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Investigation of Surface Wave–Current Interactions with Vertical Shear and Viscosity

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Preface

This report is submitted as part of the course **TFY4510** at the Norwegian University of Science and Technology in Trondheim. The work serves as a preparatory study for my upcoming master's thesis and has been carried out during the spring semester of 2025.

The majority of the theoretical framework presented in this report is based on the lecture notes from the course, MR8303 - Kinematics and Dynamics of Ocean Surface Waves, taught by the supervisor of this project, Professor **Lars Erik Holmedal**, whose insights and pedagogical clarity have been instrumental in shaping the direction of the project. I would like to thank him, and Professor Emeritus, **Dag Myrhaug** for their guidance and continuous support throughout the course of this work.

Abstract

This study presents a theoretical investigation of surface wave–current interactions (WCI) in the presence of viscosity and shear, aiming to extend linear wave theory to more realistic oceanographic conditions. Existing literature on linear wave theory and weakly nonlinear WCI theory is thoroughly reviewed. Building on linear wave theory and perturbative techniques, a weakly nonlinear model is developed using a WKB approximation to describe the evolution of surface gravity waves over a vertically sheared, viscous current. The analysis begins with the incompressible Navier–Stokes equations and derives a system of partial differential equations that incorporates viscous boundary conditions and surface stresses. Consistency checks against classical linear theory confirm the model's validity in the appropriate limits. The resulting framework offers a foundation for future studies on wave propagation in complex marine environments, where vertical shear and viscous dissipation play a significant role.

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1 Introduction

Linear wave theory, commonly referred to as Airy theory, is a foundational framework for describing surface gravity waves. Derived from the incompressible, inviscid, and irrotational form of the Navier–Stokes equations, it provides a first-order approximation of wave motion under the assumptions of small amplitude and slowly varying or constant depth (U.S. Army Corps of Engineers, 2002). One of the key conveniences of linear theory is its foundation in potential flow, which allows for elegant and tractable analytical solutions. Despite its idealized nature, linear wave theory has proven to be remarkably robust, offering predictions that closely match experimental observations in large-scale natural systems such as oceans and deep lakes (Zhang et al., 2022).

However, the theory's assumptions break down in many scenarios of practical importance, particularly in coastal and nearshore environments. In shallow water, near obstacles, or under strong forcing, wave amplitudes increase and nonlinear effects become significant. In such regimes, linear theory fails to capture critical dynamics such as wave breaking, energy dissipation, and complex interactions with currents and boundaries. Accurately modeling these systems requires either higher-order analytical models or full numerical simulations, both of which are computationally demanding and often sensitive to parameterization.

Due to the complexity of nonlinear wave motion, linear theory remains a valuable starting point. Through perturbative techniques, weakly nonlinear corrections can be introduced to extend its applicability without abandoning its analytical tractability. These methods allow for systematic exploration of effects such as slow modulations, refraction, and wave-current interactions, provided the nonlinear terms remain sufficiently small.

The interaction between surface waves and ocean currents, commonly referred to as wave–current interaction (WCI), is important in coastal engineering and oceanography as it influences sediment transport, hydrodynamic forces on offshore structures, pollutant dispersion, and wave transformation. As reviewed by Zhang et al. (2022), WCI has been the subject of extensive theoretical, experimental, and numerical studies over the past decades. Most classical models focus either on linear wave transformation in vertically uniform currents using the dispersion relation, or on boundary-layer dynamics using approximate methods to account for turbulence. These studies typically neglect viscosity in the derivation of wave kinematics, or treat it mainly through empirical corrections. However, in a real fluid, viscosity introduces internal shear stresses, enables boundary layer development, and plays a role in energy dissipation. Incorporating it into wave models with background currents could thus better the model accuracy, particularly when the currents exhibit strong vertical shear. The advantage of expressing the flow using potential theory, however, is lost when viscosity is introduced, thus also requiring more complex formulations.

This report aims to establish a theoretical foundation for studying WCI in the presence of both vertical shear and viscosity. It begins with a review of linear wave theory and viscous boundary conditions (Section 2), followed by a derivation of a viscous two-dimensional WCI model using a WKB approximation. The model is validated against known solutions from inviscid linear wave theory and by demonstrating compatibility with an eddy viscosity formulation (Sections 3–4). This work is intended to lay the groundwork for future investigations into how strong shear currents and viscosity influence wave propagation in more realistic oceanographic settings.

2 Theory

2.1 Equations for incompressible flow

In this report, we are interested in water waves. For the temporal and spatial scales of real world applications, the variation of water density is for most purposes insignificant (Mei et al., 2005). Under this assumption, water becomes *incompressible* and its motion is described by

- conservation of mass

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{and} \tag{1}$$

- conservation of momentum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}. \tag{2}$$

Equation (1) is the volume continuity equation, and (2) is the Navier–Stokes equation in tensor notation, named after Claude-Louis Navier and George Gabriel Stokes.

From the equations (1) and (2) we can deduce the vorticity vector

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{u}. \tag{3}$$

In fluid dynamics, a common class of problems is when the vorticity vector (3) equals zero. A flow which has zero vorticity is called an *irrotational flow* (Mei et al., 2005). In cases where we can also neglect the viscosity ν, we can simplify the description of the flow by expressing the velocity as the gradient of a scalar potential (Mei et al., 2005), i.e.

$$\mathbf{u} = \nabla\Phi. \tag{4}$$

We call Φ, *the velocity potential*. Using the velocity potential, the conservation of mass satisfies Laplace's equation,

$$\nabla^2\Phi = 0. \tag{5}$$

2.2 Boundary conditions

For a fluid in a gravity field, there are two interfaces on the macro scale that are of interest; the air-fluid interface at the upper boundary, and the fluid-solid interface at the bottom. Along both of these boundaries, we assume only tangential flow. We will now follow the derivation of the boundary conditions at the surface and bottom described by Mei et al. (2005, p. 4). We express the instantaneous equation of the boundary as

$$F(\mathbf{x}, t) = z - \eta(x, y, z) = 0, \tag{6}$$

where the xy-plane denotes the horizontal plane, and z is the vertical axis. η is the height measured from $z = 0$. We denote a point $\boldsymbol{x}$ on the *free surface*, which is the moving upper boundary of the fluid, as $\boldsymbol{q}$, so that when considering an infinitesimal time element dt, we can describe the surface as

$$F(\mathbf{x} + \mathbf{q}\,\mathrm{d}t,\, t + \mathrm{d}t) = 0 = F(\mathbf{x}, t) + \frac{\partial F}{\partial t}\,\mathrm{d}t + \mathbf{q} \cdot \nabla F\,\mathrm{d}t + \mathcal{O}(\mathrm{d}t^2) \tag{7}$$

Considering (6), it follows that

$$\frac{\partial F}{\partial t} + \mathbf{q} \cdot \nabla F = 0. \tag{8}$$

To fulfill the requirement of tangential motion only, we must have that

$$\mathbf{u} \cdot \nabla F = \mathbf{q} \cdot \nabla F, \quad \text{which implies} \quad \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0, \quad \text{at } z = \eta. \tag{9}$$

This is the *kinematic boundary condition*, which physically states that the particles at the surface stay on the surface. We find the bottom boundary condition from the special case where equation (9) is the wetted surface of a stationary solid constraining the system at its bottom.

On the free surface, both the elevation and the velocities are unknown. For macroscale water systems, the wavelength of the surface is long enough that the surface tension is uninteresting, so it follows that the pressure just below the free surface equals the pressure above, p_a. From the momentum equation (2), we can derive the pressure field. Applying it on the free surface gives us the *dynamic surface condition*.

In the first section of this report, we will focus on inviscid irrotational flow. We inwoke equation (4) and express the surface kinematic condition (9), the bottom kinematic condition, and the dynamic surface condition in terms of the velocity potential Φ:

- The kinematic surface

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = \eta, \tag{10}$$

- The kinematic bottom

$$\frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = -h, \tag{11}$$

- The dynamic surface

$$-\frac{p_a}{\rho} = g\eta + \frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla \Phi|^2 \quad \text{at } z = \eta, \tag{12}$$

where η is the elevation of the free surface, Φ is the velocity potential of the fluid and h is the depth of the system. ρ and g are respectively the density of the fluid and the gravitational constant.

We can combine the surface conditions into one equation by taking the material derivative of the dynamic condition and assuming that the atmospheric pressure p_a is constant. The resulting expression reads

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial t} + \frac{\partial \vec{u}^2}{\partial t} + \frac{1}{2}\vec{u} \cdot \nabla \vec{u}^2 = 0 \quad \text{at } z = \eta. \tag{13}$$

2.3 Linearization

In this report, we will consider the linearized versions of the surface wave equations. Linearization is a powerful tool that enables us to consider surface waves and currents as independent entities, while investigating their interactions as superpositions of their initial properties. Linearized systems have certain limitations and are only valid for small surface elevations around the mean surface elevation, $z = 0$, and at low Reynolds numbers.

The assumptions for a linear surface wave-current system are that convective terms disappear and that the depth of the system varies slowly or remains constant, i.e.

$$\begin{aligned} U &\ll U^2, Uv, Uw \\ \eta &\ll \eta^2, U\eta \\ \nabla h &\ll 1. \end{aligned}$$

Under these assumptions we find the continuity equation, the kinematic bottom condition, and the dynamic surface condition to be

$$\nabla^2 \Phi = 0, \tag{14}$$

$$\frac{\partial \Phi}{\partial z} = 0, \quad \text{at } z = -h(x, y), \tag{15}$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} = 0, \quad \text{at } z = 0. \tag{16}$$

We note that (14) satisfies Laplace's equation. In (16) we have used the linear kinematic boundary condition, which says that at the surface, $z = 0$, the time evolution of the free surface equals the vertical velocity of the wave, i.e.,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z}, \tag{17}$$

and that the surface pressure balances the gravitational force and the time evolution of the velocity potential

$$-\frac{P_a}{\rho} = g\eta + \frac{\partial \Phi}{\partial t}. \tag{18}$$

2.3.1 Important results from linear wave theory

When solving the equations for linearized wave theory, we find some important results. For simplicity we consider a wave propagating in the positive x-direction at a finite water depth. Figure 1 presents this idealized system.

