Riddler August 12, 2022:
Escape the casino!

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Riddler classic

From Andrew Lin comes a game for getting yourself home:

Question 1. *You are stranded in a casino (lucky you!) and need to purchase a flight home. Flights cost \$250, but you have only \$100 at the moment. However, as I just said, you're in a casino! Surely, you can gamble your way to \$250.*

The casino has a game called "Riddler's Delight," in which you can bet any amount of money in your possession for an even greater amount of money. You can even bet fractional (i.e., you can bet fractions of a penny), irrational or infinitesimal amounts if you so desire.

The catch is that the odds are not in your favor. In Riddler's Delight, whenever you bet A dollars in an attempt to win B dollars (with $B > A$), your probability of winning is not A/B, which you would expect from a fair game. Instead, your probability of winning is always 10 percent less, or $0.9(A/B)$.

What should your betting strategy be to maximize your probability of getting home, and what is that probability?

Solution

The naive strategy is to go big or go home—simply bet everything you have and go for broke by attempting to win the required \$250. With this strategy, you'll win your bet with probability

$$0.9 \times (100/250) = 0.36$$

(i.e., 36%) after which you'd have enough money to go home. But with probability 0.54 you lose all your money and you're out on the streets.

Can we do any better? Well, one thing we can try would be to split your starting money in half and make a bet of \$50. This time, we don't need to win \$250 with our bet. By betting only \$50, we still have \$50 remaining in our pockets, so we only need to win the difference of \$200. If we lose our first bet, we can bet the remaining \$50 for the entire sum of \$250. Let's analyze what the odds of winning are with this strategy.

- First bet \$50 in attempt to gain \$200. Your odds of winning the requisite funds to go home on your first bet are

  $$0/9 \times (50/200) = 0.225,$$

  while your odds of losing this first bet are $1 - 0.225 = 0.775$.

- If you lose your first bet, you still have \$50 remaining to bet to win the whole \$250. Making this bet, your odds of winning are

$$0/9 \times (50/250) = 0.18.$$

Putting this altogether, your odds of winning after either bet can be computed as

$$0.225 + (1 - 0.225)0.18 = 0.3645,$$

or 36.45%, which is *slightly higher* than your odds of betting it all in one go!

What if we made (up to) 100 smaller bets of \$1 each? After n such bets we have $\$(100 - n)$, and so on the n^{th} we attempt to win the difference between \$250 and this remaining value. The odds of winning the n^{th} bet (after having lost the first $n - 1$ bets) are therefore given by

$$p_n = 0.9 \frac{100 - n}{250 - 150 + n}.$$

To compute the probability that we win on at least one of these bets, examine the probability that we lose on every bet, which is given by

$$\begin{aligned} \Pr(\text{lose}) &= (1 - p_1)(1 - p_2) \cdots (1 - p_{100}) \\ &= \prod_{n=1}^{100} \left(1 - 0.9 \frac{100 - n}{250 - 150 + n}\right) \\ &\approx 0.6315218 \end{aligned}$$

and thus the odds of going home are $1 - \Pr(\text{lose}) \approx 0.3684782$ (roughly 36.9%), which are even slightly better odds!

Can we do even better with even smaller bets? The rules allow us to make *any* sized bet in fractional dollars, regardless how small. To analyze this, suppose your strategy is to split the original \$100 into N equally sized smaller amounts of $100/N$ dollars. You continue to make bets of this size, aiming to earn the remaining sum you need to reach \$250 every time you bet. After n bets, for some $n \in \{1, 2, \ldots, N\}$, you'd have $100(1 - n/N)$ dollars and you'd be targeting to win the remaining $250 - 100(1 - n/N)$ dollars, so your odds of winning the n^{th} bet are

$$p_n = 0.9 \frac{100/N}{250 - 100(1 - n/N)} = 0.9 \frac{1}{\frac{250-100}{100} + n} = \frac{0.9}{(r-1)N + n}$$

