Assignment 1

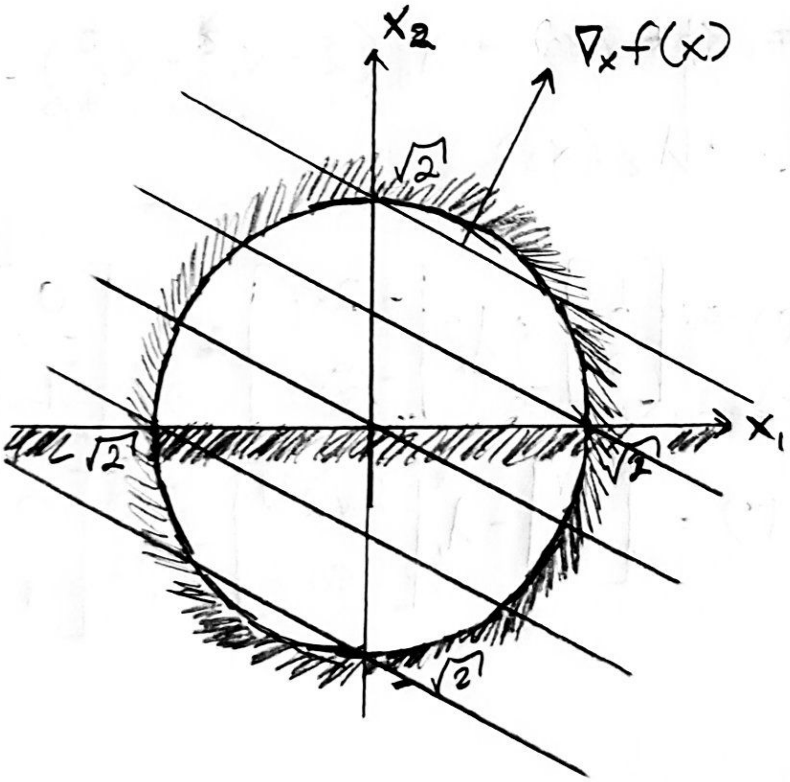
Problem 1: Example 12.3

$$\min \quad f(x) = x_1 + 2x_2$$

s.t

$$\left.\begin{aligned} c_1(x) &= 2 - x_1^2 - x_2^2 \geq 0 \\ c_2(x) &= x_2 \geq 0 \end{aligned}\right\} \Rightarrow I = \{1, 2\}$$

a) $$\nabla_x f(x) = \left[\frac{\partial f}{\partial x}\right]^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



By inspection the optimal point is :

$$\underline{\underline{x^* = [-\sqrt{2}, 0]^T}}$$

b) KKT conditions :

(I) $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$,

(II) $c_i(x^*) = 0$, $i \in \mathcal{E}$

(III) $c_i(x^*) \geq 0$, $i \in \mathcal{I}$

(IV) $\lambda_i^* \geq 0$, $i \in \mathcal{I}$

(V) $\lambda_i^* c_i(x^*) = 0$, $i \in \mathcal{E} \cup \mathcal{I}$

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

$$= x_1 + 2x_2 - \lambda_1 (2 - x_1^2 - x_2^2)$$

$$- \lambda_2 (x_2)$$

$$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \lambda_1 \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} - \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \lambda_1^* \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} - \lambda_2^* \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$$

$$\Longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \lambda_1^* \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} - \lambda_2^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 1 - \lambda_1^* 2\sqrt{2} = 0$$

