TTK4135 - Assignment 3

Problem 1: LP, and KKT condition

$$\min_{x \in \mathbb{R}^n} f(x) = c^T x$$

s.t.

$$Ax = b$$
$$x \geq 0$$

KKT conditions for LP problems:

$$\nabla L(x^*, \lambda^*) = c - A^T \lambda^* - s^* = 0$$
$$Ax^* = b$$
$$x^* \geq 0$$
$$s^* \geq 0$$
$$s_i^* x_i^* = 0, \quad i \in \{1, \dots, n\}$$

a) Newton direction: $p_k^N = -(\nabla^2 f_k)^{-1} \nabla f_k$

$$\nabla f_k = c$$

$$\nabla^2 f_k = 0 \implies (\nabla^2 f_k)^{-1} \text{ does not exist}$$

$\implies$ The Newton path is not defined for LP probs.

b) Definition of a convex function:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

Convexity of the objective function:

$$f(x) = c^T x$$

$$f(\alpha x + (1-\alpha)y) = c^T(\alpha x + (1-\alpha)y)$$
$$= \alpha c^T x + (1-\alpha)c^T y$$
$$= \alpha f(x) + (1-\alpha)f(y)$$

$\Longrightarrow$ The objective function is convex.

Convexity of the constraints

$$c_1(x) = Ax$$

$$c_1(\alpha x + (1-\alpha)y) = A(\alpha x + (1-\alpha)y)$$
$$= \alpha Ax + (1-\alpha)Ay$$
$$= \alpha c_1(x) + (1-\alpha)c_1(y)$$

$\longrightarrow$ The equality constraints are convex

$$c_2(x) = x$$
$$c_2(\alpha x + (1-\alpha)y) = \alpha x + (1-\alpha)y = \alpha c_2(x) + (1-\alpha)c_2(y)$$

$\Longrightarrow$ The inequality constraints are convex

$\Longrightarrow$ The LP problem is convex.

c) $\max_{\lambda \in \mathbb{R}^m} \; b^T\lambda$

s.t.

$A^T\lambda \leq c$

$\Updownarrow$

$\min_{\lambda \in \mathbb{R}^m} \; g(\lambda) = -b^T\lambda$

s.t.

$c - A^T\lambda \geq 0$

$L(x, \lambda) = -b^T\lambda - x^T(c - A^T\lambda)$

$\nabla_\lambda L(x, \lambda) = -b + Ax$

KKT conditions :

(I) $\nabla_\lambda L(x^*, \lambda^*) = 0 \Longrightarrow Ax^* = b$

(II) $c - A^T\lambda^* \geq 0 \Longrightarrow A^T\lambda^* \leq c$

(III) $x^* \geq 0$

(IV) $x_i^*(c - A^T\lambda_i^*)_i = 0 \qquad i = 1, \dots, n$

$s = c - A^T\lambda$

$\Longrightarrow$ (II) $s^* \geq 0$

(IV) $x_i^* s_i^* = 0 \qquad i = 1, \dots, n$

Hence the KKT conditions for the dual problem are equal to the KKT conditions of the original problem.

d)

$$x_i^*(c - A^T\lambda^*)_i = 0 \implies (c - A^T\lambda^*)^T x^* = 0$$

$$Ax^* = b$$

$$(c - A^T\lambda^*)^T x^* = (c^T - (A^T\lambda^*)^T)x^*$$

$$= c^T x^* - \lambda^{*T} A x^*$$

$$= c^T x^* - \lambda^{*T} b = 0$$

$$\implies c^T x^* = \lambda^{*T} b$$

Since the expressions are scalar, they can be transposed independently.

$$\implies c^T x^* = (\lambda^{*T} b)^T = b^T \lambda^*$$

$$c^T x^* = b^T \lambda^*$$

e) Basic feasible point:

$\beta \subseteq \{1, 2, \dots, n\}$, β contains m indices

$i \notin \beta \longrightarrow x_i = 0$

$B = [A_i]_{i \in \beta} \in \mathbb{R}^{m \times m}$

f) A has full rank

The constraint $Ax = b$ can be written as

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$a_i^T x = b_i \quad , \quad i \in \mathcal{E}$$

$$c_i(x) = a_i^T x - b_i \quad , \quad i \in \mathcal{E}$$

$$\nabla c_i(x) = a_i \quad , \quad i \in \mathcal{E}$$

Since A has full rank a_i will be linearly independent, hence the constraint gradients $\nabla c_i(x)$ will be linearly independent.

$\Longrightarrow$ LICQ is satisfied.

Problem 2: LP

Reactors R_I & R_{II}

Products A and B

1 ton of A requires 2 hours of R_I, 1 hour of R_{II}

1 ton of B requires 1 hour of R_I, 3 hours of R_{II}

Production cannot be negative.

$$\Longrightarrow \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

Available hours of the reactors:

8 hours for R_I

15 hours for R_{II}

$$\Longrightarrow \quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

Selling price of A is $\frac{3}{2}$ of the selling price of B:

$$\Longrightarrow \quad -c^T = \begin{bmatrix} 3 & 2 & 0 & 0 \end{bmatrix}$$

a) The problem can the be formulated as:

$$\min_{x \in \mathbb{R}^4} f(x) = c^T x$$

s.t

$$Ax = b$$
$$x \geq 0$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \quad c = \begin{bmatrix} -3 & -2 & 0 & 0 \end{bmatrix}^T$$

c) $x_0 = \begin{bmatrix} 0 & 0 & 8 & 15 \end{bmatrix}^T$

$x_1 = \begin{bmatrix} 4 & 0 & 0 & 11 \end{bmatrix}^T$

$x^* = \begin{bmatrix} 1{,}8 & 4{,}4 & 0 & 0 \end{bmatrix}^T$

$c_1(x) = 2x_1 + x_2 - x_3 = 8$

$c_2(x) = x_1 + 3x_2 - x_4 = 15$

$c_3(x) = x_3 \geq 0$

$c_4(x) = x_4 \geq 0$

$c_1(x^*) = 2 \cdot 1{,}8 + 4{,}4 = 8$

$c_2(x^*) = 1{,}8 + 3 \cdot 4{,}4 = 15$

The solution is at the intersection of $c_1(x)$ and $c_2(x)$.

All the constraints are active.

e) Looking the iterations it complies with the theory in chapter 13.3.

Problem 3 : QP and KKT conditions

$$\min_{x \in \mathbb{R}^n} q(x) = \frac{1}{2}x^T G x + c^T x$$

s.t.

$$a_i^T x = b_i \,, \quad i \in \mathcal{E}$$
$$a_i^T x \geq b_i \,, \quad i \in \mathcal{I}$$

a) Active set:

$$\underline{\underline{\mathcal{A}(x^*) = \{ i \in \mathcal{E} \cup \mathcal{I} \mid a_i^T x = b_i \}}}$$

b)

$$L(x, \lambda) = \frac{1}{2}x^T G x + c^T x - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i (a_i^T x - b_i)$$

$$\nabla_x L(x, \lambda) = Gx + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i a_i^T$$

$$\nabla_x L(x^*, \lambda^*) = Gx^* + c - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* a_i^T = 0$$

$$a_i^T x^* = b_i \,, \quad i \in \mathcal{A}(x^*)$$

$$a_i^T x^* \geq b_i \,, \quad i \in \mathcal{I} \setminus \mathcal{A}(x^*)$$

$$\underline{\underline{\lambda_i^* \geq 0 \,, \quad i \in \mathcal{I} \cap \mathcal{A}(x^*)}}$$