ECE 206 – Fall 2019

Final Exam

December 16, 2019 at 12:30

Instructor: Mark Girard

University of Waterloo

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Name: ____

Notes:

1. Fill in your name (first and last) and student ID number in the space above.
2. This midterm contains 14 pages (including this cover page) and 9 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided at the end. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
5. You are allowed to use **either the formula sheet provided or your own formula sheet that you've prepared yourself**. No other notes, books, calculators, or personal electronic devices of any kind may be used.

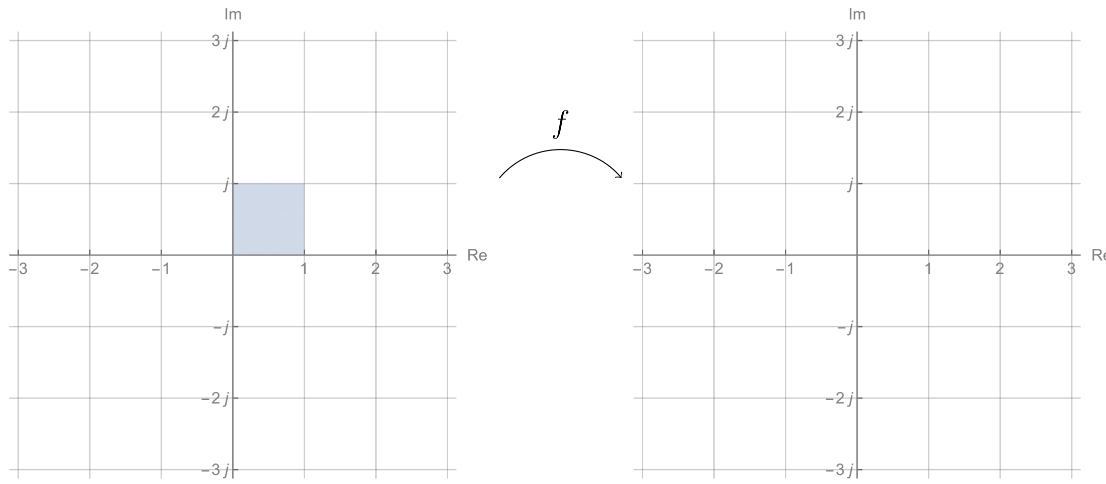
Question:	1	2	3	4	5	6	7	8	9	Total
Points:	11	4	12	7	5	10	9	7	11	76
Score:										

[3] 1. (a) Find all possible values of $z \in \mathbb{C}$ that satisfy $\cos z = -2$.

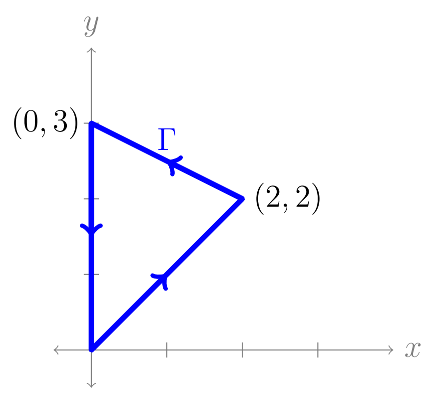
[3] (b) Find all possible values of $(1-j)^j$.

[2] (c) Expand out the first four nonzero terms of the Taylor series of $f(z) = -1/z$ about $z = 1$. What is the radius of convergence of this Taylor series? Explain.

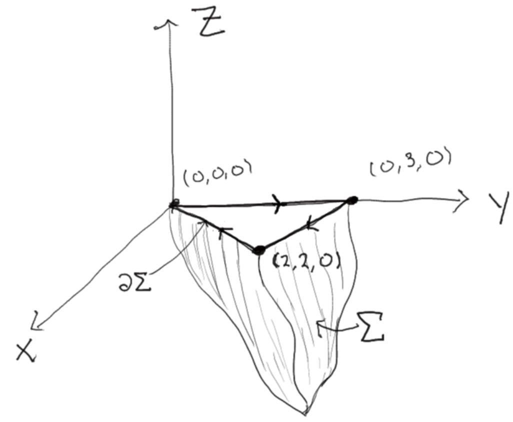
[3] (d) Consider the region $D = \{x + jy : 0 < x < 1 \text{ and } 0 < y < 1\}$ (shaded below). Sketch the image of D under the mapping f defined by $f(z) = e^z$.



[4] 2. For the curve Γ in $\mathbb{R}^2$ (shown in the figure below) and the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\boldsymbol{F}(x, y) = \big(\ln(\sin^2(x) + 1),\, \cos(\sin y) + xy\big)$, evaluate $\displaystyle\oint_\Gamma \boldsymbol{F} \cdot \boldsymbol{r}$.



3. Consider the surface Σ in $\mathbb{R}^3$ (depicted at right) with outward facing normal, and whose closed boundary curve $\partial\Sigma$ is the triangle in the xy-plane (oriented clockwise when viewed from above) with vertices at the points $(0,0,0)$, $(0,3,0)$, and $(2,2,0)$.



[4] (a) Consider the vector field defined by $\boldsymbol{F}(x,y,z) = y\,\hat{\boldsymbol{\imath}}$. Compute $\displaystyle\iint_\Sigma (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dA$.