$$\lambda_1^* = \frac{1}{2\sqrt{2}}$$

$$\Longrightarrow 2 - \lambda_2^* = 0$$

$$\lambda_2^* = 2$$

(I) satisfied
(IV) satisfied
(II) satisfied since $\mathcal{E} = \emptyset$

$$c_1(x^*) = 2 - (-\sqrt{2})^2 - (0)^2$$

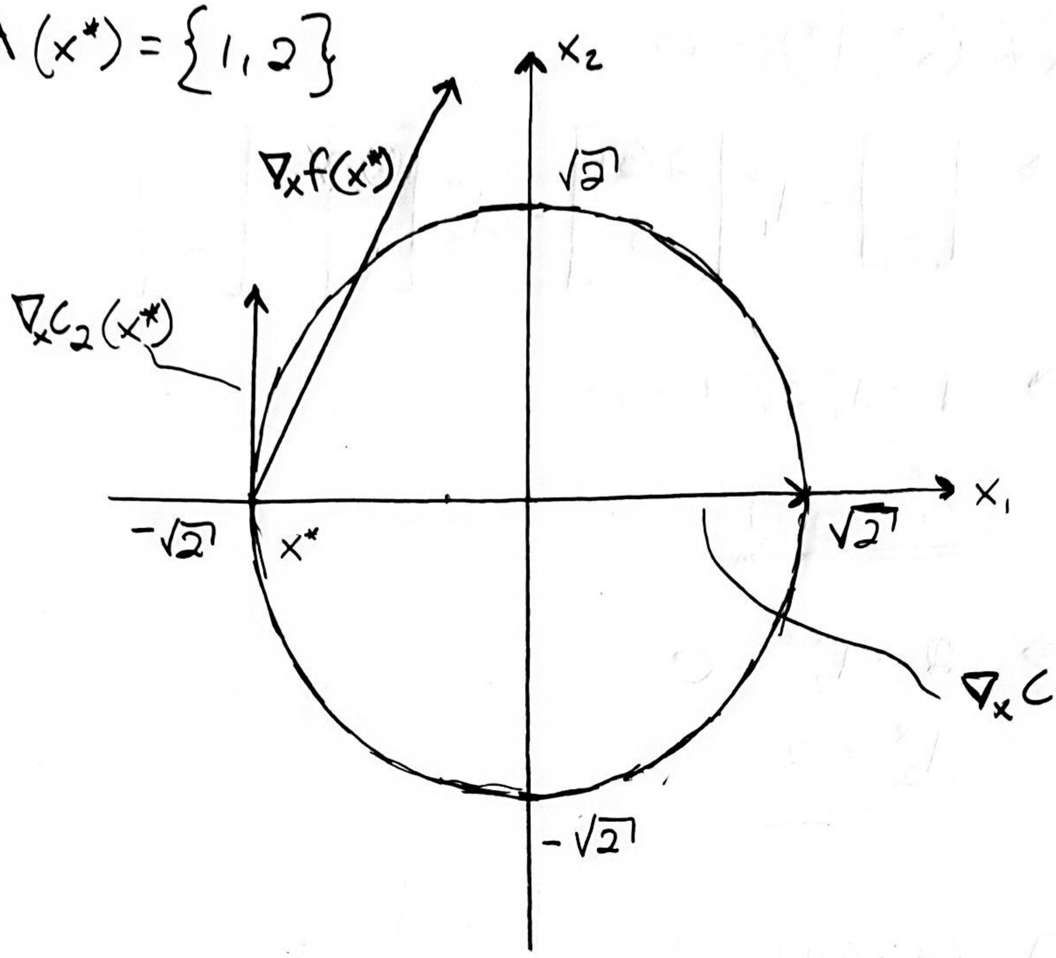
$$= 2 - 2 - 0 = 0$$

$$c_2(x^*) = 0$$

(III) satisfied
(V) satisfied

All of the KKT conditions are satisfied at $x^* = [-\sqrt{2}, 0]^T$

c) $A(x^*) = \{1, 2\}$



$$\nabla_x c_1(x^*) = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \qquad \nabla_x c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla_x f(x^*) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

d) The Lagrangean multipliers are positive because the gradients of the constraints points in a similar direction as that of the gradient of the objective function.

e) Ω is convex since a line can be drawn between any two points $x' \in \Omega$ and $x'' \in \Omega$.

Checking if the objective function is convex:

f is convex

$\Updownarrow$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

$$f(x) = x_1 + 2x_2$$

$$\longrightarrow f(\alpha x + (1-\alpha)y) = \alpha x_1 + (1-\alpha)y_1 + 2\alpha x_2 + 2(1-\alpha)y_2$$