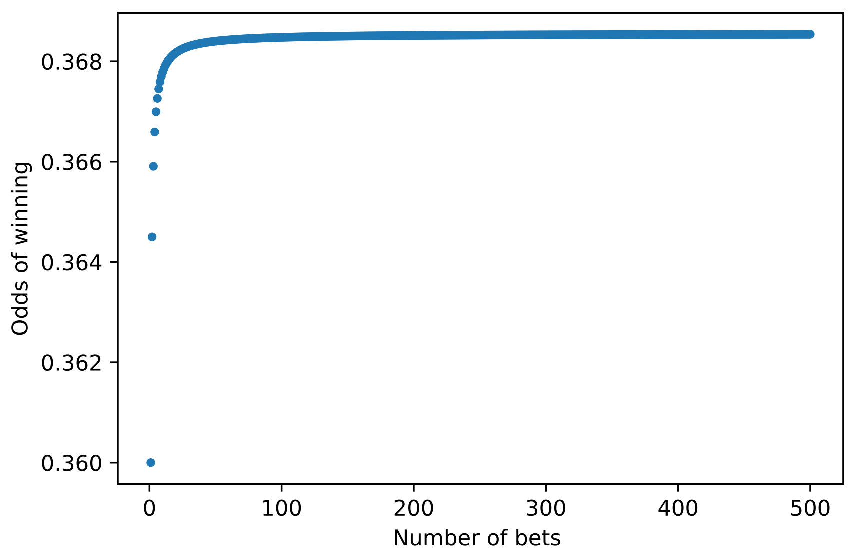
where $r = \frac{250}{100}$ is the ratio of the target amount over your starting amount of money. Your odds of winning are

$$f(N) = 1 - \Pr(\text{lose}) = 1 - \prod_{n=1}^{N} \left(1 - \frac{0.9}{(r-1)N + n}\right).$$

Summarizing these results (and computing the probability of winning for a few more values of N), we obtain the following table:

N	$f(N)$ (odds of winning)
1	0.36
2	0.3645
100	≈ 0.36847824
1000	≈ 0.36854655
10000	≈ 0.36855337
100000	≈ 0.36855405
1000000	≈ 0.36855412
10000000	≈ 0.36855413

We see that this slowly converges to ≈ 0.3685541325 in the limit of large N.



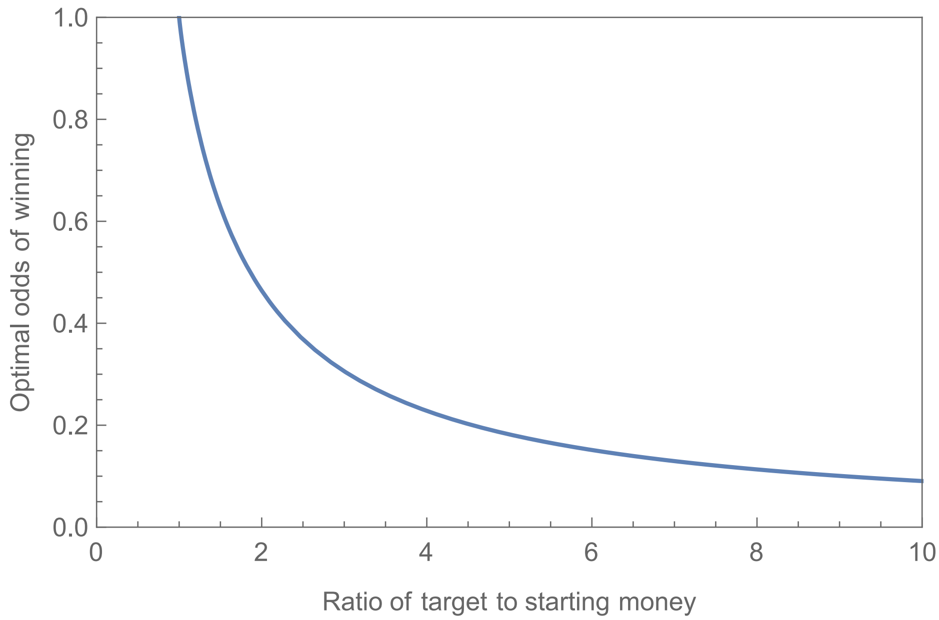
I tried numerous different strategies, but nothing else seemed to get better than this, so I assume that this is the best you can do! Perhaps someone smarter than I can prove that splitting your original sum into arbitrarily small bits is the best you can do.

Generalizing to different ticket prices

Notice that the odds of winning only depends on the ratio $r = \frac{250}{150}$ of ticket price amount to starting funds as well as the number of smaller bets N. If we call $g(r)$ the optimal winning strategy by defining

$$g(r) = 1 - \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{0.9}{(r-1)N + n}\right),$$

we can plot this function to see what the optimal odds would be for winning if the ticket price changed.



When $r \leq 1$, we don't have to gamble anything and we can simply go home. For larger values of r (i.e., increasing cost of plane ticket home), the odds of making it home decrease. Interestingly this behaves as $.9/r$ for large r.