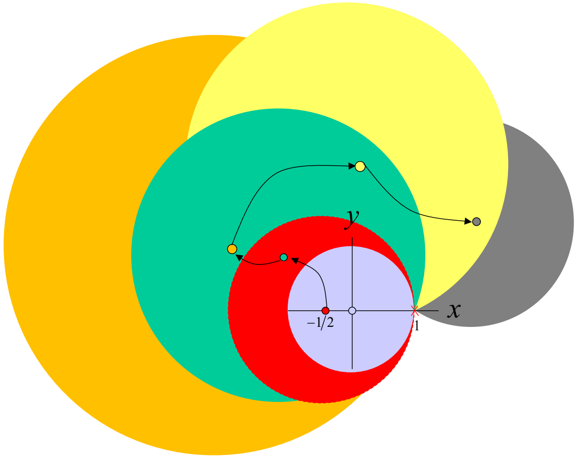


ECE 6382

Fall 2022

David R. Jackson

Notes 8

Analytic Continuation

Notes are adapted from D. R. Wilton, Dept. of ECE

Analytic Continuation of Functions

- We use analytic continuation to extend a function <u>off of the real axis</u> and into the complex plane such that the resulting function is analytic.
- More generally, analytic continuation extends the representation of a function in <u>one region</u> of the complex plane into <u>another region</u>, where the original representation may not have been valid.

For example, consider the Bessel function $J_n(x)$.

- How do we define $J_n(z)$ so that it is computable in some region and agrees with $J_n(x)$ when z is real?

Analytic Continuation of Functions (cont.)

- One approach to extend the domain of a function is to use Taylor series.
  - We start with a Taylor series that is valid in some region.
  - We extend this to a Taylor series that is valid in another region.

Note:
This may not be the easiest way in practice, but it always works in theory, and it illustrate the principle of analytic continuation.

Analytic Continuation of Functions (cont.)

Example

$$f(z) = \underbrace{\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n}_{\text{two alternative representations}}, \quad |z| < 1$$

- **Expand about $z = -\frac{1}{2}$. Since both series are valid there, find coefficients of a new series :**

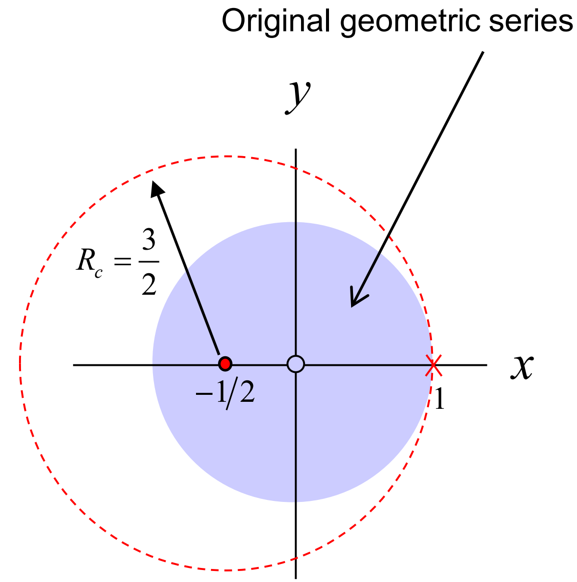
$$f(z) = \sum_{m=0}^{\infty} b_m \left(z + \tfrac{1}{2}\right)^m \quad , \quad \left|z + \tfrac{1}{2}\right| < \tfrac{3}{2}$$

by using

$$b_m = \frac{1}{m!} \frac{d^m f}{dz^m}\bigg|_{z=-\frac{1}{2}} = \frac{1}{m!} \frac{d^m}{dz^m}\bigg| \underbrace{\left[\sum_{n=0}^{\infty} z^n\right]_{z=-\frac{1}{2}}}_{\text{use the above series representation}} = \frac{1}{m!}\left[\sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-m+1)\, z^{n-m}\right]_{z=-\frac{1}{2}}$$

$$= \frac{1}{m!}\left[\sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\left(-\tfrac{1}{2}\right)^{n-m}\right]$$

$$\left(\textbf{Note} \quad \frac{d^m}{dz^m} z^n = 0, \quad m > n\right)$$



- **The coefficients of the new series — with extended region of convergence — are determined from the coefficients of the original series, even though that series did not converge in the extended region. The information to extend the convergence region is contained in the coefficients of the original series — even if it was divergent in the new region!**

Analytic Continuation of Functions (cont.)

Example (cont.)

Another way to get the Taylor series expansion:

$$f(z) = \sum_{m=0}^{\infty} b_m \left(z+\tfrac{1}{2}\right)^m \quad , \quad \left|z+\tfrac{1}{2}\right| < \tfrac{3}{2} \qquad\qquad b_m = \frac{1}{m!}\frac{d^m}{dz^m}\left(\frac{1}{1-z}\right)\Bigg|_{z=-1/2}$$

so that

$$b_m = \frac{1}{m!}(1)(2)(3)\cdots(m)\frac{1}{(1-z)^{m+1}}\Bigg|_{z=-1/2} = \frac{1}{m!}m!\left(\frac{2}{3}\right)^{m+1} = \left(\frac{2}{3}\right)^{m+1}$$

$$\Rightarrow f(z) = \sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^{m+1}\left(z+\tfrac{1}{2}\right)^m \quad , \quad \left|z+\tfrac{1}{2}\right| < \tfrac{3}{2}$$

Note:
This is sort of "cheating" in the sense that we assume we already know a closed form expression for the function.

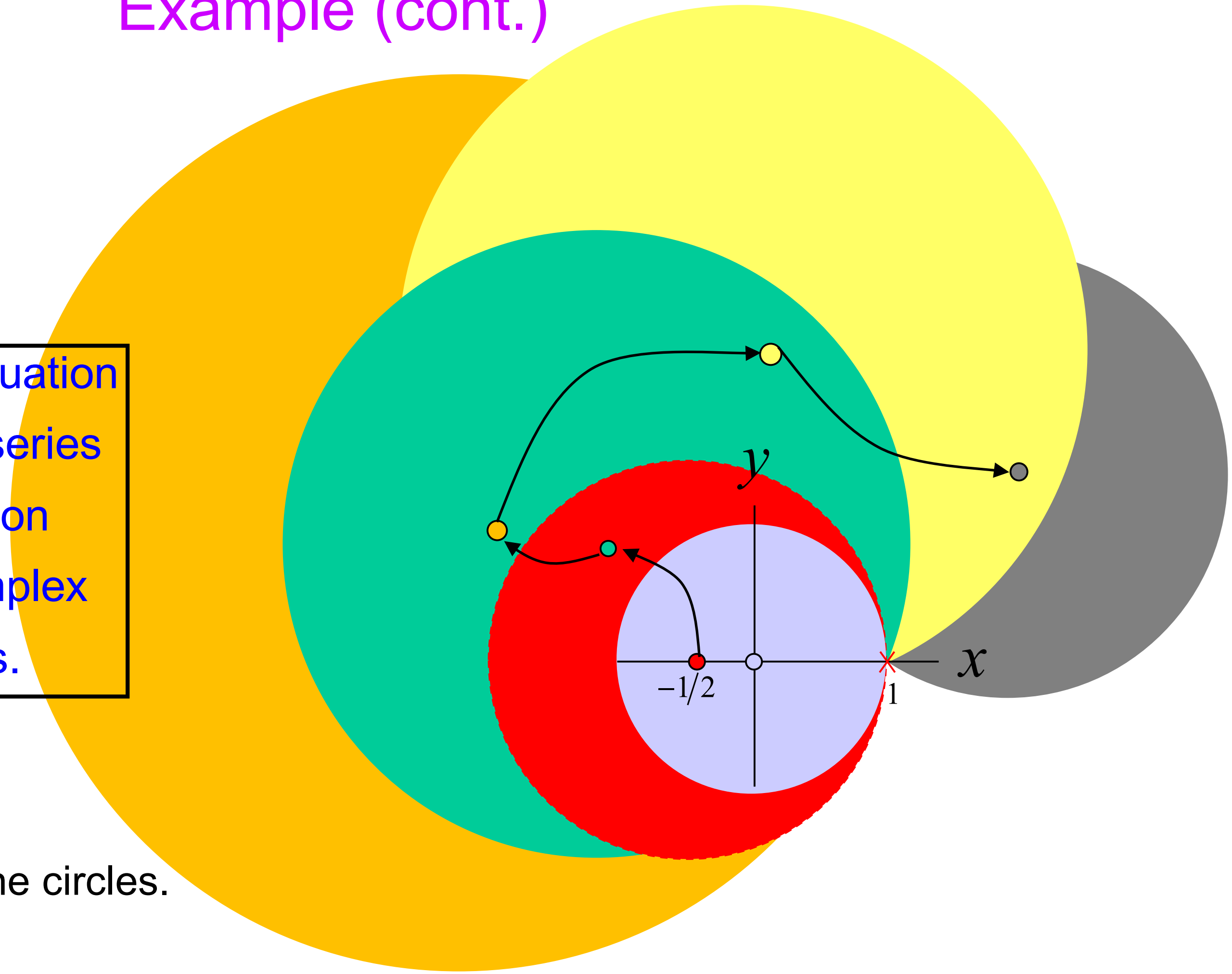
Note : $f(z) = \dfrac{1}{1-z}; \;\; f'(z) = \dfrac{1}{(1-z)^2}; \;\; f''(z) = \dfrac{2}{(1-z)^3}; \;\; f'''(z) = \dfrac{2\cdot 3}{(1-z)^4}$

Analytic Continuation of Functions (cont.)

Example (cont.)

$$f(z) \equiv \sum_{n=0}^{\infty} z^n$$

Here we show the continuation of $f(z)$ from its power series representation in the region $|z|<1$ into the entire complex plane using Taylor series.



We have a "leapfrogging" of the circles.

- If the singularities are isolated, we can continue any function into the entire complex plane via a sequence of continuations using Taylor and / or Laurent series!

Theorem

The Zeros of an Analytic Function are Isolated

(The zeros cannot be arbitrarily close together.)

Proof of theorem:

- Assume that $f(z)$ is analytic in a connected region A, and suppose $f(z_0)=0$ (simple zero). Then $f(z)$ has a Taylor series $f(z_0) = a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ with $a_0 = f(z_0)=0$ at z_0.
- More generally, we may have a zero with *multiplicity* N $(N<\infty)$ such that $f(z)$ has a Taylor series as:

$$f(z)=a_N(z-z_0)^N+a_{N+1}(z-z_0)^{N+1}+\cdots$$

$$=(z-z_0)^N\left[a_N+a_{N+1}(z-z_0)^1+\cdots\right] \qquad \left(a_N=\frac{1}{N!}f^{(N)}(z_0)\neq 0\right)$$

$$=(z-z_0)^N\ g(z), \qquad g(z_0)\neq 0,$$ and $g(z)$ is analytic in A (it is represented by a converging Taylor series).