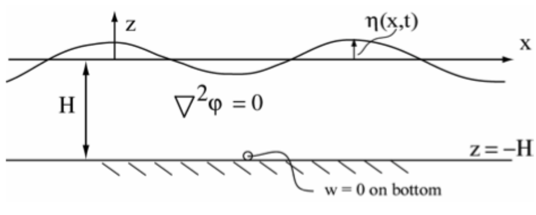

Figure 1: Linear wave on the free surface over a flat bottom. The flow satisfies Laplace's equation.

Note. Adapted from Techet (n.d.).

To solve the equations, we assume that the velocity potential takes the linearized form

$$\Phi(x, z, t) = \Phi(z)e^{i(kx - \omega t)}, \tag{19}$$

where k is the wavenumber of the velocity potential and ω its frequency. The conditions in equations (14)-(16) can then be written

$$-k^2\Phi(z) + \Phi''(z) = 0, \tag{20}$$

$$\Phi'(-h) = 0, \tag{21}$$

$$-\omega^2\Phi(0) + g\Phi'(0) = 0, \tag{22}$$

where $\Phi'(z)$ denotes the partial derivative of Φ wrt. z. We make the ansatz that the solution to the continuity equation (20) for the velocity potential can be written in the form

$$\Phi(z) = C_1 e^{kz} + C_2 e^{-kz}, \tag{23}$$

corresponding to the vertical structure of the potential. Applying the boundary conditions (21), (22) at the free surface and the seabed, we obtain a system of linear equations for C_1 and C_2. This system has a nontrivial solution if and only if its determinant equals zero, i.e.,

$$\begin{vmatrix} e^{-kh} & e^{kh} \\ gk - \omega^2 & gk + \omega^2 \end{vmatrix} = 0. \tag{24}$$

When solving the above-mentioned system of ODE's with respect to ω, we find

$$\boxed{\omega^2 = gk \tanh kh,} \tag{25}$$

which is the dispersion relation for linear waves. When solving for $\Phi(z)$ we find that

$$\Phi(z) = B \cosh k(z + h), \tag{26}$$

which results in the general solutions for the velocity potential and surface elevation of the wave,

$$\boxed{\Phi(x, z, t) = B \cosh k(z + h) e^{i(kx - \omega t)}} \tag{27}$$

$$\boxed{\eta = A e^{i(kx - \omega t)} = \frac{i\omega}{g} B \cosh kh e^{i(kx - \omega t)}} \tag{28}$$

where A is the amplitude of the surface elevation, and $B = -\frac{igA}{\omega}\frac{1}{\cosh kh}$. In the general solution for the free surface elevation (28) we have neglected the atmospheric pressure, which is a valid assumption for a local problem in which the atmospheric pressure would be constant, thus only adding a reference pressure to the dominant pressure gradients governing the wave dynamics.

2.3.2 Special cases

The general solutions of the linear wave equations simplify under a set of special cases, which are important for practical applications. To say something about the dispersion of the waves in each case, we will consider the linear group velocity,

$$\boxed{c_g \equiv \frac{\partial \omega}{\partial k} = \frac{c_p}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right),} \tag{29}$$

where $c_p = \omega/k$ is the phase velocity.

Shallow water

When the wavelength is long compared to the depth of the system, $kh \ll 1$, and $\tanh kh \approx kh$. We then see from the group velocity of the wave that

$$c_g^2 = gh, \tag{30}$$

which means that the wave is non-dispersive.

Deep water

When the wavelenght is short compared to the depth, $kh \gg 1$ and $\tanh kh \approx 1$. The group velocity then reads

$$c_g^2 = g/k = \frac{g}{2\pi}\lambda, \tag{31}$$

which gives dispersive waves, where the long waves are faster than the shorter.

Intermediate depth

When wavelength and depth are in the same order of magnitude, $kh \approx 1$, and we need the full solution of the equations to say anything about the wave dispersion.

2.3.3 Wave energy

To discuss the energy of linear surface waves, we will consider a cross section of the wave in the yz-plane. As the wave vector $\vec{k}$ is perpendicular to this plane, we can express the kinetic energy of the wave as

$$E_k = \rho \int_{-h}^{0} \frac{u^2}{2} + \frac{w^2}{2} dz, \tag{32}$$

where u, w are the vertical and horizontal velocity components of the wave. Inserting the general expression for the velocity potential (27), we find the mean kinetic energy for linear waves,

$$\overline{E_k} = \frac{1}{4}\rho g A^2, \quad \text{where } A \text{ is the amplitude.} \tag{33}$$

In the same way, we find the potential energy by integrating from the mean free surface, $z = 0$, to the surface elevation η.

$$E_p = \int_0^{\eta} \rho g z dz \quad \Rightarrow \quad \left| \overline{E_p} = \frac{\rho g}{2}\overline{\eta}^2 \right|, \tag{34}$$

which when making the before-mentioned assumption that η is a sinusoidal wave reduces to

$$\overline{E}_p = \frac{1}{4}\rho g A^2 = \overline{E}_k, \tag{35}$$

thus giving the total wave energy,

$$E = \frac{1}{2}\rho g A^2. \tag{36}$$

2.3.4 Wave packets and group velocity

The group velocity (29) describes the velocity of a wave packet of sinusoidal waves with continuous wavenumbers in the range $k_0 + \Delta k < k < k_0 + \Delta k$. We then assume the surface elevation to be a superposition of each singular wave so that we can express this as the integral

$$\eta = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k)e^{i(kx-\omega t)}dk, \tag{37}$$

where $A(k)$ is the amplitude as a function of k, and $kx - \omega t$ is the phase function. As the dispersion relation ω (25) is a function of k we must expand it to account for the band of ks, so that

$$\omega(k) = \omega_0 + \frac{\partial \omega}{\partial k}|_{k_0}\Delta k + r(\Delta k)^2, \quad \Delta k = k - k_0. \tag{38}$$

The free surface integral (37) thus becomes

$$\eta = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k)e^{i(kx-\omega t - \frac{\partial \omega}{\partial k}|_{k_0}\Delta k + r(\Delta k)^2)}dk. \tag{39}$$

We make the key assumption that the amplitude of the wave packet is slowly varying to a degree where we can consider it constant across the band of ks. We can then approximate the integral as

$$\eta \approx A(k_0)e^{i(k_0x-\omega_0 t)} \int_{k_0-k}^{k_0+k} e^{i(k-k_0)(x-c_gt)}dk. \tag{40}$$

Conducting integration by substitution, we find that

$$\eta \approx \tilde{A}(x,t)e^{i(k_0x-\omega t)}, \quad \text{where} \quad \tilde{A} = 2A(k_0)\frac{\sin \Delta k(x-c_gt)}{x-c_gt}. \tag{41}$$

Physical interpretation

The total surface elevation η, caused by the superposition of a linear wave group, is thus a sinusoidal wave with slowly varying amplitude Ã. Ã itself is a wave depending on the group velocity c_g. We see the physical representation of this result in Figure 2, where the envelope is the slowly varying Ã. The concept of separating a system of surface waves into slowly- and rapidly varying parts is a powerful technique, which facilitates the description of more complex systems of polychromatic waves, and interactions with currents and bottom topography. We will return to this topic in later discussions.

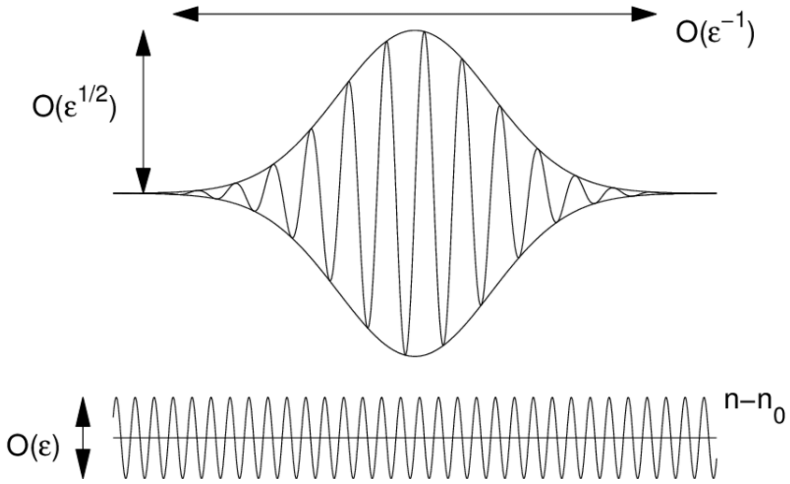

Figure 2: Wavepacket with oscillations in different scales.

Note. Adapted from Goodman et al. (2001).

2.4 Perturbation Methods and Multiscale development

Until now, we have mostly considered strictly linear wave theory, where exact solutions can be found (see (27), (25)). However, when complicating our system, introducing background currents and bottom topography, we enter a theoretical landscape where no exact solutions are yet to be known. On the other hand, we can still approximate our variables as linear entities, for example, assuming the bottom or the background current to be slowly varying sinusoidal waves. The problem lies in the difference of scale. As the small oscillations on the free surface are considered many orders of magnitude smaller than the oscillations of the bottom or the current, we will have to search for approximate solutions to describe the physics of the system.

2.4.1 Perturbation Methods

Perturbation theory provides powerful tools for finding approximate solutions close to known, exact solutions. The idea is that in the vicinity of a solved problem, we can find good approximate solutions to related problems by expanding the exact solution as a power series of a small parameter μ:

$$A \equiv A_0 + \mu A_1 + \mu^2 A_2 + \mathcal{O}(\mu^3) \tag{42}$$

The first term of the series will then be the solvable part, while successive terms will be small approximate variations to the known solution. By truncating the series we are left with a perturbation solution consisting of a known solution with a perturbation correction (Zheng & Zhang, 2017).

