TTK4135 - Assignment 4

Problem 1: Quadratic programming

QP problem:

$$\min_{x \in \mathbb{R}^n} q(x) = \frac{1}{2}x^T G x + c^T x$$

s.t.

$$a_i^T x = b_i \quad , \quad i \in \mathcal{E}$$
$$a_i^T x \geq b_i \quad , \quad i \in \mathcal{I}$$

a) A QP problem is convex if the Hessian matrix G is positive semidefinite.
Convexity is an important property because for convex problems every local minima is a global minima.

b) $p = x^* - x \Longrightarrow x = x^* - p$

$$q(x) = \frac{1}{2}x^T G x + c^T x$$

$$= \frac{1}{2}(x^* - p)^T G (x^* - p) + c^T (x^* - p)$$

$$= \frac{1}{2}x^{*T} G x^* - \frac{1}{2}x^{*T} G p - \frac{1}{2}p^T G x^* + \frac{1}{2}p^T G p$$
$$+ c^T x^* - c^T p$$

$$q(x^*) = \frac{1}{2} x^{*T} G x^* + c^T x^*$$

$$\longrightarrow q(x) = q(x^*) - p^T G x^* + \frac{1}{2} p^T G p - c^T p$$

From the KKT conditions we have:

$$Ax = b \longrightarrow Ap = A(x^* - x) = Ax^* - Ax$$
$$= b - b = 0$$

$$Gx^* = -c + A^T \lambda^*$$

$$\Longrightarrow -p^T G x^* = -p^T(-c + A^T \lambda^*) = p^T c - p^T A^T \lambda^*$$
$$= p^T c - (Ap)^T \lambda^* = p^T c$$

$$\Longrightarrow q(x) = q(x^*) + p^T c - c^T p + \frac{1}{2} p^T G p$$
$$= q(x^*) + \frac{1}{2} p^T G p$$

$$p^T G p \geq 0$$

$$\Longrightarrow \underline{\underline{q(x) \geq q(x^*)}}$$

Reformulation of theorem 16.2:

Let A have full row rank and assume that the reduced-Hessian matrix $Z^T G Z$ is positive semidefinite. Then the vector x^* satisfying (16.4) is <u>a</u> global solution of (16.3).

c) Example 16.4
Algorithm 16.3

$$\min_{x \in R^2} q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

s.t

$$
\begin{aligned}
x_1 - 2x_2 + 2 &\geq 0 & (1)\\
-x_1 - 2x_2 + 6 &\geq 0 & (2)\\
-x_1 + 2x_2 + 2 &\geq 0 & (3)\\
x_1 &\geq 0 & (4)\\
x_2 &\geq 0 & (5)
\end{aligned}
$$

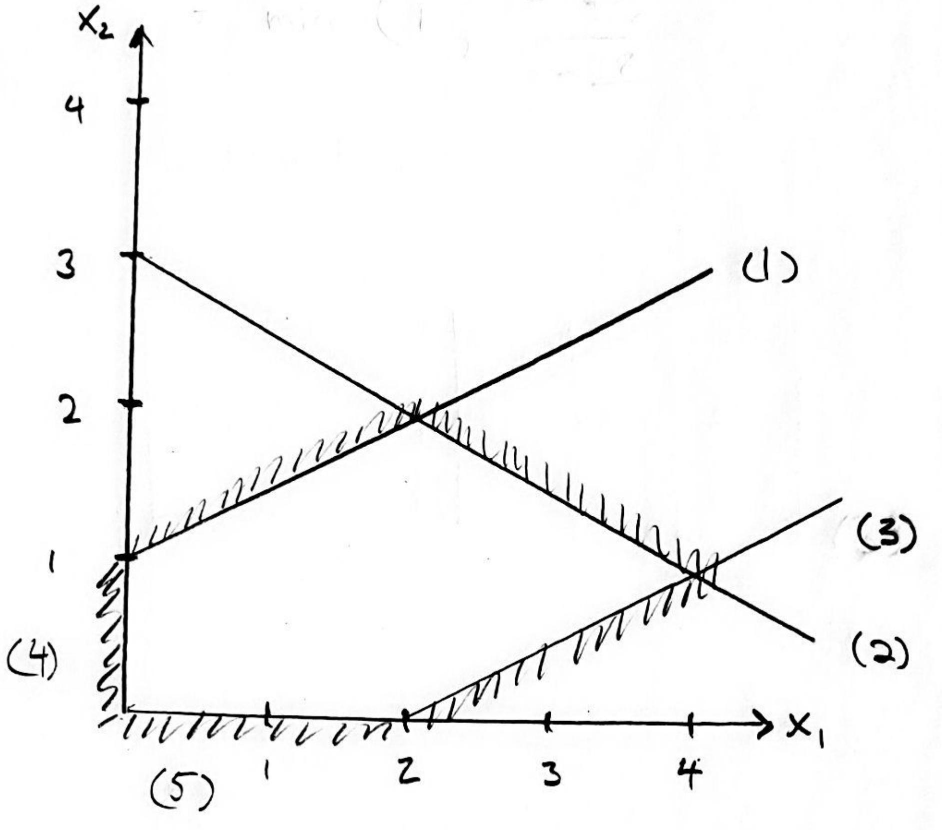
$$q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$= x_1^2 - 2x_1 + 1 + x_2^2 - 5x_2 + 2.5^2$$

