Assignment 3

MATH 667 – Quantum Information Theory

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Problem 1

Problem 1. Consider the Werner states

$$\rho = F|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}(|\psi^+\rangle\langle\psi^+| + |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|)$$

where $0 \leq F \leq 1$.

(a) Show that $U \otimes U \rho^{AB} U^* \otimes U^* = \rho^{AB}$ for all unitary matrices U.

(b) Determine the range of the parameter F for which ρ is separable.

Solution. Note that we can write the Werner states as

$$\rho = F|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}(I - |\psi^-\rangle\langle\psi^-|)$$

and recall that $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

(a) We first consider the general case of operators on system of two qudits, i.e. operators on $\mathbb{C}^d \otimes \mathbb{C}^d$. We denote the *swap* operator on this space by

$$W = \sum_{i,j} |ij\rangle\langle ji|.$$

This is the unique operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ that satisfies $W(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$. The swap operator is a permutation operator in the sense that its entries are taken from the set $\{0, 1\}$, and $W^* = W$ and $W^2 = I$ hold. Hence all of the eigenvalues of W are ± 1. It follows that W can be decomposed as $W = W_+ - W_-$, where W_+ and W_- are the unique positive semi-definite operators (whose eigenvalues are 1 and 0) satisfying

$$W = W_+ - W_- \quad \text{and} \quad W_+ W_- = W_- W_+ = 0.$$

Note that W_+ and W_- are projection operators onto orthogonal subspaces.

Furthermore, note that $[A \otimes A, W] = 0$ holds for all operators $A \in \mathrm{L}(\mathbb{C}^d)$, where $[\cdot, \cdot]$ denotes the commutator. Indeed, we have

$$\begin{aligned}(A \otimes A)W - W(A \otimes A) &= \sum_{i,j,k,l} \sum_{i',j'} a_{ij} a_{kl} (|ik\rangle\langle jl|)(|i'j'\rangle\langle j'i'|) - (|i'j'\rangle\langle j'i'|)(|ik\rangle\langle jl|) \\ &= \sum_{i,j,k,l} a_{ij} a_{kl} (|ik\rangle\langle lj| - |ki\rangle\langle jl|)\end{aligned}$$

$$= \sum_{i,j,k,l} (a_{ij}a_{kl} - a_{kl}a_{ij})|ik\rangle\langle lj| = 0.$$

Since $[A \otimes A, I] = 0$ holds trivially, it follows that $[A \otimes A, X] = 0$ for any $X \in \text{span}\{I, W\}$. Hence, for any $X \in \text{span}\{I, W\}$, it holds that $A \otimes AXA^* \otimes A^* = (AA^* \otimes AA^*)X$. In the case when A is unitary, it follows that $AA^* = I$ and thus $A \otimes AXA^* \otimes A^* = X$. Note that W_+ and W_- are in the span of $\{I, W\}$, since $W_+ = \frac{1}{2}(I + W)$ and $W_- = \frac{1}{2}(I - W)$.

We now go back to considering the case of two qubits with $d = 2$. We have that

$$W_- = |\psi^-\rangle\langle\psi^-| \quad \text{and} \quad W_+ = |\psi^+\rangle\langle\psi^+| + |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|$$

so that the 2×2-Werner states can be written as

$$\rho = FW_- + \frac{1-F}{3}W_+.$$

In the more general case with two d-dimensional systems, the $d \times d$-Werner states can be written as

$$\rho = F\frac{1}{\text{Tr}(W_-)}W_- + (1-F)\frac{1}{\text{Tr}(W_+)}W_+.$$

The desired result that $U \otimes U\rho U^* \otimes U^*$ holds for all d-dimensional unitaries U and all $d \times d$-Werner states ρ follows from the observations above. Indeed, since $U \otimes UXU^* \otimes U^* = X$ holds for any $X \in \text{span}\{I, W\}$

(b) We can use the partial transpose criterion to detect when the state is entangled. In matrix form, we have

$$|\psi^-\rangle\langle\psi^-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$|\psi^+\rangle\langle\psi^+| + |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

