1 Two-Dimensional Simulation

1.1 State

A simulation is considered complete when it can predict the state X of the drone for the next discrete time segment.

$$X = [z, y, \phi, \dot{z}, \dot{y}, \dot{\phi}] \tag{1}$$

The state X consists of the coordinates on the z-axis, which points positively towards the center of the Earth, the y-axis, and the vector ϕ, which describes the rotation angle about the x-axis. Additionally, the velocities, i.e., the rates of change, $\dot{z}, \dot{y}, \dot{\phi}$ of these quantities are also part of the vector.

The thrust F_1 and F_2 of the two rotors, in combination with environmental influences, produce all forces that result in changes to the state vector X.

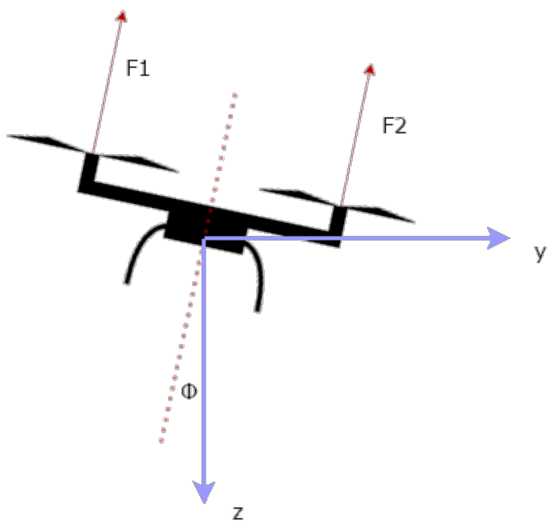

Figure 1: Two-dimensional drone

The thrusts F_1 and F_2 are directed at an angle ϕ to the z-axis.

1.2 Dynamics

Dynamics describe how the state X changes based on F_1 and F_2. To work with fundamental quantities, F_1 and F_2 are expressed by their underlying rotor velocities ω_1 and ω_2. The relationship k_f between rotor velocity and force is quadratic as shown in [1, S. 4]. Hence, we have:

$$F_i = k_f \omega_i^2 \tag{2}$$

This relationship k_f is determined experimentally as explained in Chapter 3. The changes in state in the positions z, y, and ϕ result from the integration of the velocities $\dot{z}, \dot{y}$, and $\dot{\phi}$. The state changes of these result from the integration of the accelerations $\ddot{z}, \ddot{y}$, and $\dot{\phi}$. These are derived from $F = ma$ and the force of gravity $F_g = mg$ with $g = 9.81\frac{m}{s^2}$:

$$\ddot{z} = \frac{F_g - F_z}{m} = g - F_z \qquad \ddot{y} = \frac{F_y}{m} \tag{3}$$

F_z is parallel to the z-axis, and F_y is parallel to the y-axis. Thus, there is a right angle between F_z and F_y. Additionally, it holds true that

$$F_{total} = \sum_{1}^{i} F_i = F_1 + F_2 \tag{4}$$

The total force of both rotors F_{total} is oriented at an angle ϕ to the z-axis.

