ECE 206 – Fall 2019

Final Exam

December 16, 2019 at 12:30

Instructor: Mark Girard

University of Waterloo

.

Solutions

Name: ____

Notes:

1. Fill in your name (first and last) and student ID number in the space above.
2. This midterm contains 14 pages (including this cover page) and 9 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided at the end. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
5. You are allowed to use **either the formula sheet provided or your own formula sheet that you've prepared yourself**. No other notes, books, calculators, or personal electronic devices of any kind may be used.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	11	4	12	7	5	10	9	7	11	76
Score:										

[3] 1. (a) Find all possible values of $z \in \mathbb{C}$ that satisfy $\cos z = -2$.

$z = x + jy$

$\cos(x+jy) = \cos x \cos(jy) + \sin x \sin(jy)$

$= \cos x \cosh y + j \sin x \sinh y$

$= -2$

$$\boxed{\begin{array}{c} z = n\pi + j\ln(2 \pm \sqrt{3}) \\ n \in \mathbb{Z} \text{ odd} \end{array}}$$

Imaginary part:

$\sin x \sinh y = 0$

$\Rightarrow \sin x = 0$ or $\sinh y = 0$

Real part:

- $\sinh y = 0 \Rightarrow \cosh y = 1$

$\Rightarrow \cos x = -2$

no solutions

- $\sin x = 0 \Rightarrow x = \pi n$ for $n \in \mathbb{Z}$.

$\Rightarrow \cos x = \cos(\pi n) = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

- n even $\Rightarrow \cosh y = -2 \rightarrow$ no solutions
- n odd $\Rightarrow \cosh y = 2$

$\Rightarrow \dfrac{e^y + e^{-y}}{2} = 2 \quad \Rightarrow e^y - 4 + e^{-y} = 0$

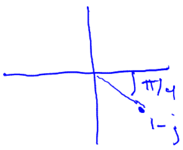
$\Rightarrow e^{2y} - 4e^y + 1 = 0$

$\Rightarrow e^y = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{4-1}$

$\Rightarrow y = \ln(2 \pm \sqrt{3})$

[3] (b) Find all possible values of $(1-j)^j$.

$1 - j = \sqrt{2}\, e^{-j\pi/4}$



$\log(1-j) = \log(\sqrt{2}\, e^{-j\pi/4}) = \ln\sqrt{2} - j(\pi/4 + 2n\pi) \quad n \in \mathbb{Z}$

$(1-j)^j = e^{j\log(1-j)} = e^{j(\ln\sqrt{2} - j(\pi/4 + 2n\pi))}$

$= e^{\pi/4 + 2n\pi}\, e^{j\ln\sqrt{2}}$

$= \underbrace{e^{\pi/4} e^{j\ln\sqrt{2}}}_{\text{Principal part}} e^{2\pi n} \quad n \in \mathbb{Z}$

[2] (c) Expand out the first four nonzero terms of the Taylor series of $f(z) = -1/z$ about $z = 1$. What is the radius of convergence of this Taylor series? Explain.

$-\dfrac{1}{z} = -\dfrac{1}{z-1+1} = -\dfrac{1}{1-(1-z)}$

$= -\dfrac{1}{1-w} \qquad w = 1 - z$

$= -(1 + w + w^2 + w^3 + \dots)$

$= -1 - (1-z) - (1-z)^2 - (1-z)^3 - \dots$

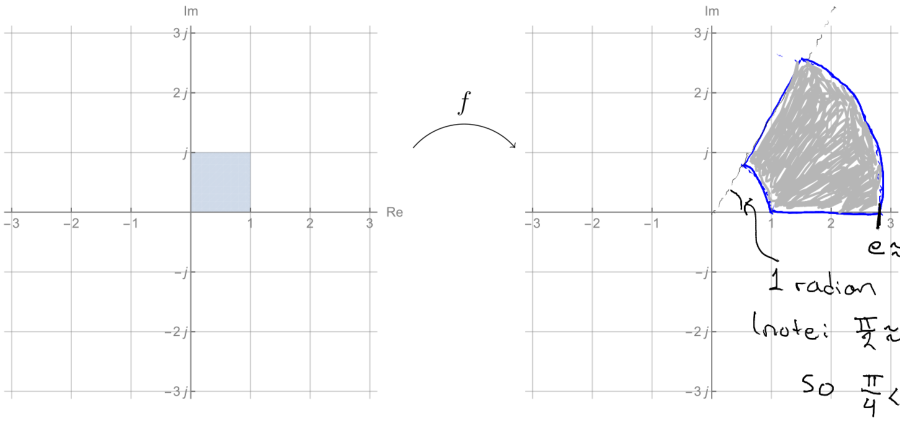
- Only singularity of $f(z) = -\frac{1}{z}$ is at $z = 0$.
- Distance from $z = 1$ to $z = 0$ is 1
- Radius of convergence is 1.