Since $g(z)$ is analytic it is also continuous. Since $g(z_0)\neq 0$, $g(z)$ *cannot vanish within a sufficiently small neighborhood of* z_0. That is, the zeros of $f(z)$ must be *isolated*.

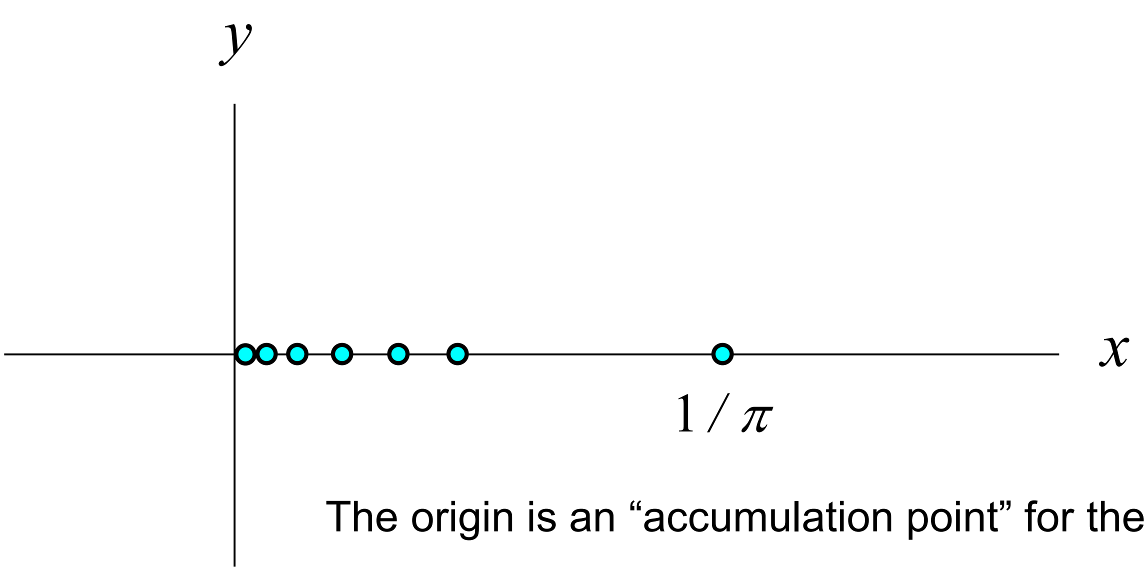
The only exception is if the function $f(z)$ is identically zero in a neighborhood of z_0 $(N=\infty)$.

The Zeros of an Analytic Function are Isolated

Example

The function $f(z) = \sin(1/z)$ has zeros at $z = \dfrac{1}{n\pi}, \; n = 1, 2, \cdots$

This function cannot be analytic at $z = 0$ since the zeros accumulate there and hence are not isolated there.



The origin is an “accumulation point” for the zeros.

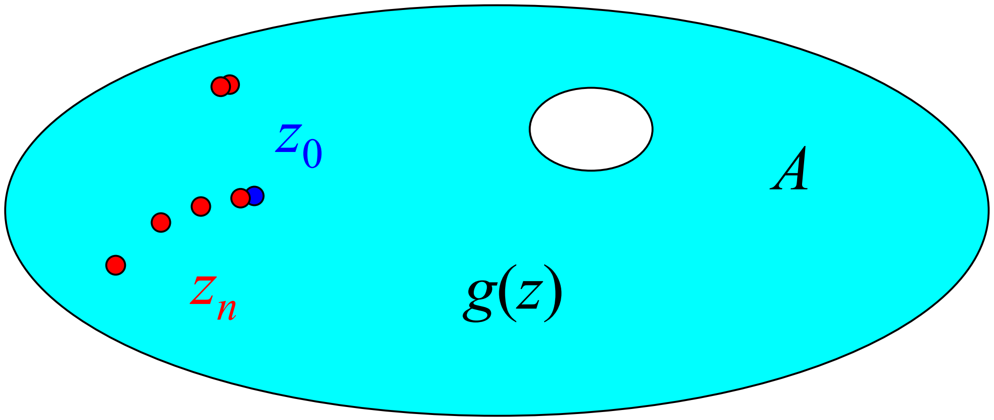
Analytic Continuation Principle

Theorem of analytic continuation:

Assume that $f(z)$ and $g(z)$ are analytic in a connected region A, and $f(z_n) = g(z_n)$ on a set of points z_n in A that converge to a point z_0 in A.

Then $g(z) = f(z)$ in A.

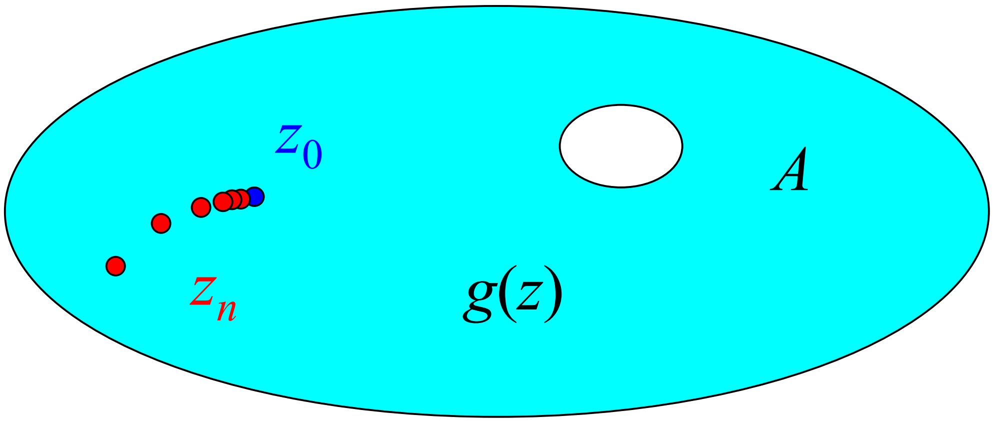
In other words, there can be only one function that is analytic in A and has a defined set of values at the converging points z_n.



Note:
If f and g agree on a contour inside A or in a region that is inside of A, this will make the functions agree everywhere in A.

Analytic Continuation Principle

Proof of theorem



Proof:

- Construct the <u>difference</u> function $F(z) = f(z)\text{-}g(z)$, which in analytic in A. This function must have a Taylor series at z_0.

- This Taylor series for $F(z)$ has all zero coefficients and thus is zero in its region of convergence about z_0; otherwise, the function must have isolated zeros – which it does not, by assumption.

- By continuing ("leapfrogging") the Taylor series that has zero coefficients (analytic continuation), the difference function must be zero throughout A.

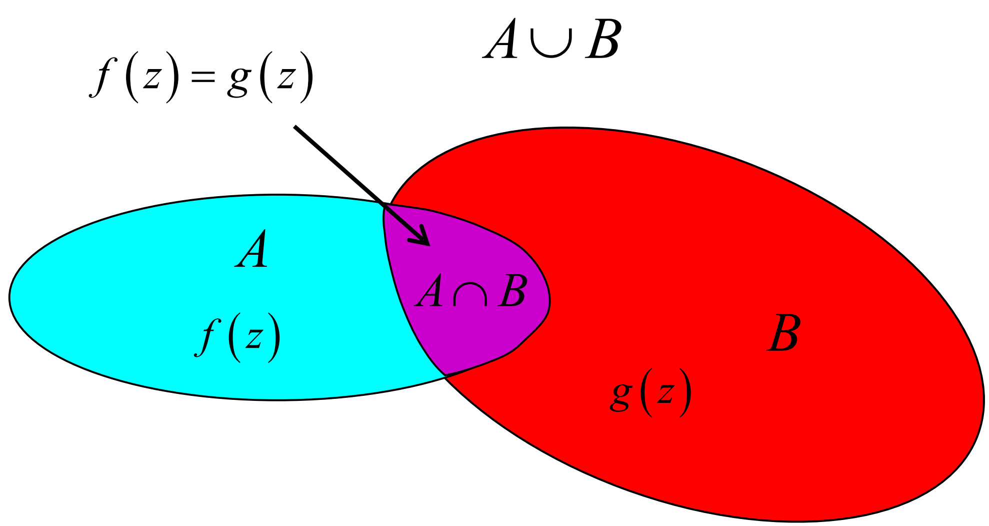
Analytic Continuation Principle (cont.)

Corollary (extending a domain from A to $A \cup B$)

Assume that $f(z)$ is analytic in A and $g(z)$ is analytic in B, and the two domains overlap in a region $A \cap B$, and $f(z) = g(z)$ in $A \cap B$.

Define $$h(z) = \begin{cases} f(z), & z \in A \\ g(z), & z \in B \end{cases}$$

Then $h(z)$ is the <u>only</u> analytic function in $A \cup B$ that equals $f(z)$ on A.



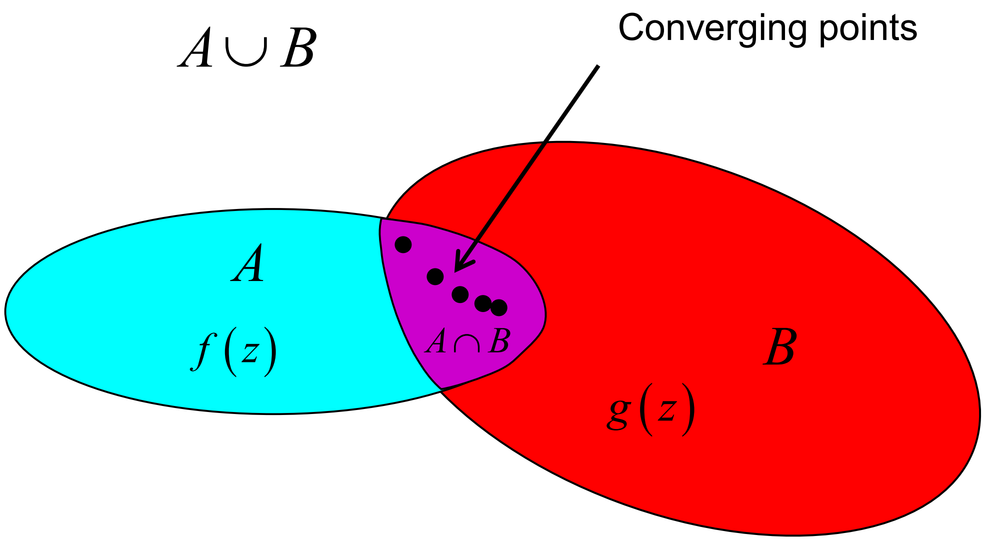
The function $h(z)$ <u>uniquely</u> extends the domain of $f(z)$ from A to $A \cup B$.

Analytic Continuation Principle (cont.)

Corollary: $h(z)$ is the only analytic function in $A \cup B$ that equals $f(z)$ on A.

