Assignment 2

Problem 1 : The Mean Value Theorem

$$f(x) = x_1^3 + 3x_1x_2^2$$
$$x \in R^2$$

a)

$$\nabla_x f(x) = \begin{bmatrix} 3x_1^2 + 3x_2^2 \\ 6x_1x_2 \end{bmatrix}$$

$$x = [0, 0]^T$$
$$p = [2, 1]^T$$

$$f(x) = 0^3 + 3 \cdot 0 \cdot 0^2 = 0$$
$$f(x+p) = 2^3 + 3 \cdot 2 \cdot 1^2 = 14$$

$$\nabla_x f(x + \alpha p) = \begin{bmatrix} 3(0 + 2\alpha)^2 + 3(0+\alpha)^2 \\ 6(0+2\alpha)(0+\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 12\alpha^2 + 3\alpha^2 \\ 12\alpha^2 \end{bmatrix} = \begin{bmatrix} 15\alpha^2 \\ 12\alpha^2 \end{bmatrix}$$

The mean value theorem:

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$$

$$\nabla f(x+\alpha p)^T p = \begin{bmatrix} 15\alpha^2 & 12\alpha^2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 30\alpha^2 + 12\alpha^2$$

$$= 42\alpha^2$$

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$$

$$\Longrightarrow \quad 14 = 0 + 42\alpha^2$$

$$\alpha^2 = \frac{14}{42} = \frac{1}{3}$$

$$\alpha = \pm \frac{1}{\sqrt{3}}$$

$$\underline{\underline{\alpha = \frac{1}{\sqrt{3}} \in (0,1)}}$$

b) $f(x) = x^{\frac{1}{2}}$

$$\nabla f(x) = \frac{1}{2} x^{-\frac{1}{2}} \qquad \lim_{x \to 0^+} \nabla f(x) = \infty$$

Hence there is no Lipschitz constant L such that

$$\| f(x_1) - f(x_0) \| \leq L \| x_1 - x_0 \|$$

and so the function is not Lipschitz at $x = 0$.

Problem 2: LP and KKT-conditions

$$\min \quad c^T x$$

$x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

s.t

$$Ax = b$$
$$x \geq 0$$

By partitioning the Lagrangean multipliers into the multipliers λ for the equality constraints and s for the inequality constraints the Lagrangean function is given by

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$$

KKT conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = 0$$

$$\longrightarrow \quad c - A^T \lambda^* - s^* = 0$$

$$\underline{A^T \lambda^* + s^* = c}$$

$$c_i(x^*) = 0, \quad i \in \mathcal{E}$$

$$\Longrightarrow \quad Ax^* - b = 0$$

$$\underline{Ax^* = b}$$

$$c_i(x^*) \geq 0, \quad i \in \mathcal{I}$$

$$\Longrightarrow \quad \underline{x^* \geq 0}$$

$$\lambda_i^* \geq 0, \quad i \in \mathcal{I}$$

$$\Longrightarrow \quad \underline{s^* \geq 0}$$

$$\lambda_i^* c_i(x_i^*) = 0, \quad i \in \mathcal{E} \cup \mathcal{I}$$

$$\Longrightarrow \quad \lambda^{*T} \underbrace{(Ax^* - b)}_{=0} + s^{*T} x^* = 0$$

$$s^{*T} x^* = 0$$

KKT conditions for a LP problem:

$$A^T \lambda^* + s^* = c$$

$$Ax^* = b$$

$$x^* \geq 0$$

$$s^* \geq 0$$

$$s^{*T} x^* = 0$$

Problem 3: Linear programming

Two stages: A and B

Time available: A - 7200
B - 6000

Three products: R, S and T

Time required: R - 3 in A, 2 in B
S - 2 in A, 2 in B
T - 1 in A, 3 in B

Profits:
R - 100
S - 75
T - 55

a) We want to maximize the profit.

x_1 = Tonnes produced of R
x_2 = Tonnes produced of S
x_3 = Tonnes produced of T

$$x = [x_1, x_2, x_3]^T$$

$$\Longrightarrow c' = [100, 75, 55]^T$$

Maximize profit

$$\max \ c'^T x \quad \Longleftrightarrow \quad \min \ c^T x$$

$$c^T = -c'^T$$

$c = [-100, -75, -55]^T$

Time constraints:

$$3x_1 + 2x_2 + 1x_3 \leq 7200$$
$$2x_1 + 2x_2 + 3x_3 \leq 6000$$

$$\Longrightarrow \quad 3x_1 + 2x_2 + 1x_3 - s_1 \leq 7200$$
$$2x_1 + 2x_2 + 3x_3 - s_2 \leq 6000$$

$$\Longrightarrow \quad 3x_1 + 2x_2 + 1x_3 = 7200$$
$$2x_1 + 2x_2 + 3x_3 = 6000$$

$$s_1 \geq 0$$
$$s_2 \geq 0$$

$$\Longrightarrow \quad Ax = b$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

$$s \geq 0$$

Cannot have negative production:

$$x \geq 0$$

Thus the problem can be written as

$$\min_{x \in \mathbb{R}^3} c^T x$$

s.t.

