

Problem 1.

Let X be a topological space, give $X \times X$ the product topology, and let the “diagonal” $\Delta \subset X \times X$ be defined by

$$\Delta = \{(x, x) : x \in X\}.$$

- (a) Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.
- (b) Use this fact to give another proof of the fact proved in a previous homework problem: If Z is another topological space, $f, g : Z \rightarrow X$ are continuous, and X is Hausdorff, then

$$E(f, g) = \{x \in Z \mid f(x) = g(x)\}$$

is closed in Z . Make sure your proof takes at most two lines.

Solution**Part (a)**

Suppose X is Hausdorff. $\Delta^c = \{(x, y) \mid x \neq y, x, y \in X\}$. X being Hausdorff provides U, V open $\subset X$ with $x \in U, y \in V$. $U \times V$ is thus a basis element with $(x, y) \in U \times V$. Because U and V are disjoint from the Hausdorff condition, they contain no common points, so they contain nothing in Δ : $U \times V \subset \Delta^c \implies \Delta^c$ open $\implies \Delta$ closed in $X \times X$.

Suppose Δ is closed in $X \times X$. $\forall (x, y) \in \Delta^c \exists U \times V$ basis element with $(x, y) \in U \times V \subset \Delta^c$. $U \times V \subset \Delta^c \implies U \cap V = \emptyset$. Any $x \neq y \in X \implies (x, y) \in \Delta^c$ with a basis element containing (x, y) which provides open, disjoint U, V with $x \in U, y \in V$, thus X is Hausdorff.

Part (b)

$$\Delta = \{(f(x), g(y)) \mid f(x) = g(y)\} \subset X \times X \text{ s.t. } E(f, g) = F^{-1}(\Delta) \quad (1)$$

$$X \text{ Hausdorff} \implies \Delta \text{ closed. } f, g \text{ cont.} \implies (\Delta \text{ closed} \implies E(f, g) \text{ closed}). \quad (2)$$

Problem 2.

Give an example of a topological space X and a compact subset $C \subset X$ with C not closed in X .

Solution

Let $X = \{0, 1\}$ and give this space the topology $\mathcal{T} = \{\emptyset, \{1\}, X\}$. The open subset $C = \{1\}$ is compact because the only open cover $\mathcal{B} = \{0, 1\}$ has the trivial finite subcover of $U_{\alpha_1} = 0, U_{\alpha_2} = 1$.

Problem 3.

Let X be a compact Hausdorff space, and let $A, B \subset X$ be closed sets which are disjoint : $A \cap B = \emptyset$. Prove that there are open sets $U, V \subset X$ with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Solution

Let $b \in B$. For each $a \in A$ we have $a \in U_a$, $b \in V_a$ with U_a, V_a open such that $U_a \cap V_a = \emptyset$ because X is Hausdorff. The collection $\{U_a\}$ forms an open cover of A , so as A is compact there exists a finite subcover of A $U_{a_1}, U_{a_2}, \dots, U_{a_k}$. Let $U_b = U_{a_1} \cup U_{a_2} \cup \dots \cup U_{a_k}$ which is finite union of open sets, so it is open. Now let $V_b = V_{a_1} \cap V_{a_2} \cap \dots \cap V_{a_k}$. The $\{V_b\}$ form an open cover of B , and for each $b \in B$, $V_b \cap U_b = \emptyset$. Note that for every b , $A \subset U_b$.

Because V_b forms an open cover of B and B compact, there exists a finite number of them such that $\bigcup V_{b_i}$ is a finite subcover of B . This union of open sets is also open. The union of $\{U_{b_i}\}$ is likewise open. A is contained in each U_{b_i} , so their union also contains A . We know that each U_{b_i} is disjoint from V_{b_i} so it follows that $\bigcup U_{b_i} \cap \bigcap V_{b_i} = \emptyset$, thus $\bigcup U_{b_i}$ and $\bigcap V_{b_i}$ are the open sets that are disjoint and contain $A \subset \bigcup U_{b_i}$, $B \subset \bigcap V_{b_i}$.

Problem 4.

- (a) Let (X, \mathcal{T}) be a topological space and let \mathcal{B} be a basis for \mathcal{T} . Prove that (X, \mathcal{T}) is compact if and only if every cover of X by elements of \mathcal{B} has a finite sub-cover.
- (b) Let X and Y be compact topological spaces and let $X \times Y$ be their product, with the product topology. Prove that $X \times Y$ is compact.

Solution**Part (a)**

Suppose that (X, \mathcal{T}) is compact. Using the definition for a basis \mathcal{B} of a topological space that the elements $B_i \in \mathcal{B}$ form a cover of \mathcal{T} , which has $X \in \mathcal{T}$, we can form a cover of X from elements B_i . Because \mathcal{B} is a basis for \mathcal{T} , each element B_i is an open set, which means that the cover of X is open, and because X is compact, the cover has a finite subcovering.

