Optimization and Optimal Control

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Summary from Deterministic LQR

- For the LTI system $\dot{x}(t) = Ax(t) + Bu(t)$ find $u(t)$ such that it minimizes

$$J_{LQR} = \int_0^\infty \left[x^T Qx + u^T Ru + 2x^T Nu\right] dt$$

- Solve the the algebraic Riccati equation (ARE) P

$$(A^T P + PA + Q - (PB + N)R^{-1}(B^T P + N^T)) = 0.$$

  (Under what conditions ARE has solution?)
- Select $u(t)$ as

$\boxed{\texttt{u(t)=-Kx(t)}}$ (where $K = R^{-1}(B^T P + N^T)$)

The state feedback LQR formulation considered above suffers from the drawback that the optimal control law $u(t) = -Kx(t)$ requires the whole state $x(t)$ of the system to be measured.

Do we measure the speed of a marine vessel?

A possible approach to overcome this difficulty is to construct an estimate $\hat{x}(t)$ of the state of the process based solely on the past values of the measured output y and control signal u, and then use $u(t) = -K\hat{x}(t)$.

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Summary from Deterministic LQR (Contd.)

The configuration for the linear quadratic regulation (LQR) problem.

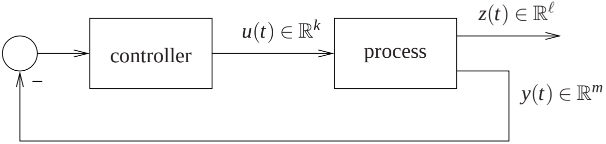


Abbildung:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ z(t) = Gx(t) + Hu(t) \\ u(t) = -Kx(t) \end{cases}$$

The configuration for the linear quadratic gaussian (LQG) problem.

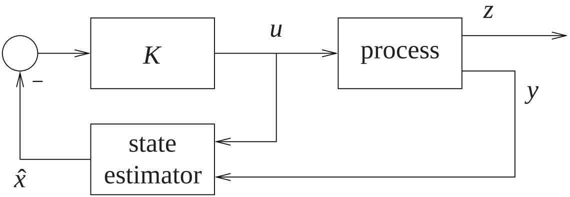


Abbildung:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ z(t) = Gx(t) + Hu(t) \\ x(t),\ u(t) \xrightarrow{??} \hat{x}(t) \\ u(t) = -K\hat{x}(t) \end{cases}$$

General Question and Focus of this Session

Consider a continuous-time LTI system of the form

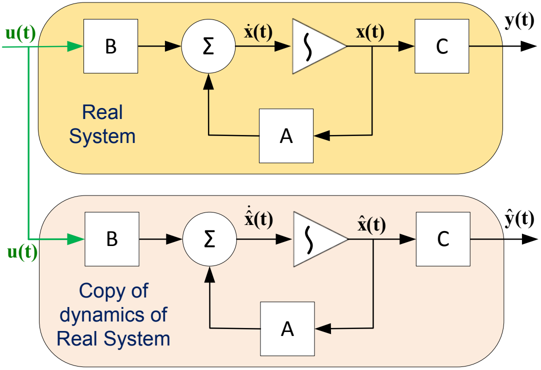
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases}$$

where $u(t)$ is the control signal and $y(t)$ is the measured output. Find estimate of the state x at some time t based on the past values of $u(t)$ and $y(t)$.

Why not Open-loop Observer?

Open-loop observer

Build an artificial copy of the system ($\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t)$), fed in parallel by the same input signal $u(t)$.



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Why not Open-loop Observer? (Contd.)

Let $\tilde{x}(t)$ denote estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) = A\left[x(t) - \hat{x}(t)\right]$$

$$\Longrightarrow \boxed{\dot{\tilde{x}}(t) = A\tilde{x}(t)} \Longrightarrow \boxed{\tilde{x}(t) = e^{A(t-t_0)}\tilde{x}(t_0)}$$

This is not ideal, because

- The dynamics of the estimation error are fixed by the eigenvalues of A and cannot be modified
- The estimation error vanishes asymptotically if and only if A is asymptotically stable
- There is no robustness to modeling error.

Note that we are not exploiting $y(t)$ to compute the state estimate $\hat{x}(t)$!

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Note that we are not exploiting $y(t)$ to compute the state estimate $\hat{x}(t)$!

Can we reconstruct $x(t)$ by having $u(t)$ and $y(t)$

Consider a continuous-time LTI system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

Given $u(\tau)$, $y(\tau)$, $\tau \leq t$, can we solve above equation for the unknown x at some time t?