$$\Longrightarrow \quad G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad c = \begin{bmatrix} -2 & -5 \end{bmatrix}^T$$

$$L(x,\lambda) = \frac{1}{2} x^T G x + c^T x - \sum_{i \in A(x)} \lambda_i a_i^T x_i$$

$$\nabla_x L(x,\lambda) = Gx + c - \sum_{i \in A(x^*)} \lambda_i a_i^T$$



x_2
4
3
2
1
(1)
(3)
(2)
(4)
(5)
1
2
3
4
x_1

Iteration 1:

$x_0 = [2 \quad 0]^T \implies g_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

$W_0 = \{3\}$

$= \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$\min \ \frac{1}{2}p^T G p + g_0^T p$

s.t.

$[-1 \quad 2] p_0 = 0$

$$\begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} p \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} -g_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} p_{01} \\ p_{02} \\ \tilde{\lambda}_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$$

$2p_{01} + \tilde{\lambda}_3 = -2 \implies p_{01} = -1 - \frac{1}{2}\hat{\lambda}_3$

$2p_{02} - 2\hat{\lambda}_3 = 5 \implies p_{02} = \frac{5}{2} + \hat{\lambda}_3$

$-p_{01} + 2p_{02} = 0 \implies p_{02} = \frac{1}{2}p_{01}$

$\implies \tilde{\lambda}_3 + \frac{5}{2} = \frac{1}{2}p_{01}$

$\tilde{\lambda}_3 = \frac{1}{2}p_{01} - \frac{5}{2}$

$\implies p_{01} = -1 - \frac{1}{2}\left(\frac{1}{2}p_{01} - \frac{5}{2}\right)$

$p_{01} = -1 - \frac{1}{4}p_{01} + \frac{5}{4}$

$$\frac{5}{4} p_{01} = \frac{1}{4}$$

$$\underline{p_{01} = \frac{1}{5}}$$

$$\underline{p_{02} = \frac{1}{10}} \implies \underline{p_0 = \left[\frac{1}{5} \quad \frac{1}{10}\right]^T}$$

(2) is the blocking constraint:

$$\alpha_0 = \min\left(1, \frac{-6-([-1 \ -2]x_0)}{[-1 \ -2]p_0}\right)$$

$$= \min\left(1, \frac{-6-(-2)}{-\frac{1}{5}-\frac{1}{5}}\right) = \underline{1}$$

Iteration 2:

$$x_1 = x_0 + \alpha_0 p_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{1}{5} \\ \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{22}{10} \\ \frac{1}{10} \end{bmatrix}$$

$$W_1 = \{3\}$$

$$g_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{22}{10} \\ \frac{1}{10} \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{44}{10} - \frac{20}{10} \\ \frac{2}{10} - \frac{50}{10} \end{bmatrix} = \begin{bmatrix} \frac{24}{10} \\ -\frac{48}{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{5} \\ -\frac{24}{5} \end{bmatrix}$$

$$\min \ \frac{1}{2} p_1^T G p_1 + g_1^T p_1$$

s.t.

$$\begin{bmatrix} -1 & 2 \end{bmatrix} p_1 = 0$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ \hat{\lambda}_3 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{24}{5} \\ 0 \end{bmatrix}$$

$$2p_{11} + \hat{\lambda}_3 = -\frac{12}{5}$$

$$2p_{12} - 2\hat{\lambda}_3 = \frac{24}{5}$$

$$-1\, p_{11} + 2p_{12} = 0$$

$$\Longrightarrow \ p_{11} = 2p_{12}$$

$$\Longrightarrow \ \hat{\lambda}_3 = -\frac{12}{5} - 2p_{11} = -\frac{12}{5} - 4p_{12}$$

$$\Longrightarrow \ p_{12} = \frac{12}{5} + \hat{\lambda}_3$$

$$= \frac{12}{5} + \left(-\frac{12}{5} - 4p_{12}\right)$$

$$p_{12} = 0$$

$$p_{11} = 0$$

$$\hat{\lambda}_3 = -\frac{12}{5}$$

$$\underline{p_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T}$$

Iteration 3:

$$x_2 = \begin{bmatrix} \frac{22}{10} & \frac{1}{10} \end{bmatrix}^T$$

$$W_2 = \emptyset$$

$$g_2 = g_1 = \begin{bmatrix} \frac{12}{5} & -\frac{24}{5} \end{bmatrix}^T$$

$$\min \ \frac{1}{2} p_2^T G p_2 + g_2^T p_2$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{24}{5} \end{bmatrix}$$

$$\Longrightarrow \quad p_{21} = -\frac{6}{5} \qquad \Longrightarrow \qquad p_2 = \begin{bmatrix} -\frac{6}{5} & \frac{12}{5} \end{bmatrix}^T$$

$$p_{22} = \frac{12}{5}$$

(1) is the blocking constraint.

$$\alpha_2 = \min\left(1, \frac{-2 - (\begin{bmatrix} 1 & -2 \end{bmatrix} x_2)}{\begin{bmatrix} 1 & -2 \end{bmatrix} p_2}\right)$$

$$= \min\left(1, \frac{-2 - \left(\frac{22}{10} - \frac{2}{10}\right)}{-\frac{6}{5} - \frac{24}{5}}\right)$$

$$= \min\left(1, \frac{-4}{-\frac{30}{5}}\right) = \min\left(1, \frac{2}{3}\right) = \frac{2}{3}$$

Iteration 4:

$$x_3 = \begin{bmatrix} \frac{22}{10} & \frac{1}{10} \end{bmatrix}^T + \frac{2}{3}\begin{bmatrix} -\frac{6}{5} & \frac{12}{5} \end{bmatrix}^T = \begin{bmatrix} \frac{7}{5} & \frac{17}{10} \end{bmatrix}$$

$$W_3 = \{1\}$$

$$g_3 = \frac{1}{10}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 14 \\ 17 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{28}{10} \\ \frac{34}{10} \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{10} & -\frac{16}{10} \end{bmatrix}^T$$

$$\min \quad \frac{1}{2} p_3^T G p_3 + g_3^T p_3$$

s.t.

$$p_{31} - 2p_{32} = 0$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} p_{31} \\ p_{32} \\ \hat{\lambda}_1 \end{bmatrix} = \begin{bmatrix} -\frac{8}{10} \\ \frac{16}{10} \\ 0 \end{bmatrix}$$

$$\Longrightarrow 2p_{31} - \tilde{\lambda}_1 = -\frac{8}{10} \Longrightarrow \tilde{\lambda}_1 = 2p_{31} + \frac{8}{10}$$

$$\Longrightarrow 2p_{32} + 2\tilde{\lambda}_1 = \frac{16}{10} \Longrightarrow p_{32} = \frac{8}{10} - \hat{\lambda}_1$$

$$\Longrightarrow p_{31} - 2p_{32} = 0 \Longrightarrow p_{31} = 2p_{32}$$

$$\longrightarrow \tilde{\lambda}_1 = 4p_{32} + \frac{8}{10}$$

$$\longrightarrow p_{32} = \frac{8}{10} - \left(4p_{32} + \frac{8}{10}\right)$$

$p_{32} = 0$

$\Longrightarrow p_{31} = 0$

$\Longrightarrow \hat{\lambda}_T = \frac{8}{10} > 0$

$$\Longrightarrow \underline{\underline{x^* = \left[\frac{7}{5} \quad \frac{17}{10}\right]^T}}$$

d)

QP problem:

$$\min \quad q(x) = \frac{1}{2} x^T G x + c^T x$$

s.t

$$Ax - b \geq 0$$

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad c = \begin{bmatrix} -2 & -5 \end{bmatrix}^T$$

The Lagrangean function is given by:

$$L(x, \lambda) = Gx + c - \lambda^T (Ax - b)$$

The dual objective is given by

$$q(\lambda) = \inf_x \left(\frac{1}{2} x^T G x + c^T x - \lambda^T (Ax - b) \right)$$

