Riddler June 18, 2022:
Elevensies and urns

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Riddler express

From Ben Weiss and David Butler comes a what is presumably Eleven's favorite puzzle to think about in her sensory deprivation tank.

Question 1. *Anna loves multiples of 11, but her friend Jane is not quite so keen. One day, Anna is flipping idly through the yellow pages (remember those?), which is full of 10-digit numbers. She notices that every 10-digit number seems to have an interesting property: It is either a multiple of 11, or it can be made a multiple of 11 by changing a single digit. For example, there are several ways to make the 10-digit number 5551234567 into a multiple of 11, such as changing the first digit to 4.*

This gets the two friends wondering: Does every counting number have this property? Either prove it's true for every number, or find the smallest counting number that is not a multiple of 11 and cannot be made a multiple of 11 by changing one digit.

Solution. No! It turns out that the smallest counter example is 545.

To see this, first note that every one-digit number can trivially be made into a multiple of 11 by switching the only digit to a 0, while every two-digit number can be seen to have this property by switching one of the digits to match the other. Hence the smallest possible counter example must be at least three digits.

Now suppose there is a three-digit integer N that does not have this property. We can express the digit representation of this number as abc so that

$$N = c + 10b + 100a.$$

Let $m \in \{0, 1, \dots 10\}$ be the remainder of N modulo 11 such that

$$N \equiv a - b + c \equiv m \pmod{11}.$$

If $m = 0$ then the number is divisible by 11, so we may suppose that $m \neq 0$.

- If it were the case that $a \not\equiv m - 1 \pmod{11}$ then we could replace a with the digit $d \in \{0, 1, \dots, 9\}$ such that $d \equiv a - m \pmod{11}$. (Indeed, as long as $a \not\equiv m - 1 \pmod{11}$ then $a - m \not\equiv 10 \pmod{10}$ and thus $a - m$ is congruent to one of $0, 1, \dots, 9$.) The number constructed this way is necessarily a multiple of 11, as

$$100d + 10b + c \equiv d - b + c \equiv (a - m) - b + c \equiv 0 \pmod{11}.$$

- Similarly, if it were the case that $c \not\equiv m-1 \pmod{11}$ then we could replace c with the digit $d \in \{0, 1, \ldots, 9\}$ such that $d \equiv c - m \pmod{11}$.
- Finally, if it were the case that $b \not\equiv 10 - m \pmod{11}$ then we could replace b with the digit $d \in \{0, 1, \ldots, 9\}$ such that $d \equiv b + m \pmod{11}$. (As before, in this case one necessarily has $d \equiv b + m \not\equiv 10$.) And the resulting number satisfies

$$100a + 10d + c \equiv a - d + c \equiv a - (b+m) + c \equiv 0 \pmod{11}.$$

As it is assumed that N does not have the desired propery, we conclude that

$$a \equiv c \equiv m - 1 \pmod{11} \qquad \text{and} \qquad b \equiv 10 - m \pmod{11}.$$

It follows that

$$m \equiv N \equiv a - b + c \equiv (m-1) - (10-m) + (m-1) \equiv 3m - 12 \pmod{11}$$

and thus $2m \equiv 12 \pmod{11}$, or simply $m = 6$. Hence a counter example may be formed by taking $a = c = 6 - 1 = 5$ and $b = 10 - 6 = 4$. Moreover, this is the *only* three-digit number that does not have the desired property, so it must be the smallest such number!

Riddler classic

This week's Classic may seem nonsensical at first. But surely there's more to it. . .

Question 2. *You have an urn with an equal number of red balls and white balls, but you have no information about what that number might be. You draw 19 balls at random, without replacement, and you get eight red balls and 11 white balls. What is your best guess for the original number of balls (red and white) in the urn?*

Solution. For this problem, we need to do a bit of *maximum likelihood estimation*. Suppose there are n red balls and n white balls in the urn. Assuming you then draw 19 balls out of the urn without replacement, the probability of obtaining exactly r red balls and w white balls is equal to

$$p(r, w|n) = \frac{\binom{n}{r}\binom{n}{w}}{\binom{2n}{r+w}}.$$

(If either $n < r$ or $n < w$ then this probability is zero.) The goal now is to determine the integer n that maximizes $p(8, 11|n)$.

