Nonlinear dynamical systems

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Based on the lecture from Prof. George Haller at ETH Zürich

February 1, 2023

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1 Introduction

Definition 1 (Dynamical System). *A triple* $(P, E, \mathcal{F})$ *where*

- P *is a set called the phase space*
- E *is the function space of the evolutionary variable (usually time)*
- $\mathcal{F}$ *is the deterministic evolution rule that defines how the state* $x(t) \in P$ *evolves*

is called dynamical system. If $E = \mathbb{Z}^d$ *the D.S. is discrete, if* $E = \mathbb{R}^d$ *the D.S. is continuous.*

Discrete D.S. can be described by iterated mappings

$$x_{n+1} = F(x_n, n) \tag{1}$$

Continuous D.S. can be described by ODEs. Any ODE can be formulated as a system of first order ODEs:

$$\dot{x} = f(x, t) \tag{2}$$

with $x \in P$ and $t \in E$, which yields an initial value problem

$$\begin{cases} \dot{x} = f(x, t) \\ x(t_0) = x_0 \end{cases} \tag{3}$$

Assuming unisolvence (see theorem 7), we can define the following map:

Definition 2 (Flow map). *We call the map*

$$F_{t_0}^t : (t; t_0, x_0) \mapsto \varphi(t; t_0, x_0)$$

the flow map of a continuous D.S. like described in (1) *if* $\dot{\varphi} = f(\varphi, t)$ *and* $\varphi(t_0) = x_0$.

Lemma 3 (Properties of the flow map). *It holds that:*

- $F_{t_0}^t$ *is as smooth as* $f(x, t)$
- $F_{t_0}^{t_0} = \mathrm{Id}$ *and* $F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_1}^{t_0}$
- $\left(F_t^{t_0}\right)^{-1} = F_{t_0}^t$ *exists and is smooth*

Property 2 and 3 are called group property.

Proof. TODO ☐

Definition 4 (Autonomous dynamical system). *A continous D.S. whose evolution rule* f *is not dependent on time is called autonomous. The system is then described by*

$$\dot{x} = f(x)$$

From this definition, the flow map can be easily formulated as a special case of Definition 2:

$$F_{t_0}^{t} : x_0 \mapsto x(t, x_0)$$

Definition 5 (Separatrix)**.** *Codimension-1 surface in phase space separating regions of different long term behaviour*

2 Fundamentals

Theorem 6 (Peano)**.** *Let $f(t,x) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a continous function, then for the initial value problem* (3) *there exists a solution*

$$\varphi(t) : I \mapsto \mathbb{R}$$

on a time interval I, which is a neighbourhood of the initial time t_0.
Further φ is continuously differentiable.

Proof. see english wikipedia □

Theorem 7 (Picard-Lindelöf)**.** *Let $f(t,x) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a locally Lipschitz continous function, then the initial value problem* (3) *has a unique solution*

$$\varphi(t) : I \mapsto \mathbb{R}$$

on a time interval I, which is a neighbourhood of the initial time t_0.
Further φ is continuously differentiable.

Proof. see english wikipedia □

Example 8 (Non-unique solution)**.** *Consider*

$$\begin{cases} x' = |x|^{\frac{1}{2}} \\ x(t_0) = 0 \end{cases} \tag{4}$$

The right hand side, which is the evolution rule, is continuous, but not Lipschitz continuous in the interval $[0,1]$*. Therefore we can only conclude solvability (Theorem 6), and not unisolvence (Theorem 7).*

A consequence of these theorems is that trajectories of autonomous, unisolvent systems never intersect. This also holds for the mathematical pendulum, which seems to create intersecting trajectories at the unstable fixed points.
These trajectories do not actually intersect:

- One trajectory is the unstable fixed point itself.
- Another one is a trajectory pointing outwards of the fixed point, not including the fixed point itself.
- And yet another one is the trajectory leading into the fixed point - again not including the fixed point, as the state never reaches the fixed point in finite time.

