Problem 1.

A topological space X is said to be *locally connected* if it has a basis consisting of connected open sets.

(a) Prove that if X is locally connected, then its connected components are open.

(b) Prove that if X is locally connected and Y is the space of its connected components, with the identification topology, then Y is discrete.

Solution

Part (a)

Let $x \in C$, where C is a connected component of X. By definition, x is contained in some open connected subset U of X. Since C is a maximal connected set containing x, $x \in U \subseteq C$. Because C is thus the union of all possible U and openness is closed under arbitrary union, this shows that C is open in X.

Part (b)

In the identification topology on Y, the open sets are defined as sets of which the pre-image is open. If Y is to be discrete, then every set must be open. Because openness is closed under arbitrary union, if we show that all points are open then every set will be open. A point in the identification topology will be open if and only if the pre-image of that point is open. The pre-image of a point is a connected component. All connected components are open by (a). Thus points in Y are open, thus Y is discrete.

Problem 2.

Recall that a topological space X is said to be *totally disconnected* if for all $x \in X$, the connected component C_x of X containing x is simply $\{x\}$.

(a) Let X be a topological space, and suppose that for all $x, y \in X, x \neq y$, there exists a continuous function $f : X \to \{0,1\}$ with $f(x) \neq f(y)$. Prove that X is totally disconnected.

(b) Prove that $\{0,1\}^{\mathbb{N}}$ (with the product topology) is totally disconnected.

(c) Let $x \in \{0,1\}^{\mathbb{N}}$ be an arbitrary point. Prove that x is an accumulation point, that is, given any nbd U of x there exists $y \in U$ such that $y \neq x$.

Solution

Part (a)

Because a connected subset has a continuous, constant function that maps that subset to $\{0,1\}$ (Thm 5.2), we know that there are no connected subsets that contain both x and y. (If there were such a connected subset, then a continuous constant function must exist for that set, which would violate $f(x) \neq f(y)$). Thus connected subsets cannot contain more than one point $\implies$ Connected subsets of X are single points $\implies$ X is totally disconnected.

Part (b)

Here we apply (a) because the product topology gives continuous projection functions which, in this case, map to $\{0,1\}$. For any $x, y \in \{0,1\}^{\mathbb{N}}$, pick the projection to a factor that is different between the two points. This function satisfies requirements for (a), thus $\{0,1\}^{\mathbb{N}}$ is totally disconnected.

Part (c)

A neighborhood U is the set of points that are equal up until some factor n, and differ in at least one factor after the first n factors. Then y can be the point that is equal to x for the first n factors and differ in one or more factors after n. Thus x is an accumulation point.

Problem 3.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin, and let $N = (0, 0, 1)$ be the north pole. *Stereographic projection from the north pole* is the map $f : S^2 \setminus \{N\} \to \mathbb{R}^2$ defined by letting $f(p)$ be the point of intersection with $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ of the straight line through N and p.

(a) Find a formula for f

(b) Find a formula for the inverse map $g : \mathbb{R}^2 \to S^2 \setminus \{N\}$.

(c) Use stereographic projections from both the north and south poles to cover S^2 by the domain of two coordinate charts to $\mathbb{R}^2$ with a smooth transition function. Conclude that S^2 is a smooth surface.

Solution

Part (a)

Based on radially invariant similar triangles,

$$f_N(x, y, z) = \left(\frac{\sqrt{x^2+y^2}}{1-z} \frac{y}{x^2+y^2}, \frac{\sqrt{x^2+y^2}}{1-z} \frac{x}{x^2+y^2} \right) \tag{1}$$

Part (b)

$$g_N(x, y) = \left(\sqrt{1 - \left(\frac{x^2+y^2-1}{x^2+y^2+1}\right)^2} \frac{y}{x^2+y^2}, \sqrt{1 - \left(\frac{x^2+y^2-1}{x^2+y^2+1}\right)^2} \frac{x}{x^2+y^2}, \frac{x^2+y^2-1}{x^2+y^2+1} \right) \tag{2}$$

Part (c)

$$f_S(x, y, z) = \left(\frac{\sqrt{x^2+y^2}}{1+z} \frac{y}{x^2+y^2}, \frac{\sqrt{x^2+y^2}}{1+z} \frac{x}{x^2+y^2} \right) \tag{3}$$

$$g_S(x, y) = \left(\sqrt{1 - \left(\frac{x^2+y^2-1}{x^2+y^2+1}\right)^2} \frac{y}{x^2+y^2}, \sqrt{1 - \left(\frac{x^2+y^2-1}{x^2+y^2+1}\right)^2} \frac{x}{x^2+y^2}, -\frac{x^2+y^2-1}{x^2+y^2+1} \right) \tag{4}$$

The compositions of these functions $f_N \circ g_S$ and $f_S \circ g_N$ are the transition functions. These functions are infinitely differentiable so the surface is smooth.