[3] (d) Consider the region $D = \{x + jy : 0 < x < 1 \text{ and } 0 < y < 1\}$ (shaded below). Sketch the image of D under the mapping f defined by $f(z) = e^z$.

$e^{x+jy} = e^x e^{jy}$

$0 < x < 1 \Rightarrow 1 < e^x < e$

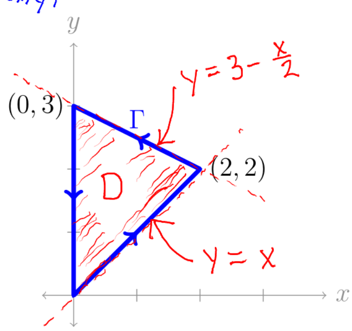
$0 < \operatorname{Arg}(e^{x+jy}) < 1$



(note: $\frac{\pi}{2} \approx 1.5$ and $\frac{\pi}{4} \approx .8$

so $\frac{\pi}{4} < 1 < \frac{\pi}{2}$)

[4] 2. For the curve Γ in $\mathbb{R}^2$ (shown in the figure below) and the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\boldsymbol{F}(x, y) = \big(\underbrace{\ln(\sin^2(x) + 1)}_{F_1(x,y)}, \underbrace{\cos(\sin y) + xy}_{F_2(x,y)}\big)$, evaluate $\oint_\Gamma \boldsymbol{F} \cdot \boldsymbol{r}$.



Set D to be the region such that $\partial D = \Gamma$.

The region can be given by

$D = \{(x,y) \mid 0 \le x \le 2 \quad x \le y \le 3 - \frac{x}{2}\}$

Since $\vec{F}$ is C^2 (as its components have continuous second derivatives) we can use Green's Theorem.

$$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_2}{\partial x} = y$$

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \iint_D y \, dA$$

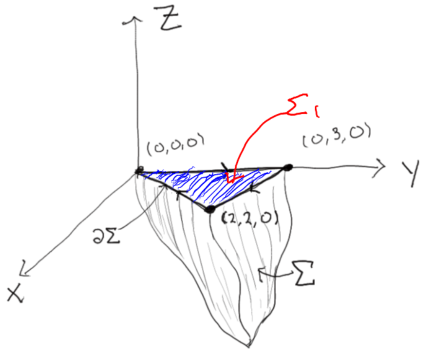
$$= \int_0^2 \int_x^{3 - x/2} y \, dy \, dx = \int_0^2 \frac{1}{2} y^2 \Big|_x^{3-\frac{x}{2}} dx$$

$$= \frac{1}{2} \int_0^2 \left(\left(3 - \frac{x}{2}\right)^2 - x^2\right) dx = \frac{1}{2} \int_0^2 \left(9 - 3x + \frac{x^2}{4} - x^2\right) dx$$

$$= \frac{1}{2} \int_0^2 \left(9 - 3x - \frac{3}{4} x^2\right) dx = \frac{1}{2} \left(9x - \frac{3}{2} x^2 - \frac{1}{4} x^3\right)\Big|_0^2$$

$$= \frac{1}{2} (18 - 6 - 2) = \frac{10}{2} = \boxed{5}$$

3. Consider the surface Σ in $\mathbb{R}^3$ (depicted at right) with outward facing normal, and whose closed boundary curve $\partial\Sigma$ is the triangle in the xy-plane (oriented clockwise when viewed from above) with vertices at the points $(0, 0, 0)$, $(0, 3, 0)$, and $(2, 2, 0)$.



