

# Building Probabilistic Models

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# Questions

Last time, I gave examples of probabilistic reasoning, e.g. colour perception:

$$P(C|L) = \sum_i P(C|L, I = i) P(I = i)$$
$$P(C|L, I = i) = \frac{P(L|C, I = i)P(C)}{P(L|I = i)}$$

- ▶ But why this structure?
- ▶ What assumptions were made?
- ▶ How should assumptions be expressed and communicated?
- ▶ How do we manipulate distributions to the form we want?

How to systematically approach  
a probabilistic modelling problem?

# Mathematical Modelling

Often, we can pose a mathematical model of a phenomenon:

- Reflection (Phong model, different symbol convention)

$$I_p = k_a i_a + \sum_{m \in \text{lights}} (k_d (\hat{L}_m \cdot \hat{N}) i_{m,d} + k_s (\hat{R}_m \cdot \hat{V})^\alpha i_{m,s})$$

- Movement of an object under gravity (Newton's laws)

$$s_t = \frac{1}{2} \frac{F}{m} t^2 + v_0 t \quad (1)$$

So if you are **given** some quantities, you can make a prediction about another.

# From Mathematical Models to Probabilistic Models

- ▶ A mathematical model expresses deterministic relationships.
- ▶ A probabilistic model expresses relationships with uncertainty.
- ▶ Often, probabilistic models are specified starting with a mathematical model.
- ▶ Mathematical relationships can help specify conditional distributions.

Mathematical models are a *special case* of deterministic models.

Probability can still express certainty!

E.g. from Newton's laws:

$$s_t = \frac{1}{2} \frac{F}{m} t^2 + v_0 t \quad (2)$$

$$p(s_t | v_0, m, F) = \delta(s_t - \frac{1}{2} \frac{F}{m} t^2 + v_0 t) \quad (3)$$

(Remember:  $\int_{\mathcal{R}} \delta(\mathbf{x} - \mathbf{y}) d\mathbf{x} = 1$  if  $\mathbf{y} \in \mathcal{R}$ , 0 otherwise.)

# Probabilistic Models: Uncertain Quantities

Given a mathematical model.

$$p(s_t|v_0, m, F) = \delta(s_t - \frac{1}{2} \frac{F}{m} t^2 + v_0 t) \quad (4)$$

A certain relationship like this can be used to work back to uncertain quantities. Imagine we are uncertain about certain quantities:

$$p(v_0, m, F) = \mathcal{N}(v_0; \mu_v, 1.0) \delta(m - 1.0) \mathcal{N}(F; \mu_F, 0.1) \quad (5)$$

We can find how our uncertainty over the initial velocity  $v_0$  changes by finding  $p(v_0|s_t)$ !

# Probabilistic Models: Uncertain Relationships

- ▶ Is the relationship  $p(s_t|v_0, m, F) = \delta(s_t - \frac{1}{2}\frac{F}{m}t^2 + v_0t)$  **realistic**?
- ▶ If we did an initial value experiment, would we really measure **exactly** the predicted value?
- ▶ Adding uncertainty makes predictions more **realistic**, by allowing **errors**.

$$p(s_t|v_0, m, F) = \mathcal{N}\left(s_t; \frac{1}{2}\frac{F}{m}t^2 + v_0t, \sigma^2\right) \quad (6)$$

# Probability of Everything

How to make these intuitions **systematic**?

It's a good idea to start from a representation that:

1. clearly expresses the assumptions made in our model,
2. allows us to derive any distribution that we are interested in.

The joint distribution over all variables.

The **Probability of Everything**.

$$p(x, y, z) \tag{7}$$

Any question we may want to answer corresponds to finding:

$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} \quad \text{or} \quad p(x|y) = \int \frac{p(x, y, z)}{p(y)} dz \tag{8}$$

- Observe a variable? Conditioning (i.e. divide and renormalise).
- Not interested in a variable? Marginalise.

# Building a Probabilistic Model: Statistical Approach

Understanding how your variables causally interact gives you a factorisation of the joint.

1. Identify all the variables that are relevant to your problem.
2. Start with a variable that you will observe.
3. Determine which variables it depends on, and choose a sensible conditional distribution.
4. Repeat previous step for one of the variables that you conditioned on.

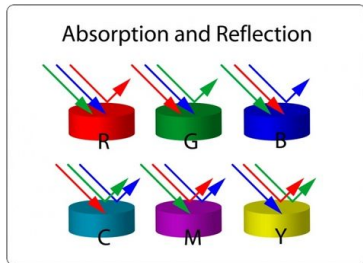


# Example: Lighting

**Step 1:** Identify all variables:

- ▶ Object colour  $C$ .
- ▶ Reflected light  $L$ .
- ▶ Illumination  $I$ .

Joint:  $p(C, L, I)$ .



**Step 2:** We observe the reflected light  $L$ . We have a model for the  $L$  given  $I$  and  $C$ :  $p(L, C, I) = p(L|C, I)p(C, I)$ .

**Step 3:** Let's pick  $C$ . Colour does not depend on illumination, so  $p(C|I) = p(C)$ , showing that  $C$  and  $I$  are independent.

While we can use our knowledge for choosing  $p(L|C, I)$ , we need to choose subjective priors for  $p(C)$  and  $p(I)$ .

