Midterm

MATH 621

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Problem 1 (Problem 5, Chapter VI.1, p. 170)**.**
Suppose $f(z)$ is analytic on the punctured plane $D = \mathbb{C} \setminus \{0\}$. Show that there is a constant c such that the function $f(z) - c/z$ has a primitive in D. Give a formula for the constant c in terms of an integral of $f(z)$ (Recall: a *primitive* for $f(z)$ is a function $F(z)$ such that $F'(z) = f(z)$.)

Solution.
Since f is analytic for $z \neq 0$, it has a Laurent decomposition $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ where the a_n's can be given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}} dz.$$

If we choose $c = a_{-1}$, then the function $f(z) - \frac{c}{z}$ has a primitive given by

$$F(z) = \sum_{\substack{k=-\infty \\ k\neq 1}}^{\infty} \frac{a_k}{k+1} z^k$$

and the constant c is given by

$$c = \frac{1}{2\pi i} \int_{|z|=1} f(z) dz.$$

Problem 2 (Problem 3 (a) and (d), Chapter VI.2, p. 176)**.**
Consider the function $f(z) = \tan z$ in the annulus $\{3 < |z| < 4\}$. Let $f(z) = f_0(z) + f_1(z)$ be the Laurent decomposition of $f(z)$, so that $f_0(z)$ is analytic for $|z| < 4$, and $f_1(z)$ is analytic for $|z| > 3$ and vanishes at ∞.

(a) Obtain an explicit expression for $f_1(z)$.

(d) What is the radius of convergence of the power series expansion for $f_0(z)$?

Solution. Since $f(z) = \tan z = \frac{\sin z}{\cos z}$, the function $f(z)$ has singularities at the roots of $\cos(z)$, which are at $\pi\left(m + \frac{1}{2}\right)$ for each $m \in \mathbb{Z}$. In particular, $\cos z$ has two roots in the disk $\{|z| < 4\}$ at $\pm\frac{\pi}{2}$, so $\tan z$ has two singularities at $z = \pm\frac{\pi}{2}$.

(a) We first find an explicit expression for $f_1(z)$. Expanding out the power series for $\cos z$ and $\sin z$ at $z = \frac{\pi}{2}$, we have

$$\begin{aligned}
\tan z = \frac{\sin z}{\cos z} = -\frac{\cos\left(z - \frac{\pi}{2}\right)}{\sin\left(z - \frac{\pi}{2}\right)} &= -\frac{1 - \frac{1}{2!}\left(z - \frac{\pi}{2}\right)^2 + \cdots}{\left(z - \frac{\pi}{2}\right) - \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^3 + \cdots} \\
&= -\frac{1}{z - \frac{\pi}{2}}\left[\frac{1 - \frac{1}{2!}\left(z - \frac{\pi}{2}\right)^2 + \cdots}{1 - \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^2 + \cdots}\right] \\
&= -\frac{1}{z - \frac{\pi}{2}}\left[\left(1 - \frac{1}{2!}\left(z - \frac{\pi}{2}\right)^2 + \cdots\right)\left(1 - \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^2 + \cdots\right)\right] \\
&= -\frac{1}{z - \frac{\pi}{2}} + \underbrace{\frac{1}{3}\left(z - \frac{\pi}{2}\right) + \cdots}_{\text{analytic}}.
\end{aligned}$$

So $\tan z$ has a simple pole at $z = \frac{\pi}{2}$ thus $\tan z + \frac{1}{z - \frac{\pi}{2}}$ is analytic near $z = \frac{\pi}{2}$. Performing a similar analysis at $z = -\frac{\pi}{2}$, we see that

