Linear maps and matrices

Notes by Markus Renoldner
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from Dr. Menny Akka Ginosar at ETH Zürich

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3.1 Change of basis

Let V be a vector space over K (usually $K = \mathbb{R}$ or $K = \mathbb{C}$) with a basis $B = \{\boldsymbol{b}_1, ...\boldsymbol{b}_n\}$ then every $\boldsymbol{v} \in V$ can be expressed as a linear combination of "coordinates" $\lambda_i \in K$ and basis vectors $\boldsymbol{b}_i$:

$$\boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \cdot \boldsymbol{v}_i \tag{1}$$

Example 1 (Vector expressed in a basis)**.**
Let $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2$*, let* $B_1 = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ *be the canonical basis and* $B_2 = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ *with* $\boldsymbol{b}_1 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\boldsymbol{b}_2 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Then:

1. *Of course* $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

3.2 asdf

Let V be an n-dimensional vector space (finite dimensional) over K (usually $K = \mathbb{R}$ or $K = \mathbb{C}$) with a basis $B = \{\boldsymbol{b}_1, ...\boldsymbol{b}_n\}$ then every $\boldsymbol{v} \in V$ can be expressed as a linear combination of "coordinates" $\lambda_i \in K$ and basis vectors $\boldsymbol{b}_i$

$$\boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \cdot \boldsymbol{v}_i \tag{2}$$

In general, a vector space has infinitely many bases B, and $\boldsymbol{v}$ looks different in each of them. Consider the example:
Let $\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2$, let $B_1 = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the canonical basis and $B_2 = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ with $\boldsymbol{b}_1 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{b}_2 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Then $\boldsymbol{x}$ in the canonical basis is of course

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and $\boldsymbol{x}$ in the basis B_2 is

$$\boldsymbol{x} = \lambda_1 b_1 + \lambda_2 b_2 = \lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ -\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Now one can compute the coefficients/coordinates and obtain

$$\lambda_1 = \frac{5}{2}, \quad \lambda_2 = \frac{1}{2}$$

Now we define the coordinate map ("Koordinatenabbildung"): Let V be a finite dimensional vector space over K and $B = \{b_1, b_2, ...b_n\}$. Then we call

$$K_B : V \to K^n \tag{3}$$

$$v = \sum_{i=1}^{n} v_i^B b_i \mapsto v_B := \begin{pmatrix} v_1^B \\ v_2^B \\ \vdots \\ v_n^B \end{pmatrix} \tag{4}$$

the coordinate map of V and v_B the coordinate vector of v with respect to the basis B. It is bijective, i.e.

$$K_B^{-1} : K^n \to V$$

exists.
Example: vector space of 2×2- matrices denoted as $M(2 \times 2, \mathbb{R})$. Consider the basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Find $K_B : M(2 \times 2, \mathbb{R}) \to \mathbb{R}^4$.

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = v_1^B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + v_2^B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v_3^B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + v_4^B \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} v_1^B & v_2^B + v_3^B \\ v_2^B - v_3^B & v_4^B \end{pmatrix} \tag{6}$$

Solving this for the coefficients/coordinates gives:

$$v_1^B = a_{11}, \quad v_2^B = \frac{a_{12} + a_{21}}{2}, \quad v_3^B = \frac{a_{12} - a_{21}}{2}, \quad v_4^B = a_{22}$$

and then

$$K_B(A) = K_B \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}-a_{21}}{2} \\ a_{44} \end{pmatrix}$$

Now we compute the inverse map. Consider a general coordinate vector

$$x_B := \begin{pmatrix} x_1^B \\ x_2^B \\ x_3^B \\ x_4^B \end{pmatrix}$$

It is the coordinate vector of the matrix

$$x_1^B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2^B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3^B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_4^B \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1^B & x_2^B + x_3^B \\ x_2^B - x_3^B & x_4^B \end{pmatrix}$$

which gives the inverse map of K_B:

$$K_B^{-1} : \mathbb{R}^4 \to M(2 \times 2, \mathbb{R}) : \begin{pmatrix} x_1^B \\ x_2^B \\ x_3^B \\ x_4^B \end{pmatrix} \mapsto \begin{pmatrix} x_1^B & x_2^B + x_3^B \\ x_2^B - x_3^B & x_4^B \end{pmatrix}$$

Clearly, as K_B and K_B^{-1} are inverses, one gets

$$(K_B^{-1} \circ K_B)A = A, \quad \text{and} \quad (K_B \circ K_B^{-1})x_B = x_B$$