Lesson 11: multiblock data and introduction to sensor-fusion

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Today's lesson

- Scalable-learning (here or later?)

- recap about PCA, PCR and PLS
- peer instructions (part 1)

- multi-block schemes
- Kalman filtering essentials
- examples
- connecting the pieces together
- peer instructions (part 2)

Scalable-learning?

recap

SVD

$$A = U\Sigma V^T \qquad\qquad Av_i = \sigma_i u_i \tag{1}$$

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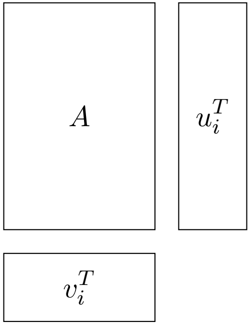
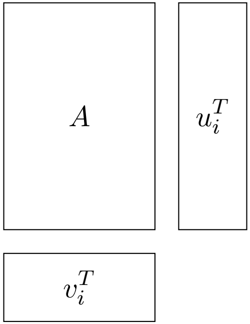


SVD

$$A = U\Sigma V^T \qquad Av_i = \sigma_i u_i \qquad A^T = V\Sigma^T U^T \qquad A^T u_i = \sigma_i v_i \tag{1}$$



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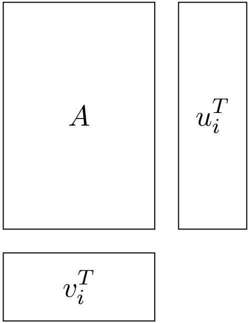

- what is the meaning of each component in u_i and v_i?

PCA - fast review

	Maths	Science	English	Music
John	80	85	60	55
Mike	90	85	70	45
Kate	95	80	40	50

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First 2 PCs:

	Maths	Science	English	Music
PC1	0.28	-0.17	-0.94	0.07
PC2	0.77	-0.08	0.19	-0.60

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- are these scores or loadings?

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- are these scores or loadings?
- how can we interpret these loadings?

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- are these scores or loadings?
- how can we interpret these loadings?
- how shall we interpret the scores?

	Maths	Science	English	Music
John	80	85	60	55
Mike	90	85	70	45
Kate	95	80	40	50
Ann	70	85	55	*???*

... and now?

From PCA to PLS - fast review

	Maths	Science	English	Music
John	80	85	60	55
Mike	90	85	70	45
Kate	95	80	40	50
Ann	70	85	55	*???*

. . . and now?

Algorithm:

1. train on J+M+K using M+S+E as x and Music as y
2. forecast A's y given A's x

From PCA to PLS - fast review

	Maths	Science	English	Music
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. . . and now?

Algorithm:

1. train on J+M+K using M+S+E as x and Music as y
2. forecast A's y given A's x

- which problems do you see with this approach?

From PCA to PCR - fast review

$$y = \theta U + e \tag{2}$$

From PCA to PCR - fast review

$$y = \theta U + e \tag{2}$$

- what is the trade-off between $\|\theta\|_0$ and the generalization capabilities of the estimator?

multi-block algorithms

PCA - from single block to multi-block

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John	80	85	60	55
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PCA - from single block to multi-block

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	Maths	Biology	Chemistry
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Kate	90	70	60

PCA - from single block to multi-block

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John	80	85	60	55
Mike	90	85	70	45
Kate	95	80	40	50

	Maths	Biology	Chemistry
Lee	85	70	65
Steve	95	75	95
Ann	90	70	60

Would Consensus-PCA be the right tool here?

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

PCA - from single block to multi-block

	Maths	Science	English	Music
John	80	85	60	55
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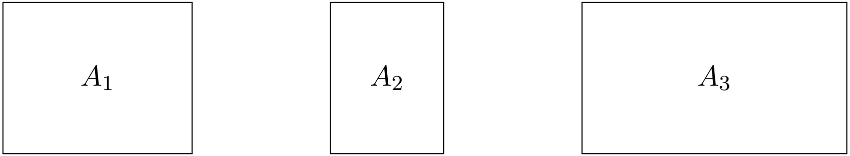
multi-block: comparing several blocks of descriptor variables measured on the *same objects*

Consensus-PCA, graphically



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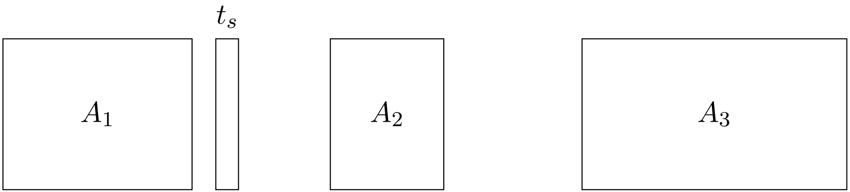
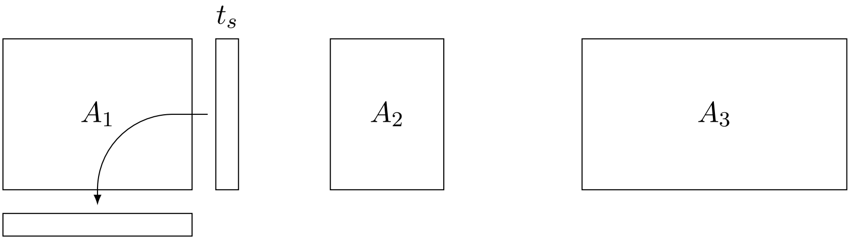