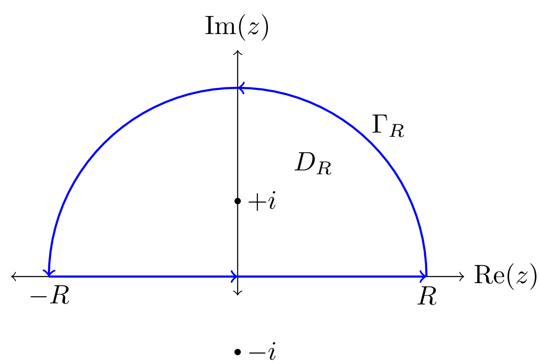
$$\begin{aligned}
\tan z = \frac{\sin z}{\cos z} &= -\frac{\cos\left(z + \frac{\pi}{2}\right)}{\sin\left(z + \frac{\pi}{2}\right)} \\
&= -\frac{1}{z + \frac{\pi}{2}} + \text{analytic},
\end{aligned}$$

so $\tan z + \frac{1}{z + \frac{\pi}{2}}$ is analytic near $z = -\frac{\pi}{2}$. Hence, choosing $f_1(z) = -\left(\frac{1}{z - \frac{\pi}{2}} + \frac{1}{z + \frac{\pi}{2}}\right)$, we see that $\tan z - f_1(z)$ is analytic for $|z| < 4$, whereas $f_1(z)$ is analytic for $|z| > 3$.

(d) Note that $f_0(z) = \tan z - f_1(z)$. The only singularities of $\tan z$ are $\pi\left(m + \frac{1}{2}\right)$, and from the above analysis we see that each singularity is a simple pole. The radius of convergence of $f_0(z)$ at $z = 0$ will be the distance to the nearest pole of $f_0(z)$, which is at $\pm\frac{3\pi}{2}$. So the radius of convergence is $\frac{3\pi}{2}$.

Problem 3 (Problem 9, Chapter VII.2, p. 203)**.**
Show that
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \frac{\pi}{2}\left[1 - \frac{1}{e^2}\right].$$

Solution. Note that $\sin^2 x = \frac{1-\cos(2x)}{2}$. We may split the integral in question into two as
$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos(2x)}{x^2+1} dx = \frac{1}{2}\left(I_1 - I_2\right),$$
where the integrals that we now need to evaluate are $I_1 = \int \frac{1}{x^2+1} dx$ and $I_2 = \int \frac{\cos(2x)}{x^2+1} dx$. The value of the integral I_2 may be taken to be the real part of the integral $\int \frac{e^{2ix}}{x^2+1} dx$. In both cases, we evaluate the integral by integrating over the boundary of the half disk D_R of radius R in the upper half-plane. Since the degree of the polynomial in the denominator is at least two, the integral over the semi circular path Γ_R of radius R in the upper half-plane will tend to zero as $R \to \infty$. So the integrals can be evaluated as
$$I_1 = \lim_{R\to\infty} \int_{\partial D_r} \frac{1}{z^2+1} dz \qquad \text{and} \qquad I_1 = \text{Re}\left[\lim_{R\to\infty} \int_{\partial D_r} \frac{e^{2iz}}{z^2+1} dz\right].$$
The only poles of the integrands are at $\pm i$, so the residues of interest are
$$\text{Res}\left[\frac{1}{z^2+1}, i\right] = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i} \qquad \text{and} \qquad \text{Res}\left[\frac{e^{i2z}}{z^2+1}, i\right] = \left.\frac{e^{i2z}}{2x}\right|_{z=i} = \frac{e^{-2}}{2i}.$$



Hence we have
$$I_1 = 2\pi i \,\text{Res}\left[\frac{1}{z^2+1}, i\right] = \pi \qquad \text{and} \qquad I_2 = \text{Re}\left[2\pi i \,\text{Res}\left[\frac{e^{2iz}}{z^2+1}, i\right]\right] = \frac{\pi}{e^2}.$$
Putting this all together yields the desired result:
$$I = \frac{1}{2}\left(I_1 + I_2\right) = \frac{\pi}{2}\left[1 - \frac{1}{e^2}\right].$$

Problem 4 (Problem 3, Chapter VII.3, p. 205)**.**
Show using residue theory that

$$\int_0^\pi \frac{\sin^2\theta}{a+\cos\theta}d\theta = \pi\left[a-\sqrt{a^2-1}\right] \qquad \text{for } a>1.$$