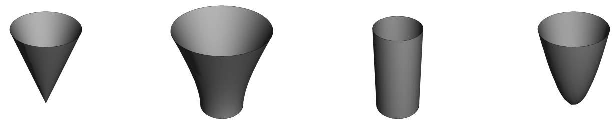
(This problem continues on the next page.)

[2] (b) Now consider the vector field defined by $\boldsymbol{G}(x, y, z) = (2xz - x,\, 2y,\, 2y - z^2)$. Show that $\nabla \cdot \boldsymbol{G}$ is a constant scalar field.

[6] (c) Suppose you know that the volume of the region contained inside the surface Σ and below the xy-plane is equal to 8. Compute $\displaystyle\iint_\Sigma \boldsymbol{G} \cdot d\boldsymbol{A}$.

4. Consider the surface Σ in $\mathbb{R}^3$ that is defined by $x^2 + y^2 = z + 1$ in the region where $-1 \leq z \leq 1$.

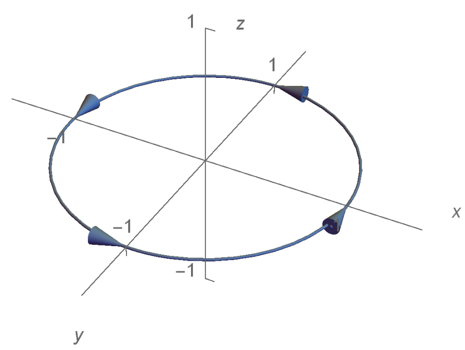
[1] (a) Circle the correct visualization of Σ below.



[2] (b) Provide a parameterization for the surface Σ. Make sure to include bounds on the variables.

[4] (c) Use your parameterization in part (b) to compute the surface area of Σ.

[5] 5. Suppose a static current density in a region of space is given by $\boldsymbol{J}(x,y,z) = b\,e^{-a(x^2+y^2)}\hat{\boldsymbol{k}}$, where $a > 0$ and $b > 0$ are constant, and let Γ denote the unit circle $x^2 + y^2 = 1$ on the xy-plane that is oriented counterclockwise when viewed from above. Compute the circulation of the magnetic field around Γ, assuming that the electric field is static.



[4] 6. (a) Write out the Laurent series expansion of the mapping $f(z) = z^4 e^{j/z}$ about the point $z = 0$.

[2] (b) Use the Laurent series you found in part (a) to evaluate $\oint_\Gamma z^4 e^{j/z}\, dz$, where Γ is the positively oriented unit circle defined by the equation $|z| = 1$.

[4] (c) Let Γ be the straight line segment connecting $-1-j$ to $1+j$. Evaluate the integral

$$\int_\Gamma (3jz^2 + \overline{z})\, dz.$$

7. Let $R > 2$ and let Γ_R be the closed contour consisting of the semicircular arc of radius R centered at the origin that goes counterclockwise from R to $-R$, followed by the line segment on the real axis from $-R$ to R. Consider the function defined by $f(z) = \frac{z^2}{z^4+5z^2+4}$.

[6] (a) Sketch Γ_R and indicate the location of the singularities of f. Then compute $\displaystyle\oint_{\Gamma_R} f(z)\,dz$.

You may use the fact that $z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$.

[3] (b) Use your answer from part (a) to compute the value of the real improper integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx.$$

Make sure to explain your reasoning at all steps.

8. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $u(x, y) = e^{-2y} \sin(2x)$.

[1] (a) Check that u is harmonic.

[3] (b) Find the unique mapping $f : \mathbb{C} \to \mathbb{C}$ satisfying $\mathrm{Re}(f(x + jy)) = u(x, y)$ and $f(0) = 0$.

[2] (c) Express $f(z)$ you found in part (b) purely in terms of z (and not x or y).

[1] (d) What is $f'(j)$?

9. All contours are assumed to be positively oriented.

[5] (a) Answer the following true/false questions by writing either 'T' or 'F' in the blank.

(i) ___ A mapping f is analytic at a point z_0 if and only if f can be expanded in a power series that converges in some disk centered at z_0.

(ii) ___ Consider the contours $\Gamma_1 = \{z \,|\, |z| = 1\}$ and $\Gamma_1 = \{z \,|\, |z + 2j| = 1\}$. Then

$$\oint_{\Gamma_1} \frac{1}{z}\, dz = \oint_{\Gamma_2} \frac{1}{z}\, dz.$$

(ii) ___ If a mapping f is analytic everywhere, then

$$\oint_{\Gamma} \frac{f'(z)}{z + j}\, dz = \oint_{\Gamma} \frac{f(z)}{(z + j)^2}\, dz$$

where Γ is the contour defined by $|z| = 3$.

(iv) ___ The mapping $f(z) = \mathrm{Log}(z)$ is analytic everywhere where it is defined.

(v) ___ If f is any mapping with an isolated singularity at a point z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} [(z - z_0)f(z)]$$

[6] (b) Compute the following integrals around the contour Γ defined by $|2z - 1| = 2$.

(i) $\displaystyle\oint_{\Gamma} \frac{1}{(z-1)^3(z+1)}\, dz$

(ii) $\displaystyle\oint_{\Gamma} \frac{\cos z}{(z-i)^4}\, dz$

(ii) $\displaystyle\oint_{\Gamma} \frac{\cos z}{(z-1)^4}\, dz$

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