Consensus-PCA, graphically



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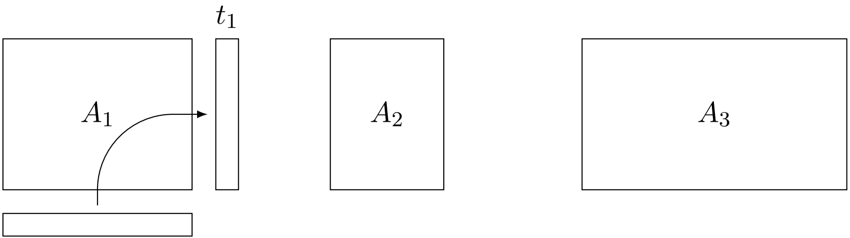
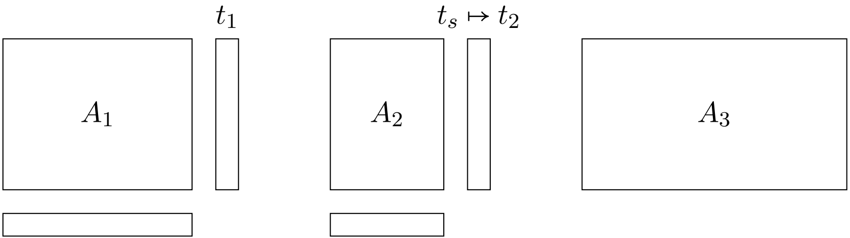
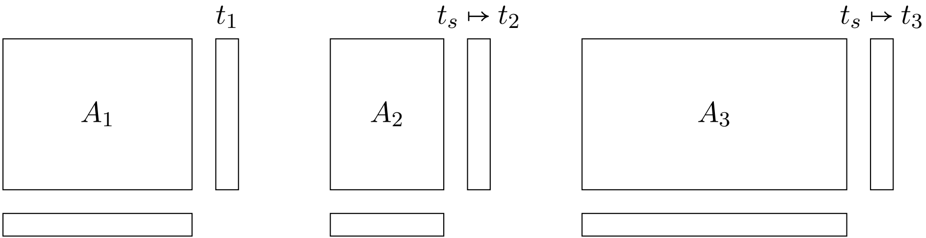


Consensus-PCA, graphically



Consensus-PCA, graphically



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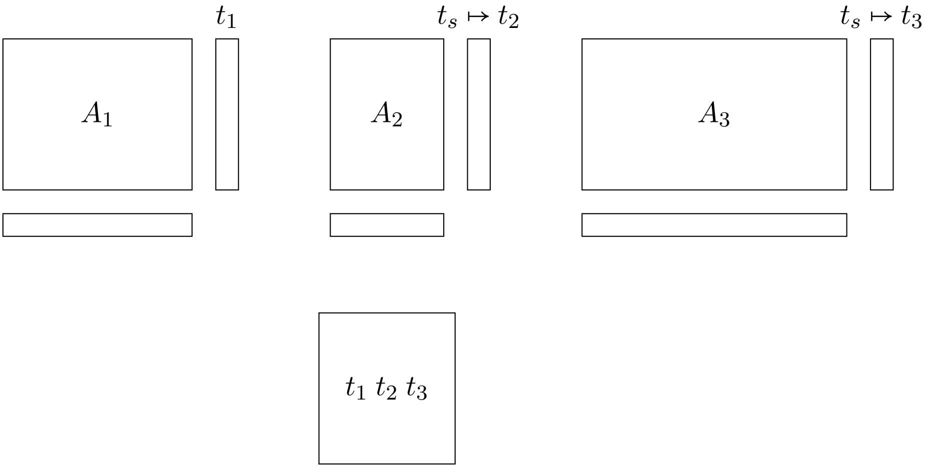

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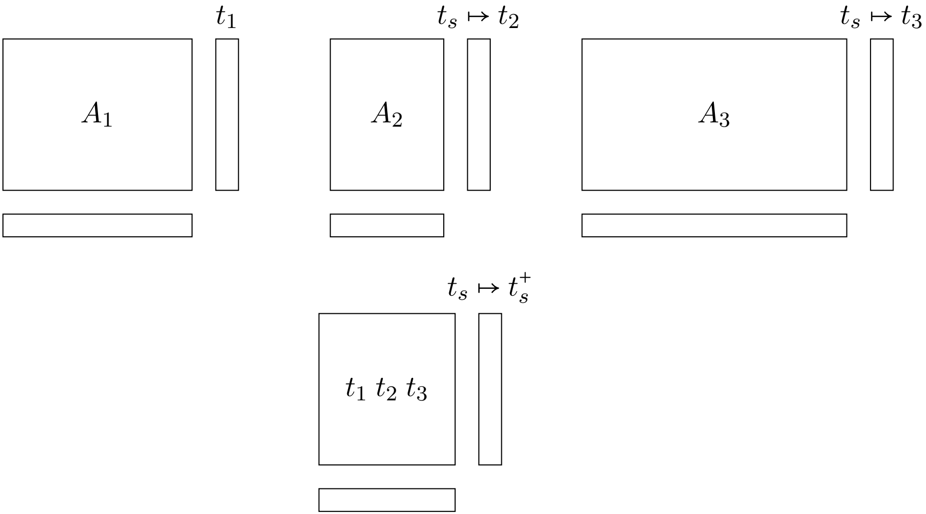
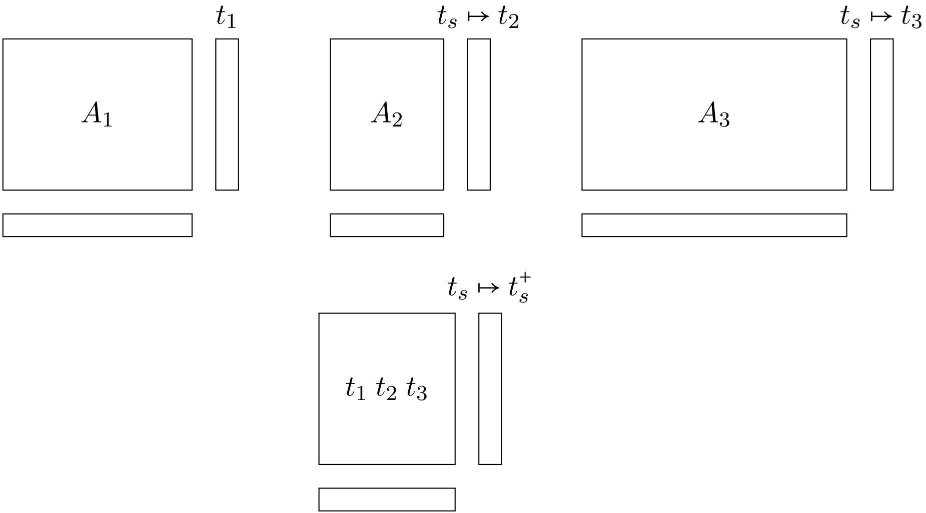
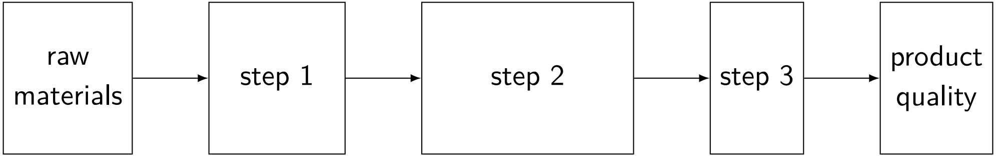