Solution. We first note that the integral in question remains the same if we instead integrate from π to 2π:

$$I = \int_0^\pi \frac{\sin^2\theta}{a+\cos\theta}d\theta = \int_\pi^{2\pi} \frac{\sin^2\theta}{a+\cos\theta}d\theta.$$

Indeed, performing the change of variables $\varphi = 2\pi-\theta$, we have $d\theta = -d\varphi$ and thus

$$\begin{aligned} I &= -\int_{2\pi}^{\pi} \frac{\sin^2(2\pi-\varphi)}{a+\cos(2\pi-\varphi)}d\varphi \\ &= \int_\pi^{2\pi} \frac{\sin^2(-\varphi)}{a+\cos(-\varphi)}d\varphi = \int_\pi^{2\pi} \frac{\sin^2(\varphi)}{a+\cos(\varphi)}d\varphi \end{aligned}$$

as desired, since $\cos(2\pi-\varphi) = \cos(-\varphi) = \cos(\varphi)$ and $\sin^2(2\pi-\varphi) = \sin^2(-\varphi) = \sin^2\varphi$. Thus, the integral I is

$$I = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta}{a+\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2}{a+\frac{e^{i\theta}+e^{-i\theta}}{2}}d\theta = -\frac{1}{4}\int_0^{2\pi} \frac{\left(e^{i\theta}-e^{-i\theta}\right)^2}{2a+e^{i\theta}+e^{-i\theta}}d\theta,$$

and we can apply residue theory. Using the change of variables $z = e^{i\theta}$, and thus $d\theta = \frac{1}{iz}dz$, this becomes

$$I = -\frac{1}{4}\oint_{|z|=1} \frac{\left(z-\frac{1}{z}\right)^2}{\left(2a+z+\frac{1}{z}\right)}\frac{dz}{iz} = -\frac{1}{4i}\oint_{|z|=1} \frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)}dz.$$

The integrand has a double pole at $z=0$ and simple poles at

$$z = \frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$$

Of the latter poles, note that only one sits within the unit circle, since $\left|-a-\sqrt{a^2-1}\right| = a+\sqrt{a^2-1} > a > 1$, whereas

$$\left|-a+\sqrt{a^2-1}\right| = a-\sqrt{a^2-1} < 1,$$

since $a-1<\sqrt{a^2-1}$. The residues of the poles of the integrand within the unit circle are

$$\begin{aligned} \operatorname{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)},0\right] &= \lim_{z\to 0}\left[\frac{d}{dz}\left(z^2\frac{(z^2-1)^2}{z^2(z^2+2az+1)}\right)\right] \\ &= \lim_{z\to 0}\left[\frac{4z(z^2+2az+1)(z^2-1)-(2z+2a)(z^2-1)^2}{(z^2+2az+1)^2}\right] = -2a. \end{aligned}$$

and (where we note that $\frac{1}{-a+\sqrt{a^2+1}} = -(a+\sqrt{a^2-1})$ and $\frac{(z^2-1)^2}{z^2} = (z-\frac{1}{2})^2$)

$$\begin{aligned} \operatorname{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)},-a+\sqrt{a^2-1}\right] &= \left.\frac{\left(z-\frac{1}{z}\right)^2}{2z+2a}\right|_{z=-a+\sqrt{a^2-1}} \\ &= \frac{\left(-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}\right)^2}{2\sqrt{a^2-1}} = \frac{2(a^2-1)}{\sqrt{a^2-1}} = 2\sqrt{a^2-1}. \end{aligned}$$

Putting this together, we see that the integral in question is

$$I = -\frac{1}{4i}2\pi i\left(\operatorname{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)},0\right]+\operatorname{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)},-a+\sqrt{a^2-1}\right]\right) = -\frac{\pi}{2}(2\sqrt{a^2-1}-2a).$$

as desired.