$$= \alpha(x_1 + 2x_2) + (1-\alpha)(y_1 + 2y_2)$$

$$= \alpha f(x) + (1-\alpha)f(y)$$

$\Longrightarrow$ f(x) is convex

The problem is a convex problem

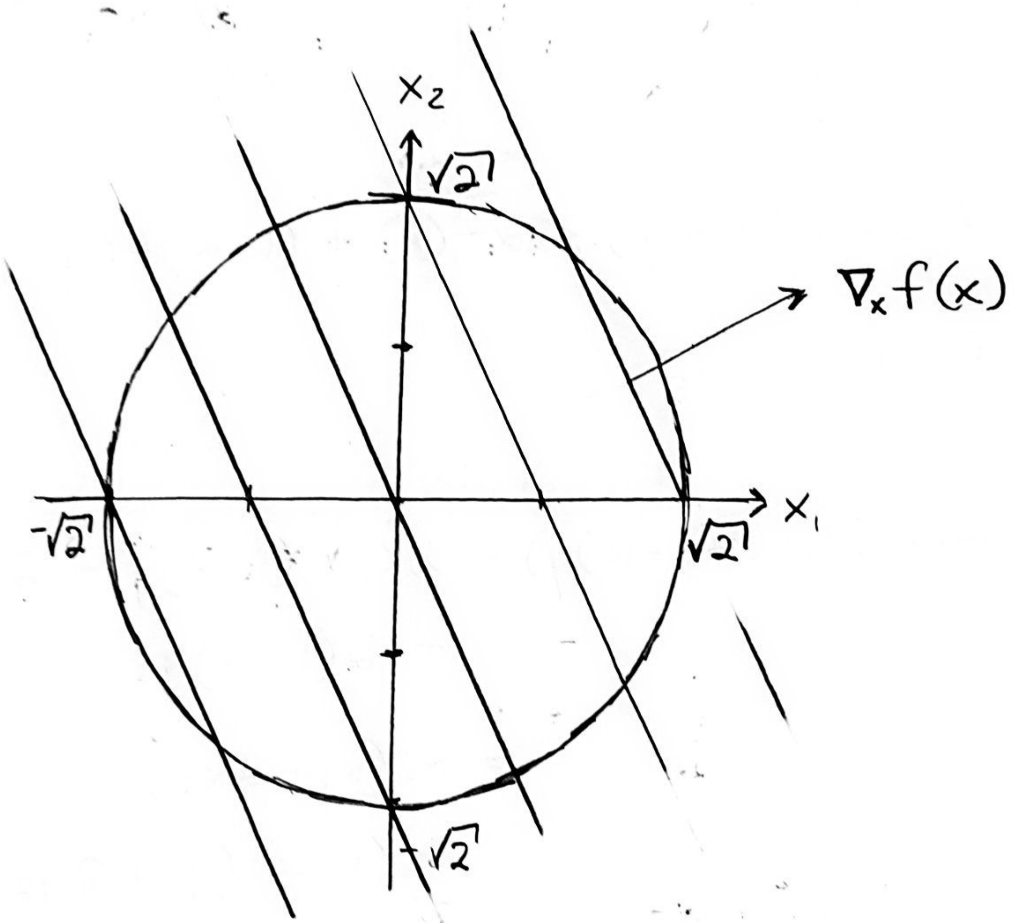
Problem 2: Example 12.1

$$\min \quad f(x) = 2x_1 + x_2$$

s.t.

$$c_1(x) = x_1^2 + x_2^2 - 2 = 0 \longrightarrow \mathcal{E} = \{1\}$$

$$\mathcal{I} = \emptyset$$



$$\nabla_x f(x) = \left[\frac{\partial f}{\partial x}\right]^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

a) $\mathcal{L}(x,\lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) = f(x) - \lambda_1 c_1(x)$

$= 2x_1 + x_2 - \lambda_1 (x_1^2 + x_2^2 - 2)$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \lambda_1 \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

Finding extreme points by setting $\nabla_x \mathcal{L}(x,\lambda) = 0$

$$\Longrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \lambda_1 \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2 - 2\lambda_1 x_1 = 0 \longrightarrow \lambda_1 x_1 = 1$

$1 - 2\lambda_1 x_2 = 0 \longrightarrow \lambda_1 x_2 = \frac{1}{2}$

$$\longrightarrow \left.\begin{array}{l} \lambda_1 = \frac{1}{x_1} \\ \lambda_1 = \frac{1}{2x_2} \end{array}\right\} \Longrightarrow 2x_2 = x_1$$

$c_1(x) = x_1^2 + x_2^2 - 2 = 0$

$\Longrightarrow (2x_2)^2 + x_2^2 - 2 = 0$

$5x_2^2 = 2$

$x_2^2 = \frac{2}{5}$

$x_2 = \pm\sqrt{\frac{2}{5}}$

Gives me two extremas:

$$x_2 = \sqrt{\frac{2}{5}} \Longrightarrow x_1 = \frac{2\sqrt{2}}{\sqrt{5}}$$

$$x_2 = -\sqrt{\frac{2}{5}} \longrightarrow x_1 = -\frac{2\sqrt{2}}{\sqrt{5}}$$

$$\underline{\underline{x' = \left[\frac{2\sqrt{2}}{\sqrt{5}}, \sqrt{\frac{2}{5}}\right]^T}}$$

$$\underline{\underline{x'' = \left[-\frac{2\sqrt{2}}{\sqrt{5}}, -\sqrt{\frac{2}{5}}\right]^T}}$$

b) KKT conditions:

(I) $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$

(II) $c_i(x^*) = 0$, $i \in \mathcal{E}$

(III) $c_i(x^*) \geq 0$, $i \in \mathcal{I}$

(IV) $\lambda_i^* \geq 0$, $i \in \mathcal{I}$

(V) $\lambda_i^* c_i(x^*) = 0$, $i \in \mathcal{E} \cup \mathcal{I}$

$\mathcal{I} = \emptyset \longrightarrow$ (III) and (IV) satisfied at both points

(I) satisfied at both points by explicity setting $\nabla_x \mathcal{L}(x, \lambda) = 0$ when the points were found.

$$x' = \left[\frac{2\sqrt{2}}{\sqrt{5}}, \sqrt{\frac{2}{5}}\right]^T \qquad x'' = \left[-\frac{2\sqrt{2}}{\sqrt{5}}, -\sqrt{\frac{2}{5}}\right]^T$$

$$c_1(x') = \left(\frac{2\sqrt{2}}{\sqrt{5}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 - 2$$

$$= 4\cdot\frac{2}{5} + \frac{2}{5} - 2 = \cancel{5}\cdot\frac{2}{\cancel{5}} - 2 = \underline{0}$$

$$c_1(x'') = \left(-\frac{2\sqrt{2}}{\sqrt{5}}\right)^2 + \left(-\sqrt{\frac{2}{5}}\right)^2 - 2$$

$$= 4\cdot\frac{2}{5} + \frac{2}{5} - 2 = \cancel{5}\cdot\frac{2}{\cancel{5}} - 2 = \underline{0}$$

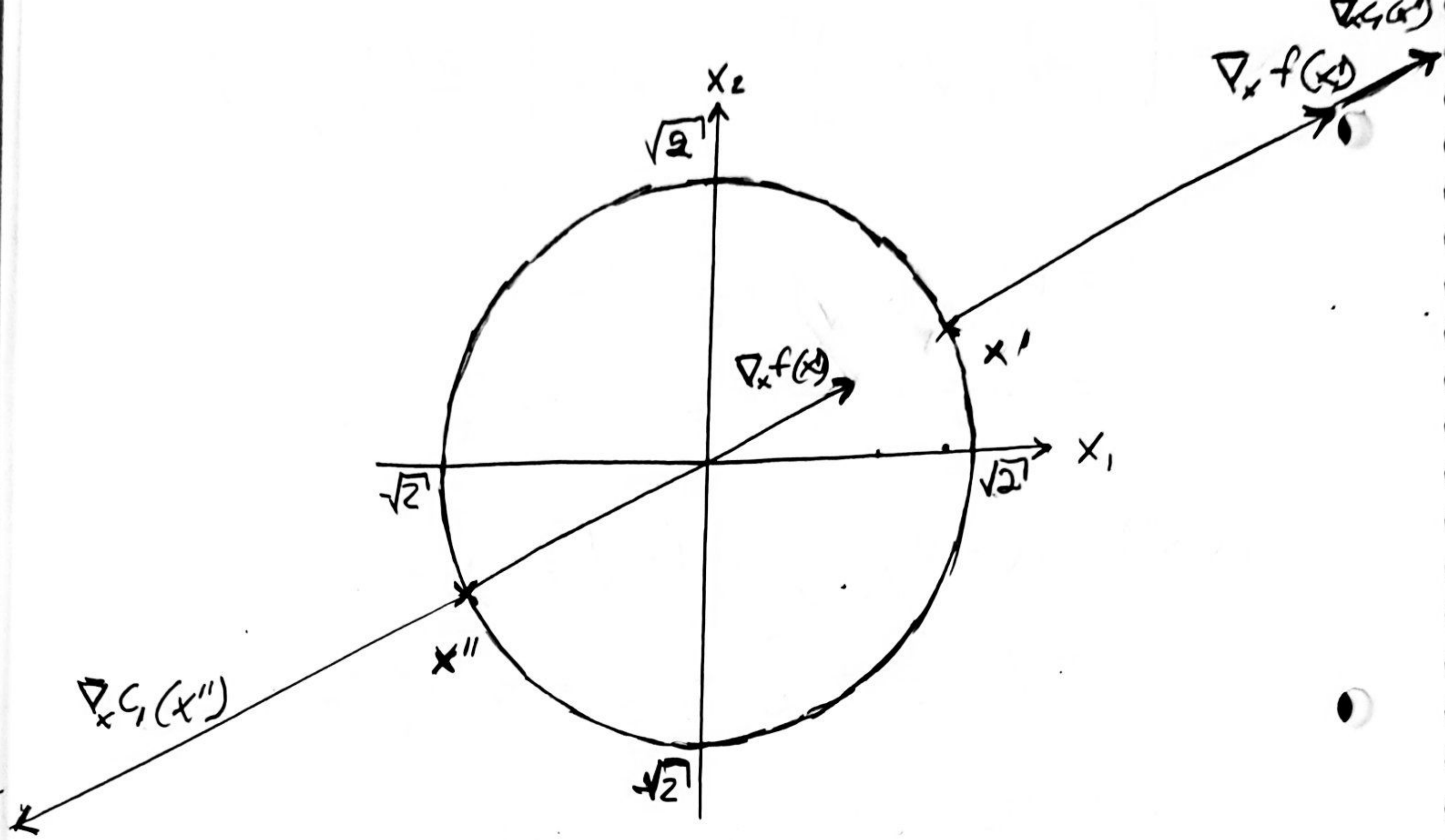
(II) and (V) satisfied at both points.