2.4.2 Multiscale development

To further discuss the theory of interactions between surface waves, bottom topography and currents, we need to introduce the mathematical concept of multiscale expansion. To describe multiscale expansion, we will first consider the wave packet in Figure 2. The rapidly oscillating wave has wavelength and period λ, T, while the envelope has the characteristic length and time scale L, τ. We assume that the properties of the rapidly oscillating wave is small, compared to those of the envelop, so that $\lambda \ll L$ and $T \ll \tau$. Secondly we assume that the the length and time scales are independent of one another. Using these assumption, we can define a new set of slowly varying variables

$$\frac{x}{L} = \frac{\lambda}{L}\frac{x}{\lambda} = \mu\frac{x}{\lambda} \Rightarrow \bar{x}_L = \mu\bar{x}_\lambda, \tag{43}$$

$$\frac{t}{\tau} = \frac{T}{\tau}\frac{t}{T} = \mu\frac{t}{T} \Rightarrow \bar{t}_\tau = \mu\bar{t}_T, \tag{44}$$

where $\mu \ll 1$.

2.4.3 The WKB Method

The WKB method, named after Gregor Wentzel, Hendrik Anthony Kramers, and Léon Brillouin, is a special case of multiscale expansion used to approximate linear differential equations with spatially varying coefficients. It is a useful perturbation method because, rather than assuming an idealized potential and then introducing perturbative terms, it directly assumes a slowly varying potential (Li, 2017). In the context of this report, where we seek to investigate interactions between linear surface waves and currents, this feature is particularly useful. Instead of a potential, we consider slowly varying currents or bottom topography, upon which linear surface waves are superimposed.

The general idea of the WKB method is to assume a wavefield of the form

$$\Psi(x_i, t) = A\, e^{\frac{i}{\mu}S(x_i)}, \tag{45}$$

where A is the amplitude and $S(x_i, t)$ the phase function. As described in section 2.4.2, the variable $\mu \ll 1$ is the relation between the two time- and spatial scales. The phase function $S(x, t)$ determines the local wavenumber and frequency of the wave, while the amplitude $A(x, t)$ varies slowly and governs the magnitude of the wavefield (Zwiebach, 2018).

To find approximate solutions for linear surface waves propagating over a slowly varying current or bottom topography, we substitute the WKB ansatz for $\Psi(x_i, t)$ into the governing linear differential equations and perform an asymptotic expansion. Each term in the expansion is organized according to its power of μ. We then solve the resulting hierarchy of equations in order of increasing power of μ, using lower-order solutions to construct higher-order corrections. Due to the presence of imaginary terms in the phase function, it is sometimes algebraically convenient to expand in powers of $(-i\mu)$ instead of μ. Throughout this report, we clearly indicate which convention is being used in each case.

Although the WKB method can, in principle, be carried out to any desired order, the assumption $\mu \ll 1$ implies that if the leading-order term $\mathcal{O}(1)$ dominates the next-order correction $\mathcal{O}(\mu)$, then it is typically sufficient to retain only the first two terms (Zwiebach, 2018).

2.5 Wave refraction and radiation theory

2.5.1 Refraction

The general definition of wave refraction is the change in direction, amplitude or wave length as the wave move from one medium to another (University of Hawaiʻi at Mānoa, 2014). In the case of water waves, refraction takes place when facing changes in bottom topography or background current. We will use the WKB method discussed in Section 2.4.3 to derive the refraction equations for the case of slow variations in bottom topography (see Figure 3 for configuration).

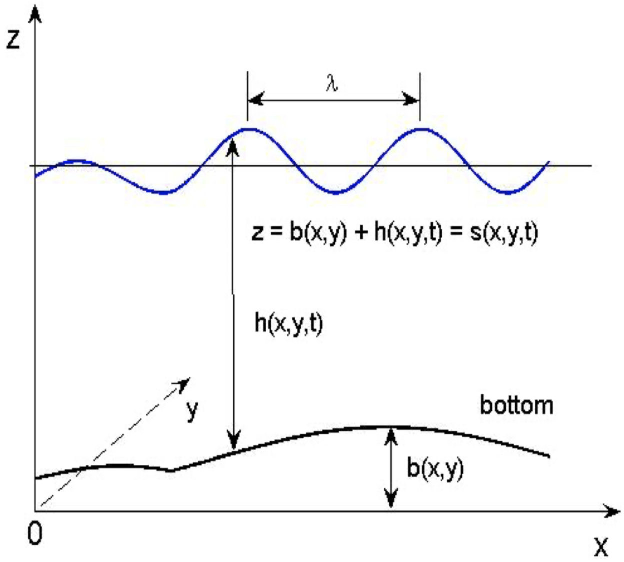

Figure 3: Linear wave over a slowly varying bottom.

Note. Adapted from Hereman (2009).

The key assumption for this derivation is that the bottom varies much more slowly than the variation in wavelength *caused by* the varying bottom itself, i.e.

$$\boxed{\frac{|\nabla h|}{kh} \ll 1.} \tag{46}$$

As this is the relation between the rapidly varying surface wave and the slowly varying bottom, we say that

$$\mu = \frac{|\nabla h|}{kh}, \tag{47}$$

and define the new slowly varying variables

$$\bar{x}_i = \mu x_i \tag{48}$$
$$\bar{t} = \mu t, \tag{49}$$

and their corresponding partial derivatives

$$\frac{\partial}{\partial x_i} = \mu \frac{\partial}{\partial \bar{x}_i} \tag{50}$$
$$\frac{\partial}{\partial t} = \mu \frac{\partial}{\partial \bar{t}}. \tag{51}$$

The slowly varying linearized wave equations and the boundary conditions can then be written as functions of the slowly varying variables

$$\mu^2 \Phi_{\bar{x}_i \bar{x}_i} + \Phi_{zz} = 0, \quad \text{at } -h < z < 0, \tag{52}$$

$$\mu^2\Phi_{\overline{t}\overline{t}} + g\Phi_z = 0, \quad \text{at } z = 0, \tag{53}$$

$$\Phi_z = -\mu^2\Phi_{\bar{x}_i}h_{\bar{x}_i}, \quad \text{at } z = -h. \tag{54}$$

We now assume progressive waves, so that we can asymptotically expand the velocity potential Φ as

$$\Phi = [\Phi_0 + (-i\mu)\Phi_1 + (-i\mu)^2\Phi_2 + ...]e^{i\frac{S}{\mu}} \tag{55}$$

(note that we are using the $(-i\mu)$-convention) where,

$$\Phi_j = \Phi_j(\bar{x}_i, z, \bar{t}); \quad j = 0, 1, 2, ... \tag{56}$$

$$S = S(\bar{x}_i, \bar{t}) = k\bar{x} - \omega\bar{t}. \tag{57}$$

We insert the asymptotic expansion into (52)–(54) and collect terms by powers of μ. The leading-order equations are:

$\mathcal{O}((-i\mu)^0)$:

$$\Phi_{0zz} - k^2\Phi_0 = 0, \quad -h < z < 0, \tag{58}$$

$$\Phi_{0z} - \frac{\omega^2}{g}\Phi_0, \quad z = 0, \tag{59}$$

$$\Phi_{0z} = 0, \quad z = -h. \tag{60}$$

These are equal to the linear wave equations, (20)-(22), thus giving the solution (27), and the dispersion relation (25). The velocity potential and the dispersion relation are related to the slowly varying depth $h(\bar{x}_i)$. The amplitude A remains undetermined.

We now consider the next order. $\overline{\nabla}$ is the slowly varying del-operator, $\left(\frac{\partial}{\partial_{\bar{x}_i}}\right)$.

$\mathcal{O}(-i\mu)$:

$$\Phi_{1zz} - k^2\Phi_1 = \vec{k}\cdot\overline{\nabla}\Phi_0 + \overline{\nabla}\cdot(\Phi_0\vec{k}), \quad -h < z < 0 \tag{61}$$

$$\Phi_{1x} - \frac{\omega^2}{g}\Phi_1 = -\frac{1}{g}[\omega\Phi_{0\bar{t}} + (\omega\Phi_0)_{\bar{t}}], \quad z = 0 \tag{62}$$

$$\Phi_{1z} = \Phi_0\vec{k}\cdot\overline{\nabla}h, \quad z = -h \tag{63}$$

We find the solution of (58)-(63) by multiplying (58) and (59) by Φ_1, and (61) and (62) by Φ_0. Subtracting the resulting pairs of equations (i.e., (61) from (58), and similarly for the surface condition) eliminates the non-differentiated terms, which simplifies the expression for vertical integration.

We then apply the boundary conditions and integrate over the full depth of the water column:

$$\int_{-h}^{0} \overline{\nabla}\cdot(\vec{k}\Phi_0^2)\,dz = -\frac{1}{g}\left(\frac{\partial}{\partial\bar{t}}(\omega\Phi_0)^2\right)_{z=0} - \left(\Phi_0^2\vec{k}\cdot\overline{\nabla}h\right)_{z=-h}. \tag{64}$$

However $h(\bar{x},\bar{y})$ is not constant, as the bottom is slowly varying. The integrands of (64) are therefore also not constant and we apply Leibniz' rule to the left-hand side, thus resulting in the final integral

$$\boxed{\overline{\nabla}\cdot\int_{-h}^{0}\vec{k}\Phi_0^2dz+\frac{1}{g}\frac{\partial}{\partial\bar{t}}(\omega\Phi_0)^2=0.} \tag{65}$$

Physical interpretation

Equation (65) is the full solution for surface wave refraction over a slowly varying bottom, but is difficult to physically interpret. By inserting the general linear solution (27) for Φ_0 and expressing the result in terms of the group velocity (29) and wave energy (36), we obtain

$$\boxed{\overline{\nabla}\cdot\left(\frac{E}{\omega}\vec{c}_g\right)+\frac{\partial}{\partial\bar{t}}\left(\frac{E}{\omega}\right)=0,} \tag{66}$$

which describes the conservation of wave action $\mathcal{A}=E/\omega$ in the absence of mean flow or dissipative effects.

