ADAPTING NEURON COUNT DURING TRAINING

A Bayesian Nonparametric View

Mark van der Wilk Invited Talk 14th International Conference on Bayesian Nonparametrics





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Based on a True Story

Adjusting Model Size in Continual Gaussian Processes: How Big is Big Enough?

Guiomar Pescador-Barrios ¹ Sarah Filippi ¹ Mark van der Wilk ²

Spotlight at ICML 2025.

A Bayesian Nonparametric View on Adapting Neuron Count During Training

In submission, soon to be on arxiv.

Thesis of the Talk



Ideas from Bayesian Nonparametrics may help with new capabilities in deep learning



How big should a model be?

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Why is this a relevant question?

Reason 1:

- Network size determines compute and energy costs.
- It is possible to shrink models after training.
- Current advice is to make models as large as possible.
 - ? Can we find weights and size in a unified way?

To avoid unnecessary computation.

Reason 2:

- Data can arrive in a streaming fashion.
- We don't know a priori how large a dataset we have.
- ? Can we grow a NN's size as we see more data?

To avoid poor performance, from constant/restricted model size.



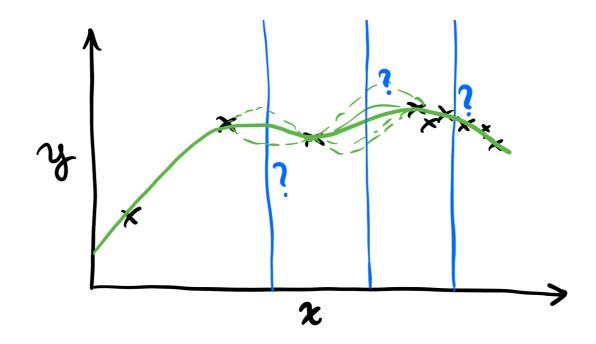
Minimise model size, while predictions are near-optimal

Most of Machine Learning is just Curve Fitting

Dataset: $(x_n, y_n)_{n=1}^N$.

Inputs $x_n \in \mathcal{X}$, outputs $y_n \in \mathcal{Y}$.

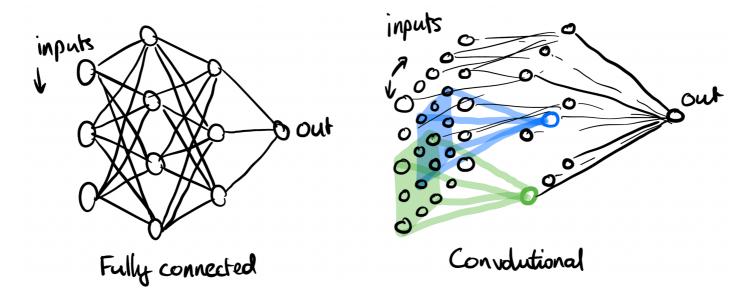
Goal: Find $f: \mathcal{X} \to \mathcal{Y}$, that predicts well for new x.



Neural networks just parameterise functions $f_w(x)$.

Designing a Neural Network

• Inductive bias: **connectivity structure** (architecture)



- Choose network **size** (how *many* neurons)
- Choose **weights**, using *backpropagation*

$$w_{t+1} \leftarrow w_t + \nabla_w \ell(f_w(x_t), y_t)$$

These problems should be tackled together.

Problem Formulation (let's walk before we run)

Predictor is a *single layer* neural network:

$$f(x) = \sum_{m=1}^{M} \varphi(x; Z_m, \theta) w_m$$

Goal is to find:

- Hyperparameters heta Inductive bias.
- The size of the model $\hspace{1cm} M$ Number of neurons.

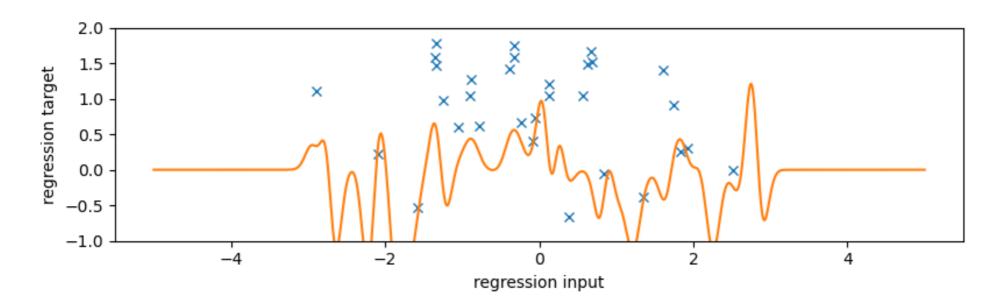
Start by finding clear answers for single-layer NNs.