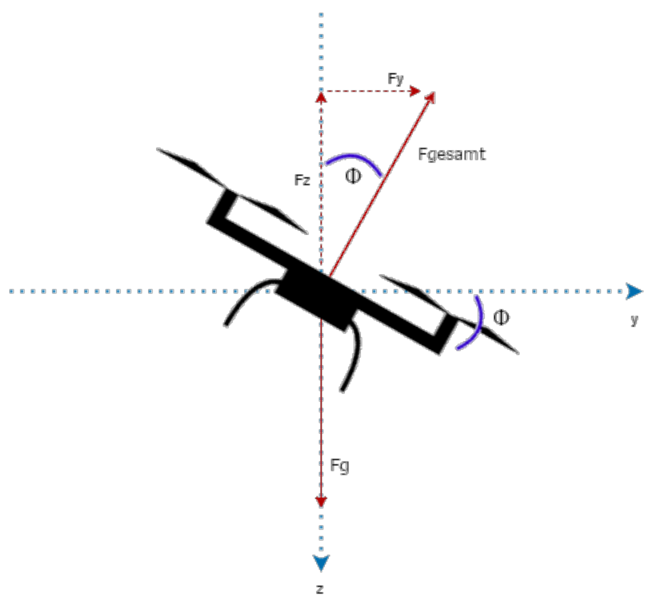

Figure 2: Forces on a 2D drone

Thus, the forces F_z and F_y result from the triangle's trigonometry as

$$F_z = F_{total} \cos\phi \qquad F_y = F_{total} \sin\phi \tag{5}$$

and the accelerations along the z- and y-axes by combining Equations 1.2, 1.2, and 1.2:

$$\ddot{z} = g - \frac{(F_1 + F_2)\cos\phi}{m} \qquad \ddot{y} = \frac{(F_1 + F_2)\sin\phi}{m} \tag{6}$$

$\ddot{\phi}$ is derived through the combination of the equations for torque M. l is the distance from the center of mass of the dual-rotor drone to one of the motors.

$$M_x = F_\phi l \qquad M_x = I_x \ddot{\phi} \tag{7}$$

The moment of inertia I_x has to be determined experimentally, as described in Chapter 3. The force around the x-axis F_ϕ is the difference in the two motor forces $F1$ and $F2$ from Figure 1. A positive F_ϕ corresponds to clockwise rotation, and a negative value indicates counter-clockwise rotation. In summary, we have:

$$F_\phi = F_1 - F_2 \tag{8}$$

Therefore, the angular acceleration $\ddot{\phi}$ can be described as

$$\ddot{\phi} = \frac{(F_1 - F_2) \cdot l}{I_x} \tag{9}$$

Hence, the dual-rotor drone's state at time t can be determined by the sum of the state $X(t)$ and the integral of the time derivative of the state $\dot{X}$.

$$X(t + \Delta t) = X(t) + \int_t^{t+\Delta t} \dot{X}(\tau) d\tau \tag{10}$$

2 Three-Dimensional Simulation

2.1 State

In the three-dimensional case, the state vector X consists of twelve quantities and is formally defined as

$$X = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, p, q, r] \tag{11}$$

where

x, y, z represent positions,

$\dot{x}, \dot{y}, \dot{z}$ represent velocities in their respective axes,

ϕ, θ, ψ, also known as Euler angles, represent rotations about the x-axis, y-axis, and z-axis, and

p, q, r are the angular velocities about the axes kx, ky, and kz in the body coordinate system.

Except for the last three parameters, all information resides in the world coordinate system, which was the only one used in the two-dimensional case from Chapter 2. The body coordinate system, however, is a fixed coordinate system attached to the quadrotor.

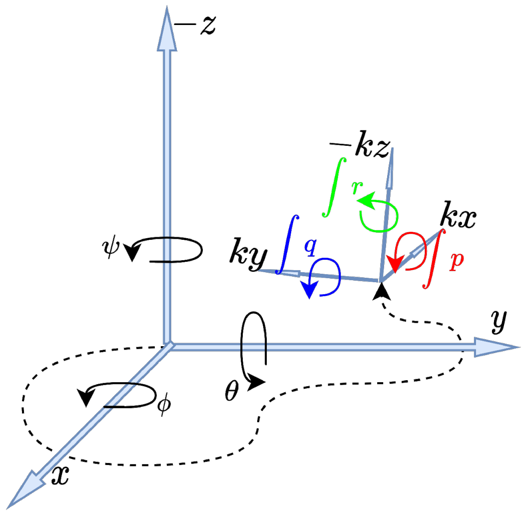

Figure 3: World and body coordinate systems

2.2 Dynamics

Equation (1.2) still applies here as before. Analogous to Equation 1.2, in the three-dimensional case, we also require the derivative $\dot{X}$ of the state vector to move the state forward in time. Equations 1.2 still apply but now, analogous to $\ddot{y}$, we add the formula for $\ddot{x}$. In summary, still derived from Newton's $F = ma$, we have

$$\ddot{z} = \frac{F_g - F_z}{m} = g - F_z \qquad \ddot{y} = \frac{F_y}{m} \qquad \ddot{x} = \frac{F_x}{m} \tag{12}$$

