Optimization and Optimal Control

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General Concept

General form:

minimize	cost function(al)
subject to	constraints

Optimal Control & Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0\big(x(t), t_0, t_F\big) \\ \text{subject to} & \left\{ \begin{array}{ll} f_i\big(x(t)\big) \leq b_i, & \quad i = 1, \ldots, m \\ \dot{x}(t) = g\big(x(t)\big) & \end{array} \right. \end{array}$$

- $x = \big(x_1(t), \ldots, x_n(t)\big)$: optimization variables
- $f_0 : \mathbb{R}^{n+2} \to \mathbb{R}$: objective function
- $f_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \ldots, m$: static constraint functions
- $g : \mathbb{R}^n \to \mathbb{R}^n$: dynamic constraint function (vector field)

optimal solution $x^\star(t)$ has smallest value of f_0 among all vectors that satisfy the constraints

Focus of this Session

At the end of this session we will be able to solve the following problem.

$$\text{minimize} \quad J_{LQR} = \int_0^{\infty} \left[x(t)^T Q x(t) + u(t)^T R u(t) + 2x(t)^T N u(t) \right] dt$$

$$\text{subject to} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases}$$

- $u(t) \in \mathbb{R}^k$: optimization variables
- $J_{LQR}(x(.), u(.)) \in \mathbb{R}$: objective functional
- $x(t) \in \mathbb{R}^n$: state of the system
- $y(t) \in \mathbb{R}^m$: output of the system

optimal solution $u^\star(t)$ has smallest value of J_{LQR} among all signals that satisfy the constraints.

Deterministic Linear Quadratic Regulation (LQR)

Following Fig. shows the feedback configuration for the linear quadratic regulation (LQR) problem.

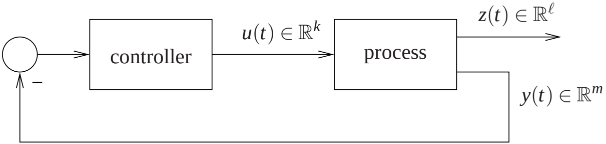

Abbildung:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ z(t) = Gx(t) + Hu(t) \end{cases}$$

It has two distinct outputs

- The measured output $y(t)$ corresponds to the signal(s) that can be measured and are therefore available for control.
- The performance output z(t) corresponds to a signal(s) that one would like to make as small as possible in the shortest possible time.

Optimal Regulation

The LQR problem is defined as follows. Find the control input $u(t)$, $t \in [0, \infty)$ that makes the following criterion as small as possible:

$$J_{LQR} = \int_0^{\infty} \left[\|z(t)\|^2 + \rho \|u(t)\|^2 \right] dt$$

where

- The the term $\int_0^{\infty} \|z(t)\|^2 dt$ corresponds to the energy of the performance output.
- The the term $\int_0^{\infty} \|u(t)\|^2 dt$ corresponds to the energy of the control signal.

By changing ρ in the cast function we can decide to have *cheap control* or *expensive control*.

Optimal Regulation (Contd.)

In the rest of this session we shall consider the most general form for a quadratic criterion, which is

$$J_{LQR} = \int_0^\infty \left[x(t)^T Qx(t) + u(t)^T Ru(t) + 2x(t)^T Nu(t)\right] dt$$

Feedback Invariants

In order to solve our problem we need the following definition:

Definition: Feedback Invariants

Given the LTI system $\dot{x}(t) = Ax(t) + Bu(t)$, we say that a functional $H(x(.), u(.))$ is a feedback invariant when computed along a solution to the system, its value depends only on the initial condition $x(0)$ and not on the specific input signal $u(.)$.

Proposition

For every symmetric matrix P, the functional

$$H(x(.), u(.)) = -\int_0^\infty \left[\left(Ax(t)+Bu(t)\right)^T Px(t) + x(t)^T P \left(Ax(t)+Bu(t)\right) \right] dt$$

is a feedback invariant for the system $\dot{x}(t) = Ax(t) + Bu(t)$, as long as $\lim_{t\to\infty} x(t) = 0$.

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Feedback Invariants (Contd.)

