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Name: ______________________________

Notes:

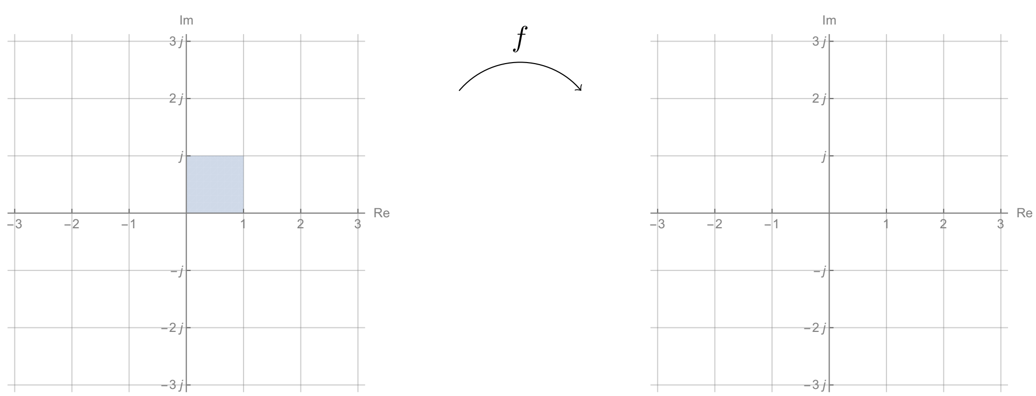
1. Fill in your name (first and last) and student ID number in the space above.
2. This midterm contains 20 pages (including this cover page) and 11 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided at the end. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	9	10	4	12	11	10	5	6	8	9	6	90
Score:												

[3] 1. (a) Find all possible values of $z \in \mathbb{C}$ that satisfy $\cos z = e^{jz}$.

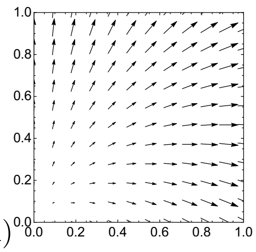
[3] (b) Find all possible values of $(1 - j)^{4j}$.

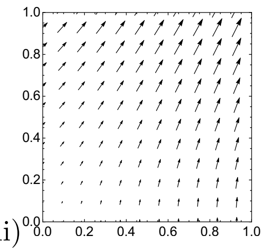
[3] (c) Consider the domain $D = \{x + jy : 0 < x < 1 \text{ and } 0 < y < 1\}$ (shaded below). Sketch the image of the domain D under the function $f(z) = e^z$.

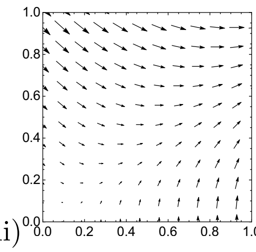


2. Consider the vector field $\vec{\boldsymbol{F}} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\vec{\boldsymbol{F}}(x, y) = (y, x - y)$.

[2] (a) Which of the following is the correct picture for the field?

(i) 

(ii) 

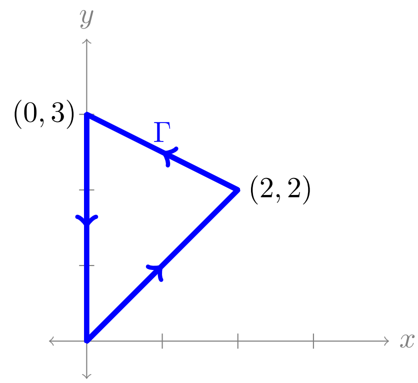
(iii) 

[4] (b) Consider the curve Γ that is part of the parabola $x = y^2$ from $x = 0$ to $x = 1$. Sketch the curve on the picture above, and compute $\displaystyle\int_\Gamma \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ explicitly by finding a parameterization.

