Num Comp Repetition

Polynomial Interp.

Lagrange $p(x) = \sum_i f_i \cdot \left(\prod_j \frac{x-x_j}{x_i - x_j} \right)$ Cost $O(n^2)$

Neville Schema $P_{j,0} := f_j$, $P_{j,m} := \frac{(x-x_j)\cdot P_{j+1,m-1}(x) - (x - x_{j+m}) P_{j,m-1}(x)}{x_{j+m} - x_j}$

Cost $O(n^2 \cdot \frac{1}{2})$

Newton Poly $p(x) = \sum_i d_i \cdot \left(\prod_j x - x_j \right) \Rightarrow d_i = f[x_0, \dots, x_i]$

with $\begin{cases} f[x_0, \dots x_i] := \frac{f[x_1, \dots x_i] - f[x_0, \dots x_{i-1}]}{x_i - x_0} \\ f[x_i] := f_i \end{cases}$

Horner scheme $p(x) = d_0 + (x - x_0) \cdot \left[d_1 + (x - x_1) \left[d_2 + \dots \right] \right]$

Cost of finding coeff $O(n^2)$

Cost of eval $O(n)$!!!

Neville for Extrapol zB: $u(x) = e^x$, $h_i := 2^{-i}$, $i = 0, 1, \dots$

$$\begin{array}{c|c|l} h_0 & u(h_0) & =: u_{00} \to u_{01} \to u_{02} \\ h_1 & u(h_1) & =: u_{10} \nearrow u_{11} \nearrow \\ \vdots & \vdots & \quad u_{20} \nearrow \end{array}$$

Interp. error $h_i := q^i$, p_m interpolates f at $x_0 + h_{i+j}$, $j = 0, \dots m$

$|f(x_0) - p_m(x_0)| \leq C \cdot h_i^{m+1}$

Extrapol w. add. struct. ~~[illegible]~~ if f is function of $x^2 \Rightarrow$ interpolate $(x^2, f(x^2))$

$\Rightarrow$ error: $|f - p| \leq C \cdot h^{2m}$

Chebyshev. choose $x_i := \frac{a+b}{2} + \frac{b-a}{2} \cdot \cos\left(\pi \frac{2i+1}{2n+2}\right) \to \left\| \prod_i (x - x_i) \right\|_\infty \leq \left\| \prod_k (x - x_k) \right\|_\infty$

$\forall x_k$

Error $\begin{cases} \| I^{cheb} f \| \leq \Lambda_n \|f\|_\infty \\ \| f - I^{cheb} f \| \leq (1 + \Lambda) \min_q \| f - q \| \\ \Lambda \leq C \cdot \ln(n) \end{cases}$

$\Lambda := \max_x \sum_i \left(\prod_j \frac{x - x_j}{x_i - x_j} \right)$

Splines space $S^{p,r} := \{ u \in C^r,\ u|_{I_i} \in \mathcal{P}^p \}$ $(r \geq p \Rightarrow s \in \mathcal{P}^p)$

- linear: $p=1$, $r=0$
- cubic: $p=3$, $r=2$

but: $\begin{cases} \text{cond. } s(x_i) \overset{!}{=} f(x_i) \text{ yields } n+1 \text{ equations} \\ \dim(S^{3,2}) = n(p+1) - (n-1)(r+1) = n+3 \end{cases}$ $\Rightarrow$ 2 missing cond.

$\Rightarrow$
1. compl./clamp. $s'_{0,n} = f'_{0,n}$ BC.
2. period. $s'_0 = s'_n$, $s''_0 = s''_n$
3. nat. $s''_0 = s''_n = 0$
4. not-a-knot no ~~…~~ jump of s'''_0 / s'''_n

Cubic Spline Error 1./2/4. $\Rightarrow$ s unique and $\|f-s\| \leq C \cdot h^4 \|f^{(4)}\|$

Energy min ~~…~~

$\begin{cases} 1. \text{ compl./clamp. } \|s''\|_{L^2} \leq \|y''\|_{L^2} \ \forall y \in C_{compl} \\ \qquad C_{compl} := \{ y \in C^2 : y_i = f_i,\ y'_0 = f'_0,\ y'_n = f'_n \} \\ 2. \text{ nat. } \|s''\| < \|y''\| \ \forall y \in C_{nat} \\ 3. \text{ per. } \|s''\| < \|y''\| \ \forall y \in C_{per} \end{cases}$