Proof of corollary:

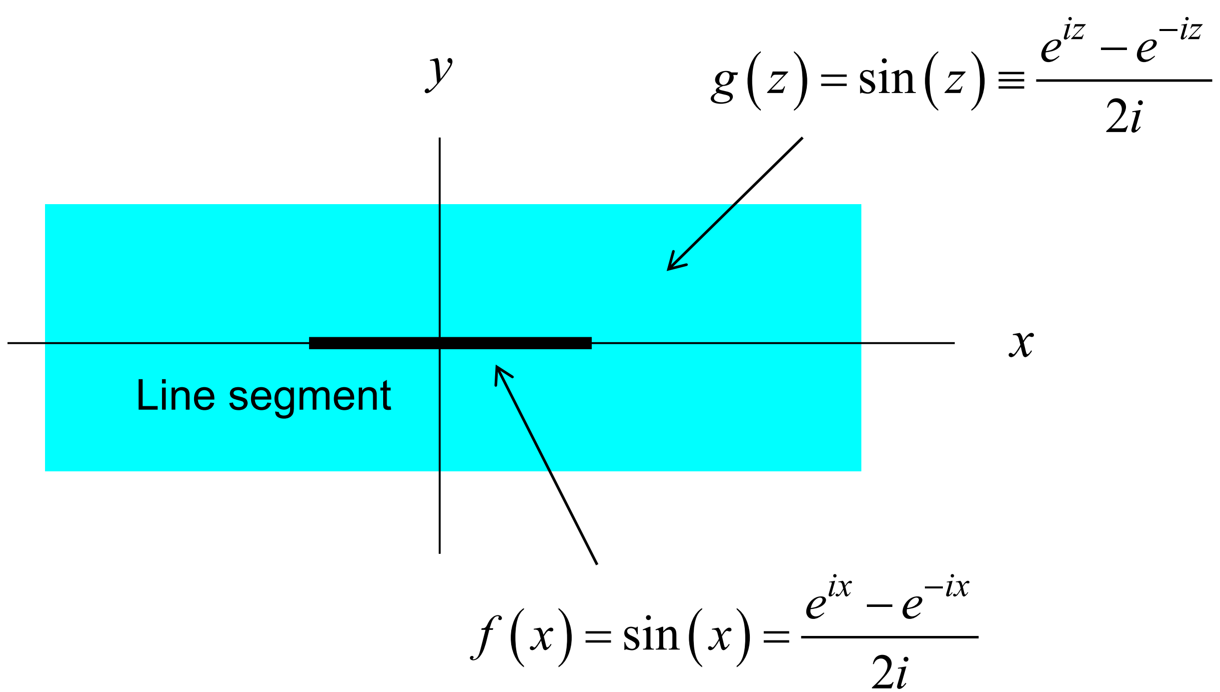
- The function h is analytic in the region $A \cup B$ and also equals $f(z_n)$ on any set of converging points in the intersection region.
- The theorem of analytic continuation thus ensures that h is unique in $A \cup B$.



Analytic Continuation Principle (cont.)

Example (example of theorem)

The function $\sin(x)$ is continued off the real axis.



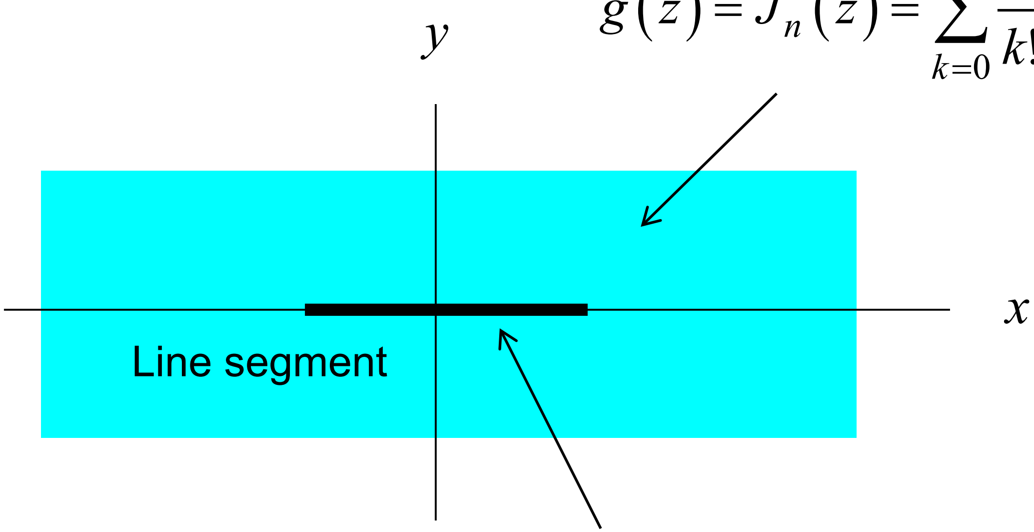
The function $g(z)$ is the <u>only one</u> that is analytic in the blue region of the complex plane and agrees with $\sin(x)$ on any segment of the real axis.

Analytic Continuation Principle (cont.)

Example (example of theorem)

The Bessel function $J_n(x)$ is continued off the real axis.

$$g(z) = J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{n+2k}$$



$$f(x) = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

The function $g(z)$ is the <u>only</u> one that is analytic in the blue region of the complex plane and agrees with $J_n(x)$ on any segment of the real axis.

Analytic Continuation Principle (cont.)

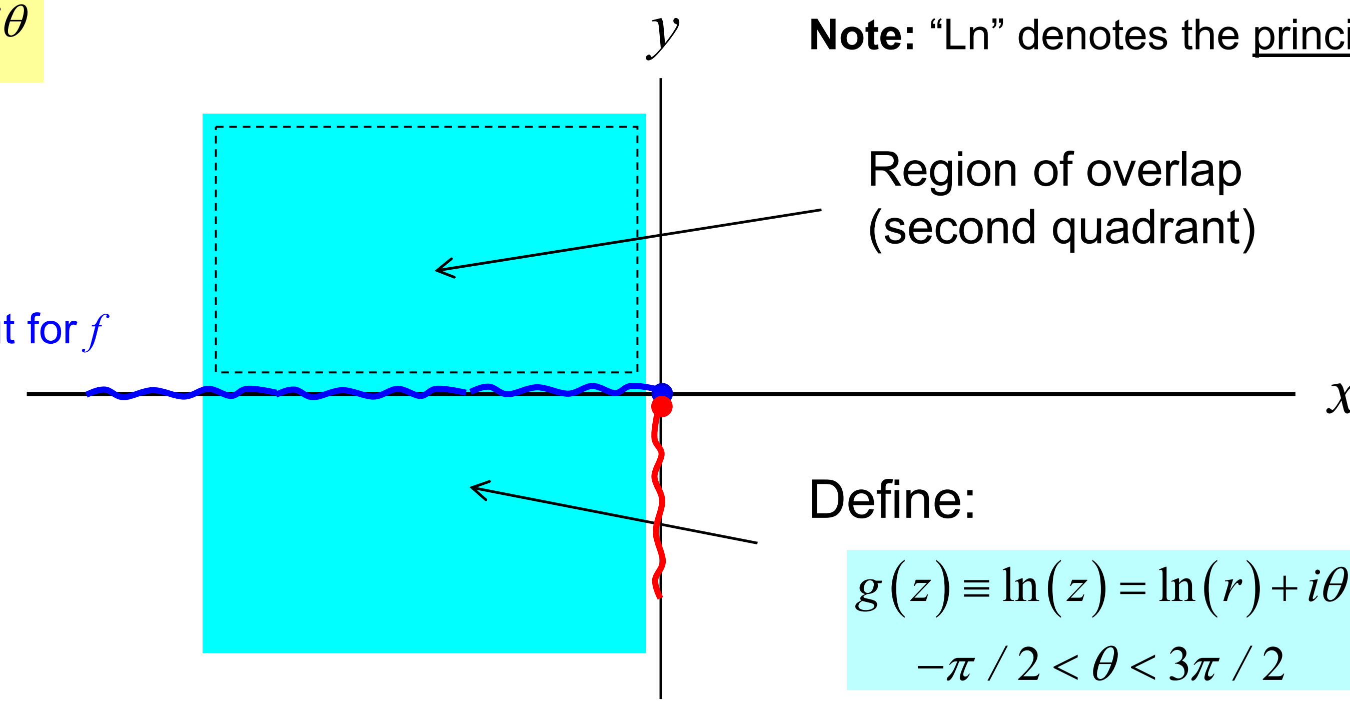
Example (of corollary)

The function $\mathrm{Ln}(z)$ (principal branch) is continued beyond a branch cut.

$$f(z) \equiv \mathrm{Ln}(z) = \ln(r) + i\theta, \qquad -\pi < \theta < \pi$$

$$z = re^{i\theta}$$

Note: "Ln" denotes the <u>principal</u> branch.



Define:

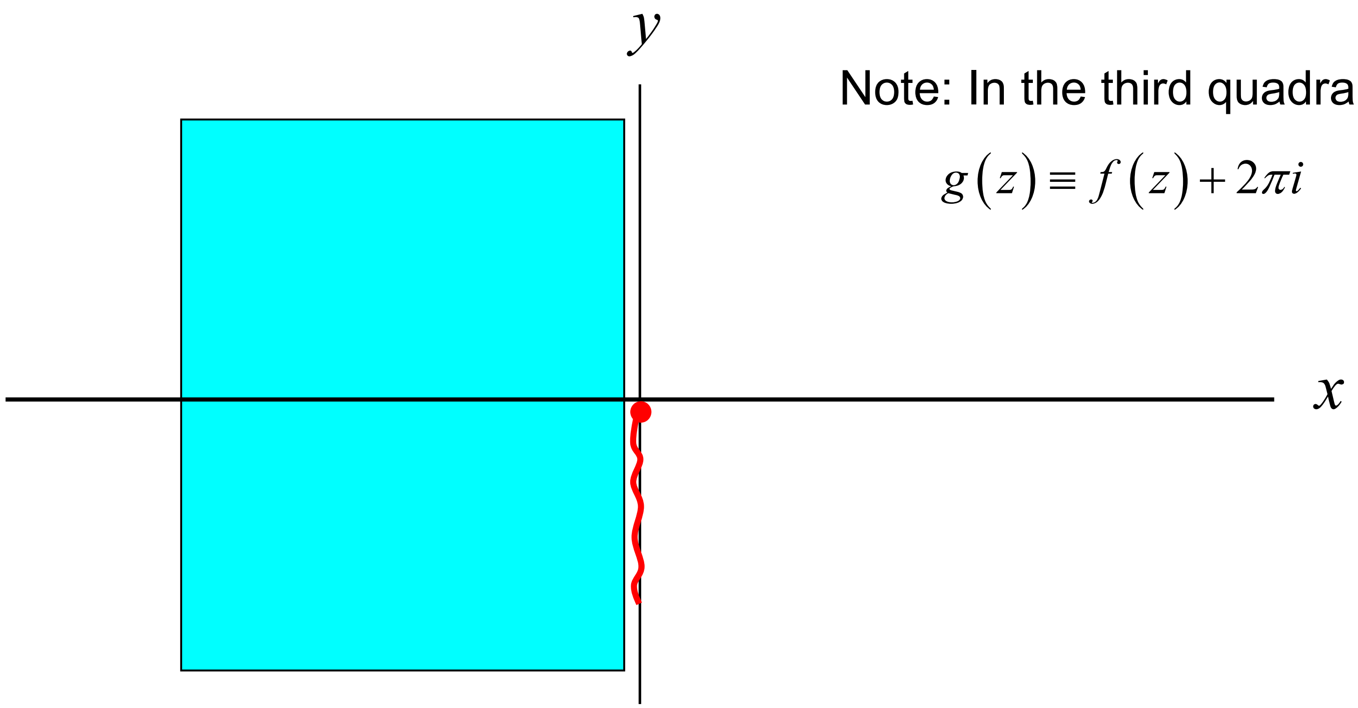
$$g(z) \equiv \ln(z) = \ln(r) + i\theta$$
$$-\pi/2 < \theta < 3\pi/2$$

In the blue region (second and third quadrants), $g(z)$ is analytic. Also, $g(z)$ agrees with the function $f(z)$ in the second quadrant. The original function $f(z)$ is not analytic in the entire blue region.