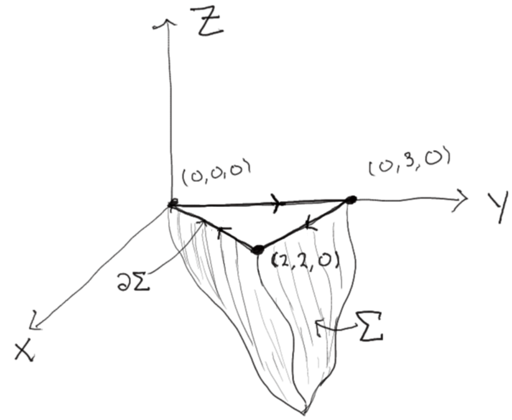
[2] (c) The vector field $\vec{\boldsymbol{F}}$ is conservative. Find a scalar potential function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ for $\vec{\boldsymbol{F}}$.

[2] (d) Use the potential you found in part (c) to verify your answer in part (b).

[4] 3. For the curve Γ in $\mathbb{R}^2$ (shown in the figure below) and the vector field $\vec{\boldsymbol{F}} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\vec{\boldsymbol{F}}(x, y) = \big(\ln(\sin^2(x) + 1),\, \cos(\sin y) + x\big)$, evaluate $\displaystyle\oint_\Gamma \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}}$.



4. Consider the surface Σ in $\mathbb{R}^3$ (depicted at right) with outward facing normal, and whose closed boundary curve $\partial\Sigma$ is the triangle in the xy-plane (oriented clockwise when viewed from above) with vertices at the points $(0,0,0)$, $(0,3,0)$, and $(2,2,0)$.



[4] (a) Consider the vector field defined by $\vec{\boldsymbol{F}}(x,y,z) = y\,\hat{\boldsymbol{\imath}}$. Compute $\displaystyle\iint_\Sigma (\nabla \times \vec{\boldsymbol{F}}) \cdot \hat{\boldsymbol{n}}\, dA$.

(Hint: Find a surface with the same boundary curve and make use of an important theorem.)

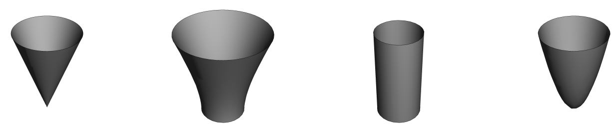
(This problem continues on the next page.)

[2] (b) Now consider the vector field defined by $\vec{\boldsymbol{G}}(x, y, z) = (2xz - x,\ 2y,\ 2y - z^2)$. Show that $\nabla \cdot \vec{\boldsymbol{G}}$ is a constant scalar field.

[6] (c) Suppose you know that the volume of the region contained inside the surface Σ and below the xy-plane is equal to 8. Compute $\displaystyle\iint_\Sigma \vec{\boldsymbol{G}} \cdot \hat{\boldsymbol{n}}\, dA$.

5. Consider the surface Σ in $\mathbb{R}^3$ that is defined by $x^2 + y^2 = z^2 + 1$ in the region where $0 \leq z \leq 1$.

[2] (a) Circle the correct visualization of Σ below.



[3] (b) Provide a parameterization for the surface Σ. Make sure to include bounds on the variables.

[6] (c) Let $\vec{\boldsymbol{F}}$ be the vector field defined by $\vec{\boldsymbol{F}}(x, y, z) = (xz, 0, -1)$. Find the flux of $\vec{\boldsymbol{F}}$ through Σ with respect to the outward facing normal.

6. Let a be a constant, let f be the function defined by $f(r) = \dfrac{a}{r^3}$ for all $r > 0$, and let $\vec{\boldsymbol{F}}$ be the radial vector field defined by $\vec{\boldsymbol{F}}(x, y, z) = f(r)\vec{\boldsymbol{r}}$, where $\vec{\boldsymbol{r}} = (x, y, z)$ and $r = \|\vec{\boldsymbol{r}}\| = \sqrt{x^2 + y^2 + z^2}$. (Note that $\vec{\boldsymbol{F}}$ is $\mathcal{C}^1$ everywhere except at $\vec{\mathbf{0}}$, where it is not defined.)

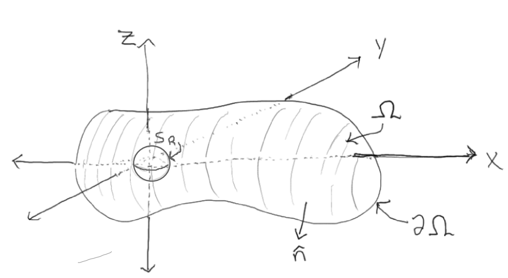
[2] (a) Show that $\nabla \cdot \vec{\boldsymbol{F}} = 0$ on all points where $\vec{\boldsymbol{F}}$ is defined.