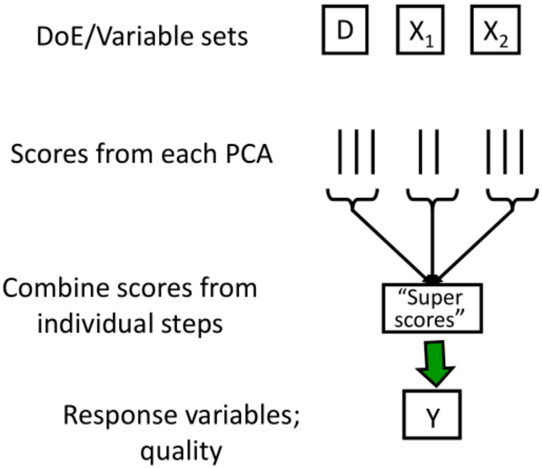
Consensus-PCA, graphically - what does this produce?



Example 1 - Process modelling



Example 2 - Multiblock model with dimension reduction



Procedure:

1.PCA on each block

2.Run PLSR for the combined tables of scores

NB! Validation must be done for all steps!

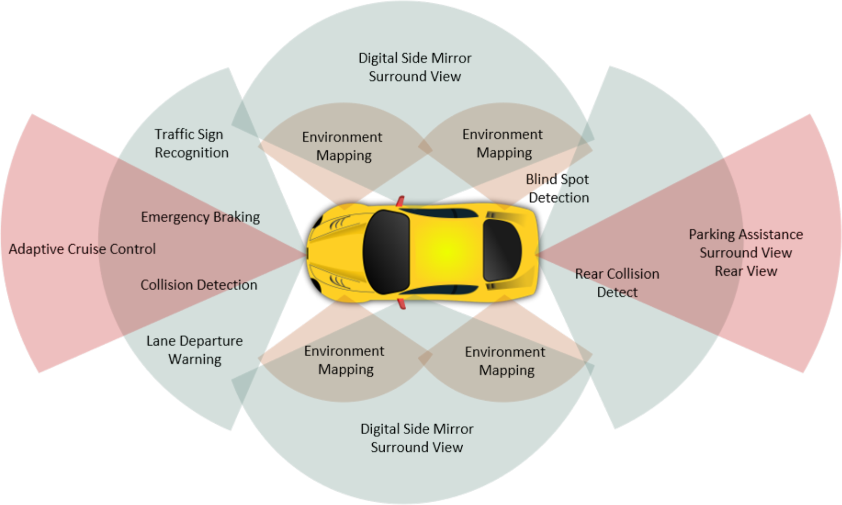
Example 3 - Unscrambler

Scalable-learning?

peer instructions (part 1)

essentials of sensor fusion through Kalman filtering

Example of sensor fusion



Time for "just in time" adaptations

- priors / likelihoods / posteriors / conjugate priors?
- state-space models?
- reachability, controllability, observability?

Priors / likelihoods / posteriors / conjugate priors

$$y = f(u, \theta) + e \quad \mapsto \quad y \sim p(y; u, \theta) \tag{3}$$

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$$p(\theta) \quad \Longrightarrow \quad p(\theta | u, y) \tag{4}$$

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conjugate priors: "the posterior has the same structure of the prior"

Examples of likelihood

- classical one: $y(k) = ax(k) + b + e(k), \quad e \sim \mathcal{N}\left(0, \sigma^2\right)$

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- an other classical one: flipping a coin with $p_H = \alpha$

Example of posterior

- a school has 60% boys and 40% girls students
- the girls wear trousers or skirts in equal numbers; all boys wear trousers
- an observer sees a random student from a distance that is wearing trousers
- question: what is $P(G|T)$?

Example of posterior

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- the girls wear trousers or skirts in equal numbers; all boys wear trousers
- an observer sees a random student from a distance that is wearing trousers
- question: what is $P(G|T)$?

$$
\begin{aligned}
P(G|T) &= \frac{P(T|G)P(G)}{P(T)} \\
P(T|G) & \\
P(G) & \\
P(T) &= P(T|G)P(G) + P(T|B)P(B) \\
P(B) &
\end{aligned}
\tag{5}
$$

Example of conjugate prior

$$p(\theta|x) = \frac{p(x|\theta)\, p(\theta)}{p(x)} \tag{6}$$

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$$\begin{aligned} &\mu \sim \mathcal{N}\left(\mu_0, \sigma_0^2\right) && \text{(prior)} \\ &y = \mu + e, \quad e \sim \mathcal{N}\left(0, \sigma_y^2\right) && \text{(likelihood)} \end{aligned} \tag{7}$$