Analytic Continuation Principle (cont.)

Example (cont.)

The function $g(z)$ is the only function that is analytic in the entire left-half plane and agrees with $\mathrm{Ln}(z)$ in the second quadrant.



Note: In the third quadrant:

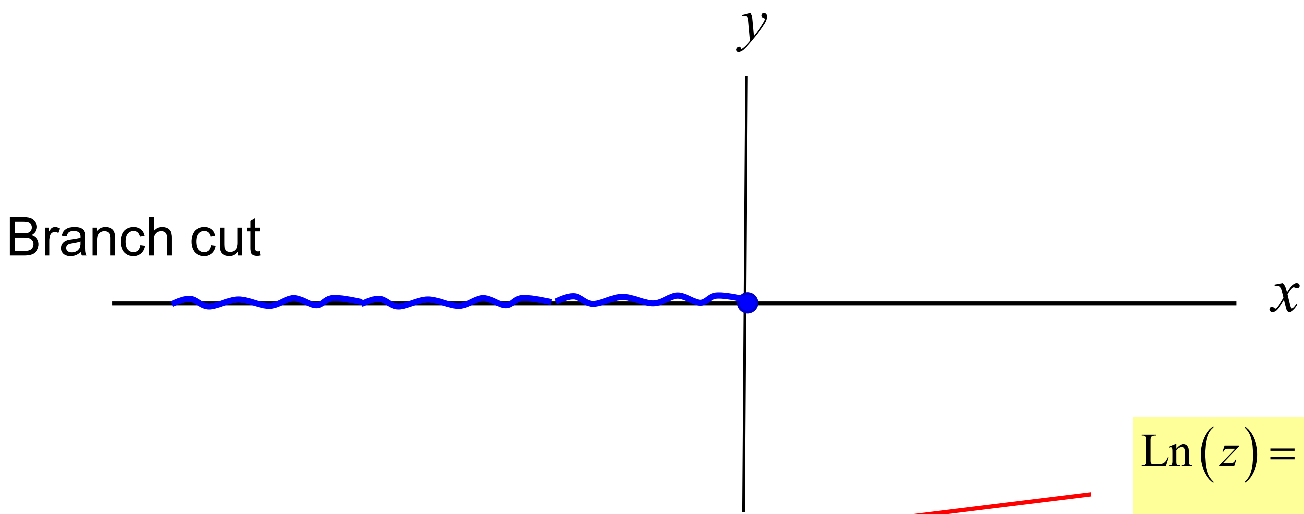
$$g(z) \equiv f(z) + 2\pi i$$

$$g(z) \equiv \ln(z) = \ln(r) + i\theta, \quad -\pi/2 < \theta < 3\pi/2$$

Analytic Continuation Principle (cont.)

Example (of corollary)

The function $Y_n(z)$ (Bessel function of the second kind) is continued beyond a branch cut.



$$\mathrm{Ln}(z) = \ln(r) + i\theta$$
$$-\pi < \theta < \pi$$

$$Y_n(z) = \frac{2}{\pi} J_n(z)\left[\mathrm{Ln}\left(\frac{z}{2}\right) + \gamma\right] - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\Phi(k) + \Phi(n+k)\right]\frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}$$

$\gamma \doteq 0.577216$

(Euler's constant)

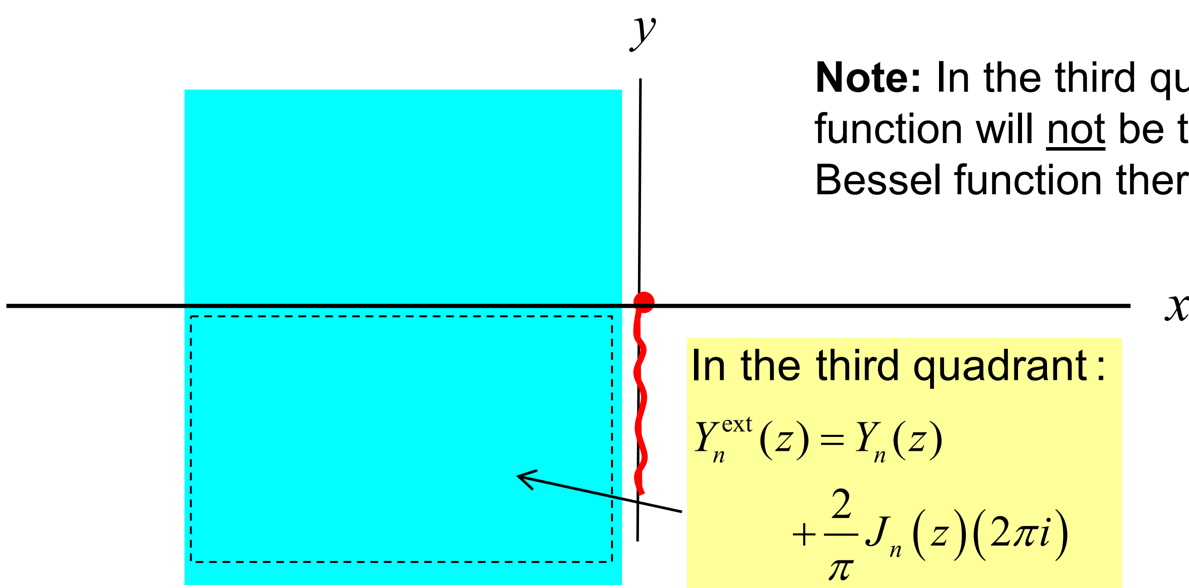
$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} \quad (p > 0)$$

Analytic Continuation Principle (cont.)

Example (cont.)

The “extended Bessel function” is analytic within the blue region.

Note: In the third quadrant, this “extended” function will <u>not</u> be the same as the usual Bessel function there.



In the third quadrant:

$$Y_n^{\text{ext}}(z) = Y_n(z) + \frac{2}{\pi} J_n(z)(2\pi i)$$

$$\ln(z) = \ln(r) + i\theta$$
$$-\pi/2 < \theta < 3\pi/2$$

$$Y_n^{\text{ext}}(z) \equiv \frac{2}{\pi} J_n(z)\left[\ln\left(\frac{z}{2}\right) + \gamma\right] - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\Phi(k) + \Phi(n+k)\right]\frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}$$

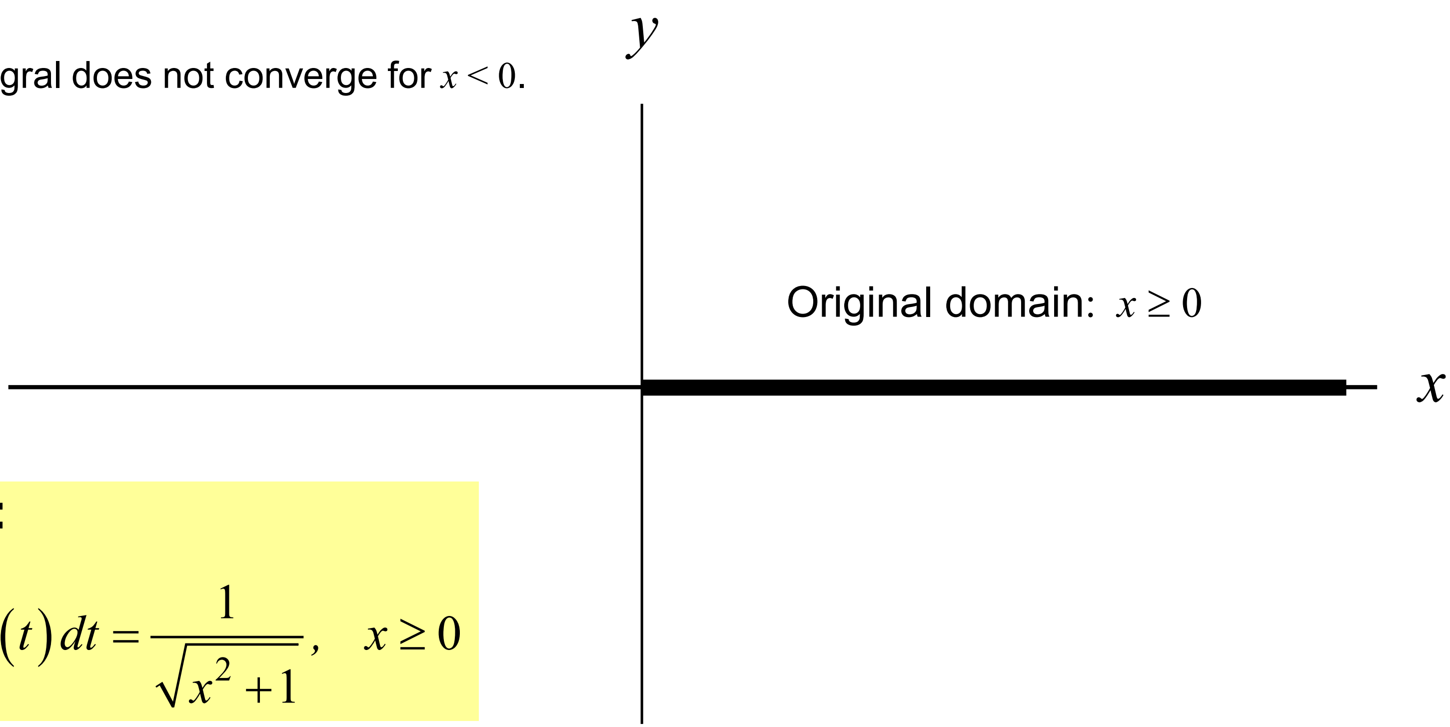
Analytic Continuation Principle (cont.)

