UCL CSML Seminar Series

Imperial College London

Causal Discovery through Bayesian Model Selection

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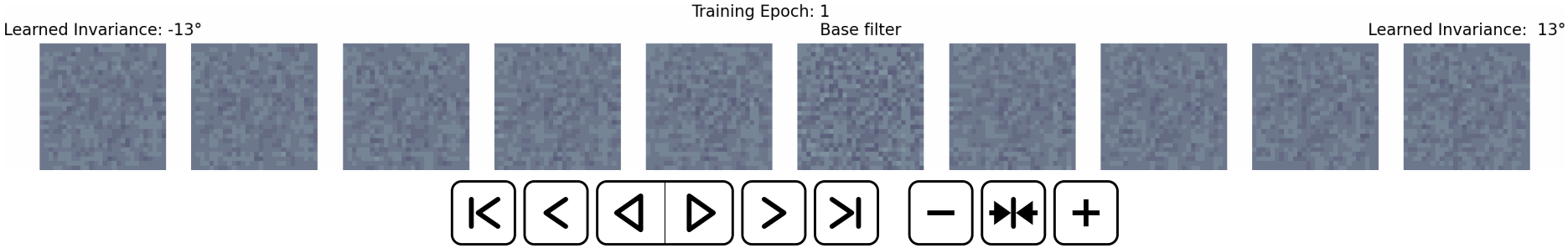
My Research

Research interests:

- Automatically learning inductive biases (automatic + computational)
- Adaptivity & robustness: Continual learning, uncertainty
- Applications: Decision making, BayesOpt, Model-based RL
- Methods:
  - Bayesian model selection / Occam's razor / MDL
  - Meta-learning
  - Approximate inference
  - (Deep) Gaussian processes & relations to DNNs
  - Interesting architectures (capsule networks)



Bayesian Model Selection for Causality

Causal Discovery using Marginal Likelihood
CML4Impact workshop @ NeurIPS 2022
Anish Dhir · MvdW

Extended version soon to be on arxiv.

Overview

Causality: Overview and Background

Causality: Approaches and Problems

Bayesian Model Selection for Causality and its Properties

An Actual Method

Discussion & Conclusion

Causality: Motivation

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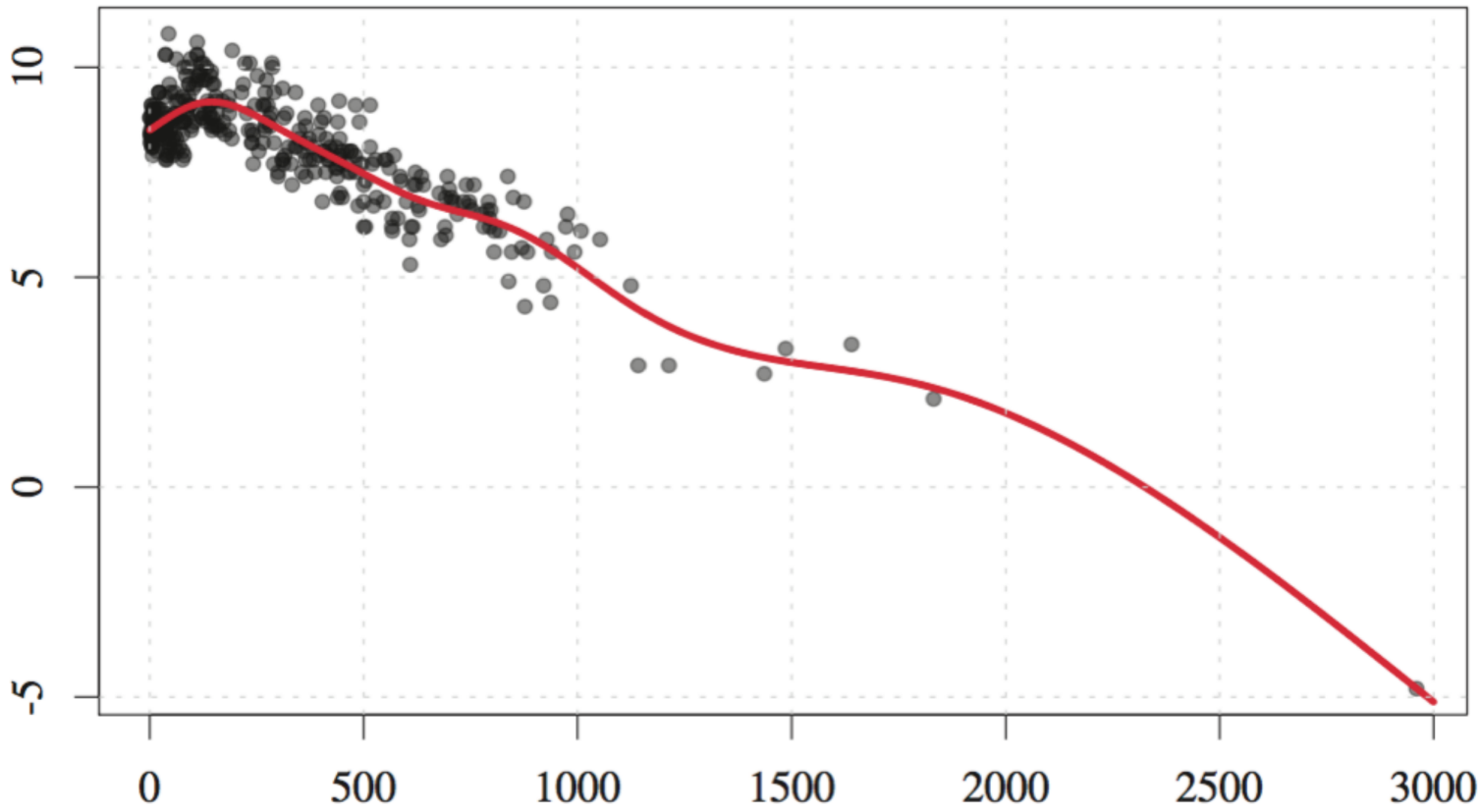
One of the big questions in the causal inference community:

How far can we get with observational data alone?