Example of conjugate prior

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$$\begin{aligned} \mu &\sim \mathcal{N}\left(\mu_0, \sigma_0^2\right) && \text{(prior)} \\ y &= \mu + e, \quad e \sim \mathcal{N}\left(0, \sigma_y^2\right) && \text{(likelihood)} \end{aligned} \tag{7}$$

posterior:

$$\mu_{|y} \sim \mathcal{N}\left(\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{y}{\sigma_y^2}\right), \quad \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_y^2}\right)^{-1}\right) \tag{8}$$

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

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- inputs, outputs and state variables
- first-order differential equations

State space representations - Definition

mathematical model (typically but not limited to of a physical system) as a finite set of inputs, outputs and state variables related by first-order differential equations satisfying the separation principle

Ingredients

- inputs, outputs and state variables
- first-order differential equations
- *separation principle:* current value of states contains all the information necessary to forecast the future evolution of the outputs and of the states given the inputs

State space representations - Facts

- the future output depends only on the current state and the future input

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- the future output depends on the past input only through the current state

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- the future output depends only on the current state and the future input
- the future output depends on the past input only through the current state
- the state summarizes the effect of past inputs on future output
  (sort of a "memory" of the system)

Example

Rechargeable flashlight:

- input = on / off button
- state = level of charge of the battery & being on, off, or in re-charging
- output = how much light the device is producing

State space representations - Notation

$$u_1, \ldots, u_m \;= \text{inputs}$$

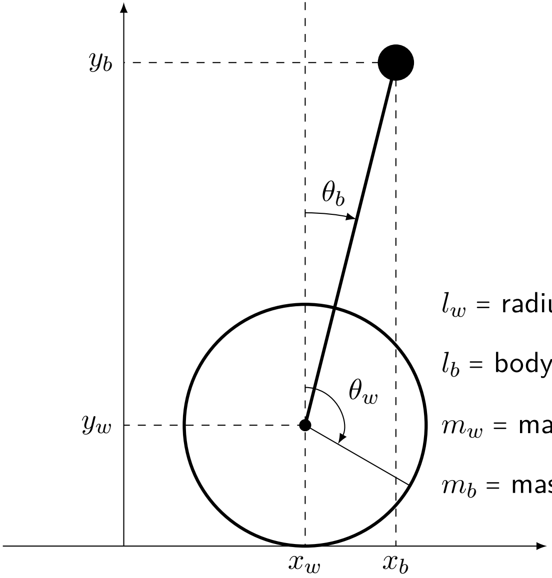
$$y_1, \ldots, y_p \;= \text{outputs}$$

$$x_1, \ldots, x_n \;= \text{states}$$

State space representations - Continuous time

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right) \end{aligned}$$

- $f =$ state transition map
- $g =$ output map



l_w = radius of the wheel

l_b = body-wheel's center distance

m_w = mass of the wheel

m_b = mass of the body

State space representations - Continuous time - Example

$$
\begin{cases}
\left(I_b + m_b l_b^2\right)\ddot{\theta}_b & = +m_b l_b g \sin\left(\theta_b\right) - m_b l_b \ddot{x}_w \cos\left(\theta_b\right) - \dfrac{K_t}{R_m} v_m + \left(\dfrac{K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\
\left(\dfrac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w & = -m_b l_b l_w \ddot{\theta}_b \cos\left(\theta_b\right) + m_b l_b l_w \dot{\theta}_b^2 \sin\left(\theta_b\right) + \dfrac{K_t}{R_m} v_m - \left(\dfrac{K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right)
\end{cases}
\tag{9}
$$

State space representations - Discrete time

$$
\begin{aligned}
\boldsymbol{x}(k+1) &= \boldsymbol{f}\big(\boldsymbol{x}(k), \boldsymbol{u}(k)\big) \\
\boldsymbol{y}(k) &= \boldsymbol{g}\big(\boldsymbol{x}(k), \boldsymbol{u}(k)\big)
\end{aligned}
$$

State space representations - Discrete time

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"sensor fusion": include in $\boldsymbol{y}$ different sensors

Question 1

is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right) \\ \boldsymbol{y} &= \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)\end{aligned}$$

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

Question 1

is any $\boldsymbol{f}$ and $\boldsymbol{g}$ ok?

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned}$$

- yes (1 finger)
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No; for every given $\boldsymbol{x}_0$ and $\boldsymbol{u}$ the solution must be:

- unique
- guaranteed to exist for $t \in [0, \bar{t})$ for some $\bar{t} > 0$

(check "existence and uniqueness of solutions of ODEs")

Question 2

can every physical system be described with a state space representation?

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

Question 2

can every physical system be described with a state space representation?

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

No; noticeable exceptions:

- non-lumped parameters model (i.e., PDEs)
- continuous-time systems with delays

Discrete-time linear systems

$$\begin{aligned}
\boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\
\boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k)
\end{aligned}$$

Discrete-time linear systems

$$\begin{aligned}\boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k)\\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k)\end{aligned}$$

what we would like to do: estimate the unknown x using the known y and u

Discrete-time linear systems - Example

First step = linearize, e.g.,

$$\begin{cases} \left(I_b + m_b l_b^2\right)\ddot{\theta}_b & = +m_b l_b g \theta_b - m_b l_b \ddot{x}_w - \dfrac{K_t}{R_m} v_m + \left(\dfrac{K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \\ \left(\dfrac{I_w}{l_w} + l_w m_b + l_w m_w\right)\ddot{x}_w & = -m_b l_b l_w \ddot{\theta}_b + \dfrac{K_t}{R_m} v_m - \left(\dfrac{K_e K_t}{R_m} + b_f\right)\left(\dfrac{\dot{x}_w}{l_w} - \dot{\theta}_b\right) \end{cases} \tag{10}$$

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Second step, discretize, e.g., using forward Euler (or something else)

