


# Decision Theory

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# Taking Action

What do you want to **use** a posterior for?

- ▶ Posterior is a belief over what will happen.
- ▶ You use beliefs to take action / make decisions.
- ▶ Uncertainty helps you to balance risk, reward, and new information.

We **learn** about the world,  
so we can **act** in it,  
to get outcomes that we **desire**.

# Axiomatisation

As with uncertain reasoning, we can get maths to tell us how to make rational decisions, if we write down our wishlist of axioms.

Paraphrased axioms:

1. Comparing probability distributions A and B over outcomes, you either prefer one over the other, or you have no preference.
2. If you prefer A over B, and B over C, then you prefer A over C.
3. Reduction of compound lotteries: If you prefer A over B, then you also prefer a 50% chance of getting A over a 50% chance of getting B, with a 50% chance of getting C.

See Ch 16 “Artificial Intelligence: A Modern Approach”, Russell [1]

# Von Neumann-Morgenstern Utility Theorem

- ▶ If you behave according to these axioms, then there exists a **utility function** that you maximise.
- ▶ In other words, if you believe you want to act according to those axioms, you should follow the principle of **Maximum Expected Utility**.

## Principle of Maximum Expected Utility

1. Define a utility function  $U : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$   
 $\mathcal{X}$ : Space of outcomes.  $\mathcal{A}$ : Space of actions. Quantifies how good an outcome is, if you take a particular action.
2. Compute expected utility for your actions.  
Your beliefs are a distribution over outcomes given action.

$$u(a) = \mathbb{E}_{p(x|\mathcal{D},a)}[U(x,a)] \quad (1)$$

3. At the time of decision making, choose action which maximises expected utility.

$$a^* = \operatorname{argmax}_a u(a) \quad (2)$$

## Exercise: Why do we predict the mean?

Your regression model gives you  $p(y|X, \mathbf{y})$ . You need to give a “best guess”  $y_p$ , and your utility will be  $U(y, y_p) = -(y - y_p)^2$ .

► Find  $y_p$  in terms of properties of  $p(y|X, \mathbf{y})$ .

Solution: We want  $\frac{d}{dy_p} u(y_p) = \frac{d}{dy_p} \mathbb{E}_{p(y|X, \mathbf{y})} [U(y, y_p)] = 0$ .

$$\frac{d}{dy_p} u(y_p) = \frac{d}{dy_p} \int_{-\infty}^{\infty} p(y|X, \mathbf{y}) U(y, y_p) dy$$

$$= - \int p(y|X, \mathbf{y}) \frac{\partial}{\partial y_p} (y - y_p)^2 dy$$

Leibniz Integral Rule

$$= 2 \int p(y|X, \mathbf{y}) (y - y_p) dy$$

$$0 = 2 \mathbb{E}_{p(y|X, \mathbf{y})} [y] - 2y_p$$

$$y_p = \mathbb{E}_{p(y|X, \mathbf{y})} [y], \quad \text{i.e. } y_p \text{ should be the mean!}$$

## Exercise: Absolute error loss.

Your regression model gives you  $p(y|X, \mathbf{y})$ . You need to give a “best guess”  $y_p$ , and your utility will be  $U(y, y_p) = -|y - y_p|$ .

► Find  $y_p$  in terms of properties of  $p(y|X, \mathbf{y})$ .

Solution: We want  $\frac{d}{dy_p} u(y_p) = \frac{d}{dy_p} \mathbb{E}_{p(y|X, \mathbf{y})} [U(y, y_p)] = 0$ .

$$\frac{d}{dy_p} u(y_p) = \frac{d}{dy_p} \int_{-\infty}^{\infty} p(y|X, \mathbf{y}) U(y, y_p) dy$$

$$= \int p(y|X, \mathbf{y}) \frac{\partial}{\partial y_p} U(y, y_p) dy$$

Leibniz Integral Rule

$$= \int p(y|X, \mathbf{y}) \begin{cases} -1 & \text{if } y > y_p \\ 1 & \text{if } y < y_p \end{cases} dy$$

$$= - \int_{-\infty}^{y_p} p(y|X, \mathbf{y}) dy + \int_{y_p}^{\infty} p(y|X, \mathbf{y}) dy$$

$$0 = -F(y_p) + (1 - F(y_p))$$

$F$  is cdf

$$\implies y_p = F^{-1}(0.5), \quad \text{i.e. } y_p \text{ should be the median!}$$

# Applications and Applicability

Hugely influential theory:

- ▶ Philosophy: Utilitarianism effective altruism
- ▶ Psychology: Rational behaviour as a model for human behaviour.
- ▶ Economics: How to optimise investments.
- ▶ Game theory: Analyse implications of rational choice.
- ▶ Politics: Voting systems (Arrow's impossibility theorems).

Discussions about desirability/realism of axioms:

- ▶ Naïve application leads to bad behaviour (St Petersburg Paradox)
- ▶ Can you express desires as utility functions?
- ▶ The Law of Perverse Optimization: Whenever a desired behaviour is formulated as a utility, optimising for the utility will give you behaviour you didn't want. (AI Paperclip factory)
- ▶ Human's don't behave according to the axioms. But perhaps for good reason?
- ▶ Bounded rationality: It assumes you can compute the optimal decision.



# Conclusion

- ▶ Decision theory is easy (in principle): See slide 5.
- ▶ The implications of decision theory are vast, but sadly we don't have time for this.
- ▶ Next: Things that are easy in principle, are often very hard in practice.

# References I

- [1] S. J. Russell. *Artificial intelligence a modern approach*. Pearson Education, Inc., 2010.