Causality: Example



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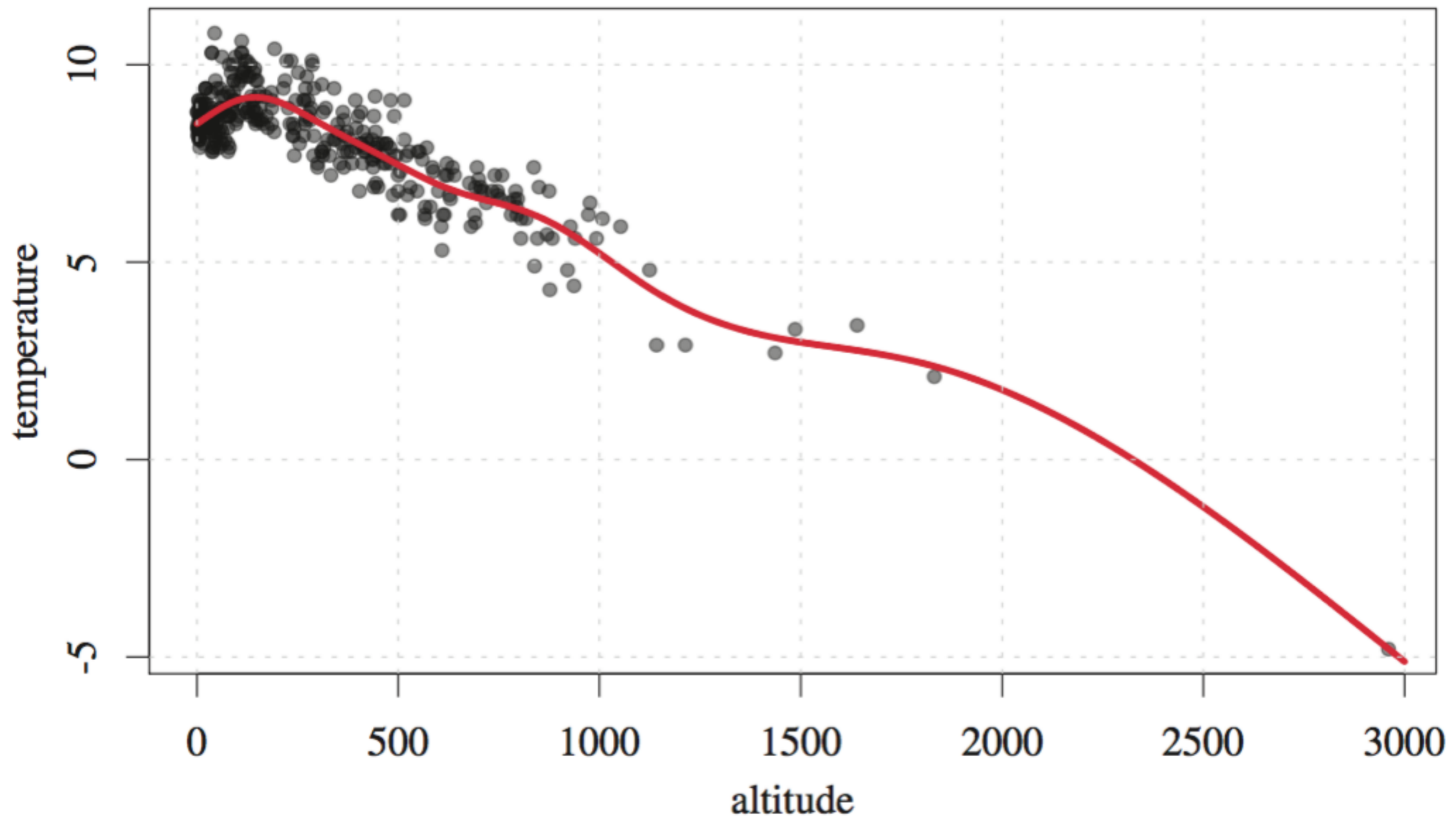

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Structural Causal Models

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For bivariate problems, we simply have:

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We consider behaviour over many datasets, each of which is generated by different functions f_X, f_Y.

- A distribution on datasets arises from distributions $\Pi(F_X), \Pi(F_Y)$
- For dataset $\mathcal{D} = (X, Y) = \{(X_i, Y_i)\}$ we get distribution $\Pi(X, Y)$

Factorisations & Interventions

Assuming $X \to Y$, we can factorise our joint in two ways:

- $\Pi(X, Y) = \Pi(X)\Pi(Y|X)$: Causal factorisation
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An intervention on a variable changes its value, generation function, or noise input, **while leaving those of all other variables unchanged**. E.g. if $X \to Y$, an intervention on X

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Assumptions known as **Independent Causal Mechanism** (ICM)

Causal Models

Our model should directly parameterise the causal factorisation:

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where the marginals are chosen from $\mathcal{R}$ and conditionals from $\mathcal{C}$

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Allows part of the model to be re-used after an intervention
without degradation of accuracy

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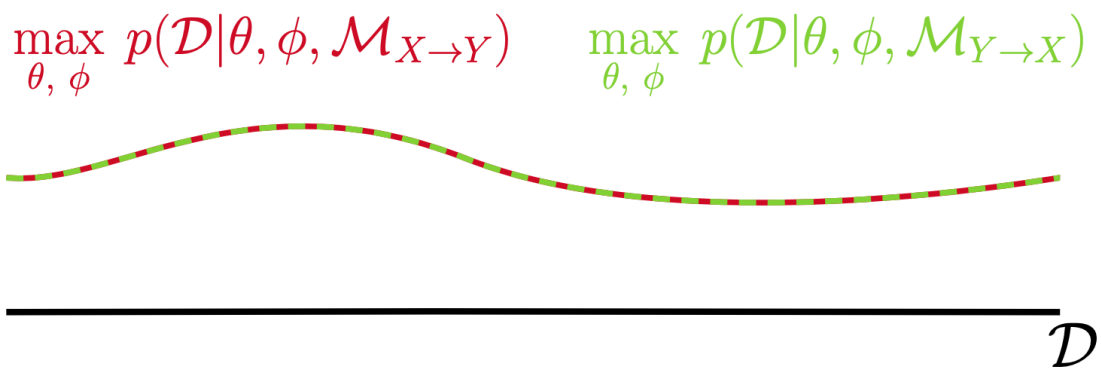

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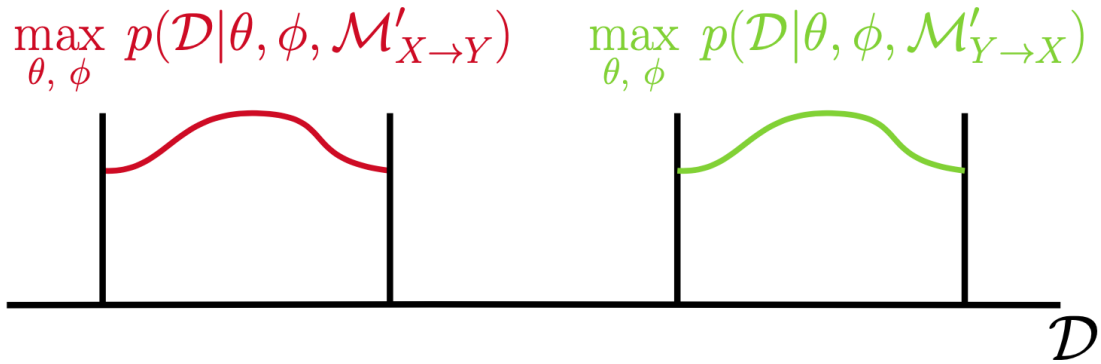

