Nonlinear dynamical systems

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Contents

1 Introduction

Definition 1 (Dynamical System). *A triple* $(P, E, \mathcal{F})$ *with*

- P *is the phase space*
- E *is the space of the evolutionary variable, e.g. time* t
- $\mathcal{F}$ *is the deterministic evolution rule that defines how the state* $x(t) \in P$ *evolves*

is called dynamical system. If $E = \mathbb{Z}^d$ *the D.S. is discrete, if* $E = \mathbb{R}^d$ *the D.S. is continuous.*

Discrete D.S. can be described by iterated mappings

$$x_{n+1} = F(x_n, n) \tag{1}$$

Continuous D.S. can be described by ODEs. Any ODE can be formulated as a system of first order ODEs:

$$\dot{x} = f(x,t) \tag{2}$$

with $x \in P$ and $t \in E$, which yields an initial value problem

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases} \tag{3}$$

Assuming unisolvence, we can define the following map:

Definition 2 (Flow map). *We call the map*

$$F_{t_0}^t : (t; t_0, x_0) \mapsto \varphi(t; t_0, x_0)$$

the flow map of a continuous D.S. like described in (1) *if* $\dot{\varphi} = f(\varphi, t)$ *and* $\varphi(t_0) = x_0$.

Lemma 3 (Properties of the flow map)**.** *It holds that:*

- $F_{t_0}^t$ *is as smooth as* $f(x,t)$
- $F_{t_0}^{t_0} = \mathrm{Id}$ *and* $F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_1}^{t_0}$
- $\left(F_t^{t_0}\right)^{-1} = F_{t_0}^t$ *exists and is smooth*

Property 2 and 3 are called group property.

Proof. TODO □

Definition 4 (Autonomous dynamical system)**.** *A continous D.S. whose evolution rule f is not dependent on time is called autonomous. The system is then described by*

$$\dot{x} = f(x)$$

From this definition, the flow map can be easily formulated as a special case of Definition 2:

$$F_{t_0}^t : x_0 \mapsto x(t, x_0)$$

Definition 5 (Separatrix)**.** *Codimension-1 surface in phase space separating regions of different long term behaviour*

2 Fundamentals

Theorem 6 (Peano)**.** *Let* $f(t,x) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be a continous function, then for the initial value problem* (3) *there exists a solution*

$$\varphi(t) : I \mapsto \mathbb{R}$$

on a time interval I*, which is a neighbourhood of the initial time* t_0*.*
Further φ *is continuously differentiable.*

Proof. see english wikipedia □

Theorem 7 (Picard-Lindelöf)**.** *Let* $f(t,x) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be a locally Lipschitz continous function, then the initial value problem* (3) *has a unique solution*

$$\varphi(t) : I \mapsto \mathbb{R}$$

on a time interval I*, which is a neighbourhood of the initial time* t_0*.*
Further φ *is continuously differentiable.*

Proof. see english wikipedia ☐

Example 8 (Non-unique solution)**.** *Consider*

$$\begin{cases} x' = |x|^{\frac{1}{2}} \\ x(t_0) = 0 \end{cases} \tag{4}$$

The right hand side, which is the evolution rule, is continuous, but not Lipschitz continuous in the interval $[0, 1]$*. Therefore we can only conclude solvability (Theorem 6), and not unisolvence (Theorem 7).*

A consequence of these theorems is that trajectories of autonomous, unisolvent systems never intersect. This also holds for the mathematical pendulum, which seems to create intersecting trajectories at the unstable fixed points.
These trajectories do not actually intersect:

- One trajectory is the unstable fixed point itself.
- Another one is a trajectory pointing outwards of the fixed point, not including the fixed point itself.
- And yet another one is the trajectory leading into the fixed point - again not including the fixed point, as the state never reaches the fixed point in finite time.

Non-autonomous, unisolvent systems do have intersecting trajectories. This can be avoided by extening the phase space:

$$X := \begin{pmatrix} x \\ t \end{pmatrix}, \quad F(X) := \begin{pmatrix} f(x,t), \\ 1 \end{pmatrix}$$

This yields

$$\dot{X} = F(X)$$

.

3 Stability of fixed points