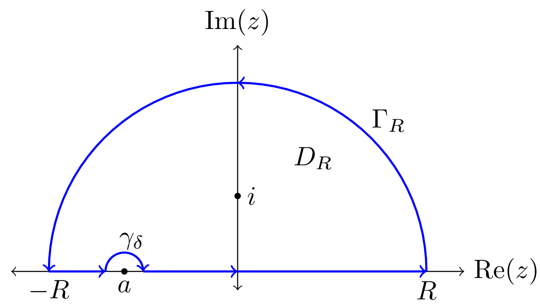
Problem 5 (Problem 3, Chapter VII.6, p. 215).
By integrating around the boundary of an indented half-disk in the upper half-plane, show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}dx = -\frac{\pi a}{a^2+1} \qquad \text{for } -\infty < a < \infty.$$

Solution. The integral may be evaluated by taking the limit of the sum of the integrals

$$\int_{-R}^{a-\delta} \frac{1}{(x^2+1)(x-a)}dx + \int_{a+\delta}^{R} \frac{1}{(x^2+1)(x-a)}dx$$

as $R \to \infty$ and $\delta \to 0$. These can be evaluated by integrating around the boundaries of the indented half-disks D_R as shown in the following figure.



The indented half-disk contains only one pole at $z = i$, so integrating around the boundary of D_R yields

$$\begin{aligned}\int_{\partial D_R} \frac{1}{(z^2+1)(z-a)}dz = 2\pi i \operatorname{Res}\left[\frac{1}{(z^2+1)(z-a)}, i\right] &= 2\pi i \left.\frac{1}{2z(z-a)}\right|_{z=i} \\ &= \frac{2\pi i}{2i(i-a)} = \frac{\pi}{i-a} = \frac{-\pi(a+i)}{a^2+1}.\end{aligned}$$

The integral along the semicircular path Γ_R of radius R will tend to zero as $R \to \infty$, since the polynomial in the denominator has degree three. So we only need to worry about the integral along the semicircular path γ_δ in the clockwise direction around the point $z = a$. By the Fractional Residue Theorem, taking the limit of this as $\delta \to 0$ yields

$$\lim_{\delta\to 0}\int_{\gamma_\delta} \frac{1}{(z^2+1)(z-a)}dz = -\pi i \operatorname{Res}\left[\frac{1}{(z^2+1)(z-a)}, a\right] = -\pi i \left.\frac{1}{z^2+1}\right|_{z=a} = \frac{-\pi i}{a^2+1}.$$

Putting this together in the limit as $R \to \infty$ yields

$$\left(\int_{\partial D_R}\right) - \left(\lim_{\delta\to 0}\int_{\gamma_\delta}\right) = -\frac{\pi a}{a^2+1}$$

as desired.

Problem 6 (Problem 5, Chapter VII.7, p. 218)**.**
Show that

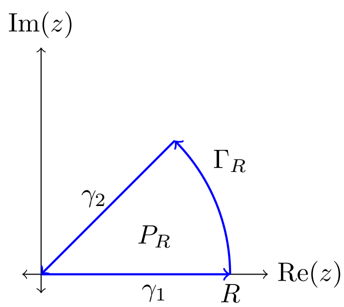
$$\lim_{R\to\infty}\int_0^R \sin(x^2)dx = \lim_{R\to\infty}\int_0^R \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

by integrating e^{iz} around the boundary of the pie-slice domain determined by $0 < \arg z < \frac{\pi}{4}$ and $|z| < R$. (*Remark*: These improper integrals are called the **Fresnel integrals**.)

Solution. We can evaluate these integrals by examining the real and imaginary parts of the integrals

$$\int_0^R e^{ix^2}dx,$$

and integrating this around the boundary of the pie-slice domain P_R as shown in the following figure.



The integral around the outer arc of radius R tends to zero as $R \to \infty$. Indeed, making the change of variables $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$ yields

$$\begin{aligned}\int_{\Gamma_R} e^{iz^2}dz = iR\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}e^{i\theta}d\theta &= iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos(2\theta)+i\sin(2\theta))}e^{i\theta}d\theta \\ &= iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}e^{i\left(R^2\cos(2\theta)+\theta\right)}d\theta.\end{aligned}$$

Along the lines of the proof of Jordan's lemma, note that $\sin(2\theta) \geq \frac{4\theta}{\pi}$ on the interval $0 \leq \theta \leq \frac{\pi}{4}$, and thus

$$\left|\int_{\Gamma_R} e^{iz^2}dz\right| \leq R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}d\theta \leq R\int_0^{\frac{\pi}{4}} e^{-4R^2\theta/\pi}d\theta = R\frac{\pi}{4R^2}\int_0^{R^2} e^{-t}dt \leq \frac{\pi}{4R}\int_0^{\infty} e^{-t}dt = \frac{\pi}{4R}.$$

Hence $\int_{\Gamma_R} e^{iz^2}dz \to 0$ as $R \to \infty$.