1. What is wrong with minimising losses?

- 2. Bayesian Model Selection
- 2. Model Selection over Model Size? Or Nonparametrics?
- 3. A principle for selecting size

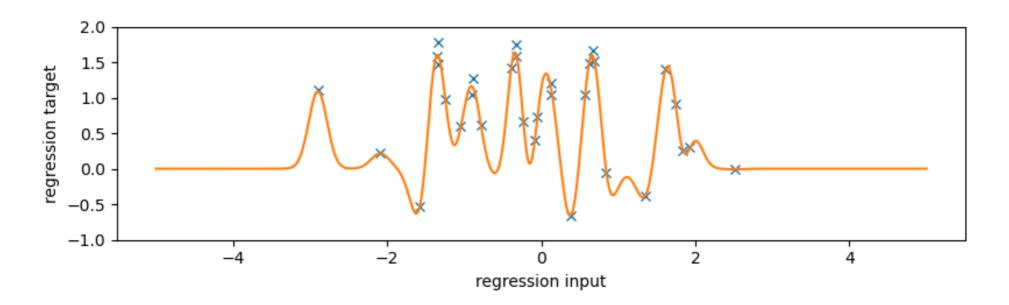
If we train weights W only, given (θ, M) by

$$f^* = \underset{W}{\operatorname{argmin}} \ \operatorname{const} + \sum_n (f(x_n) - y_n))^2$$



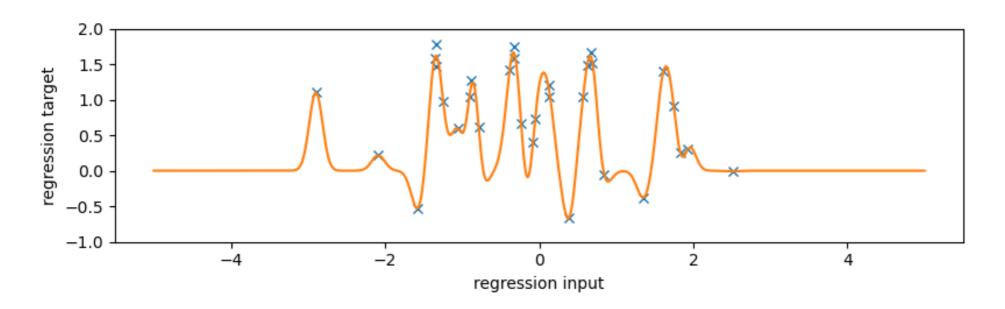
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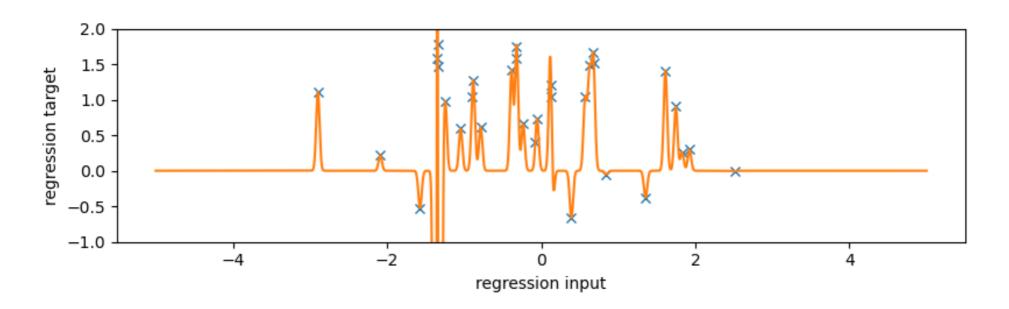
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- Restricting model size never improves loss $\Rightarrow M \to N$
- Narrower basis functions to allow more flexible functions
- "Overfitting"

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- 2. The Bayesian answer to model size: Nonparametrics.
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Let's accept the "large" number of basis functions for now, and solve the overfitting problem.

Bayesian inference is rumoured to be "robust to overfitting".

General procedure: Just do Bayes rule on your unknowns!

