Project 2

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1) $\ddot{y} + 2y^3\dot{y} + y = u \rightarrow \dot{x}_2 + 2x_1^3 x_2 + x_1 = u$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \dot{y} = -2x_1^3 x_2 - x_1 + u$$

$$\underline{f}(\underline{x}, u) = \begin{cases} x_2 \\ -2x_1^3 x_2 - x_1 + u \end{cases}$$

2) $\dot{x}_1 = 0 = x_2$

$$\dot{x}_2 = 0 = -2x_1^3 \cancelto{0}{x_2} - x_1 + \cancelto{0}{u}$$

$$0 = -x_1$$

$$x_1 = 0$$

$$x_e = (0, 0)$$

3) $\nabla_x f(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6x_1^2 x_2 - 1 & -2x_1^3 \end{bmatrix}$

$$\dot{\underline{x}} = A\underline{x} + bu$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Eigenvalues

4) $\delta = \begin{bmatrix} j \\ -j \end{bmatrix} \leftarrow$ center

Center is NOT asymptotically stable because real parts of eigenvalues are NOT negative. No conclusion can be made for the non-linear (original) model.

5)

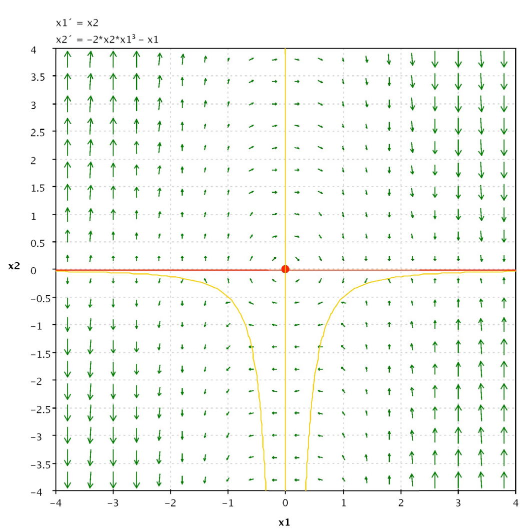

Figure 1: Nonlinear Plant Model Phase Portrait

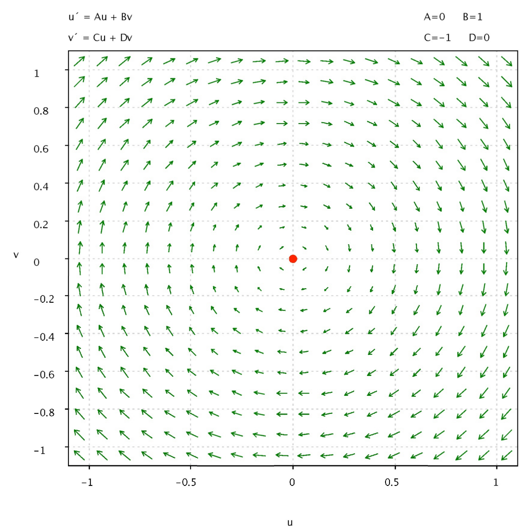

Figure 2: Linearized Phase Portrait

6) $u = -Kx$ $\qquad K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$= -k_1 x_1 - k_2 x_2$$

$$A - bK \rightarrow \lambda = \begin{bmatrix} -1 + j1 \\ -1 - j1 \end{bmatrix}$$

Using place(A, b, P) where $P = \lambda$,

$k_1 = 1$

$k_2 = 2$

7) $$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= A\underline{x} + BKx$$

$$= (A - BK)\underline{x}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \end{bmatrix} \right) \underline{x}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \underline{x}$$

$\lambda = -1 \pm j$

(spiral sink)

- asymptotically stable since real parts are nonzero.

