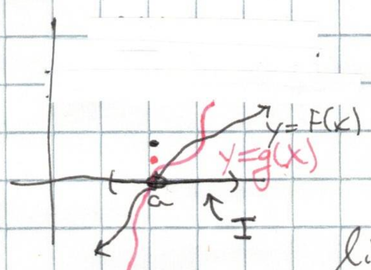
§4.3 L'hôpital's Rule

Theorem (L'hôpital's Rule – 1st form)
Let I be an open interval containing a and suppose that f and g are differentiable everywhere on I except possibly at a. If

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$$

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ (if this limit exists).

Proof idea: We may suppose that $f(a)=0=g(a)$ since f and g have at most a removeable discontinuity at a.



Now:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} =$$

$$= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

Idea: If f and g are polynomials with $f(0)=g(0)=0$,

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{a_1x+a_2x^2+\cdots+a_nx^n}{b_1x+b_2x^2+\cdots+b_mx^m}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0}\frac{a_1+a_2x+\cdots+a_nx^{n-1}}{b_1+b_2x+\cdots+b_mx^{m-1}} = \frac{a_1}{b_1} = \frac{f'(0)}{g'(0)}.$$

and $f'(x) = a_1 + 2a_2x + \cdots$ so $f'(0)=a_1$
$g'(x) = b_1 + 2b_2x + \cdots$ so $g'(0)=b_1$.

Note L'hôpital's Rule also works for one-sided limits $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$ and limits at infinity $\lim_{x\to\infty}$ $\lim_{x\to-\infty}$

Examples:

- $\lim_{x\to 2} \frac{2-x}{\sqrt{2}-\sqrt{x}}$ looks like $\frac{0}{0}$ at $x=2$

Now $\frac{d}{dx}(2-x) = -1$

and $\frac{d}{dx}(\sqrt{2}-\sqrt{x}) = -\frac{1}{2\sqrt{x}}$

so by L'hôpital's Rule

$$\lim_{x\to 2} \frac{2-x}{\sqrt{2}-\sqrt{x}} = \lim_{x\to 2} \frac{-1}{-\frac{1}{2\sqrt{x}}} = \lim_{x\to 2} 2\sqrt{x} = \boxed{2\sqrt{2}}$$

- $\lim_{x\to 0} \frac{\tan x}{x}$ $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\sec^2 x}{1} = \sec^2(0) = \boxed{1.}$$

- $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$ $\left(= \frac{0}{0}\right)$

$LR = \lim_{x\to 0} \frac{-\sin x}{2x}$ $\left(= \frac{0}{0}\right)$

$LR = \lim_{x\to 0} \frac{-\cos x}{2}$

$= \frac{-\cos 0}{2} = \boxed{-\frac{1}{2}}$

Can Repeatedly apply LR. Must check conditions each time.

Thm (L'hôpital's Rule, 2^{nd} form)

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \quad \text{(If limit exists)}.$$

[Also works for one-sided limits and $x\to\pm\infty$]

Idea: Suppose $L = \lim_{x\to a} \frac{f(x)}{g(x)}$. ~~Then~~

Note that $\lim_{x\to a} \frac{1}{f(x)} = 0 = \lim_{x\to a} \frac{1}{g(x)}$. So

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} \quad \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{-\frac{g'(x)}{g(x)^2}}{-\frac{f'(x)}{f(x)^2}} = \lim_{x\to a} \left[\frac{f(x)^2}{g(x)^2} \frac{g'(x)}{f'(x)}\right]$$

$$= L^2 \frac{1}{\lim \frac{f'(x)}{g'(x)}}$$

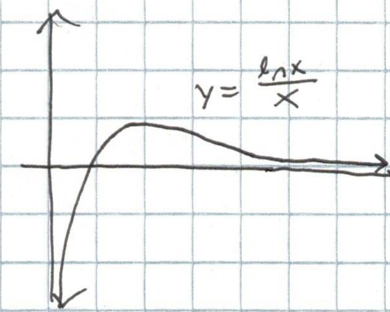
$$\Rightarrow \lim_{x\to a} \frac{f'(x)}{g'(x)} = L = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

Examples:

- $\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{1/x}{1}$ by L'hôpital's

$= \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$



Note · $\lim_{x\to 0^+} \frac{\ln x}{x} = \frac{-\infty}{0}$ NOT an indeterminate form!
Don't use L'hôpital's here.

$= \lim_{x\to 0^+} \left[\frac{1}{x} \ln x\right] = (+\infty)(-\infty) = \boxed{-\infty}$

Note: If you tried using L'hôpital's here, you get the wrong answer!

$f(x) = \ln x$ $f'(x) = 1/x$

$g(x) = x$ $g'(x) = 1$

$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{1}{x} = \boxed{+\infty}$

$+\infty$ not the same as $-\infty$.

