Problem 1.

Let X be a topological space, give $X \times X$ the product topology, and let the "diagonal" $\Delta \subset X \times X$ be defined by

$$\Delta = \{(x, x) : x \in X\}.$$

(a) Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) Use this fact to give another proof of the fact proved in a previous homework problem: If Z is another topological space, $f, g : Z \to X$ are continuous, and X is Hausdorff, then

$$E(f, g) = \{x \in Z \mid f(x) = g(x)\}$$

is closed in Z. Make sure your proof takes at most two lines.

Solution

Part (a)

Problem 2.

Give an example of a topological space X and a compact subset $C \subset X$ with C not closed in X.

Solution

Let $X = \{0, 1\}$, and give this space the indiscrete topology $\mathcal{T} = \{\emptyset, X\}$.

Problem 3.

Let X be a compact Hausdorff space, and let $A, B \subset X$ be closed sets which are disjoint : $A \cap B = \emptyset$. Prove that there are open sets $U, V \subset X$ with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Solution

Problem 4.

(a) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Prove that $(X, \mathcal{T})$ is compact if and only if every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.

(b) Let X and Y be compact topological spaces and let $X \times Y$ be their product, with the product topology. Prove that $X \times Y$ is compact.

Solution

Part (a)

Part (b)

Problem 5.

We have seen that the Cantor set can be described as the set $\{0, 2\}^{\mathbb{N}}$ of infinite sequences of zeros and twos, which is in bijective correspondence with the more convenient set $\{0, 1\}^{\mathbb{N}}$ of infinite sequences of zeros and ones. This choice has the advantage that $\{0, 1\}$ can be naturally identified with $\mathbb{Z}/2$, the integers modulo two, which forms a group under addition: $0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1$. In this way the Cantor set becomes a group, by pointwise addition of sequences: $\{a_i\} + \{b_i\} = \{a_i + b_i\}$.

   Prove that this operation is continuous. This means that the addition map

$$A : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$$

defined by

$$A(\{a_1, a_2, \ldots, a_i, \ldots\}, \{b_1, b_2, \ldots, b_i, \ldots\}) = \{a_1 + b_1, a_2 + b_2, \ldots, a_i + b_i, \ldots\}$$

is continuous, where $\{0, 1\}^{\mathbb{N}}$ is given the (infinite) product topology, and $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ the product (two factors) of the infinite product topologies in each factor.

   Suggestion: Fix $i_0 \in \mathbb{N}$, and fix an open set $U \subset \{0, 1\}$ (so U is one of $\emptyset, \{0\}, \{1\}, \{0, 1\}$). The sets $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ form a sub-basis (see notes, v1, 3.4.3) for the topology of $\{0, 1\}^{\mathbb{N}}$, so it is enough to show that $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ is open.

Solution