[4] (a) Consider the vector field defined by $\boldsymbol{F}(x, y, z) = y\,\hat{\boldsymbol{\imath}}$. Compute $\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA$.

As the vector field is C^2 we can use Stoke's theorem.

$$\iint_\Sigma (\nabla\times\vec{F})\cdot d\vec{A} = \int_{\partial\Sigma} \vec{F}\cdot d\vec{r}$$

$$= \int_{\partial\Sigma_1} \vec{F}\cdot d\vec{r}$$

$$= \iint_{\Sigma_1} \nabla\times\vec{F}\cdot d\vec{A}$$

$$= \iint_{\Sigma_1} (-\hat{k})\cdot(-\hat{k})\, dA$$

$$= \iint_{\Sigma_1} dA$$

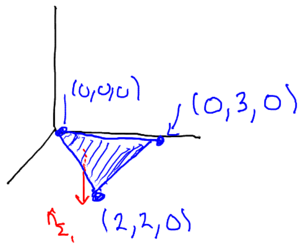
$$= \text{area}(\Sigma_1)$$

$$= \frac{1}{2}(3\cdot 2)$$

$$= \boxed{3}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\hat{k}$$

However this is integral is still complicated. We can find a simpler surface with the same boundary. Choose Σ_1 to be the surface inside the triangle on the xy-plane such that $\partial\Sigma_1 = \partial\Sigma$.
Note: This is the same region as in question 2!



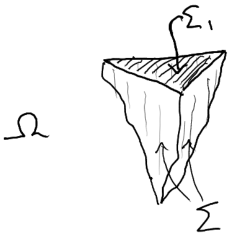
This surface has normal vector $\hat{n}_{\Sigma_1} = -\hat{k}$

(This problem continues on the next page.)

[2] (b) Now consider the vector field defined by $\boldsymbol{G}(x, y, z) = (2xz - x,\, 2y,\, 2y - z^2)$. Show that $\nabla \cdot \boldsymbol{G}$ is a constant scalar field.

$$\nabla \cdot \vec{G} = \frac{\partial}{\partial x}(2xz - x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2y - z^2)$$
$$= 2z - 1 + 2 - 2z$$
$$= 1 \checkmark$$

[3] ~~[6]~~ (c) Suppose you know that the volume of the region contained inside the surface Σ and below the xy-plane is equal to 8. Compute $\iint_\Sigma \boldsymbol{G} \cdot d\boldsymbol{A}$.

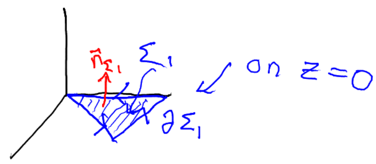


Boundary of region Ω has two parts:

$$\partial \Omega = \Sigma \cup \Sigma_1$$

Where Σ_1 is same surface as earlier!

Use divergence Theorem



This time $\hat{n}_{\Sigma_1} = +\hat{k}$

$$\text{vol}(\Omega) = \iiint_\Omega 1\, dV$$
$$= \iiint_\Omega \nabla \cdot \vec{G}\, dV \quad (\text{since } \nabla \cdot \vec{G} = 1)$$
$$= \oiint_{\partial\Omega} \vec{G} \cdot d\vec{A}$$
$$= \underbrace{\iint_{\Sigma_1} \vec{G} \cdot d\vec{A}}_{I_1} + \underbrace{\iint_{\Sigma} \vec{G} \cdot d\vec{A}}_{I}$$

To compute $I_1 = \iint_{\Sigma_1} \vec{G} \cdot dA$

$\vec{G}(x, y, 0) = (-x, 2y, 2y)$

$\hat{n}_{\Sigma_1} = \hat{k}$

$\vec{G}(x,y,0) \cdot \hat{n}_{\Sigma_1} = 2y$

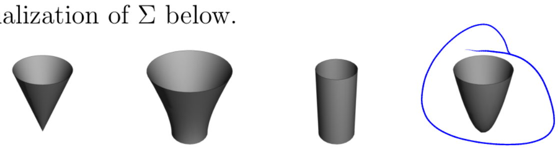
So $I = \iint_{\Sigma_1} \vec{G} \cdot d\vec{A}$