- $\lim_{x\to\infty} \frac{x^3 - 2x + 7}{3x^3 + x^2 + x + 1}$ $\left[= \frac{\infty}{\infty}\right]$

$[LR] = \lim_{x\to\infty} \frac{3x^2 - 2}{9x^2 + 2x + 1}$ $\left[= \frac{\infty}{\infty}\right]$

$[LR] = \lim_{x\to\infty} \frac{6x}{18x + 2}$ $= \left[\frac{\infty}{\infty}\right]$

$[LR] = \lim_{x\to\infty} \frac{6}{18} = \boxed{\frac{1}{3}}$

Other Indeterminate Forms

0^0 1^∞ $0 \cdot \infty$ $\infty - \infty$ ∞^0

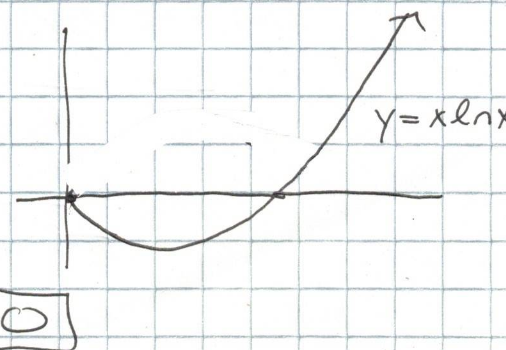
Manipulate into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and apply L'hôpital's Rule!

Examples $(0)\cdot(\infty)$

- $\lim_{x\to 0^+} x \ln(x)$ $\left\{= (0)(-\infty)\right\}$

$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x}$ $\left[= \frac{-\infty}{+\infty}\right]$

$[LR] = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x = \boxed{0}$



• $\lim_{x\to\infty} x e^{-x}$ (looks like $\infty \cdot 0$)

$= \lim_{x\to\infty} \frac{x}{e^x}$ $\left(\frac{\infty}{\infty}\right)$

L'hopital's $= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty}$

$= \boxed{0}$

$\lim_{x\to\infty} x\ln(1+\frac{1}{x})$

$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{1/x}$ $\cdot\left(\frac{0}{0}\right)$

$[LR] = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{-\frac{1}{x^2}}}{\cancel{-1/x^2}}$

$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{\lim_{x\to\infty} 1+\frac{1}{x}} = \frac{1}{1}$

$= \boxed{1}$

For indeterminate forms 0^0, 1^∞ or ∞^0

use : $f(x)^{g(x)} = e^{g(x)\ln(f(x))}$

and consider $g(x)\ln(f(x))$.

Ex • $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)}$

$= e^{\lim_{x\to 0^+}[x\ln(x)]}$ since exponential function is continuous

Now

$= e^0$ (from before : $\lim_{x\to 0^+} x\ln x = 0$)

• $\lim_{x\to\infty} (1+\frac{1}{x})^x$

$= \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$

from before

$= e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e^1 = \boxed{e}$

• $\lim_{x\to\frac{\pi}{2}^-} \left[(\sec x)^{\cos x}\right] = \lim_{x\to\frac{\pi}{2}^-} \left(\frac{1}{\cos x}\right)^{\cos x}$ ∞^0

$= \lim_{x\to\frac{\pi}{2}^-} e^{\cos(x)\ln(\sec x)} = e = \boxed{e^0} = \boxed{1}$

$\lim_{x\to\frac{\pi}{2}^-} \cos x\ln(\sec x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln\sec x}{\sec x}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\sec(x)}\cancel{\sec'(x)}}{\cancel{\sec'(x)}} = \lim_{x\to\frac{\pi}{2}^-} \cos x = \boxed{0}$

Indeterminate forms $\underline{\infty - \infty}$

<u>Ex</u> $\lim_{x \to \frac{\pi}{2}^-} \{\sec x - \tan x\}$ $\quad \infty - \infty$

$$= \lim_{x \to \frac{\pi}{2}^-} \left\{\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right\}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x} \quad = \frac{0}{0}$$

$$\{LR\} = \lim_{x \to \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = \frac{0}{-1} = \boxed{0}$$

<u>Ex</u> $\lim_{x \to \infty} \{\ln(x) - \ln(3x+1)\}$ $\quad \{\infty - \infty\}$

$$\boxed{\ln a - \ln b = \ln \frac{a}{b}}$$

$$= \lim_{x \to \infty} \ln\left(\frac{x}{3x+1}\right)$$

$$= \ln\left(\lim_{x \to \infty} \frac{x}{3x+1}\right)$$

$$= \boxed{\ln\left(\frac{1}{3}\right)}$$

Ex

<u>Note</u> to use L'hôpital's Rule, we <u>MUST</u> have an indefinite form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Make sure to check that $\lim f(x) = 0 = \lim g(x)$

or $\lim f(x) = \pm\infty = \lim g(x)$

FIRST to find $\lim_{x \to a} \frac{f(x)}{g(x)}$.

<u>Ex</u> $\lim_{x \to 0} \frac{\cos(\pi x)}{x + 4} = \frac{\cos 0}{4} = \frac{1}{4}$

$f(x) = \cos \pi x \qquad f'(x) = -\pi \sin(\pi x)$

$g(x) = x + 4 \qquad g'(x) = 1$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = -\pi \sin(\pi) = 0$$