Different from the two-dimensional case, however, are the formulas for F_x, F_y, and F_z. Here a rotation matrix is defined [2, S. 12], which transforms any vector from the body coordinate system into a vector in the world coordinate system through multiplication. Essentially, the matrix is still based on triangle trigonometry. The matrix R_{Body}^{World} is

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c(\psi)c(\theta) & c(\psi)s(\theta)s(\phi) - s(\psi)c(\phi) & s(\psi)s(\phi) + c(\psi)s(\theta)c(\phi) \\ s(\psi)c(\theta) & c(\psi)c(\phi) + s(\psi)s(\theta)s(\phi) & s(\psi)s(\theta)c(\phi) - c(\psi)s(\phi) \\ -s(\theta) & c(\theta)s(\phi) & c(\theta)c(\phi) \end{bmatrix}
\end{aligned} \tag{13}$$

So, from Equations 2.2 and 2.2, we derive the formulas for axis accelerations in the world coordinate system as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \frac{1}{m} R_{Body}^{World} \begin{bmatrix} 0 \\ 0 \\ -F_{total} \end{bmatrix} = \begin{bmatrix} R_{13}\frac{1}{m}(-F_{total}) \\ R_{23}\frac{1}{m}(-F_{total}) \\ g + R_{33}\frac{1}{m}(-F_{total}) \end{bmatrix} \tag{14}$$

Here it is also shown that the propellers generate thrust in the direction $-kz$. On Earth, $g = 9.81\frac{m}{s^2}$. And the total force now results from the sum of all four rotors, so that

$$F_{total} = \sum_{1}^{i} F_i = F_1 + F_2 + F_3 + F_4 \tag{15}$$

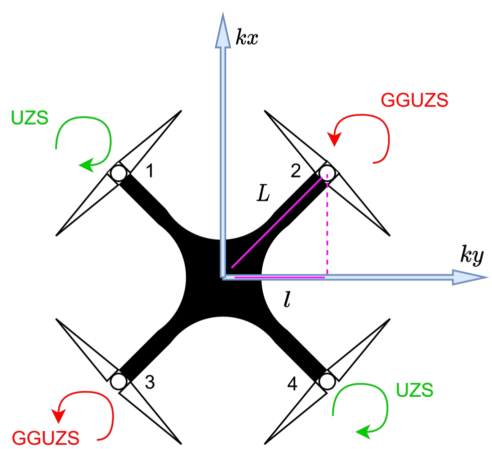

Figure 4: 3D Drone from above: Propeller rotation directions and numbers. kz goes into the plane of the drawing.

The accelerations around the body axes $\dot{p}$, $\dot{q}$, and $\dot{r}$ are calculated by the Euler's equation of motion, which is

$$\begin{aligned} M &= I\dot{\omega} + \omega \times (I\omega) = \\ \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} &= \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \left(\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \end{aligned} \tag{16}$$

The moments of inertia I_x, I_y, and I_z are determined by experiments, as explained in Chapter 3. By rearranging, we then get the accelerations as

$$
\begin{aligned}
\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} &= \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} qI_xr - rI_yq \\ rI_xp - pI_zr \\ pI_yq - qI_xp \end{bmatrix} \\
\begin{bmatrix} M_x - (qI_xr - rI_yq) \\ M_y - (rI_xp - pI_zr) \\ M_z - (pI_yq - qI_xp) \end{bmatrix} &= \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} \\
\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \begin{bmatrix} M_x - (qI_xr - rI_yq) \\ M_y - (rI_xp - pI_zr) \\ M_z - (pI_yq - qI_xp) \end{bmatrix} \\
\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} \frac{M_x - rq(I_x - I_y)}{I_x} \\ \frac{M_y - rp(I_x - I_z)}{I_y} \\ \frac{M_z - pq(I_y - I_x)}{I_z} \end{bmatrix}
\end{aligned}
\tag{17}
$$