[4] (b) Let S_R denote the solid sphere of radius R centered at the origin. Compute the flux of $\vec{\boldsymbol{F}}$ through the boundary sphere ∂S_R with respect to the outward pointing normal of ∂S_R.

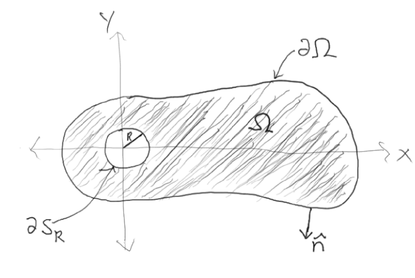
(You may use the fact that the outward unit normal vector on the surface of the sphere of radius R is given by $\hat{\boldsymbol{n}} = \frac{\vec{r}}{R}$ and that the surface area of a sphere of radius R is given by $4\pi R^2$.)

(This problem continues on the next page.)

[3] (c) Let Ω be a solid region of $\mathbb{R}^3$ (depicted below) with outward facing normal such that the origin lies inside Ω. For some $R > 0$ small enough, the solid sphere S_R lies entirely inside Ω. We can cut out of Ω the sphere S_R to leave a small spherical "hole" of radius R inside Ω, as depicted below.



View of Ω with "hole" S_R cut out.



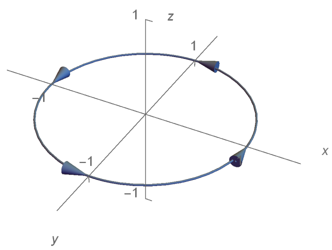
Slice view on xy-plane.

Explain why $\displaystyle\iint_{\partial\Omega} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dA = \iint_{\partial S_R} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dA$. (You should reference an important theorem.)

Conclude that $\displaystyle\iint_{\partial\Omega} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dA = 4\pi a$.

[1] (d) What is $\displaystyle\iint_{\partial\Omega} \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dA$ if the region Ω instead does not contain the origin? Explain.

[5] 7. Suppose a static current density in a region of space is given by $\vec{\boldsymbol{J}}(x, y, z) = b\, e^{-(x^2+y^2)}\,\hat{\boldsymbol{k}}$, where b is a constant, and let Γ denote the unit circle $x^2 + y^2 = 1$ on the xy-plane that is oriented counterclockwise when viewed from above. Compute the circulation of the magnetic field around Γ, assuming that the electric field is static.



8. Consider the function defined by $f(z) = \dfrac{1}{z(z^2+2)}$.

[4] (a) Write out the two possible Laurent series expansions of f about the point $z = 0$, and specify the region of validity for each.

[2] (b) Use one of the Laurent series you found in part (a) to evaluate $\displaystyle\oint_{\Gamma} \frac{1}{z(z^2+2)}\, dz$, where Γ is the positively oriented unit circle $|z| = 1$.

9. Let $R > 2$ and let Γ_R be the closed contour consisting of the semicircular arc of radius R centered at the origin that goes counterclockwise from R to $-R$, followed by the line segment on the real axis from $-R$ to R. Consider the function defined by $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$.

[6] (a) Sketch Γ_R and indicate the location of the singularities of f. Then compute $\displaystyle\oint_{\Gamma_R} f(z)\,dz$.

[2] (b) Explain how your answer from part (a) can be used to compute the value of the real improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx$.

10. Consider the function defined by $f(z) = |z|^2$ and the integral $\displaystyle\int_\Gamma f(z)\,dz$.

[2] (a) Use the Cauchy-Riemann equations to find all points where the function f is analytic.