$\Rightarrow$ minim. $\frac{1}{2}\|y''\|^2_{L^2}$

Trigonometric polyn. $p = \sum_{j=-m}^{m} c_j e^{ijx}$, $p_k \overset{!}{=} f_k$

Four. series $f = \sum_{j=-\infty}^{\infty} f_j e^{ijx}$, $f_j = \frac{1}{2\pi}\int_0^{2\pi} f e^{-ijx} dx$

Modified trigon. poly. $e^{imx} \cdot p = \ldots = \sum_{l=0}^{n-1} c_l e^{ilx}$ (new index starting at 0)

Solve interp. prob. $\left.\begin{array}{l} z_j := e^{ix_j} \\ p = \sum_{k=0}^{n-1} c_k e^{ikx} \end{array}\right\} \Rightarrow \underbrace{\begin{pmatrix} z_0^0 & z_0^1 & \cdots & z_0^{n-1} \\ z_1^0 & & & \\ \vdots & & & \end{pmatrix}}_{=: \tilde{V}} \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}$

$\det(\tilde{V}) \neq 0$!!

Now: $\omega_n := e^{-\frac{2\pi i}{n}}$, ~~…~~ matrix V: ~~…~~ $V_{ij} := \omega_n^{j \cdot k}$

$\Rightarrow \begin{cases} 1. \ \frac{1}{n} V_n = \tilde{V}^{-1} \text{ or } \frac{1}{n} V_n \cdot c = y \\ 2. \ \frac{1}{\sqrt{n}} V_n \text{ symm + unitary} \end{cases}$

DFT $\mathcal{F}_n : \mathbb{C}^n \to \mathbb{C}^n$

$\vec{y} \mapsto V_n \cdot \vec{y} = \vec{c}$ Cost $\mathcal{O}(n^2)$

$\left(\mathcal{F}_n^{-1} \vec{y} = \frac{1}{n} \overline{\mathcal{F}_n \overline{y}} \right)$

FFT divide and conquer: instead of DFT $c_k = \sum_{j=0}^{n-1} y_j \omega^{kj}$

$$\begin{cases} m := \frac{n}{2}, & c_{2l} = \sum_j^{m-1} (y_j + y_{j+m}) \omega^{2lj} \\ & c_{2l+1} = \sum_j^{m-1} (y_j - y_{j+m}) \omega^j \omega^{2lj} \end{cases}$$

reduce $\mathcal{F}_n(y)$ to 2 times $\mathcal{F}_{\frac{n}{2}}$. Do this recursively! $n \stackrel{!}{=} 2^p, p \in \mathbb{N}$

Cost: $A :=$ cost of $FFT(n,y) \Rightarrow A(n) \leq 2 \cdot A(\frac{n}{2}) + C \cdot n$ ← computing $(y_j + y_{j+m}), \ldots$

$\ldots \Rightarrow A(n) \leq n \cdot \log(n) \cdot C$ Cost $\mathcal{O}(n \log n)$

Conv $(f * g)_k := \sum_j f_{k-j} g_j$, f, g .. n-per. sequ.

$\Rightarrow \mathcal{F}(f*g) = \hat{f} \cdot \hat{g}$, $\hat{f}, \hat{g}$ ... Four. coeff. Cost $\mathcal{O}(n^2)$

Cost using FFT $\mathcal{O}(n \log n)$

Product of large nr. $x = \sum x_j b^j$, $y = \sum y_j b^j \Rightarrow x \cdot y = \sum_j z_j b^j$, $z_j = \sum_k x_{j-k} y_k$

Num Integration

Def Quad. formula $Q(f) = \sum_i w_i f_i$

Newton-Cotes interpolate $f \approx \sum f_i l_i(x) \Rightarrow Q_{NC} = \sum f_i \underbrace{\int l_i(x) dx}_{w_i}$

Error/Accuracy n odd : exact for $f \in \mathcal{P}^n$

n even : $f \in \mathcal{P}^{n+1}$

Ex: $n = 1$, $w_1 = w_2 = \frac{1}{2} \Rightarrow$ trapezoidal rule $T(f)$

$n = 2$, $w_1 = w_3 = \frac{1}{6}$, $w_2 = \frac{4}{6} \Rightarrow$ Simpson rule $S(f)$

Error $f \in C^1 \Rightarrow \left| \int f dx - T(f) \right| \leq 2 \sum h_i \cdot \min_{v \in P_1} \|f - v\|_\infty$

$f \in C^2 \Rightarrow \left| \int f dx - T(f) \right| \leq C h^2 \|f''\|_\infty$