Proof of the Proposition

For every symmetric matrix P, the functional

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\begin{aligned}
& H(x(.), u(.)) \\
&= -\int_0^\infty \left[\big(Ax(t) + Bu(t)\big)^T Px(t) + x(t)^T P\big(Ax(t) + Bu(t)\big) \right] dt \\
&= -\int_0^\infty \left[\big(\dot{x}(t)\big)^T Px(t) + x(t)^T P\big(\dot{x}(t)\big) \right] dt \\
&= -\int_0^\infty \frac{d}{dt}\big(x(t)^T Px(t)\big) dt \\
&= x(0)^T Px(0) - \lim_{t\to\infty} x(t)^T Px(t) \\
&= x(0)^T Px(0) \qquad (\text{as long as } \lim_{t\to\infty} x(t) = 0)
\end{aligned}
$$

Feedback Invariants in Optimal Control

Suppose that we are able to express a criterion J_{LQR} by an appropriate choice of the input $u(.)$ in the following form:

$$J_{LQR} = \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t) + 2x(t)^T N u(t)\right] dt$$
$$= H(x(t), u(t)) + \int_0^\infty \left[\Lambda(x(t), u(t))\right] dt$$

where H is a feedback invariant and the function $\Lambda(x, u)$ has the property that for every $x \in \mathbb{R}^n$, $\min_{u \in \mathbb{R}^k} \Lambda(x, u) = 0$.

Optimal Control Solution

In this case, the control $u^\star(t) = \arg\min_{u \in \mathbb{R}^k} \Lambda(x, u)$ will minimize the criterion J_{LQR}, and the optimal value of J_{LQR} is equal to the feedback invariant $J_{LQR} = H(x(.), u(.))$.

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Optimal State Feedback

Now we will try to express the LQR criterion as $J_{LQR} = H(x(.), u(.)) + \int_0^\infty [\Lambda(x(t), u(t))]\, dt$. To this effect, we add and subtract the introduced feedback invariant to the LQR criterion.

$$J_{LQR} = \int_0^\infty [x^T Qx + u^T Ru + 2x^T Nu]\, dt + H(x(.), u(.)) - H(x(.), u(.))$$

$$= H(x(.), u(.)) + \int_0^\infty [x^T Qx + u^T Ru + 2x^T Nu]\, dt + \int_0^\infty [(Ax + Bu)^T Px + x^T P(Ax + Bu)]\, dt$$

$$= H(x(.), u(.)) + \int_0^\infty [x^T (A^T P + PA + Q)x + u^T Ru + 2u^T (B^T P + N^T)x]\, dt$$

By completing the square, we can group the quadratic term in u with the cross-term in u times x: $(u + Kx)^T R(u + Kx) = u^T Ru + x^T (PB + N)R^{-1}(B^T P + N^T)x + 2u^T (B^T P + N^T)x$ where $K = R^{-1}(B^T P + N^T)$. By adding and subtracting these terms to the criterion we conclude that

$$J_{LQR} = H(x(.), u(.)) + \int_0^\infty [x^T (A^T P + PA + Q - (PB + N)R^{-1}(B^T P + N^T))x + (u + Kx)^T R(u + Kx)]\, dt$$

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$$J_{LQR} = H(x(.), u(.))$$
$$+ \int_0^\infty [x^T (A^T P + PA + Q - (PB + N)R^{-1}(B^T P + N^T))x + (u + Kx)^T R(u + Kx)]\, dt$$

Optimal State Feedback (Contd.)

$$
\begin{aligned}
&J_{LQR} = H(x(.), u(.)) \\
&+ \int_0^{\infty} \left[x^T\left(A^T P + PA + Q - (PB+N)R^{-1}(B^T P + N^T)\right)x + (u+Kx)^T R(u+Kx)\right] dt
\end{aligned}
$$

If we are able to select the matrix P so that $(A^T P + PA + Q - (PB+N)R^{-1}(B^T P + N^T)) = 0$, we obtain precisely

$$
\begin{aligned}
J_{LQR} &= \int_0^{\infty} \left[x(t)^T Qx(t) + u(t)^T Ru(t) + 2x(t)^T Nu(t)\right] dt \\
&= H(x(t), u(t)) + \int_0^{\infty} \left[\Lambda(x(t), u(t))\right] dt
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where

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H(x(t), u(t)) = -\int_0^{\infty} \left[\left(Ax(t) + Bu(t)\right)^T Px(t) + x(t)^T P\left(Ax(t) + Bu(t)\right)\right] dt,
$$

$$
\Lambda(x(t), u(t)) = (u + Kx)^T R(u + Kx).
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Optimal State Feedback (Contd.)

