

Building Probabilistic Models

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$$P(C|L) = \sum_i P(C|L, I = i) P(I = i)$$
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How to systematically approach
a probabilistic modelling problem?

Mathematical Modelling

Often, we can pose a mathematical model of a phenomenon:

- Reflection (Phong model, different symbol convention)

$$I_p = k_a i_a + \sum_{m \in \text{lights}} (k_d (\hat{L}_m \cdot \hat{N}) i_{m,d} + k_s (\hat{R}_m \cdot \hat{V})^\alpha i_{m,s})$$

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So if you are **given** some quantities, you can make a prediction about another.

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E.g. from Newton's laws:

$$s_t = \frac{1}{2} \frac{F}{m} t^2 + v_0 t \quad (2)$$

$$p(s_t | v_0, m, F) = \delta(s_t - \frac{1}{2} \frac{F}{m} t^2 + v_0 t) \quad (3)$$

(Remember: $\int_{\mathcal{R}} \delta(\mathbf{x} - \mathbf{y}) d\mathbf{x} = 1$ if $\mathbf{y} \in \mathcal{R}$, 0 otherwise.)

Probabilistic Models: Uncertain Quantities

Given a mathematical model.

$$p(s_t|v_0, m, F) = \delta(s_t - \frac{1}{2} \frac{F}{m} t^2 + v_0 t) \quad (4)$$

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$$p(v_0, m, F) = \mathcal{N}(v_0; \mu_v, 1.0) \delta(m - 1.0) \mathcal{N}(F; \mu_F, 0.1) \quad (5)$$

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We can find how our uncertainty over the initial velocity v_0 changes by finding $p(v_0|s_t)$!

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- ▶ If we did an initial value experiment, would we really measure **exactly** the predicted value?
- ▶ Adding uncertainty makes predictions more **realistic**, by allowing **errors**.

$$p(s_t|v_0, m, F) = \mathcal{N}\left(s_t; \frac{1}{2} \frac{F}{m} t^2 + v_0 t, \sigma^2\right) \quad (6)$$

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Any question we may want to answer corresponds to finding:

$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} \quad \text{or} \quad p(x|y) = \int \frac{p(x, y, z)}{p(y)} dz \tag{8}$$

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- ▶ Observe a variable? Conditioning (i.e. divide and renormalise).
- ▶ Not interested in a variable? Marginalise.

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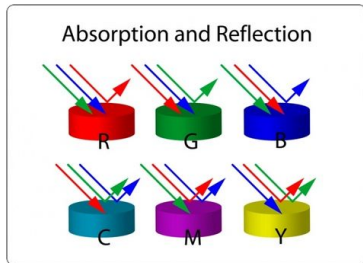
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4. Repeat previous step for one of the variables that you conditioned on.

Example: Lighting

Step 1: Identify all variables:

- ▶ Object colour C .
- ▶ Reflected light L .
- ▶ Illumination I .

Joint: $p(C, L, I)$.



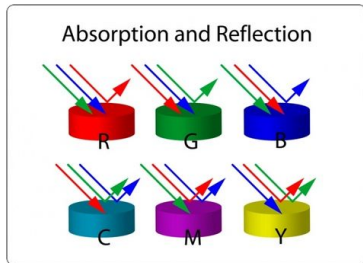
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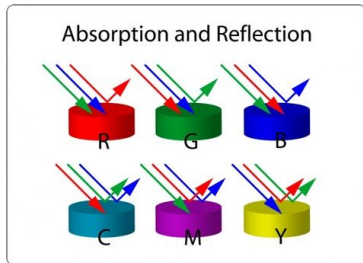


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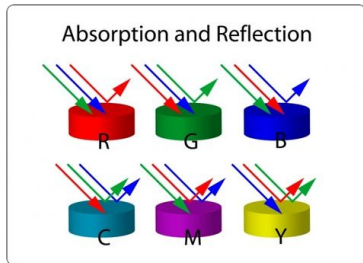
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While we can use our knowledge for choosing $p(L|C, I)$, we need to choose subjective priors for $p(C)$ and $p(I)$.

Finding the right posterior

- ▶ Now that we have the joint, how do we find $p(C|L)$?
- ▶ Remember: We need to find it in terms of the conditional distributions **which we can actually evaluate**.
- ▶ This is why starting with the joint is such a good idea! Given the definitions from the previous slide, we can evaluate it!

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We can take many different routes!

$$p(C|L) = \frac{p(L|C) p(C)}{p(L)} = \frac{[\sum_I p(L|C, I) p(I)] p(C)}{p(L)} \quad (11)$$

Many roads lead to Rome, but starting from the joint highlights assumptions

Example: Burglars, Earthquakes, and Alarms

“Fred lives in Los Angeles and commutes 60 miles to work. Whilst at work, he receives a phone-call from his neighbour saying that Fred’s burglar alarm is ringing. What is the probability that there was a burglar in his house today? While driving home to investigate, Fred hears on the radio that there was a small earthquake that day near his home. ‘Oh’, he says, feeling relieved, ‘it was probably the earthquake that set off the alarm’. What is the probability that there was a burglar in his house?” (MacKay, 2003, §21.1)

Q: How does the joint factorise? What conditionals should we define?

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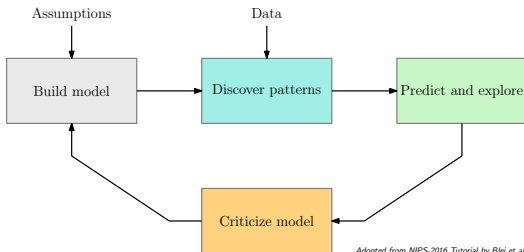
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- ▶ After selecting sensible conditionals, we have the joint.

Probabilistic Pipeline

If your assumptions are good/correct, inference will give accurate results and good predictions.



- ▶ Good models **generate** data that is similar to the data we observe.
- ▶ **Predict and explore**: Sample from the prior, assess predictions.
- ▶ **Criticize/revise the model**.

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All we really need is $p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})$, so we can find the predictive distribution:

$$p(\mathcal{D}_{\text{future}} | \mathcal{D}_{\text{observed}}) = \frac{p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})}{p(\mathcal{D}_{\text{observed}})}. \quad (13)$$

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- ▶ Induces correlations between data, that can help to predict

$$p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}) = \int p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}, \mathbf{z}) d\mathbf{z} \quad (16)$$

Example: Linear Basis Function Regression

Linear regression falls under this!

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Similar, VAE (which we will also discuss).

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The ML philosophy: if you predict well, you understand.

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Reading & exercises:

- ▶ Chapter 3 (MacKay, 2003).
- ▶ Exercise: the burglar alarm (MacKay, 2003, ch.21)
- ▶ Exercise: bent coin (MacKay, 2003, §3.2)
- ▶ Exercise: legal evidence (MacKay, 2003, §3.4)

References I

MacKay, D. J. C. (2003). *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press.