Problem 1.

(a) Let X, Y be topological spaces and let $q : X \to Y$ be a continuous open surjection. Define $Q : X \times X \to Y \times Y$ by $Q(x_1, x_2) = (q(x_1), q(x_2))$. Prove that Q is a continuous open surjection. In particular, Q is an identification.

(b) Let X be a Hausdorff space, let $q : X \to Y$ be an open map and an identification, and let $R \subset X \times X$ be the equivalence relation defined by q, that is,

$$R = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\}.$$

Prove that Y is Hausdorff if and only if R is closed in $X \times X$.

Suggestion Observe that $R = Q^{-1}(\Delta_Y)$ where $\Delta_Y \subset Y \times Y$ is the diagonal. Refer to Problem (1) of Homework 4.

(c) How does this apply to the example $X = (0\times\mathbb{R})\cup(1\times\mathbb{R})$ and identification $0\times x \sim 1\times x$ for all $x < 0$ discussed in class?

Solution

Part (a)

how can we apply q to Q?

Are basis the right approach to build the product topology on YxY?

Do we also have to 'deconstruct' the XxX topology with continuous projections?

Part (b)

Suppose Y is Hausdorff. $\Delta_Y^{\mathbf{c}} = \{(x,y) \mid x \neq y,\ x, y \in Y\}$. Y being Hausdorff provides U, V open $\subset Y$ with $x \in U$, $y \in V$. $U \times V$ is thus a basis element with $(x,y) \in U \times V$. Because U and V are disjoint (from the Hausdorff condition), they contain no common points, so they contain nothing in Δ_Y: $U \times V \subset \Delta_Y^{\mathbf{c}} \implies \Delta_Y^{\mathbf{c}}$ open $\implies \Delta_Y$ closed in $Y \times Y$.

Suppose Δ_Y is closed in $Y \times Y$. $\forall\, (x,y) \in \Delta_Y^{\mathbf{c}}\ \exists\, U \times V$ basis element with $(x,y) \in U \times V \subset \Delta_Y^{\mathbf{c}}$. $U \times V \subset \Delta_Y^{\mathbf{c}} \implies U \cap V = \emptyset$. Any $x \neq y \in Y \implies (x,y) \in \Delta_Y^{\mathbf{c}}$ with a basis element containing (x,y) which provides open, disjoint $U,\ V$ with $x \in U$, $y \in V$, thus Y is Hausdorff.

Part (c)

X not Hausdorff, because Diagonal cannot be closed since it is open at 0?

Problem 2.

In the following diagram suppose that the maps with solid arrows are given, that q_1 and q_2 are identifications and that f sends fibers of q_1 to fibers of q_2. This means: if $x_1, x_2 \in X_1$ and $q_1(x_1) = q_1(x_2)$, then $q_2(f(x_1)) = q_2(f(x_2))$.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ q_1 \downarrow & & q_2 \downarrow \\ Y_1 & \overset{g}{\dashrightarrow} & Y_2 \end{array}$$

If $y \in Y_1$ choose $x \in X_1$ with $q_1(x) = y$ and define $g(y) = q_2(f(x))$. Prove that g is well-defined, the whole diagram commutes, and, if f is continuous, so is g.

Solution

What does it mean to say "g is well-defined"? How is well-defined different from the definition given in the problem?

How do we show f continuous $\implies$ g continuous? Is this just by the usual definition of pre-images of open sets are open?

Problem 3.

Using the following models $Y = X/\sim$ for the Torus, Klein bottle, Möbius strip and circle,

$$\begin{array}{rclcl} T &=& [0,2]\times[-1,1] &/& (x,-1)\sim(x,1),(0,y)\sim(2,y) \\ K &=& [0,1]\times[-1,1] &/& (x,-1)\sim(x,1),(0,y)\sim(1,-y) \\ M &=& [0,1]\times[-1/2,1/2] &/& (0,y)\sim(1,-y) \\ S^1 &=& [0,1] &/& 0\sim 1 \end{array}$$

and letting $q : X \to Y$ be the map that sends each $x \in X$ to its equivalence class, check that the following maps $f : X_1 \to X_2$ satisfy the condition of problem (2), thus define continuous maps $g : Y_1 \to Y_2$. We describe this process briefly as "g is defined from f":

(a) $g : K \to S^1$ defined from $f(x,y) = x$ (Here $X_1 = [0,1]\times[-1,1], X_2 = [0,1]$)

(b) $g : M \to K$ defined from $f(x,y) = (x,y)$
(Here $X_1 = [0,1]\times[-1/2,1/2], X_2 = [0,1]\times[-1,1]$)

(c) $g : T \to K$ defined from $f(x,y) = (x,y)$ if $0 \le x \le 1$, and $f_3(x,y) = (x-1,-y)$ if $1 \le x \le 2$. (Here $X_1 = [0,2]\times[-1,1], X_2 = [0,1]\times[-1,1]$)

Solution

Need we show all conditions? or just that f is continuous? Are pictures sufficient as proofs?

Part (a)

Part (b)

Can the preimage be the empty set? for the regions in the plane that are outside of [-1/2,1/2].

Part (c)

Problem 4.

(a) Describe the fibers (= pre-images of points) of g in parts (a) and (c).

(b) Prove that in (b), the closure of the complement of $g(M)$ in K is homeomorphic to M.

Solution

Part (a)

The fibers of g_1, the map of the Klein bottle to the circle, are circles on the klein bottle surface that wrap around the "smaller" axis of the Klein bottle. These circles do not wrap through the "mismatched" identified "edge" of the klein bottle, although one fiber does exist exactly along this "reversely identified edge".

The fibers of g_3, the map of the Torus to the Klein bottle, are each two points on the torus. These points are equidistant from the upper and lower identified edges in the plane (and equidistant from the horizontal zero) and both shifted an equal amount from zero of 1 in the first dimension. Thus in the torus, these points are radially symmetric but on upper or lower halves of the torus (however this is dependent on the identified edge of the torus being the most outer or inner points and not on the top, bottom, or anywhere in between).

Part (b)

Do we do this in the plane or on the bottle?