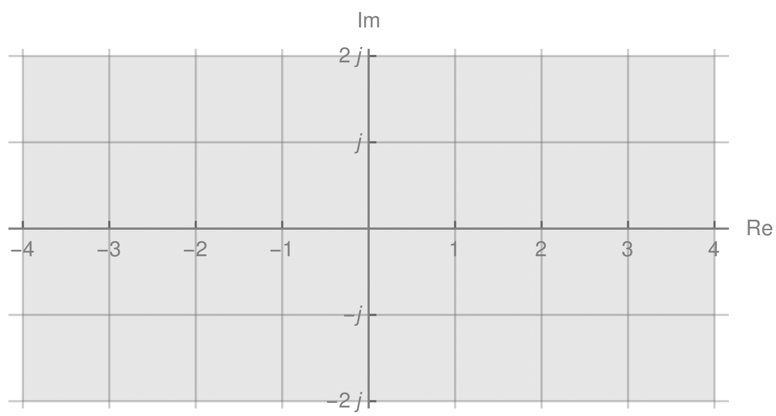
[3] (b) Evaluate the integral where Γ is the line segment with initial point -1 and terminal point j.

[3] (c) Evaluate the integral where Γ is the arc of the unit circle $|z| = 1$ traversed clockwise with initial point -1 and terminal point j.

[1] (d) Do your answers from (b) and (c) agree? Why or why not?

11. Let D be the domain (indicated below) defined by $D = \{x + jy : -4 < x < 4 \text{ and } -2 < y < 2\}$ and consider the function defined by

$$f(z) = \frac{4z}{z^4 - 1} + \frac{1}{\sin z} + 2(2z - j)^3.$$



[4] (a) Indicate the locations of the singularities of f in D above and compute the residue of f at each.

[2] (b) Draw an oriented simple closed contour Γ in D such that $\oint_\Gamma f(z)\, dz = -8\pi j$, or explain why such a contour cannot exist.

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Change of variable formula If $\Phi(u,v) = (x(u,v), y(u,v))$ is a transformation then

$$\iint_{D_{xy}} f(x,y)\,dx\,dy = \iint_{D_{uv}} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the region D_{xy} is mapped to D_{uv} under Φ. The Jacobian is defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

The Jacobian satisfies the following property: $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$.

- **Polar coordinates:** $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian is $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$
- **Cylindrical coordinates:** $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian is $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$
- **Spherical coordinates:** $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$. The Jacobian is $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin\varphi$

Vector calculus identities

$$\nabla \times (f\vec{\boldsymbol{F}}) = (\nabla f) \times \vec{\boldsymbol{F}} + f(\nabla \times \vec{\boldsymbol{F}})$$
$$\nabla \cdot (f\vec{\boldsymbol{F}}) = (\nabla f) \cdot \vec{\boldsymbol{F}} + f(\nabla \cdot \vec{\boldsymbol{F}})$$
$$\nabla \times (\nabla f) = \vec{\mathbf{0}}$$
$$\nabla \cdot (\nabla \times \vec{\boldsymbol{F}}) = 0$$
$$\nabla \times (\nabla \times \vec{\boldsymbol{F}}) = \nabla(\nabla \cdot \vec{\boldsymbol{F}}) - \nabla^2\vec{\boldsymbol{F}}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \vec{\boldsymbol{E}} &= \frac{\rho}{\epsilon_0} \\
\nabla \times \vec{\boldsymbol{E}} &= -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \\
\nabla \cdot \vec{\boldsymbol{B}} &= 0 \\
\nabla \times \vec{\boldsymbol{B}} &= \mu_0 \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t} + \mu_0 \vec{\boldsymbol{J}}
\end{aligned}
$$

Complex trigonometric and hyperbolic identities

$$
\cos z = \frac{e^{jz} + e^{-jz}}{2} \qquad \sin z = \frac{e^{jz} - e^{-jz}}{2j} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}
$$

For all complex numbers $z \in \mathbb{C}$:

$$
\cos jz = \cosh z \qquad \sin jz = j \sinh z
$$

Cauchy-Riemann equations Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions. The Cauchy-Riemann equations are:

$$
\boxed{u_x = v_y \qquad u_y = -v_x}
$$

The Cauchy-Riemann equations in *polar form* are:

$$
\boxed{u_r = \frac{1}{r} v_\theta \qquad v_r = -\frac{1}{r} u_\theta}
$$

If $f = u + jv$ is analytic, its derivative in Cartesian coordinates is given by

$$
f'(z) = f(x + jy) = u_x + jv_x
$$

and in polar coordinates is given by

$$
f'(z) = f'(re^{j\theta}) = e^{-j\theta}(u_r + jv_r)
$$

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