

Problem 1.

- (a) Let X, Y be topological spaces and let $q : X \rightarrow Y$ be a continuous open surjection. Define $Q : X \times X \rightarrow Y \times Y$ by $Q(x_1, x_2) = (q(x_1), q(x_2))$. Prove that Q is a continuous open surjection. In particular, Q is an identification.
- (b) Let X be a Hausdorff space, let $q : X \rightarrow Y$ be an open map and an identification, and let $R \subset X \times X$ be the equivalence relation defined by q , that is,

$$R = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\}.$$

Prove that Y is Hausdorff if and only if R is closed in $X \times X$.

Suggestion Observe that $R = Q^{-1}(\Delta_Y)$ where $\Delta_Y \subset Y \times Y$ is the diagonal. Refer to Problem (1) of Homework 4.

- (c) How does this apply to the example $X = (0 \times \mathbb{R}) \cup (1 \times \mathbb{R})$ and identification $0 \times x \sim 1 \times x$ for all $x < 0$ discussed in class?

Solution**Part (a)**

Continuous: Any open set $U \in Y \times Y$ has continuous projections to each factor in the product, thus $\exists V, W$ open $\in Y$ s.t. $U = V \times W$. These open sets are in the codomain of q , which is also continuous, so by taking another pre-image there are corresponding open sets in X : $\exists S, T$ open $\in X$. As there is the product topology on $X \times X$, these open sets in $S, T \in X$ are an element of the basis for the topology on $X \times X$, so $S \times T$ open $\in X \times X$. Thus the pre-image through Q of any open set U is $S \times T$ which is open, so Q is continuous.

Open: For any open set $U \in X \times X$, there are projections to each factor of the product which are also open. As q is an open map, these open sets in X are mapped to open sets in Y . As $Y \times Y$ has the product topology, these open sets are in the basis for the topology on $Y \times Y$ so their product is also open in $Y \times Y$, thus Q is an open map. (This is the same argument as for continuity but in reverse, except using q as an open map preserves open-ness instead of continuity.)

Surjection: Given any $(y_1, y_2) \in Y \times Y$, $\exists (x_1, x_2) \in X \times X$ s.t. $Q((x_1, x_2)) = (y_1, y_2)$. This is because each factor in the product is independently mapped by a surjective map q , thus the product is also surjective.

Thus by Theorem 4.17 from the course notes, Q is an identification.

Part (b)

Suppose Y is Hausdorff. $\Delta_Y^c = \{(x_1, x_2) \mid q(x_1) \neq q(x_2), (x_1, x_2) \in X \times X\}$. Y being Hausdorff provides U, V open $\subset Y$ with $x_1 \in U, x_2 \in V$. $U \times V$ is thus a basis element with $(x_1, x_2) \in U \times V$. Because U and V are disjoint (from the Hausdorff condition), they contain no common points, so they contain nothing in Δ_Y : $U \times V \subset \Delta_Y^c \implies \Delta_Y^c$ open $\implies \Delta_Y$ closed in $Y \times Y$. As $R = Q^{-1}(\Delta_Y)$ and Q continuous, R must be closed in $X \times X$.

Suppose R is closed in $X \times X$. ???

Part (c)

From our discussion in class, we know X is not Hausdorff. This is mainly because there do not exist open neighborhoods around the ‘two separate’ zeros. This can also be shown since R is not closed in $X \times X$: R is the set of all equivalence points, which is not closed in this case because the lines are equivalent for $x < 0$ which is open at 0.

Problem 2.

In the following diagram suppose that the maps with solid arrows are given, that q_1 and q_2 are identifications and that f sends fibers of q_1 to fibers of q_2 . This means: if $x_1, x_2 \in X_1$ and $q_1(x_1) = q_1(x_2)$, then $q_2(f(x_1)) = q_2(f(x_2))$.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ q_1 \downarrow & & q_2 \downarrow \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

If $y \in Y_1$ choose $x \in X_1$ with $q_1(x) = y$ and define $g(y) = q_2(f(x))$. Prove that g is well-defined, the whole diagram commutes, and, if f is continuous, so is g .