$\Longrightarrow$ Both x' and x'' satisfies the KKT conditions.

c) $\nabla_x c_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$

$$\nabla_x c_1(x') = \begin{bmatrix} 2\cdot\frac{2\sqrt{2}}{\sqrt{5}} \\ 2\cdot\sqrt{\frac{2}{5}} \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{2}}{\sqrt{5}} \\ \frac{2\sqrt{2}}{\sqrt{5}} \end{bmatrix}$$

$$\nabla_x c_1(x'') = \begin{bmatrix} 2\cdot\left(-\frac{2\sqrt{2}}{\sqrt{5}}\right) \\ 2\cdot\left(-\sqrt{\frac{2}{5}}\right) \end{bmatrix} = \begin{bmatrix} -\frac{4\sqrt{2}}{\sqrt{5}} \\ -\frac{2\sqrt{2}}{\sqrt{5}} \end{bmatrix}$$



d) $\lambda_1 = \frac{1}{x_1}$

$$\lambda_1' = \frac{1}{x_1'} = \frac{1}{\frac{2\sqrt{2}}{\sqrt{5}}} = \underline{\underline{\frac{\sqrt{5}}{2\sqrt{2}}}}$$

$$\lambda_1'' = \frac{1}{x_1''} = \frac{1}{-\frac{2\sqrt{2}}{\sqrt{5}}} = \underline{\underline{-\frac{\sqrt{5}}{2\sqrt{2}}}}$$

The Lagrangean multipliers are consistent with the KKT conditions since they belong to an equality constraint and thus have no requirement in terms of sign.

e) Second order condition:

$$z^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*)\, z \quad \text{is positive definite}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \lambda_1^* \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} 2 - 2\lambda_1^* x_1^* \\ 1 - 2\lambda_1^* x_2^* \end{bmatrix}$$

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -2\lambda_1^* & 0 \\ 0 & -2\lambda_1^* \end{bmatrix}$$

$$\nabla^2_{xx} \mathcal{L}(x', \lambda_1') = \begin{bmatrix} -\cancel{2} \cdot \frac{\sqrt{5}}{\cancel{2}\sqrt{2}} & 0 \\ 0 & -\cancel{2} \cdot \frac{\sqrt{5}}{\cancel{2}\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{\frac{5}{2}} & 0 \\ 0 & -\sqrt{\frac{5}{2}} \end{bmatrix}$$

$\Longrightarrow$ $z^T \nabla^2_{xx} \mathcal{L}(x', \lambda')\, z$ is <u>negative definite</u>.

$$\nabla^2_{xx} \mathcal{L}(x'', \lambda_1'') = \begin{bmatrix} +\cancel{2} \cdot \left(+\frac{\sqrt{5}}{\cancel{2}\sqrt{2}}\right) & 0 \\ 0 & +\cancel{2} \cdot \left(+\frac{\sqrt{5}}{\cancel{2}\sqrt{2}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{5}{2}} \end{bmatrix}$$

$\Longrightarrow$ $z^T \nabla_{xx}^2 \mathcal{L}(x'', \lambda_,'') z$ is positive definite.

