Decision Theory

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Taking Action

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We **learn** about the world,
so we can **act** in it,
to get outcomes that we **desire**.

Axiomatisation

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2. If you prefer A over B, and B over C, then you prefer A over C.
3. Reduction of compound lotteries: If you prefer A over B, then you also prefer a 50% chance of getting A over a 50% chance of getting B, with a 50% chance of getting C.

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See Ch 16 “Artificial Intelligence: A Modern Approach”, Russell [2]

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Principle of Maximum Expected Utility

1. Define a utility function $U : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$
   $\mathcal{X}$: Space of outcomes. $\mathcal{A}$: Space of actions. Quantifies how good an outcome is, if you take a particular action.

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2. Compute expected utility for your actions.
   Your beliefs are a distribution over outcomes given action.

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3. At the time of decision making, choose action which maximises expected utility.

$$a^* = \operatorname*{argmax}_a u(a) \tag{2}$$

Exercise: Why do we predict the mean?

Your regression model gives you $p(y|X, \mathbf{y})$. You need to give a "best guess" y_p, and your utility will be $U(y, y_p) = -(y - y_p)^2$.

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y_p &= \mathbb{E}_{p(y|X,\mathbf{y})}[y], \qquad \text{i.e. } y_p \text{ should be the } \textbf{mean}!
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$$0 = -F(y_p) + (1 - F(y_p)) \qquad F \text{ is cdf}$$

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$\implies y_p = F^{-1}(0.5)$, i.e. y_p should be the **median**!

Reject option (todo)

Applications and Applicability

Hugely influential theory:

- Philosophy: Utilitarianism effective altruism
- Psychology: Rational behaviour as a model for human behaviour.
- Economics: How to optimise investments.
- Game theory: Analyse implications of rational choice.
- Politics: Voting systems (Arrow's impossibility theorems).

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Discussions about desirability/realism of axioms:

- Naïve application leads to bad behaviour (St Petersburg Paradox)
- Can you express desires as utility functions?
- The Law of Perverse Optimization: Whenever a desired behaviour is formulated as a utility, optimising for the utility will give you behaviour you didn't want. (AI Paperclip factory)
- Human's don't behave according to the axioms. But perhaps for good reason?
- Bounded rationality: It assumes you can compute the optimal decision.

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Recommended reference: MacKay [1] chapter 36.
Further reading: Russell [2] chapter 16.

References I

[1] D. J. C. MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003.

[2] S. J. Russell. *Artificial intelligence a modern approach*. Pearson Education, Inc., 2010.