Example

How do we extend $F(x)$ to arbitrary z?

$$F(x) \equiv \int_0^{\infty} e^{-xt} J_0(t)\, dt, \quad x \geq 0$$

Note: The integral does not converge for $x < 0$.



Identity :

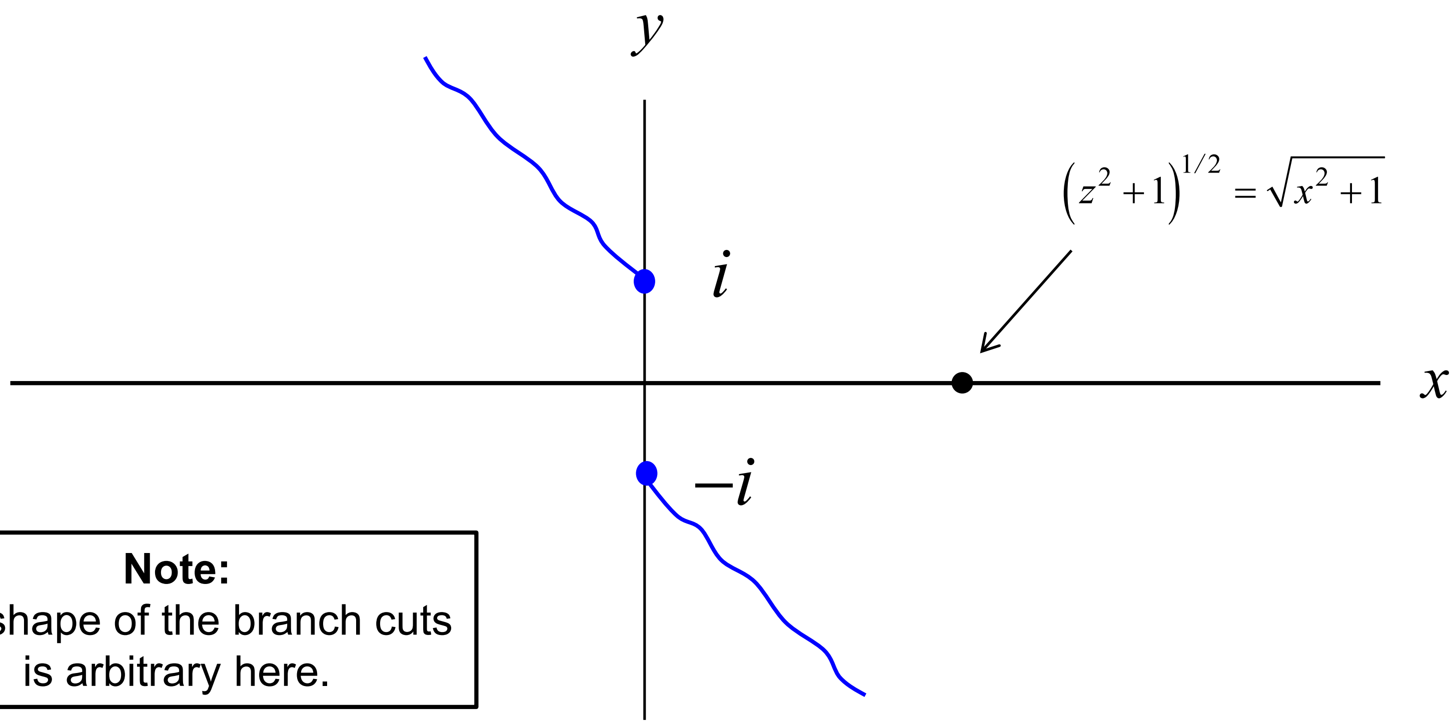
$$\int_0^{\infty} e^{-xt} J_0(t)\, dt = \frac{1}{\sqrt{x^2+1}}, \quad x \geq 0$$

Analytic Continuation Principle (cont.)

Example (cont.)

Here we define: $$F(z) \equiv \frac{1}{\left(z^2+1\right)^{1/2}}$$

This is defined <u>everywhere</u> in the complex plane (except on the branch cuts).



Note:
The shape of the branch cuts is arbitrary here.

Analytic Continuation Principle (cont.)

Example (cont.)

Here we define: $$F(z) \equiv \frac{1}{\sqrt{z^2+1}}$$

Note: $\operatorname{Re}\sqrt{z^2+1} \geq 0$

This corresponds to using vertical branch cuts.

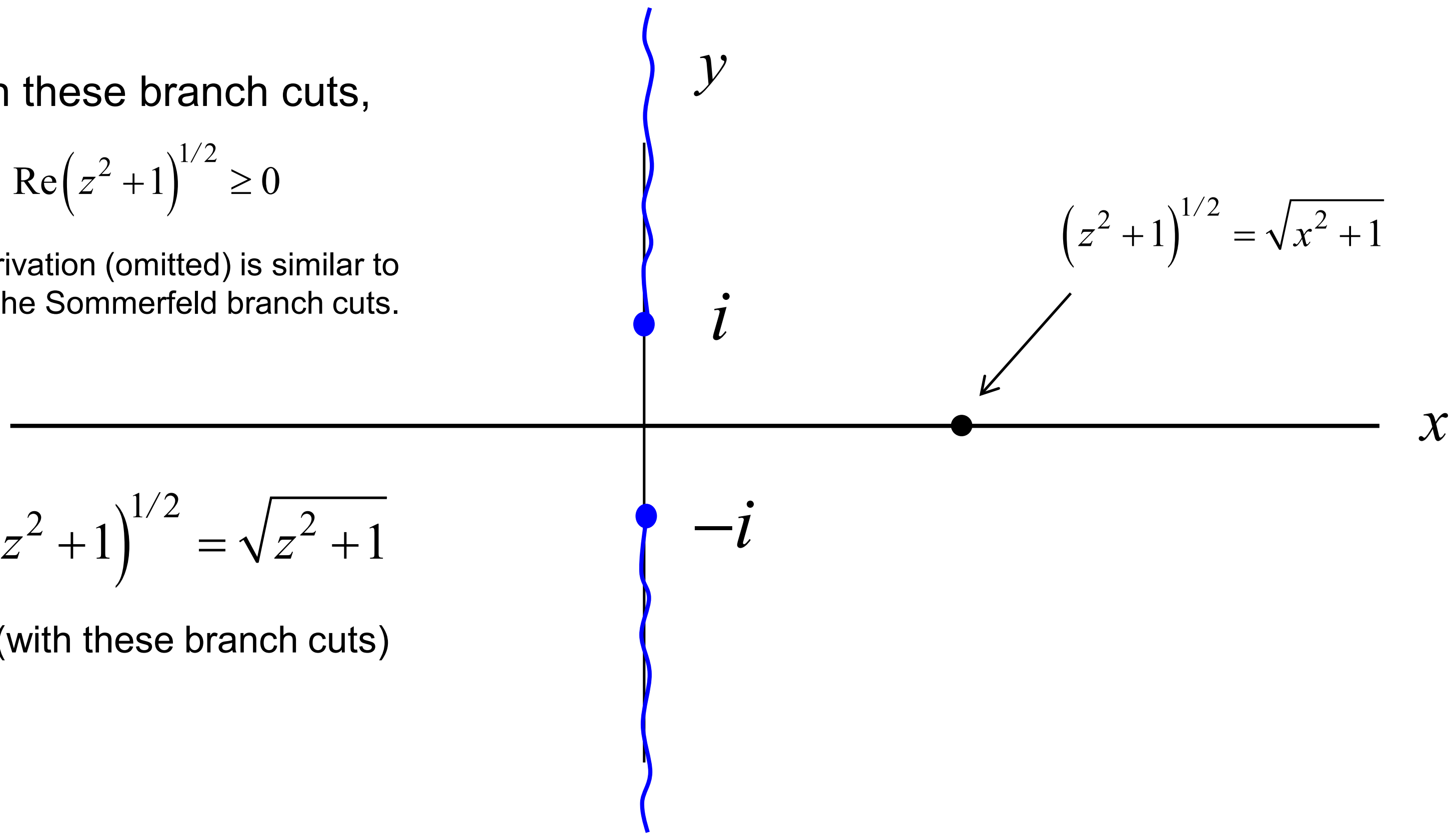
With these branch cuts,

$$\operatorname{Re}\left(z^2+1\right)^{1/2} \geq 0$$

The derivation (omitted) is similar to that of the Sommerfeld branch cuts.

$$\Rightarrow \left(z^2+1\right)^{1/2} = \sqrt{z^2+1}$$

(with these branch cuts)

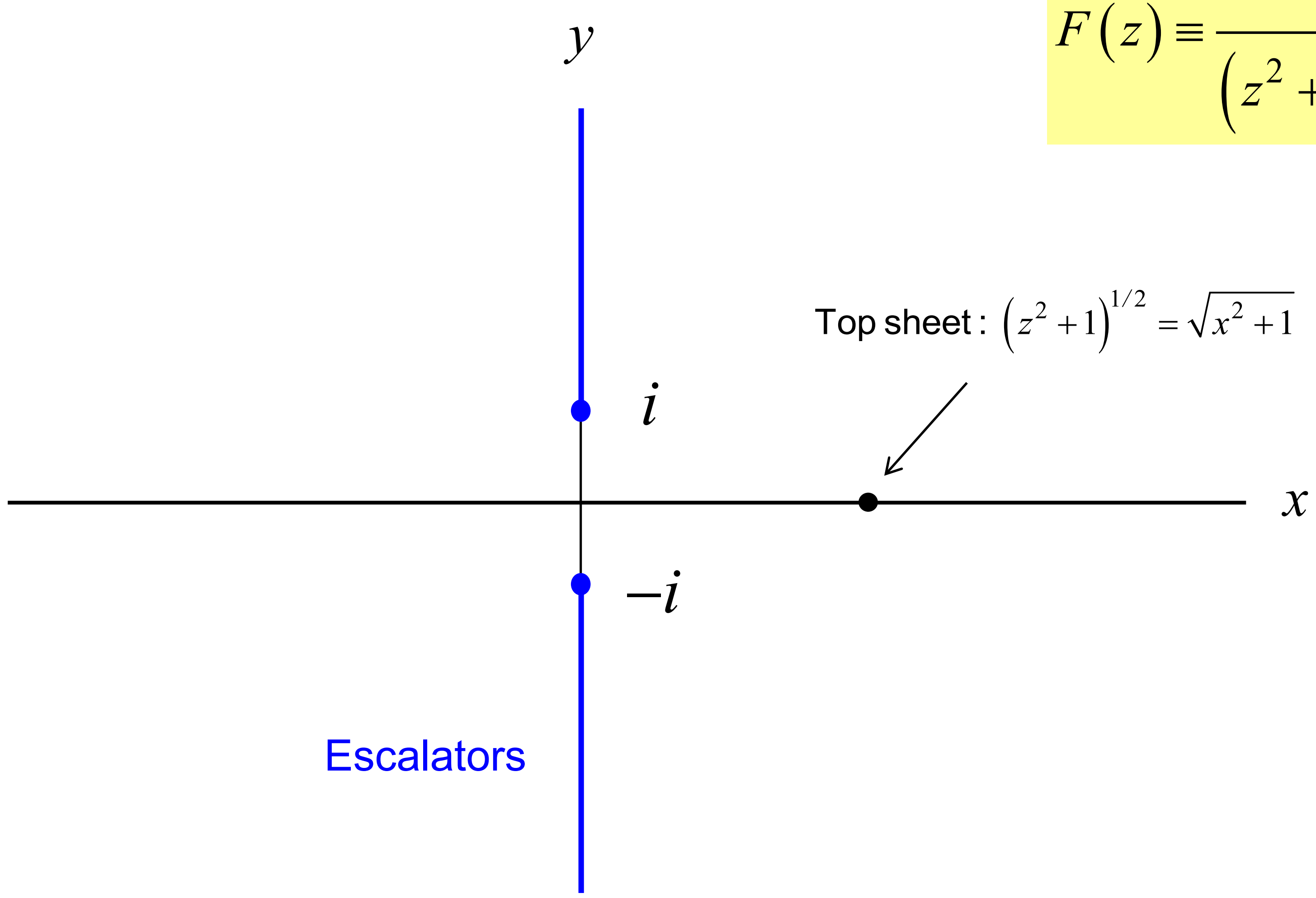


Analytic Continuation Principle (cont.)

