Decision Theory

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Taking Action

What do you want to **use** a posterior for?

- Posterior is a belief over what will happen.
- You use beliefs to take action / make decisions.
- Uncertainty helps you to balance risk, reward, and new information.

We **learn** about the world,
so we can **act** in it,
to get outcomes that we **desire**.

Axiomatisation

As with uncertain reasoning, we can get maths to tell us how to make rational decisions, if we write down our wishlist of axioms.
Paraphrased axioms:

1. Comparing probability distributions A and B over outcomes, you either prefer one over the other, or you have no preference.
2. If you prefer A over B, and B over C, then you prefer A over C.
3. Reduction of compound lotteries: If you prefer A over B, then you also prefer a 50% chance of getting A over a 50% chance of getting B, with a 50% chance of getting C.

See Ch 16 "Artificial Intelligence: A Modern Approach", Russell [1]

Von Neumann-Morgenstern Utility Theorem

- If you behave according to these axioms, then there exists a **utility function** that you maximise.
- In other words, if you believe you want to act according to those axioms, you should follow the principle of **Maximum Expected Utility**.

Principle of Maximum Expected Utility

1. Define a utility function $U : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$
   $\mathcal{X}$: Space of outcomes. $\mathcal{A}$: Space of actions. Quantifies how good an outcome is, if you take a particular action.
2. Compute expected utility for your actions.
   Your beliefs are a distribution over outcomes given action.

$$u(a) = \mathbb{E}_{p(x|\mathcal{D},a)}[U(x,a)] \tag{1}$$

3. At the time of decision making, choose action which maximises expected utility.

$$a^* = \underset{a}{\operatorname{argmax}}\, u(a) \tag{2}$$

Exercise: Why do we predict the mean?

Your regression model gives you $p(y|X, \mathbf{y})$. You need to give a "best guess" y_p, and your utility will be $U(y, y_p) = -(y - y_p)^2$.

⏩ Find y_p in terms of properties of $p(y|X, \mathbf{y})$.

Solution: We want $\frac{\mathrm{d}}{\mathrm{d}y_p} u(y_p) = \frac{\mathrm{d}}{\mathrm{d}y_p} \mathbb{E}_{p(y|X,\mathbf{y})}\big[U(y, y_p)\big] = 0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y_p} u(y_p) &= \frac{\mathrm{d}}{\mathrm{d}y_p} \int_{-\infty}^{\infty} p(y|X, \mathbf{y}) U(y, y_p) \mathrm{d}y \\
&= -\int p(y|X, \mathbf{y}) \frac{\partial}{\partial y_p} (y - y_p)^2 \mathrm{d}y && \text{Leibniz Integral Rule} \\
&= 2 \int p(y|X, \mathbf{y})(y - y_p) \mathrm{d}y \\
0 &= 2\mathbb{E}_{p(y|X,\mathbf{y})}[y] - 2y_p \\
y_p &= \mathbb{E}_{p(y|X,\mathbf{y})}[y], \qquad \text{i.e. } y_p \text{ should be the } \textbf{mean}!
\end{aligned}$$

Exercise: Absolute error loss.

Your regression model gives you $p(y|X, \mathbf{y})$. You need to give a "best guess" y_p, and your utility will be $U(y, y_p) = -|y - y_p|$.

⏩ Find y_p in terms of properties of $p(y|X, \mathbf{y})$.

Solution: We want $\frac{\mathrm{d}}{\mathrm{d}y_p} u(y_p) = \frac{\mathrm{d}}{\mathrm{d}y_p} \mathbb{E}_{p(y|X,\mathbf{y})}\big[U(y, y_p)\big] = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}y_p} u(y_p) = \frac{\mathrm{d}}{\mathrm{d}y_p} \int_{-\infty}^{\infty} p(y|X, \mathbf{y}) U(y, y_p) \mathrm{d}y$$

$$= \int p(y|X, \mathbf{y}) \frac{\partial}{\partial y_p} U(y, y_p) \mathrm{d}y \qquad \text{Leibniz Integral Rule}$$

$$= \int p(y|X, \mathbf{y}) \begin{Bmatrix} -1 & \text{if } y > y_p \\ 1 & \text{if } y < y_p \end{Bmatrix} \mathrm{d}y$$

$$= -\int_{-\infty}^{y_p} p(y|X, \mathbf{y}) \mathrm{d}y + \int_{y_p}^{\infty} p(y|X, \mathbf{y}) \mathrm{d}y$$

$$0 = -F(y_p) + (1 - F(y_p)) \qquad F \text{ is cdf}$$

$$\implies y_p = F^{-1}(0.5)\,, \qquad \text{i.e. } y_p \text{ should be the } \textbf{median}!$$

Reject option (todo)

Applications and Applicability

Hugely influential theory:

- Philosophy: Utilitarianism effective altruism
- Psychology: Rational behaviour as a model for human behaviour.
- Economics: How to optimise investments.
- Game theory: Analyse implications of rational choice.
- Politics: Voting systems (Arrow's impossibility theorems).

Discussions about desirability/realism of axioms:

- Naïve application leads to bad behaviour (St Petersburg Paradox)
- Can you express desires as utility functions?
- The Law of Perverse Optimization: Whenever a desired behaviour is formulated as a utility, optimising for the utility will give you behaviour you didn't want. (AI Paperclip factory)
- Human's don't behave according to the axioms. But perhaps for good reason?
- Bounded rationality: It assumes you can compute the optimal decision.

Conclusion

- Decision theory is easy (in principle): See slide 5.
- The implications of decision theory are vast, but sadly we don't have time for this.
- Next: Things that are easy in principle, are often very hard in practice.

Recommended reference: MacKay [**?**] chapter 36.
Further reading: Russell [1] chapter 16.

References I

[1] D. J. C. MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003.

[2] S. J. Russell. *Artificial intelligence a modern approach*. Pearson Education, Inc., 2010.