

**Problem 1.**

Consider two topologies on  $\mathbb{R}$ : the Euclidean (or usual) topology  $\mathcal{T}_E$ , and the topology  $\mathcal{T}_{CF}$  in which the *closed* sets are  $\mathbb{R}$  and all the finite subsets of  $\mathbb{R}$ .

(a) Suppose  $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$  is continuous. Prove that it is also continuous as a map  $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$ .

(b) Give an example of a continuous map  $g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$  that is *not* continuous as a map  $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ .

*Suggestion:* Try strictly increasing piecewise linear maps.

(c) Find all continuous maps  $f : (\mathbb{R}, \mathcal{T}_{CF}) \rightarrow (\mathbb{R}, \mathcal{T}_E)$ .

**Solution****Part (a)**

Let  $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$ .  $f$  is continuous iff  $\forall U \subset \mathbb{R}$  when  $U$  closed in  $\mathcal{T}_{CF} \implies f^{-1}(U)$  is closed in  $\mathcal{T}_E$ . Let

$$U = \bigcup_{i=1}^N x_i \quad (1)$$

be a closed set in the topology  $\mathcal{T}_{CF}$ , then this set is also closed in  $\mathcal{T}_E$  because a single point  $x_i$  is closed and the closed sets are closed under finite union. The mapping  $f : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$  is continuous  $\implies f^{-1}(U)$  is closed, thus  $f$  is continuous.

**Part (b)**

A continuous map  $g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_{CF})$  that is *not* continuous as  $(\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$  is

$$g(x) = \begin{cases} x & x \leq 0 \\ x + 1 & x > 0 \end{cases} \quad (2)$$

This map is continuous  $\mathcal{T}_E \rightarrow \mathcal{T}_{CF}$  as the preimage of any finite set of points (which is closed in  $\mathcal{T}_{CF}$ ) is closed in  $\mathcal{T}_E$ :

$$\text{Let } U = \bigcup_{i=1}^N x_i \quad (3)$$

$$f^{-1}(U) = \left\{ \bigcup x_i \mid \forall x_i \leq 0 \right\} \cup \left\{ \bigcup x_i + 1 \mid \forall x_i > 0 \right\} \quad (4)$$

where  $x_i \in \mathbb{R}$ , thus the preimage in  $\mathcal{T}_E$  is a finite union of closed sets, which is also closed.

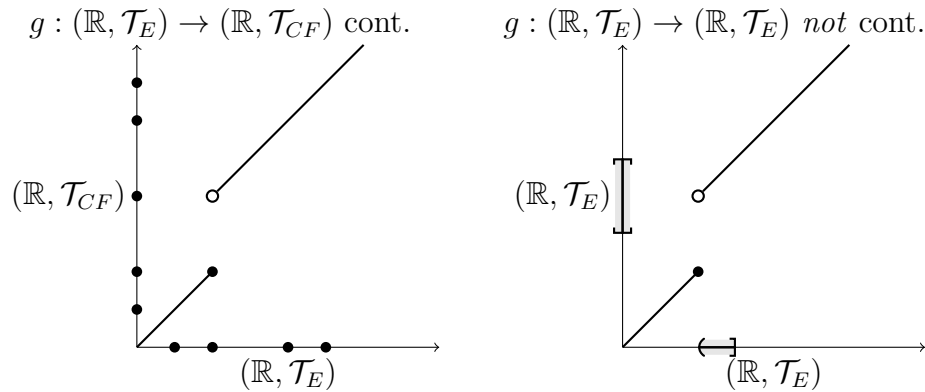
$g : (\mathbb{R}, \mathcal{T}_E) \rightarrow (\mathbb{R}, \mathcal{T}_E)$  is *not* continuous because the preimage of a  $\mathcal{T}_E$ -closed set is not  $\mathcal{T}_E$ -closed:

$$\text{Let } U = \left[ \frac{1}{2}, \frac{3}{2} \right] \quad (5)$$

$$f^{-1}(U) = \left( 0, \frac{1}{2} \right] \quad (6)$$

This preimage is not  $\mathcal{T}_E$ -closed, so the map is not continuous, because a map is continuous iff the preimage of every closed set is closed.

These diagrams illustrate these cases:



Note that for this  $g$ ,  $g^{-1}((1, 2]) = \emptyset$ , which is still closed.

### Part (c)

Constant mappings.

**Problem 2.**

- (a) Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces let  $f, g : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$  be continuous maps, and suppose that  $(Y, \mathcal{T}_Y)$  is Hausdorff. Prove that

$$E(f, g) = \{x \in X \mid f(x) = g(x)\}$$

is *closed* in  $(X, \mathcal{T}_X)$ .

