Dynamic Hypothesis Testing

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Dynamic Hypothesis Testing

Consider a sampled random Process {y(0), y(1), y(2), … y(t), y(t+1), …}

- **Assume that some properties of y(t) depends on an unknown parameter** θ
- **Consider the follwing N hypotheses:**
  - $\mathcal{H}_1 = \{\theta = \theta_1\}$
  - $\mathcal{H}_2 = \{\theta = \theta_2\}$
  - **…**
  - $\mathcal{H}_i = \{\theta = \theta_i\}$
  - **…**
  - $\mathcal{H}_N = \{\theta = \theta_N\}$
- **How can we assess the above-mentioned hypotheses?**

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Statistical hypothesis testing

A statistical hypothesis is a hypothesis that is testable on the basis of observing a process that is modeled via a set of random variables.

A statistical hypothesis test is a method of statistical inference.

Commonly, two statistical data sets are compared, or a data set obtained by sampling is compared against a synthetic data set from an idealized model.

A hypothesis is proposed for the statistical relationship between the two data sets, and this is compared as an alternative to an idealized null hypothesis that proposes no relationship between two data sets.

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Dynamic Hypothesis Testing

What do we want?

- Fast algorithm, preferably iterative in time
- Low computational power
- Identifiability and distinguishability criteria

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The core idea:

At each sampling time, the signal y(t) is used to generate the conditional probability of each hypothesis being true.
These conditional probabilities are then used to evaluate which hypothesis is more probable to comply with the collected measurements.

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Now we can reformulate the problem as the case of N hypotheses $\mathcal{H}_i \quad i = 1, 2, \cdots, N$ among which only one is true:

- let $\mathrm{Y}(\mathrm{t}) \equiv \{y(1), y(2), ..., y(t)\}$ denote the time history of the observed signal.
- Let us take $\mathcal{H}$ as the DHT random variable which takes the value $\mathcal{H} = \mathcal{H}_i$ on the event that the hypothesis $\mathcal{H}_i$ is true.
- Let us further assume that probability of any hypotheses being true at t=0 is equal, i.e.
$Pr\{\mathcal{H} = \mathcal{H}_i\} = \frac{1}{N} \quad for \quad i = 1, 2, ..., N$

In what follows, we establish an algorithm to calculate, in real time, the conditional probability of each hypothesis based on the observation vector

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we look to calculate, in real time, the conditional probability of each hypothesis based on the observation vector

$$h_i(t) := Pr\{\mathcal{H} = \mathcal{H}_i | Z(t)\} \qquad i = 1, 2, \cdots, N.$$

- **Rationally, as more measurement are available, one should become able to to calculate conditional hypothesis probabilities $h_i(t)$ with more accuracy, which in turn leads to more certainty in the choice of the true hypothesis.**

- **To calculate the conditional probability of each hypothesis in an iterative manner, we take a Bayesian approach such that at any sampling time instant, by relying on our current belief about all the hypotheses, we only study the information hidden in the most recent measurement.**

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Bayes' theorem $$p(A|B) = \frac{p(B\&A)}{p(B)} = \frac{p(B|A)p(A)}{p(B)}$$

- **To this end, by using Bayes rule, it follows that**

$$Pr\{\mathcal{H} = \mathcal{H}_i \big| Y(t+1)\} = \frac{Pr\{\mathcal{H} = \mathcal{H}_i\,,\, Y(t+1)\}}{Pr\{Y(t+1)\}}$$
$$= \frac{Pr\{\mathcal{H} = \mathcal{H}_i\,,\, y(t+1)\,,\, Y(t)\}}{Pr\{y(t+1)\,,\, Y(t)\}}$$

- **Equivalently,** $$Pr\{\mathcal{H} = \mathcal{H}_i \big| Y(t+1)\} = \frac{Pr\{y(t+1)\,,\mathcal{H}{=}\mathcal{H}_i \big| Y(t)\} Pr\{Y(t)\}}{Pr\{y(t+1) \big| Y(t)\} Pr\{Y(t)\}}$$
$$= \frac{Pr\{y(t+1)\,,\mathcal{H}{=}\mathcal{H}_i \big| Y(t)\}}{Pr\{y(t+1) \big| Y(t)\}}$$

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Bayes' theorem (cont.)

- **Recall from last slide** $$Pr\{\mathcal{H}=\mathcal{H}_i\big|Y(t+1)\}=\frac{Pr\{y(t+1)\,,\,\mathcal{H}=\mathcal{H}_i\big|Y(t)\}}{Pr\{y(t+1)\big|Y(t)\}}$$

- **Applying the conditional probability theorem to the numerator yields**

(p(x,y|z)=p(x|y,z)p(y|z) **)**

$$Pr\{\mathcal{H}=\mathcal{H}_i\big|Y(t+1)\}=\frac{Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Y(t)\}Pr\{\mathcal{H}=\mathcal{H}_i|Y(t)\}}{Pr\{y(t+1)\big|Y(t)\}}.$$

- **Furthermore, applying the total probability theorem to the denominator yields**

$$p(x)=\int_Y p(x|y)p(y)dy, \qquad p(x|y)=\int_Z p(x|y,z)p(z|y)dz,$$

$$Pr\{\mathcal{H}=\mathcal{H}_i\big|Y(t+1)\}=\frac{Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Y(t)\}Pr\{\mathcal{H}=\mathcal{H}_i|Y(t)\}}{\int_H Pr\{y(t+1)\big|Y(t)\,,\,\mathcal{H}\}Pr\{\mathcal{H}\big|Y(t)\}\,d\mathcal{H}}$$

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Bayes' theorem (cont.)