This means that x' maximizes $f(x)$, while x'' minimizes $f(x)$.

f) Ω is convex since any line s between ~~the two~~ points $x' \in \Omega$ and $x'' \in \Omega$ is also in Ω.

$$f(x) = 2x_1 + x_2$$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= 2(\alpha x_1 + (1-\alpha)y_1) \\ &\quad + (\alpha x_2 + (1-\alpha)y_2) \\ &= 2\alpha x_1 + 2(1-\alpha)y_1 \\ &\quad + \alpha x_2 + (1-\alpha)y_2 \\ &= \alpha(2x_1 + x_2) + (1-\alpha)(2y_1 + y_2) \\ &= \alpha f(x) + (1-\alpha) f(y) \end{aligned}$$

$\Longrightarrow$ $f(x)$ is convex

$\longrightarrow$ The problem is a convex problem

Problem 3 : Exercise 2.19

$$\min \; f(x) = -2x_1 + x_2$$

s.t.

$$c_1(x) = (1-x_1)^3 - x_2 \geq 0$$

$$c_2(x) = x_2 + 0.25x_1^2 - 1 \geq 0$$

$$x^* = (0,1)^T$$

$$A(x^*) = \{1,2\}$$

a) $$\nabla_x c_1(x) = \begin{bmatrix} -3(1-x_1)^2 \\ -1 \end{bmatrix}$$

$$\nabla_x c_2(x) = \begin{bmatrix} 0{,}5x_1 \\ 1 \end{bmatrix}$$

$$\nabla_x c_1(x^*) = \begin{bmatrix} -3(1-0)^2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$\nabla_x c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ $\nabla_x c_1(x^*)$ and $\nabla_x c_2(x^*)$ are linearly independent

$\Longrightarrow$ <u>LICQ holds for x^*</u>

b)

$$\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$$
$$= -2x_1 + x_2$$
$$- \lambda_1((1-x_1)^3 - x_2)$$
$$- \lambda_2(x_2 + 0{,}25x_1^2 - 1)$$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \lambda_1 \begin{bmatrix} -3(1-x_1)^2 \\ -1 \end{bmatrix} - \lambda_2 \begin{bmatrix} 0{,}5x_1 \\ 1 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*,\lambda^*) = 0$$

$$\Longrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \lambda_1^* \begin{bmatrix} -3 \\ -1 \end{bmatrix} - \lambda_2^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow -2 + 3\lambda_1^* = 0$$

$$\underline{\lambda_1^* = \frac{2}{3}}$$

$$\longrightarrow \quad 1 + \lambda_1^* - \lambda_2^* = 0$$

$$\lambda_2^* = \lambda_1^* + 1 = \frac{2}{3} + 1 = \underline{\frac{5}{3}}$$

KKT conditions :

(I) $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$

(II) $c_i(x^*) = 0 \quad , \quad i \in \mathcal{E}$

(III) $c_i(x^*) \geq 0 \quad , \quad i \in \mathcal{I}$

(IV) $\lambda_i^* \geq 0 \quad , \quad i \in \mathcal{I}$

(V) $\lambda_i^* c_i(x^*) = 0 \quad , \quad i \in \mathcal{E} \cup \mathcal{I}$

$$c_1(x^*) = (1-0)^3 - 1 = 1 - 1 = \underline{0}$$

$$c_2(x^*) = 1 + 0 - 1 = \underline{0}$$

$\Longrightarrow$ The KKT conditions are satisfied at $x^* = [0, 1]^T$

d) Second-order necessary conditions:

x^* is a local solution

KKT holds

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w \geq 0 \ , \quad w \in \mathcal{C}(x^*, \lambda^*)$$

$$\nabla c_1(x^*) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$\nabla c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{F}(x^*) = \left\{ d \;\middle|\; \begin{array}{ll} d^T \nabla c_i(x^*) = 0 \ , & i \in \mathcal{E} \\ d^T \nabla c_i(x^*) \geq 0 \ , & i \in \mathcal{A}(x^*) \cap \mathcal{I} \end{array} \right\}$$