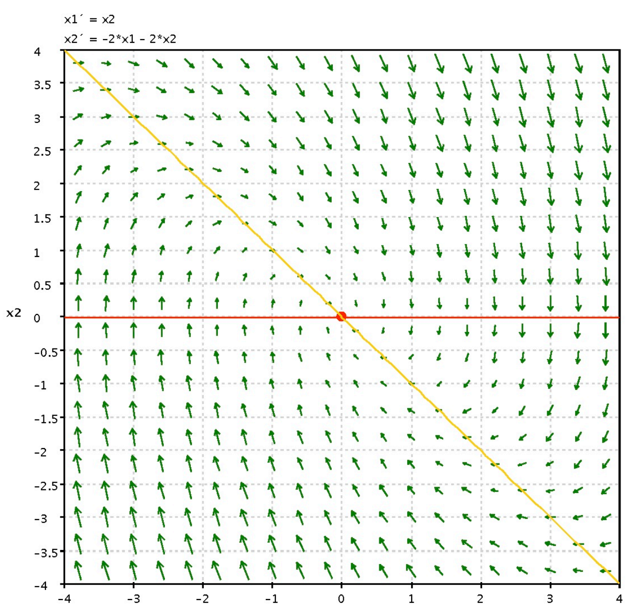

Figure 3: Phase portrait around equilibrium of closed loop system from exercise 7.

$$8)\ \underline{f}(\underline{x}, u) = \begin{cases} x_2 \\ -2x_1^3 x_2 - x_1 + u \end{cases} \Bigg|_{u = k\underline{x}}$$

$$= \begin{cases} x_2 \\ -2x_1^3 x_2 - x_1 - x_1 - 2x_2 \end{cases}$$

$$= \begin{cases} x_2 \\ -2x_2(x_1^3 - 1) - 2x_1 \end{cases}$$

$$= \begin{cases} x_2 \\ -2\left[x_2(x_1^3 - 1) - x_1\right] \end{cases}$$

The linearized model:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\lambda = 1 \pm \sqrt{3} = \begin{bmatrix} -0.73 \\ 2.73 \end{bmatrix}$$

This is a saddle since both eigenvalues are real with opposite signs.
The control system is unstable for the nonlinear case even though the linearized closed loop control system was shown to be asymptotically stable.

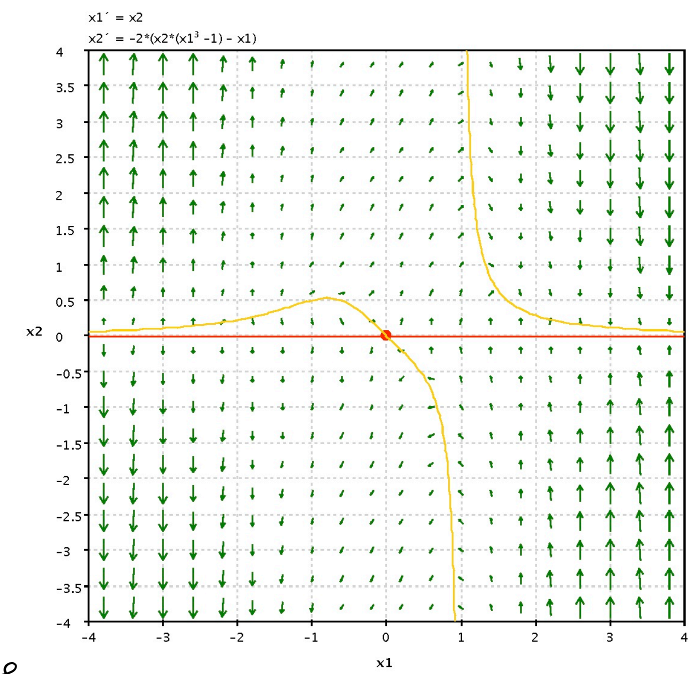

Figure 4: Phase portrait of nonlinear model with linear control.