Suppose that every covering of X by B_i has a finite subcover. Let U_α , $\alpha \in A$ be an open covering of X . By definition, each element of U_α is a union of elements of \mathcal{B} . Now take the set U_α such that $\mathcal{B} \subset U_\alpha$. This U_α , by definition, is an open covering of X by elements $B_i \in \mathcal{B}$. Since we assumed that every cover has a finite subcover, this U_α has a finite subcovering B_1, B_2, \dots, B_i with each one contained in at least one element of U_α , $\alpha \in A$. For each B_i , we can choose a U_i such that $B_i \subset U_i$, which means that $\{U_i\}$ is a finite subcovering of X , which implies that X is compact.

Part (b)

Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of $X \times Y$. For each $(x, y) \in X \times Y$ we can choose some $\alpha = \alpha(x, y)$ such that $(x, y) \in O_{\alpha(x, y)}$. Because of how it was constructed, $O_{\alpha(x, y)}$ is open, which means (x, y) is contained in some open rectangle $R \subset O_{\alpha(x, y)}$ where $R = U_{(x, y)} \times V_{(x, y)}$ with $U_{(x, y)} \subset X$ and $V_{(x, y)} \subset Y$.

Fix x , and allow y to vary. For every point (x, y) the point is contained in an open rectangle in the product $X \times Y$, and that rectangle is the product of a subset of X with a subset of Y . Choosing several points, we see that the collection of sets $\{V_{(x, y)}\}_{y \in Y}$ is an open cover of Y . Because Y is compact, we can find a finite cover $\{V_{(x, y_i(x))}\}$ of Y that consists of finitely many open sets containing points $\{(x, y_i(x))\}$.

Let $U_x = \bigcap_i U_{(x, y_i(x))}$. Because U_x is the intersection of finitely many open sets, it is itself open. Using that X is compact, there are finitely many x_j such that $\{U_{x_j}\}$ forms an open cover of X . It follows that $\{O_{x_i, y_i(x)}\}$ for any i, j combination is a finite cover of $X \times Y$, which means that $X \times Y$ is compact.

Problem 5.

We have seen that the Cantor set can be described as the set $\{0, 2\}^{\mathbb{N}}$ of infinite sequences of zeros and twos, which is in bijective correspondence with the more convenient set $\{0, 1\}^{\mathbb{N}}$ of infinite sequences of zeros and ones. This choice has the advantage that $\{0, 1\}$ can be naturally identified with $\mathbb{Z}/2$, the integers modulo two, which forms a group under addition: $0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1$. In this way the Cantor set becomes a group, by pointwise addition of sequences: $\{a_i\} + \{b_i\} = \{a_i + b_i\}$.

Prove that this operation is continuous. This means that the addition map

$$A : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$$

defined by

$$A(\{a_1, a_2, \dots, a_i, \dots\}, \{b_1, b_2, \dots, b_i, \dots\}) = \{a_1 + b_1, a_2 + b_2, \dots, a_i + b_i, \dots\}$$

is continuous, where $\{0, 1\}^{\mathbb{N}}$ is given the (infinite) product topology, and $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ the product (two factors) of the infinite product topologies in each factor.

Suggestion: Fix $i_0 \in \mathbb{N}$, and fix an open set $U \subset \{0, 1\}$ (so U is one of $\emptyset, \{0\}, \{1\}, \{0, 1\}$). The sets $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ form a sub-basis (see notes, v1, 3.4.3) for the topology of $\{0, 1\}^{\mathbb{N}}$, so it is enough to show that $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ is open.

Solution

Fix $i_0 \in \mathbb{N}$. Let $U \subset \{0, 1\}$ be open. We show that the preimage, $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$, is open for each possible open set U :

For $a_{i_0} \in \emptyset$:

$$\text{Trivially, } A^{-1}(\emptyset) = \emptyset \text{ which is open.} \quad (3)$$

For $a_{i_0} \in \{0\}$:

$$A^{-1}(\{0\}) = (\{0\}, \{0\}) \cup (\{1\}, \{1\}) \quad (4)$$

$$= (\{0, 1\}, \{0, 1\}) \text{ which is open.} \quad (5)$$

For $a_{i_0} \in \{1\}$:

$$A^{-1}(\{1\}) = (\{0\}, \{1\}) \cup (\{1\}, \{0\}) \quad (6)$$

$$= (\{0, 1\}, \{1, 0\}) \quad (7)$$

$$= (\{0, 1\}, \{0, 1\}) \text{ which is open.} \quad (8)$$

For $a_{i_0} \in \{0, 1\}$:

$$A^{-1}(\{0, 1\}) = (\{0, 1\}, \{0, 1\}) \cup (\{0, 1\}, \{0, 1\}) \quad (9)$$

$$= (\{0, 1\}, \{0, 1\}) \text{ which is open.} \quad (10)$$

This shows the preimage of all open sets remains open, thus A is continuous.

Note that the value at the i_0 position depends *only* on the corresponding values at the i_0 position in the product space – position gets preserved.