Moreover, from the WKB definitions (50), (51), (57) we have that

$$\vec{k}=\overline{\nabla}S,\qquad \omega=-\frac{\partial S}{\partial\bar{t}}. \tag{67}$$

We thus obtain the condition for conservation of wave crests,

$$\frac{\partial\vec{k}}{\partial t}+\overline{\nabla}\omega=0. \tag{68}$$

This implies that the wave crests propagate without distortion due to dispersion or spatial inhomogeneities in the absence of external forcing. The local wavenumber $\vec{k}$ evolves only due to changes in ω, preserving the spatial density of wave crests.

2.5.2 Radiation

Up to the order $\mathcal{O}(-i\mu)$ we found that the linear solution remains valid for a locally constant depth $h(\bar{x}_i)$. In order to say something about the change of direction of the waves we now assume that A and k vary slowly with the changes in $h(\bar{x}_i)$. Since we are now interested in the slowly varying wave envelope rather than individual oscillations, we say that

$$\bar{x},\bar{y},\bar{t}\to x,y,t;\quad \overline{\nabla}\to\nabla. \tag{69}$$

However, we assume that the bottom does not vary in time, so that

$$\nabla\omega=0, \tag{70}$$

$$\nabla\cdot\left(\frac{E}{\omega}\vec{c_g}\right)=0. \tag{71}$$

Equation (70) indicates that for the linear dispersion relation (25) to be constant, there should be some inverse relation between k and h. We will elaborate on this relation in the following discussion.

We consider two rays containing a water column of volume V.

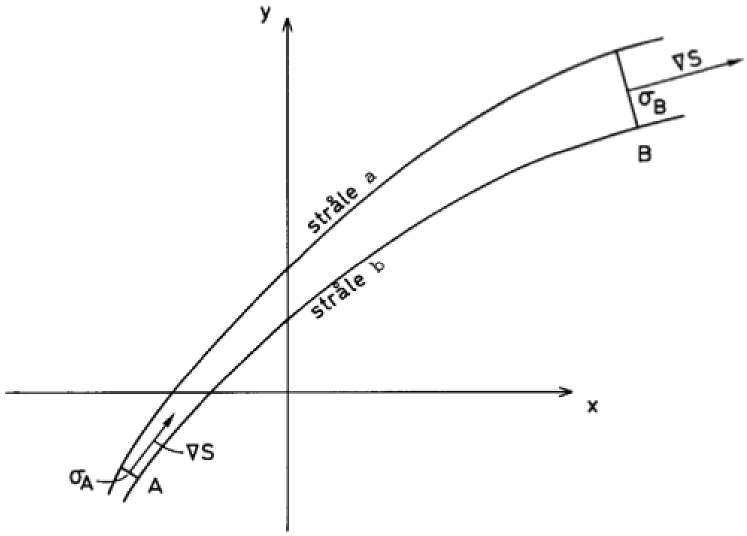

Figure 4: Two rays showing the paths of the wave crests, A and B. $\vec{k}$ and $\vec{c}_g$ lies in the direction of ∇S.

Note. Adapted from Gjevik et al. (2021).

We apply Gauss' theorem on the wave action in the closed volume,

$$\int_{\mathcal{V}} \nabla \cdot \left(\frac{E}{\omega} \vec{c_g} \right) = \int_{\mathcal{S}} \frac{E}{\omega} \vec{c_g} \cdot \vec{n} ds, \tag{72}$$

thus making it a surface integral, where $\vec{n}$ is the outward-pointing normal vector of each surface element ds. As $\vec{c}_g \parallel \vec{k}$ for all sides, and $\vec{n} \perp \vec{k}$ everywhere but on the wave crests, A and B, these are the only sides that contribute to the integral.

$$\int_{\mathcal{S}} \frac{E}{\omega} \vec{c}_g \cdot \vec{n} ds = \left(\frac{E}{\omega} c_g ds \right)_B - \left(\frac{E}{\omega} c_g ds \right)_A = 0, \quad \text{for any } ds. \tag{73}$$

The energy flux ($Ec_g ds$) is therefore constant along the rays.

From this result, we can find the amplitude relation by inserting (36) for E into

$$Ec_g ds = E_0 c_{g0} ds_0 = \rho g A_0^2/2, \tag{74}$$

so that

$$\frac{A}{A_0} = \left(\frac{c_{g0}}{c_g} \frac{ds_0}{ds} \right)^{1/2}. \tag{75}$$

Equation (75) shows that the amplitude varies with the group velocity and the spreading of the two ray paths, represented by the surface element ds. Since both depend on the wave number, which is set by the bottom topography through the dispersion relation, we must first calculate the ray path to evaluate the amplitude along it.

To find the rays we consider the Eikonal equation,

$$\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 = k^2 \tag{76}$$

which is a well-known relation within geometrical optics. After some mathematical manipulation, (76) results in the Euler-Lagrange equation,

$$\boxed{\frac{d}{dx}\left(\frac{ky'}{(1+y'^2)^{1/2}}\right) = (1+y'^2)^{1/2}\frac{dk}{dy},} \tag{77}$$

where

$$y' = \frac{\dfrac{\partial S}{\partial y}}{\dfrac{\partial S}{\partial x}}. \tag{78}$$

The Euler-Lagrange equation (77) determines the ray $y(x)$, when an initial position is given. $y(x)$ must be calculated numerically.

Physical interpretation

If we consider depth contours parallel to a shore-line, we have that $h = h(x)$ and $k = k(x)$, so that $dk/dy = 0$ and the RHS of (77) vanishes, thus reducing it to

$$\frac{ky'}{(1+y'^2)^{1/2}} = \text{constant}. \tag{79}$$

Since the slope of the ray is $y' = \tan\varphi$ (see Figure 5), we compute

$$\frac{y'}{\sqrt{1+y'^2}} = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}} = \sin\varphi. \tag{80}$$

Thus, equation (79) becomes

$$k\sin\varphi = \text{constant}, \tag{81}$$

which is Snell's law. We saw from the dispersion relation that k and c_g are inversely related, so we can express Snell's law as

$$\frac{\sin\varphi}{c_g} = \frac{\sin\varphi_0}{c_{g0}}. \tag{82}$$

This shows that as $c_g = \sqrt{gh}$ decreases in shallower water, the angle φ also decreases. In other words, the ray bends toward the normal to the depth contours, and hence toward the shoreline (see Figure 5).

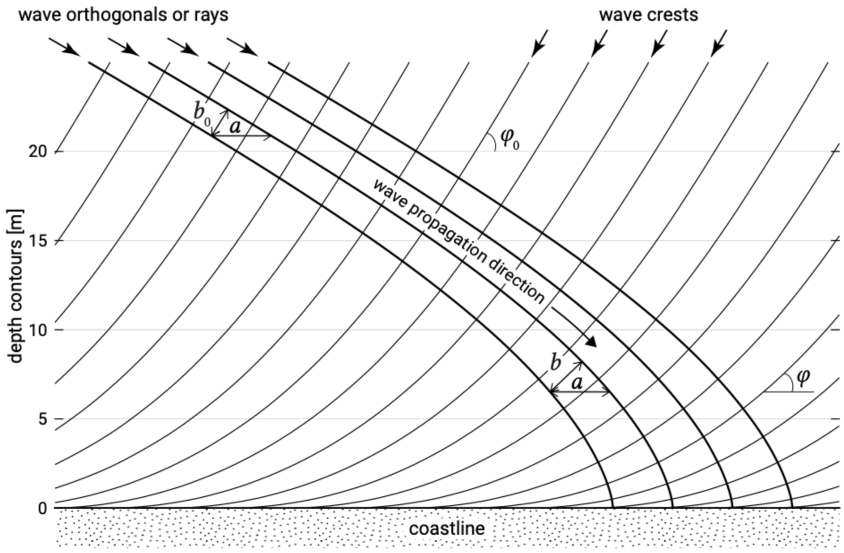

Figure 5: Incoming wave rays bending toward the shoreline. The coastline lies in the y-direction, with the x-direction pointing directly away from it.

Note. Adapted from Bosboom and Stive (2020).

Now we see that refraction also affects the wave amplitude. When accounting for both the variation in group velocity and the angular spreading of the wavefronts, the amplitude relation (75) becomes

$$\frac{A}{A_0} = \left(\frac{c_{g0}}{c_g} \cdot \frac{\cos \varphi_0}{\cos \varphi} \right)^{1/2}, \tag{83}$$

which shows that the amplitude increases as $\varphi \to 0$.

2.6 The Combined Dynamics of Waves and Current

A two-dimensional system of a slowly varying background current and a free surface is considered (see Figure 6).

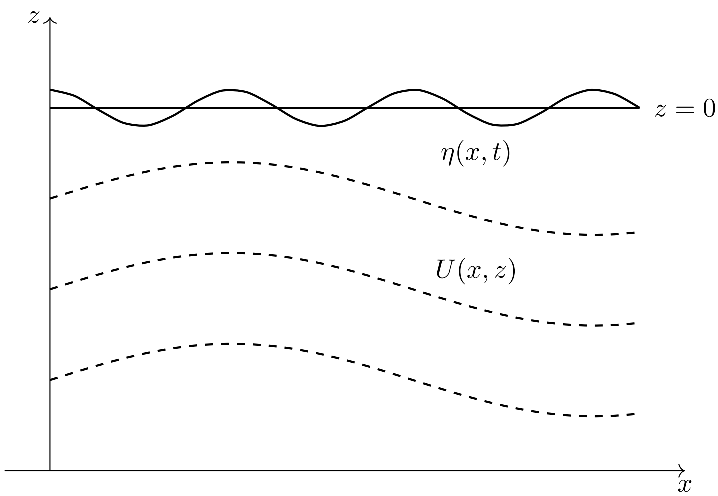

Figure 6: Linear surface wave $\eta(x,t)$ over a slowly varying background current $U(x,z)$.

The background current is treated as a slowly varying linear flow field, independent of the surface waves. We can then consider the total flow field as a linear combination of the surface waves and the background current. We write the continuity equation as

$$\frac{\partial}{\partial x}(U+u)+\frac{\partial}{\partial z}(W+w)=0, \tag{84}$$

where U, W are the current velocity components and u, w the surface wave velocity components.