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\end{cases}
\tag{10}
$$

Second step, discretize, e.g., using forward Euler (or something else)

$$\boxed{\text{problem} = \text{estimate the state given the measurements}}$$

Main intuition (both for recursive least squares and Kalman filters)

$$\text{new estimate} = \text{previous estimate} + \text{gain} \cdot \text{“innovation”}$$

Path

1. least squares
2. recursive least squares
3. Luenberger observers
4. linear Kalman filters
5. non-linear Kalman filters

Least squares

$$\begin{cases} y_1 & = a_1\theta + e_1 \\ & \vdots \\ y_1 & = a_n\theta + e_n \end{cases} \implies \boldsymbol{y} = A\theta + \boldsymbol{e} \implies \widehat{\theta} = \left(A^T A\right)^{-1} A^T \boldsymbol{y} \quad (11)$$

From least squares to recursive least squares

$$\begin{cases} y_1 &= a_1\theta + e_1 \\ & \vdots \\ y_n &= a_n\theta + e_n \end{cases} \implies \boldsymbol{y} = A\theta + \boldsymbol{e} \implies \widehat{\theta} = \left(A^T A\right)^{-1} A^T \boldsymbol{y} \quad (12)$$

$$\left(A^T A\right)^{-1} A^T \boldsymbol{y} = \left(\sum_{i=1}^{n-1} a_i a_i^T + a_n a_n^T\right)^{-1} \left(\sum_{i=1}^{n-1} a_i y_i + a_n y_n\right) \quad (13)$$

$$\left(\Sigma + uv^T\right)^{-1} = \Sigma^{-1} - \frac{\Sigma^{-1} u v^T \Sigma^{-1}}{1 + v^T \Sigma^{-1} u} \quad (14)$$

Luenberger observers for linear systems

$$\begin{aligned} \boldsymbol{x}^+ &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned}$$

Idea: use feedback

$$\begin{cases} \widehat{\boldsymbol{x}}(0) = \boldsymbol{x}(0) \\ \widehat{\boldsymbol{x}}^+ = A\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L\left(\boldsymbol{y} - C\widehat{\boldsymbol{x}}\right) \end{cases}$$

Luenberger observers for linear systems

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\boldsymbol{x}^+ &= A\boldsymbol{x} + B\boldsymbol{u} \\
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\widehat{\boldsymbol{x}}(0) = \boldsymbol{x}(0) \\
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\end{cases}
$$

how shall we tune L?

Kalman filters - main quantities

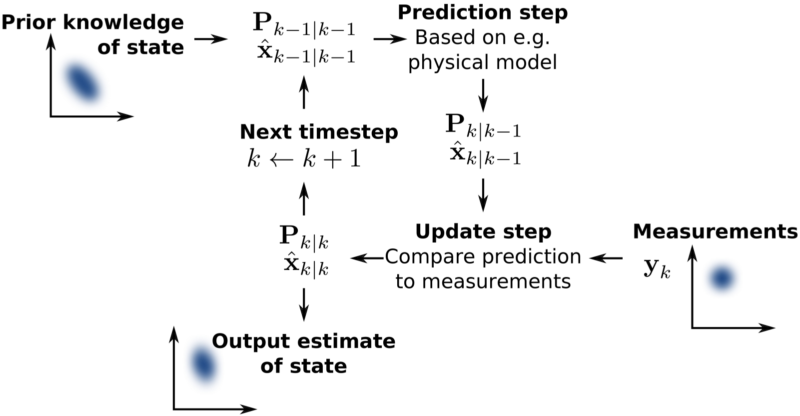
$$\begin{aligned} \boldsymbol{x}^+ &= A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{w} \\ \boldsymbol{y} &= C\boldsymbol{x} + \boldsymbol{v} \end{aligned}$$

- Q
- R

- $\widehat{x}_{k|k}$
- $P_{k|k}$

- $\widehat{x}_{k+1|k}$
- $P_{k+1|k}$

Kalman filters - main intuition



Kalman filters - main equations

$$
\begin{aligned}
\widehat{x}_{k|k-1} &= A_k \widehat{x}_{k-1|k-1} + B_k \boldsymbol{u}_{k-1} \\
P_{k|k-1} &= A_k P_{k-1|k-1} A_k^T + Q_k \\
\widetilde{\boldsymbol{y}}_k &= \boldsymbol{y}_k - H_k \widehat{x}_{k|k-1} \\
S_k &= H_k P_{k|k-1} H_k^T + R_k \\
K_k &= P_{k|k-1} H_k^T S_k^{-1} \\
\widehat{x}_{k|k} &= \widehat{x}_{k|k-1} + K_k \widetilde{\boldsymbol{y}}_k \\
P_{k|k} &= (I - K_k H_k) P_{k|k-1} \\
\widetilde{\boldsymbol{y}}_{k|k} &= \boldsymbol{y}_k - H_k \widehat{x}_{k|k}
\end{aligned}
\tag{15}
$$

Kalman filters - main equations

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\end{aligned}
\tag{15}
$$

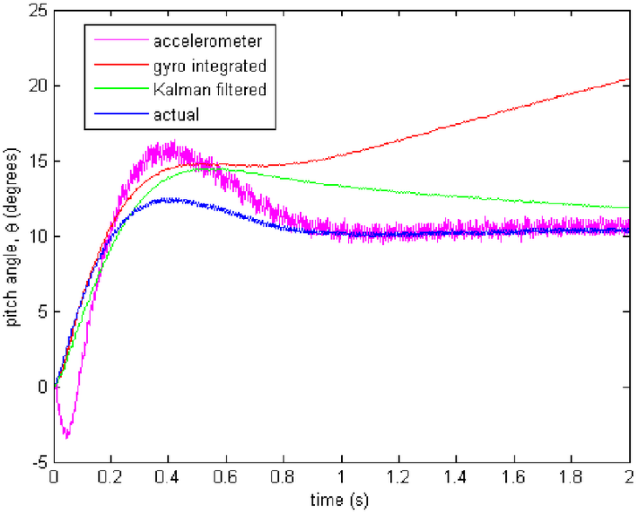
In practice,

$$
p\left(\boldsymbol{x}_k | Y_k\right) = \frac{p\left(\boldsymbol{z}_k | \boldsymbol{x}_k\right) p\left(\boldsymbol{x}_k | Y_{k-1}\right)}{p\left(\boldsymbol{z}_k | Y - 1\right)} \tag{16}
$$

