

Closed-Form Inference

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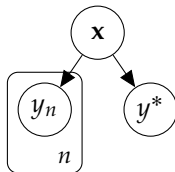
$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} \quad (1)$$

- ▶ As before, the model is specified by the full joint, often in terms of tractable densities, i.e.

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \quad (2)$$

Terminology

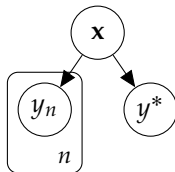
$$p(\mathbf{y}, y^*, \mathbf{x}) = p(y^* | \mathbf{x}) \prod_{n=1}^N p(y_n | \mathbf{x}) p(\mathbf{x}) \quad (3)$$



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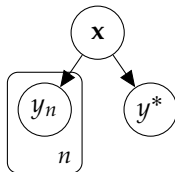


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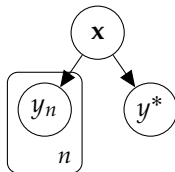


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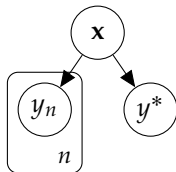


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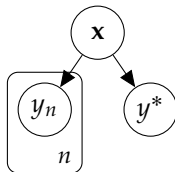


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- ▶ When solving an inference problem, what is fixed and what is variable in $p(\mathbf{x} | \mathbf{y})$? ►► observation \mathbf{y} is fixed, \mathbf{x} varies.
- ▶ What variable is the likelihood a function of? ►► \mathbf{x} only!
We say “likelihood of parameters / latent variable \mathbf{x} ”.

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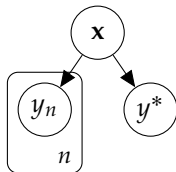
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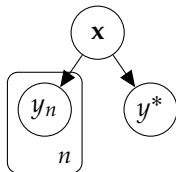


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- If we aren't talking about a fixed observed dataset, we can investigate properties of distributions as a function of \mathbf{y} . If we do so we may refer to $p(\mathbf{y}) = \int p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}$ as the *prior predictive distribution*.

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- ▶ We use different terminology for these settings to indicate whether \mathbf{y} is observed and fixed, or whether we investigate how the probability changes for different possible outcomes \mathbf{y} .

Example: One-Armed Bandit

- ▶ Each time you run a “one-armed bandit”, you get a random return of Y_n .
- ▶ Y_n is distributed according to density $p(y_n|x)$, with $\mathbb{E}_{p(y_n|x)}[y_n] = x$.
- ▶ The mean return is assigned by the manufacturer by sampling from $p(x)$.



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You are interested in computing for example:

- ▶ $p(x|\mathbf{y})$ to understand your belief about the average return. In particular

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- ▶ In general, we are interested in **summary statistics** of posterior distributions ► **integrals**.

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- ▶ Do we really need $p(\mathbf{y})$? It's just a constant... Are relative probabilities not enough?
- ▶ ► No hope of computing $p(X > 0|\mathbf{y})$ without $p(\mathbf{y})$.

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$$= \frac{(2\pi\sigma^2)^{-\frac{N}{2}} (2\pi v)^{-\frac{1}{2}}}{p(\mathbf{y})} \exp \left[-\frac{1}{2\sigma^2} \sum_n (y_n - x)^2 - \frac{1}{2v} x^2 \right] \quad (11)$$

$$= \frac{(2\pi)^{-\frac{N+1}{2}} \sigma^{-N} v^{-\frac{1}{2}}}{p(\mathbf{y})} \exp \left[-\frac{1}{2\tau} (x - \mu)^2 - \frac{1}{2} \left(\sum_n y_n^2 - \frac{\mu^2}{\tau} \right) \right] \quad (12)$$

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Equate coefficients to obtain

$$\tau = \frac{v\sigma^2}{vN + \sigma^2}, \quad \mu = \frac{v}{vN + \sigma^2} \sum_n y_n. \quad (14)$$

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2. Let someone else do the integral, by using knowledge that

$$\mathcal{N}(x; \mu, \tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left[-\frac{1}{2\tau}(x - \mu)^2\right]. \quad (18)$$

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We could integrate the unnormalised posterior.

$$p(x|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \quad (19)$$

$$Z = \int p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x} \quad \text{in this case} \quad (20)$$

$$p(x|\mathbf{y}) = \frac{1}{Z}p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \quad \text{in this case} \quad (21)$$

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- ▶ This was the case because $p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$ as a function of \mathbf{x} implies a Gaussian distribution.
- ▶ We know how to do the integral to normalise a Gaussian.