$$Ax = b$$
$$x \geq 0$$

$$c = [-100, -75, -55]^T$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

b) $m = 2 \quad n = 3$

$\beta \subseteq \{1, 2, 3\}$

β contains $m = 2$ indices

$\Longrightarrow$ 3 possible permutations of β

$i \notin \beta \Longrightarrow x_i = 0$

$\beta_1 = \{1, 2\}$

$\Longrightarrow x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

$x_3 = 0$

$$B x_B = b \longrightarrow x_B = B^{-1} b$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 \cdot 7200 - 2 \cdot 6000 \\ -2 \cdot 7200 + 3 \cdot 6000 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1800 \end{bmatrix}$$

$$x = [1200, 1800, 0]^T$$

$$\beta_2 = \{1, 3\}$$

$$\Longrightarrow x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \quad x_2 = 0, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{7} \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 15600 \\ 3600 \end{bmatrix}$$

$$x = \left[\frac{15600}{7}, 0, \frac{3600}{7} \right]^T$$

$$\beta_3 = \{2, 3\}$$

$$\Longrightarrow x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad x_1 = 0, \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7200 \\ 6000 \end{bmatrix} = \begin{bmatrix} 3900 \\ -600 \end{bmatrix}$$

$$x = [0, 3900, -600]^T$$

c) KKT conditions for LP problems:

$$A^T\lambda^* + s^* = c$$
$$Ax^* = b$$
$$x^* \geq 0$$
$$s^* \geq 0$$
$$s^{*T}x^* = 0$$

$x = [1200,\ 1800,\ 0]^T$

$$s^{*T}x^* = 0 \longrightarrow s^* = [0,\ 0,\ s_3]^T$$

$$A^T\lambda^* + s^* = c$$

$$\Longrightarrow \begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix}$$

$$3\lambda_1 + 2\lambda_2 = -100 \quad \text{(I)}$$
$$2\lambda_1 + 2\lambda_2 = -75 \quad \text{(II)}$$

$$\longrightarrow 3\lambda_1 + (-75 - 2\lambda_1) = -100$$
$$\lambda_1 = -100 + 75 = -25$$

$$\lambda_2 = -\frac{75}{2} - \lambda_1 = -\frac{75}{2} + 25 = -\frac{25}{2}$$

$$\lambda_1 + 3\lambda_2 + s_3 = -55 \qquad (\text{III})$$

$$\Longrightarrow -25 + 3\left(-\frac{25}{2}\right) + s_3 = -55$$

$$s_3 = -55 + 25 + \frac{75}{2} = \underline{\frac{15}{2}}$$

$\underline{\underline{x^* = [1200, 1800, 0] \text{ satisfies the KKT cond.}}}$

$$\underline{x = \left[\frac{15600}{7}, 0, \frac{3600}{7}\right]^T}$$

$$s^{*T} x^* = 0$$

$$\Longrightarrow s^* = [0, s_2, 0]^T$$

$$A^T \lambda^* + s^* = c$$

$$\Longrightarrow \begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix}$$

$$\Longrightarrow 3\lambda_1 + 2\lambda_2 = -100 \qquad (\text{I})$$

$$2\lambda_1 + 2\lambda_2 + s_2 = -75 \qquad (\text{II})$$

$$\lambda_1 + 3\lambda_2 = -55 \qquad (\text{III})$$

$(\text{III}) \Longrightarrow (\text{I})$

$$\longrightarrow 3(-55 - 3\lambda_2) + 2\lambda_2 = -100$$

$$-7\lambda_2 = 65$$

$\lambda_2 = -\frac{65}{7}$

(III)

$\Longrightarrow \lambda_1 = -55 - 3\lambda_2 = -55 - 3 \cdot \left(-\frac{65}{7}\right)$

$\lambda_1 = -\frac{190}{7}$

(II)

$\longrightarrow s_2 = -75 - 2\lambda_1 - 2\lambda_2$

$s_2 = -75 - 2\left(-\frac{190}{7}\right) - 2\left(-\frac{65}{7}\right)$

$s_2 = -\frac{15}{7}$

$s < 0$

$\Longrightarrow x = \left[\frac{15600}{7}, 0, \frac{3600}{7}\right]^T$ does not satisfy the KKT conditions

$x = [0, 3900, -600]^T < 0$ does not satisfy the KKT conditions

d) The dual problem:

$$\max \; b^T \lambda$$

s.t.

$$A^T \lambda \leq c$$

e) From the KKT conditions of the LP problem we have:

$$A^T \lambda^* + s^* = c$$

$$A x^* = b$$

$$s^{*T} x^* = 0$$

$$\Rightarrow b^T \lambda^* = (Ax^*)^T \lambda^* = x^{*T} A^T \lambda^*$$

$$= x^{*T}(c - s^*) = x^{*T} c - x^{*T} s^*$$

$$= c^T x^* - s^{*T} x^* = c^T x^* - 0$$

$$= \underline{\underline{c^T x^*}}$$

f) $x^* = [1200, 1800, 0]^T$ $\lambda^* = [-25, -\frac{25}{2}]^T$

Using the property that the Lagrangean multipliers represent the sensitivity of the objective function wrt. the constraints.

$$|\lambda_1^*| > |\lambda_2^*|$$

Hence I would change the capacity of A.

$$f(x^*) = [-100, -75, -55]\begin{bmatrix}1200\\1800\\0\end{bmatrix} = \underline{-255\,000}$$

$$b' = \begin{bmatrix}7201\\6000\end{bmatrix}$$

$$x_B' = \frac{1}{2}\begin{bmatrix}2 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}7201\\6000\end{bmatrix} = \begin{bmatrix}1201\\1799\end{bmatrix}$$

$$f(x^{*'}) = [-100, -75, -55]\begin{bmatrix}1201\\1799\\0\end{bmatrix} = \underline{-255\,025}$$

$$b'' = \begin{bmatrix}7200\\6001\end{bmatrix}$$

$$x_B'' = \frac{1}{2}\begin{bmatrix}2 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}7200\\6001\end{bmatrix} = \begin{bmatrix}1199\\1801{,}5\end{bmatrix}$$

$$f(x^{*''}) = [-100, -75, -55]\begin{bmatrix}1199\\1801{,}5\\0\end{bmatrix} = \underline{-255\,012{,}5}$$