Water fraction uncertainty, part 2

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Previously I showed that the posterior distribution for water fraction, α, is Inv-Gamma distributed, having expectation $\frac{\frac{N_L}{N_L-1} \quad - \quad {}_l}{{}_w - \quad {}_l}$ and variance $\frac{{}^2_L \;\; {}^2}{(\quad {}_L-1)^2(\quad {}_L-2)(\quad {}_w - \quad {}_l)^2}$. Here I'll use those equations to propagate uncertainty from the $\mu\quad$ and $\mu\quad$ terms.

Using the law of total variance we have

$$
\begin{aligned}
Var(\alpha) &= Var(E[\alpha|\mu \quad, \mu\ ]) + E[Var(\alpha|\mu \quad, \mu\ )] \\
&= Var\Big(\frac{\frac{L}{L-1}p - \mu}{\mu \quad - \mu}\Big) + \frac{N^2p^2}{(N \quad -1)^2(N \quad -2)}E\Big[\frac{1}{(\mu \quad - \mu\ )^2}\Big]
\end{aligned}
$$

Next, defining $W = \mu \quad - \mu\ $, continue with another total-variance decomposition of the first term.

$$
\begin{aligned}
Var\Big(\frac{\frac{L}{L-1}p - \mu}{W}\Big) &= Var\Big(E\Big[\frac{\frac{L}{L-1}p - \mu}{W}\Big|W\Big]\Big) + E\Big[Var\Big(\frac{\frac{L}{L-1}p - \mu}{W}\Big|W\Big)\Big] \\
&= Var\Big(\frac{\frac{L}{L-1}p - E[\mu\ ]}{W}\Big) + E\Big[\frac{1}{W^2}Var(\mu\ )\Big] \\
&= \Big(\frac{N}{N \quad -1}p - E[\mu\ ]\Big)^2 Var(\frac{1}{W}) + Var(\mu\ )E[\frac{1}{W^2}]
\end{aligned}
$$

Combining with the first result, this gives a formula for $Var(\alpha)$ that is a funciton of the mean and variance of two variables, $\mu\ $ and $\frac{1}{\ }$ (since $E[\frac{1}{\ ^2}] = Var(\frac{1}{\ }) + (E[\frac{1}{\ }])^2$).

To go further, we need to express the moments of $\frac{1}{\ }$ in terms of the moments of $\mu\quad$ and $\mu\ $. The only way I can think to do that is to approximate W as a gamma-distributed random variable, so that $\frac{1}{\ }$ is inverse-gamma distributed. I don't know exactly how well this approximation will work, but generally it will be better if $SD(\mu \quad - \mu\ )$ is small compared to $E(\mu \quad - \mu\ )$.

Define $\mu_* = E[\mu \quad] - E[\mu\ ]$ and $\sigma_*^2 = Var(\mu \quad) + Var(\mu\ )$. Then, method of moments gives the k, θ parameters of W's approximated Gamma distribution as $\frac{{}^2_*}{{}^2_*}$ and $\frac{{}^2_*}{{}_*}$, respectively. Thus by the inverse gamma distribution we arrive at

$$
E[\frac{1}{W}] = \frac{\mu_*}{\mu_*^2 - \sigma_*^2}; Var(\frac{1}{W}) = \frac{\mu_*^2\sigma_*^2}{(\mu_*^2 - \sigma_*^2)^2(\mu_*^2 - 2\sigma_*^2)}
$$

Conclusion

The final reassembled result for variance of water fraction is as follows:

$$
\begin{aligned}
Var(\alpha) = &\Big(\frac{N}{N \quad -1}p - E[\mu\ ]\Big)^2 Var(\frac{1}{W}) \\
&+ Var(\mu\ )E[\frac{1}{W^2}] \\
&+ \frac{N^2p^2}{(N \quad -1)^2(N \quad -2)}E\Big[\frac{1}{W^2}\Big]
\end{aligned}
$$

where $Var\frac{1}{\ }$ is computed as above and $E\Big[\frac{1}{\ ^2}\Big] = Var(\frac{1}{\ }) + (E[\frac{1}{\ }])^2 = \frac{{}^2_*}{(\ {}^2_* - \ {}^2_*)(\ {}^2_* - 2\ {}^2_*)}$.