Problem 1.

Let X be a topological space, give $X \times X$ the product topology, and let the "diagonal" $\Delta \subset X \times X$ be defined by

$$\Delta = \{(x, x) : x \in X\}.$$

(a) Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) Use this fact to give another proof of the fact proved in a previous homework problem: If Z is another topological space, $f, g : Z \to X$ are continuous, and X is Hausdorff, then

$$E(f, g) = \{x \in Z \mid f(x) = g(x)\}$$

is closed in Z. Make sure your proof takes at most two lines.

Solution

Part (a)

Suppose X is Hausdorff. $\Delta^{\mathbf{c}} = \{(x, y) \mid x \neq y,\ x, y \in X\}$. X being Hausdorff provides U, V open $\subset X$ with $x \in U$, $y \in V$. $U \times V$ is thus a basis element with $(x, y) \in U \times V$. Because U and V are disjoint from the Hausdorff condition, they contain no common points, so they contain nothing in Δ: $U \times V \subset \Delta^{\mathbf{c}} \implies \Delta^{\mathbf{c}}$ open $\implies \Delta$ closed in $X \times X$.

Suppose Δ is closed in $X \times X$. $\forall (x, y) \in \Delta^{\mathbf{c}} \exists U \times V$ basis element with $(x, y) \in U \times V \subset \Delta^{\mathbf{c}}$. $U \times V \subset \Delta^{\mathbf{c}} \implies U \cap V = \emptyset$. Any $x \neq y \in X \implies (x, y) \in \Delta^{\mathbf{c}}$ with a basis element containing (x, y) which provides open, disjoint U, V with $x \in U$, $y \in V$, thus X is Hausdorff.

Part (b)

$$\Delta = \{(f(x), g(y)) \mid f(x) = g(y)\} \subset X \times X \quad \text{s.t.} \quad E(f, g) = F^{-1}(\Delta) \tag{1}$$

$$X \text{ Hausdorff} \implies \Delta \text{ closed. } f, g \text{ cont.} \implies (\Delta \text{ closed} \implies E(f, g) \text{ closed}). \tag{2}$$

Problem 2.

Give an example of a topological space X and a compact subset $C \subset X$ with C not closed in X.

Solution

Let $X = \{0, 1\}$ and give this space the topology $\mathcal{T} = \{\emptyset,\ \{1\},\ X\}$. The open subset $C = \{1\}$ is compact because the only open cover $\mathcal{B} = \{0, 1\}$ has the trivial finite subcover of $U_{\alpha_1} = 0, U_{\alpha_2} = 1$.

Problem 3.

Let X be a compact Hausdorff space, and let $A, B \subset X$ be closed sets which are disjoint : $A \cap B = \emptyset$. Prove that there are open sets $U, V \subset X$ with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Solution

Let $b \in B$. For each $a \in A$ we have $a \in U_a,\ b \in V_a$ with U_a, V_a open such that $U_a \cap V_a = \emptyset$ because X is Hausdorff. The collection $\{U_a\}$ forms an open cover of A, so as A is compact there exists a finite subcover of A $U_{a_1}, U_{a_2}, \ldots, U_{a_k}$. Let $U_b = U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_k}$ which is finite union of open sets, so it is open. Now let $V_b = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_k}$. The $\{V_b\}$ form an open cover of B, and for each $b \in B, V_b \cap U_b = \emptyset$. Note that for every b, $A \subset U_b$.

Because V_b forms an open cover of B and B compact, there exists a finite number of them such that $\bigcup V_{b_i}$ is a finite subcover of B. This union of open sets is also open. The union of $\{U_{b_i}\}$ is likewise open. A is contained in each U_{b_i}, so their union also contains A. We know that each U_{b_i} is disjoint from V_{b_i} so it follows that $\bigcup U_{b_i} \cap (\cap V_{b_i}) = \emptyset$, thus $\bigcup U_{b_i}$ and $\bigcap V_{b_i}$ are the open sets that are disjoint and contain $A \subset \bigcup U_{b_i},\ B \subset \bigcap V_{b_i}$.

Problem 4.

(a) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Prove that $(X, \mathcal{T})$ is compact if and only if every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.

(b) Let X and Y be compact topological spaces and let $X \times Y$ be their product, with the product topology. Prove that $X \times Y$ is compact.

Solution

Part (a)

Suppose that $(X, \mathcal{T})$ is compact. Using the definition for a basis $\mathcal{B}$ of a topological space that the elements $B_i \in \mathcal{B}$ form a cover of $\mathcal{T}$, which has $X \in \mathcal{T}$, we can form a cover of X from elements B_i. Because $\mathcal{B}$ is a basis for $\mathcal{T}$, each element B_i is an open set, which means that the cover of X is open, and because X is compact, the cover has a finite subcovering.