So the Werner states with parameter F can be written as

$$\rho_F = F\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (1-F)\frac{1}{6}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2-2F & 0 & 0 & 0 \\ 0 & 1+2F & 1-4F & 0 \\ 0 & 1-4F & 1+2F & 0 \\ 0 & 0 & 0 & 2-2F \end{pmatrix}.$$

Taking the partial transpose of these states yields

$$(T \otimes \hat{I})(\rho_F) = \frac{1}{6}\begin{pmatrix} 2-2F & 0 & 0 & 1-4F \\ 0 & 1+2F & 0 & 0 \\ 0 & 0 & 1+2F & 0 \\ 1-4F & 0 & 0 & 2-2F \end{pmatrix} = \frac{1}{6}\left((1+2F)I + 2(1-4F)|\phi^+\rangle\langle\phi^+|\right).$$

Hence the eigenvalues of $(T \otimes \hat{I})(\rho_F)$ will be $\frac{1+2F}{6}$ and $\frac{1+2F+2(1-4F)}{6} = \frac{1-2F}{2}$. The first eigenvalue is always positive, but the second of these eigenvalues will be negative if and only if $F > \frac{1}{2}$. Therefore, all Werner states with $1 \geq F > \frac{1}{2}$ are certainly entangled.

It remains to show that all Werner states with $0 \leq F \leq \frac{1}{2}$ are separable. For every $\theta \in \mathbb{R}$ we can define the unit vector

$$|u_\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle).$$

Consider the density operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by $|u_\theta\rangle\langle u_\theta| \otimes |u_\theta\rangle\langle u_\theta|$, which are clearly separable. In matrix form, these look like

$$|u_\theta\rangle\langle u_\theta| \otimes |u_\theta\rangle\langle u_\theta| = \frac{1}{4}\begin{pmatrix} 1 & e^{-i\theta} & e^{-i\theta} & e^{-2i\theta} \\ e^{i\theta} & 1 & 1 & e^{-i\theta} \\ e^{i\theta} & 1 & 1 & e^{-i\theta} \\ e^{2i\theta} & e^{i\theta} & e^{i\theta} & 1 \end{pmatrix}.$$

We now take the sum of four of these separable density operators with $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$:

$$\sum_{a \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}} |u_a\rangle\langle u_a| \otimes |u_a\rangle\langle u_a| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Taking the convex mixture of the separable states above along with the separable states $|00\rangle\langle 00|$ and $|11\rangle\langle 11|$ yields

$$\frac{1}{6}|00\rangle\langle 00| + \frac{1}{6}|11\rangle\langle 11| + \sum_{a \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}} \frac{1}{6}|u_a\rangle\langle u_a| \otimes |u_a\rangle\langle u_a| = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

which is the Werner state with $F = \frac{1}{2}$, hence this state is separable.

Now consider the density operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by $|u_\theta\rangle\langle u_\theta| \otimes |u_{\pi+\theta}\rangle\langle u_{\pi+\theta}|$, which are clearly separable. In matrix form, these look like

$$|u_\theta\rangle\langle u_\theta| \otimes |u_{\pi+\theta}\rangle\langle u_{\pi+\theta}| = \frac{1}{4}\begin{pmatrix} 1 & e^{-i\theta} & -e^{-i\theta} & -e^{-2i\theta} \\ e^{i\theta} & 1 & -1 & -e^{-i\theta} \\ -e^{i\theta} & -1 & 1 & e^{-i\theta} \\ -e^{2i\theta} & -e^{i\theta} & e^{i\theta} & 1 \end{pmatrix}$$

We now take the sum of four of these separable density operators with $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$:

$$\sum_{a \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}} |u_\theta\rangle\langle u_\theta| \otimes |u_{\pi+\theta}\rangle\langle u_{\pi+\theta}| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Taking the convex mixture of the separable states above along with the separable states $|01\rangle\langle 01|$ and $|10\rangle\langle 10|$ yields

$$\frac{1}{6}|01\rangle\langle 01| + \frac{1}{6}|10\rangle\langle 10| + \sum_{a \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}} \frac{1}{6}|u_a\rangle\langle u_a| \otimes |u_a\rangle\langle u_a| = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is the Werner state with $F = 0$, and so this Werner state is also separable.

Since each Werner state with $0 < F < \frac{1}{2}$ can be written as a convex combination of the Werner states with $F = 0$ and $F = \frac{1}{2}$, it follows that each of these Werner states is separable, since they can be written as a convex combination of separable product states.

