Assignment 6 - Martin Kvisvik Larsen

Problem 1: Finite-horizon LQR

a) x_1 - position
x_2 - velocity

N2: $ma = F$

$m = 1 \quad , \quad F = u$

$$\Longrightarrow \quad a = u$$
$$\dot{x}_2 = u$$

$$\dot{x}_1 = x_2$$

$$\Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= A_c} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= b_c} u$$

$$\Longrightarrow \quad \underline{\underline{\dot{x} = A_c x + b_c u}} \qquad (\mathrm{I})$$

b) The solution to (I) on the time interval $t \in [kT, (k+1)T]$ is:

$$x_{k+1} = e^{A_c((k+1)T - kT)} x_k + \int_{kT}^{(k+1)T} e^{A_c(\tau - kT)} b_c u(\tau) d\tau$$

Assuming tha $u(\tau)$ is produced by a zero order hold, i.e. $u(\tau)$ is constant on the time interval $\tau \in [kT, (k+1)T)$:

$$x_{k+1} = e^{A_c T} x_k + \int_{kT}^{(k+1)T} e^{A_c(\tau - kT)} d\tau \, b_c u_k$$

Change of variable:

$$\gamma = \tau - kT$$

$$\tau = kT \Longrightarrow \gamma = 0$$

$$\tau = (k+1)T \longrightarrow \gamma = T$$

$$\Longrightarrow x_{k+1} = e^{A_c T} x_k + \left(\int_0^T e^{A_c \gamma} d\gamma \right) b_c u_k$$

$$e^{A_c T} = I + TA_c + \frac{T^2}{2!} A_c^2 + \ldots$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_c^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow e^{A_c T} = I + TA_c$$

$$\int_0^T e^{A_c \tau} d\tau = \int_0^T (I + \tau A_c) d\tau$$

$$= \int_0^T \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix} \right) d\tau$$

$$= \int_0^T \left(\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right) d\tau = \begin{bmatrix} \tau & \frac{\tau^2}{2} \\ 0 & \tau \end{bmatrix} \Bigg|_{\tau = 0}^{\tau = T}$$

$$= \begin{bmatrix} T & \frac{T^2}{2} \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0.5 & 0.125 \\ 0 & 0.5 \end{bmatrix}$$

$$\Longrightarrow \int_0^T e^{A_c \tau} d\tau \, b_c = \begin{bmatrix} 0.5 & 0.125 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}$$

$$e^{A_c T} = I + TA_c = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \quad x_{k+1} = \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}}_{=A} x_k + \underbrace{\begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}}_{=b} u_k$$

c) $$f(z) = \frac{1}{2} \sum_{k=0}^{N-1} \left(x_{k+1}^T Q x_{k+1} + u_k^T R u_k \right)$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 2$$

$$z = \left[x_1^T, \dots, x_N^T, u_0^T, \dots, u_{N-1}^T \right]^T$$

The Riccati equation for the problem is:

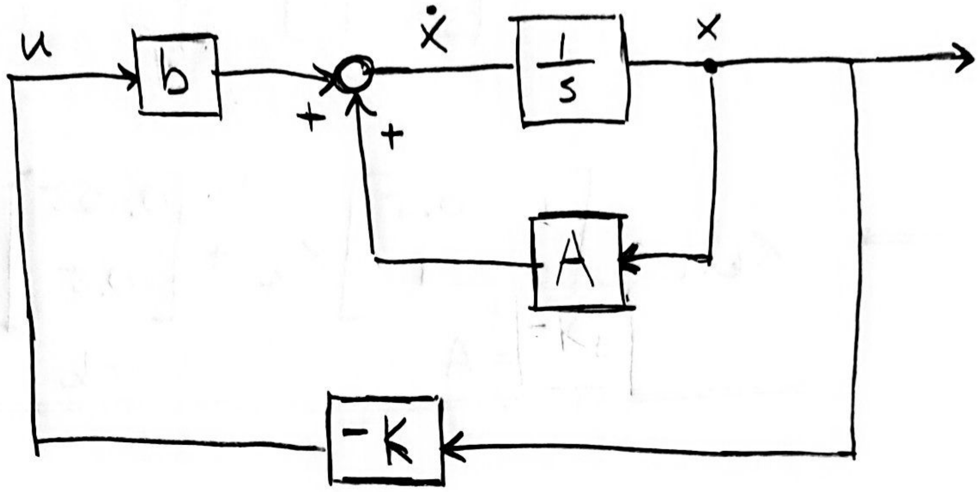
$$P_t = Q + A^T P_{t+1} \left(I + bR^{-1} b^T P_{t+1} \right)^{-1} A, \quad t = 0, \dots, N-1$$