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Benefit #1: Uncertainty estimates on your parameters

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$$p(\mathcal{D}|\theta) = \int p(\mathcal{D}|W, \theta)p(W|\theta) \,dW$$

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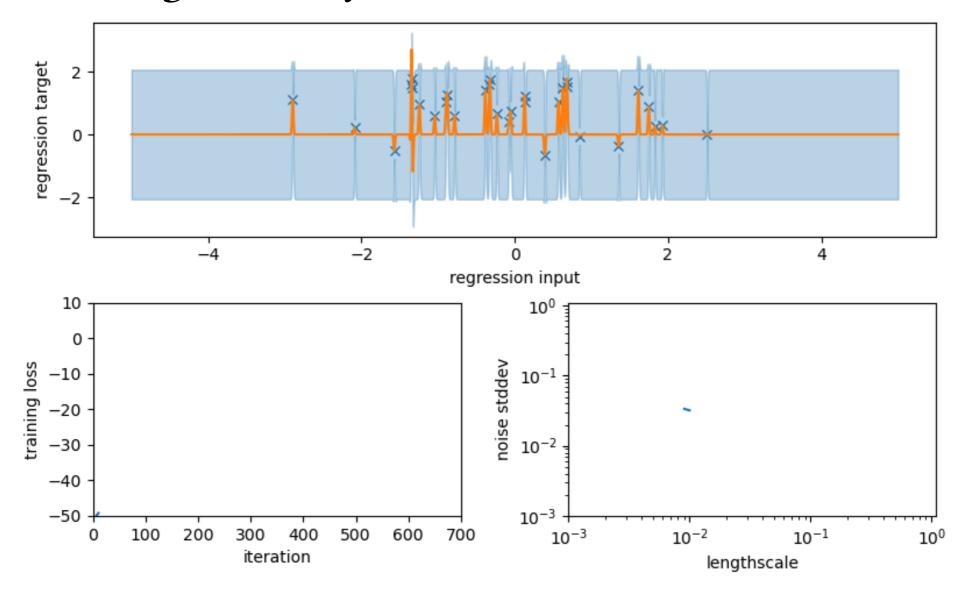
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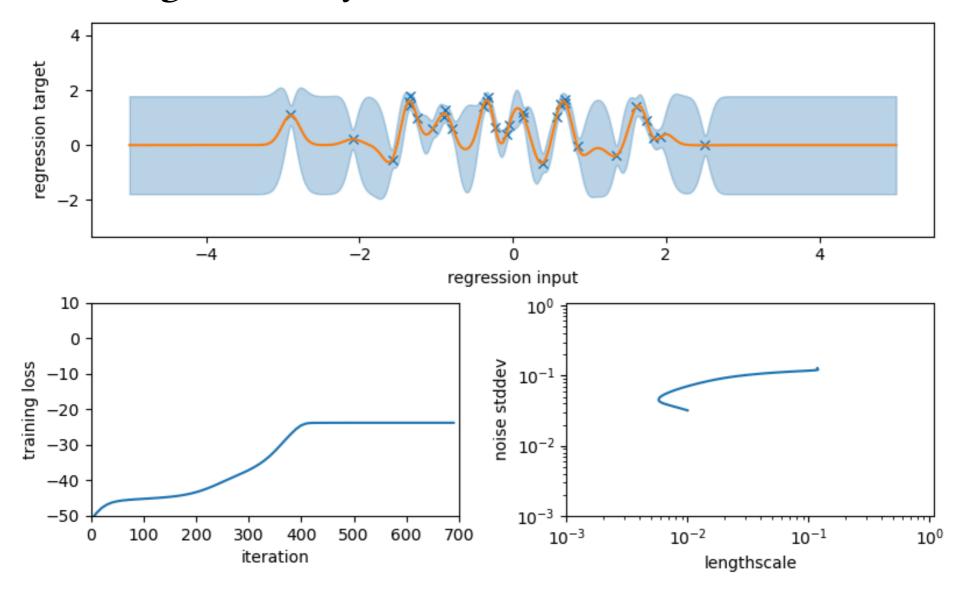
Bayesian computations are often intractable. \Rightarrow Approximating $p(\mathcal{D}|\theta)$ is hard enough, let alone for many different values of θ !

$$\begin{split} \theta^* &= \operatorname*{argmin} \log p(\mathcal{D} \mid \theta) \\ p(W | \mathcal{D}, \theta) &= \frac{p(\mathcal{D} | W, \theta) p(W | \theta)}{p(\mathcal{D} | \theta)} \end{split}$$

The Pragmatic Bayesian Answer



The Pragmatic Bayesian Answer



To Summarise

- We used a "large" number of basis functions.
- We performed Bayesian inference over the weights

$$p(W|\mathcal{D}, \theta) = \frac{p(\mathcal{D}|W, \theta)p(W|\theta)}{p(\mathcal{D}|\theta)}$$

• Estimated **inductive bias** (hyperparams) using Type-II MaxLik

$$\operatorname*{argmin}_{ heta} \log p(\mathcal{D} \mid heta)$$

- You may have noticed this was a Gaussian process.
- Interestingly, form of predictor is still single-layer NN:

$$f(x) = \sum_{m=0}^{N} \varphi(x; \theta, Z_m) w_m$$

$$\boldsymbol{\varphi}(x; \theta, Z_m) = k_{\theta}(x, X_m)$$
 $\boldsymbol{w} = \left(K(X, X) + \sigma^2 I\right)^{-1} \boldsymbol{y}$

Predictor is a *single layer* neural network:

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- Hyperparameters Inductive bias.
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 Control the function.
- The size of the model Number of neurons.