Problem 2

Problem 2. In this question all vectors are bipartite pure states in $\mathbb{C}^n \otimes \mathbb{C}^n$. Show that $|\psi\rangle$ can be converted to $|\phi_j\rangle$ with probability r_j if and only if

$$E_k(|\psi\rangle) \geq \sum_j r_j E_k(|\phi_j\rangle), \quad \forall k = 1, \ldots, n$$

where

$$E_k(|\psi\rangle) = \sum_{\ell=k}^{n} p_\ell$$

where $\{p_\ell\}$ are the Schmidt coefficients of $|\psi\rangle$ in decreasing order.

Solution. We first note that there exists an operator A such that $|\psi\rangle = (A \otimes I)|\phi^+\rangle$, where $|\phi^+\rangle$ is the unnormalized maximally entangled state

$$|\phi^+\rangle = \sum_{i=1}^{n} |ii\rangle.$$

Suppose that $|\psi\rangle$ can be converted into states $|\phi_j\rangle$ with probabilities r_j. That is to say that there is a measurement $\{M_j\}$ that can be performed on system A with corresponding unitaries $\{U_j\}$ to be performed on system B conditioned on the measurement outcome j such that

$$r_j|\phi_j\rangle = (M_j \otimes U_j)|\psi\rangle$$

holds for each j, and $r_j = \|(M_j \otimes U_j)|\psi\rangle\|$. Using the observation above, we see that we can write

$$|\phi_j\rangle = \frac{1}{\|M_j A\|}(M_j A U_j^T \otimes I)|\phi^+\rangle$$

where we see that $r_j = \|M_j A\|^2$ using the Hilbert-Schmidt matrix norm defined by $\|X\| = \sqrt{\mathrm{Tr}(X^* X)}$. It follows that

$$\mathrm{Tr}_{\mathsf{B}}(|\phi_j\rangle\langle\phi_j|) = \frac{1}{\|M_j A\|^2} M_j A A^* M_j^*$$

and so the Schmidt coefficients of $|\phi_j\rangle$ are

$$p_\ell(|\phi_j\rangle) = \frac{1}{\|M_j A\|^2}\lambda_\ell(\mathrm{Tr}_{\mathsf{B}}|\phi_j\rangle\langle\phi_j|) = \frac{1}{\|M_j A\|^2}\lambda_\ell\left(M_j A A^* M_j^*\right).$$

Moreover, note that $\lambda_\ell\left(M_j A A^* M_j^*\right) = \lambda_\ell\left(A^* M_j^* M_j A\right)$ and that these eigenvalues are just the squares of the singular values of $M_j A$.

From the convexity of the Ky-Fan norms, for each k we have that

$$\begin{aligned}
\sum_j r_j E_k(|\phi_j\rangle) = \sum_j \sum_{\ell=k}^{n} r_j \frac{1}{\|M_j A\|^2}\lambda_\ell(M_j A A^* M_j^*) &= \sum_j \sum_{\ell=k}^{n} \lambda_\ell(M_j A A^* M_j^*) \\
&\leq \sum_{\ell=k}^{n} \lambda_\ell\big(A^* \underbrace{\sum_j M_j^* M_j}_{I} A\big) \\
&= \sum_{\ell=k}^{n} \lambda_\ell(A^* A) = \sum_{\ell=k}^{n} p_\ell(|\psi\rangle) = E_k(|\psi\rangle),
\end{aligned}$$

and thus $\sum_j r_j E_k(|\phi_j\rangle) \leq E_k(|\psi\rangle)$ as desired.