- **Recall from last slide**

$$Pr\{\mathcal{H} = \mathcal{H}_i \big| Y(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H} = \mathcal{H}_i \, , \, Y(t)\}Pr\{\mathcal{H} = \mathcal{H}_i|Y(t)\}}{\int_H Pr\{y(t+1)\big|Y(t) \, , \, \mathcal{H}\}Pr\{\mathcal{H}\big|Y(t)\} \, d\mathcal{H}}$$

- **Recall that H is the sample space of the random variable $\mathcal{H}$ and since H represents a finite set of variable, i.e. $H = \{\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N\}$, which means the equation above reduces to**

$$Pr\{\mathcal{H} = \mathcal{H}_i \big| Y(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H} = \mathcal{H}_i \, , \, Y(t)\}Pr\{\mathcal{H} = \mathcal{H}_i|Y(t)\}}{\sum_{k=1}^{N} Pr\{y(t+1)\big|Y(t) \, , \, \mathcal{H} = \mathcal{H}_k\}Pr\{\mathcal{H} = \mathcal{H}_k\big|Y(t)\}} .$$

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Conditional Probabilities: We are almost there!

- **Recall from last slide**

$$Pr\{\mathcal{H}=\mathcal{H}_i\big|Y(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Y(t)\}}{\sum_{k=1}^{N} Pr\{y(t+1)\big|Y(t)\,,\,\mathcal{H}=\mathcal{H}_k\}Pr\{\mathcal{H}=\mathcal{H}_k\big|Y(t)\}}Pr\{\mathcal{H}=\mathcal{H}_i|Y(t)\}$$

- **Before continuing our discussion on calculation of the conditional probability of each hypothesis let us highlight an important property of \the equation above.**

- **In fact, the proposed DHT algorithm relies on its current ``credence'' to adaptively settle on the most ``factual'' action by exploiting the information coded in the most recent measurement.**

- **The iterative nature of the equation above allows us to calculate, at each sampling time, the conditional probability of each hypothesis only based on the most recent measurement and our current credence (belief) about each hypothesis, i.e. conditional probability of the hypothesis at previous sampling time.**

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Conditional Probabilities

$$Pr\{\mathcal{H}=\mathcal{H}_i \big| Y(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Y(t)\}}{\sum_{k=1}^{N} Pr\{y(t+1)\big|Y(t)\,,\,\mathcal{H}=\mathcal{H}_k\}Pr\{\mathcal{H}=\mathcal{H}_k\big|Y(t)\}} Pr\{\mathcal{H}=\mathcal{H}_i|Y(t)\}$$

- **We only need to calculate** $Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Y(t)\}$ **to have the conditional probabilities.**

The example that follows
next will shed more light
to our approach.

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Fault Detection using Dynamic Hypothesis Testing

Detection of Mooring Line Failure using Dynamic Hypothesis Testing

- **Problem Formulation:**
- **Consider thruster assisted position mooring (PM) systems,**
- **Use only input and output data (plus dynamical model of the system)**
- **Try to detect any line breakage as soon as it occurs to compensate for the lost tension by proper use of DP thruster assistance**

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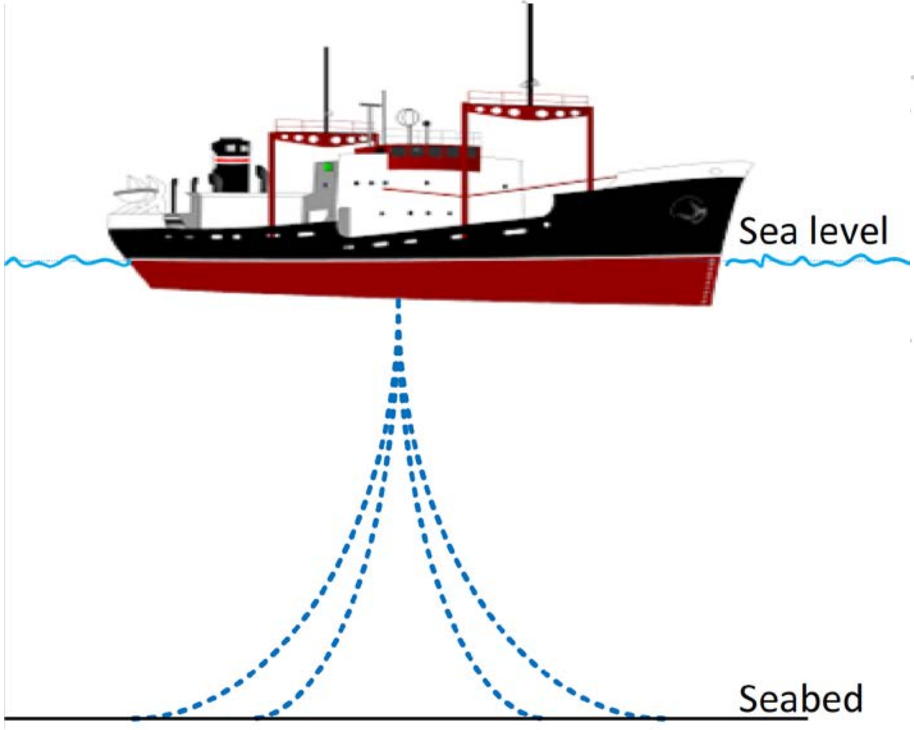
Fault Detection using Dynamic Hypothesis Testing

What is the thruster assisted position mooring (PM) systems?