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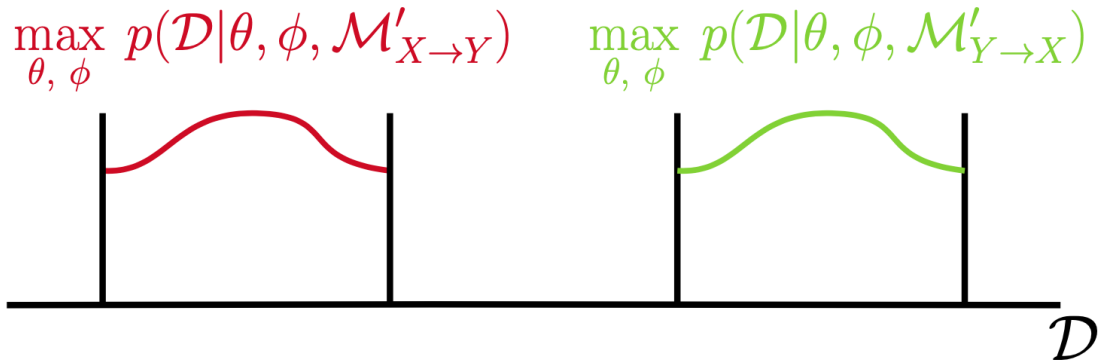

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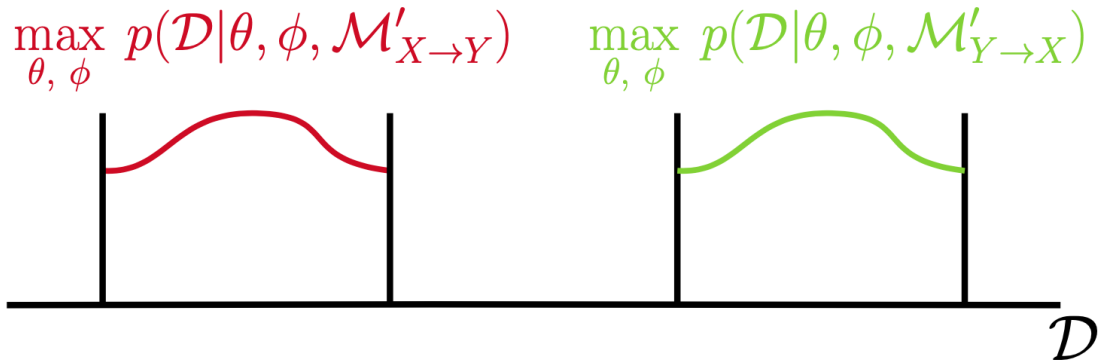

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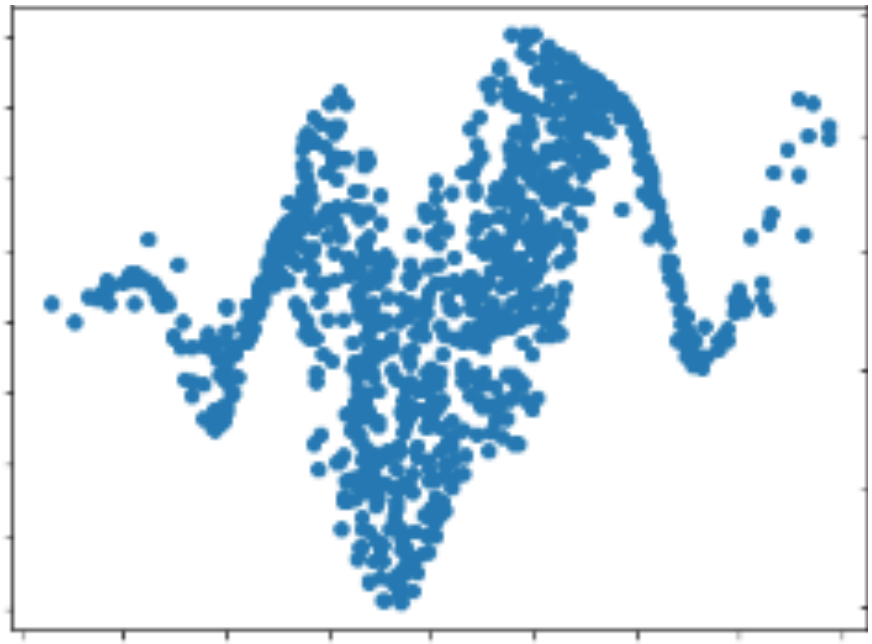
- Other classes of models are similarly identifiable.

Additive Noise Model: Problem

- What to do for a dataset like this?



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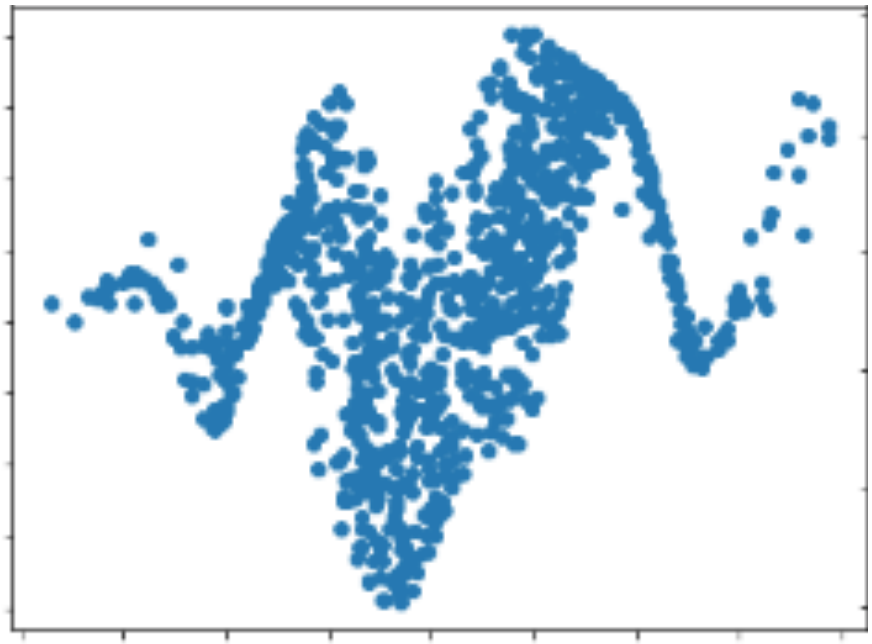