For the other direction, suppose now that $\sum_j r_j E_k(|\phi_j\rangle) \leq E_k(|\psi\rangle)$ holds for all k. Without loss of generality, we may write the states $|\psi\rangle$ and $|\phi_j\rangle$ in Schmidt form as

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}|ii\rangle \qquad \text{and} \qquad |\phi_j\rangle = \sum_{i=1}^{n} \sqrt{t_{i|j}}|ii\rangle.$$

Define the quantities $q_i = \sum_j r_j t_{i|j}$ for each $i = 1, \ldots, n$ and define the vector

$$|\phi\rangle = \sum_{i=1}^{n} \sqrt{q_i}|ii\rangle.$$

It holds that $|\psi\rangle$ can be converted into $|\phi\rangle$ by LOCC. Indeed, we have that

$$E_k(|\phi\rangle) = \sum_{\ell=k}^{n} q_i = \sum_j \sum_{\ell=k}^{n} r_j t_{\ell|j} = \sum_j r_j \sum_{\ell=k}^{n} t_{\ell|j} = \sum_j r_j E_k(|\phi_j\rangle).$$

Since $\sum_j r_j E_k(|\phi_j\rangle) \leq E_k(|\psi\rangle)$ holds by assumption, it follows that $E_k(|\phi\rangle) \leq E_k(|\psi\rangle)$ and thus $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$. It now suffices to show that $|\phi\rangle$ can be converted into the ensemble $\{r_j, |\phi_j\rangle\}_j$ via some LOCC measurement. For each j, consider the operator

$$M_j = \sum_{i=1}^{n} \sqrt{\frac{r_j t_{i|j}}{q_i}}|i\rangle\langle i|.$$

This collection of operators $\{M_j\}_j$ comprises a valid measurement, since

$$\begin{aligned}
\sum_j M_j^* M_j &= \sum_{i,i'=1}^{n} \sqrt{\frac{r_j t_{i|j}}{q_i}} \sqrt{\frac{r_j t_{i'|j}}{q_{i'}}}|i\rangle\langle i|i'\rangle\langle i'| \\
&= \sum_{i=1}^{n} \frac{1}{q_i} \underbrace{\sum_j r_j t_{i|j}}_{=q_j} |i\rangle\langle i| = I.
\end{aligned}$$

Finally, performing this measurement on system A to the state $|\phi\rangle$ from above, we have

$$\begin{aligned}
(M_j \otimes I)|\phi\rangle &= \sum_{i,i'=1}^{n} \sqrt{\frac{r_j t_{i|j}}{q_i}} (|i\rangle\langle i| \otimes I)\sqrt{q_{i'}}|i'\rangle \otimes |i'\rangle \\
&= \sqrt{r_j} \sum_{i=1}^{n} \sqrt{t_{i|j}}|ii\rangle = \sqrt{r_j}|\phi_j\rangle.
\end{aligned}$$

Hence state $|\phi_j\rangle$ is obtained with probability r_j, as desired.

Problem 3

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Problem 3.

Solution.

Problem 4

Problem 4. Let $|\psi\rangle_{\mathsf{AB}} = \sum_{x=1}^{n} \sqrt{p_x}|x\rangle_{\mathsf{A}}|x\rangle_{\mathsf{B}} \in \mathbb{C}^n \otimes \mathbb{C}^n$ be a bipartite entangled state. Consider its reduced density matrix on system A:

$$\rho_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{B}}|\psi\rangle_{\mathsf{AB}}\langle\psi| = \sum_{x=1}^{n} p_x|x\rangle_{\mathsf{A}}\langle x|$$

Show that any pure state decomposition of ρ_{A} is realizable by a measurement on system B. Specifically, show that for any ensemble of pure states $\{q_y, |\phi_y\rangle_{\mathsf{A}}\}$ with the property that $\rho_{\mathsf{A}} = \sum_y q_y|\phi_y\rangle_{\mathsf{A}}\langle\phi_y|$, there exists a corresponding measurement $\{M_{\mathsf{B}}^y\}_y$ on system B, such that the measurement $\{I_{\mathsf{A}} \otimes M_{\mathsf{B}}^y\}_y$ on $|\psi\rangle_{\mathsf{AB}}$ produce the state $|\phi_y\rangle_{\mathsf{A}}$ on system A with probability q_y.