- (b) Give an example of two continuous maps  $f, g$  with  $E(f, g)$  not closed. (If possible, try to find an example with  $(X, \mathcal{T}_X)$  Hausdorff.)

**Solution****Part (a)**

Suppose  $f(x) \neq g(x)$ , because  $Y$  is Hausdorff  $\exists U = \text{nbhd of } f(x), V = \text{nbhd of } g(x)$  where  $U \cap V = \emptyset$ . The preimages of these open sets remain open as  $f, g$  continuous, and since they both contain  $x$ :  $f^{-1}(U), g^{-1}(V) \subset X$  are open nbds of  $x$ . Let  $W = f^{-1}(U) \cap g^{-1}(V) \implies x \in W$ .  $W$  is still open as  $\mathcal{T}_X$  is closed under finite intersection. These  $x$  are such that  $f(x) \neq g(x)$ , so they are in the complement of  $E$ :  $x \in W \subset E^c \implies E^c \text{ open} \implies E \text{ closed}$ .

**Part (b)**

**Problem 3.**

Let  $(X, d)$  be a metric space, let  $x \in X$  and let  $A \subset X$  be non-empty. Define the *distance between  $x$  and  $A$* , denoted  $d(x, A)$ , by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

- (a) Prove that  $d(x, A)$  is a continuous function of  $x$ .

*Suggestion:* Prove more: it is a Lipschitz function, with Lipschitz constant 1.

- (b) Prove that  $d(x, A) = 0 \iff x \in \overline{A}$ .

- (c) Prove that, if  $A$  is closed, then there exists a continuous function  $f : X \rightarrow \mathbb{R}$  so that  $A = \{x \in X \mid f(x) = 0\}$ .

- (d) Suppose  $A, B \subset X$  are *closed* sets, non-empty, and *disjoint*:  $A \cap B = \emptyset$ . Prove that there exists a continuous function  $g : X \rightarrow [0, 1]$  such that

$$g(x) = 0 \iff x \in A \text{ and } g(x) = 1 \iff x \in B.$$

*Suggestion:* Experiment with functions with  $d(x, A) + d(x, B)$  as denominator.

- (e) Prove that if  $A$  and  $B$  are disjoint, non-empty, closed sets as above, there exist open sets  $U, V \subset X$  so that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

**Solution****Part (a)****Part (b)****Part (c)**

The distance function given satisfies these conditions for  $A$  closed:

$$f(x) = d(x, A) : X \rightarrow \mathbb{R} \tag{7}$$

$f$  is continuous as shown in (a). As shown in (b),  $A = \{x \in X \mid f(x) = 0\}$  as  $A$  is its own closure.

**Part (d)**

The proposed function exists and can be defined as

$$g(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} \tag{8}$$

To show  $x \in A \implies g(x) = 0$ :

$$(b) \implies d(x, A) = 0 \quad (9)$$

$$A \cap B = \emptyset \implies d(x, B) = \varepsilon, \varepsilon > 0 \quad (10)$$

$$g(x) = \frac{0}{0 + \varepsilon} = 0 \quad (11)$$

To show  $x \in B \implies g(x) = 1$ :

$$(b) \implies d(x, B) = 0 \quad (12)$$

$$A \cap B = \emptyset \implies d(x, A) = \varepsilon, \varepsilon > 0 \quad (13)$$

$$g(x) = \frac{\varepsilon}{\varepsilon + 0} = 1 \quad (14)$$

To show  $g(x) = 0 \implies x \in A$ :

$$g(x) = 0 \implies d(x, A) = 0 \quad (15)$$

$$(b) \implies x \in A \quad (16)$$

To show  $g(x) = 1 \implies x \in B$ :

$$g(x) = 1 \implies d(x, A) = d(x, A) + d(x, B) \quad (17)$$

$$d(x, A) - d(x, A) = d(x, B) \implies d(x, B) = 0 \implies x \in B \quad (18)$$

### Part (e)

As  $X$  is a metric space, it is metrizable, thus the topology on  $X$  is Hausdorff. This allows the following constructions:

$$U = \bigcup \{L \subset X \mid \forall x \in A, y \in B \text{ (so } x, y \in X), L = \text{nbhd}(x) \text{ open, } M = \text{nbhd}(y) \text{ open, s.t. } L \cap M = \emptyset\} \quad (19)$$

$$V = \bigcup \{M \subset X \mid \forall x \in A, y \in B \text{ (so } x, y \in X), L = \text{nbhd}(x) \text{ open, } M = \text{nbhd}(y) \text{ open, s.t. } L \cap M = \emptyset\} \quad (20)$$

so the sets never intersect and meet the conditions.