Non-autonomous, unisolvent systems do have intersecting trajectories. This can be avoided by extening the phase space:

$$X := \begin{pmatrix} x \\ t \end{pmatrix}, \quad F(X) := \begin{pmatrix} f(x,t), \\ 1 \end{pmatrix} \tag{5}$$

This yields

$$\dot{X} = F(X) \tag{6}$$

2.1 Local and global existence

Example 9 (Exploding solution).

$$\begin{cases} \dot{x} = x^2 \\ x(t_0) = 1 \end{cases} \tag{7}$$

The solution is $x(t) = \frac{1}{1-(t-t_0)}$ *which blows up at* $t^* = t_0 + 1$.

The solution of this example does not exist globally. To characterize that, we need the following

Definition 10 (Analytic continuation). *Let f be an analytic (or for real functions: smooth) function with a target set U and F be an analytic function on V sucht that $U \subset V$. We call F the analytic continuation of f if*

$$F(z) = f(z) \quad \forall z \in U$$

Sometimes f is then called the restriction of F to U.

Theorem 11 (Analytic continuation of ODE solutions). *If local solutions to ODEs can not be continued to a time T*

$$\implies \lim_{t \to T} ||x(t)|| = \infty$$

Again, the solution does not exist globally in this case.

Proof. Can be found in [AS92]. $\square$

Linear dynamical systems, i.e systems where f is linear in the state vector, can always be written using linear maps, which can be written as matrices:

$$\dot{x} = f(x) =: A(t)x \tag{8}$$

with $x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$.

Lemma 12 (Global existence of linear D.S.). *Consider a linear D.S. $\dot{x} = A(t)x$. Global solutions for all t exist as long as*

$$\int_{t_0}^{t} \lambda_{max}(s) ds < \infty$$

Proof. Let $S = \frac{1}{2}(A + A^T)$ and $\Omega = \frac{1}{2}(A - A^T)$ (symmetric, and non-symmetric part). Thus, the eigenvalues λ_i of S are real and their eigenvectors e_i are orthogonal.

Consider

$$\begin{aligned}
\langle x, \dot{x}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2 = \langle x, A(t)x\rangle = \langle x, (S(t) + \Omega(t))\rangle \\
\overset{*}{=} \langle x, S(t)x\rangle + \underbrace{\langle x, \Omega(t)x\rangle}_{=0} \overset{**}{=} \sum_{i=1}^{n} \lambda_i(t) x_i^2 \\
\leq \lambda_{max}(t) \sum_{i=1}^{n} x_i^2 = \lambda_{max}(t)\|x(t)\|^2 = \lambda_{max}(t)\|x(t)\|^2
\end{aligned}$$

In step (*) we used that skew-symmetric matrices in quadratic forms are zero.
In step (**) we used the expansion of x in the orthonormal eigenvector basis:
$Sx = S\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i S e_i = \sum_{i=1}^{n} x_i \lambda_i$.
Now we integrate and exponentiate the resulting inequality:

$$\begin{aligned}
\int_{t_0}^{t} \log\left(\frac{\|x(s)\|^2}{\|x(t_0)\|^2}\right)\mathrm{d}s \leq \int_{t_0}^{t} \lambda_{\max}(s)\mathrm{d}s \\
\Longleftrightarrow \|x(t)\| \leq \|x(t_0)\| \exp\left(\int_{t_0}^{t} \lambda_{\max}(s)\mathrm{d}s\right)
\end{aligned}$$

Applying theorem 11 completes the proof. □

A remark on S and Ω:
S in continuum mechanics is often called rate-of-strain tensor and Ω is often called spin or vorticity tensor if A is the jacobian of a velocity field: $A = \nabla v(x(t), t)$.

2.2 Dependence on initial conditions

Consider again

$$\begin{cases} \dot{x} = f(x, t) \\ x(t_0 = x_0) \end{cases} \tag{9}$$

with f r-times differentiable.

Theorem 13 (Dependence on initial conditions)**.** *Let f be r-times differentiable in x and t. Then $x(t; t_0, x_0)$ is r-times differentiable in x_0 and t_0*

Proof. See in [AS92]. □

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3 Stability of fixed points