Rhomberg extrapol $\left| \int f dx - T(f) \right| \xrightarrow{N \to \infty} 0$

$\Rightarrow$ use Neville extrapol of (h_i, T_{h_i}) for $h_i := \frac{a-b}{2^i}$

Adaptive Quad. e.g. $f = x^{0.1}$, to get $\mathcal{O}(h^2)$ converg. with Trapec. refine points towards 0: $x_i = \left(\frac{i}{N}\right)^2$ or $\left(\frac{i}{N}\right)^3$

estimate rule with higher rule → refine if necessary

Legendre Poly $L_n \in \mathcal{P}_n$ s.t.
- $\{L_n\}$ is a basis of $\mathcal{P}_n$
- L_n is orthog. to $\mathcal{P}_{n-1}$
- $L_n(1) = 1$

construct: $\begin{cases} \text{Rodrig. formula (explicit)} \\ \text{Recursion 3-term (implicit)}: \; L_{n+1}\cdot(n+1) = L_n \cdot x(2n+1) - L_{n-1}\cdot n \end{cases}$

Gauss Quad $Qf = \sum w_i f_i$, $w_i = \int \prod_j \left(\frac{x - x_j}{x_i - x_j}\right) dx$, x_j ... zeros of L_{n+1}

$\Rightarrow$ Qf is exact for $f \in \mathcal{P}_{2n+1}$!!! (best possible Quad.)

Proof: write f as $f(x) = L_{n+1}(x)\cdot q_n(x) + r_n(x)$ (polyn. division)

degree of polyn.

$$\Rightarrow \int_{-1}^{1} f\,dx = \int \underbrace{L_{n+1}\, q_n}_{=0 \text{ (orthog.)}} + r_n\,dx = Q(r_n) = \sum w_i \cdot r_n(x_i)$$

$$= \sum w_i \cdot [r_n(x_i) + \underbrace{L_{n+1}(x_i)\, q_n(x_i)}_{=0 \text{ (zeros of } L_{n+1})}] = Q(f) \quad \text{and } f \in \mathcal{P}_{2n+1} \text{ !!!} \quad \blacksquare$$

Conv/Error $\left|\int f dx - Qf\right| \leq 4 \min_v \|f - v\|_\infty$, $v \in \mathcal{P}_{2n+1}$

Gauss > Trapez, except for periodic func!!

Quadr. in 2D $Q = \sum_{i,j} w_i w_j F(x_i, x_j)$ s.t. Qf is exact for $f \in \{x^i y^j : i,j = 0, \ldots p\}$

Gauss Q. in practice: choose $\begin{cases} 1. \text{ Newton method} \Rightarrow \text{zeros of } L_{n+1} \\ 2. \text{ zeros as eval of matrix} \end{cases}$

2.: recurrence relation $\Leftrightarrow$ $x \cdot \vec{L} = T \cdot \vec{L} + \frac{1}{\alpha_n} L_n \vec{e}_n$

$\Rightarrow$ $\vec{L}(\xi)$ is evec of T iff. $L_n(\xi) = 0$

$\Rightarrow$ evals of T are zeros of L_n with evec $\vec{L}(\xi)$

(Given evals of T finding w_i is easy, see lec. notes)

Quad. with weight $\int f \cdot w\,dx = \sum w_i f_i \quad \forall f \in \mathcal{P}_p$

$\Rightarrow$ $\begin{cases} w = 1: \text{ Legendre poly to find zeros for } w_i \\ w = (1-x^2)^{-1}: \text{ Chebyshev poly to find zeros for } w_i \end{cases}$

Conditioning + error ana

Norm: 1. triangle ineq 2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homog.) 3. definitness

Cond.: amplific. of input perturbations of func. evaluation

OUTPUT INPUT

$$\begin{cases} 1.\ \text{abs cond: smallest } K \text{ s.t.: } \|f(x) - f(x+\Delta x)\| \leq K \|\Delta x\| \\ 2.\ \text{rel. cond: smallest } K \text{ s.t. } \frac{\|\Delta f\|}{\|f\|} \leq K \frac{\|\Delta x\|}{\|x\|} \end{cases}$$

OUTPUT INPUT

in practise: $\begin{cases} K_{abs} \approx |f'| \\ K_{rel} \approx |f'| \cdot \frac{|x|}{|f|} \end{cases}$

Ex. Add $f(x,y) := x+y \Rightarrow \frac{|x+\Delta x + y + \Delta y - x - y|}{|x+y|} \leq \ldots \leq \frac{|\Delta x|}{|x|}$ well cond.