$$= \iint_{\Sigma_1} 2y\, dA$$
$$= 2 \underbrace{\iint_{\Sigma_1} y\, dA}_{= 5 \text{ from problem 2}}$$
$$= 10.$$

$$\Rightarrow I = \text{vol}(\Omega) - I_1$$
$$= 8 - 10$$
$$= \boxed{-2}$$

4. Consider the surface Σ in $\mathbb{R}^3$ that is defined by $x^2+y^2=z+1$ in the region where $-1 \le z \le 1$.

[1] (a) Circle the correct visualization of Σ below.



[2] (b) Provide a parameterization for the surface Σ. Make sure to include bounds on the variables.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2 = z+1$

$\vec{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2)$

$\theta \in [0, 2\pi]$

$r \in [0, \sqrt{2}]$

[4] (c) Use your parameterization in part (b) to compute the surface area of Σ.

$$\frac{\partial \vec{\Phi}}{\partial \theta} = (-r\sin\theta, \ r\cos\theta, 0)$$

$$\frac{\partial \vec{\Phi}}{\partial r} = (\cos\theta, \sin\theta, -2r)$$

$$\text{Area}(\Sigma) = \iint_\Sigma dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left\| \frac{\partial \vec{\Phi}}{\partial \theta} \times \frac{\partial \vec{\Phi}}{\partial r} \right\| d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left\| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & -2r \end{matrix} \right\| d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left\| \left(-2r^2\cos\theta, -2r^2\sin\theta, -r\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\right)\right\| d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2}\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\, dr\right)$$

$u = 4r^2+1$

$$= 2\pi \, \frac{1}{8}\int_1^9 \sqrt{u}\, du$$

$du = 8r\,dr$

$\Rightarrow r\,dr = \frac{1}{8}du$

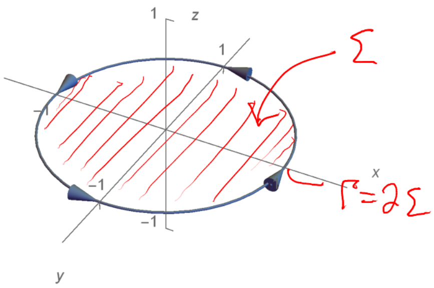
$$= 2\pi \frac{1}{8}\, \frac{2}{3} u^{3/2}\Big|_1^9 = \frac{\pi}{6}(9^{3/2}-1)$$

$u(r=0) = 1$

$u(r=\sqrt{2}) = 9$

$$= \frac{\pi}{6}(27-1) = \frac{26\pi}{6} = \boxed{\frac{13\pi}{3}}$$

[5] 5. Suppose a static current density in a region of space is given by $\boldsymbol{J}(x,y,z) = b\,e^{-a(x^2+y^2)}\hat{\boldsymbol{k}}$, where $a > 0$ and $b > 0$ are constant, and let Γ denote the unit circle $x^2+y^2=1$ on the xy-plane that is oriented counterclockwise when viewed from above. Compute the circulation of the magnetic field around Γ, assuming that the electric field is static.



Let Σ be the disk of radius 1 centred on the z-axis on the xy-plane such that $\partial\Sigma = \Gamma$. It has normal $\hat{n}_\Sigma = \hat{k}$

$$\oint_\Gamma \vec{B}\cdot d\vec{r} = \oint_{\partial\Sigma} \vec{B}\cdot d\vec{r}$$

Use Stokes' Theorem

$$= \iint_\Sigma \nabla\times\vec{B}\cdot d\vec{A}$$

Use Maxwell's equation

$$\nabla\times\vec{B} = \mu_0\varepsilon_0 \underbrace{\frac{\partial\vec{E}}{\partial t}}_{0} + \mu_0\vec{J} = \mu_0\vec{J}$$

