Problem 1.

Let X be a topological space, give $X \times X$ the product topology, and let the "diagonal" $\Delta \subset X \times X$ be defined by

$$\Delta = \{(x, x) : x \in X\}.$$

(a) Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) Use this fact to give another proof of the fact proved in a previous homework problem: If Z is another topological space, $f, g : Z \to X$ are continuous, and X is Hausdorff, then

$$E(f, g) = \{x \in Z \mid f(x) = g(x)\}$$

is closed in Z. Make sure your proof takes at most two lines.

Solution

Part (a)

Suppose X is Hausdorff. $\Delta^{\mathbf{c}} = \{(x, y) \mid x \neq y,\ x, y \in X\}$. X being Hausdorff provides U, Vopen $\subset X$ with $x \in U$, $y \in V$. $U \times V$ is thus a basis element with $(x, y) \in U \times V$. Because U and V are disjoint from the Hausdorff condition, they contain no common points, so they contain nothing in Δ: $U \times V \subset \Delta^{mathbfc} \implies \Delta^{\mathbf{c}}$ open $\implies \Delta$ closed in $X \times X$.

Suppose Δ is closed in $X \times X$. $\forall (x, y) \in \Delta^{\mathbf{c}} \exists U \times V$ basis element with $(x, y) \in U \times V \subset \Delta^{\mathbf{c}}$. $U \times V \subset \Delta^{\mathbf{c}} \implies U \cap V = \emptyset$. Any $x \neq y \in X \implies (x, y) \in \Delta^{\mathbf{c}}$ with a basis element containing (x, y) which provides open, disjoint U, V with $x \in U$, $y \in V$, thus X is Hausdorff.

Part (b)

Problem 2.

Give an example of a topological space X and a compact subset $C \subset X$ with C not closed in X.

Solution

Let $X = \{0, 1\}$ and give this space the discrete topology $\mathcal{T} = \{\emptyset,\ \{0\},\ \{1\},\ X\}$. The open subset $C = \{1\}$ is compact as only open cover $\mathcal{B} = \{0, 1\}$ has the trivial finite subcover of $U_{\alpha_1} = 0, U_{\alpha_2} = 1$.

Problem 3.

Let X be a compact Hausdorff space, and let $A, B \subset X$ be closed sets which are disjoint : $A \cap B = \emptyset$. Prove that there are open sets $U, V \subset X$ with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Solution

Let $b \in B$. For each $a \in A$, $U_a \ni a$, $V_a \ni b$ open s.t. $U_a \cap V_a = \emptyset$. The collection of U_a form an open cover of A, which implies a finite subcover $U_{a_1}, \ldots, U_{a_n}$.

Problem 4.

(a) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Prove that $(X, \mathcal{T})$ is compact if and only if every cover of X by elements of $\mathcal{B}$ has a finite sub-cover.

(b) Let X and Y be compact topological spaces and let $X \times Y$ be their product, with the product topology. Prove that $X \times Y$ is compact.

Solution

Part (a)

First assume X is compact then take a covering of X by members of B since each member of B is open we have a open covering of X, then because X is compact the covering by members of B have a finite subcovering.

Second assume every covering by members of B have a finite subcovering. Take a open covering of X call it $\{\mathrm{U}_\alpha, \alpha \in A\}, bydefinitioneachU_\alpha isunionofelementsofB, thentakesetformedforallthe$
$A\}, foreachB_i, i = 1..npickU_i, i = 1..nwiththepropertyB_i \subset U_i, thenU_1, U_2, ..., U_nisafinitesubcoveringofX$

Yeah, maybe rewrite that....

Part (b)

Problem 5.

We have seen that the Cantor set can be described as the set $\{0,2\}^{\mathbb{N}}$ of infinite sequences of zeros and twos, which is in bijective correspondence with the more convenient set $\{0,1\}^{\mathbb{N}}$ of infinite sequences of zeros and ones. This choice has the advantage that $\{0,1\}$ can be naturally identified with $\mathbb{Z}/2$, the integers modulo two, which forms a group under addition: $0+0=1+1=0, 0+1=1+0=1$. In this way the Cantor set becomes a group, by pointwise addition of sequences: $\{a_i\}+\{b_i\}=\{a_i+b_i\}$.

Prove that this operation is continuous. This means that the addition map

$$A : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$$

defined by

$$A(\{a_1, a_2, \ldots, a_i, \ldots\}, \{b_1, b_2, \ldots, b_i, \ldots\}) = \{a_1+b_1, a_2+b_2, \ldots, a_i+b_i, \ldots\}$$

is continuous, where $\{0,1\}^{\mathbb{N}}$ is given the (infinite) product topology, and $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ the product (two factors) of the infinite product topologies in each factor.

Suggestion: Fix $i_0 \in \mathbb{N}$, and fix an open set $U \subset \{0,1\}$ (so U is one of $\emptyset, \{0\}, \{1\}, \{0,1\}$). The sets $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ form a sub-basis (see notes, v1, 3.4.3) for the topology of $\{0,1\}^{\mathbb{N}}$, so it is enough to show that $A^{-1}(\{\{a_i\} \mid a_{i_0} \in U\})$ is open.

Solution