We will now discuss how the background current affects the linear surface waves. When doing this, the difference in the scale of the two components is essential. We assume that

$$u, w, p \sim \mathcal{O}(kA), \tag{85}$$

$$U, W \sim \mathcal{O}(1), \tag{86}$$

where k and A are the wavenumber and the amplitude of the linear surface waves. In accordance with linear theory, we assume that $A \ll 1$.

Expanding Euler's equations, we can see which terms depend on the current only. As we are only interested in the interactions, these terms vanish. Similarly, terms of the order $\mathcal{O}(kA)^2$ vanish, as they are nonlinear, and not relevant in this discussion. We make the same assumptions for the boundary conditions (10)-(12) and are left with the following system of differential equations describing pressure p,

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2}=-2p\left(\frac{\partial u}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial W}{\partial z}\right), \quad \text{at } -h<z<\bar{\eta}, \tag{87}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{\partial p}{\partial z}=\frac{1}{\rho}\frac{\partial h}{\partial x}\frac{\partial p}{\partial x}+\mathcal{O}(\mu^2 kA), \quad \text{at } z=-h(x), \tag{88}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 p+2W\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}+U\frac{\partial p}{\partial x}\right)-g\frac{\partial \bar{\eta}}{\partial x}\frac{\partial p}{\partial x}=0, \quad \text{at } z=\bar{\eta}. \tag{89}$$

Similarly to how surface waves refract over a slowly varying bottom (see Section 2.5.1), we approximate (87)–(89) using the WKB method, under the assumption that the background current varies slowly compared to the surface waves. As before, we assume conservation of wave crests (see equation (68)). Following the procedure described in Sections 2.4.3 and 2.5.1, we introduce the intrinsic frequency,

$$\boxed{\sigma=\omega-kU,} \tag{90}$$

which is the wave frequency observed in a reference frame moving with the current velocity U. At leading order $\mathcal{O}(1)$, we then find that p has the solution

$$p_0=\rho g A\frac{\cosh k(z+h)}{\cosh k\bar{h}}, \tag{91}$$

where $A=A(\bar{x},t)$ is the slowly varying amplitude, $\bar{h}=\bar{\eta}+h$ is the total mean depth, and the intrinsic frequency satisfies $\sigma^2=gk\tanh k\bar{h}$. In (91), $\sigma(\bar{x},t)$ and $k(\bar{x},\bar{t})$ are related to the local depth $\bar{h}(\bar{x},\bar{t})$, but $A(\bar{x},\bar{t})$ remains unknown.

From this result, we find the group velocity relative to the current,

$$c_g = \left.\frac{\partial \sigma}{\partial k}\right|_{\bar{h}} = \frac{\sigma}{2k}\left(1 + \frac{2k\bar{h}}{\sinh 2k\bar{h}}\right), \tag{92}$$

which has the direction $\vec{c}_g = c_g\hat{k}$. As (92) shows, we calculate the group velocity for a given height, $\bar{h}$. σ and c_g are rapidly varying, so that in their frame of reference, $\bar{h}$ appears approximately constant. This is an important aspect, as it indicates that as $\bar{h}$ is varying, so does c_g and σ.

Physical interpretation

From the leading-order kinematics, we identify two important results.

Firstly, we note that as $\omega = \sigma\left(k(\bar{x},\bar{t}), \bar{h}(\bar{x},\bar{t})\right) + k(\bar{x},\bar{t})U(\bar{x},\bar{t})$, we can use this and (92) to show that

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial \bar{t}} + (U + c_g)\frac{\partial \omega}{\partial \bar{x}} = \frac{\partial \sigma}{\partial \bar{h}}\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial U}{\partial \bar{t}}k, \quad \text{at } \bar{h} = h + \bar{\eta}. \tag{93}$$

This shows that in the general case, the absolute wave frequency changes when interacting with a background current. However, if the current is steady and the free surface η is time-dependent, the absolute frequency ω does not change when observed in a frame of reference moving with the total group velocity $C_g \equiv U + c_g$. This is an important result, as the linear wave theory described in previous sections thus will remain valid when moving with the total velocity from the current and the surface wave spectrum.

The second observation we make is that when expressing the conservation of wave crests (68) in terms of the intrinsic frequency σ, we find the equation determining the ray path,

$$\frac{\partial k}{\partial t} + (U + c_g)\frac{\partial k}{\partial x} = -\frac{\partial \sigma}{\partial \bar{h}}\frac{\partial \bar{h}}{\partial \bar{t}} - \frac{\partial U}{\partial \bar{t}}k, \quad \text{at } \bar{h} = h + \bar{\eta}. \tag{94}$$

Considering the configuration of a constant current U, and a flat bottom $\bar{h}$, the wavenumber remains constant seen from a frame of reference moving with velocity C_g. The surface waves will under this configuration propagate in the direction $\vec{c}_g$, meanwhile the *wave energy* will propagate in the direction of $\vec{C}_g$ (see Figure 7).

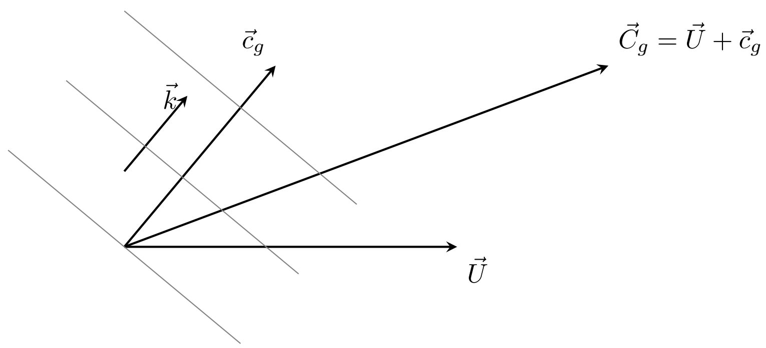

Figure 7: The background current $\vec{U}$, the intrinsic group velocity $\vec{c}_g$, the wavevector $\vec{k}$, and the total group velocity $\vec{C}_g$. Wave crests are orthogonal to $\vec{k}$.

To determine the amplitude A, we need to consider the second leading order of magnitude $\mathcal{O}(-i\mu)$. This set of equations is very complex and we will not solve them in this report. However, we will conclude that it can be shown that the wave action in the direction of $\vec{C}_g$ is conserved, i.e.

$$\boxed{\frac{\partial}{\partial \bar{t}}\left(\frac{E}{\sigma}\right) + \frac{\partial}{\partial \bar{x}}\left(C_g\frac{E}{\omega}\right) = 0,} \quad \text{where } E = \frac{1}{2}\rho g A^2. \tag{95}$$

This conservation law governs the evolution of the wave amplitude A through the conservation of wave action density. This result is similar to the case in Section 2.5.1

2.7 Surface Stress and Viscous Boundary Conditions

Under hydrostatic assumptions and in the absence of surface tension, the free surface of water must satisfy a condition of continuous stress, ensuring that internal fluid stresses balance the stresses from the surrounding air.

Physically, stress is force per unit area, so for an infinitesimal surface area

$$\mathbf{A} = A_0\hat{\mathbf{n}},$$

we can define the stress vector,

$$\vec{f} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \frac{\vec{F}}{A_0}, \tag{96}$$

where the components of $\boldsymbol{\sigma}$ is the stress tensor,

$$\sigma_{ij} = -P\delta_{ij} + \rho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{97}$$

Here, δ_{ij} is the Kronecker delta and ν, the viscosity. The viscosity of a fluid is its resistance to shearing flows, i.e., the friction between adjacent layers that move parallel to each other with different speeds (Jenkins, 2013). The stress tensor appears in the Navier-Stokes equation on the tensor form (linearized version)(Mei et al., 2005),

$$\frac{\partial u_i}{\partial t} = -g\delta_{i3} - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}. \tag{98}$$

From the condition of continuity of stress through the surface, we will derive the *full viscous boundary conditions* at the free surface. This derivation of the boundary conditions follows roughly that of Mei et al. (2005), and we will work under the linear assumption discussed in Section 2.3.

We divide the stresses into a normal stress, perpendicular to the surface, and the tangential stress $\vec{F_t} = \vec{f_x} + \vec{f_y}$ (see Figure 8).

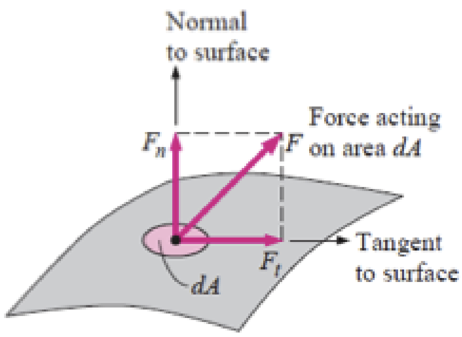

Figure 8: Decomposition of surface stress upon unit surface area dA. The tangential stress $\vec{F_t}$ can be decomposed into $\vec{f_x}, \vec{f_y}$.

Note. Adapted from Güner (2021).

The Helmholtz decomposition (also called the fundamental theorem of vector calculus), says that we can separate any vector field into an irrotational and a divergence-free component, i.e.

$$u_i = \frac{\partial \Phi}{\partial x_i} + U_i. \tag{99}$$

This separation allows us to treat potential flow and viscous effects independently in the bulk. From the incompressibility condition, $\nabla \cdot \vec{u} = 0$, we find that

$$\nabla^2 \Phi = 0, \tag{100}$$

so the potential Φ must satisfy Laplace's equation, showing consistency with the inviscid linear wave theory discussed in Section 2.3. Substituting into the momentum equation, the divergence-free component U_i evolves independently according to the simple diffusion equation:

$$\frac{\partial U_i}{\partial t} = \nu \nabla^2 U_i. \tag{101}$$

This indicates that in the linear regime, and without external forcing, the vorticity decays by viscous diffusion.