Intractable Inference

Example where things don't work out so nicely. Take $y_n \in \{0, 1\}$.

$$p(x) = \mathcal{N}(x; 0, v) \quad (22)$$

$$\ell(x) = \frac{1}{1 + e^{-x}} \quad \text{Logistic function} \quad (23)$$

$$p(y_n|x) = \ell(x)^{y_n} \cdot (1 - \ell(x))^{1-y_n}. \quad (24)$$

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$$p(x|\mathbf{y}) = \frac{1}{Z} \frac{e^{-N_1 x - \frac{1}{2v} x^2}}{(1 + e^{-x})^N} \quad (25)$$

$$Z = \int \frac{e^{-N_1 x - \frac{1}{2v} x^2}}{(1 + e^{-x})^N} dx \quad (26)$$

Intractable Inference

$$Z = \int \frac{e^{-N_1 x - \frac{1}{2v} x^2}}{(1 + e^{-x})^N} dx \quad (27)$$

- ▶ No known “**closed-form**” solution to this integral.
- ▶ Closed-form: Combination of finite number of terms of standard functions (exp, sin, log, sqrt...). Sometimes includes special functions (e.g. Gamma, Bessel...)
- ▶ If no closed-form solution is known, a quantity is also said to be **intractable**.
- ▶ Inference is intractable if it requires computing intractable quantities.

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Definition

A prior and likelihood are **conjugate** if their resulting posterior is of the same family as the prior.

If your prior was tractable,
then your posterior will be as well!

Example: Gaussian-Gaussian conjugacy

The Gaussian example we saw earlier was an example of conjugacy.

- Likelihood formed from Gaussian with unknown mean:

$$L(x) = p(\mathbf{y}|x) = \prod_n \mathcal{N}(y_n; x, \sigma^2) \quad (28)$$

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- ▶ Likelihood formed from Gaussian with unknown mean:

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- ▶ Prior from the Gaussian family of distributions:

$$p(x) = \mathcal{N}(x; 0, v) \quad (29)$$

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The Gaussian example we saw earlier was an example of conjugacy.

- ▶ Likelihood formed from Gaussian with unknown mean:

$$L(x) = p(\mathbf{y}|x) = \prod_n \mathcal{N}(y_n; x, \sigma^2) \quad (28)$$

- ▶ Prior from the Gaussian family of distributions:

$$p(x) = \mathcal{N}(x; 0, v) \quad (29)$$

- ▶ Posterior is also Gaussian!

$$p(x|\mathbf{y}) \propto L(x)p(x) \quad (30)$$

$$p(x|\mathbf{y}) = \mathcal{N}\left(x; \frac{v \sum_n y_n}{vN + \sigma^2}, \frac{v\sigma^2}{vN + \sigma^2}\right) \quad (31)$$

Exponential Family

This is no coincidence. The Gaussian distributions are part of the **exponential family**:

$$p(x|\eta) = h(x) \exp(\eta^\top t(x) - A(\eta)) \quad \eta, t \in \mathbb{R}^D \quad (32)$$

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Example: Gaussian

$$p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) \quad (33)$$

$$t(x) = [x \quad x^2]^\top, \quad \eta = [\mu/\sigma^2 \quad -\frac{1}{2\sigma^2}]^\top, \quad (34)$$

$$A(\eta) = -\frac{\eta_1^2}{4\eta_2^2} - \frac{1}{2} \log(-2\eta_2), \quad h(x) = (2\pi)^{-\frac{1}{2}}. \quad (35)$$

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Example: Bernoulli

$$p(x) = \theta^x \cdot (1 - \theta)^{1-x} \quad x \in \{0, 1\} \quad (37)$$

$$= \exp(x(\log \theta - \log 1 - \theta) + \log 1 - \theta) \quad (38)$$

$$t(x) = x, \quad \eta = \log \frac{p}{1-p}, \quad (39)$$

$$A(\eta) = \log 1 - p, \quad h(x) = 1. \quad (40)$$

Conjugate Prior for Exponential Family

Exponential families have conjugate priors!
For the likelihood:

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Exam skills (NOT THIS YEAR)

Previous years:

- ▶ Convert distributions that are exponential families into their **natural form** (i.e. parameterised by η).
- ▶ Recognise when a likelihood and prior are conjugate, and when they are not.
- ▶ Find conjugate prior to a likelihood in exponential family.

See examples sheet for practice.

Exam skills (THIS YEAR)

You must be able to:

- ▶ do closed-form inference when distributions are Gaussian,
- ▶ do closed-form inference for discrete distributions,
- ▶ recognise when integrals w.r.t. Gaussians are possible,
- ▶ do integrals if an identity is given.

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The procedure of drawing conclusions from observations.

In Bayesian statistics: Computing some conditional distribution (posterior).

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Closed-form Inference

An inference problem where all relevant quantities (e.g. posteriors) can be computed in closed-form.

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- ▶ Integrals appear when making predictions
- ▶ Integrals can only be done in special cases
- ▶ Conjugate models is a (big) family of these special cases, which helps you recognise when you can do the closed-form inference (but this isn't examined this year)

Reading

Recommended reading:

- ▶ §6.6 of Mathematics for Machine Learning [1].

Further reading:

- ▶ §9.2 of ML: a Probabilistic Perspective [2].

References I

- [1] M. P. Deisenroth, A. A. Faisal, and C. S. Ong. *Mathematics for machine learning*. Cambridge University Press, 2020.
- [2] K. P. Murphy. *Machine learning: a probabilistic perspective*. MIT press, 2012.