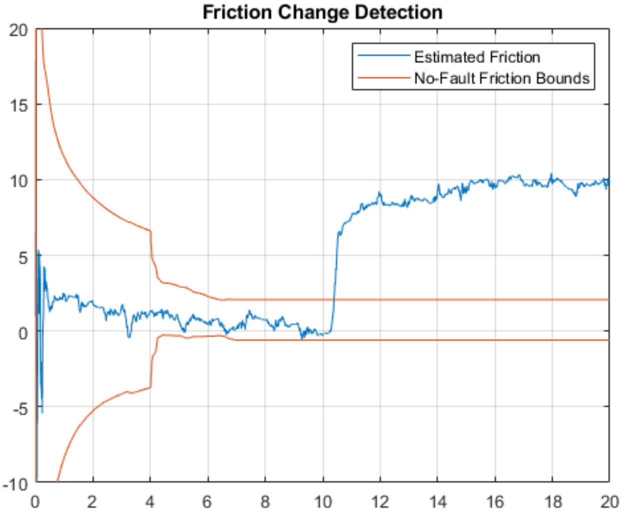
Example 1

`https://www.youtube.com/watch?v=Jq8HcIar68Y` (from 4:28)

Example 2


accelerometer
gyro integrated
Kalman filtered
actual
pitch angle, θ (degrees)
time (s)
25
20
15
10
5
0
-5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Example 3



Friction Change Detection
Estimated Friction
No-Fault Friction Bounds
20
15
10
5
0
-5
-10
0
2
4
6
8
10
12
14
16
18
20

Nonlinear Kalman filtering

1. extended KF
2. unscented KF
3. ensemble KF

Kalman filtering - convergence properties

Question 3

Are KFs working for any A, B, C or $f(\cdot)$, $g(\cdot)$?

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

Towards Controllability: Discrete-time state evolution

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

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$$\boldsymbol{x}(k+1) = A^{k+1}\boldsymbol{x}(0) + \sum_{\sigma=0}^{k} A^{k-\sigma}Bu(\sigma)$$

Forced evolution, i.e., $\boldsymbol{x}(0) = \mathbf{0}$

$$\boldsymbol{x}(k+1) = \begin{bmatrix} B & AB & A^2B & \cdots & A^kB \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

(some matrix algebra. . .)

Intuition:

$$\operatorname{rank}\left(\begin{bmatrix} B & AB & A^2B & \cdots & A^kB \end{bmatrix}\right) = n$$

$$\Downarrow$$

$$\boldsymbol{x}(k+1) = \begin{bmatrix} B & AB & A^2B & \cdots & A^kB \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

can be anything I want

Definition 1 (reachable subspace at time k)

$$\mathcal{RS}(k) := span\left\langle B \quad AB \quad A^2B \quad \cdots \quad A^{k-1}B \right\rangle$$

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Definition 2 (reachability matrix at time k)

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Definition 3 (reachability index at time k)

$$r(k) := rank\Big(\mathcal{R}(k)\Big) = \dim\Big(\mathcal{RS}(k)\Big)$$

Chains of reachable subspaces

$$\mathcal{RS}(1) = \operatorname{span}\langle B\rangle$$

Chains of reachable subspaces

$$\mathcal{RS}(1) = \text{span}\left\langle B \right\rangle \subseteq \mathcal{RS}(2) = \text{span}\left\langle B \quad AB \right\rangle$$

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Theorem 1

$\mathcal{RS}(k) \subseteq \mathcal{RS}(k+1)$ *(and thus* $r(k) \leq r(k+1)$*) for every* k

Chains of reachable subspaces

Important fact:

$$\mathcal{RS}(k) = \mathcal{RS}(k+1) \quad \Longrightarrow \quad \mathcal{RS}(k) = \mathcal{RS}(k+\tau) \quad \forall \tau \in \mathbb{N}_+$$

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- every $\boldsymbol{x} \in \mathcal{RS}(k+2)$ can be reached in $k+1$ steps and with the last step from an element in $\mathcal{RS}(k)$
- i.e., $\mathcal{RS}(k+2) = \mathcal{RS}(k+1)$

Chains of reachable subspaces

Essential fact:

$$\mathcal{RS}(n) = \mathcal{RS}(n+1)$$

(i.e., after n steps with n = dimension of $\boldsymbol{x}$ we are sure that the reachable subspace stops growing)

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  - $r(1)$ = at most 1

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  - $\ldots$
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- if the reachable subspace does not stop growing before n, then:
  - $r(1)$ = at most 1
  - $r(2)$ = at most 2
  - $\ldots$
  - $r(n)$ = at most n
  - the dimension of $\mathcal{RS}(n)$ cannot exceed n, thus QED

Reachability and controllability, intuitions

Reachability := my initial state is $\boldsymbol{x}(0) = \mathbf{0}$, where may I end up by choosing opportunely $u(0), u(1), \ldots$?

Reachability and controllability, intuitions

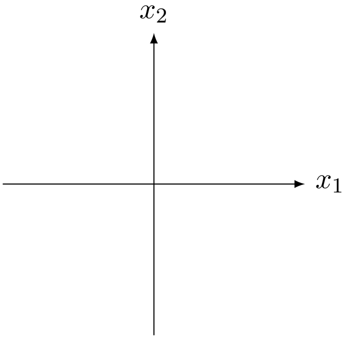
Reachability := my initial state is $\boldsymbol{x}(0) = \mathbf{0}$, where may I end up by choosing opportunely $u(0), u(1), \ldots$?