Example (cont.)

Analytic continuation onto Riemann surface:

$$F(z) \equiv \frac{1}{\left(z^2+1\right)^{1/2}}$$



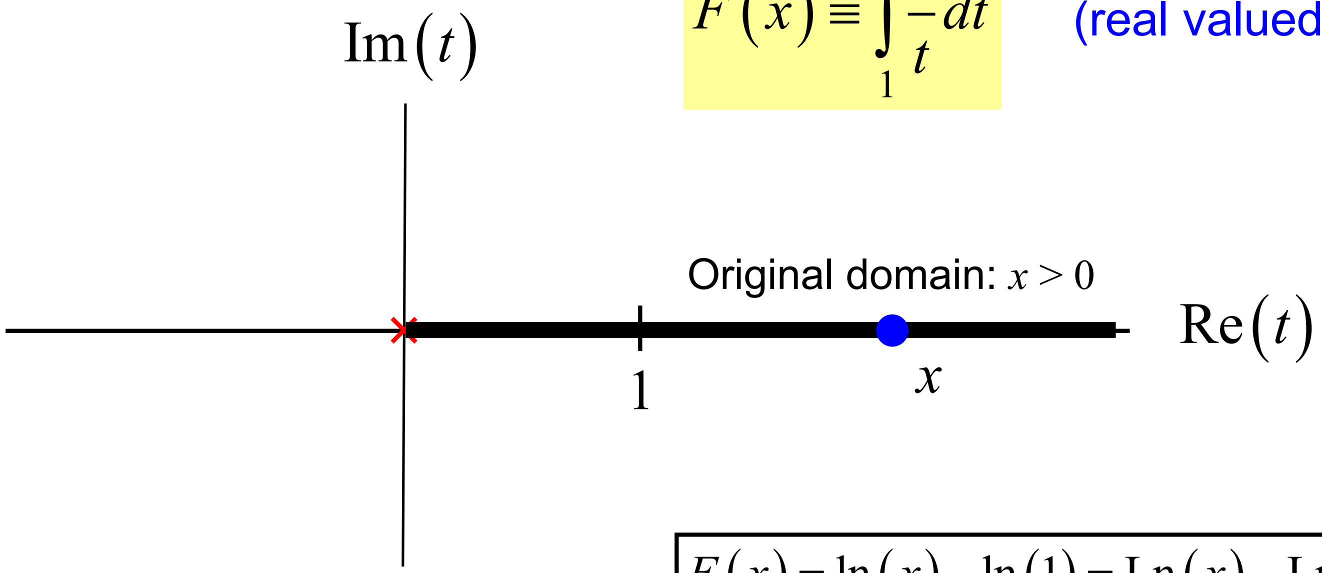
Analytic Continuation Principle (cont.)

Example

How do we extend $F(x)$ to arbitrary z?

$$F(x) \equiv \int_1^x \frac{1}{t}\,dt$$

(real valued for real $x > 0$)



$$\boxed{F(x) = \ln(x) - \ln(1) = \mathrm{Ln}(x) - \cancel{\mathrm{Ln}(1)}}$$

Recall: $\ln(z) = \mathrm{Ln}(z) + i(2\pi n)$

Note: $\mathrm{Ln}(z)$ means $-\pi < \arg(z) < \pi$

$$\Rightarrow \quad F(x) = \mathrm{Ln}(x)$$

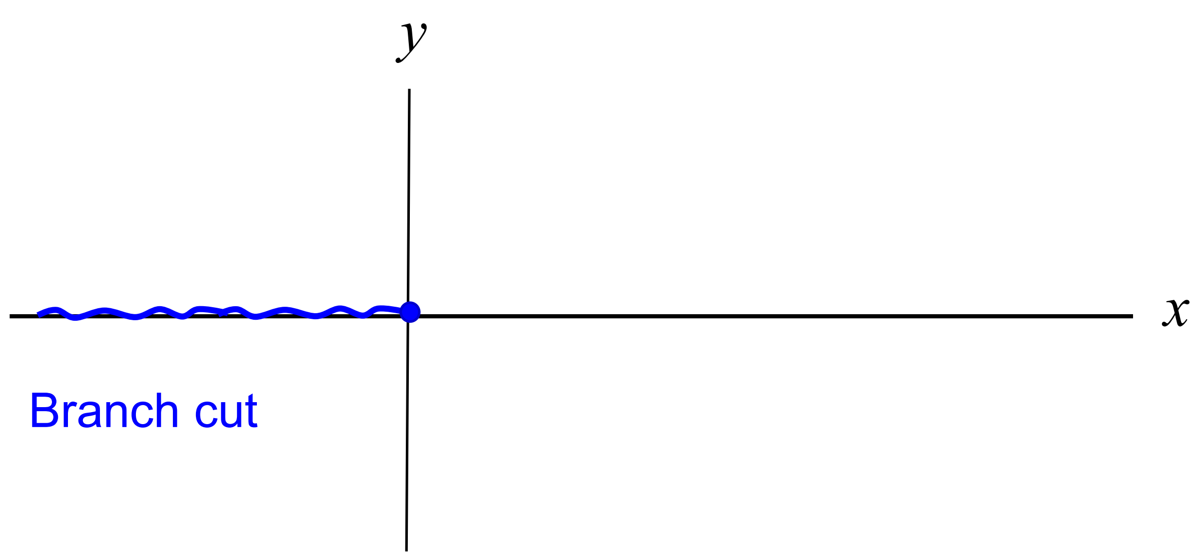
Analytic Continuation Principle (cont.)

Example (cont.)

Analytic continuation:

$$F(z) \equiv \mathrm{Ln}(z)$$

(This agrees with $F(x)$ on the real axis.)

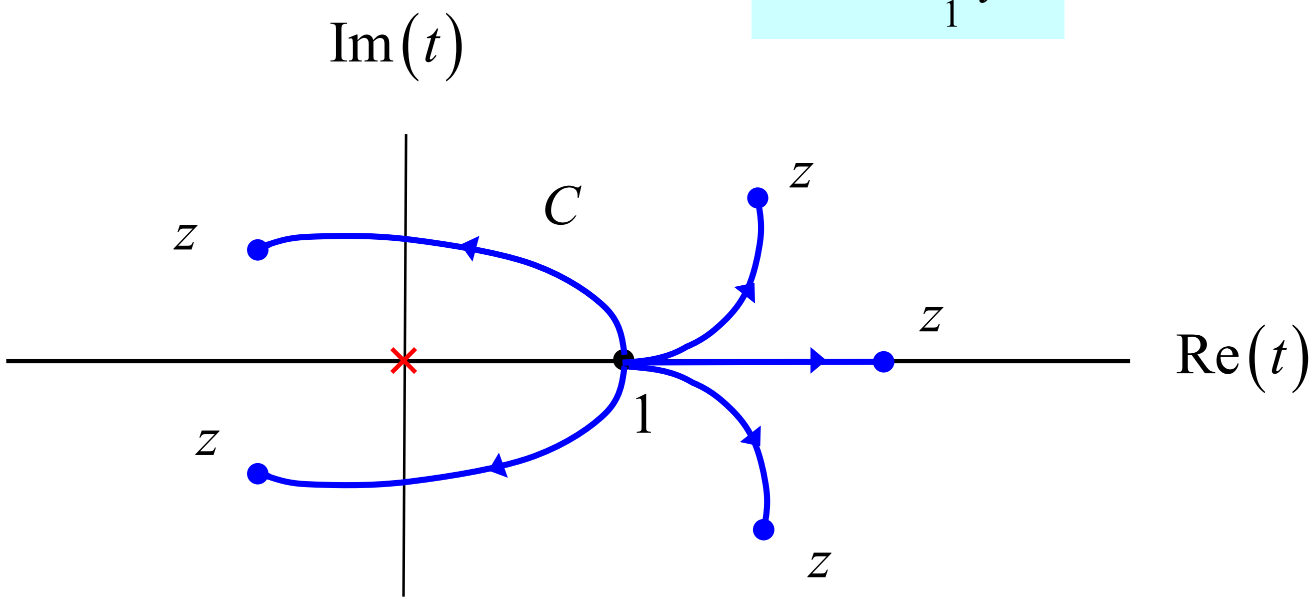


Analytic Continuation Principle (cont.)

Example (cont.)