Solution. Assume that each p_x is strictly positive[1]. Note that we can write $|\psi\rangle_{\mathsf{AB}}$ as

$$|\psi\rangle_{\mathsf{AB}} = (\sqrt{\rho_{\mathsf{A}}} \otimes I_{\mathsf{B}}) \sum_{x=1}^{n} |x\rangle \otimes |x\rangle.$$

Let $\{q_y, |\phi_y\rangle\}$ be an ensemble of pure states satisfying

$$\sum_y q_y|\phi_y\rangle\langle\phi_y| = \rho_A.$$

Consider the operators $M_y : \mathbb{C}^n \longrightarrow \mathbb{C}^1$ defined by

$$M_y = \sqrt{q_y}|1\rangle|\phi_y\rangle^T \rho_{\mathsf{A}}^{-1/2} = \sqrt{q_y}|1\rangle\overline{\langle\phi_y|}\rho_{\mathsf{A}}^{-1/2},$$

where $\overline{|\phi\rangle}$ and $\overline{\langle\phi|}$ denote the complex conjugate of $|\phi\rangle$ and $\langle\phi|$ in the $\{|x\rangle\}$ basis. These operators form a valid measurement, since

$$\begin{aligned}
\sum_y M_y^* M_y &= \rho_{\mathsf{A}}^{-1/2}\left(\sum_y q_y \overline{|\phi_y\rangle}\langle 1||1\rangle\overline{\langle\phi_y|}\right)\rho_{\mathsf{A}}^{-1/2} \\
&= \rho_{\mathsf{A}}^{-1/2}\overline{\underbrace{\left(\sum_y q_y|\phi_y\rangle\langle\phi_y|\right)}_{\rho_{\mathsf{A}}}}\rho_{\mathsf{A}}^{-1/2} \\
&= \rho_{\mathsf{A}}^{-1/2}\overline{\rho_{\mathsf{A}}}\rho_{\mathsf{A}}^{-1/2} \\
&= I,
\end{aligned}$$

where we note that $\overline{\rho_{\mathsf{A}}} = \rho_{\mathsf{A}}$ since this operator is diagonal.

We now show that this measurement satisfies the desired criterion. Recall that $(A \otimes I)\sum_{x=1}^{n}|x\rangle \otimes |x\rangle = (I \otimes A^T)\sum_{x=1}^{n}|x\rangle \otimes |x\rangle$ holds for any operator A. It follows that

$$(I_{\mathsf{A}} \otimes M_y)|\psi\rangle = (I_{\mathsf{A}} \otimes M_y)(I_{\mathsf{A}} \otimes \sqrt{\rho_{\mathsf{A}}})\sum_{x=1}^{n}|x\rangle \otimes |x\rangle$$

[1] If $p_x = 0$ for some x, we can simply restrict our attention to the subspace spanned by all $|x\rangle$ such that $p_x \neq 0$ and include the operator $M_0 = \Pi_0$ in our collection of measurement operators. Here Π_0 is the projector onto the space spanned by all $|x\rangle$ such that $p_x = 0$. The measurement would then correspond to the operators $\{M_0\} \cup \{M_y\}_y$ and we would have

$$M_0^* M_0 + \sum_y M_y^* M_y = \Pi_0 + (I - \Pi_0) = I$$

where outcome 0 would occur with probability equal to zero.

$$
\begin{aligned}
&= (\sqrt{\rho_{\mathsf{A}}} M_y^T \otimes I_{\mathsf{B}}) \sum_{x=1}^{n} |x\rangle \otimes |x\rangle \\
&= (\sqrt{q_y} |\phi_y\rangle\langle 1| \otimes I_{\mathsf{B}}) \sum_{x=1}^{n} |x\rangle \otimes |x\rangle \\
&= \sqrt{q_y} |\phi_y\rangle \otimes |1\rangle.
\end{aligned}
$$

Hence, if the measurement corresponding to $\{M_y\}$ is performed on system B, the outcome y is obtained with probability q_y and the resulting state on system A is $|\phi_y\rangle$, as desired.

Problem 5

Problem 5. Consider the Choi isomorphism

$$\rho_{\mathsf{RA}} = \left(\hat{I}_{\mathsf{R}} \otimes \mathcal{E}\right)(|\phi^+\rangle\langle\phi^+|) \quad \text{and} \quad \mathcal{E}(\tau) = \mathrm{Tr}_{\mathsf{R}}\left[\rho_{\mathsf{RA}}\left(\tau^T \otimes I_{\mathsf{A}}\right)\right]$$

where

$$|\phi^+\rangle = \sum_{i=1}^{d_{\mathsf{A}}} |i\rangle_{\mathsf{R}}|i\rangle_{\mathsf{A}}.$$