- PM systems have been available since the late 1980s.
- PM systems are built upon DP systems.
- Key differences between PM and DP.
- The main function of thruster assistance in PM systems is to keep the heading angle at a desired value and add damping in the surge, sway and yaw motions while the mooring lines keep the position of the vessel in a predefined admissible region.



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Fault Detection using Dynamic Hypothesis Testing

Modeling the thruster assisted position mooring (PM) systems

- **Vehicle dynamics:**

$$
\begin{aligned}
\dot{\xi}_\omega &= A_\omega(\omega_0)\xi_\omega + E_\omega w_\omega \\
\eta_\omega &= R(\psi_L)C_\omega\xi_\omega \\
\dot{b} &= -T^{-1}b + E_b w_b \\
\dot{\eta}_L &= R(\psi_L)\nu \\
M\dot{\nu} + D\nu &= \tau_m + \tau_c + R^T(\psi_{tot})b \\
\eta_{tot} &= \eta_L + \eta_\omega \\
\eta_y &= \eta_{tot} + v
\end{aligned}
$$

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Fault Detection using Dynamic Hypothesis Testing

Modeling the thruster assisted position mooring (PM) systems (contd.)

- **Mooring line**

$$\tau_m = -R^T(\psi_{tot})g_{mo}(\eta_L) - d_{mo}(\nu)$$

$$\tau_m \approx -G_{mo}R^T(\psi_{tot})\eta_L - D_{mo}\nu$$

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Fault Detection using Dynamic Hypothesis Testing

Modeling the thruster assisted position mooring (PM) systems (contd.)

- **Total system**

$$
\begin{aligned}
\dot{\xi}_\omega &= A_\omega(\omega_0)\xi_\omega + E_\omega w_\omega \\
\eta_\omega &= R(\psi_L)C_\omega\xi_\omega \\
\dot{b} &= -T^{-1}b + E_b w_b \\
\dot{\eta}_L &= R(\psi_L)\nu \\
M\dot{\nu} + (D+ & D_{mo})\nu + G_{mo}R^T(\psi_{tot})\eta_L \\
&= \tau_c + R^T(\psi_{tot})b \\
\eta_{tot} &= \eta_L + \eta_\omega \\
\eta_y &= \eta_{tot} + v.
\end{aligned}
$$

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Fault Detection using Dynamic Hypothesis Testing

Modeling the thruster assisted position mooring (PM) systems (contd.)

- **Linearizing the model (under certain assumptions)**

$$
\begin{aligned}
\dot{\xi}_\omega &= A_\omega(\omega_0)\xi_\omega + E_\omega w_\omega \\
\eta^b_\omega &= C_\omega \xi_\omega \\
\dot{b}^p &= -T^{-1} b^p + w^f_b \\
\dot{\eta}^p_L &= \nu \\
M\dot{\nu} + (D + D_{mo})\nu + G_{mo}\eta^p_L &= \tau_c + b^p \\
\eta^f_y &= \eta^p_L + \eta^b_\omega
\end{aligned}
$$

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Modeling the thruster assisted position mooring (PM) systems (contd.)

- **Linearizing the model (under certain assumptions)**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Lw(t), \\ y(t) &= Cx(t) + v(t), \end{aligned}$$

$$A = \begin{bmatrix} A_\omega(\omega_0) & O & O & O \\ O & -T^{-1} & O & O \\ O & O & O & I \\ O & I & -M^{-1}G_{mo} & -M^{-1}(D+D_{mo}) \end{bmatrix} \quad B = \begin{bmatrix} O \\ O \\ O \\ I \end{bmatrix} \quad L = \begin{bmatrix} E_\omega & O \\ O & I \\ O & O \\ O & O \end{bmatrix}$$

$$C = \begin{bmatrix} C_\omega & O & I & O \end{bmatrix},$$

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Modeling the thruster assisted position mooring (PM) systems (contd.)

- **In what follows, without loss of generality, we consider the case of only single line failure.**
- **Let us assume the PM system consists of N-1 mooring lines;**
- **Hence, the A matrix can take one of N possible configurations (N -1 possible line break configuration plus one configuration without any fault).**

$$A = \begin{bmatrix} A_\omega(\omega_0) & O & O & O \\ O & -T^{-1} & O & O \\ O & O & O & I \\ O & I & -M^{-1}G_{mo} & -M^{-1}(D+D_{mo}) \end{bmatrix}$$

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Now we can reformulate the problem as the case of N hypotheses $\mathcal{H}_i \;\; i = 1, 2, \cdots, N$ among which only one is true:

- let $Z(t) \equiv \{U(t), Y(t)\}$ denote the time history of the observed signal, where $\mathrm{Y}(\mathrm{t}) \equiv \{y(1), y(2), ..., y(t)\}$ $U(t) \equiv \{u(1), u(2), \cdots, u(t)\}$
- Let us take $\mathcal{H}$ as the DHT random variable which takes the value $\mathcal{H} = \mathcal{H}_i$ on the event that the hypothesis $\mathcal{H}_i$ is true.
- Let us further assume that probability of any hypotheses being true at t=0 is equal, i.e.

$$Pr\{\mathcal{H} = \mathcal{H}_i\} = \frac{1}{N} \;\; for \;\; i = 1, 2, ..., N$$

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Fault Detection using Dynamic Hypothesis Testing

