Project 1

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Project 1
Elec-7560

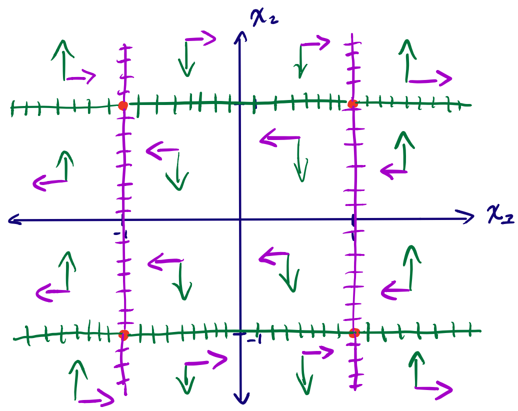


Figure 1: Equilibrium Points and Nullclines

Finding equilibrium points and nullclines:

$$f_1(x) = x_2^2 - 1$$
$$f_2(x) = x_1^2 - 1$$
$$x_2^2 - 1 = 0 = \dot{x}_1$$
$$* \; x_2 = \pm\sqrt{1} \quad \text{(purple on fig 1)}$$
$$= \pm 1$$

$$x_1^2 - 1 = 0 = \dot{x}_2$$
$$* \; x_1 = \pm\sqrt{1} \quad \text{(green on fig 2)}$$
$$= \pm 1$$

The nullclines are lines where,

$$x_1 = 1$$
$$x_1 = -1$$
$$x_2 = 1$$
$$x_2 = -1$$

as indicated on the graph.

Therefore, the equilibrium points are located at, $(-1, 1), (1, 1), (-1, -1), (1, -1)$ as indicated on the graph above by red • (dots)

Linearization near equilibrium points (where state is not changing.)

Jacobian:

$$f(x) = \begin{bmatrix} x_2 - 1 \\ x_1 - 1 \end{bmatrix}$$

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} x_2^2 - 1 & 2x_2 \\ 2x_1 & x_1^2 - 1 \end{bmatrix}$$

$$\delta x = x - x_e$$

$$\delta \dot{x} = A \cdot \delta x$$

$$= \nabla_x f(x)\Big|_{x = x_e = (-1,-1)} \cdot \delta x \qquad = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Repeating for the rest,

$$x_e = (1,-1)$$

$$\delta \dot{x}\Big|_{x=x_e} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$x_e = (-1,1)$$

$$\delta \dot{x}\Big|_{x=x_e} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$x_e = (1,1)$$

$$\delta \dot{x}\Big|_{x=x_e} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Linear Analysis

Finding the eigenvalues of the jacobian matrix gives us an indication to the behavior of the field around each equilibrium point.

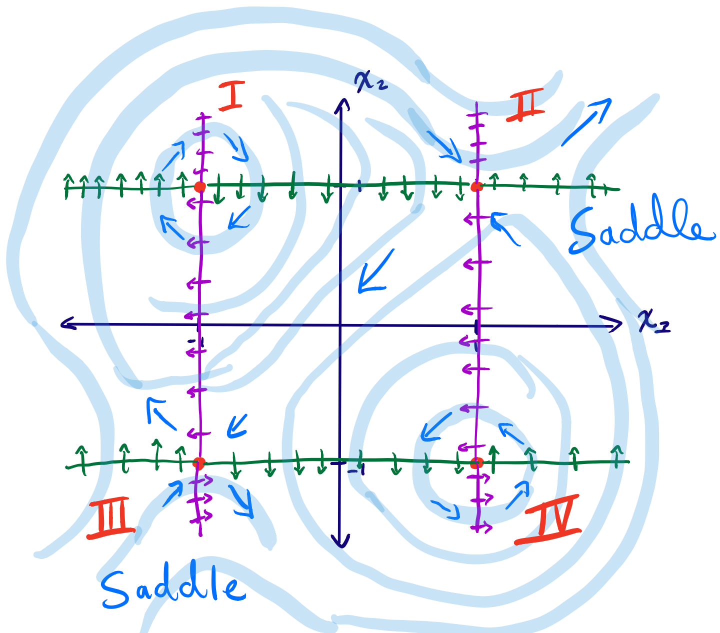


Figure 2: Sketching vector field by hand

Using Wolfram Alpha the eigenvalues were found for each point:

$(-1, 1) \quad A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$\lambda = |\lambda I - A| = 0$ (eigenvalues)

$\lambda = \begin{bmatrix} 2j \\ -2j \end{bmatrix} \leftarrow$ center

$v = \begin{bmatrix} -j & 1 \\ j & 1 \end{bmatrix}$ (eigenvectors)

* Center
  - purely imaginary eigenvalues
* Saddle
  - Two distinct real eigenvalues of opposite sign.