9) $V(\underline{x}) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(\underline{x}) = x_1\dot{x}_1 + x_2\dot{x}_2$

$= x_1x_2 + x_2(-2x_1^3x_2 - x_1 + g(\underline{x}))$

$= \cancel{x_1x_2} - 2x_1^3x_2^2 - \cancel{x_1x_2} + x_2g(\underline{x})$

$= -2x_1^3x_2^2 + x_2g(\underline{x})$

So we want to design $g(\underline{x})$ to meet the Lyapunov criteria

- neg def
- radially unbounded

$\dot{V}(\underline{x}) < 0$

$-2x_1^3x_2^2 + x_2g(\underline{x}) < 0$

$x_2g(\underline{x}) < 2x_1^3x_2^2$

$g(\underline{x}) < 2x_1^3x_2$

$g(x) = 2(x_1^3x_2 - x_2) = 2x_2(x_1^3 - 1)$

$\dot{V}(\underline{x}) \overset{?}{<} 0 \Big|_{g(x) = 2(x_1^3x_2 - x_2)}$

$\dot{V}(\underline{x}) = -2x_1^3x_2^2 + x_2(2x_1^3x_2 - 2x_2)$

$= \cancel{-2x_1^3x_2^2} + \cancel{2x_1^3x_2^2} - 2x_2^2$

$\dot{V}(\underline{x}) = -2x_2^2 < 0$ ✓

so, negative <u>semi</u>-definite, so locally stable.

Per La Salle's theorem, if $\dot{V} = 0$ has only one equilibrium,

$$\dot{V} = 0 = -2x_2^2$$

$$x_2 = 0$$

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = \cancelto{0}{-2x_1^3x_2} - x_1 + 2(\cancel{x_1^3x_2} - x_2)x_2$$

$$= -x_1 - \cancelto{0}{2x_2^2}$$

$$= -x_1$$

So the only equilibrium is when $x_1 = 0$ and therefore $\underline{\dot{x}}$ is asymptotically stable! This is confirmed by the figure below.

10) When $g(\underline{x}) = 2x_2(x_3^3 - 1)$ the system's eigenvalues represent a spiral sink as seen in Figure 4. Compared to a linearly controlled model, the system is globally stable. Compared to the linearized model, the appear very similar.

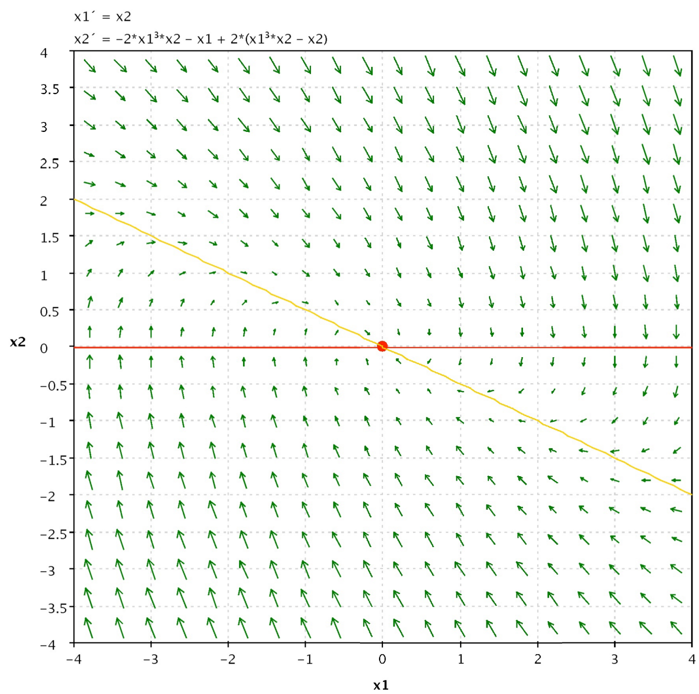

Figure 4: Nonlinear System with g(x) controller.

I purposefully added a factor of "2" into $g(\underline{x})$ to have eigenvalues that match the linearized model more closely.