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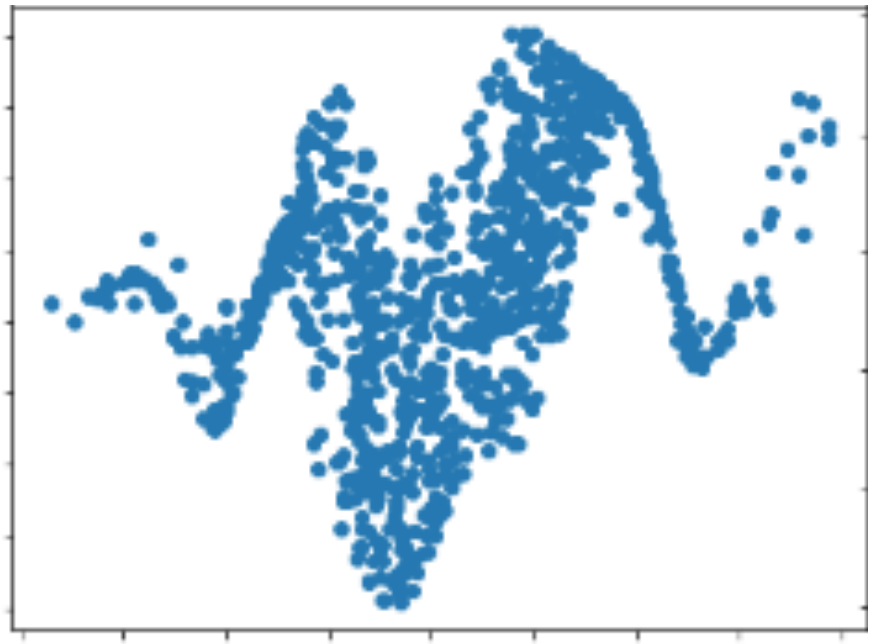

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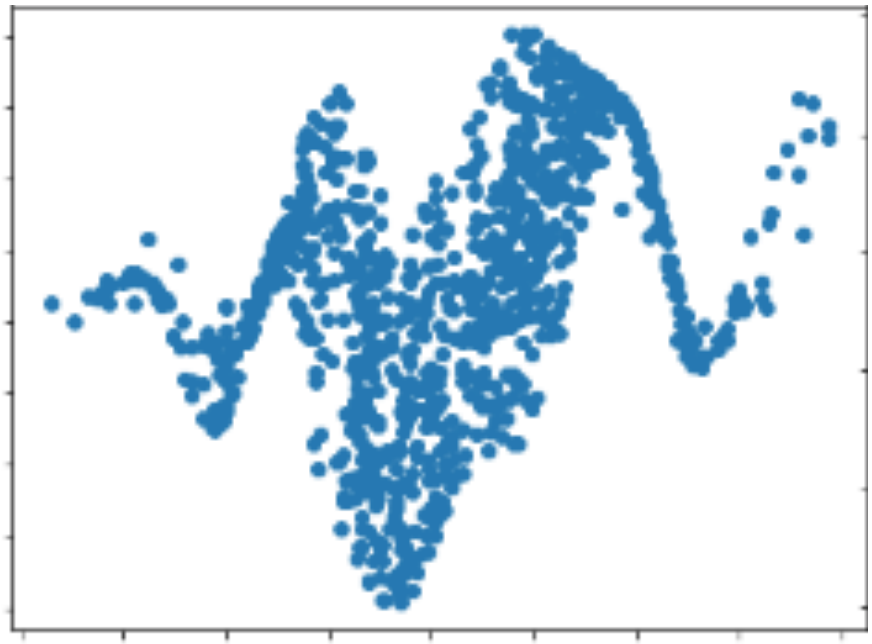

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- Guarantees don't hold anymore!

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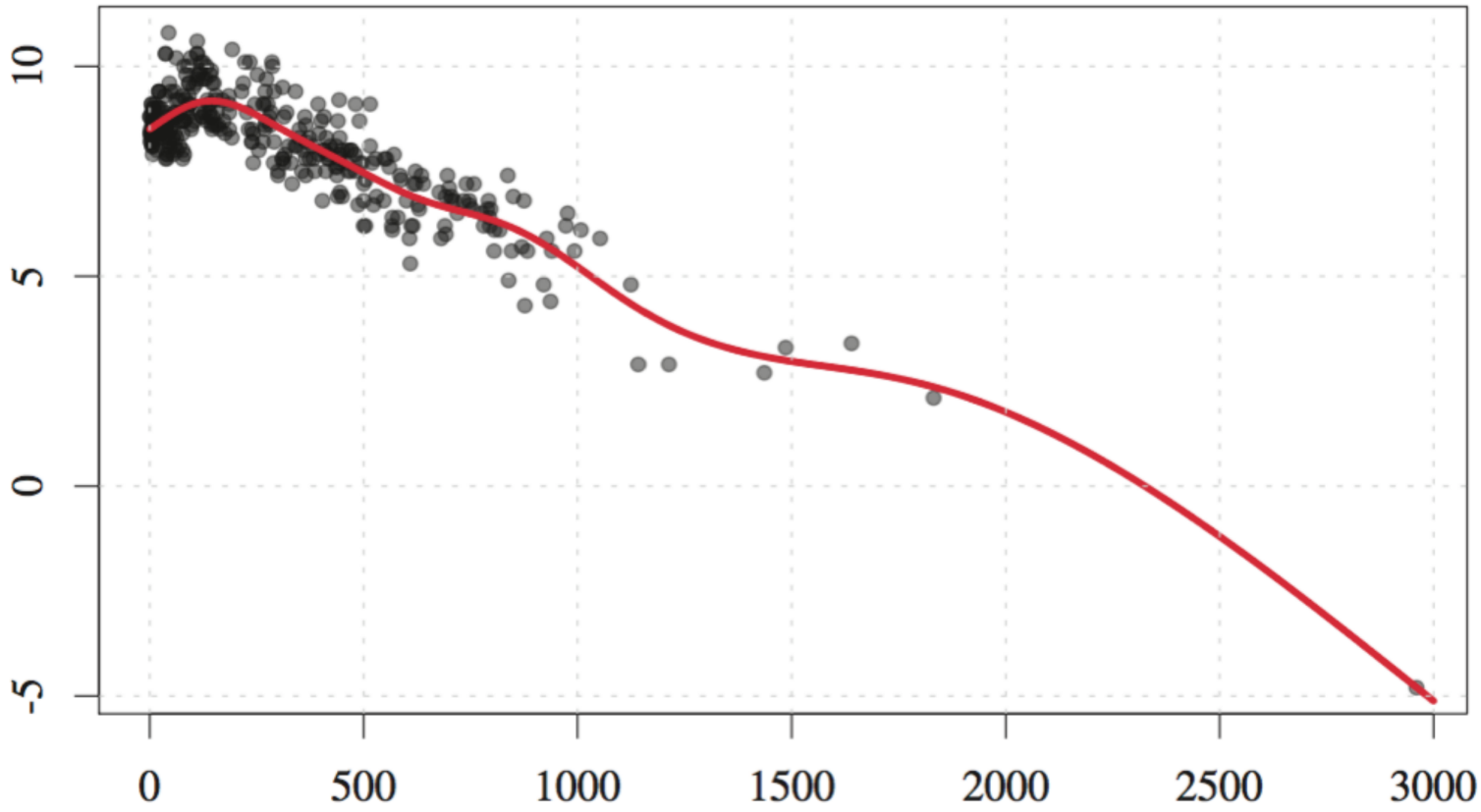