- **In what follows, without loss of generality, we consider the case of only single line failure.**
- **Let us assume the PM system consists of N-1 mooring lines;**
- **Hence, the A matrix can take one of N possible configurations (N -1 possible line break configuration plus one configuration without any fault).**

$$A = \begin{bmatrix} A_\omega(\omega_0) & O & O & O \\ O & -T^{-1} & O & O \\ O & O & O & I \\ O & I & -M^{-1}G_{mo} & -M^{-1}(D+D_{mo}) \end{bmatrix}$$

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Dynamic Hypothesis Testing

Conditional Probabilities in Dynamic Hypothesis Testing

$$Pr\{\mathcal{H} = \mathcal{H}_i \big| Z(t+1)\} = \frac{Pr\{z(t+1)|\mathcal{H} = \mathcal{H}_i\,,\, Z(t)\}}{\sum_{k=1}^{N} Pr\{z(t+1)\big|Z(t)\,,\, \mathcal{H} = \mathcal{H}_k\}Pr\{\mathcal{H} = \mathcal{H}_k\big|Z(t)\}} Pr\{\mathcal{H} = \mathcal{H}_i|Z(t)\}$$

- **Recall that the control action is a deterministic signal, so**

$$Pr\{\mathcal{H} = \mathcal{H}_i \big| Z(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H} = \mathcal{H}_i\,,\, Z(t)\}}{\sum_{k=1}^{N} Pr\{y(t+1)\big|Z(t)\,,\, \mathcal{H} = \mathcal{H}_k\}Pr\{\mathcal{H} = \mathcal{H}_k\big|Z(t)\}} Pr\{\mathcal{H} = \mathcal{H}_i|Z(t)\}$$

- **We only need to calculate** $Pr\{y(t+1)|\mathcal{H} = \mathcal{H}_i\,,\, Y(t)\}$ **to have the conditional probabilities.**

It is very easy to calculate!
Any idea how?

Flash back to Linear Systems

Good to remember that:

The linearity of

a) the state equation

b) the measurement equation

and the gaussian nature of

a) the initial state, $x(0)$

b) the plant white noise $\xi(t)$

c) the measurement white noise $\theta(t)$

imply that $p\left(x(t) \mid Y(t), U(t-1)\right)$ is gaussian!

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Back to Dynamic Hypothesis Testing

Conditional Probabilities in Dynamic Hypothesis Testing

$$Pr\{\mathcal{H}=\mathcal{H}_i \big| Z(t+1)\} = \frac{Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Z(t)\}}{\sum_{k=1}^{N} Pr\{y(t+1)\big|Z(t)\,,\,\mathcal{H}=\mathcal{H}_k\}Pr\{\mathcal{H}=\mathcal{H}_k \big| Z(t)\}} Pr\{\mathcal{H}=\mathcal{H}_i|Z(t)\}$$

- **We only need to calculate** $Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Z(t)\}$ **to have the conditional probabilities; and to do so, since random variable** $\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Z(t)\}$ **is Gaussian it only suffice to finds its mean and covariance**

The probability density function

$$Pr\{y(t+1)|\mathcal{H}=\mathcal{H}_i\,,\,Z(t)\}$$

can be iteratively calculated by a Kalman filter incorporating the assumption that

$$\mathcal{H}=\mathcal{H}_i$$

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Fault Detection using Dynamic Hypothesis Testing

Consider the system and incorporate the assumption that $\mathcal{H} = \mathcal{H}_i$

$$x(t+1) = A(t)x(t) + B(t)u(t) + L(t)\xi(t)$$

$$y(t+1) = C(t+1)x(t+1) + \theta(t+1)$$

- **Design a Kalman filter for the above-mentioned system:**

$$\hat{x}_{\mathcal{H}_i}(t+1) = A_{\mathcal{H}_i}\hat{x}_{\mathcal{H}_i}(t) + Bu(t) + K_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t),$$

$$\tilde{y}_{\mathcal{H}_i}(t) = y(t) - \hat{y}_{\mathcal{H}_i}(t)$$

$$\hat{y}_{\mathcal{H}_i}(t) = C\hat{x}_{\mathcal{H}_i}(t),$$

$$K_{\mathcal{H}_i} = \Sigma_{\mathcal{H}_i}C^T[C\Sigma_{\mathcal{H}_i}C^T + R]^{-1}$$

- **Solve the Riccati Eq:**

$$A_{\mathcal{H}_i}\Sigma_{\mathcal{H}_i}A_{\mathcal{H}_i}^T + LQL^T = \Sigma_{\mathcal{H}_i} + A_{\mathcal{H}_i}^T\Sigma_{\mathcal{H}_i}C^T[C\Sigma_{\mathcal{H}_i}C^T + R]^{-1}C\Sigma_{\mathcal{H}_i}A_{\mathcal{H}_i},$$

- **the covariance of the output estimation vector can be found as**

$$S_{\mathcal{H}_i} = C\Sigma_{\mathcal{H}_i}C^T + R,$$

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Fault Detection using Dynamic Hypothesis Testing

Detection of Mooring Line Failure using Dynamic Hypothesis Testing

- **Conditional Probabilities in Dynamic Hypothesis Testing:**

$$Pr\{\mathcal{H}=\mathcal{H}_i \big| Z(t+1)\} = \frac{\dfrac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_i}|}}}{\sum_{k=1}^{N} h_k(t)\dfrac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_k}(t+1)S^{-1}_{\mathcal{H}_k}\tilde{y}_{\mathcal{H}_k}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_k}|}}} Pr\{\mathcal{H}=\mathcal{H}_i|Z(t)\}$$