# Finding the right posterior

- ▶ Now that we have the joint, how do we find  $p(C|L)$ ?
- ▶ Remember: We need to find it in terms of the conditional distributions **which we can actually evaluate**.
- ▶ This is why starting with the joint is such a good idea! Given the definitions from the previous slide, we can evaluate it!

$$p(C|L) = \frac{p(C, L)}{p(L)} = \frac{\sum_I p(C, L, I)}{\sum_{C, I} p(C, L, I)} \quad (9)$$

$$= \frac{\sum_I p(L|C, I) p(C) p(I)}{\dots} \quad (10)$$

We can take many different routes!

$$p(C|L) = \frac{p(L|C) p(C)}{p(L)} = \frac{[\sum_I p(L|C, I) p(I)] p(C)}{p(L)} \quad (11)$$

Many roads lead to Rome, but starting from the joint highlights assumptions

## Example: Burglars, Earthquakes, and Alarms

*“Fred lives in Los Angeles and commutes 60 miles to work. Whilst at work, he receives a phone-call from his neighbour saying that Fred’s burglar alarm is ringing. What is the probability that there was a burglar in his house today? While driving home to investigate, Fred hears on the radio that there was a small earthquake that day near his home. ‘Oh’, he says, feeling relieved, ‘it was probably the earthquake that set off the alarm’. What is the probability that there was a burglar in his house?” (MacKay, 2003, §21.1)*

Q: How does the joint factorise? What conditionals should we define?

- Variables: **p**honecall, **a**larm, **b**urglar, **r**adio, **e**arthquake

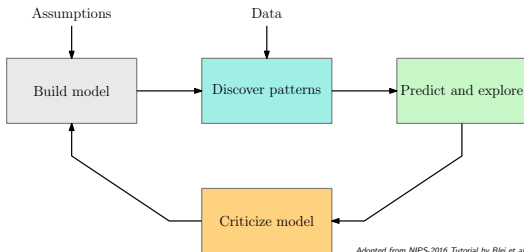
$$p(p, a, b, r, e) = p(p, a, b, r, e) \quad (12)$$

$$p(p, a, b, r, e) = p(p|a, b, r, e)p(a, b, r, e) \quad (13)$$

$$p(p, a, b, r, e) = p(p|a)p(a, b, r, e) \quad (14)$$

# Probabilistic Pipeline

If your assumptions are good/correct, inference will give accurate results and good predictions.



- ▶ Good models **generate** data that is similar to the data we observe.
- ▶ **Predict and explore**: Sample from the prior, assess predictions.
- ▶ **Criticize/revise the model**.

# Building a Probabilistic Model: ML Approach

Q: What happens if we don't have a mathematical/mechanistic model?

- ▶ For some problems, little is known about the process.
- ▶ No known latent variables to use for creating a model.
- ▶ We mainly want good prediction!

All we really need is  $p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})$ , so we can find the predictive distribution:

$$p(\mathcal{D}_{\text{future}} | \mathcal{D}_{\text{observed}}) = \frac{p(\mathcal{D}_{\text{future}}, \mathcal{D}_{\text{observed}})}{p(\mathcal{D}_{\text{observed}})}. \quad (20)$$

# Latent Variables

How do we create a joint with interesting relationships between the observed and future data?

- ▶ Invent **latent variables** that are **common** to all data.

$$p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}, \mathbf{z}) \quad (21)$$

- ▶ For simplicity, often data is iid given latent variables.

$$p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}, \mathbf{z}) = \prod_i p(\mathcal{D}_i | \mathbf{z}) p(\mathbf{z}) \quad (22)$$

- ▶ May not have a direct physical basis initially, but can turn out to be interpretable after training.
- ▶ Induces correlations between data, that can help to predict

$$p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}) = \int p(\mathcal{D}_{\text{fut}}, \mathcal{D}_{\text{obs}}, \mathbf{z}) d\mathbf{z} \quad (23)$$

# Example: Linear Basis Function Regression

Linear regression falls under this!

$$p(\mathbf{y}, \mathbf{y}^*) = \int \prod_i p(y_i | \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (24)$$

$$p(y_i | \boldsymbol{\theta}) = \mathcal{N}(y_i; \boldsymbol{\theta}^\top \boldsymbol{\phi}(\mathbf{x}), \sigma^2) \quad (25)$$

- ▶ Do we really believe that the data we obtained was generated by sampling some parameters?
- ▶ Do we really believe that the relationship is a sum of polynomials?
- ▶ No, but it's useful for predicting!

Similar, VAE (which we will also discuss).

# How do we know we have a good model?

Model criticism is crucial!

Luckily, we have an objective metric on how well it's doing:

## Predictive accuracy!

- ▶ Hold-out test set.
- ▶ Check where it is overconfident and underconfident.
- ▶ Does it predict well when you change the setting?
- ▶ Bayesian model selection (soon).

The ML philosophy: if you predict well, you understand.



# Conclusion

## Summary:

- ▶ You can do anything with the joint.
- ▶ Can create joints from understanding of the world.
- ▶ Can create joints by just hypothesising relationships.  
Just make sure you validate your model...

## What you should be able to do:

- ▶ Create probabilistic model (i.e. joints) by composing conditionals.
- ▶ Apply sum and product rules to find desired posteriors.

## Reading & exercises:

- ▶ Chapter 3 (MacKay, 2003).
- ▶ Exercise: the burglar alarm (MacKay, 2003, ch.21)
- ▶ Exercise: bent coin (MacKay, 2003, §3.2)
- ▶ Exercise: legal evidence (MacKay, 2003, §3.4)

# References I

MacKay, D. J. C. (2003). *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press.