Next, we evaluate the integral along the path γ_2 from the point $Re^{i\frac{\pi}{4}}$ to the origin. This may be parameterized as $z = re^{i\frac{\pi}{4}}$ where we take r from R to zero, and note that $dz = e^{i\frac{\pi}{4}}dr$. This becomes

$$\int_{\gamma_2} e^{iz^2}dz = e^{i\frac{\pi}{4}}\int_R^0 e^{ir^2e^{i\frac{\pi}{2}}}dr = -e^{i\frac{\pi}{4}}\int_0^R e^{-r^2}dr,$$

and taking the limit of this as $R \to \infty$ yields the well-known Gaussian integral $\int_0^\infty e^{-r^2}dr = \frac{\sqrt{\pi}}{2}$. Since there are no poles within the pie-slice domain, in the limit as $R \to \infty$ we have that

$$\int_{\gamma_1} e^{iz^2}dz = -\int_{\gamma_1} e^{iz^2}dz = e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2\sqrt{2}}\left(1+i\right).$$

Taking the real and imaginary parts of this integral yields the desired results.

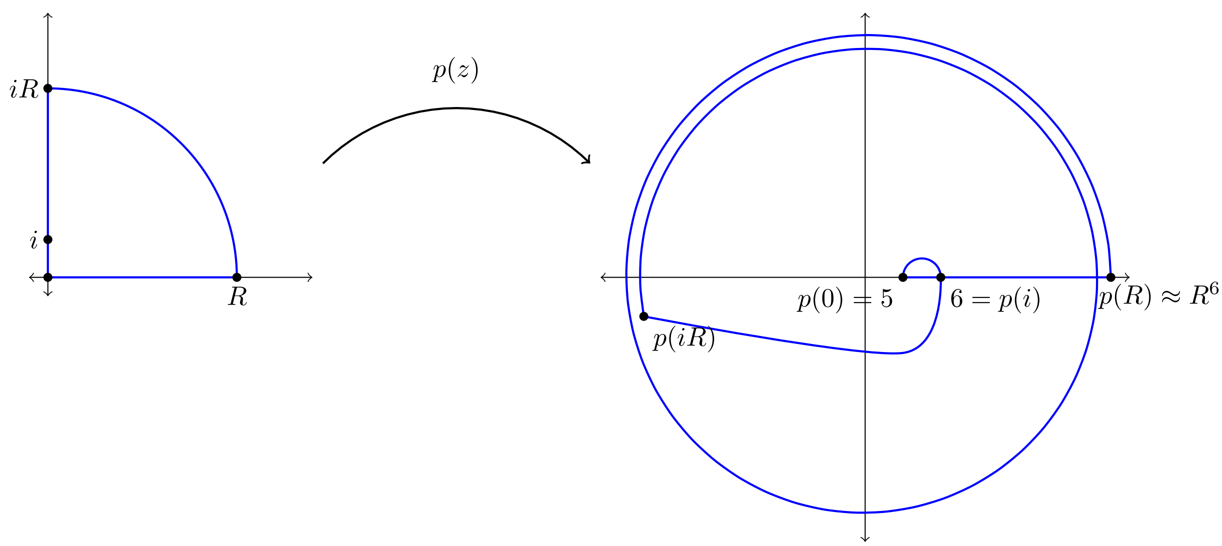
Problem 7 (Problem 3, Chapter VIII.1, p. 228)**.**
Find the number of zeros of the polynomial $p(z) = z^6 + 4z^4 + z^3 + 2z^2 + z + 5$ in the first quadrant ($\mathrm{Re}\, z > 0$, $\mathrm{Im}\, z > 0$).