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We stumbled into using "large" models, but *why* do we use nonparametric models?

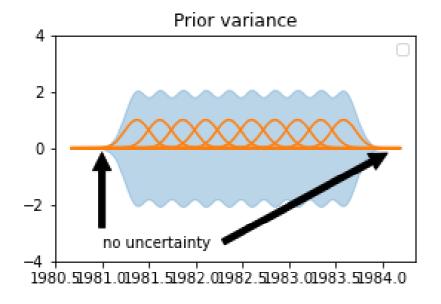
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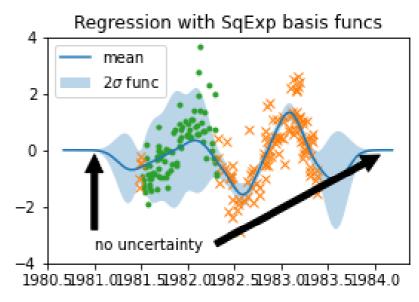
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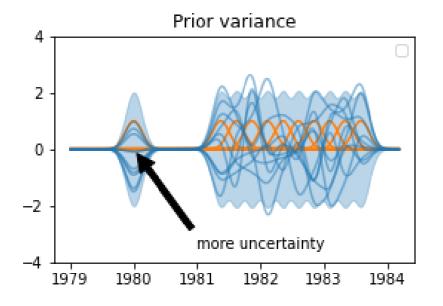
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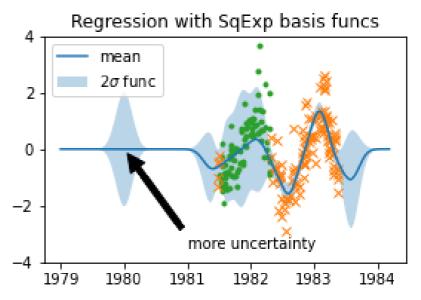




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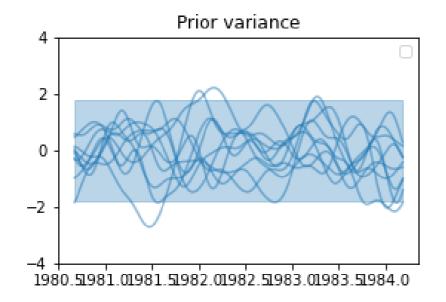
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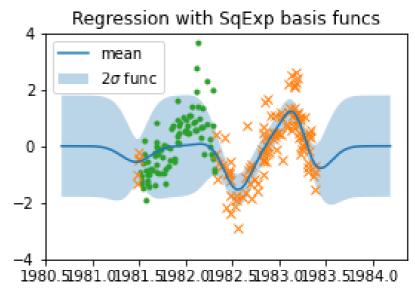




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Can Bayes answer the Size Question?

⚠ Using "infinite" models leads to using N neurons.

This requires memorising the data, which is too many!

? Can we use Bayesian model selection to determine model size?

Or are we stuck with memorising all the data?

We could do model selection over the model size...

$$\begin{split} p(W,\theta,M|\mathcal{D}) &= \frac{p(\mathcal{D}|W,\theta,M)p(W|\theta,M)}{p(\mathcal{D}|\theta,M)} \frac{p(\mathcal{D}|\theta,M)p(\theta)}{p(\mathcal{D})} \\ \theta^*, M^* &= \operatorname*{argmax} \log p(\mathcal{D}|\theta,M) \\ \theta, M \end{split}$$

Bayesian Model Selection of Model Size is BAD

- 1. We would lose the good uncertainty estimation properties!
- 2. If you set up your model correctly,

Bayes doesn't even distinguish between models of different sizes!