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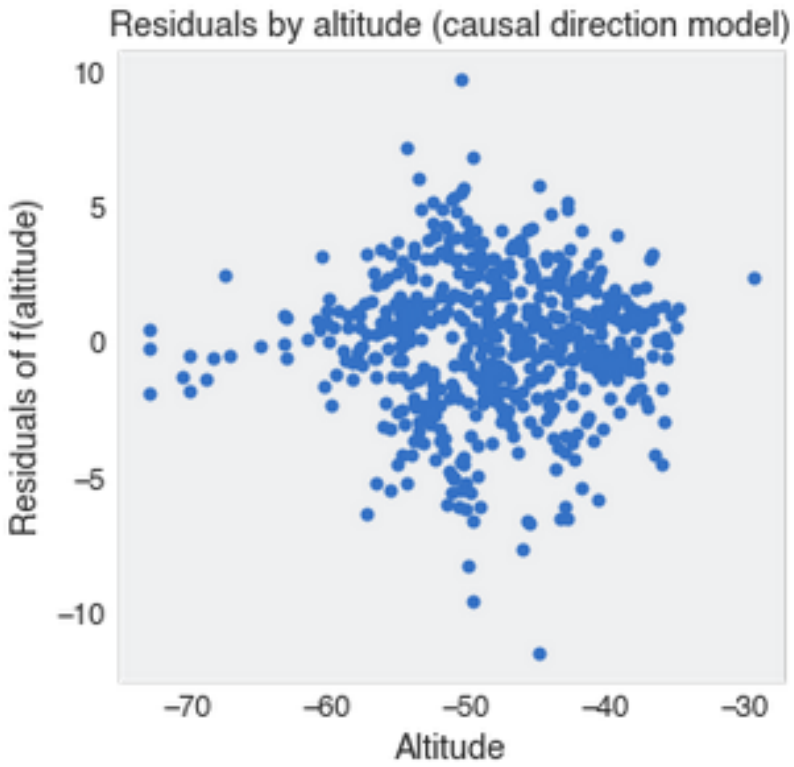

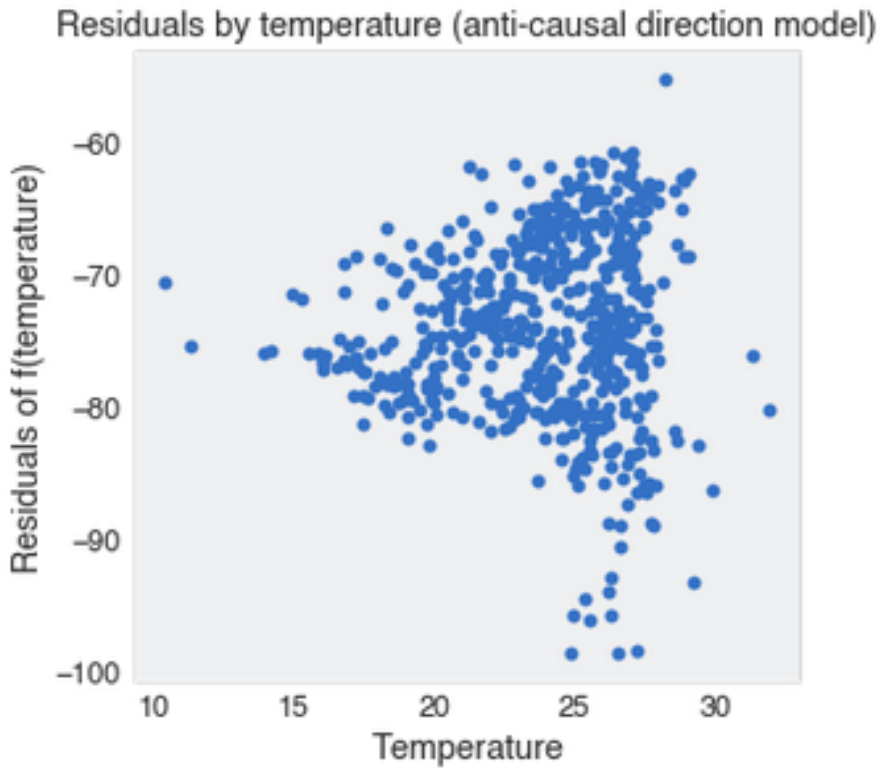

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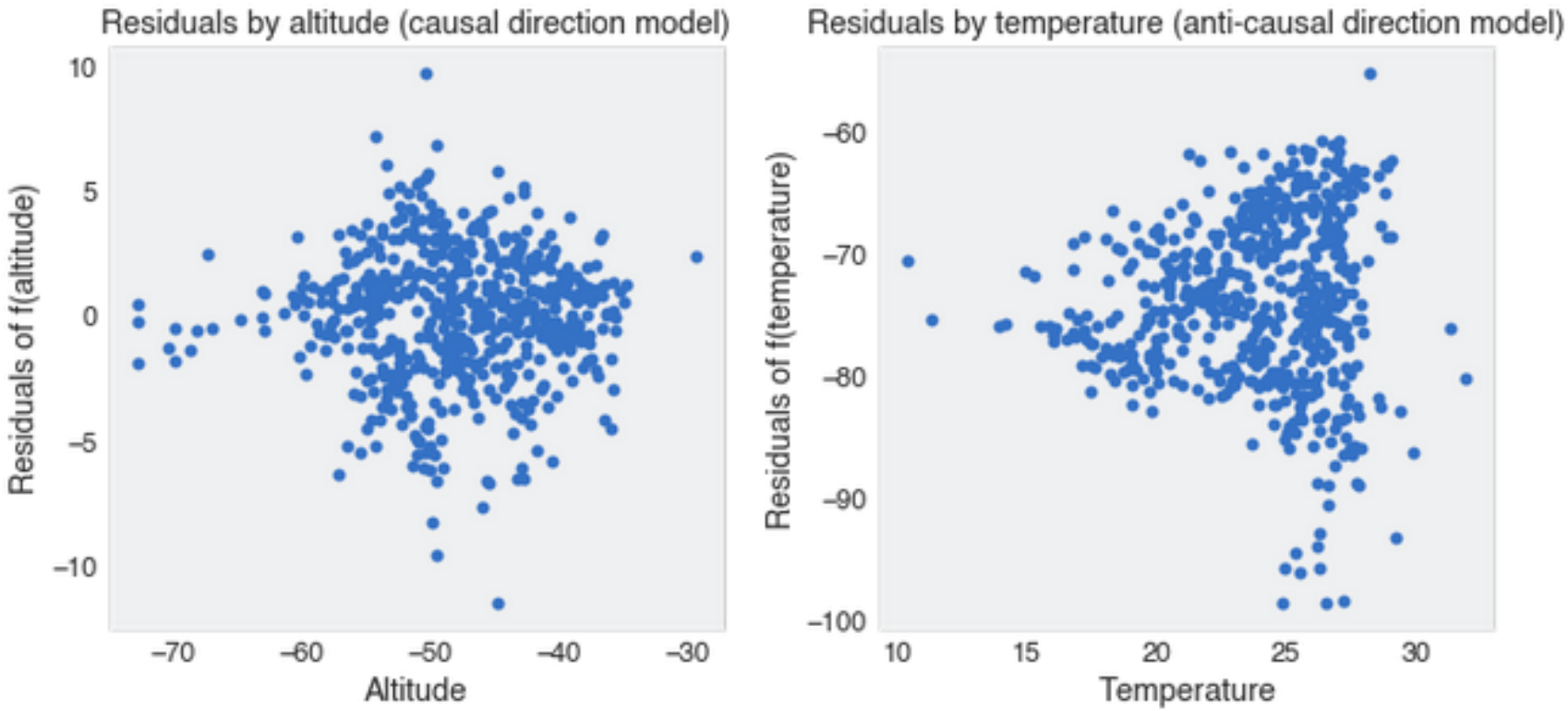