Now that we can express the LQR criterion as

$$J_{LQR} = H(x(t), u(t)) + \int_0^{\infty} \left[\Lambda(x(t), u(t))\right] dt$$

and from the fact that H(x(.),u(.)) does not depend on selection of $u(.)$, in order to minimize the cost function we recall that $u^{\star}(t) = \arg\min_{u \in \mathbb{R}^k} \Lambda(x, u)$. Hence,

$$\Lambda(x(t), u(t)) = 0 \Rightarrow (u + Kx)^T R(u + Kx) = 0 \Rightarrow u = -Kx \quad (\text{where } K = R^{-1}(B^T P + N^T))$$

Optimal State Feedback (Summary)

- For the LTI system $\dot{x}(t) = Ax(t) + Bu(t)$ find $u(t)$ such that it minimizes

$$J_{LQR} = \int_0^\infty \left[x^T Qx + u^T Ru + 2x^T Nu\right] dt$$

- Find the matrix P such that

$$(A^T P + PA + Q - (PB + N)R^{-1}(B^T P + N^T)) = 0.$$

  (This called algebraic Riccati equation)
- Select $u(t)$ as

  `u(t)=-Kx(t)` (where $K = R^{-1}(B^T P + N^T)$)

Why don't you just use the command [K,P,E]=lqr(A,B,Q,R,N) in MATLAB?

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LQR and PID

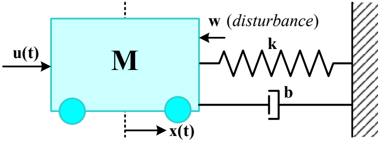

Abbildung:

$$\begin{cases} M\,a = \sum f \\ M\ddot{x} = u - w - b\dot{x} - kx \\ M\dot{v} = u - w - bv - kx \\ \text{Let state vector be } X = [x \;\; v]^T \end{cases}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u + \begin{bmatrix} 0 \\ -\frac{1}{M} \end{bmatrix} w$$

For the time being, forget the disturbance part ($w(t) = 0$). $\Rightarrow$ $\boxed{\dot{X}(t) = AX(t) + Bu(t)}$

Furthermore, let us apply a PD controller to regulate the position $x(t)$ about origin, i.e.

$$u(t) = -Px(t) - Dv(t) \quad \text{which can be written as} \quad u(t) = -[P \;\; D] \begin{bmatrix} x \\ v \end{bmatrix} = -[P \;\; D]X(t)$$

Recall the LQR format $\boxed{u(t) = -KX(t)}$ and compare it with PD controller $\boxed{u(t) = -[P \;\; D]X(t)}$.

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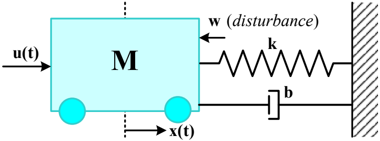

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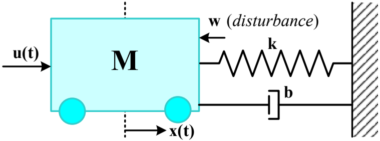

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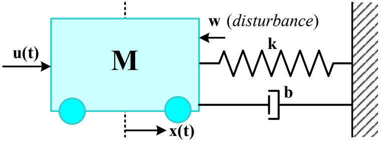

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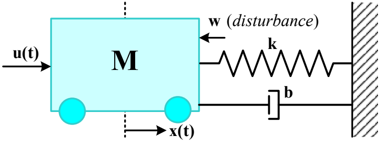

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LQR and PID (Summary)

- For the LTI system $\dot{x}(t) = Ax(t) + Bu(t)$ find $u(t)$ such that it minimizes

$$J_{LQR} = \int_0^\infty \left[x^T Q x + u^T R u\right] dt$$

- Find the matrix P such that ir solves the algebraic Riccati equation

$$A^T P + PA + Q = PBR^{-1} B^T P.$$

- Select $u(t)$ as

$$\boxed{u(t) = -KX(t) = -[P \;\; D]X(t)} \quad (\text{where } K = [P \;\; D] = R^{-1}(B^T P^T))$$

Remember that you can use the command [K,P,E]=lqr(A,B,Q,R) in MATLAB.

You can augment the state vector to include integral of position, i.e. $X = [\int x \quad x \quad v]^T$, to compute the integral gain in PID controller. However, due to unmodelled dynamics and other uncertainties in modelings, it is recommended to find and tune the integral term by setting the steady state error to zero.

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Remember that you can use the command [K,P,E]=lqr(A,B,Q,R) in MATLAB.

You can augment the state vector to include integral of position, i.e. $X = [\int x \quad x \quad v]^T$, to compute the integral gain in PID controller. However, due to unmodelled dynamics and other uncertainties in modelings, it is recommended to find and tune the integral term by setting the steady state error to zero.

LQR and PID (Summary)

- For the LTI system $\dot{x}(t) = Ax(t) + Bu(t)$ find $u(t)$ such that it minimizes

$$J_{LQR} = \int_0^\infty \left[x^T Q x + u^T R u\right] dt$$

- Find the matrix P such that ir solves the algebraic Riccati equation

$$A^T P + PA + Q = PBR^{-1}B^T P.$$

- Select $u(t)$ as

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