$$= \iint_\Sigma \mu_0\vec{J}\cdot d\vec{A}$$

$$= \iint_\Sigma \mu_0 b e^{-a(x^2+y^2)}\,dx\,dy$$

$$= \mu_0 b \int_0^{2\pi}\int_0^1 e^{-ar^2}\, r\,dr\,d\theta$$

Use polar coordinates

$x = r\cos\theta$ $\quad \theta\in[0,2\pi]$

$y = r\sin\theta$ $\quad r\in[0,1]$

$dx\,dy = r\,dr\,d\theta$

$$= \mu_0 b\int_0^{2\pi} d\theta \int_0^1 r e^{-ar^2}\,dr$$

$u = ar^2$

$du = 2ar\,dr$

$r\,dr = \frac{1}{2a}du$

$$= \mu_0 b 2\pi \int_0^a \frac{1}{2a} e^{-u}\,du$$

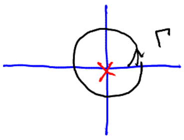
$$= \frac{\mu_0 b\pi}{a}\left(-e^{-u}\right)\Big|_0^a = \boxed{\frac{\mu_0\pi b}{a}\left(1-e^{-a}\right)}$$

[4] 6. (a) Write out the Laurent series expansion of the mapping $f(z) = z^4 e^{j/z}$ about the point $z = 0$.

$$z^4 e^{j/z} = z^4\left(1 + \frac{j}{z} + \frac{1}{2}\left(\frac{j}{z}\right)^2 + \frac{1}{6}\left(\frac{j}{z}\right)^3 + \frac{1}{4!}\left(\frac{j}{z}\right)^4 + \frac{1}{5!}\left(\frac{j}{z}\right)^5 + \cdots\right)$$

$$= z^4 + jz^3 - \frac{1}{2}z^2 - \frac{j}{6}z + \frac{1}{4!} + \underbrace{\frac{j}{5!}\frac{1}{z}}_{C_{-1} = \frac{j}{5!}} - \frac{1}{6!}\frac{1}{z^2} + \cdots$$

[2] (b) Use the Laurent series you found in part (a) to evaluate $\oint_\Gamma z^4 e^{j/z}\, dz$, where Γ is the positively oriented unit circle defined by the equation $|z| = 1$.



$$\text{Res}(z^4 e^{j/z}, z=0) = \frac{j}{5!}$$

$$\oint_\Gamma z^4 e^{j/z}\, dz = 2\pi j\, \text{Res}(z^4 e^{jz}, z=0)$$

$$= 2\pi j\left(\frac{j}{5!}\right) = -\frac{2\pi}{5!} = \boxed{-\frac{\pi}{60}}$$

[4] (c) Let Γ be the straight line segment connecting $-1-j$ to $1+j$. Evaluate the integral

$$I = \int_\Gamma (3jz^2 + \bar{z})\, dz.$$

$$= \underbrace{\int_\Gamma 3jz^2\, dz}_{I_1} + \underbrace{\int_\Gamma \bar{z}\, dz}_{I_2}$$

$$I_1 = \int_\Gamma 3jz^2\, dz$$

$$= \int_{-1-j}^{1+j} 3jz^2\, dz = jz^3\Big|_{-1-j}^{1+j}$$

$$= j\left((1+j)^3 - (-1-j)^3\right) = j\left((1+j)^3 + (1+j)^3\right)$$

$$= j2(1+j)^3 = j2(1+3j-3-j) = j2(2j-2)$$

$$= -4-4j$$

I_2: parameterize! $\gamma(t) = t(1+j)$ for $t \in [-1, 1]$

$\gamma'(t) = 1+j$

$$I_2 = \int_\Gamma \bar{z}\, dz = \int_{-1}^{1} \overline{\gamma(t)}\, \gamma'(t)\, dt = \int_{-1}^{1} t(1-j)(1+j)\, dt = \int_{-1}^{1} t\, 2\, dt = t^2\Big|_{-1}^{1}$$

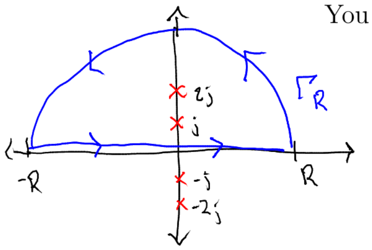
$$= 0$$

$$I = I_1 + I_2 = -4(1+j) + 0 = \boxed{-4(1+j)}$$

7. Let $R > 2$ and let Γ_R be the closed contour consisting of the semicircular arc of radius R centered at the origin that goes counterclockwise from R to $-R$, followed by the line segment on the real axis from $-R$ to R. Consider the function defined by $f(z) = \frac{z^2}{z^4+5z^2+4}$.