$(1,1)$ $\lambda = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $u = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

-saddle (always unstable)

$(1,-1)$

$\lambda = \begin{bmatrix} j2 \\ -j2 \end{bmatrix}$ $v = \begin{bmatrix} j & 1 \\ -j & 1 \end{bmatrix}$

-center

$(-1,-1)$

$\lambda = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

-saddle (unstable)

Limit Cycle Theorems

Poincaré-Hopf (Index Theory)

For the two centers $(-1,1)$, $(1,-1)$, the index is 1, for each.

For the two saddles, $(1,1)$, $(-1,-1)$ the index is -1, for each.

Bendixon's Theorem can rule out periodic solutions

Applying to the given system,

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2x_1 + 2x_2 \neq 0$$

Therefore, when $x_1 = -x_2$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0$

If we look at the previous equilibrium points, it clear that there is a zero crossing at the two centers $(-1, 1)$ and $(1, -1)$ and $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0$

So we can't rule out period behavior using bendixon's theorem

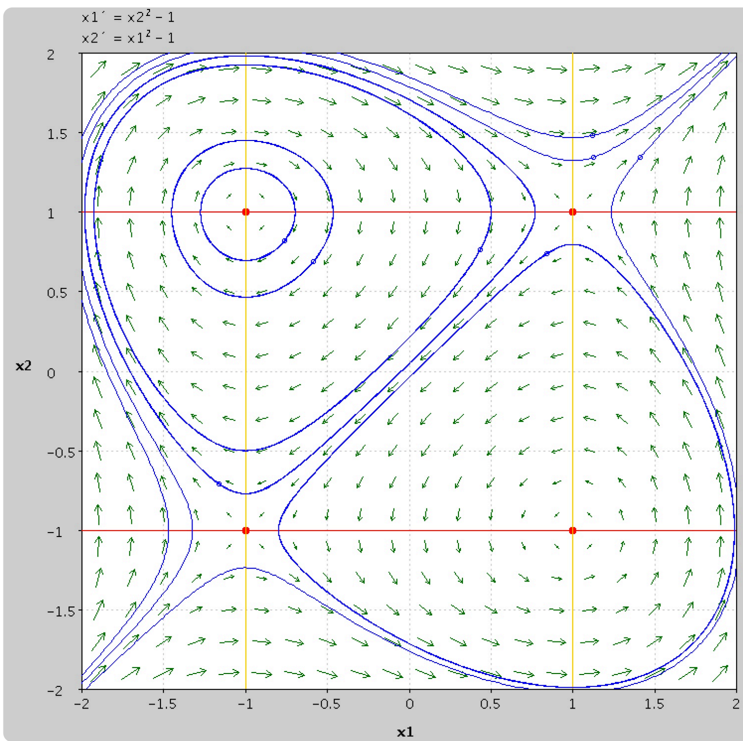
However, at the two saddle points $(-1,-1)$, $(1,1)$, there is no zero crossing and $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$ so we can rule out periodic behavior.

Combining this information, we investigate the equilibria and combinations of them using Poincaré-Bendixon's Theorem

We know the system contains 2 centers and 2 saddles so therefore, enclosing all four equilibria will result $N - S = 2 - 2 = 0$ and we can thus not confirm a limit cycle exists. Only by bounding our region around 2 centers and 1 saddle or the centers can we confirm either a limit cycle or equilibrium point. Since saddles are unstable and centers are asymptotically unstable, we can conclude that a limit cycle escists, given the appropriate combination.

Computer Based Analysis

For the model studied in this project, we can conclude that linearizing non-linear models, when being careful to note for discontinuities in its derivative, etc. The analytical linearization represented the nonlinear model drawn with the pplane tool accurately as seen below.



Our limit cycle theorems are confirmed for all four equilibrium points by the computer simulation which shows two saddles (unstable) and two centers/spirals (neutrally stable).