Ex. Subtr. $x_1 = 0{,}123467*$ perturbation
$x_2 = 0{,}123456*$

perturbation in 3rd digit!!

$\Rightarrow x_1 - x_2 = 0{,}11* \cdot 10^{-4}$ ill cond.

Ex. avoid canc. zeros of $x^2 - 2px - q = 0$?

ill cond: $\begin{rcases} x_0 = p - \sqrt{p^2+q} \\ x_1 = p + \sqrt{p^2+q} \end{rcases} \Leftrightarrow \begin{cases} x_0 = \frac{q}{x_1} \\ x_1 = p + \sqrt{p^2+q} \end{cases}$ well cond. (no subtraction!!)

Gauss Elim

goal: solve $A\vec{x} = \vec{b}$

Triangular A A upper or lower triang $\Rightarrow$ solve in $\mathcal{O}(n^2)$

Gauss elim

```
for k = 1, ..., n-1
  for i = k+1, n-1
    l_ik = a_ik / a_kk
    A[i, [k+1, ..., n]] += -l_ik · A[k, [k+1, ..., n]]
```

Gauss as LU updates on A can be described by L:

$$L^{(k)} := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & l & \ddots & \\ & l & & 1 \\ & l & & \end{pmatrix}, \quad L := L^{(1)} \cdot L^{(2)} \cdots L^{(n-1)}$$

$\Rightarrow A = L \cdot U$, U is upper triang.

$$\begin{cases} 1.\ \text{find } L, U \\ 2.\ \text{solve } L\vec{y} = \vec{b} \\ 3.\ \text{solve } U\vec{x} = \vec{y} \end{cases}$$ cost $\mathcal{O}(\frac{2}{3}n^3)$

Crout n^2 equations to factorize $A = LU$. Idea: use structure of L, U

$\Rightarrow A = \begin{pmatrix} 2 & 1 & 3 \\ & 4 & 5 \end{pmatrix}$ order of equations!!

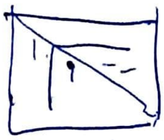
still $\mathcal{O}(n^3)$

Remark: diff. LU algos have diff. memory access!

Banded / Skyline

$O(npq)$ \\ band width

skyline



non-skyline

Cholesky if A spd $\Rightarrow$ $\exists C$ st. $A = C \cdot C^T$, C... triangular   Cost $O(\frac{4}{3}n^3)$

Gauss with pivot   1. $A = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$ has no LU-fact. as $a_{11} = 0$

2. $\exists P$ st. $PA = LU$ $\forall A$

P is a permutation matrix $\Leftrightarrow$ switches colums/rows ($a_{11} \neq 0$)

3. also helps if $a_{11} << a_{ij}$ $\forall i,j \neq 1,1$

Condition nr. $A\vec{x} = \vec{b}$, perturb input $(\Delta \vec{b})$ $\leadsto$ is $\Delta \vec{x}$ large?

$A(x + \Delta x) = b + \Delta b$

$\|\Delta x\| = \|A^{-1} \Delta b\| \leq \underbrace{\|A^{-1}\|}_{=:k} \|\Delta b\|$.

$\frac{\|\Delta x\|}{\|x\|} = \dots \leq \underbrace{\|A\| \|A^{-1}\|}_{=:k} \frac{\|\Delta b\|}{\|b\|}$

Orth matr. $Q^T = Q^{-1}$, $\|Qx\|_2 = \|x\|_2$, multipl. by Q is num. stable

$\exists Q, R$ st. $A = QR$ $\forall A$ invertible

QR by Gram Schmidt $A \in \mathbb{R}^{n \times n}$, $A = (\vec{a}_1, \dots \vec{a}_n)$

Gram Schmidt $\begin{cases} Q := (\vec{q}_1, \dots, \vec{q}_n) \text{ with} \\ \vec{q}_1 = \vec{a}_1 \\ \vec{q}_2 = \vec{a}_2 + \alpha_2 \vec{a}_1 \quad \text{st. } \vec{q}_2 \perp \vec{q}_1 \end{cases}$

$\Rightarrow Q = A \cdot \tilde{R}$, $\tilde{R}$ depends on α_i $\Rightarrow R := \tilde{R}^{-1}$

QR by Householder $H := \mathbb{I} - 2\vec{v}\vec{v}^T$ with $\|v\|_2 = 1$ is a reflection "along $\vec{v}$"

H ... orthog.