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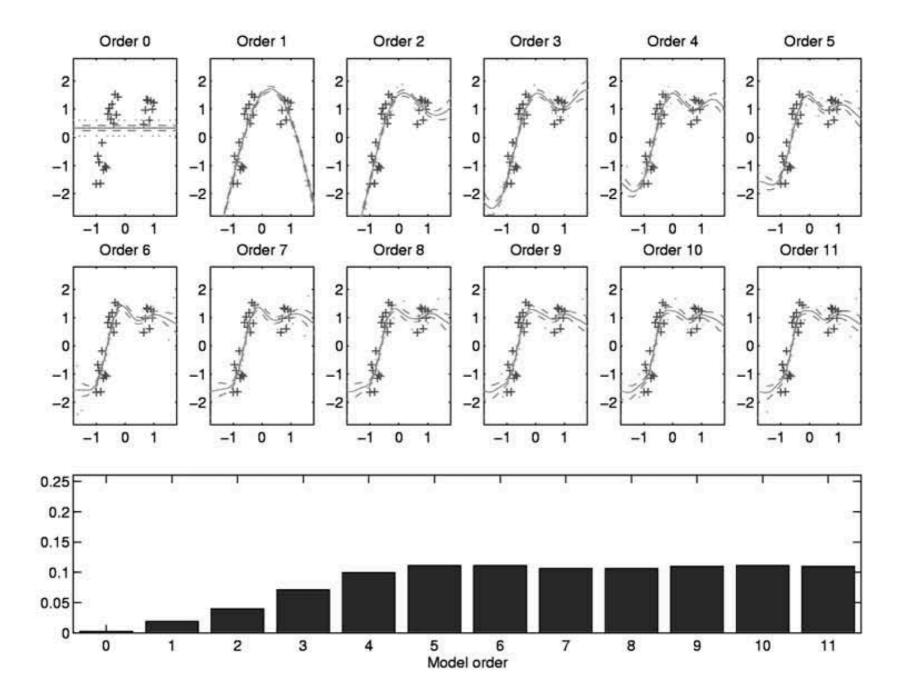
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See *Occam's Razor* (Rasmussen & Ghahramani, 2000). One of my favourite papers.



Bayes selects a nonparametric model!

- Bayes itself is pushing us to use "large" nonparametric models!
- Cannot rely on Bayes to choose a "small" model!

- 1. What is wrong with minimising losses?
- 2. Bayesian Model Selection
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What principle can determine a compressed model size, without removing the benefits of nonparametrics?



Define a nonparametric model, then approximate it with M < N basis funcs.

Step 1: Introduce family of approximate predictors

$$q(f(x)) = \mathcal{N}\left(f(x); \sum_{m=1}^{M} \varphi(x; Z_m, \theta) w_m, \ldots\right)$$

! Predictor is finite neural network!

At least in the mean... Covariance is still nonparametric!

$$\mathrm{ELBO}(\boldsymbol{w}, Z, M, \boldsymbol{\theta}) = \log(\mathcal{D}|\boldsymbol{\theta}) - \mathrm{KL}[q(f) \parallel p(f|\mathcal{D}, \boldsymbol{\theta})]$$

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Step 2: Introduce objective function

$$\mathrm{ELBO}(\boldsymbol{w}, Z, M, \boldsymbol{\theta}) = \log(\mathcal{D}|\boldsymbol{\theta}) - \mathrm{KL}[q(f) \parallel p(f|\mathcal{D}, \boldsymbol{\theta})]$$

Since KL > 0... ELBO $\leq \log p(\mathcal{D}|\theta)$.

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- (i) ELBO is a unified objective for all our questions!
- Optimising w.r.t. w, Z: finds weights (min KL)
- Optimising w.r.t. θ : finds hyperparameters (max $\log p(\mathcal{D}|\theta)$)
- Select M large enough, that more gives diminishing returns!

More basis functions is always better:

$$\mathrm{KL}\big[q_{M+1}(f) \parallel p(f|\mathcal{D},\theta)\big] \leq \mathrm{KL}[q_{M}(f) \parallel p(f|\mathcal{D},\theta)]$$

• In single-layer models, we can also compute an upper bound to the marginal likelihood

$$\begin{split} & \text{ELBO} \leq \log p(D|\theta) \leq \text{EUBO} \\ & \text{$:$} \text{$\text{KL}[q(f) \parallel p(f|\mathcal{D}, \theta)] \leq \text{EUBO} - \text{ELBO} \end{split}$$

