MATH 135 — Fall 2021
Sample Proofs from Lecture 6

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March 24, 2025

Proving existentially quantified statements

To prove a existentially statement (“$\forall x \in S,\ P(x)$”):

- Do the scratchwork investigation to determine a value of x that works.
- Start your proof with “Let $x = \ldots$” and state clearly that x is in fact an element of S (or demonstrate that x is an element of S if it is not clear).
- Then proceed to show that $P(x)$ is true for this particular value of x.
- Do not include your scratchwork as part of your proof! However, your proof must be self-contained, so include any explanations necessary to justify your argument.

Claim. $\exists m \in \mathbb{Z},\ \frac{m-7}{2m+4} = 5$.

Proof. Let $m = -3$, which is an integer. Now

$$\frac{m-7}{2m+4} = \frac{-3-7}{-6+4} = \frac{-10}{-2} = 5,$$

which proves the claim.

□

Claim. *There exists a perfect square k such that $k^2 - \frac{31}{2}k = 8$.*

Proof. Let $k = 16$, which is a perfect square as $16 = 4^2$. Then

$$\begin{aligned} k^2 - \frac{31}{2}k &= 16^2 - 31 \cdot 8 \\ &= 256 - 248 \\ &= 8, \end{aligned}$$

as desired. □

Disproving statements

To prove that a statement is false:

- Negate the statement.
- Prove that the negation is true.

> **Claim.** $\forall x \in \mathbb{R},\ (x^2-1)^2 > 0.$

We prove this statement is false by stating its negation and proving that. The negation of this statement is:

$$\exists x \in \mathbb{R},\ (x^2-1)^2 \leq 0.$$

Proof (of the negation). Let $x = 1$, which is a real number. Then

$$(x^2-1)^2 = (1-1)^2 = 0^2 = 0 \leq 0,$$

as desired. □

> **Claim.** *There exists a real number θ for which it holds that* $\sin(2\theta) + \cos(2\theta) = 3$.

We prove this statement is false by stating its negation and proving that. The negation of this statement is:

$$\forall \theta \in \mathbb{R},\ \sin(2\theta) + \cos(2\theta) \neq 3.$$

Proof (of the negation). Let θ be a real number. Note that

$$\begin{aligned} -1 &\leq \sin(2\theta) \leq 1 \\ \text{and} \quad -1 &\leq \cos(2\theta) \leq 1, \end{aligned}$$

and thus

$$-2 \leq \sin(2\theta) + \cos(2\theta) \leq 2 < 3.$$

Hence $\sin(2\theta) + \cos(2\theta) < 3$. □

Proving statements with nested quantifiers

> Prove or disprove the following statements:
>
> $$\begin{aligned} A&\colon \text{“}\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1\text{”} \\ B&\colon \text{“}\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x^3 - y^3 = 1\text{”} \end{aligned}$$

Statement A is true but statement B is false. In statement A, one decides on a value for y *after* a value for x is given. In statement B, a value of y is picked first and this value must work for every possible choice of x.

We first prove statement A.

Proof. Let x be a real number. Choose $y = (x^3 - 1)^{1/3}$, which is a real number. Then

$$x^3 - y^3 = x^3 - \left((x^3-1)^{1/3}\right)^3 = x^3 - (x^3 - 1) = 1,$$

which completes the proof. □

Now state the negation of B:

$$\neg B\colon \text{“}\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x^3 - y^3 \neq 1\text{”}$$

Proof (of negation of B). Let y be a real number and choose $x = (y^3 - 2)^{1/3}$. Then

$$x^3 - y^3 = \left((y^3-2)^{1/3}\right)^3 - y^3 = y^3 - 2 + y^3 = 2,$$

which is not equal to 1. □

Proving implications

An implication is a statement of the form “$A \implies B$” or “$\forall x \in S,\ P(x) \implies Q(x)$.”

1. Assume that the hypothesis (i.e., A or $P(x)$) is true.
2. Prove the conclusion (i.e., B or $Q(x)$) using only what you know to be true.
3. Do not worry about instances where the hypothesis is false!

Claim. *For every integer k, if k^5 is a perfect square then $9k^{19}$ is a perfect square.*

Proof. Let k be an integer. Assume that k^5 is a perfect square. There exists an integer m such that $m^2 = k$. Now,

$$\begin{aligned} 9k^{19} &= 3^2 \cdot k^{14} \cdot k^5 \\ &= 3^2 \cdot (k^7)^2 \cdot m^2 \\ &= (3mk^7)^2 \end{aligned}$$

which is a perfect square as $3mk^7$ is an integer. □

Claim. *For every integer n, if 2^{2n} is odd then 2^{-2n} is odd.*

Proof. Let n be an integer. There are three possible cases to consider: $n < 0$, $n = 0$, and $n > 0$.

Case 1: Suppose $n < 0$. Then 2^{2n} is not an integer and thus not odd.

Case 2: Suppose $n = 0$. Then $2^{2n} = 2^0 = 1$, which is odd. In this case, one has $2^{-2n} = 2^0 = 1$, which is again odd.

Case 2: Suppose $n > 0$. Then $n - 1 \geq 0$ and thus

$$2^{2n} = 2^{2n-2+2} = 2^2 \cdot 2^{2(n-1)} = 2(2 \cdot 4^{n-1})$$

which is even as $2 \cdot 4^{n-1}$ is an integer, and thus 2^{2n} is not odd.

This proves the claim, as the implication has been shown to be true in every case where the hypothesis holds. $\square$