We want to map columns of A to multiples of unit vectors!!

Choose: $\vec{v} = \frac{x + \lambda \vec{e}_i}{\|x + \lambda \vec{e}_i\|}$ with $\lambda = \text{sign}(x_i) \cdot \|x_i\|_2$

$\Rightarrow Hx_i = \dots = -\lambda \vec{e}_i$

(No proof)

$\Rightarrow Q = Q_1^T Q_2^T \dots Q_n^T = H_{\vec{v}_1}^T H_{\vec{v}_2}^T \dots H_{\vec{v}_n}^T$ st. $A = Q \cdot R$

And $A = QR \Leftrightarrow QRx = b \Rightarrow Rx = Q^T b$ Cost $O(\frac{4}{3}N^3)$

Pivoting $rk(A) < n \Rightarrow$ pivot colum with largest 2-norm to first column

QR by Givens rotations

$$G(i,j,\theta) := \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & s & \\ & & s & c & \\ & & & & \ddots\ 1 \end{pmatrix} \begin{array}{c} i \\ j \end{array} \\ \quad\quad i \quad j \end{array} \quad \begin{cases} c := \cos\theta \\ s := \sin\theta \end{cases}$$

$\exists\, G(i,j,\theta)$ st. $(GA)_{ij} = 0$

It touches only colum i and j $\Rightarrow$ parallelize , Cost $O(\frac{8}{3}n^3)$

For Hessenberg matrix : $O(n^2)$

Least Squares

solve $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $m \neq n$

LS solution of $Ax=b$: find x st. $\|b - Ax\|_2 = \min \{ \|b - Ay\| \}$

Normal eq. x also solves $(\Leftrightarrow)$ $A^T A x = A^T b$

LS by QR $A \in \mathbb{R}^{m \times n}$, $m > n$, $R := \begin{pmatrix} R^* \\ 0 \end{pmatrix}$, $Q^T b = \begin{pmatrix} b^* \\ \tilde{b} \end{pmatrix}$

$\min \|Ay - b\|^2 = \|Ry - Q^T b\|^2 = \|R^* y - b^*\|^2 + \|\tilde{b}\|^*$

$\Longrightarrow x = y = R^{*-1} b^*$

Algo: 1. $QR =: A$, 2. $b^* = Q^T b$, 3. $x = R^{*-1} b^{*-1}$

Underdetermined $A \in \mathbb{R}^{m \times n}$, $m < n$ (sol not unique)

find x st. $\|x\|_2 = \min_y \{ \|y\|_2 : Ay = b \}$ .. min norm sol

SVD $\begin{cases} \exists\, \sigma_1 \geq \dots \geq \sigma_{\min(m,n)}, \ \exists\, U \in \mathbb{R}^{m \times m} \text{ orth.}, \ \exists\, V \in \mathbb{R}^{n \times n} \text{ orth} \\ \text{and } \Sigma_{ij} := \delta_{ij} \sigma_i \in \mathbb{R}^{m \times n} \text{ st.} \end{cases}$

$$A = U \Sigma V^T$$

$\Rightarrow$ non-zero $\sigma_i \rightsquigarrow$ rank of A

$\Rightarrow$ colums of U: basis of $Im(A)$

$\rightarrow$ $\underbrace{\text{colums}}_{r+1,\dots n}$ of V: basis of $Ker(A)$, colum $1..r$ of $(\ker A)^\perp$

$\rightarrow$ evals (A) = evals $(\Sigma^T \Sigma)$

Min norm sol by SVD $\tilde{V} = V[:, 1{:}r]$, $V' = V[:, r{+}1{:}n]$, $\tilde{\Sigma} = \Sigma[1{:}r, 1{:}r]$

reduced SVD: $A = U \tilde{\Sigma} \underbrace{\tilde{V}^T}$ storing all non-zero entries!