Assuming that the model is exact and observable, it can be shown that $x(t)$ can be reconstructed exactly using the constructibility Gramian:

$$x(t) = W_{Cn}(t_0, t)^{-1}\Big(\int_{t_0}^{t} e^{A^T(\tau-t)}C^T y(\tau)d\tau + \int_{t_0}^{t}\int_{\tau}^{t} e^{A^T(\tau-t)}C^T C e^{A(\tau-s)}Bu(s)dsd\tau\Big)$$

where $W_{Cn}(t_0, t) = \int_{t_0}^{t} e^{A^T(\tau-t)}C^T C e^{A(\tau-t)}d\tau$.

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Can we reconstruct $x(t)$ by having $u(t)$ and $y(t)$ (Contd.)

In practice, the dynamic model is never exact, and the measured output y is generated by a system of the form

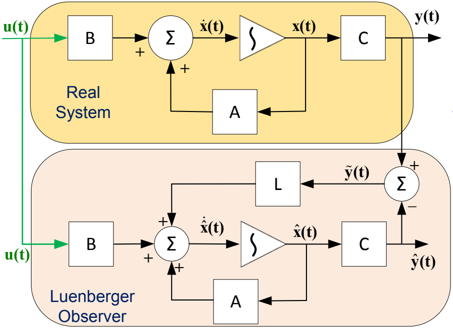
$$\dot{x}(t) = Ax(t) + Bu(t) + \bar{B}d(t), \quad y(t) = Cx(t) + n(t),$$

where $d(t)$ represents a disturbance and $n(t)$ measurement noise. Since neither d nor n are known, solving system equation for x no longer yields a unique solution, since essentially any state value could explain the measured output for sufficiently large noise/disturbances.

Luenberger Observer

Luenberger Observer

Correct the estimation equation with a feedback from the output estimation error $\tilde{y}(t) = y(t) - \hat{y}(t)$.



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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t)\right] \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

Luenberger Observer (Contd.)

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$$= [Ax(t) + Bu(t) - Ly(t)] - [A\hat{x}(t) + Bu(t) - LC\hat{x}(t)]$$

$$= [Ax(t) - LCx(t)] - [A\hat{x}(t) - LC\hat{x}(t)] = (A - LC)\big[x(t) - \hat{x}(t)\big]$$

$$\Rightarrow \boxed{\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t)} \Rightarrow \boxed{\tilde{x}(t) = e^{(A-LC)(t-t_0)}\tilde{x}(t_0)}$$

- The dynamics of the estimation error is not fixed and can be shaped by changing observer gain L.
  (If the pair (A,C) is observable, then the eigenvalues of $(A - LC)$ can be placed arbitrarily).
- The estimation error vanishes asymptotically if and only if $A - LC$ is asymptotically stable (A can be unstable!).

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In MATLAB, the command *place* (or acker) can be used for Pole Placement.

Given the single- or multi-input system $\dot{x}(t) = Ax(t) + Bu(t)$ and a vector p of desired self-conjugate closed-loop pole locations, *place* computes a gain matrix K such that the state feedback $u(t) = -Kx(t)$ places the closed-loop poles poles at the locations p. In other words, the eigenvalues of $A - BK$ match the entries of p (The pair (A,B) should be controllable).

Since taking the transpose of $A - LC$ leaves the eigenvalues unchanged and produces a result $A' - C'L'$ that exactly matches the form of $A - BK$, we can use the *place* commands: L = place(A',C',P)'

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Back to Optimization

Can we do better and formulate the problem of estimating $\hat{x}(t)$ from $u(t)$ and $y(t)$ in optimization framework?

Suppose we are given the signals $y(\tau)$ and $u(\tau)$ over the interval $\tau \in [t_0,\ t]$. It is anticipated that the signals $y(\tau)$ and $u(\tau)$ are related to a process of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

In practice, the above model is never exact; to account for model uncertainty, plant disturbance, and measurement noise, the process model used is modified to:

$$\begin{aligned} \dot{\bar{x}}(t) &= A\bar{x}(t) + Bu(t) + B_d d(t) \\ y(t) &= C\bar{x}(t) + n(t) \end{aligned}$$

where d and n are the process and measurement noises that are assumed to be unknown. Furthermore, the final time t is assumed to be fixed.

Find the least amount of noise n, disturbance d, and initial state $\bar{x}(t_0)$, measured by

$$J_{LQR} = \bar{x}(t_0)^T S\bar{x}(t_0) + \int_{t_0}^{t} \left[n(\tau)^T Qn(\tau) + d(\tau)^T Rd(\tau)\right] d\tau$$

such that the past measured output is consistent, i.e. $y(\tau) = C\bar{x}(\tau) + n(\tau) \quad \text{for all } \tau \in [t_0,\ t]$.