• We select M such that

$$EUBO - ELBO \le tolerance$$

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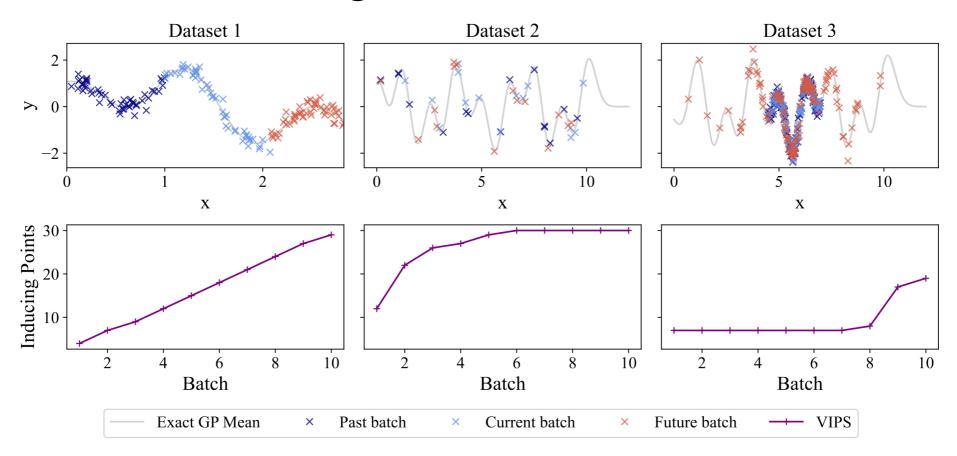
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• Can achieve arbitrarily exact approximation with $M \ll N!$ (Burt et al., 2019; 2020)

(i) Simple Rule, Interesting Adaptive Behaviour!

Continual Learning (Pescador-Barrios et al., 2024)



Growth of neurons depends on *novelty* in data.

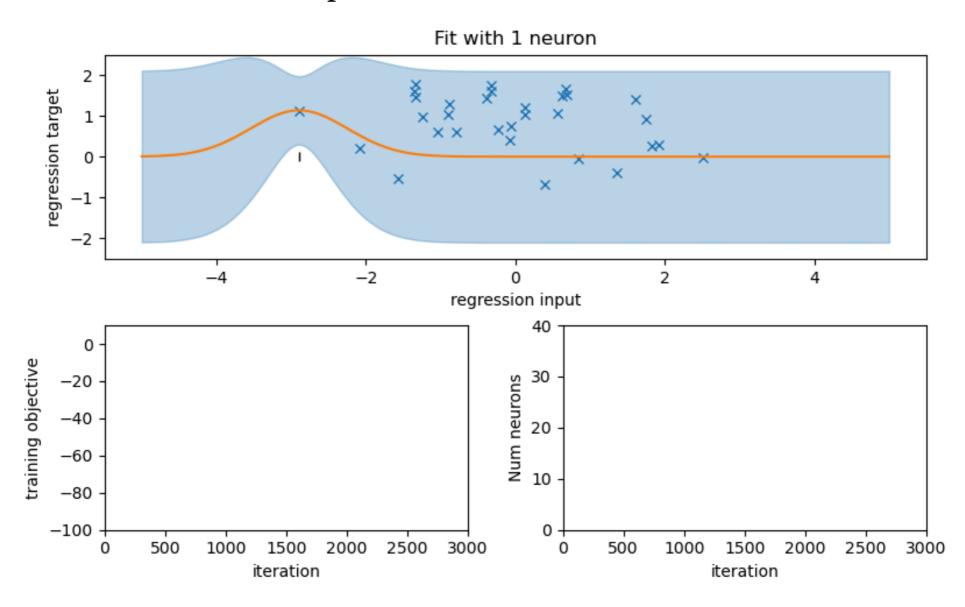
- Input range grows with N (constant novelty)
- Input range constant (diminishing novelty)
- Heavy tailed inputs (occasional novelty)

Growing Neurons, Grokking, Pruning

Number of neurons depends on inductive bias!

