Water fraction uncertainty, part 2

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7/16/2019

Previously I showed that the posterior distribution for water fraction, α, is Inv-Gamma distributed, having expectation $\frac{\frac{N_L}{N_L-1}p-\mu_l}{\mu_w-\mu_l}$ and variance $\frac{N_L^2p^2}{(N_L-1)^2(N_L-2)(\mu_w-\mu_l)^2}$. Here I'll use those equations to propagate uncertainty from the μ_l and μ_w terms.

Using the law of total variance we have

$$\begin{aligned} Var(\alpha) &= Var(E[\alpha|\mu_w,\mu_l]) + E[Var(\alpha|\mu_w,\mu_l)] \\ &= Var\Big(\frac{\frac{N_L}{N_L-1}p-\mu_l}{\mu_w-\mu_l}\Big) + \frac{N_L^2p^2}{(N_L-1)^2(N_L-2)}E\Big[\frac{1}{(\mu_w-\mu_l)^2}\Big] \end{aligned}$$

Next, defining $W = \mu_w - \mu_l$, continue with another total-variance decomposition of the first term.

$$\begin{aligned} Var\Big(\frac{\frac{N_L}{N_L-1}p-\mu_l}{W}\Big) &= Var\Big(E\Big[\frac{\frac{N_L}{N_L-1}p-\mu_l}{W}\Big|W\Big]\Big) + E\Big[Var\Big(\frac{\frac{N_L}{N_L-1}p-\mu_l}{W}\Big|W\Big)\Big] \\ &= Var\Big(\frac{\frac{N_L}{N_L-1}p-E[\mu_l]}{W}\Big) + E\Big[\frac{1}{W^2}Var(\mu_l)\Big] \\ &= \Big(\frac{N_L}{N_L-1}p-E[\mu_l]\Big)^2 Var(\frac{1}{W}) + Var(\mu_l)E[\frac{1}{W^2}] \end{aligned}$$

Combining with the first result, this gives a formula for $Var(\alpha)$ that is a funciton of the mean and variance of two variables, μ_l and $\frac{1}{W}$ (since $E[\frac{1}{W^2}] = Var(\frac{1}{W}) + (E[\frac{1}{W}])^2$).

To go further, we need to express the moments of $\frac{1}{W}$ in terms of the moments of μ_w and μ_l. The only way I can think to do that is to approximate W as a gamma-distributed random variable, so that $\frac{1}{W}$ is inverse-gamma distributed. I don't know exactly how well this approximation will work, but generally it will be better if $SD(\mu_w-\mu_l)$ is small compared to $E(\mu_w-\mu_l)$.

Define $\mu_* = E[\mu_w] - E[\mu_l]$ and $\sigma_*^2 = Var(\mu_w) + Var(\mu_l)$. Then, method of moments gives the k, θ parameters of W's approximated Gamma distribution as $\frac{\mu_*^2}{\sigma_*^2}$ and $\frac{\sigma_*^2}{\mu_*}$, respectively. Thus by the inverse gamma distribution we arrive at

$$E[\frac{1}{W}] = \frac{\mu_*}{\mu_*^2-\sigma_*^2}; Var(\frac{1}{W}) = \frac{\mu_*^2\sigma_*^2}{(\mu_*^2-\sigma_*^2)^2(\mu_*^2-2\sigma_*^2)}$$

Conclusion

The final reassembled result for variance of water fraction is as follows:

$$\begin{aligned} Var(\alpha) = &\Big(\frac{N_L}{N_L-1}p-E[\mu_l]\Big)^2 Var(\frac{1}{W}) \\ &+ Var(\mu_l)E[\frac{1}{W^2}] \\ &+ \frac{N_L^2p^2}{(N_L-1)^2(N_L-2)}E\Big[\frac{1}{W^2}\Big] \end{aligned}$$

where $Var\frac{1}{W}$ is computed as above and $E\Big[\frac{1}{W^2}\Big] = Var(\frac{1}{W}) + (E[\frac{1}{W}])^2 = \frac{\mu_*^2}{(\mu_*^2-\sigma_*^2)(\mu_*^2-2\sigma_*^2)}$.