Once the trajectory $\bar{x}(\tau)$ has been found based on the data collected on the interval $[t_0,\ t]$ (i.e. $y(\tau)$ and $u(\tau)$ over the interval $\tau \in [t_0,\ t]$), the minimum-energy state estimate is simply the most recent value of $\bar{x}$,

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Minimum Energy Observer

Consider an LTI system

$$\dot{x}(t) = Ax(t) + Gw(t),$$
$$y(t) = Cx(t) + v(t),$$

where initial condition $x(0) \in \mathbb{R}^n$, disturbance $w(t) \in \mathbb{R}^r$ and measurement noise $v(t) \in \mathbb{R}^q$ are bounded.

The filtering problem for a given observation $y(s),\ 0 \leq s \leq t$ is, then, that of finding the trajectory with minimum energy needed to cause the same observation. The uncertainties $x(0)$, w, and v are assumed be an element of a Hilbert space and bounded in norm.

In fact we seek for uncertainties, $x(0)$, w, and v, that have a minimum energy

$$x^T(0)P^{-1}(0)x(0) + \int_0^T w^T(s)Q^{-1}w(s) + v^T(s)R^{-1}v(s)\ ds,$$

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subject to $\dot{x}(t) = Ax(t) + Gw(t), \quad y(t) = Cx(t) + v(t).$

Minimum Energy Observer (Contd.)

Since I know the solution, I cheat and instead of solving the optimization problem I show that the solution works!

Let P be a solution of Differential Riccati Equation

$$\dot{P}(t) = AP + PA^T + GQG^T - PC^T R^{-1} CP,$$

under the controllability assumption, $P(t)$ is invertible and considering the fact that $\frac{d}{dt} P_i^{-1} = -P_i^{-1} \dot{P}_i P_i^{-1}$, it is given by

$$\dot{P}^{-1} = -P^{-1}A - A^T P^{-1} - P^{-1} GQG^T P^{-1} + C^T R^{-1} C.$$

Let $\beta(t) \in \mathbb{R}^n$ be state of adjoint system satisfying

$$\dot{\beta}(t) = -A^T \beta(t) - C^T R^{-1} y(t) - P^{-1} GQG^T \beta(t),$$

where $\beta(0) = 0$, and let $\alpha(t)$ be a scalar satisfying

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Minimum Energy Observer (Contd.)

Since I know the solution, I cheat and instead of solving the optimization problem I show that the solution works!

Let P be a solution of Differential Riccati Equation

$$\dot{P}(t) = AP + PA^T + GQG^T - PC^T R^{-1} CP,$$

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Minimum Energy Observer (Contd.)

In order to make a perfect squares in cost function, we add a zero term

$$\left(x^T(t)P^{-1}(t)x(t) + 2\beta^T(t)x(t) + \alpha(t)\right)\Big]_0^\tau$$
$$-\int_0^\tau \frac{d}{dt}\left(x^T(t)P^{-1}(t)x(t) + 2\beta^T(t)x(t) + \alpha(t)\right)dt = 0$$

to the cost function.

Considering $v(t) = y(t) - Cx(t)$, we obtain

$$\left(x^T(\tau)P^{-1}(\tau)x(\tau) + 2\beta^T(\tau)x(\tau) + \alpha(\tau)\right)$$
$$+\int_0^\tau \|Q^{\frac{-1}{2}}(t)w(t) - Q^{\frac{T}{2}}G^TP^{-1}(t)x(t) - Q^{\frac{T}{2}}G^T\beta(t)\|^2 dt$$

where $Q = Q^{\frac{T}{2}}Q^{\frac{1}{2}}$.

It is easy to see that the trajectory that minimize this cost function at time τ is the argument which minimizes

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and $\dot{\beta}(t) = -A^T\beta(t) - C^T R^{-1} y(t) - P^{-1}GQG^T\beta(t),$

we obtain

$$\dot{\hat{x}}(t) = A\hat{x}(t) + P(t)C^T R^{-1}\left(y(t) - C\hat{x}(t)\right).$$

The Minimum Energy Estimator introduced before also has a stochastic interpretation. It is the famous Kalman Filter.

Why don't you investigate what does "kalman" command do in MATLAB?

Minimum Energy Observer (Contd.)

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References:

- “Optimal Filtering”, Brian D. O. Anderson and John B. Moore, Prentice-Hall, 1979
  Available online at
  `http://users.cecs.anu.edu.au/~john/papers/BOOK/B02.PDF`
- “Minimum covariance, minimax and minimum energy linear estimators”, Journal of Stochastic Control Theory and Stochastic Differential systems, 1979
- “Introduction to Dynamic Systems: Theory, Models, and Applications”, David Luenberger, New York: John Wiley & Sons, 1979
- “Linear Systems Theory” Joaõ Hespanha, Princeton University Press, 2009