[6] (a) Sketch Γ_R and indicate the location of the singularities of f. Then compute $\oint_{\Gamma_R} f(z)\,dz$.

You may use the fact that $z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$.



Singularities at $z = j$ and $z = 2j$

$$f(z) = \frac{z^2}{(z-j)(z+j)(z-2j)(z+2j)}$$

- $$\text{Res}(f, j) = \lim_{z\to j}\left[(z-j)f(z)\right] = \lim_{z\to j}\left(\frac{z^2}{(z+j)(z-2j)(z+2j)}\right) = \frac{j^2}{(2j)(-j)(3j)} = \frac{-1}{6j} = \frac{j}{6}$$

- $$\text{Res}(f, 2j) = \lim_{z\to 2j}\left[(z-2j)f(z)\right] = \lim_{z\to 2j}\left[\frac{z^2}{(z-j)(z+j)(z+2j)}\right] = \frac{4j^2}{(j)(3j)(4j)} = \frac{-4}{-12j} = \frac{1}{3j} = -\frac{j}{3}$$

$$\Rightarrow \oint_{\Gamma_R} f(z)\,dz = 2\pi j\left(\text{Res}(f,j) + \text{Res}(f,2j)\right) = 2\pi j\left(\frac{j}{6} - \frac{j}{3}\right) = -2\pi\left(\frac{1}{6} - \frac{1}{3}\right) = \boxed{\frac{\pi}{3}}$$

[3] (b) Use your answer from part (a) to compute the value of the real improper integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx.$$

Make sure to explain your reasoning at all steps.

On the semicircular contour defined by $|z| = R$,

$$|f(z)| = \left|\frac{z^2}{(z^2+1)(z^2+4)}\right| = \frac{R^2}{|z^2+1||z^2+4|} \leq \frac{R^2}{(R^2-1)(R^2-4)}$$

and thus $$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| \leq \lim_{R\to\infty}\frac{\pi R^3}{(R^2-1)(R^2-4)} = 0.$$

Therefore $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{+R} f(x)\,dx = \lim_{R\to\infty}\left(\oint_{\Gamma_R} f(z)\,dz - \underbrace{\int_{C_R} f(z)\,dz}_{\to 0}\right)$$

$$= \frac{\pi}{3} - 0 = \boxed{\frac{\pi}{3}}$$

8. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $u(x, y) = e^{-2y}\sin(2x)$.

[1] (a) Check that u is harmonic.

[3] (b) Find the unique mapping $f : \mathbb{C} \to \mathbb{C}$ satisfying $\mathrm{Re}(f(x + jy)) = u(x, y)$ and $f(0) = 0$.

[2] (c) Express $f(z)$ you found in part (b) purely in terms of z (and not x or y).

[1] (d) What is $f'(j)$?

a) $u_x = 2e^{-2y}\cos 2x \qquad u_{xx} = -4e^{-2y}\sin 2x$

$u_y = -2e^{-2y}\sin 2x \qquad u_{yy} = 4e^{-2y}\sin 2x \qquad \Rightarrow u_{xx} + u_{yy} = 0$

b) $v_x = -u_y \qquad v_y = u_x$

$v = \int v_x\,dx = -\int u_y\,dx = -\int -2e^{-2y}\sin 2x\,dx = -e^{-2y}\cos 2x + g(y)$

$v = \int v_y\,dy = \int u_x\,dy = \int 2e^{-2y}\cos 2x\,dy = -e^{-2y}\cos 2x + h(x)$

$\Rightarrow g(y) = h(x) = c$ constant

$f(z) = f(x+jy) = u(x,y) + jv(x,y) = e^{-2y}\sin 2x - je^{-2y}\cos 2x + jc$

$= e^{-2y}(\sin 2x - j\cos 2x) + jc$

$= -je^{-2y}(\cos 2x + j\sin 2x) + jc$

$= -je^{-2y}e^{j2x} + jc = -je^{j2(x+jy)} + jc$

$= -je^{j2z} + jc$

c) $0 = f(0) = -je^0 + jc \Rightarrow 0 = -j + jc \Rightarrow \underline{c = 1}$

$\boxed{f(z) = -je^{j2z} + j}$

d) $f'(z) = 2e^{j2z} \qquad f'(j) = 2e^{j2j} = \boxed{2e^{-2}}$

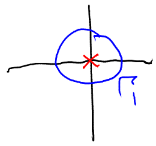
9. All contours are assumed to be positively oriented.