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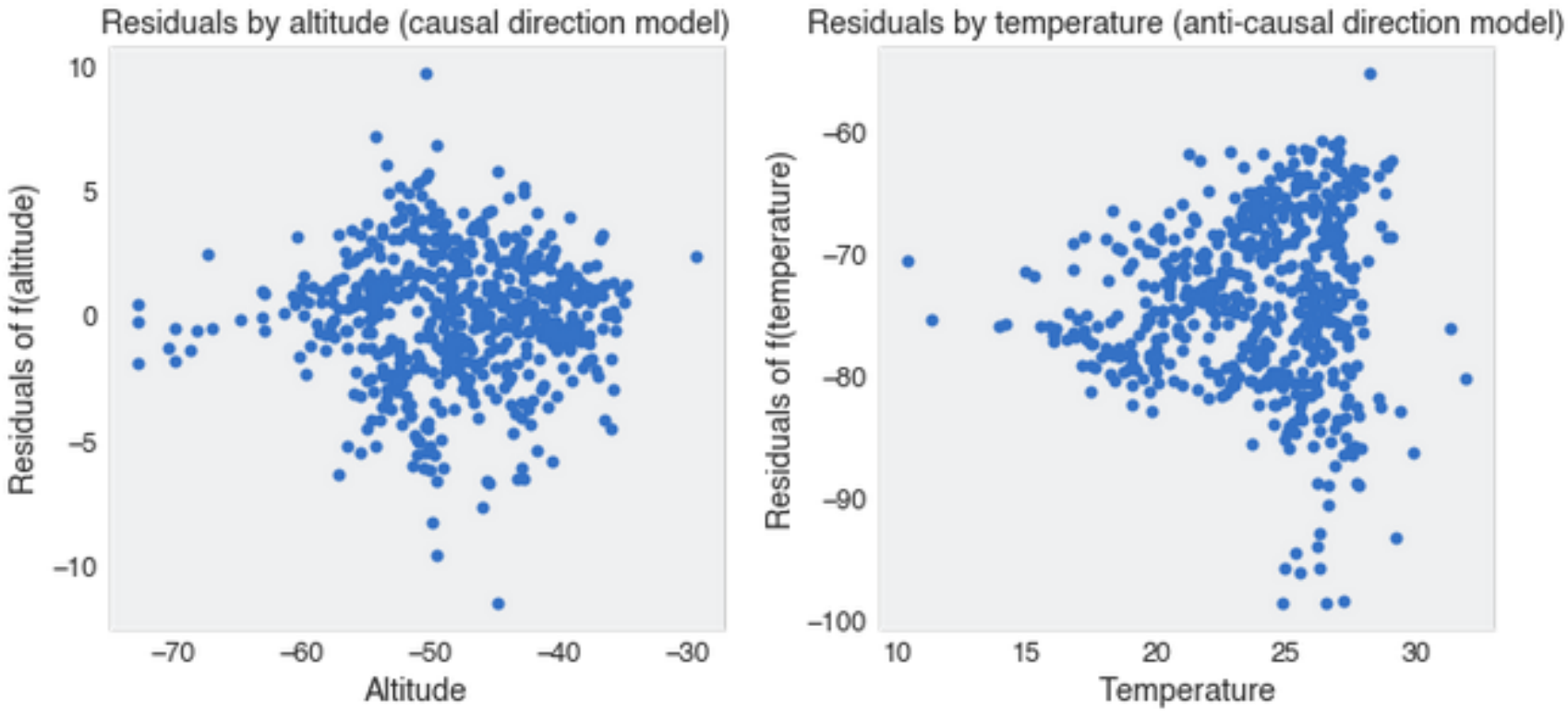