$$\mathcal{E} = \emptyset \ , \quad \mathcal{I} = \{1, 2\} \ , \quad \mathcal{A}(x^*) = \{1, 2\}$$

$$\Longrightarrow \mathcal{F}(x^*) = \left\{ d \;\middle|\; \begin{array}{l} d^T \nabla c_1(x^*) \geq 0 \\ d^T \nabla c_2(x^*) \geq 0 \end{array} \right\}$$

$$d^T \nabla c_2(x^*) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = d_2 \geq 0$$

$$d^T \nabla c_1(x^*) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = -3d_1 - d_2 \geq 0$$

$$\Longrightarrow \quad d_1 \leq -\frac{1}{3} d_2$$

$$\mathcal{F}(x^*) = \left\{ d \in \mathbb{R}^2 \mid d_1 \leq -\frac{1}{3} d_2 \, ; \; d_2 \geq 0 \right\}$$

$$\mathcal{C}(x^*, \lambda^*) = \left\{ w \in \mathcal{F}(x^*) \mid \nabla c_i(x^*)^T w = 0, \right.$$
$$\left. \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0 \right\}$$

$$\lambda_1^* > 0 \, , \quad \lambda_2^* > 0$$

$$\Longrightarrow \quad \mathcal{C}(x^*, \lambda^*) = \left\{ w \in \mathcal{F}(x^*) \mid \nabla c_i(x^*)^T w = 0, \right.$$
$$\left. \text{for } i \in \{1, 2\} \right\}$$

$$\nabla c_2(x^*)^T w = 0 \quad \Longrightarrow \quad w_2 = 0$$
$$\nabla c_1(x^*)^T w = 0 \quad \Longrightarrow \quad w_1 = 0$$

$$\Longrightarrow \quad \mathcal{C}(x^*, \lambda^*) = \{ w = \underline{0} \}$$

$$\Longrightarrow \quad \underline{w^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) w = 0 \, , \quad w \in \mathcal{C}(x^*, \lambda^*)}$$

<u>Hence the second-order necessary conditions are satisfied.</u>

Second-order sufficient conditions:

KKT holds

$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0$

Since $w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w = 0$ the second-order sufficient conditions are not satisfied

Problem 4: Exercise 12.21

max $f(x) = x_1 x_2$

s.t.

$c_1(x) = 1 - x_1^2 - x_2^2 \geq 0$

$\Updownarrow$

min $f(x) = -x_1 x_2$

s.t.

$c_1(x) = 1 - x_1^2 - x_2^2 \geq 0$

$$\mathcal{L}(x,\lambda) = -x_1 x_2 - \lambda_1 (1 - x_1^2 - x_2^2)$$

$$\nabla_x \mathcal{L}(x,\lambda) = -\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} - \lambda_1 \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$$

$$\Longrightarrow \quad -x_2^* + 2\lambda_1^* x_1^* = 0 \longrightarrow x_2^* = 2\lambda_1^* x_1^*$$

$$\Longrightarrow \quad -x_1^* + 2\lambda_1^* x_2^* = 0$$

$$-x_1^* + 2\lambda_1^* (2\lambda_1^* x_1^*) = 0$$

$$-x_1^* + 4\lambda_1^{*2} x_1^* = 0$$

$$x_1^* (4\lambda_1^{*2} - 1) = 0$$

$$4\lambda_1^{*2} - 1 = 0$$

$$\lambda_1^{*2} = \frac{1}{4}$$

$$\lambda_1^* = \pm \frac{1}{2}$$

From the KKT conditions we have $\lambda_i^* > 0, i \in I$.

$$\Longrightarrow \underline{\lambda_1^* = \frac{1}{2}}$$

$$\longrightarrow \quad x_2^* = 2\lambda_1^* x_1^* = \cancel{2} \cdot \frac{1}{\cancel{2}} x_1^* = x_1^*$$