Controllability := I have the final state $\boldsymbol{x}_f = \mathbf{0}$ as my given target. Where can I start so that I end in $\boldsymbol{x}_f$ by choosing opportunely $u(0), u(1), \ldots$?

Intuition: reachability $\neq$ controllability

Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Controllability

Definition 4 (controllable state)

$\boldsymbol{x}(0)$ *is controllable in* $k+1$ *steps if* $\exists u(0), \ldots, u(k)$ *s.t.*

$$\boldsymbol{x}(k+1) = \mathbf{0} = A^{k+1}\boldsymbol{x}(0) + \sum_{\sigma=0}^{k} A^{k-\sigma} B u(\sigma)$$

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$$\boldsymbol{x}(k+1) = \mathbf{0} = A^{k+1}\boldsymbol{x}(0) + \sum_{\sigma=0}^{k} A^{k-\sigma} B u(\sigma)$$

alternatively, $\boldsymbol{x}(0)$ is controllable in $k+1$ steps if

$$\operatorname{range}\left[A^{k+1}\right] \subseteq \operatorname{range}\begin{bmatrix} B & AB & A^2 B & \cdots & A^k B \end{bmatrix}$$

Definition 5 (controllability matrix)

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Definition 6 (Controllable system)

A system (A, B, C, D) *is controllable if* $rank\,(\mathcal{C}) = n$ *(sufficient condition)*

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Definition 6 (Controllable system)

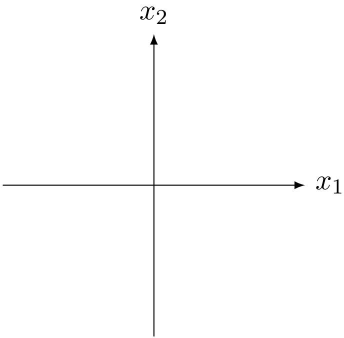
A system (A, B, C, D) *is controllable if* $rank\,(\mathcal{C}) = n$ *(sufficient condition)*

if a system is controllable then independently of the initial condition I can bring its state to zero by choosing opportunely the inputs

Remember: reachability $\neq$ controllability

if a system is controllable then it may not be reachable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Question 4

Is this system controllable?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
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Question 5

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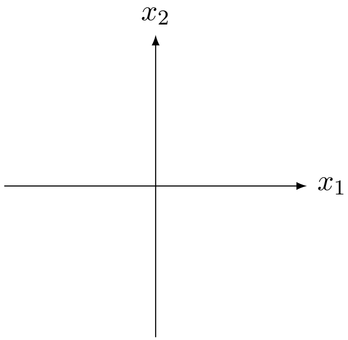

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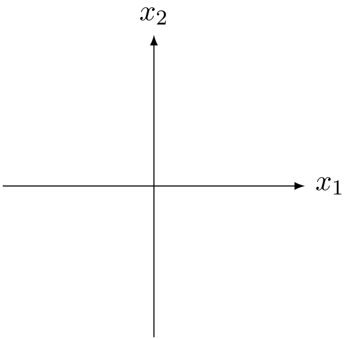

and for continuous time systems?

Discussion

is controllability something that depends on the chosen basis for $\mathbb{R}^n$?

Discrete-time state evolution

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

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$$\boldsymbol{x}(k+1) = A^{k+1}\boldsymbol{x}(0) + \sum_{\sigma=0}^{k} A^{k-\sigma}Bu(\sigma)$$

$$y(k) = C\boldsymbol{x}(k)$$

control need: reconstruct the state from partial information

Free evolution, i.e., $u(k) = 0 \quad \forall k$

$$\boldsymbol{x}(k+1) = A^k \boldsymbol{x}(0)$$

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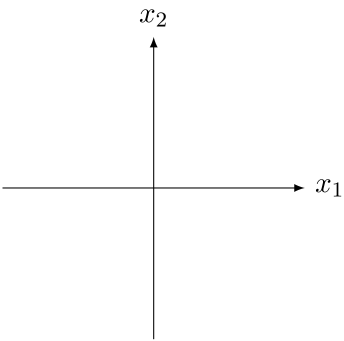
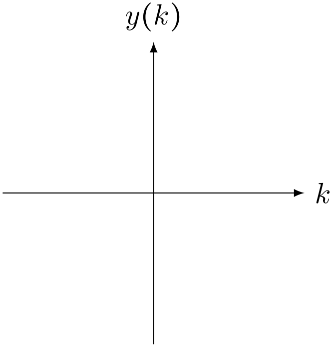
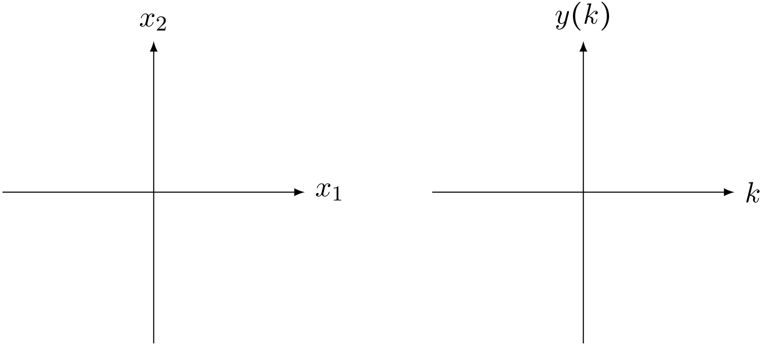




Distinguishability - intuitions

$$\boldsymbol{x}(0) = \boldsymbol{x}_a \quad \Longrightarrow \quad \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \boldsymbol{x}_a$$