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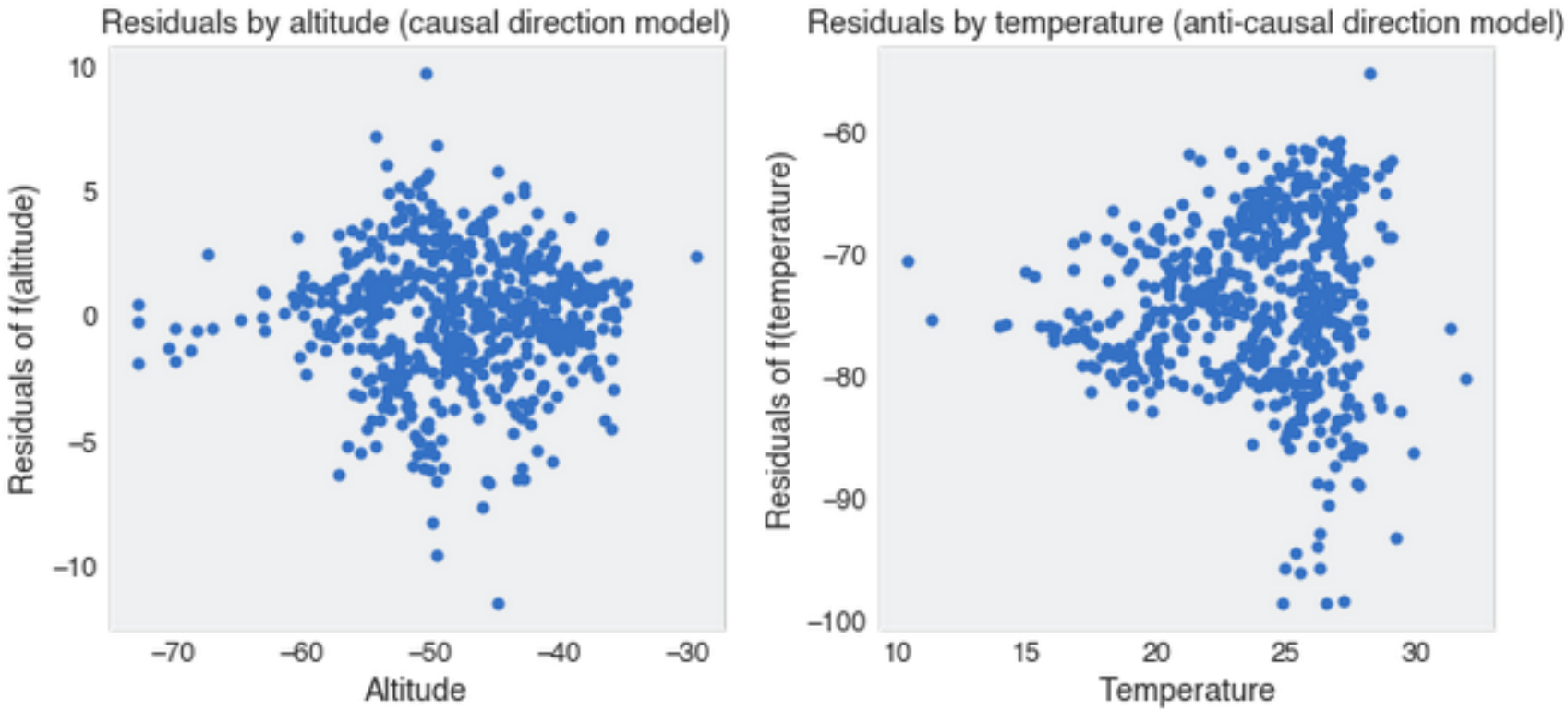

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- But noise in correct causal direction not independent of cause
- Still, one direction more complex than the other?

Complexity Metrics for Causality

The relation between simplicity and causality has been noted and used before. We have methods that

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- ... and has the same (lack) of guarantees

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$\implies$ We must be indifferent, so choose 0.5.

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So we just need to compute the **marginal likelihoods**

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- Consistent with earlier constraint approaches: Zero mass in prior.

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$$\mathcal{M}^* = \begin{cases} \mathcal{M}_{\mathbf{X}\to Y} \text{ if } p(\mathcal{D}|\mathcal{M}_{\mathbf{X}\to Y}) > p(\mathcal{D}|\mathcal{M}_{X\leftarrow\mathbf{Y}}) \\ \mathcal{M}_{X\leftarrow\mathbf{Y}} \text{ if } p(\mathcal{D}|\mathcal{M}_{\mathbf{X}\to Y}) < p(\mathcal{D}|\mathcal{M}_{X\leftarrow\mathbf{Y}}) \end{cases}, \tag{8}$$

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Probability of error in both causal directions are equal:

$$\begin{aligned} P(E) &= P(E|\mathcal{M}_{\mathbf{X}\to Y})P(\mathcal{M}_{\mathbf{X}\to Y}) + P(E|\mathcal{M}_{X\leftarrow\mathbf{Y}})P(\mathcal{M}_{X\leftarrow\mathbf{Y}}) = P(E|\mathcal{M}_{\mathbf{X}\to Y}) \\ &= \frac{1}{2}(1 - \mathrm{TV}[P_{\mathcal{D}}(\cdot|\mathcal{M}_{\mathbf{X}\to Y}), P_{\mathcal{D}}(\cdot|\mathcal{M}_{X\leftarrow\mathbf{Y}})]) \end{aligned}$$

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Overview

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A More Realistic Model

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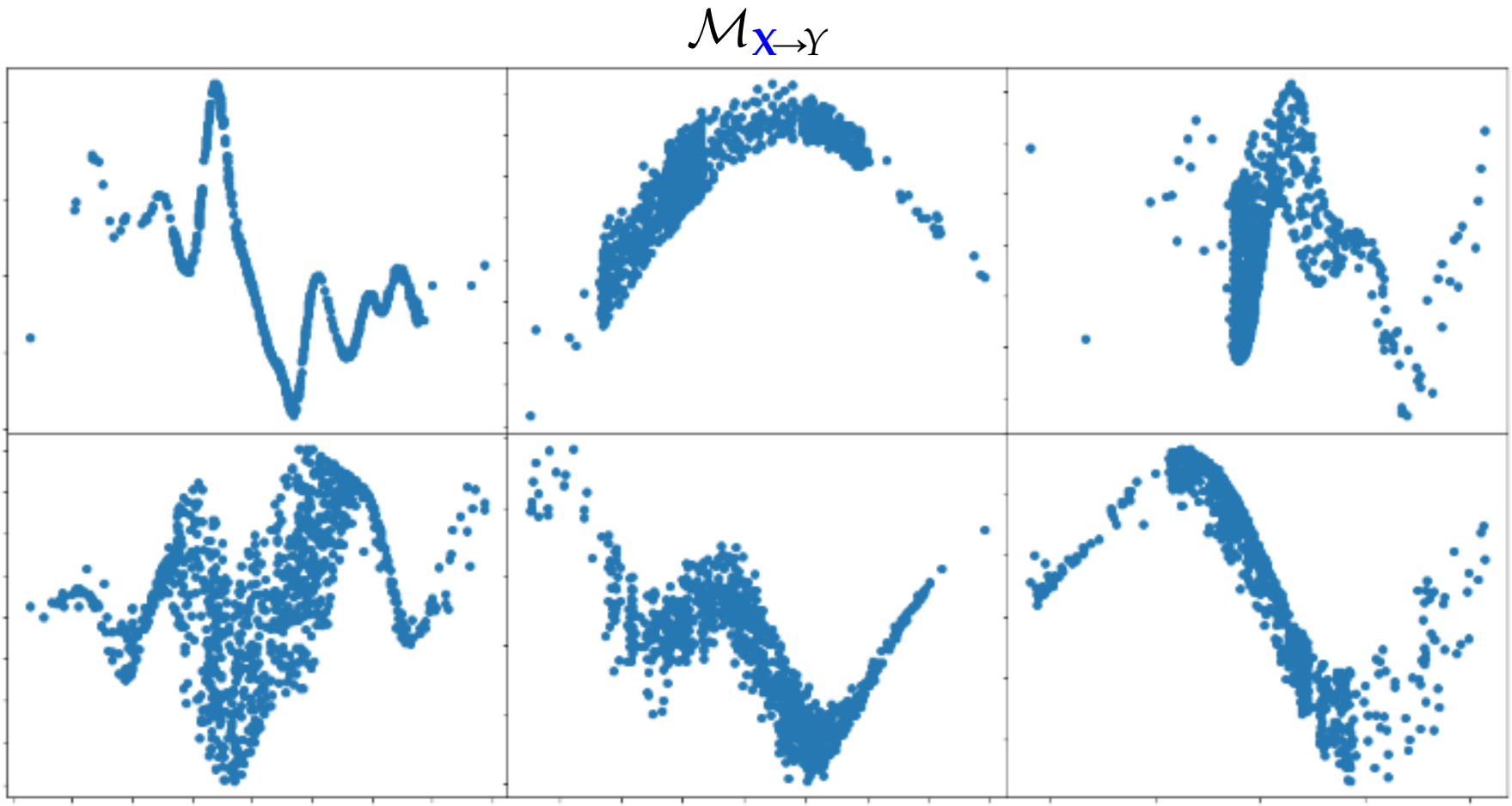