Solution. Note that there are no zeros on the positive real axis, since $p(x)$ is the sum of positive numbers when $x > 0$. Furthermore, note that $p(0) = 5$. There are also no roots on the positive imaginary axis. Indeed, the imaginary part of

$$p(iy) = -y^6 + 4y^4 - 2y^2 + 5 + i(-y^3 + y)$$

is zero only when $y = 0$ or $y = \pm 1$, but the real part of $p(iy)$ is $-1 + 4 - 2 + 5 = 6$ when $y = 1$.

We can count the number of zeros in the first quadrant by considering a closed path around the first quadrant and estimating the change in argument of $p(z)$ along this path.



When $R \gg 0$, note that $p(R)$ is dominated by the R^6 term, and so $p(R) \approx R^6$. There is no change in argument along the path on the real axis from $z = 0$ to R. Along the path of the quarter circle from R to iR, the path of polynomial $p(Re^{i\theta})$ traces out a curve that goes 1.5 times around the origin, ending at $p(iR) \approx -R^6 - iR^3$. The change in argument of $p(z)$ along this path is approximately 3π. Along the path on the positive imaginary axis from iR to zero, the imaginary part of the polynomial has roots at $y = 1$ and $y = 0$. At $y = 1$, the value of the polynomial is 6. So the change in argument of $p(z)$ along the imaginary axis from iR to i is approximately π. From $z = i$ to $z = 0$, the value of the polynomial goes from 6 to 5 without crossing the real axis again, so the change in argument is zero Hence, the total change in argument is approximately 4π. Since

$$\int_{\partial D_R} d \arg(p(z)) = 2\pi(N_0 - N_\infty),$$

and $p(z)$ has no poles, the number of zeros in the first quadrant must be two.

Problem 8 (Problem 1, Chapter VIII.2, p. 230)**.**
Show that $2z^5 + 6z - 1$ has one root on the interval $0 < x < 1$ and four roots in the annulus $\{1 < |z| < 2\}$.

Solution. Let $p(z) = 2z^5 + 6z - 1$. Note that $p(0) = -1 < 0$ but $p(1) = 2 + 6 - 1 = 7 > 0$, so $p(z)$ must have at least one root on the interval $0 < x < 1$ by the Intermediate Value Theorem.

Consider $f_1(z) = 6z$ and $h_1(z) = 2z^5 - 1$. For z on the unit circle, note that

$$|h_1(z)| = \left|2z^5 - 1\right| \leq 2\,|z|^5 + 1 = 2 + 1 = 3$$

but $|f_1(z)| = 6\,|z| = 6$. Hence $|h_1| < |f_1|$ on the circle $\{|z| = 1\}$. Note that $f_1(z) = 6z$ has one root at $z = 0$, which is inside the unit disk, and thus $p(z) = f_1(z) + h_1(z)$ has exactly one root inside the unit disk by Rouché's Theorem. But we already determined that $p(z)$ has one real root on the interval $(0, 1)$, so there is exactly one root on this interval.

Now consider $f_2(z) = 2z^5$ and $h_2(z) = 6z - 1$. For z on the circle $\{|z| = 2\}$, note that

$$|h_2(z)| = |6z - 1| \leq 6\,|z| + 1 = 13$$

whereas $|f_2(z)| = 2\,|z|^5 = 2^6$. Hence $|h_2| < |f_2|$ on the circle $\{|z| = 2\}$. Note that $f_2(z) = 2z^5$ has five roots inside the disk $\{|z| = 2\}$, and thus $p(z) = f_2(z) + h_2(z)$ also has five roots inside this disk by Rouché's Theorem. Since $p(z)$ has no roots on the unit circle and exactly one root in the unit disk, $p(z)$ must have four roots inside the annulus $\{1 < |z| < 2\}$.

Problem 9 (Problem 4, Chapter X.3, p. 287)**.**
Suppose the curve γ passing through 0 is the graph of a function $y = h(x)$ that can be expressed as a convergent power series $h(x) = \sum_{k=1}^{\infty} a_k x^k$, $-r < x < r$, where the a_k's are real.