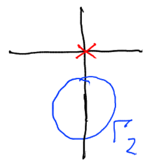
[4] ~~[5]~~ (a) Answer the following true/false questions by writing either 'T' or 'F' in the blank.

(i) ____ ~~A mapping f is analytic at a point z_0 if and only if f can be expanded in a power series that converges in some disk centered at z_0.~~

(ii) F Consider the contours $\Gamma_1 = \{z \,|\, |z| = 1\}$ and $\Gamma_1 = \{z \,|\, |z + 2j| = 1\}$. Then

$$\oint_{\Gamma_1} \frac{1}{z}\, dz = \oint_{\Gamma_2} \frac{1}{z}\, dz.$$





(ii) T If a mapping f is analytic everywhere, then

$$\oint_{\Gamma} \frac{f'(z)}{z+j}\, dz = \oint_{\Gamma} \frac{f(z)}{(z+j)^2}\, dz$$

where Γ is the contour defined by $|z| = 3$.

(iv) F The mapping $f(z) = \mathrm{Log}(z)$ is analytic everywhere where it is defined.

Log defined everywhere except $z=0$, but not continuous on negative real axis

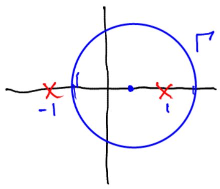
(v) F If f is any mapping with an isolated singularity at a point z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} [(z - z_0) f(z)]$$

[6] (b) Compute the following integrals around the contour Γ defined by $|2z - 1| = 2$.

$\Rightarrow |z - \frac{1}{2}| = 1$

(i) $\displaystyle \oint_{\Gamma} \frac{1}{(z-1)^3(z+1)}\, dz$

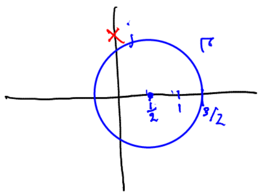


Only singularity inside Γ is at $z=1$. Let $f(z) = \frac{1}{z+1}$ $\quad f'(z) = -\frac{1}{(z+1)^2}$

$\oint_\Gamma \frac{1}{(z-1)^3(z+1)} dz = \oint_\Gamma \frac{f(z)}{(z-1)^3} dz = \frac{2\pi j}{2!} f''(1)$ $\quad f''(z) = 2\frac{1}{(z+1)^3}$

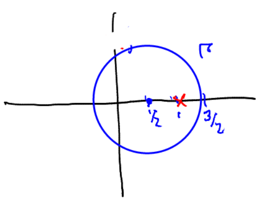
$= \pi j \left(2 \frac{1}{(1+1)^3}\right) = \boxed{\frac{\pi j}{4}}$

(ii) $\displaystyle \oint_{\Gamma} \frac{\cos z}{(z-j)^4}\, dz$ $= 0$



By Cauchy-Goursat, this integral is zero because the integrand is analytic everywhere inside Γ.

(ii) $\displaystyle \oint_{\Gamma} \frac{\cos z}{(z-1)^4}\, dz$ $\quad f(z) = \cos z \quad f^{(3)}(z) = \sin z$



$\oint_\Gamma \frac{f(z)}{(z-1)^4} dz = \frac{1}{3!} 2\pi j\, f^{(3)}(1) = \boxed{\frac{\pi j}{3} \sin 1}$

This space is for sketch work or overflow

(If you want something here marked, be sure to clearly indicate on the question page.)

This space is for sketch work or overflow

(If you want something here marked, be sure to clearly indicate on the question page.)

This space is for sketch work or overflow

(If you want something here marked, be sure to clearly indicate on the question page.)