Suppose that every covering of X by B_i has a finite subcover. Let $U_\alpha,\ \alpha \in A$ be an open covering of X. By definition, each element of U_α is a union of elements of $\mathcal{B}$. Now take the set U_α such that $\mathcal{B} \subset U_\alpha$. This U_α, by definition, is an open covering of X by elements $B_i \in \mathcal{B}$. Since we assumed that every cover has a finite subcover, this U_α has a finite subcovering $B_1, B_2, \ldots, B_i$ with each one contained in at least one element of $U_\alpha, \alpha \in A$. For each B_i, we can choose a U_i such that $B_i \subset U_i$, which means that $\{U_i\}$ is a finite subcovering of X, which implies that X is compact.

Part (b)

Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of $X \times Y$. For each $(x, y) \in X \times Y$ we can choose some $\alpha = \alpha(x, y)$ such that $(x, y) \in O_{\alpha(x,y)}$. Because of how it was constructed, $O_{\alpha(x,y)}$ is open, which means (x, y) is contained in some open rectangle $R \subset O_{\alpha(x,y)}$ where $R = U_{(x,y)} \times V_{(x,y)}$ with $U_{(x,y)} \subset X$ and $V_{(x,y)} \subset Y$.

Fix x, and allow y to vary. For every point (x, y) the point is contained in an open rectangle in the product $X \times Y$, and that rectangle is the product of a subset of X with a subset of Y. Choosing several points, we see that the collection of sets $\{V_{(x,y)}\}_{y \in Y}$ is an open cover of Y. Because Y is compact, we can find a finite cover $\{V_{(x,y_i(x))}\}$ of Y that consists of finitely many open sets containing points $\{(x, y_i(x))\}$.

Let $U_x = \bigcap_i U_{(x,y_i(x))}$. Because U_x is the intersection of finitely many open sets, it is itself open. Using that X is compact, there are finitely many x_j such that $\{U_{x_j}\}$ forms an open cover of X. It follows that $\{O_{x_i,y_i(x)}\}$ for any i, j combination is a finite cover of $X \times Y$, which means that $X \times Y$ is compact.

Problem 5.

We have seen that the Cantor set can be described as the set $\{0,2\}^{\mathbb{N}}$ of infinite sequences of zeros and twos, which is in bijective correspondence with the more convenient set $\{0,1\}^{\mathbb{N}}$ of infinite sequences of zeros and ones. This choice has the advantage that $\{0,1\}$ can be naturally identified with $\mathbb{Z}/2$, the integers modulo two, which forms a group under addition: $0+0=1+1=0, 0+1=1+0=1$. In this way the Cantor set becomes a group, by pointwise addition of sequences: $\{a_i\}+\{b_i\}=\{a_i+b_i\}$.

Prove that this operation is continuous. This means that the addition map

$$A : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$$

defined by

$$A(\{a_1, a_2, \dots, a_i, \dots\}, \{b_1, b_2, \dots, b_i, \dots\}) = \{a_1 + b_1, a_2 + b_2, \dots, a_i + b_i, \dots\}$$

is continuous, where $\{0,1\}^{\mathbb{N}}$ is given the (infinite) product topology, and $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ the product (two factors) of the infinite product topologies in each factor.

Suggestion: Fix $i_0 \in \mathbb{N}$, and fix an open set $U \subset \{0,1\}$ (so U is one of $\emptyset, \{0\}, \{1\}, \{0,1\}$). The sets $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ form a sub-basis (see notes, v1, 3.4.3) for the topology of $\{0,1\}^{\mathbb{N}}$, so it is enough to show that $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ is open.

Solution

Fix $i_0 \in \mathbb{N}$. Let $U \subset \{0,1\}$ be open. We show that the preimage, $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$, is open for each possible open set U:

For $a_{i_0} \in \emptyset$:

$$\text{Trivially, } A^{-1}(\emptyset) = \emptyset \text{ which is open.} \tag{3}$$

For $a_{i_0} \in \{0\}$:

$$\begin{aligned} A^{-1}(\{0\}) &= (\{0\},\{0\}) \cup (\{1\},\{1\}) && (4)\\ &= (\{0,1\},\{0,1\}) \text{ which is open.} && (5) \end{aligned}$$

For $a_{i_0} \in \{1\}$:

$$\begin{aligned} A^{-1}(\{1\}) &= (\{0\},\{1\}) \cup (\{1\},\{0\}) && (6)\\ &= (\{0,1\},\{1,0\}) && (7)\\ &= (\{0,1\},\{0,1\}) \text{ which is open.} && (8) \end{aligned}$$

For $a_{i_0} \in \{0,1\}$:

$$\begin{aligned} A^{-1}(\{0,1\}) &= (\{0,1\},\{0,1\}) \cup (\{0,1\},\{0,1\}) && (9)\\ &= (\{0,1\},\{0,1\}) \text{ which is open.} && (10) \end{aligned}$$

This shows the preimage of all open sets remains open, thus A is continuous.

Note that the value at the i_0 position depends *only* on the corresponding values at the i_0 position in the product space –position gets preserved.