$\tilde{x} = \tilde{V}\tilde{\Sigma}^{-1}U^T b$ satisfies $A\tilde{x} = b$ and

But any sol $x = \tilde{x} + V'y$ as $V'y \in \ker(A)$!

and $\tilde{x} \perp V'$ (as its spanned by $\tilde{V}$)

$\Rightarrow$ $\|x\|^2 = \|\tilde{x}\|^2 + \|V'y\|^2$ is minim. by $y = 0$

Moore-Penrose Inverse $A = U\tilde{\Sigma}\tilde{V}^T \Rightarrow \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T =: A^+$

$A^+ b$ .. min norm sol of $Ax = b$

Proof: $\|Ax - b\|^2 = \| U\hat{\Sigma}\tilde{V}^T x - \underbrace{\tilde{U}\tilde{U}^T b}_{\in \text{ range of } A} + \underbrace{U'U'^T b}_{\text{rest}} \|^2$

$= \|\tilde{\Sigma}\tilde{V}^T x - \tilde{U}^T b\|^2 + \|U'U'^T b\|^2$

Minimal if $\tilde{V}^T x = \tilde{\Sigma}^{-1}\tilde{U} b$

Decompose $\|x\|^2 = \|\tilde{V}\tilde{V}^T x\|^2 + \|V'V'^T x\|^2 = \|V\Sigma U b\|^2 + \|V'V'^T x\|^2$

Minimal if $V'^T x = 0$

$\Rightarrow$ $x = A^+ b$

Remark: $A^+ : b \mapsto \tilde{U}\tilde{U}^T b \longmapsto A_k^{-1}\tilde{U}\tilde{U}^T b = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T b = A^+ b$

$\mathbb{R}^m \to \text{range}(A) \longrightarrow (\ker A)^\perp$

$A_k : (\ker A)^\perp \to \text{range}(A)$.

SVD: find evec + evals of $\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$

6, Newton

find zeros of $f(x)$, f can be in $\mathbb{R}^d$

Fixed point iter $\phi(x_n) = x_{n+1}$ should contract, ie

$\|\Delta\phi\| \leq q\|\Delta x\|$ $q \in (0,1)$

Newton $\phi := x_n - \frac{f(x_n)}{f'(x_n)}$ ... derived by linearization at x_n

Converg. contraction $\Rightarrow$ $\|x^* - x_{n+1}\| \leq q\|x^* - x_n\|^1$

Faster conv $\phi \in C^P$ and $\phi^{(j)} = 0 \ \forall j = 1, \dots, P-1 \Rightarrow \|\Delta x\| \leq q\|\Delta x\|^P$!!!

Remarks
- instead of $f'^{-1} \rightsquigarrow$ LSE
- residual $f(x_n)$ as error
- tri ineq: $\|x_n - x^*\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - x^*\|$

<u>Damped</u> $x_{n+1} = x_n - \lambda \cdot \frac{f}{f'}$, $\lambda \in (0,1)$ (also: „globalized")

<u>Descent Method</u> find min of g: 1. step length d_n 2. search direc. λ_n

$\tilde{g}(\lambda) := g(x + \lambda \cdot d)$

$\tilde{g}' = \nabla g \cdot d \overset{!}{<} 0$, $d := -\nabla g$

Armijo: $\lambda := g^k$, $k = 0, 1, \ldots$

Remark: zeros of f are min of $x \mapsto \|f\|_2^2 =: g$
use descent method with Newton direction $-f_n'^{-1} f_n =: d_n$
(this is a decent direction $\Leftarrow$ Taylor expand $\tilde{g}(\lambda)$)

<u>Non-lin. least squares / Gauss Newt.</u> minimize non-lin $F(x)$

$g := \|F\|_2^2$, min satisfies $\nabla g = 0$ $\Rightarrow$ define Newt. method

for $G := \nabla g$: $G_{n+1} = G_n + G_n' \cdot \Delta x_n \overset{!}{=} 0$

$\Rightarrow G_n' \Delta x_n = -G_n$

Further: $G' = F'^T F' + \underbrace{F''^T F}_{\text{small}}$ for $F(x^*) = 0$

$\Rightarrow F'^T F' \Delta x_n = -F'^T F$ (these are the normal equ. of the lin Least. Sq. problem $\|F' \Delta x_n + F\|^2 \leq \|F' y + F\| \ \forall y$)

quadr. conv (or lin if $F(x) \neq 0$)

<u>Quasi Newton</u>: f_n' expensive $\Rightarrow$ use f_0' : lin. conv.

better: <u>Broyden</u>: $x_{n+1} = x_n - \underbrace{H_n^{-1}}_{\text{approx of } f_n'^{-1}} f(x_n)$

find H by $H_{n+1} \Delta x_n = \Delta f_n$ st. $\|\Delta H\| \to \min$
(this is uni solvable)

<u>Conv</u>: superlinear $\|\Delta x_n\| \leq \varepsilon_n \|\Delta x_{n-1}\|$ for $\varepsilon_n \to 0$

H_n^{-1} easily computable using Sherman Morrison Woodbury