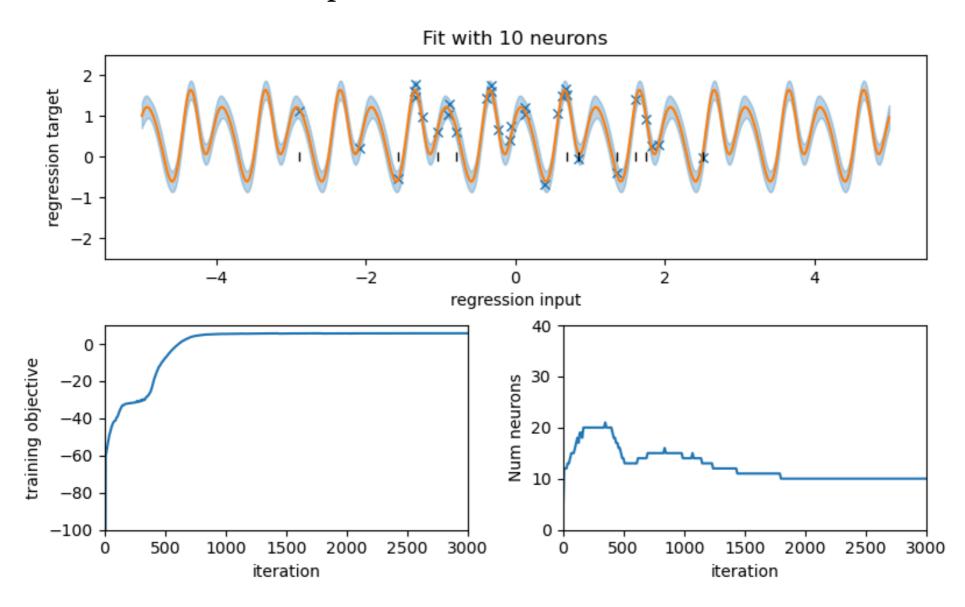
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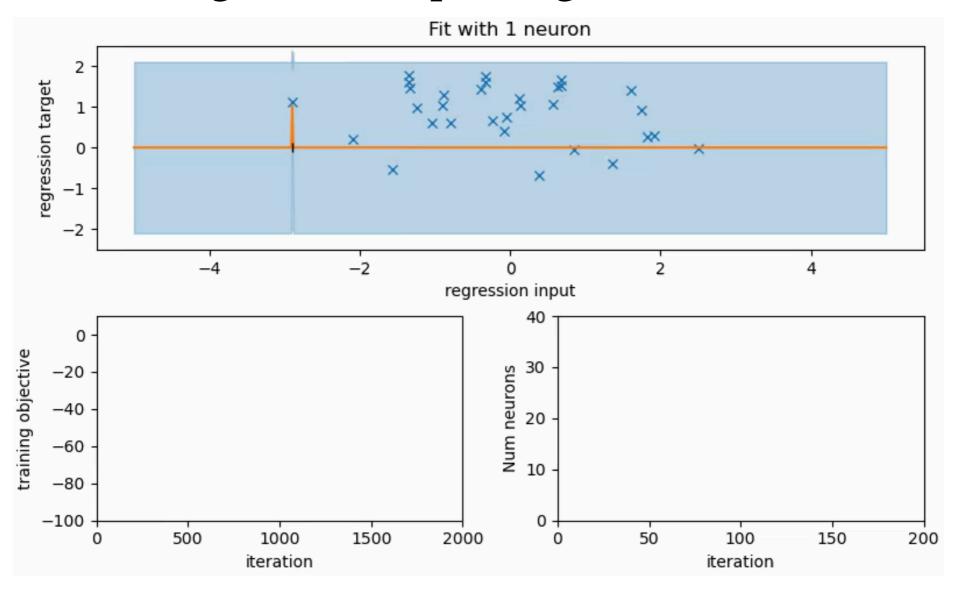


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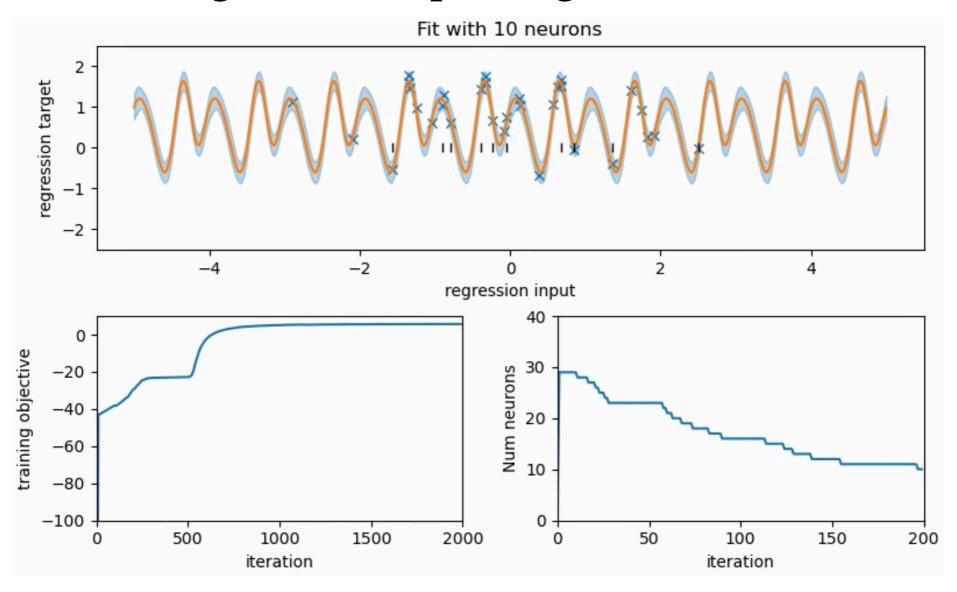
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Memorising first, then pruning



Memorising first, then pruning



- Bayesian model selection for finding *inductive bias*, but not *model size*.
- Approximate GPs can give all benefits of nonparametric models, but with *decoupled model size*.
- Bounding the approximation error, gives a principle for determining model size.
- This leads to *adaptive* behaviour of the size of the network, to the problem.