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Fault Detection using Dynamic Hypothesis Testing

Detection of Mooring Line Failure using Dynamic Hypothesis Testing

- **Conditional Probabilities in Dynamic Hypothesis Testing:**

$$h_i(t+1) = \frac{\dfrac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_i}|}}}{\sum_{k=1}^{N} h_k(t)\dfrac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_k}(t+1)S^{-1}_{\mathcal{H}_k}\tilde{y}_{\mathcal{H}_k}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_k}|}}} h_i(t)$$

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Fault Detection using Dynamic Hypothesis Testing

Detection of Mooring Line Failure using Dynamic Hypothesis Testing

- **We construct the set of hypotheses as** $H = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$ **where** $\mathcal{H}_0$ **is the hypothesis that all the mooring lines in the PM system are intact and** $\mathcal{H}_1, \cdots, \mathcal{H}_4$ **denote hypotheses of mooring line 1,…,4 brakeage, respectively.**

- **The following figures presents the results of simulations where the PM system is initially moored by all four mooring lines and at t=150 (sec) line number two breaks.**

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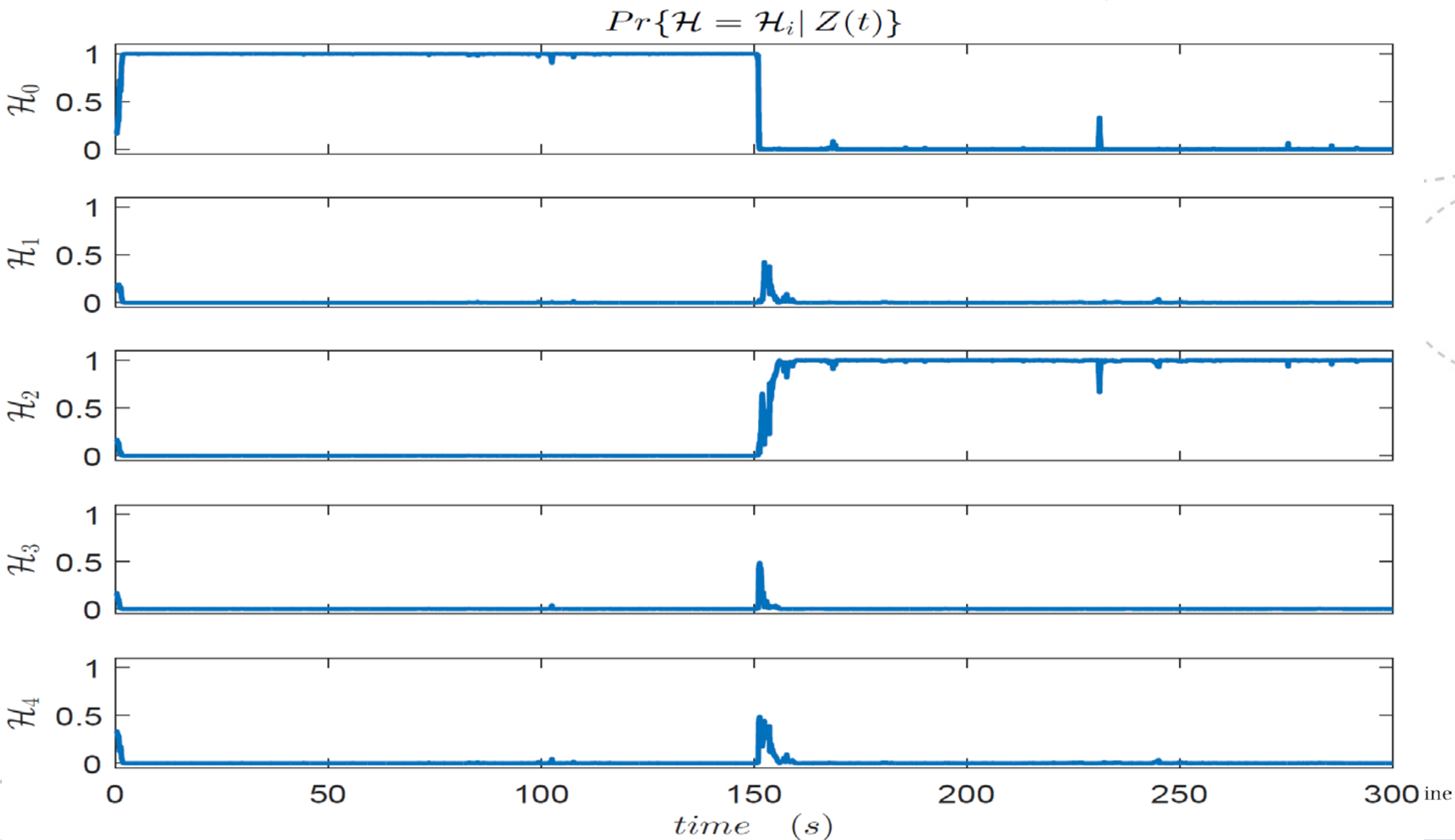
Fault Detection using Dynamic Hypothesis Testing



Dynamic Hypothesis Testing (Flash back)

What did we want?

- **Fast algorithm, preferably iterative in time**
- **Low computational power**
- **Convergence, Identifiability and distinguishability criteria**

Im following we will
study the necessary condition
under which any pair of
hypotheses are distinguishable.

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Dynamic Hypothesis Testing

Distinguishability of Hypotheses

- We would like to study under what conditions any pair of hypotheses are distinguishable from each other.
- We examine the necessary condition under which the conditional probabilities of the true hypothesis converge to one.