From another point of view:

$$F(z) \equiv \int_1^z \frac{1}{t}\,dt$$



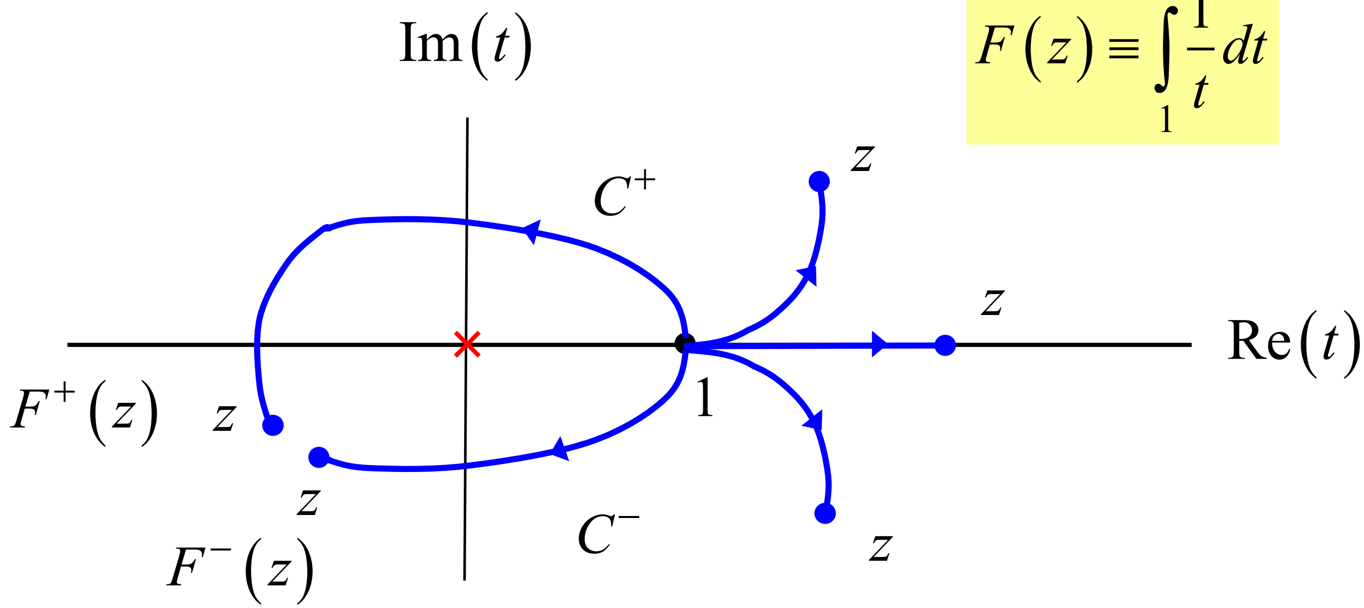
We analytically continue $F(z)$ from the real axis.

We require that the path is varied continuously as z leaves the real axis.

Analytic Continuation Principle (cont.)

Example (cont.)

$$F(z) \equiv \int_1^z \frac{1}{t}\,dt$$



As we encircle the pole at the origin, we get a different result for the function $F(z)$.

$$\oint_C \frac{1}{z}\,dz = = 2\pi i \quad \Rightarrow \quad \int_{C^+} \frac{1}{z}\,dz - \int_{C^-} \frac{1}{z}\,dz = F^+(z) - F^-(z) = 2\pi i$$

This is why we need a branch cut in the z plane, with a branch point at $z = 0$.

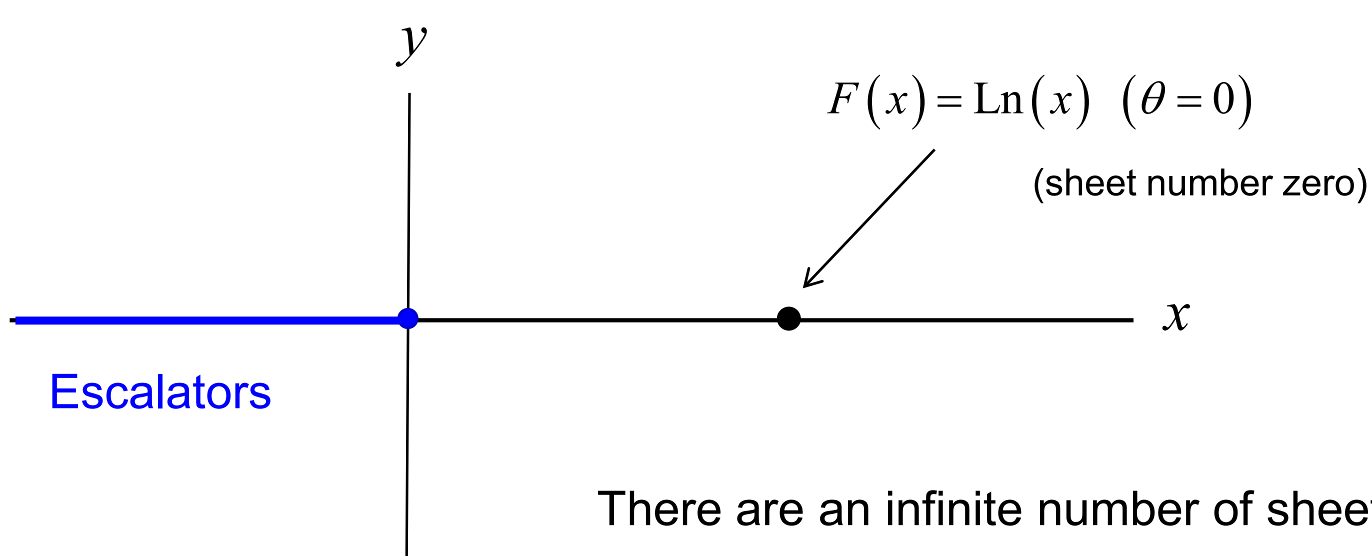
Analytic Continuation Principle (cont.)

Example (cont.)

Analytic continuation onto Riemann surface:

$$F(z) \equiv \ln(z)$$

(The function is defined so that $F(z)$ agrees with $F(x)$ on the real axis.)

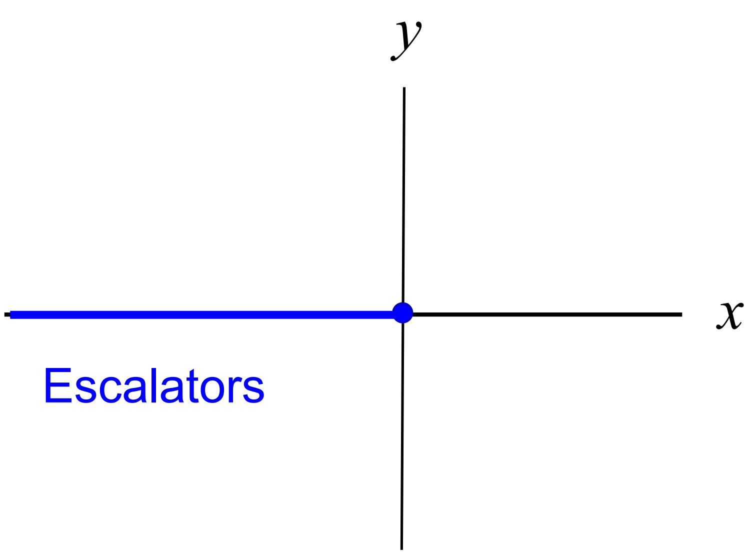


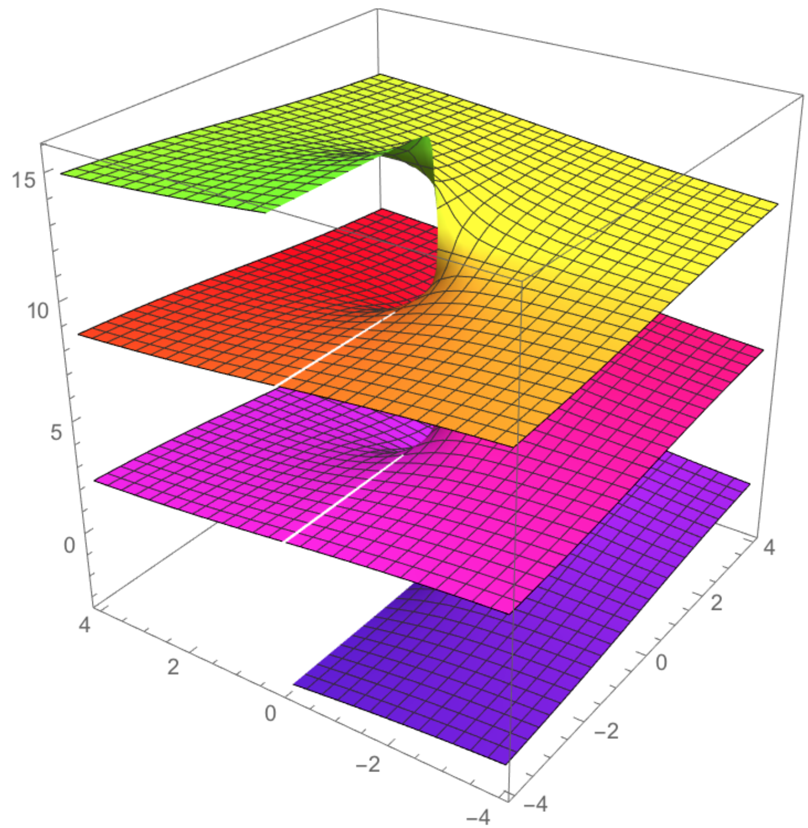
There are an infinite number of sheets!

Analytic Continuation Principle (cont.)

Example (cont.)

$$F(z) \equiv \ln(z)$$





Analytic Continuation Principle (cont.)

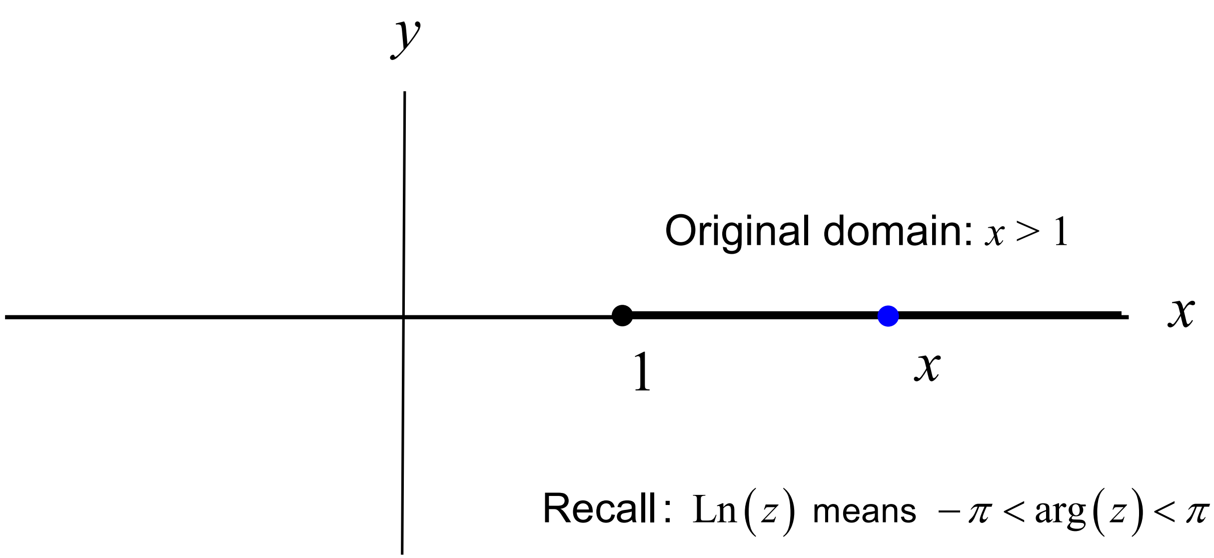
Example

How do we interpret $\cosh^{-1}(z)$ for arbitrary z?