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We saw:

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We can have our cake and eat it

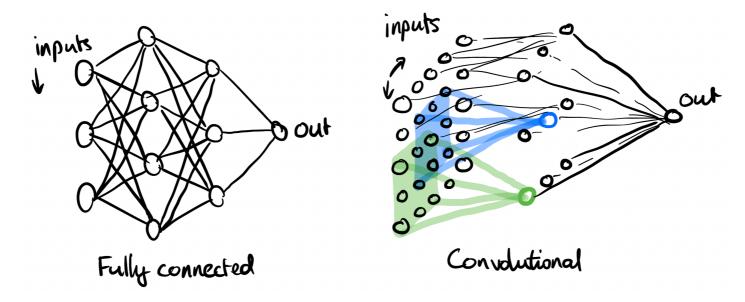
We can *define* an infinite-sized model, but near-perfectly approximate it with *just* the right amount of computational resources!

6*

New procedures for training neural networks!

Can we automatically find:

• Inductive bias / connectivity structure / architecture



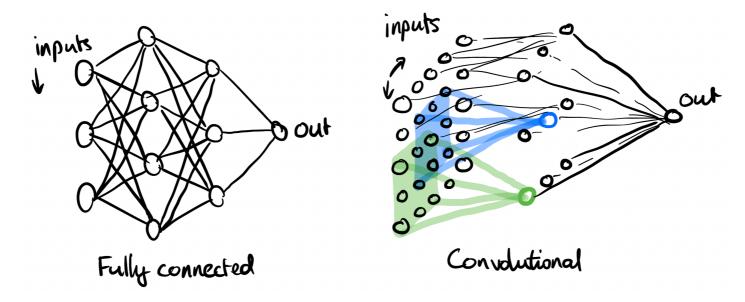
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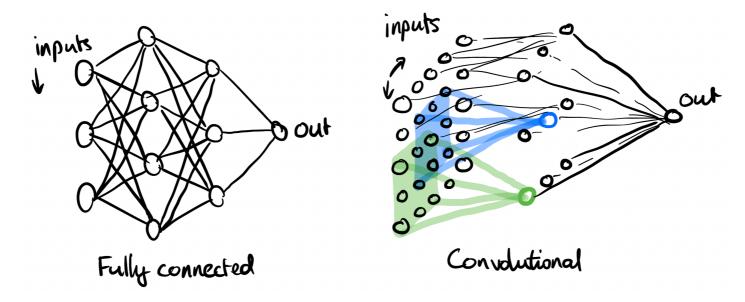
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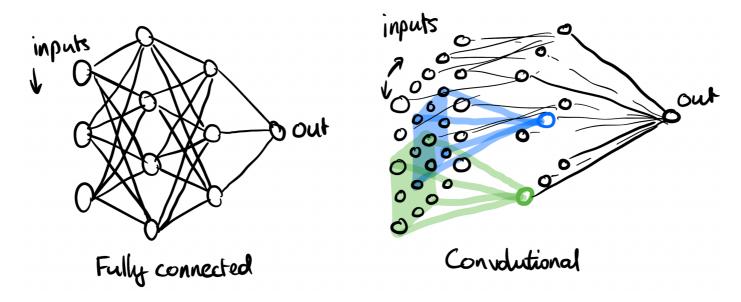
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More efficient, more adaptive, more automatic!

Papers

Gaussian processes:

- Background of some ideas described in my thesis (van der Wilk, 2019)
- Proof of accuracy of variational approximation (basis for when to stop adding inducing variables / basis functions)
 (Burt et al., 2019; 2020)
- Adaptive model size for continual learning (Pescador-Barrios et al., 2024)
- Overall narrative of this talk (online soon!)

Bayesian Model Selection in Neural Networks:

- Bayesian Model Selection (Laplace approximation) *recovers* ResNets, without explicit human design (Ouderaa et al., 2023)
- See more by Tycho van der Ouderaa!

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