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Dynamic Hypothesis Testing

The space of N hypotheses forms a finite probability space.

- **Finite probability space! What does it mean?**
  - **We need to show that** $\forall \mathcal{H} \in H, \;\; P(\mathcal{H}) > 0 \quad and \quad \sum_{\mathcal{H} \in H} P(\mathcal{H}) = 1.$
- **Defining** $P_{sum}(t) = \sum_{i=1}^{N} h_i(t)$ **and computing its time-evolution:**

$$P_{sum}(t+1) \quad = \sum_{i=1}^{N} \frac{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1) S^{-1}_{\mathcal{H}_i} \tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{(2\pi)^3 |S_{\mathcal{H}_i}|}}}{\sum_{k=1}^{N} h_k(t) \frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_k}(t+1) S^{-1}_{\mathcal{H}_k} \tilde{y}_{\mathcal{H}_k}(t+1)}}{\sqrt{(2\pi)^3 |S_{\mathcal{H}_k}|}}} h_i(t) = \frac{\sum_{i=1}^{N} h_i(t) \frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1) S^{-1}_{\mathcal{H}_i} \tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{(2\pi)^3 |S_{\mathcal{H}_i}|}}}{\sum_{k=1}^{N} h_k(t) \frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_k}(t+1) S^{-1}_{\mathcal{H}_k} \tilde{y}_{\mathcal{H}_k}(t+1)}}{\sqrt{(2\pi)^3 |S_{\mathcal{H}_k}|}}} = 1.$$

- **It is easy to verify that if the initial probabilities are selected properly then the space of N hypotheses forms a finite probability space.**

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Dynamic Hypothesis Testing

Theorem:

Let $\mathcal{H}_i$ **be the true hypothesis and let** $H^i = \{\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N\} \setminus \{\mathcal{H}_i\}$ **be the set of all remaining hypothesis (except** $\mathcal{H}_i$**). Suppose that there exist positive constants** n_1 , t_1 , and ϵ **such that for all** $t \geq t_1$, $n \geq n_1$, and $\mathcal{H}_j \in H^i$ **the following condition holds:**

$$\frac{1}{n}\sum_{\tau=t}^{t+n-1}\left(\frac{1}{2}\tilde{y}_{\mathcal{H}_i}^T(\tau)S_{\mathcal{H}_i}^{-1}\tilde{y}_{\mathcal{H}_i}(\tau)\right)+\frac{1}{2}\ln|S_{\mathcal{H}_i}|+\epsilon<\frac{1}{n}\sum_{\tau=t}^{t+n-1}\left(\frac{1}{2}\tilde{y}_{\mathcal{H}_j}^T(\tau)S_{\mathcal{H}_j}^{-1}\tilde{y}_{\mathcal{H}_j}(\tau)\right)+\frac{1}{2}\ln|S_{\mathcal{H}_j}|$$

Then, conditional probability of true hypothesis, i.e. $h_i(t)$**, converges to one as** $t \to \infty$ **.**

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Proof:

Let us define $L_i^j(t) = \dfrac{Pr\{\mathcal{H} = \mathcal{H}_j | Z(t)\}}{Pr\{\mathcal{H} = \mathcal{H}_i | Z(t)\}} = \dfrac{h_j(t)}{h_i(t)}; \quad \mathcal{H}_j \in H^i.$

Remember that

- $H^i = \{\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N\} \setminus \{\mathcal{H}_i\}$ **and** $h_i(t+1) = \dfrac{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_i}|}}}{\sum_{k=1}^N h_k(t)\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_k}(t+1)S^{-1}_{\mathcal{H}_k}\tilde{y}_{\mathcal{H}_k}(t+1)}}{\sqrt{(2\pi)^3|S_{\mathcal{H}_k}|}}} h_i(t)$

We obtain

$$L_i^j(t+1) = \frac{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_j}(t+1)S^{-1}_{\mathcal{H}_j}\tilde{y}_{\mathcal{H}_j}(t+1)}}{\sqrt{|S_{\mathcal{H}_j}|}}}{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{|S_{\mathcal{H}_i}|}}} L_i^j(t),$$

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Dynamic Hypothesis Testing

Proof: (cont.)

- **Recall from last slide**

$$L_i^j(t+1) = \frac{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_j}(t+1)S^{-1}_{\mathcal{H}_j}\tilde{y}_{\mathcal{H}_j}(t+1)}}{\sqrt{|S_{\mathcal{H}_j}|}}}{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(t+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(t+1)}}{\sqrt{|S_{\mathcal{H}_i}|}}} L_i^j(t),$$

- **from which it follows that**

$$L_i^j(t+n) = \prod_{\tau=t}^{t+n-1} \frac{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_j}(\tau+1)S^{-1}_{\mathcal{H}_j}\tilde{y}_{\mathcal{H}_j}(\tau+1)}}{\sqrt{|S_{\mathcal{H}_j}|}}}{\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(\tau+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(\tau+1)}}{\sqrt{|S_{\mathcal{H}_i}|}}} L_i^j(t).$$

- **Taking logarithms of both sides**

$$\ln \frac{L_i^j(t+n)}{L_i^j(t)} = \sum_{\tau=t}^{t+n-1} \ln\left(\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_j}(\tau+1)S^{-1}_{\mathcal{H}_j}\tilde{y}_{\mathcal{H}_j}(\tau+1)}}{\sqrt{|S_{\mathcal{H}_j}|}}\right) - \sum_{\tau=t}^{t+n-1} \ln\left(\frac{e^{-\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(\tau+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(\tau+1)}}{\sqrt{|S_{\mathcal{H}_i}|}}\right)$$

$$= -\sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}^T_{\mathcal{H}_j}(\tau+1)S^{-1}_{\mathcal{H}_j}\tilde{y}_{\mathcal{H}_j}(\tau+1)\right) + \frac{n}{2}\ln|S_{\mathcal{H}_j}| + \sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}^T_{\mathcal{H}_i}(\tau+1)S^{-1}_{\mathcal{H}_i}\tilde{y}_{\mathcal{H}_i}(\tau+1)\right) + \frac{n}{2}\ln|S_{\mathcal{H}_i}|$$

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Dynamic Hypothesis Testing

Proof: (cont.)

