Building Probabilistic Models

Mark van der Wilk

Department of Computing
Imperial College London

@markvanderwilk
m.vdwilk@imperial.ac.uk

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$$P(C|L) = \sum_i P(C|L, I = i)\, P(I = i)$$

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How to systematically approach
a probabilistic modelling problem?

Mathematical Modelling

Often, we can pose a mathematical model of a phenomenon:

- Reflection (Phong model, different symbol convention)

$$I_{\mathrm{p}} = k_{\mathrm{a}} i_{\mathrm{a}} + \sum_{m \in \text{lights}} \left(k_{\mathrm{d}} (\hat{L}_m \cdot \hat{N}) i_{m,\mathrm{d}} + k_{\mathrm{s}} (\hat{R}_m \cdot \hat{V})^{\alpha} i_{m,\mathrm{s}}\right)$$

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So if you are **given** some quantities, you can make a prediction about another.

From Mathematical Models to Probabilistic Models

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E.g. from Newton's laws:

$$s_t = \frac{1}{2}\frac{F}{m}t^2 + v_0 t \tag{2}$$

$$p(s_t|v_0, m, F) = \delta(s_t - \frac{1}{2}\frac{F}{m}t^2 + v_0 t) \tag{3}$$

(Remember: $\int_{\mathcal{R}} \delta(\mathbf{x} - \mathbf{y}) \mathrm{d}\mathbf{x} = 1$ if $\mathbf{y} \in \mathcal{R}$, 0 otherwise.)

Probabilistic Models: Uncertain Quantities

Given a mathematical model.

$$p(s_t|v_0, m, F) = \delta(s_t - \frac{1}{2}\frac{F}{m}t^2 + v_0 t) \tag{4}$$

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$$p(v_0, m, F) = \mathcal{N}(v_0; \mu_v, 1.0)\delta(m - 1.0)\mathcal{N}(F; \mu_F, 0.1) \tag{5}$$

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We can find how our uncertainty over the initial velocity v_0 changes by finding $p(v_0|s_t)$!

Probabilistic Models: Uncertain Relationships

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- If we did an initial value experiment, would we really measure **exactly** the predicted value?
- Adding uncertainty makes predictions more **realistic**, by allowing **errors**.

$$p(s_t|v_0, m, F) = \mathcal{N}\left(s_t; \frac{1}{2}\frac{F}{m}t^2 + v_0 t, \sigma^2\right) \tag{6}$$

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$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} \qquad \text{or} \qquad p(x|y) = \int \frac{p(x, y, z)}{p(y)} \mathrm{d}z \tag{8}$$

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- Observe a variable? Conditioning (i.e. divide and renormalise).
- Not interested in a variable? Marginalise.

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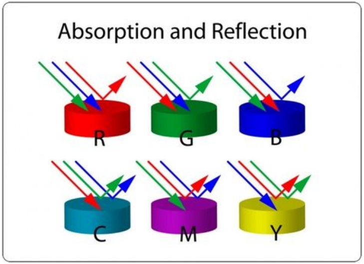
3. Determine which variables it depends on, and choose a sensible conditional distribution.
4. Repeat previous step for one of the variables that you conditioned on.

Example: Lighting

Step 1: Identify all variables:

- Object colour C.
- Reflected light L.
- Illumination I.

Joint: $p(C, L, I)$.



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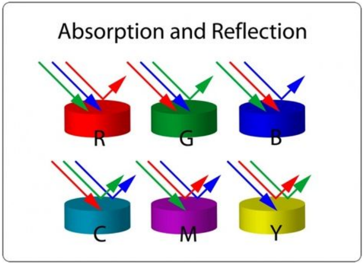

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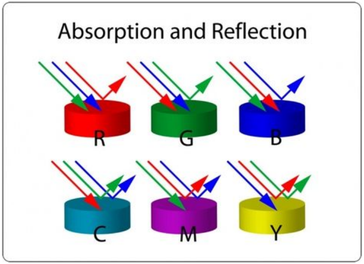

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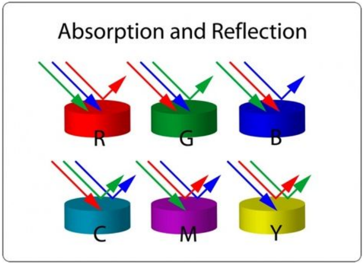

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While we can use our knowledge for choosing $p(L|C, I)$, we need to choose subjective priors for $p(C)$ and $p(I)$.

Finding the right posterior

- Now that we have the joint, how do we find $p(C|L)$?
- Remember: We need to find it in terms of the conditional distributions **which we can actually evaluate**.
- This is why starting with the joint is such a good idea! Given the definitions from the previous slide, we can evaluate it!

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We can take many different routes!

$$p(C|L) = \frac{p(L|C)\, p(C)}{p(L)} = \frac{\left[\sum_I p(L|C, I) p(I)\right] p(C)}{p(L)} \tag{11}$$

Many roads lead to Rome, but starting from the joint highlights assumptions

Example: Burglars, Earthquakes, and Alarms

"Fred lives in Los Angeles and commutes 60 miles to work. Whilst at work, he receives a phone-call from his neighbour saying that Fred's burglar alarm is ringing. What is the probability that there was a burglar in his house today? While driving home to investigate, Fred hears on the radio that there was a small earthquake that day near his home. 'Oh', he says, feeling relieved, 'it was probably the earthquake that set off the alarm'. What is the probability that there was a burglar in his house?" (MacKay, 2003, §21.1)

Q: How does the joint factorise? What conditionals should we define?

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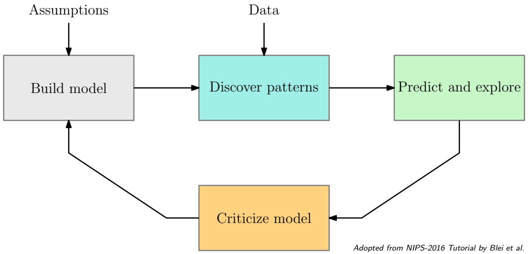
- After selecting sensible conditionals, we have the joint.

Probabilistic Pipeline

If your assumptions are good/correct, inference will give accurate results and good predictions.



Adopted from NIPS-2016 Tutorial by Blei et al.

- Good models generate data that is similar to the data we observe.
- Predict and explore: Sample from the prior, assess predictions.
- **Criticize/revise the model**.

Building a Probabilistic Model: ML Approach

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- No known latent variables to use for creating a model.
- We mainly want good prediction!

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All we really need is $p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})$, so we can find the predictive distribution:

$$p(\mathcal{D}_{\text{future}}|\mathcal{D}_{\text{observed}}) = \frac{p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})}{p(\mathcal{D}_{\text{observed}})} \,. \tag{13}$$

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- May not have a direct physical basis initially, but can turn out to be interpretable after training.
- Induces correlations between data, that can help to predict

$$p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}) = \int p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}, \mathbf{z})\mathrm{d}\mathbf{z} \tag{16}$$

Example: Linear Basis Function Regression

Linear regression falls under this!

$$p(\mathbf{y}, \mathbf{y}^*) = \int \prod_i p(y_i|\boldsymbol{\theta})p(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} \tag{17}$$

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Similar, VAE (which we will also discuss).

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The ML philosophy: if you predict well, you understand.

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Reading & exercises:

- Chapter 3 (MacKay, 2003).
- Exercise: the burglar alarm (MacKay, 2003, ch.21)
- Exercise: bent coin (MacKay, 2003, §3.2)
- Exercise: legal evidence (MacKay, 2003, §3.4)

References I

MacKay, D. J. C. (2003). *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press.