We are interested in finding the **viscous surface boundary conditions**, which determine the interaction between potential and viscous components. We remember that the dynamic boundary condition is the continuity of normal and tangential stresses, so that

$$(-P\delta_{ij} + \sigma_{ij})n_j = -(-P\delta_{ij} + \sigma_{ij})_{air} n_j, \quad \text{at } z = 0. \tag{102}$$

However, we assume that the atmospheric pressure is zero, so the RHS of (102) vanishes. We continue by splitting the pressure field of the flow into a hydrostatic background and a dynamic perturbation, thus introducing the modified pressure p';

$$P = -\rho g z + p' = -\rho g z - \rho \frac{\partial \Phi}{\partial t}, \tag{103}$$

Substituting the linearized pressure (103) into the stress balance (102) yields the normal stress condition,

$$\boxed{p' = \rho g \eta + 2\nu\rho \frac{\partial w}{\partial z} \quad \text{at } z = 0,} \tag{104}$$

where $p' = -\rho \frac{\partial \Phi}{\partial t}$ is the modified pressure, and η is the linear surface displacement. From the equation, we see that the modified pressure p' of the dynamic flow, the restoring hydrostatic pressure due to surface displacement, and the vertical viscous stress must balance at the free surface.

Since the air cannot support shear stress, the **tangential viscous stresses** must vanish at the interface:

$$\boxed{\rho\nu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0, \quad \rho\nu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0 \quad \text{at } z = 0.} \tag{105}$$

These are the σ_{xz} and σ_{yz} components of the stress tensor, representing the shear force per unit area on horizontal planes.

2.8 Weak Nonlinearity in Fluids

As we soon will proceed to the derivation of a WCI system with a shear current and viscous terms, we must comment on the concept of weak nonlinearity. Weak nonlinearity in fluid mechanics refers to situations where nonlinear effects are present but remain relatively small, allowing them to be treated as perturbations to an underlying linear theory. An example is Stokes' expansion, where wave quantities like velocity and pressure are expanded in powers of wave steepness. The leading-order terms describe linear wave motion, while higher-order terms capture progressively more nonlinear phenomena. In the case of this report, the weak nonlinearity arises from the introduction of a shear current, in which shear stress introduces horizontal friction within the water column.

Weakly nonlinear theories can explain effects such as wave drift forces, which are steady forces on floating bodies due to second-order wave interactions. These forces can be handled analytically up to a few orders in perturbation theory, which is also how we will treat the weakly nonlinear terms in our derivation.

Even though weakly nonlinear theories are useful for many engineering applications, they tend to break down when particle escape occurs (Rainey, 2007). This is often a precursor to more violent, strongly nonlinear events.

2.9 Eddy Viscosity

At high Reynolds numbers, fluid flow becomes turbulent, resulting in chaotic mixing and enhanced transport of momentum. Just as molecular viscosity transfers momentum through molecular interactions, turbulence transports momentum down the gradient of the mean flow (Jenkins, 2013). This turbulent flux of momentum can be parameterized using an *eddy viscosity*, a phenomenological coefficient that plays a role analogous to molecular viscosity.

Unlike molecular viscosity, which is a property of the fluid itself, eddy viscosity depends on the characteristics of the flow. It varies with factors such as wave activity, shear, and stratification (Shen, 2019). In the momentum equation, the turbulent momentum flux can be expressed using the eddy viscosity as (Durbin & Yang, 1992):

$$\frac{\partial}{\partial x_j}\left(\nu_e \frac{\partial u_i}{\partial x_j}\right), \tag{106}$$

where ν_e is the eddy viscosity, and u_i is the i-th component of the velocity field (u, v, w). In this report we wish to build a weakly nonlinear model that is also compatible with an eddy viscosity.

3 Method

3.1 Governing Equations and Boundary Conditions

As mentioned in Section 1, this report seeks to lay the groundwork for further investigation of the interactions between linear surface waves and a background shear current including viscous terms. The current adds shear stress and weakly non-linear properties to the system, which makes the task of describing the system much more complicated than the mostly strictly linear cases presented in Section 2. In this chapter, we present an analytical approach to this problem.

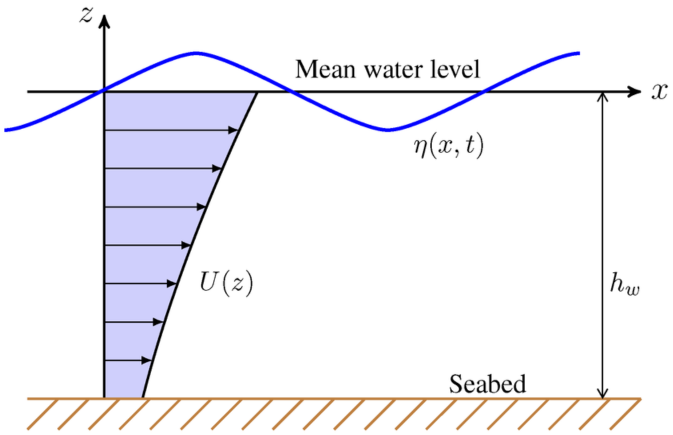

Figure 9: A free surface wave over a shear background current.

Note. Adapted from Sarkar and Fitzgerald (2021).

We consider a two-dimensional system consisting of water in motion, constrained by a horizontal free surface at $z = 0$ and a flat bottom at $z = -h$. The x-axis lies in the horizontal plane and the z-axis in the vertical. We assume $-h \ll 1$. See figure 9 for configuration. The guiding equations of motion for this problem are the incompressible continuity equation (1) and the incompressible conservation of momentum (2). We separate the horizontal and vertical components of the incompressible momentum equation into:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{107}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{108}$$

where u and w are the horizontal and vertical velocities, ρ is the constant fluid density, p is the pressure throughout the fluid, and ν is the viscosity of the fluid. The assumption of the fluid being incompressible gives the volume continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{109}$$

As we wish to investigate the wave-current interactions (WCI) in a weakly nonlinear regime, we will evaluate our system under a set of linear boundary conditions. Due to the nonlinear viscous term in (107) and (108), we consider fully viscous boundary conditions at the free surface. At the mean surface elevation, $z = 0$, this implies

- a conserved tangential surface stress,

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \text{at } z = 0, \tag{110}$$

- and the dynamical surface condition taking the form

$$p = \rho g \eta + 2\nu\rho\frac{\partial w}{\partial z}, \quad \text{at } z = 0, \tag{111}$$

where η is the surface elevation, and p is the modified pressure (103) (we have changed $p' \to p$ for consistency with the momentum equations). As in the strictly linear case, we also consider the kinematic surface condition,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w, \quad \text{at } z = 0. \tag{112}$$

In the inviscid linear theory, we required the normal velocity to vanish at the bottom to satisfy the impermeability condition. With the introduction of viscosity, the tangential velocity must also vanish due to the no-slip condition. That is,

$$u = w = 0, \quad \text{at } z = -h. \tag{113}$$

To investigate the interactions between the surface waves and the background current, we introduce the flow field $u, w \to U + u, W + w$, where U, W are the respective horizontal and vertical velocity components of the background current, and u, w the corresponding components of the free surface. We assume the current to be a horizontally uniform shear current with variation only in the vertical z-direction, i.e. $U = U(z)$. It is then evident that the vertical component $W = 0$, and that the derivatives of the current with respect to the horizontal direction, $\frac{\partial U}{\partial x} = 0$.

As we are only interested in the *interactions* between the waves and the current, we must follow the procedure described in Section 2.6. We thus evict from our equations, the terms which only involve the background current and the non-linear terms which scale as $\mathcal{O}(kA)^2$. We expand the equations of our combined wave-current system, (107)-(113), and show which terms are neglected, equal zero, or kept for further discussion:

- The horizontal momentum equation

$$\overbrace{\cancel{\frac{\partial U}{\partial t}}}^{(*)} + \frac{\partial u}{\partial t} + \overbrace{U\cancel{\frac{\partial U}{\partial x}}}^{(*)} + U\frac{\partial u}{\partial x} + \overbrace{u\cancel{\frac{\partial U}{\partial x}}}^{=0} + \overbrace{u\cancel{\frac{\partial u}{\partial x}}}^{\mathcal{O}(kA)^2} + \overbrace{W\cancel{\frac{\partial U}{\partial z}}}^{=0} + \overbrace{W\cancel{\frac{\partial u}{\partial z}}}^{=0} + w\frac{\partial U}{\partial z} + \overbrace{w\cancel{\frac{\partial u}{\partial z}}}^{\mathcal{O}(kA)^2}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\overbrace{\cancel{\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right)}}^{(*)} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{114}$$

- The vertical momentum equation

$$\overbrace{\cancel{\frac{\partial W}{\partial t}}}^{=0} + \frac{\partial u}{\partial t} + \overbrace{U\cancel{\frac{\partial W}{\partial x}}}^{=0} + U\frac{\partial w}{\partial x} + \overbrace{u\cancel{\frac{\partial W}{\partial x}}}^{=0} + \overbrace{u\cancel{\frac{\partial w}{\partial x}}}^{\mathcal{O}(kA)^2} + \overbrace{W\cancel{\frac{\partial W}{\partial z}}}^{=0} + \overbrace{W\cancel{\frac{\partial w}{\partial z}}}^{=0} + \overbrace{w\cancel{\frac{\partial W}{\partial z}}}^{=0} + \overbrace{w\cancel{\frac{\partial w}{\partial z}}}^{\mathcal{O}(kA)^2}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\overbrace{\cancel{\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right)}}^{=0} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{115}$$

- The incompressible continuity equation

$$\overbrace{\cancel{\frac{\partial U}{\partial x}}}^{=0} + \overbrace{\cancel{\frac{\partial W}{\partial z}}}^{=0} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{116}$$

- Conserved tangential surface stress

$$\overbrace{\cancel{\frac{\partial U}{\partial z}}}^{(*)} + \overbrace{\cancel{\frac{\partial W}{\partial x}}}^{=0} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \text{at } z = 0. \tag{117}$$

- The dynamical surface condition

$$p = \rho g \eta + \cancel{2\nu\rho \overbrace{\frac{\partial W}{\partial z}}^{=0}} + 2\nu\rho \frac{\partial w}{\partial z}, \quad \text{at } z = 0. \tag{118}$$

- The kinematic surface condition

$$\frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} + \overbrace{\cancel{u \frac{\partial \eta}{\partial x}}}^{\mathcal{O}(kA)^2} = \overbrace{\cancel{W}}^{=0} + w, \quad \text{at } z = 0. \tag{119}$$

- The bottom condition

$$U = u = w = 0, \quad \text{and} \quad \overbrace{\cancel{W}}^{=0} = 0, \quad \text{at } z = -h. \tag{120}$$

$(*)$ Terms only depending on the background current.