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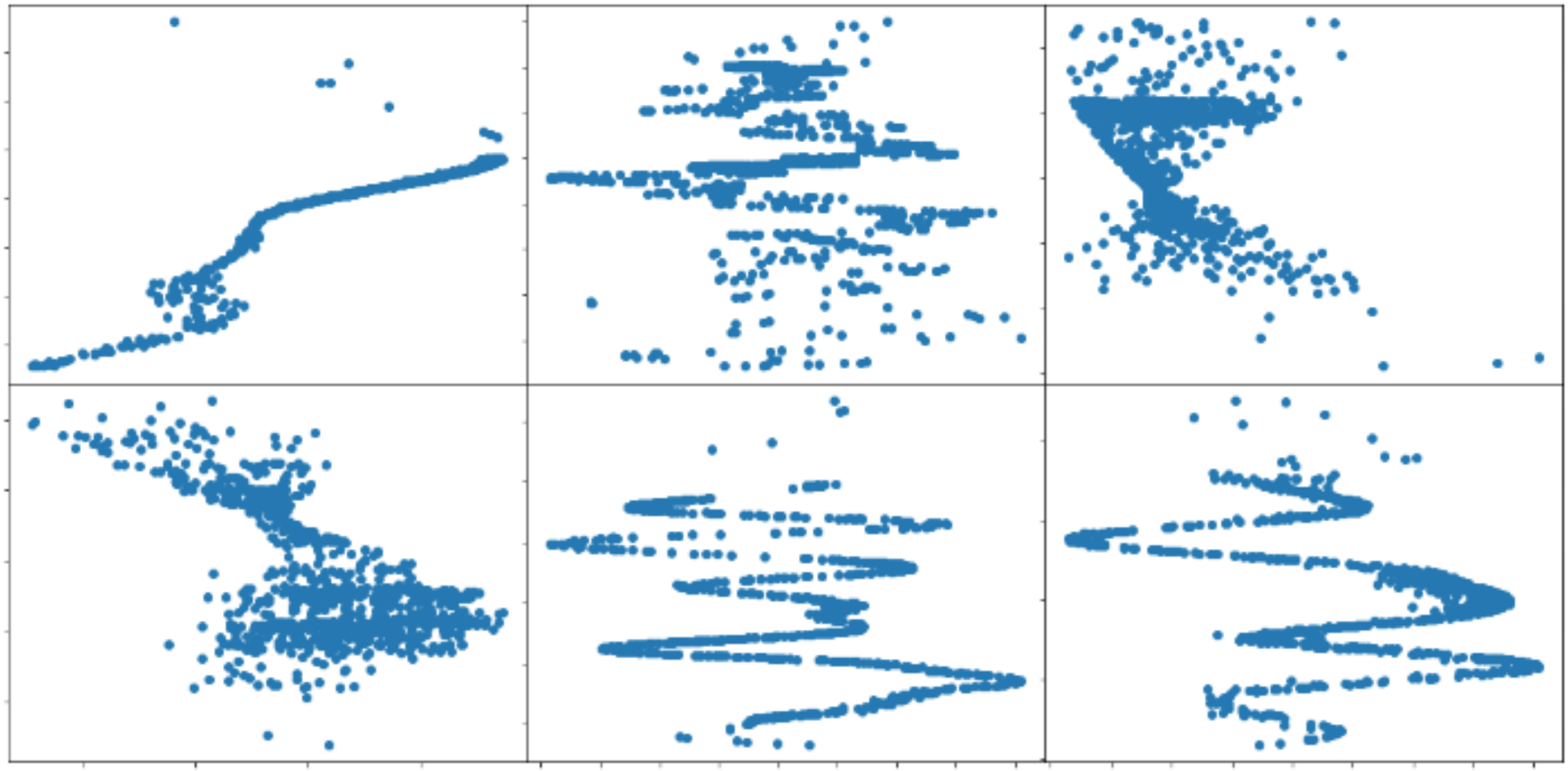

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Experiments: ANM data

- How does the non-fully-identifiable model work on fully-identifiable data?
- How does this compare to a fully-identifiable model?

Methods	ANM
Gaussian Process	100.0
GPLVM	100.0

Table: ROC AUC scores for identifying causal direction of datastes generated by an ANM (higher is better).

$\Longrightarrow$ Addition of flexibility does not (detectably) hurt performance on "easy" datasets.

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CGNN	76.2	94.7	86.3	89.3	76.6
GPI	71.5	73.8	88.1	90.2	70.6
PNL	78.6	51.7	75.6	84.7	73.8
ANM	43.7	25.5	87.8	90.7	63.9
IGCI	55.6	77.8	57.4	16.0	63.1
LiNGAM	57.8	62.3	3.3	72.2	31.1
RECI	59.0	94.7	66.0	71.0	70.5
CCS	69.3	96.0	89.7	90.5	N/A
CHD	72.0	97.6	90.5	91.4	N/A
CKL	69.8	95.5	89.3	91.0	N/A
CKM	69.7	90.6	94.3	91.6	N/A
CTV	72.2	95.8	91.9	**91.8**	N/A
GPLVM	**82.1**	**97.7**	**98.8**	90.2	**78.3**

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- E.g. more ambiguous datasets than expected by prior
- Variational inference could be inaccurate in some cases

Origins of Bayes for Causality

Discuss. References in paper.

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- Allows specification of realistic assumptions.
- Realistic assumptions lead to good performance.

Join us!


Anish Dhir


Artem Artemev


Jose Pablo Folch


Ruby Sedgwick


Seth Nabarro


Tycho van der Ouderaa

- I'm looking for PhD candidates (in Oxford from September)
- Check my website (`https://mvdw.uk/`) for tips on applying, and how to get in touch.
- And do find me to chat if you have questions.