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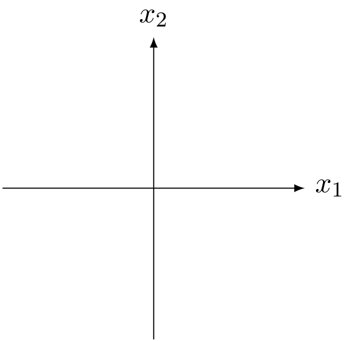

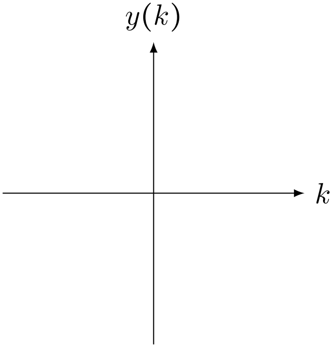

does an other $\boldsymbol{x}_b \neq \boldsymbol{x}_a$ necessarily lead to a $\neq$ trajectory?

Example

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$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Distinguishability - definition

Definition 7 (distinguishable states for discrete time systems)

given $(A, B, C, 0)$, $\boldsymbol{x}_a$ *and* $\boldsymbol{x}_b$ *are distinguishable in* k *steps if*

$$\exists \kappa \in \{0, \ldots, k-1\} \quad \textit{s.t.} \quad y_a(\kappa) = CA^{\kappa}\boldsymbol{x}_a \quad \neq \quad y_b(\kappa) = CA^{\kappa}\boldsymbol{x}_b$$

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Question 6

What are the states that are observable in 3 steps for the system

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}?$$

- $\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$
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(A, B, C, D) *is observable in* k *steps if every* $\boldsymbol{x} \neq \mathbf{0}$ *is observable in* k *steps* $(k \leq n)$

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Definition 10 (observable discrete time systems)

(A, B, C, D) *is observable in* k *steps if every* $\boldsymbol{x} \neq \mathbf{0}$ *is observable in* k *steps* $(k > n)$

Alternatively,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \boldsymbol{x} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \forall \boldsymbol{x} \neq \mathbf{0}$$

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Definition 11 (observability matrix at time k)

$$\mathcal{O}(k) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}$$

Definition 12 (observability index at time k)

$$o(k) := rank\left(\mathcal{O}(k)\right)$$

Theorem 2

- $o(k) \leq o(k+1)$ *for every* $k \geq 1$
- $o(n) = o(n+k)$ *for every* $k \geq 1$

what happens if a state is not observable?

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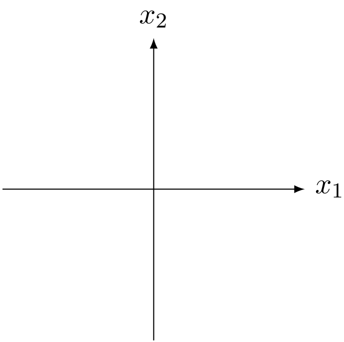
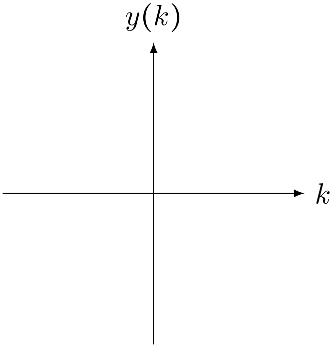
$$y(k) = CA^{k-1}\boldsymbol{x}(0)$$





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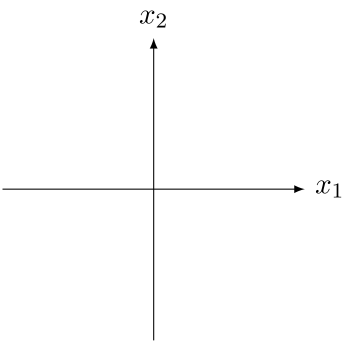

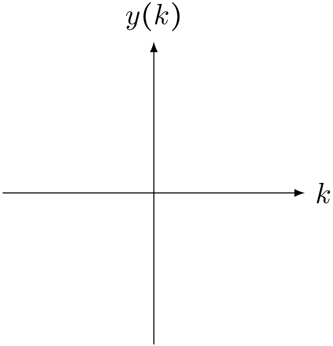

and for continuous time systems?

Question 8

Is this system observable?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}?$$

- yes (1 finger)
- yes, maybe (2 fingers)
- I don't know (3 fingers)
- no, maybe not (4 fingers)
- no (5 fingers)

Question 9

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$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}?$$

- yes (1 finger)
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Definition 13 (Stabilizability)

A system (A, B, C) *is stabilizable if all its non-controllable modes are asymptotically stable*

Definition 14 (Detectability)

A system (A, B, C) *is detectable if all its non-observable modes are asymptotically stable*

Main result

Given a system (A, B, C),

$$\begin{cases} P(k|k) \xrightarrow{k+\infty} \overline{P} \geq 0 \ \forall \ P(0|0) \geq 0 \\ \overline{P} \text{ stabilizes the KF} \end{cases} \tag{17}$$

if and only if

$$\begin{cases} (A, C) \text{ is detectable} \\ (A, Q^{\frac{1}{2}}) \text{ is stabilizable} \end{cases} \tag{18}$$

Example: when do Kalman filters diverge?

$$\begin{cases} x(k+1) &= ax(k) + w(k) \\ y(k) &= x(k) + v(k) \end{cases} \tag{19}$$

$$\Downarrow$$

$$\begin{cases} \widehat{x}(k+1) &= a\widehat{x}(k) + a\dfrac{p(k)}{p(k)+r}\left(y(k) - \widehat{x}(k)\right) \\ p(k+1) &= a^2\dfrac{r}{p(k)+r}p(k) + q \end{cases} \tag{20}$$

Interesting cases:

1. $r = 0, \ r > 0$
2. $q = 0, \ q > 0$
3. $|a| < 1, \ |a| = 1, \ |a| > 1$