4.2

$$\left.\begin{array}{l} s = \frac{1}{2}(x+y) \\ t = \frac{1}{2}(x-y) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x = s+t \\ y = s-t \end{array}\right. \qquad L\phi := \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}$$

$$\tau : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} s \\ t \end{pmatrix} \to \begin{pmatrix} s+t \\ s-t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\tilde{\Phi}(s,t) = \phi(\tau(s,t))$$

Find $\tilde{L}\tilde{\Phi} := L\phi$

We try to „guess" $\tilde{L}$ by combing $\frac{\partial \tilde{\Phi}}{\partial s}, \frac{\partial \tilde{\Phi}}{\partial t}, \ldots$

$$\frac{\partial \tilde{\Phi}}{\partial s} = \frac{\partial \phi}{\partial \tau_1} \cdot \frac{\partial \tau_1}{\partial s} + \frac{\partial \phi}{\partial \tau_2} \cdot \frac{\partial \tau_2}{\partial s}$$

$$\frac{\partial \tilde{\Phi}}{\partial t} = \ldots$$

$$\frac{\partial^2 \tilde{\Phi}}{\partial s \partial t} = \ldots = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}$$

... and find, that $\frac{\partial^2 \tilde{\Phi}}{\partial s \partial t} \equiv \tilde{L}\tilde{\Phi}$

ALTERNATIVE: $\tau^{-1} := \sigma : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$$

$$\tilde{\Phi}(\sigma(x,y)) = \phi(x,y)$$

Find $\tilde{L}\tilde{\Phi} := L\phi$

Now, we can compute constructively without guessing:

$$L\phi = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \tilde{\Phi}(\sigma)}{\partial x^2} - \frac{\partial^2 \tilde{\Phi}(\sigma)}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \tilde{\Phi}}{\partial \sigma_1} \frac{\partial \sigma_1}{\partial x} + \frac{\partial \tilde{\Phi}}{\partial \sigma_2} \frac{\partial \sigma_2}{\partial x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial \tilde{\Phi}}{\partial \sigma_1} \frac{\partial \sigma_1}{\partial y} + \frac{\partial \tilde{\Phi}}{\partial \sigma_2} \frac{\partial \sigma_2}{\partial y} \right)$$

$$= \ldots = \frac{1}{4}\left(\frac{\partial^2 \tilde{\Phi}}{\partial s^2} + 2\frac{\partial^2 \tilde{\Phi}}{\partial s \partial t} + \frac{\partial^2 \tilde{\Phi}}{\partial t^2} \right) - \frac{1}{4}\left(\frac{\partial^2 \tilde{\Phi}}{\partial s^2} - 2\frac{\partial^2 \tilde{\Phi}}{\partial s \partial t} + \frac{\partial^2 \tilde{\Phi}}{\partial t^2} \right)$$

$$= \frac{\partial^2 \tilde{\Phi}}{\partial s \partial t} \equiv \tilde{L}\tilde{\Phi}$$