We can then collect the remaining terms describing the WCI from equations (114)-(120), which will form the basis of this investigation.

Governing Equations and Boundary Conditions

- **The horizontal momentum equation**

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + w \frac{\partial U}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{121}$$

- **The vertical momentum equation**

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{122}$$

- **The incompressible continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{123}$$

- **Conserved tangential surface stress (at $z = 0$)**

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{124}$$

- **The dynamical surface condition (at $z = 0$)**

$$p = \rho g \eta + 2\nu\rho \frac{\partial w}{\partial z} \tag{125}$$

- **The kinematic surface condition (at $z = 0$)**

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = w \tag{126}$$

- **The bottom condition (at $z = -h$)**

$$U = u = w = 0 \tag{127}$$

3.2 Approximation Scheme

Finding exact solutions to the equations (121)-(127) would be a formidable task as there are no known exact solution to the Navier-Stokes equation (Clay Mathematics Institute, n.d.). We therefore seek approximate solutions using the methodological framework presented in Section 2.4. To ease the analisis of our system and find approximate solutions, we expand equations (121)-(127) using the WKB method.

Our assumption is that $U(z, t)$ is a slowly varying current, while the oscillations on the free surface are small, and rapidly varying. Following the procedure described in Section 2.5, we then define the slowly varying variables of our system in terms of the scaling parameter $\mu \ll 1$. We expand the variables using the μ-convention.

WKB Expansion Setup

- **Scaled variables**

$$\bar{x} = \mu x, \quad \bar{t} = \mu t \tag{128}$$

- **Background flow**

$$U = U(z, \bar{t}) \tag{129}$$

- **Pressure ansatz**

$$p(x, z, t) = [p_0 + \mu p_1 + \cdots]\, e^{i\frac{S}{\mu}} \tag{130}$$

- **Horizontal velocity ansatz**

$$u(x, z, t) = [u_0 + \mu u_1 + \cdots]\, e^{i\frac{S}{\mu}} \tag{131}$$

- **Vertical velocity ansatz**

$$w(x, z, t) = [w_0 + \mu w_1 + \cdots]\, e^{i\frac{S}{\mu}} \tag{132}$$

- **Free surface elevation ansatz**

$$\eta(x, z, t) = [\eta_0 + \mu \eta_1 + \cdots]\, e^{i\frac{S}{\mu}} \tag{133}$$

Where,

$$S = k\bar{x} - \omega\bar{t}, \quad p_i = p_i(\bar{x}, z, \bar{t}),$$
$$u_i = u_i(\bar{x}, z, \bar{t}), \quad w_i = w_i(\bar{x}, z, \bar{t}), \quad \eta_i = \eta_i(\bar{x}, z, \bar{t})$$

From the expanded variables in the equations (128)-(133), we define the relations for the the phase function, the derivatives with respect to the slowly varying variables and the condition for conservation of wave crests.

Relations for Phase Function and Derivatives

$$k \equiv \frac{\partial S}{\partial \bar{x}}, \qquad \omega \equiv -\frac{\partial S}{\partial \bar{t}}, \tag{134}$$

$$\frac{\partial}{\partial x} \equiv \mu \frac{\partial}{\partial \bar{x}}, \qquad \frac{\partial}{\partial t} \equiv \mu \frac{\partial}{\partial \bar{t}}, \tag{135}$$

$$\frac{\partial k}{\partial \bar{t}} + \frac{\partial \omega}{\partial \bar{x}} = 0 \tag{136}$$

Having now defined the coupling between the slowly varying current and the rapidly varying surface waves we are ready to proceed with the WKB expansion for each equation (121)-(127).

4 Results

4.1 The Leading Order of Magnitude

Similarly to the procedure in Section 2.6, we insert the expansions (128)-(133) into (121)-(127). Upon calculating the derivatives, and reorganizing the expressions, we then collect the terms with respect to their order of μ, which in this case is $\mathcal{O}(1)$. The resulting system of PDEs reads;

System of PDEs for WCI – Leading order of magnitude

- **The horizontal momentum equation**

$$\left[i(kU-\omega)-\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)\right]u_0=-i\frac{k}{\rho}p_0-w_0\frac{\partial U}{\partial z} \tag{137}$$

- **The vertical momentum equation**

$$\left[i(kU-\omega)-\nu\left(\frac{\partial^2}{\partial z^2}-k^2\right)\right]w_0=-\frac{1}{\rho}\frac{\partial p_0}{\partial z} \tag{138}$$

- **The incompressible continuity equation**

$$iku_0+\frac{\partial w_0}{\partial z}=0 \tag{139}$$

- **The kinematic surface condition (at $z=0$)**

$$i(kU-\omega)\eta_0=w_0 \tag{140}$$

- **The dynamic surface condition (at $z=0$)**

$$p_0=\rho g\eta_0+2\nu\frac{\partial w_0}{\partial z} \tag{141}$$

- **The conserved tangential stress (at $z=0$)**

$$ikw_0+\frac{\partial u_0}{\partial z}=0 \tag{142}$$

- **The bottom condition (at $z=-h$)**

$$U=u_0=w_0=0 \tag{143}$$

4.2 Model Validation

4.2.1 Recovery of Inviscid Linear Wave Theory

Through the introduction of a background shear current, we now have a system with weakly nonlinear terms. However, we are still working under the linear approximation, assuming small oscillations around the mean surface elevation $z=0$, and a flow with a low Reynolds number. Before further manipulations or investigations on our system of PDEs, we should then pause and check if the assumptions made when developing the equations and conducting the WKB expansion are consistent with the inviscid linear wave theory discussed in Section 2.

To check if our WKB expansion for the leading order is consistent, we set the shear current $U(z,\bar{t})$ and the viscosity ν equal to zero. If our WCI system then simplifies to the inviscid linear wave equations (14)-(16) and the conserved tangential stress (105), the equations are consistent.

In the absence of a background current, there is no longer a need to account for scale separation in the system. Accordingly, we revert to the original dimensional variables, so that $\bar{x}, \bar{t} \to x, t$. We continue to assume that all perturbation quantities represent progressive wave solutions, consistent with the assumptions introduced in Section 2.3.1. Specifically, we represent the variables u_0, w_0, p_0, and η_0 in the form

$$\lambda(x,z,t) = \lambda_0(z)e^{i(kx-\omega t)}, \quad \text{with} \quad \lambda_0(z) \sim e^{\kappa z}, \tag{144}$$

where x denotes the direction of wave propagation and ω is the angular frequency.

The momentum equations

We begin with the horizontal momentum, which when brought back to a strictly linear scheme reduces to

$$i\omega u_0 = i\frac{k}{\rho}p_0. \tag{145}$$

We see that this can be expressed as

$$\frac{\partial}{\partial t}u(x,z,t) = -\frac{1}{\rho}\frac{\partial}{\partial x}p(x,z,t), \tag{146}$$

which is a strictly linear version of the Navier-Stokes equation for horizontal flow with no external forcing or viscous terms. Similarly, we consider the vertical momentum equation which simplifies to

$$i\omega w_0 = \frac{1}{\rho}\frac{\partial p_0}{\partial z}, \tag{147}$$

and can thus be rewritten

$$\frac{\partial}{\partial t}w(x,z,t) = -\frac{1}{\rho}\frac{\partial}{\partial z}p(x,z,t). \tag{148}$$

Equation (148) is equal to (146), only changing the horizontal $u(x,z,t)$ for the vertical velocity $w(x,z,t)$. Thus, we conclude that the horizontal and vertical momentum equations are consistent with inviscid linear wave theory.

The Continuity Equation

As discussed in Section 2.3, the equations (14)-(16) describe linear wave theory in its simplest form, using potential flow, $\Phi(x,z,t)$. We consider the incompressible continuity equation for the leading order of magnitude in our WKB approximation (139). This expression has no terms depending on the viscosity $U(z,\bar{t})$ or ν, and under the assumption of the velocity components being monochromatic waves (144), we see directly that it can be rewritten as

$$\frac{\partial}{\partial x}u(x,z,t) + \frac{\partial}{\partial z}w(x,z,t) = 0. \tag{149}$$

As described in Section 2.1, the condition for expressing a flow field in terms of the velocity potential is that it is irrotational. To introduce a potential flow in our equations we must then show that $\omega = \nabla \times \mathbf{u} = 0$. In the xz-plane the vorticity becomes

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

The velocity field, $\mathbf{u}$, of the flow is described in the momentum equations (146), (148). We differentiate wrt. time and subtract (148) from (146) to get

$$\frac{\partial \omega_y}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\right) = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x \partial z} - \frac{\partial^2 p}{\partial z \partial x}\right) = 0.$$

Thus, if the flow is initially irrotational, it remains irrotational for all time. This justifies introducing a scalar velocity potential Φ, such that

$$\mathbf{u} = \nabla \Phi,$$

and

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi = 0. \tag{150}$$

This shows that the incompressible continuity equation is similar to equation (14), and consistent with linear wave theory.

