a Lipschitz constant of $C = 1$. $d(x, A)$ being Lipschitz $\implies$ $d(x, A)$ is uniformly continuous $\implies$ $d(x, A)$ is continuous.

Part (b)

Let $x \in \bar{A}$. Then for any $\epsilon > 0$ there exists an open ball $B_\epsilon(x)$ centered around x and of radius ϵ such that $B_\epsilon(x) \cap A \neq \emptyset$. Let $y_\epsilon \in B_\epsilon(x) \cap A$. Then $d(x, A) \leq d(x, y_\epsilon) < \epsilon$, and since $\epsilon > 0$ was arbitrary, it follows that $d(x, A) = \inf\{d(x, y_\epsilon)\} = 0$.

Conversely, if $d(x, A) = 0$ then for any $\epsilon > 0$ there exists $y_\epsilon \in A$ such that $d(x, y_\epsilon) < \epsilon$, and so $B_\epsilon(x) \cap A \neq \emptyset$. But any open ball centered at x will contain $B_\epsilon(x)$ for some $\epsilon > 0$ and hence have nonempty intersection with A.

Part (c)

The distance function given satisfies these conditions for A closed:

$$f(x) = d(x, A) : X \to \mathbb{R} \tag{9}$$

f is continuous as shown in (a). As shown in (b), $A = \{x \in X | f(x) = 0\}$ as A is its own closure.

Part (d)

The proposed function exists and can be defined as

$$g(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} \tag{10}$$

To show $x \in A \implies g(x) = 0$:

$$\text{(b)} \implies d(x, A) = 0 \tag{11}$$
$$A \cap B = \emptyset \implies d(x, B) = \varepsilon,\ \varepsilon > 0 \tag{12}$$
$$g(x) = \frac{0}{0 + \varepsilon} = 0 \tag{13}$$

To show $x \in B \implies g(x) = 1$:

$$\text{(b)} \implies d(x, B) = 0 \tag{14}$$
$$A \cap B = \emptyset \implies d(x, A) = \varepsilon,\ \varepsilon > 0 \tag{15}$$
$$g(x) = \frac{\varepsilon}{\varepsilon + 0} = 1 \tag{16}$$

To show $g(x) = 0 \implies x \in A$:

$$g(x) = 0 \implies d(x, A) = 0 \tag{17}$$
$$\text{(b)} \implies x \in A \tag{18}$$

To show $g(x) = 1 \implies x \in B$:

$$g(x) = 1 \implies d(x, A) = d(x, A) + d(x, B) \tag{19}$$
$$d(x, A) - d(x, A) = d(x, B) \implies d(x, B) = 0 \implies x \in B \tag{20}$$

Part (e)

As X is a metric space, it is metrizable, thus the topology on X is Hausdorff. This allows the following constructions:

$$U = \bigcup \{L \subset X | \forall x \in A,\ y \in B \text{ (so } x, y \in X),\ L = \text{nbd}(x) \text{ open},\ M = \text{nbd}(y) \text{ open, s.t. } L \cap M = \emptyset\} \tag{21}$$

$$V = \bigcup \{M \subset X | \forall x \in A,\ y \in B \text{ (so } x, y \in X),\ L = \text{nbd}(x) \text{ open},\ M = \text{nbd}(y) \text{ open, s.t. } L \cap M = \emptyset\} \tag{22}$$

so the sets never intersect and meet the conditions.