Solution

g is well-defined because if there exist multiple $x_1, x_2 \in X_1$ s.t. $q_1(x_1) = q_1(x_2) = y$, these two x_1, x_2 are mapped to the same point in Y_2 . This is because the map f will preserve the fibers of q_1 and q_2 , and that q_1 and q_2 's fibers/equivalence classes will reduce these multiple x to equivalent points in Y_2 .

Because $g(y) = q_2(f(x))$ is well defined, and $y = q_1(x)$ for some $x \in X_1$, the diagram commutes because $g(q_1(x)) = q_2(f(x))$.

If f is continuous, so is g ???

What does it mean to say “ g is well-defined”? How is well-defined different from the definition given in the problem?

How do we show f continuous $\implies g$ continuous? Is this just by the usual definition of pre-images of open sets are open?

Problem 3.

Using the following models $Y = X/\sim$ for the Torus, Klein bottle, Möbius strip and circle,

$$\begin{array}{lll} T & = & [0, 2] \times [-1, 1] \quad / \quad (x, -1) \sim (x, 1), (0, y) \sim (2, y) \\ K & = & [0, 1] \times [-1, 1] \quad / \quad (x, -1) \sim (x, 1), (0, y) \sim (1, -y) \\ M & = & [0, 1] \times [-1/2, 1/2] \quad / \quad (0, y) \sim (1, -y) \\ S^1 & = & [0, 1] \quad / \quad 0 \sim 1 \end{array}$$

and letting $q : X \rightarrow Y$ be the map that sends each $x \in X$ to its equivalence class, check that the following maps $f : X_1 \rightarrow X_2$ satisfy the condition of problem (2), thus define continuous maps $g : Y_1 \rightarrow Y_2$. We describe this process briefly as “ g is defined from f ”:

- (a) $g : K \rightarrow S^1$ defined from $f(x, y) = x$ (Here $X_1 = [0, 1] \times [-1, 1]$, $X_2 = [0, 1]$)
- (b) $g : M \rightarrow K$ defined from $f(x, y) = (x, y)$
(Here $X_1 = [0, 1] \times [-1/2, 1/2]$, $X_2 = [0, 1] \times [-1, 1]$)
- (c) $g : T \rightarrow K$ defined from $f(x, y) = (x, y)$ if $0 \leq x \leq 1$, and $f_3(x, y) = (x - 1, -y)$ if $1 \leq x \leq 2$. (Here $X_1 = [0, 2] \times [-1, 1]$, $X_2 = [0, 1] \times [-1, 1]$)

Solution

Need we show all conditions? or just that f is continuous? Are pictures sufficient as proofs?

Part (a)**Part (b)**

Can the preimage be the empty set? for the regions in the plane that are outside of $[-1/2, 1/2]$.

Part (c)

Problem 4.

- (a) Describe the fibers (= pre-images of points) of g in parts (a) and (c).
- (b) Prove that in (b), the closure of the complement of $g(M)$ in K is homeomorphic to M .

Solution**Part (a)**

The fibers of g_1 , the map of the Klein bottle to the circle, are circles on the Klein bottle surface that wrap around the ‘smaller’ axis of the Klein bottle. These circles do not wrap through the ‘mismatched’ identified ‘edge’ of the Klein bottle, although one fiber does exist exactly along this ‘edge of identification’.

The fibers of g_3 , the map of the Torus to the Klein bottle, are each two points on the torus. These points are equidistant from the upper and lower identified edges in the plane (and equidistant from the horizontal zero) and both shifted an equal amount from zero of 1 in the first dimension. Thus in the torus, these points are radially symmetric but on upper or lower halves of the torus (however this is dependent on the identified edge of the torus being the most outer or inner ring and not on the top, bottom, or anywhere in between).

Part (b)

Do we do this in the plane or on the bottle?