The boundary conditions

Upon setting $U(z), \nu = 0$, the kinematic surface, the dynamic surface, the conserved tangential stress and the bottom condition, reduce respectively to the following

$$-i\omega\eta_0 = w_0, \quad (\text{at } z = 0), \tag{151}$$

$$-g\eta_0 + \frac{1}{\rho}p_0 = 0, \quad (\text{at } z = 0), \tag{152}$$

$$\frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial z} = 0, \quad (\text{at } z = 0), \tag{153}$$

$$w_0 = 0, \quad (\text{at } z = -h). \tag{154}$$

We see directly that (153) and (154) are equal to (105) and (21), thus both being consistent with the inviscid linear case. Under the assumption of (144), it is also clear that the kinematic surface condition (151) equals (17). For the dynamic surface condition, we seek consistency with (16), which is expressed in terms of the surface pressure p_a. Meanwhile, (152) is expressed in terms of the modified pressure $p_0 = p_a + \rho g \eta_0$, so we differentiate (152) wrt. time and substitute for p_0, i.e.,

$$-g\frac{\partial \eta_0}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial t}(p_a + \rho g \eta_0) = 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial p_a}{\partial t} = 0, \quad (\text{at } z = -h). \tag{155}$$

We see that also this boundary condition is consistent. Thus we conclude that the leading order of the expanded viscous WCI system with a background shear current is consistent with inviscid linear wave theory.

4.2.2 Compatibility with Eddy Viscosity

As previously discussed, a motivation for deriving the equations (137)–(143) is to investigate the effect of eddy viscosity on surface wave propagation. Since the governing equations are obtained through an approximate method, we must ensure that the inclusion of eddy viscosity does not invalidate the model.

In the formulation presented in Section 3, we assume a shear current that varies only with depth, i.e., $U = U(z)$. It is then natural to assume that the eddy viscosity also varies only in the vertical direction: $\nu_e = \nu_e(z)$ (Jenkins, 2013). We consider the viscous terms in the momentum equations (107) and (108) for the full WCI model, i.e., prior to applying the WKB approximation.

For constant viscosity ν, the viscous terms take the form

$$\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{156}$$

Replacing the constant viscosity with a depth-dependent eddy viscosity $\nu_e(z)$(106), the viscous terms become

$$\frac{\partial}{\partial x}\left(\nu_e(z)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu_e(z)\frac{\partial u}{\partial z}\right), \quad \frac{\partial}{\partial x}\left(\nu_e(z)\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu_e(z)\frac{\partial w}{\partial z}\right). \tag{157}$$

Expanding the derivatives, we obtain

$$\nu_e(z)\frac{\partial^2 u}{\partial x^2} + \frac{\partial \nu_e(z)}{\partial z}\frac{\partial u}{\partial z} + \nu_e(z)\frac{\partial^2 u}{\partial z^2}, \tag{158}$$

$$\nu_e(z)\frac{\partial^2 w}{\partial x^2} + \frac{\partial \nu_e(z)}{\partial z}\frac{\partial w}{\partial z} + \nu_e(z)\frac{\partial^2 w}{\partial z^2}. \tag{159}$$

From the velocity ansatz used in the WKB expansion (131)–(132), we recall that the variables $\bar{x}, \bar{t}$ are scaled by a factor μ, while z remains of $\mathcal{O}(1)$. Thus, the expansion in terms of μ remains valid when eddy viscosity is included via the substitutions in (158) and (159).

The viscous terms in the leading-order momentum equations (137) and (138) are therefore modified as

$$-\nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)u_0 \quad \Longrightarrow \quad -\left[\frac{\partial \nu_e(z)}{\partial z}\frac{\partial}{\partial z} + \nu_e(z)\frac{\partial^2}{\partial z^2} - \nu_e(z)k^2\right]u_0, \tag{160}$$

and similarly for w_0.

Finally, we note that the dynamic boundary condition at the surface remains unchanged, since $\nu_e(z)$ is constant at $z = 0$. We conclude that the model is compatible with the inclusion of a depth-dependent turbulent eddy viscosity.

4.3 Steps Towards Solving the WCI System of PDEs for the Leading Order of Magnitude

As we have now ensured that the WCI system is consistent with inviscid linear wave theory, we proceed in our efforts towards solving the equations (137)-(142). In our considerations we will consider the viscosity ν, and the shear current $U(z)$ to be known variables. We note that the system of PDEs contains three equations for the bulk of the system (137)-(139), expressed in terms of the three unknown variables u_0, w_0, p_0, thus constituting a solvable set.

As PDEs are substantially harder to solve than ODEs, a natural approach is to reduce the system of equations to an ODE. From the incompressible continuity equation (139), we see that

$$u_0 = \frac{i}{k} w_{0z}, \tag{161}$$

which we substitute into the horizontal momentum equation and get

$$\left[i(kU - \omega) - \nu \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{i}{k} w_{0z} = -i \frac{k}{\rho} p_0 - w_0 U_z. \tag{162}$$

We then differentiate (162) with respect to z and substitute for p_{0z} in the vertical momentum equation (138). After reorganizing the equation and collecting terms of the vertical velocity component with respect to their order, we find;

$$\begin{aligned} \left[i(kU - \omega) + \nu k^2 - U_{zz}/k^2 \right] w_0 - \frac{1}{k^2} \left[ik(kU_z - \omega) - U_z \right] w_{0z} \\ - \frac{1}{k^2} \left[i(kU - \omega) + 2\nu k^2 \right] w_{0zz} + \frac{1}{k^2} \nu w_{0zzzz} = 0. \end{aligned} \tag{163}$$

Equation (163) is a fourth order ODE, describing the vertical velocity component of the surface waves. To facilitate solving the equation wrt. w_0, we rewrite it as a system of ODEs. We then define the new variables

$$\begin{aligned} y_1 &= w_0, \\ y_2 &= w_{0z}, \\ y_3 &= w_{0zz}, \\ y_4 &= w_{0zzz}, \end{aligned}$$

so that:

System of First-Order ODEs for The Leading Order of Magnitude of WCI with a Shear Current and Viscosity

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= \frac{1}{\nu} \Big[- \Big(k^2 \left[i(kU - \omega) + \nu k^2 \right] - U_{zz} \Big) y_1 + \big(ik(kU_z - \omega) - U_z \big) y_2 \\ &\qquad + \Big(i(kU - \omega) + 2\nu k^2 \Big) y_3 \Big] \end{aligned}$$

The system of ODEs (4.3) presented above can be solved numerically for w_0. However, this is not within the scope of this report.

4.4 The Second Leading Order of Magnitude

To increase the exactness of the approximate WCI solution, the next order, $\mathcal{O}(\mu)$, of the WKB approximation should be considered. For (121)-(127), this order yields complex and extensive equations, and it is beyond the scope of this preparatory report to comment any further on these. They are however presented to the reader, as they are of interest once having solved the equations of the leading order of magnitude (137)-(142).

System of PDEs for WCI – Second leading order of magnitude

- **The horizontal momentum**

$$-i\nu k_{\bar{x}} u_0 + (U - 2i\nu k) u_{0\bar{x}} + u_{0\bar{t}} + \frac{1}{\rho} p_{0\bar{x}} = -\left[i(kU - \omega) - \nu \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \right] u_1 - i\frac{k}{\rho} p_1 - w_1 U_z \tag{165}$$

- **The vertical momentum**

$$-i\nu w_0 k_{\bar{x}} + (U - 2i\nu k) w_{0\bar{x}} + w_{0\bar{t}} = -\left[i(kU - \omega) - \nu \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \right] w_1 - \frac{1}{\rho} p_{1z} \tag{166}$$

- **The incompressible continuity equation**

$$u_{0\bar{x}} = -iku_1 - w_{1z} \tag{167}$$

- **The kinetic surface condition (at $z = 0$)**

$$U\eta_{0\bar{x}} + \eta_{0\bar{t}} = -i(kU - \omega)\eta_1 + w_1 \tag{168}$$

- **The dynamic surface condition (at $z = 0$)**

$$0 = -p_1 + \rho g \eta_1 + 2\nu w_{1z} \tag{169}$$

- **The conserved tangential stress (at $z = 0$)**

$$w_{0\bar{x}} = -ikw_1 - u_{1z} \tag{170}$$

- **The bottom condition (at $z = -h$)**

$$0 = U = u_1 = w_1 \tag{171}$$

5 Discussion and Further Work

This report has presented a theoretical foundation for understanding linear wave theory and the interaction between surface waves and background currents (WCI). Building on this framework, we derived a linearized WCI model that incorporates weakly nonlinear viscous terms using a WKB approximation. As this work is intended as a preparatory study for a master's thesis, this final section briefly discusses the results and outlines potential directions for future research and the broader applicability of the results.

A key element in this study is the inclusion of viscosity and background shear in the linearized WCI framework. While the influence of viscosity is typically minor in the presence of weak shear, its role under strong shear remains insufficiently understood. The aim of the WCI model derived in this report is to facilitate investigations on the combined effects of strong vertical shear, such as that generated by wind forcing and stratification, and turbulent eddy viscosity. Improving our understanding of how shear and viscosity influence wave dynamics has important implications for coastal engineering, offshore operations, and the modeling of upper-ocean mixing.

From the model, we expect to observe some of the phenomena discussed in Section 2, such as refraction, changes in wave amplitude, and Doppler shifts. However, the inclusion of shear and viscosity introduces dissipative effects, meaning that the conservation of wave action and energy may no longer hold.

A natural continuation of this work would be to extend the current model to account for stronger shear currents and parameterized eddy viscosity profiles. The fourth-order ODE derived in this study (163) can be solved numerically and serves as a practical starting point for investigating how shear-induced turbulence modifies wave dispersion and energy transport. Analyzing particle trajectories, velocity profiles, and wave transformations could reveal physical phenomena not captured in models that neglect viscosity.

Additionally, the weakly nonlinear framework may highlight physical effects absent in linear theory, which might otherwise be obscured or difficult to interpret in fully nonlinear, potentially chaotic models. Investigating the significance of these effects could enhance our general understanding of nonlinearity in WCI. To validate the model's predictions, it would also be valuable to compare numerical results with experimental or observational data.

At this stage, however, since the equations in the model remain unsolved, it is not yet certain whether the approach will lead to valuable new insights into WCI. If pursued further in a master's thesis, the work should aim to produce results that clarify the utility of the model. Regardless of the model's eventual success, the theory and methods reviewed in this report provide a foundation upon which new or improved WCI models can be developed.

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