- **Recall from last slide**

$$\ln \frac{L_i^j(t+n)}{L_i^j(t)} = -\sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}_{\mathcal{H}_j}^T(\tau+1)S_{\mathcal{H}_j}^{-1}\tilde{y}_{\mathcal{H}_j}(\tau+1)\right) + \frac{n}{2}\ln|S_{\mathcal{H}_j}| + \sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}_{\mathcal{H}_i}^T(\tau+1)S_{\mathcal{H}_i}^{-1}\tilde{y}_{\mathcal{H}_i}(\tau+1)\right) + \frac{n}{2}\ln|S_{\mathcal{H}_i}|$$

- **Compare this with the condition in the theorem:**

$$\frac{1}{n}\sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}_{\mathcal{H}_i}^T(\tau)S_{\mathcal{H}_i}^{-1}\tilde{y}_{\mathcal{H}_i}(\tau)\right) + \frac{1}{2}\ln|S_{\mathcal{H}_i}| + \epsilon < \frac{1}{n}\sum_{\tau=t}^{t+n-1} \left(\frac{1}{2}\tilde{y}_{\mathcal{H}_j}^T(\tau)S_{\mathcal{H}_j}^{-1}\tilde{y}_{\mathcal{H}_j}(\tau)\right) + \frac{1}{2}\ln|S_{\mathcal{H}_j}|$$

- **For** $t \geq t_1$ and $n \geq n_1$, **we can conclude that there exists a positive** ϵ **such that**

$$\ln \frac{L_i^j(t+n)}{L_i^j(t)} < -n\epsilon$$ **or, equivalently,** $$L_i^j(t+n) < e^{-n\epsilon}L_i^j(t)$$

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Dynamic Hypothesis Testing

Proof: (cont.)

- **Recall from last slide(s)**
  - **We defined** $$L_i^j(t) = \frac{Pr\{\mathcal{H} = \mathcal{H}_j | Z(t)\}}{Pr\{\mathcal{H} = \mathcal{H}_i | Z(t)\}} = \frac{h_j(t)}{h_i(t)}; \quad \mathcal{H}_j \in H^i.$$

  where $H^i = \{\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N\} \setminus \{\mathcal{H}_i\}$

  - **Then, we showed that for** $t \geq t_1 \;$ and $\; n \geq n_1$, **there exists** ϵ **such that**

$$L_i^j(t+n) < e^{-n\epsilon} L_i^j(t)$$

- **Which completes our proof:**

$$L_i^j(t) = \frac{Pr\{\mathcal{H} = \mathcal{H}_j | Z(t)\}}{Pr\{\mathcal{H} = \mathcal{H}_i | Z(t)\}} \to 0 \quad \text{as} \quad n \to \infty \quad \forall \mathcal{H}_j \in H^i$$

Dynamic Hypothesis Testing (Flash back)

What did we want?

- **Fast algorithm, preferably iterative in time**
- **Low computational power**
- **Convergence, Identifiability and distinguishability criteria**

As long as the distinguishability condition

$$\frac{1}{n}\sum_{\tau=t}^{t+n-1}\left(\frac{1}{2}\tilde{y}_{\mathcal{H}_i}^T(\tau)S_{\mathcal{H}_i}^{-1}\tilde{y}_{\mathcal{H}_i}(\tau)\right)+\frac{1}{2}\ln|S_{\mathcal{H}_i}|+\epsilon<\frac{1}{n}\sum_{\tau=t}^{t+n-1}\left(\frac{1}{2}\tilde{y}_{\mathcal{H}_j}^T(\tau)S_{\mathcal{H}_j}^{-1}\tilde{y}_{\mathcal{H}_j}(\tau)\right)+\frac{1}{2}\ln|S_{\mathcal{H}_j}|$$

holds, any pair of hypotheses are distinguishable from each other and for sufficiently large n (number of collected measurements) the true hypothesis will be selected.

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Dynamic Hypothesis Testing

More on distinguishability condition:

- **For Linear Time-Invariant (LTI) systems the defined distinguishability condition can be verified in advance and off-line**
- **It can be used to define a (Pseudo) Metric Topology on space of Linear Dynamic Systems**
- **And further uses to follow!**

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Link between DHT and ML estimation

Consider the following system with parametric uncertainty

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + L(\theta)w(t),$$
$$y(t) = C(\theta)x(t) + v(t),$$

where θ belongs to finite set $\Theta = \{\theta_1, \theta_2, \cdots, \theta_N\}$

We know how we can estimate θ by using input and output measurements using DHT.

How is it related to ML estimation of θ?