Allowing r and w to be arbitrary nonnegative integers for a moment, consider the sequence $a_0, a_1, \ldots$ defined by

$$a_n = p(r, w|n) = \frac{\binom{n}{r}\binom{n}{w}}{\binom{2n}{r+w}}$$

for each nonnegative integer n. Note that $a_n = 0$ if $a < \min\{r, w\}$ and strictly positive otherwise. This sequence can obtain its maximum only if there exists an integer $\geq \min\{r, w\}$ such that $a_n \geq a_{n+1}$. Now,

$$\frac{a_{n+1}}{a_n} = \frac{\binom{n+1}{r}\binom{n+1}{w}}{\binom{n}{r}\binom{n}{w}} \frac{\binom{2n}{r+w}}{\binom{2n+2}{r+w}} = \frac{(n+1)^2}{(n-r+1)(n-w+1)} \frac{(2n-r-w+1)(2n-r-w+2)}{(2n+1)(2n+2)}.$$

If this sequence obtains its maximum at n we must have $\frac{a_{n+1}}{a_n} < 1$. After a bit of rearranging, we see that $\frac{a_{n+1}}{a_n} < 1$ holds if and only if

$$(2n+1)(1+n-r)(1+n-w)\Big[2rw - \big(r+w-(r-w)^2\big)(n+1)\Big] < 0$$

or equivalently (because $(2n+1)(1+n-r)(1+n-w) > 0$ always holds)

$$2rw < \big(r+w-(r-w)^2\big)(n+1). \tag{1}$$

Now, in the specific case when $r = 8$ and $w = 11$, this reduces to

$$176 < 10(n+1)$$

or equivalently $n > 16 + \frac{3}{5}$. In particular, as n must be an integer we see that

$$\frac{a_{n+1}}{a_n} > 1 \quad \text{whenever} \quad 11 \leq n < 17$$

and

$$\frac{a_{n+1}}{a_n} < 1 \quad \text{whenever} \quad n \geq 17.$$

Hence the maximum likelihood is obtained when there are 17 red balls and 17 white balls!

The general solution for arbitrary r and w

Let's consider now the general case when $r, w \geq 0$. If it is the case that $r = w$, then the value of n that maximzes the likelihood is just $n = r = w$. Indeed, if there are $2r$ balls (with r red and r white) and we draw $2r$ balls, one draws r red balls and w white balls with unit probability and thus $p(r, r|r) = 1$. For any other value of n, the probability of obtaining exactly the same number of red and white balls is less than 1.

Suppose now that $r \neq w$. What is the number n that maximizes the likelihood? From equation (1) above, note that

$$\frac{p(r,w|n+1)}{p(r,w|n)} < 1$$

holds if and only if

$$2rw < \big(r+w-(r-w)^2\big)(n+1).$$

There are two cases to consider.

1. If $r + w - (r-w)^2 <= 0$ then there are *no* positive solutions to this inequality. That is, the sequence of likelihoods $p(r, w|n)$ is strictly increasing in n, so there is *no finite value* of n that maximizes this likelihood. Hence, in this case, the most "likely" situation is that the urn contains infintely many balls!

2. If $r + w - (r-w)^2 > 0$ then we can simply rearrange this inequality to find that $\frac{p(r,w|n+1)}{p(r,w|n)} < 1$ holds if and only if

$$\frac{2rw}{r+w-(r-w)^2} - 1 < n.$$

Hence the sequence of likelihoods $p(r, w|n)$ is increasing for $n \leq \frac{2rw}{r+w-(r-w)^2} - 1$ and strictly decreasing afterwards. So the number n that maximizes the likelihood is

$$n = \left\lceil \frac{2rw}{r + w - (r - w)^2} - 1 \right\rceil .$$

We can collect this results into the following conclusion: If you draw r red balls and w white balls from the urn, the number n that maximizes the likelihood such that there are n red balls and n white balls in the urn is given by

$$n = \begin{cases} r, & r = w \\ \left\lceil \frac{2rw}{r+w-(r-w)^2} - 1 \right\rceil, & r + w > (r - w)^2 > 0 \\ \infty, & \text{otherwise} \end{cases}$$

where $n = \infty$ means that there is no finite value of n that maximizes the likelihood.