3) Conditioning and error analysis

Def: (Norm)
- i) triangle inequ.
- ii) homogeneity $\|\lambda x\| = |\lambda| \cdot \|x\|$
- iii) definiteness $\|x\| = 0 \Leftrightarrow x = 0$

Examples: $\|x\|_2 := \sqrt{\sum |x_i|^2}$, $\|x\|_\infty := \max_i |x_i|$, $\|x\|_1 := \sum |x_i|$

Def (cond. number) cond. nr. of problem (described as func. eval.) is the factor by which $\overset{\text{input}}{\text{perturbations}}$ are amplified. 2 cases:

1. absolute cond. nr.: K_{abs} is the smallest number s.t.

$$\underbrace{\|f(x) - f(x+\Delta x)\|}_{\text{output}} \leq K_{abs} \cdot \underbrace{\|\Delta x\|}_{\text{input}}$$

2. rel. cond. nr.:

$$\underbrace{\frac{\|f(x) - f(x+\Delta x)\|}{\|f(x)\|}}_{\text{output}} \leq K_{rel} \underbrace{\frac{\|\Delta x\|}{\|x\|}}_{\text{input}}$$

In practise: $K_{abs(x)} \approx |f'(x)|$, $K_{rel(x)} \approx |f'(x)| \cdot \frac{|x|}{|f(x)|}$

K large: „ill conditioned"
small: „well cond."

Example (Addition), i.e. $f(x,y) = x+y$

$$\Rightarrow \frac{|x+\Delta x + y + \Delta y - x - y|}{|x+y|} \leq \dots \leq \delta, \quad \delta := \frac{|\Delta x|}{x}$$

and $K_{rel} = 1$

Example (Subtraction - Cancellation)

$x_1 = 0{,}123467*$
$x_2 = 0{,}123456*$ ↑ perturbation in 7th digit

$$\Rightarrow x_1 - x_2 = 0{,}000011* = 0{,}11* \cdot 10^{-4}$$

↖ perturbation in 3rd digit in float representation

$$\Rightarrow K_{rel} \approx 10^4 \quad \text{„large"}$$

Example (avoid Cancellation) zeros of $x^2 - 2px - q = 0$

$x_0 = p - \sqrt{\dots}$
$x_1 = p + \sqrt{\dots}$

$$\Leftrightarrow x_0 = -\frac{q}{x_1} \quad \dots \text{avoids Subtraction!}$$