The solution from the Riccati equation can be used in the state feedback controller

$u_t = -K_t x_t$

where

$$K_t = R^{-1} b^T P_{t+1} \left(I + bR^{-1} b^T P_{t+1} \right)^{-1} A, \quad t = 0, \dots, N-1$$



d) $N \longrightarrow \infty$ (Infinite horizon)

The stationary Riccati equation becomes

$$P = Q + A^T P (I + bR^{-1} b^T P)^{-1} A$$

and feedback gain matrix becomes

$$K = R^{-1} b^T P (I + bR^{-1} b^T P)^{-1} A$$

Printout from Matlab:

$$P = \begin{bmatrix} 4.0350 & 2.0616 \\ 2.0616 & 4.1438 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.6514 & 1.3142 \end{bmatrix}$$

$$x_{k+1} = Ax_k + bu_k$$

$$u_k = -Kx_k$$

$$\Longrightarrow \quad x_{k+1} = Ax_k + b(-Kx_k)$$

$$x_{k+1} = (A - bK)x_k$$

Stability can be evaluated by examining the eigenvalues of $A - bK$. For discrete-time system stability is ensured if $|\lambda_i| < 1$ for $i = 1, \dots, n$.

From Matlab:

$$\lambda_{1,2} = 0.6307 \pm 0.1628j$$

$$|\lambda_{1,2}| = \sqrt{(0.6307)^2 + (0.1628)^2}$$

$$\approx 0.6514$$

$|\lambda_{1,2}| < 1 \Longrightarrow$ The system is stable

e) The LQ controller gives asymptotically stable systems if A, B is stabilizable and A, D is detectable, where $Q = D^T D$.

Problem 2: Infinite-Horizon LQ control

$$x_{k+1} = 3x_k + 2u_k \qquad x \in \mathbb{R}^1, \quad u \in \mathbb{R}^1$$

$$f^{\infty}(z) = \frac{1}{2}\sum_{t=0}^{\infty}\left(q x_{k+1}^2 + u_k^2\right) \qquad q > 0$$

a) Stationary Riccati equation:

$$P = Q + A^T P\left(I + BR^{-1}B^T P\right)^{-1} A$$

In the scalar case:

$$p = q + \frac{a^2 p}{1 + \frac{b^2 p}{r}} = q + \frac{a^2 p r}{r + b^2 p}$$

$q = 2$

From the system: $a = 3 \quad b = 2$

From the cost function: $r = 1$

$$\Longrightarrow \quad p = 2 + \frac{3^2 p}{1 + 2^2 p}$$

$$(1+4p)p = 2(1+4p) + 9p$$

$$4p^2 + p = 2 + 8p + 9p$$

$$4p^2 - 16p - 2 = 0$$

$$p = \frac{16 \pm \sqrt{16^2 - 4 \cdot 4 \cdot (-2)}}{2 \cdot 4}$$

$$p = \frac{16 \pm \sqrt{288}}{8} = \frac{16 \pm 12\sqrt{2}}{8}$$

$$p = 2 \pm \frac{3}{2}\sqrt{2}$$

$$p > 0 \implies \underline{\underline{p = 2 + \frac{3}{2}\sqrt{2}}}$$

b) Stationary feedback gain:

$$K = R^{-1}B^T P (I + BR^{-1}B^T P)^{-1} A$$

Scalar case:

$$k = \frac{abp}{r(1 + \frac{b^2 p}{r})} = \frac{abp}{r + b^2 p}$$

Inserted values:

$$k = \frac{3 \cdot 2 (2 + \frac{3}{2}\sqrt{2})}{1 + 2^2 (2 + \frac{3}{2}\sqrt{2})} = \frac{12 + 9\sqrt{2}}{9 + 6\sqrt{2}}$$

$$= \frac{4 + 3\sqrt{2}}{3 + 2\sqrt{2}} = \frac{(\cancel{2\sqrt{2} + 3})\sqrt{2}}{\cancel{3 + 2\sqrt{2}}} = \underline{\underline{\sqrt{2}}}$$

c) The LQ controller gives an asymptotically stable system if (A, B) is stabilizable and (A, D) detectable, where $Q = D^T D$.

Problem 3: MPC and input blocking

$$x_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0.1 & -0.79 & 1.78 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix} u_t$$

$$y_t = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_t$$

$$x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$f(y_1, \ldots, y_N, u_0, \ldots, u_{N-1}) = \sum_{t=0}^{N-1} \left(y_{t+1}^2 + r u_t^2 \right), \quad r > 0$$

$$-1 \leq u_t \leq 1, \quad t \in [0, N-1]$$

a) Since $y_{t+1} = x_{3,t+1}$ the optimization problem can be rewritten as:

$$f(x_1, \dots, x_N, u_0, \dots, u_{N-1}) = \sum_{t=0}^{N-1} (x_{t+1}^T Q x_{t+1} + r u_t^2)$$

with

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Introducing the following vector:

$$z = [x_1^T, \dots, x_N^T, u_0^T, \dots, u_{N-1}^T]^T$$

$\underset{\sim}{t=0}$:

$$x_1 = Ax_0 + Bu_0$$

$$x_1 - Bu_0 = Ax_0$$

$\underset{\sim}{t=1}$:

$$x_2 = Ax_1 + Bu_1$$

$$-Ax_1 + x_2 - Bu_1 = 0$$

$\underset{\sim}{t=2}$:

$$x_3 = Ax_2 + Bu_2$$

$$-Ax_2 + x_3 - Bu_2 = 0$$

The equality constraints can the be rewritten as:

$$\left[\begin{array}{cccc|ccc} I & & & & -B & & \\ -A & I & & & & -B & \\ & -A & I & & & & -B \\ & & \vdots & & \vdots & & \vdots \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} Ax_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A_{eq} z = b_{eq}$$

The cost function can be rewritten as:

$$f(z) = \frac{1}{2} z G z$$

$$G = \begin{bmatrix} Q & & & & & \\ & \ddots & & & & \\ & & Q & & & \\ & & & R & & \\ & & & & \ddots & \\ & & & & & R \end{bmatrix}$$

b) We want to make the u_t over time blocks consisting of 5 time steps.

Assuming that N is a multiple of 5: Denoting the number of time blocks $\tilde{N}$ and the number of time steps per block $\tilde{t}$.

$$\tilde{N} = \frac{N}{\tilde{t}}$$

The vector z is then modified to:

$$z = [x_1^T, \dots, x_N^T, u_0, \dots, u_{N_B-1}]^T$$

The equality constraints then become:

$$\left[\begin{array}{cccccccc|cc}
I & & & & & & & & -B & \\
-A & I & & & & & & & -B & \\
 & -A & I & & & & & & -B & \\
 & & -A & I & & & & & -B & \\
 & & & -A & I & & & & -B & \\
 & & & & -A & I & & & & -B \\
 & & & & & \vdots & & & & \vdots
\end{array}\right]
\begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N_B-1} \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

quadprog used 5 iterations to solve the QP problem.

c) quadprog still used 5 iterations to solve the QP problem.

d)

e)

f) Input blocking decreases the number of variables during optimization and thus decreases the time needed to solve the optimization problem.