(a) Show that $z = \zeta + ih(\zeta)$ can be solved for $\zeta = \zeta(z)$ as an analytic function of z for $|z| < \varepsilon$.

(b) Show that γ is an analytic curve.

(c) Show that the reflection through γ is given by $z^* = 2\overline{\zeta(z)} - \bar{z}$.

Solution. Note that the curve γ is given by the points $x + ih(x)$ for $-r < x < r$, and that $h(0) = 0$.

(a) Set $z(\zeta) = \zeta + ih(\zeta)$ and note that z is analytic in ζ. Taking the derivative yields

$$z'(\zeta) = 1 + ih'(\zeta) = 1 + i \sum_{k=1}^{\infty} k a_k \zeta^{k-1}$$

and evaluating this derivative at zero gives us $z'(0) = 1 + ia_1 \neq 0$. By the inverse function theorem, $z(\zeta)$ is invertible in some neighborhood of $\zeta = 0$. Hence, there is an $\varepsilon > 0$ and some open domain U containing 0 such that $z(\zeta)$ is an analytic isomorphism of U and the unit disk D_ε of radius ε. So for each $|z| < \varepsilon$ there is a unique $\zeta \in U$ such that $z = \zeta + ih(\zeta)$, and we can write this as $\zeta = \zeta(z)$.

(b) Let $x + ih(x)$ be a point on the curve γ for some $-r < x < r$. As above, consider the function $z(\zeta) = \zeta + ih(\zeta)$, but this time near the point $\zeta = x$. The derivative of z at x is

$$z'(x) = 1 + ih'(x),$$

which is never zero since $\mathrm{Re}\,[z'(x)] = 1$ for any real x. So there exist neighborhoods of x and $z(x)$ such that $z(\zeta)$ is an analytic isomorphism between the two. Then we can find a disk $D_\delta(x)$ of radius $\delta > 0$ centered at x such that $z(\zeta)$ is an analytic isomorphism on this domain, and $D_\delta(x) \cap \mathbb{R} = (x - \delta, x + \delta)$ is mapped into γ. This is the definition of an analytic arc.

(c) Recall that the reflection across an analytic arc is given by $z^*(\zeta) = z(\bar{\zeta})$. Since $z(\zeta) = \zeta + ih(\zeta)$, note that

$$\overline{z(\zeta)} = \bar{\zeta} - ih(\bar{\zeta})$$

and thus $ih(\bar{\zeta}) = \bar{\zeta} - \overline{z(\zeta)}$. So the reflection z^* can be given by

$$\begin{aligned} z^*(\zeta) = z(\bar{\zeta}) &= \bar{\zeta} + ih(\bar{\zeta}) \\ &= \bar{\zeta} + \bar{\zeta} - \overline{z(\zeta)}, \end{aligned}$$

which yields $z^* = 2\overline{\zeta(z)} - \bar{z}$, as desired.

Problem 10 (Problem 6, Chapter X.1, p. 279)**.**
A function $f(z)$, $z \in \mathbb{D}$, is said to have **radial limit** L at $\zeta \in \partial\mathbb{D}$ if $f(r\zeta) \to L$ as r increases to 1. Let $h\left(e^{i\theta}\right)$ be a piecewise continuous function on the unit circle. Show that $\tilde{h}(z)$ has a radial limit at each $\zeta \in \partial\mathbb{D}$, equal to the average of the limits of $h\left(e^{i\theta}\right)$ at ζ from each side.

Solution. (Don't we need to assume that $h\left(e^{i\theta}\right)$ is bounded?)

Let $\zeta \in \partial\mathbb{D}$ with $\zeta = e^{i\theta}$. Since h is piecewise continuous and bounded, the 'right'- and 'left'-sided limits (or perhaps we should call them the 'clockwise' and 'counterclockwise' limits) of h exist at each point $e^{i\theta}$ on the unit circle. We denote these one-sided limits as

$$h_-\left(e^{i\theta}\right) = \lim_{\varphi \searrow 0^+} h\left(e^{i(\theta-\varphi)}\right) \qquad \text{and} \qquad h_+\left(e^{i\theta}\right) = \lim_{\varphi \nearrow 0^-} h\left(e^{i(\theta-\varphi)}\right).$$

Hence for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\left|h\left(e^{i(\theta-\varphi)}\right) - h_-\left(e^{i\theta}\right)\right| < \varepsilon$ whenever $\varphi \in (0, \delta)$, and $\left|h\left(e^{i(\theta-\varphi)}\right) - h_+\left(e^{i\theta}\right)\right| < \varepsilon$ whenever $\varphi \in (-\delta, 0)$.