As remaining necessary quantities to establish $\dot{X}$, which is the goal in dynamics, we have the velocities about the world coordinate system axes $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$. These are obtained, like the linear accelerations in Equation 2.2 with the help of a rotation matrix specifically for Euler angle velocities [3, S. 12, Eq. 79]

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{18}
$$

Thus, the state of the quadrotor at time t can be determined by the sum of the state $X(t)$ and the integral of the state's derivative $\dot{X}$.

$$
X(t + \Delta t) = X(t) + \int_t^{t+\Delta t} \dot{X}(\tau) d\tau \tag{19}
$$

3 Determination of Simulation Parameters

In addition to the laws of dynamics, some central quantities also determine the behavior of the quadrotor. These are

m the mass,

k_f the constant for the quadratic relationship between rotor velocity and thrust,

k_m the constant for the quadratic relationship between rotor velocity and torque,

I_x, I_y, I_z the moments of inertia about each axis.

These constants must be determined experimentally.
The mass is measured with a bathroom scale. A project member weighs himself and then again while holding the drone near his upper body. The difference gives the mass m.
k_f and k_m are measured using a seesaw.

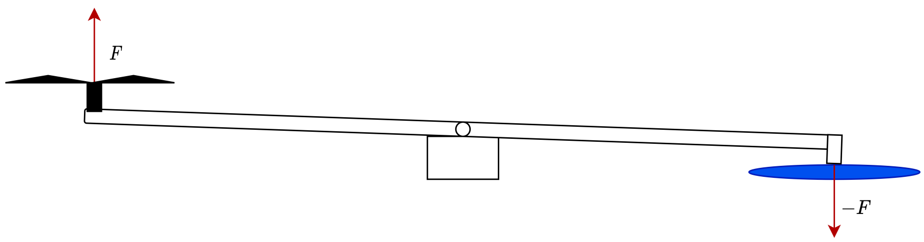

Figure 5: Diagram of Seesaw: Measurement of k_f. The kitchen scale is colored blue.

At one end, the propeller is attached, and at the other end, a pin pushes the thrust force onto a kitchen scale. By reading the weight on the scale, the resulting force can be determined.

Figure 6: Photo of Seesaw: Measurement of k_f

k_m can be measured with a slight modification of the seesaw. The propeller simply needs to stand at a 90° angle to the seesaw.

Figure 7: Photo of Seesaw: Measurement of k_m

The rotational velocity is recorded by a tachometer. To this end, a reflective strip is attached to the propeller. The tachometer emits a laser and counts the number of reflections per minute.

Figure 8: Photo Tachometer: Measurement of ω with a tachometer

Measured v [rads/s]	**Measured k_f [N]**	**Measured k_m [N]**
20.00	0.56898	0.11772
26.70	0.99081	0.30411
35.08	1.74618	0.37278
41.26	2.43288	0.6867
48.17	3.2373	0.981
53.20	4.08096	1.12815
58.85	4.93443	1.30473
62.41	5.4936	1.52055
65.97	6.0822	1.6677

Table 1: Results of Experiments

The test series yielded a $k_f = 0.00141446535$ and a $k_m = 0.0004215641$. The moments of inertia are dependent on the weight and dimensions of the drone body. This paper uses results from a publication in which a similar frame of reference was used.[4, S. 7] Thus we have $I_x = 0.0121$, $I_y = 0.0119$, and $I_z = 0.0223$.

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