Show that if instead one defines

$$\sigma_{\mathsf{RA}} = \left(\hat{I}_{\mathsf{R}} \otimes \mathcal{E}\right)(|\Psi\rangle\langle\Psi|) \quad \text{where} \quad |\Psi\rangle = \sum_{i=1}^{d_{\mathsf{A}}} \lambda_i |i\rangle_{\mathsf{R}}|i\rangle_{\mathsf{A}}, \text{ with } \lambda_i \neq 0 \text{ for all } i$$

then

$$\mathcal{E}(\tau) = \mathrm{Tr}_{\mathsf{R}}\left[\sigma_{\mathsf{RA}}\left(\sigma_{\mathsf{R}}^{-1/2} U \tau^T U^* \sigma_{\mathsf{R}}^{-1/2} \otimes I_{\mathsf{A}}\right)\right]$$

where U is some fixed diagonal unitary on system R and $\sigma_{\mathsf{R}} = \mathrm{Tr}_{\mathsf{A}}(\sigma_{\mathsf{RA}})$.

Solution. First note that we can write the vector $|\Psi\rangle$ as

$$|\Psi\rangle = M \otimes I_{\mathsf{A}}|\phi^+\rangle$$

where M is the invertible operator $M = \sum_{i=1}^{d_{\mathsf{A}}} \lambda_i |i\rangle\langle i|$. Note that

$$\begin{aligned}
\sigma_{\mathsf{RA}} &= \left(\hat{I}_{\mathsf{R}} \otimes \mathcal{E}\right)(M \otimes I_{\mathsf{A}}|\phi^+\rangle\langle\phi^+|M^* \otimes I_{\mathsf{A}}) \\
&= (M \otimes I_{\mathsf{A}})\left[\left(\hat{I}_{\mathsf{R}} \otimes \mathcal{E}\right)(|\phi^+\rangle\langle\phi^+|)\right](M^* \otimes I_{\mathsf{A}}) \\
&= (M \otimes I_{\mathsf{A}})\rho_{\mathsf{RA}}(M^* \otimes I_{\mathsf{A}})
\end{aligned}$$

and thus $\sigma_{\mathsf{R}} = M\rho_{\mathsf{R}}M^*$. However, it holds that $\rho_{\mathsf{R}} = \mathrm{Tr}_{\mathsf{A}}|\phi^+\rangle\langle\phi^+| = I_{\mathsf{R}}$, hence

$$\sigma_{\mathsf{R}} = MM^* = \sum_{i=1}^{d_{\mathsf{A}}} |\lambda_i|^2 |i\rangle\langle i| \qquad \text{and thus} \qquad \sigma_{\mathsf{R}}^{-1/2} = \sum_{i=1}^{d_{\mathsf{A}}} \frac{1}{|\lambda_i|}|i\rangle\langle i|.$$

Let U be the diagonal unitary matrix defined by

$$U = \sum_{i=1}^{d_{\mathsf{A}}} \frac{|\lambda_i|}{\overline{\lambda_i}} |i\rangle\langle i|.$$

It follows that $\sigma_{\mathsf{R}}^{-1/2} U = \sum_{i=1}^{d_{\mathsf{A}}} 1/\overline{\lambda_i}|i\rangle\langle i| = (M^*)^{-1}$. Finally, we have that

$$\begin{aligned}
\mathrm{Tr}_{\mathsf{R}}\left[\sigma_{\mathsf{RA}}\left(\sigma_{\mathsf{R}}^{-1/2} U \tau^T U^* \sigma_{\mathsf{R}}^{-1/2} \otimes I_{\mathsf{A}}\right)\right] &= \mathrm{Tr}_{\mathsf{R}}\left[(M \otimes I_{\mathsf{A}})\rho_{\mathsf{RA}}(M^* \otimes I_{\mathsf{A}})\left((M^*)^{-1}\tau^T M^{-1} \otimes I_{\mathsf{A}}\right)\right] \\
&= \mathrm{Tr}_{\mathsf{R}}\left[\rho_{\mathsf{RA}}(\tau^T \otimes I_{\mathsf{A}})\right] \\
&= \mathcal{E}(\tau)
\end{aligned}$$

as desired.