$$x_2^* = x_1^*$$

This yields two points on the boundary

$$x^* = \left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$$

$$x^{**} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$$

$$\nabla c_1(x) = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}$$

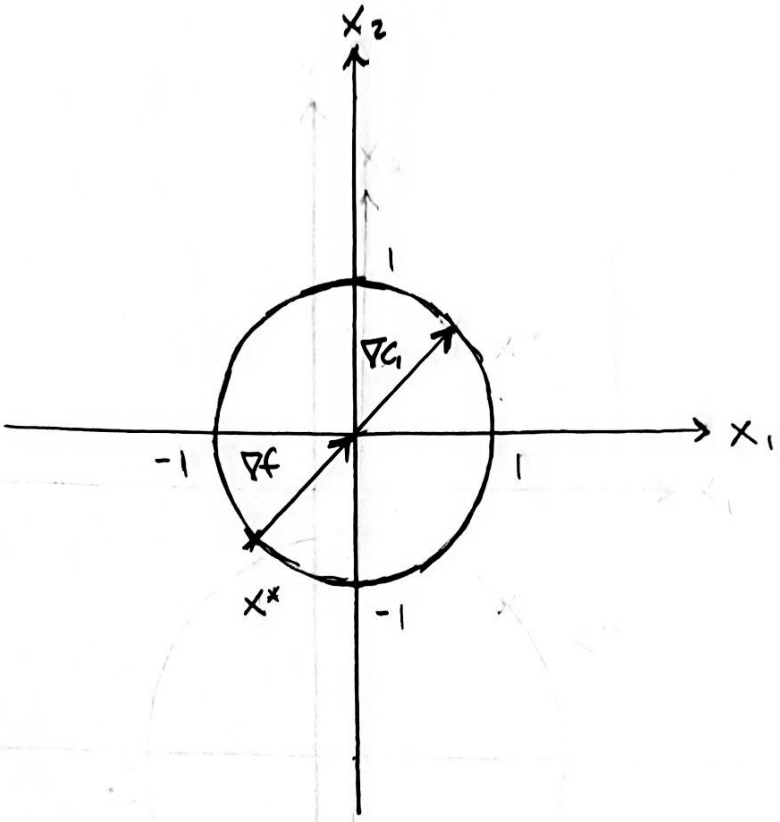
$$\nabla f(x) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$$

$$\nabla c_1(x^*) = \left[\sqrt{2}, \sqrt{2}\right]^T$$

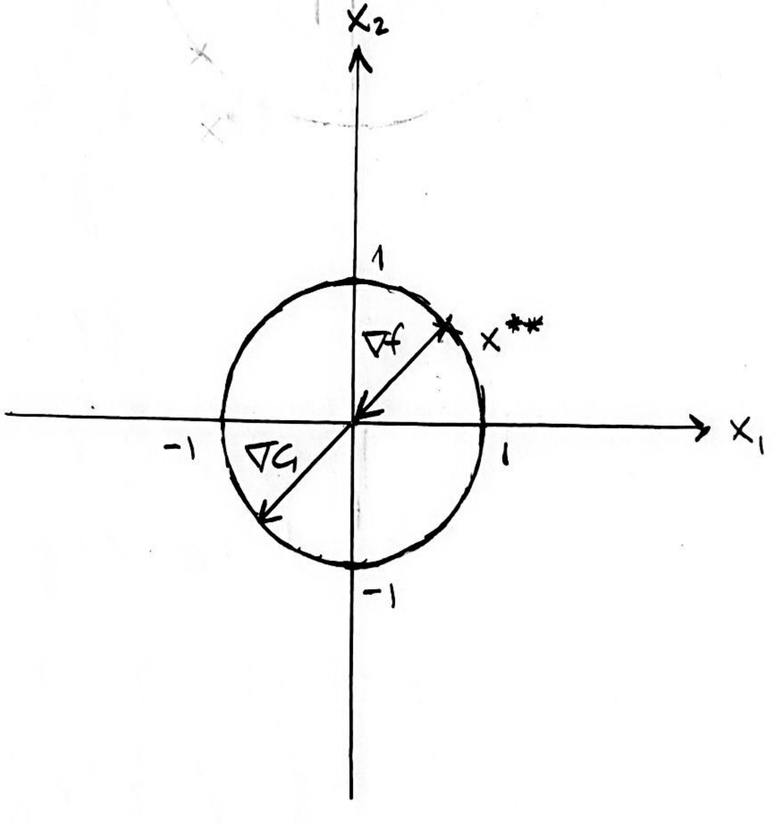
$$\nabla f(x^*) = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$$

$$\nabla c_1(x^{**}) = \left[-\sqrt{2}, -\sqrt{2}\right]^T$$

$$\nabla f(x^{**}) = \left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$$



x2
x1
1
-1
1
-1
∇c1
∇f
x*



x2
x1
1
-1
1
-1
∇f
∇c1
x**