Since G is positive definite $\inf_x L(x, \lambda)$ is achieved when $\nabla_x L(x, \lambda) = 0$

$$\nabla_x L(x, \lambda) = Gx + c - \lambda^T A = 0$$

$$\Longrightarrow \quad Gx + c = \lambda^T A$$

$$Gx = \lambda^T A - c$$

$$x = G^{-1}(\lambda^T A - c)$$

$$\Longrightarrow \quad x^T = \left(G^{-1}(\lambda^T A - c)\right)^T$$

$$= (\lambda^T A - c)^T G^{-1T}$$

$$= (A^T \lambda - c^T) G^{-1}$$

Inserting into $q(\lambda)$:

$$q(\lambda) = \frac{1}{2}(A^T\lambda - c^T) G^{-1} G G^{-1} (\lambda^T A - c)$$

$$+ c^T G^{-1} (\lambda^T A - c) - \lambda^T (A(G^{-1}(\lambda^T A - c)) - b)$$

$$= \frac{1}{2}(\lambda^T A - c)^T G^{-1} (A^T \lambda - c)$$

$$+ c^T G^{-1} \lambda^T A - c^T G^{-1} c - \lambda^T A G^{-1} \lambda^T A + \lambda^T A G^{-1} c$$

$$+ \lambda^T b$$

$$= \frac{1}{2}(\lambda^T A - c)^T G^{-1}(\lambda^T A - c) + \lambda^T A (G^{-1} c - G^{-1} \lambda^T A)$$

$$+ \lambda^T b - c^T G^{-1} c + c^T G^{-1} \lambda^T A$$

$$\lambda^T A (G^{-1} c - G^{-1} \lambda^T A) - c^T G^{-1} c + c^T G^{-1} \lambda^T A$$

$$= \lambda^T A G^{-1}(c - \lambda^T A) + c^T G^{-1}(\lambda^T A - c)$$

$$= c^T G^{-1}(\lambda^T A - c) - \lambda^T A G^{-1}(\lambda^T A - c)$$

$$= -(\lambda^T A - c^T) G^{-1}(\lambda^T A - c)$$

$$= -(A^T \lambda - c)^T G^{-1}(\lambda^T A - c) = -(\lambda^T A - c)^T G^{-1}(A^T \lambda - c)$$

$$\Rightarrow g(\lambda) = \frac{1}{2}(\lambda^T A - c)^T G^{-1}(A^T \lambda - c)$$

$$- (\lambda^T A - c)^T G^{-1}(A^T \lambda - c) + \lambda^T b$$

$$= -\frac{1}{2}(A^T \lambda - c)^T G^{-1}(\lambda^T A - c) + \lambda^T b$$

This means that the dual problem can be formulated as

$$\max \; g(\lambda) = -\frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) + \lambda^T b$$

s.t.

$$\lambda \geq 0$$

Which can be rewritten to

$$\min \; g(\lambda) = \frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) - \lambda^T b$$

s.t.

$$\lambda \geq 0$$

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e) Theorem 12.11:

For any $\bar{x}$ feasible and any $\bar{\lambda} \geq 0$, we have $q(\bar{\lambda}) \leq q(\bar{x})$.

$$q(\bar{x}) - q(\bar{\lambda}) \geq 0$$

$$q(\bar{x}) \geq q(x^*)$$

$$\longrightarrow \quad \underline{q(\bar{x}) - q(\bar{\lambda}) \geq q(x^*) - q(\bar{\lambda})}$$

Problem 2: Production planning and QP

Two reactors, R_I and R_{II}
Two products, A and B

Production:

1000 kg of A - 2 hours of R_I, 1 hour of R_{II}
1000 kg of B - 1 hour of R_I, 3 hours of R_{II}

Availability:

R_I - 8 hours
R_{II} - 15 hours

Profit:

$c_1 = 3 - 0{,}4x_1$
$c_2 = 2 - 0{,}2x_2$

a) Profit (objective):

$$-q(x) = (3 - 0{,}4x_1)x_1 + (2 - 0{,}2x_2)x_2$$
$$= -0{,}4x_1^2 + 3x_1 - 0{,}2x_2^2 + 2x_2$$

Constraints:

$$2x_1 + x_2 \leq 8$$
$$x_1 + 3x_2 \leq 15$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$\rightarrow$ min $q(x) = \frac{1}{2}x^T G x + c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$G = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix}$ $\quad c^T = \begin{bmatrix} -3 & -2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$

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c) Iterations:

$x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

$x_1 = \begin{bmatrix} 2.4 & 3.2 \end{bmatrix}^T$

$x_2 = \boxed{x^* = \begin{bmatrix} 2.25 & 3.5 \end{bmatrix}^T}$

<u>The solution is not at a point of intersection.</u>

<u>Only constraint (1) is active.</u>

d) The active set method calculates the Newton path towards the objective function minima. If no such path can be found the constraint with the most positive sensitivity is remeved from the active set.

If a Newton path is found the iteration point moves along it untill it hits a blocking constraint.

e) Since the problem in exercise 3 was a LP problem the solution was found at an intersection between constraints.
In this exercise, since the problem was a QP problem the solution was not at an intersection.