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Maximum Likelihood Estimation (Flash back)

- let $Z(t) \equiv \{U(t), Y(t)\}$ denote the time history of the observed signal, where $\mathrm{Y(t)} \equiv \{y(1), y(2), ..., y(t)\}$ $U(t) \equiv \{u(1), u(2), \cdots, u(t)\}$
- We would like to highlight that the output measurements y(t) is a random vector.
- The joint probability density function $Pr\{Y(t);\theta\}$ belongs to a finite set of joint probability density functions $Pr\{Y(t);\theta_i\}$ indexed by the parameter $\theta_i \in \Theta = \{\theta_1, \theta_2, \cdots, \theta_N\}$
- For a fixed measurement $Y(t)$, the joint probability density function can be considered as a function of θ, called likelihood function and denoted by $\mathcal{L}(\theta; Y(t)) = p(Y(t);\theta)$

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Maximum Likelihood Estimation (Flash back)

In other words, the likelihood function of the state space model is a parameterized density function of the set of observations Y(t) which reflects how likely it is to observe Y(t) if θ were the true values of the uncertain parameters.

In other words, $Pr\{Y(t);\theta\} = p(Y(t);\theta)$ **is a finite family of density functions that can be computed for different values of θ and observation history Y(t), and for a fixed set of observations Y(t), the maximum likelihood estimate** $\hat{\theta}$ **of θ, is defined as the value that maximizes** $p(Y(t);\theta)$ **(for the above mentioned fixed Y(t)):**

$$\hat{\theta} = \underset{\theta_i \in \Theta}{\operatorname{argmax}} \, \mathcal{L}(\theta; Y(t))$$

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Maximum Likelihood Estimation (Flash back)

Recall from last slide that $$\hat{\theta} = \underset{\theta_i \in \Theta}{\operatorname{argmax}}\ p(Y(t);\theta)$$

- **Using the definition of conditional probability and employing Bayes's theorem recursively, $p(Y(t);\theta)$ can be described as product of conditional densities**

$$p(Y(t);\theta) = \prod_{\tau=1}^{\tau=t} p(y(\tau)|Y(\tau-1);\theta)$$

It is very easy to calculate!
You remeber how?

Maximum Likelihood Estimation (Flash back)

Recall from last slide that $$\hat{\theta} = \underset{\theta_i \in \Theta}{\operatorname{argmax}}\ p(Y(t);\theta)$$

- **Designing a Kalman filter for the above-mentioned system, incorporating** $\theta = \theta_i$ **in the dynamic equations, we can easily calculate** $p(y(t)|Y(t-1);\theta_i)$ **as**

$$p(y(t)|Y(t-1);\theta) = \frac{e^{-\frac{1}{2}\tilde{y}_\theta^T(t)S_\theta^{-1}(t)\tilde{y}_\theta(t)}}{(2\pi)^{\frac{q}{2}}\sqrt{|S_\theta(t)|}},$$

- **It is easy to verify that**

$$\mathcal{L}(\theta;Y(t)) = p(Y(t);\theta) = \prod_{\tau=1}^{\tau=t} p(y(\tau)|Y(\tau-1);\theta) = \prod_{\tau=1}^{\tau=t} \frac{e^{-\frac{1}{2}\tilde{y}_\theta^T(\tau)S_\theta^{-1}(\tau)\tilde{y}_\theta(\tau)}}{(2\pi)^{\frac{q}{2}}\sqrt{|S_\theta(\tau)|}}$$

- **It is in several respects more convenient to work with the logarithm of the likelihood function, called the log-likelihood function, defined as**

$$\log\left(\mathcal{L}(\theta;Y(t))\right) = \sum_{\tau=1}^{\tau=t}[-\frac{q}{2}\log(2\pi) - \frac{1}{2}\log(|S_\theta(\tau)|) - \frac{1}{2}\tilde{y}_\theta^T(\tau)S_\theta^{-1}(\tau)\tilde{y}_\theta(\tau)]$$

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Maximum Likelihood Estimation (Flash back)

Recall from last slide that

$$\hat{\theta} = \underset{\theta_i \in \Theta}{\operatorname{argmax}} \mathcal{L}(\theta; Y(t)) = \underset{\theta_i \in \Theta}{\operatorname{argmax}}\ p(Y(t); \theta) = \underset{\theta_i \in \Theta}{\operatorname{argmax}} \left\{ \log \left(\mathcal{L}(\theta; Y(t)) \right) \right\}$$

And the log-likelihood function

$$\log \left(\mathcal{L}(\theta; Y(t)) \right) = \sum_{\tau=1}^{\tau=t} [-\frac{q}{2} \log(2\pi) - \frac{1}{2} \log(|S_\theta(\tau)|) - \frac{1}{2} \tilde{y}_\theta^T(\tau) S_\theta^{-1}(\tau) \tilde{y}_\theta(\tau)]$$

- **Now, recall our distinguishability condition**

$$\frac{1}{n} \sum_{\tau=t}^{t+n-1} \left(\frac{1}{2} \tilde{y}_{\mathcal{H}_i}^T(\tau) S_{\mathcal{H}_i}^{-1} \tilde{y}_{\mathcal{H}_i}(\tau) \right) + \frac{1}{2} \ln |S_{\mathcal{H}_i}| + \epsilon < \frac{1}{n} \sum_{\tau=t}^{t+n-1} \left(\frac{1}{2} \tilde{y}_{\mathcal{H}_j}^T(\tau) S_{\mathcal{H}_j}^{-1} \tilde{y}_{\mathcal{H}_j}(\tau) \right) + \frac{1}{2} \ln |S_{\mathcal{H}_j}|$$

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Questions?