The rest of the proof follows along the lines of the proof of the boundary-value problem in chapter X.1 of the book. Let $\varepsilon > 0$ and choose $\delta > 0$ as above. Since h is bounded, there is an $M > 0$ such that $\left|h\left(e^{i\theta}\right)\right| < M$ for all θ. Note that $\int_0^\pi P_r(\varphi)\frac{d\varphi}{2\pi} = \int_{-\pi}^0 P_r(\varphi)\frac{d\varphi}{2\pi} = \frac{1}{2}$, and thus comparing $h\left(e^{i(\theta-\varphi)}\right)$ to the left-sided limit of h at $e^{i\theta}$ for $\varphi > 0$ yields

$$\begin{aligned}\left|\int_0^\pi h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_-\left(e^{i\theta}\right)\right| &\leq \int_0^\pi \left|h\left(e^{i(\theta-\varphi)}\right) - h_-\left(e^{i\theta}\right)\right| P_r(\varphi)\frac{d\varphi}{2\pi} \\ &\leq \int_0^\delta \varepsilon P_r(\varphi)\frac{d\varphi}{2\pi} + M \max_{\delta \leq \varphi \leq \pi} P_r(\varphi) \\ &\leq \frac{\varepsilon}{2} + M \max_{\delta \leq \varphi \leq \pi} P_r(\varphi). \end{aligned} \tag{10.1}$$

Similarly, comparing $h\left(e^{i(\theta-\varphi)}\right)$ to the right-sided limit of h at $e^{i\theta}$ for $\varphi < 0$ yields

$$\begin{aligned}\left|\int_{-\pi}^0 h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_+\left(e^{i\theta}\right)\right| &\leq \int_{-\pi}^0 \left|h\left(e^{i(\theta-\varphi)}\right) - h_+\left(e^{i\theta}\right)\right| P_r(\varphi)\frac{d\varphi}{2\pi} \\ &\leq \int_{-\delta}^0 \varepsilon P_r(\varphi)\frac{d\varphi}{2\pi} + M \max_{-\pi \leq \varphi \leq -\delta} P_r(\varphi) \\ &\leq \frac{\varepsilon}{2} + M \max_{-\pi \leq \varphi \leq -\delta} P_r(\varphi). \end{aligned} \tag{10.2}$$

Note that we can consider $\tilde{h}\left(re^{i\theta}\right)$ by splitting the integral in two parts

$$\tilde{h}\left(re^{i\theta}\right) = \int_{-\pi}^0 h\left(re^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} + \int_0^\pi h\left(re^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi}. \tag{10.3}$$

Putting together equations (10.1), (10.2), and (10.3), we see that

$$\begin{aligned}&\left|\tilde{h}\left(re^{i\theta}\right) - \frac{h_-\left(e^{i\theta}\right) + h_+\left(e^{i\theta}\right)}{2}\right| \\ &\qquad \leq \left|\int_0^\pi h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_-\left(e^{i\theta}\right)\right| + \left|\int_{-\pi}^0 h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_+\left(e^{i\theta}\right)\right| \\ &\qquad \leq \varepsilon + 2M \max_{\delta \leq |\varphi| \leq \pi} P_r(\varphi),\end{aligned}$$

and the value of the second summand in the last line tends to zero as $r \to 1$. Hence, for fixed θ, the values of $\tilde{h}\left(re^{i\theta}\right)$ cluster to within ε of $\frac{h_-\left(e^{i\theta}\right)+h_+\left(e^{i\theta}\right)}{2}$ as $r \to 1$, and this is for any $\varepsilon > 0$. This result yields the desired limit.