Laplace in Polar

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$\frac{\partial^2}{\partial x^2}$ 

$+ \frac{\partial^2}{\partial y^2}$ 

$$\nabla^2 u = u_{xx} + u_{yy}$$

$$x = r\cdot\cos\varphi \qquad \frac{\partial x}{\partial r} = \cos\varphi \qquad \frac{\partial y}{\partial r} = \sin\varphi$$

$$y = r\cdot\sin\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\cdot\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\cdot\cos\varphi$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial r^2} = \cos\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial y}$$

$$= \cos\varphi\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r}\frac{\partial u}{\partial x}\right) + \sin\varphi\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial r}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r}\frac{\partial u}{\partial y}\right)$$

$$= \cos\varphi\left(\cos\varphi\frac{\partial^2 u}{\partial x^2} + \sin\varphi\frac{\partial^2 u}{\partial y\partial x}\right) + \sin(\varphi)\left(\cos\varphi\frac{\partial^2 u}{\partial y\partial x} + \sin\varphi\frac{\partial^2 u}{\partial y^2}\right)$$

$$= \cos^2\varphi\frac{\partial^2 u}{\partial x^2} + 2\cos\varphi\sin\varphi\frac{\partial^2 u}{\partial y\partial x} + \sin^2\varphi\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = -r\cdot\sin\varphi\frac{\partial u}{\partial x} + r\cdot\cos\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}(\ldots) = -r\cdot\cos\varphi\frac{\partial u}{\partial x} - r\cdot\sin\varphi\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial x} - r\cdot\sin\varphi\frac{\partial u}{\partial y} + r\cdot\cos\varphi\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial y}$$

$$= -r\cdot\cos\varphi\frac{\partial u}{\partial x} - r\cdot\sin\varphi\cdot\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \varphi}\right)\frac{\partial u}{\partial x} - r\cdot\sin\varphi\frac{\partial u}{\partial y} + r\cdot\cos\varphi\cdot\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \varphi}\right)\frac{\partial u}{\partial y}$$

$$= -r\cos\varphi\frac{\partial u}{\partial x} - r\cdot\sin\varphi\left(-r\cdot\sin\varphi\frac{\partial}{\partial x} + r\cdot\cos\varphi\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial x} - r\cdot\sin\varphi\frac{\partial u}{\partial y} + r\cdot\cos\varphi\left(-r\cdot\sin\varphi\frac{\partial}{\partial x} + r\cdot\cos\varphi\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial y}$$

$$= -r\cos\varphi\, u_x + r^2\sin^2\varphi\, u_{xx} - r^2\sin\varphi\cos\varphi\, u_{xy} - r\sin\varphi\, u_y - r^2\cos\varphi\sin\varphi\, u_{xy} + r^2\cos^2\varphi\, u_{yy}$$

$$\frac{\partial^2 u}{\partial \varphi^2} = -r\cos\varphi\, u_x - r\sin\varphi\, u_y + r^2\sin^2\varphi\, u_{xx} - 2r^2\sin\varphi\cos\varphi\, u_{xy} + r^2\cos^2\varphi\, u_{yy}$$

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\varphi\, u_{xx} + 2\cdot\sin\varphi\cos\varphi\, u_{xy} + \sin^2\varphi\, u_{yy}$$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = -\frac{1}{r}\left(\cos\varphi\, u_x + \sin\varphi\, u_y\right) + \sin^2\varphi\, u_{xx} - 2\sin\varphi\cos\varphi\, u_{xy} + \cos^2\varphi\, u_{yy}$$

$$+ \quad \frac{\partial^2 u}{\partial r^2} = \cos^2\varphi\, u_{xx} + 2\sin\varphi\cos\varphi\, u_{xy} + \sin^2\varphi\, u_{yy}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = -\frac{1}{r}\underbrace{\left(\cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}\right)}_{\frac{\partial u}{\partial r}} + \underline{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}}$$