$$\cosh^{-1}(x) = \mathrm{Ln}\left(x + \sqrt{x^2 - 1}\right), \quad x > 1$$

Note : $\cosh^{-1}(1) = 0$

Note : $\mathrm{Ln}\left(x + \sqrt{x^2 - 1}\right)$ is real (and positive) for $x > 1$



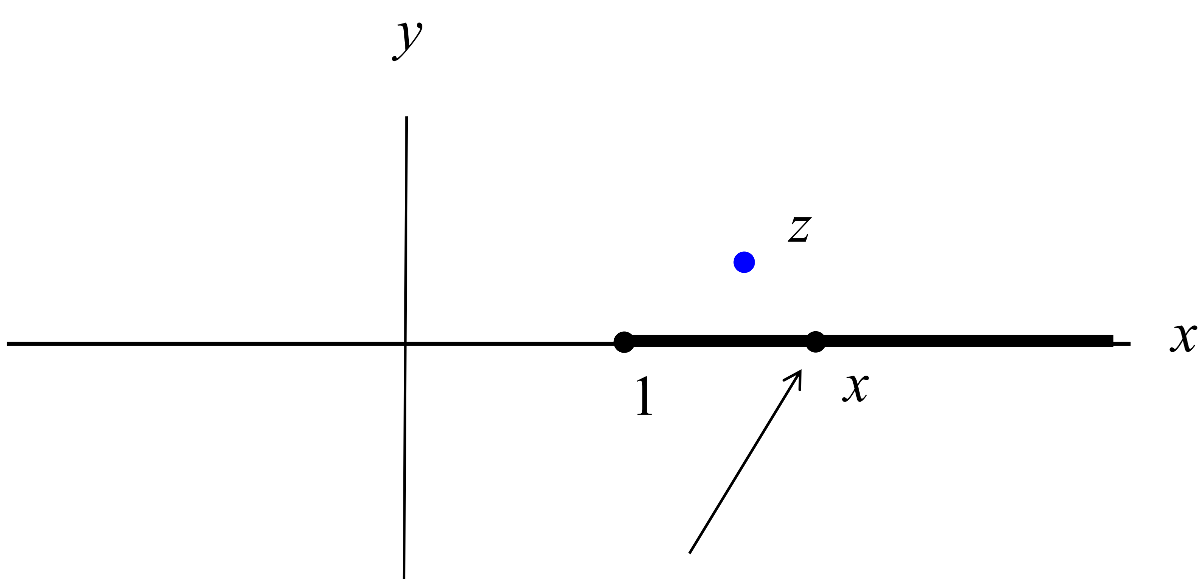
Recall : $\mathrm{Ln}(z)$ means $-\pi < \arg(z) < \pi$

Analytic Continuation Principle (cont.)

Example (cont.)

Analytic continuation:

$$\cosh^{-1}(z) \equiv \mathrm{Ln}\left(z + \sqrt{z^2 - 1}\right)$$



$$\cosh^{-1}(x) = \mathrm{Ln}\left(x + \sqrt{x^2 - 1}\right)$$

Analytic Continuation Principle (cont.)

Example (cont.)

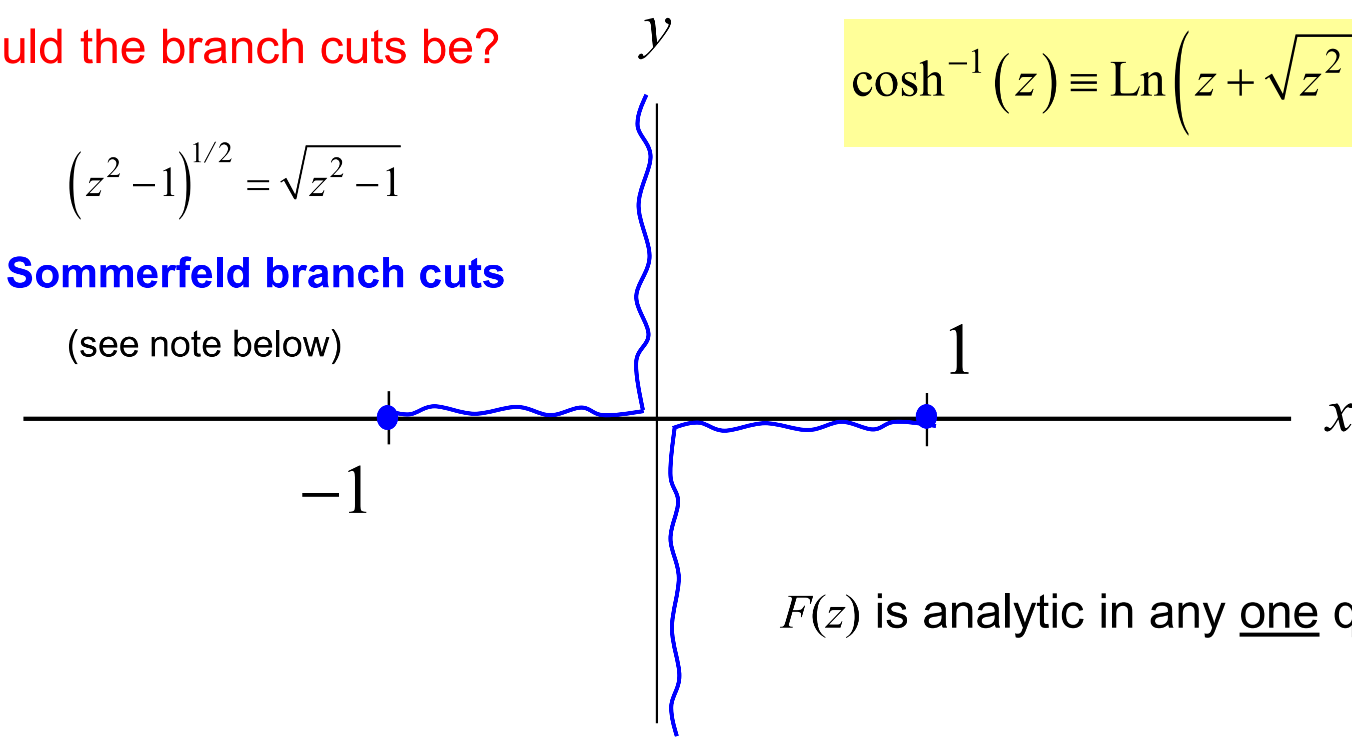
Where would the branch cuts be?

$$\cosh^{-1}(z) \equiv \mathrm{Ln}\left(z + \sqrt{z^2 - 1}\right)$$

$$\left(z^2 - 1\right)^{1/2} = \sqrt{z^2 - 1}$$

⇨ **Sommerfeld branch cuts**

(see note below)



$F(z)$ is analytic in any <u>one</u> quadrant.

Note:

$$f(z) \equiv \sqrt{z^2 - 1} \quad \Rightarrow \quad \mathrm{Re}\, f(z) > 0 \;\text{ for all } z \quad \Rightarrow \;\text{Sommerfeld branch cuts}$$

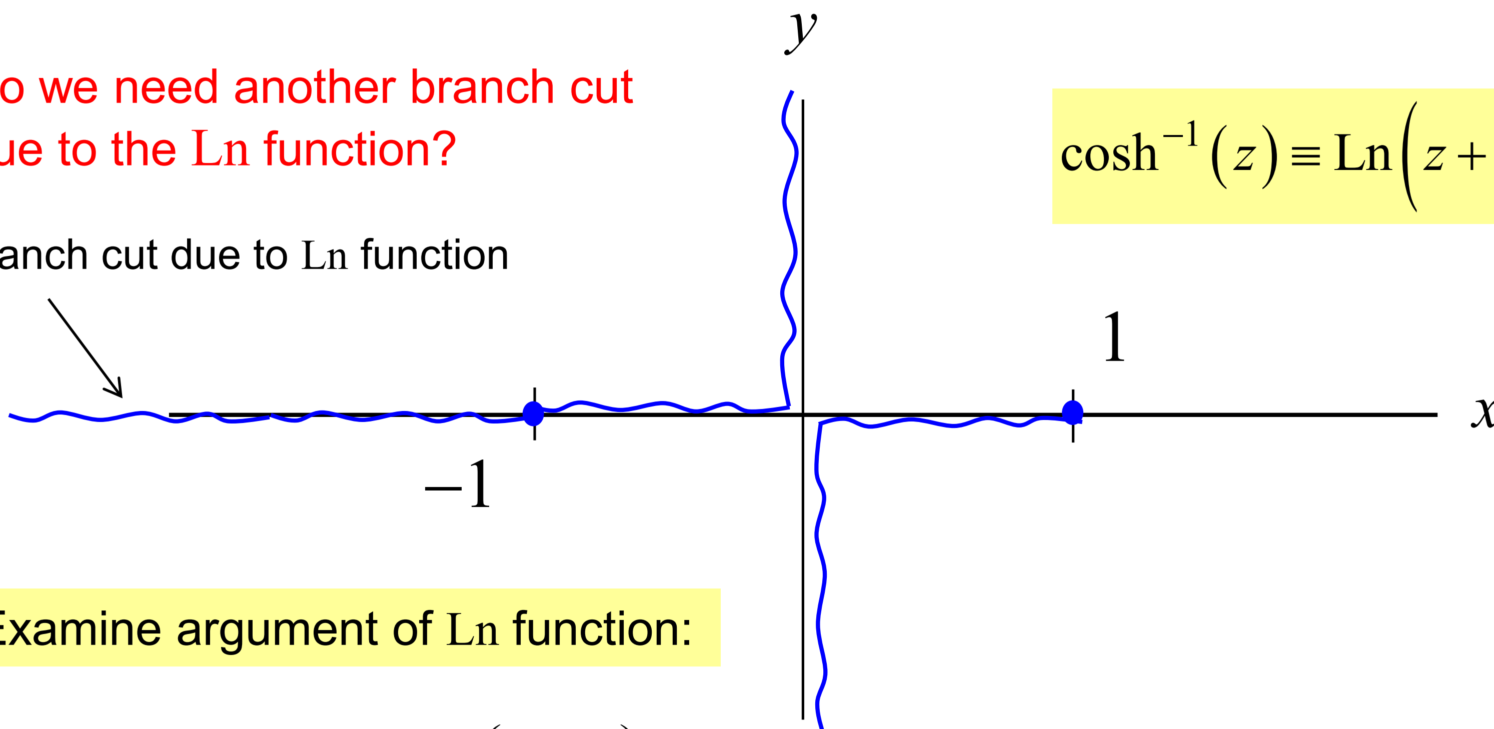
(**Recall:** $\mathrm{Re}\sqrt{w} > 0$)

Analytic Continuation Principle (cont.)

Example (cont.)

Do we need another branch cut due to the Ln function?

$$\cosh^{-1}(z) \equiv \mathrm{Ln}\left(z + \sqrt{z^2 - 1}\right)$$



Examine argument of Ln function:

$$w = z + \sqrt{z^2 - 1} \;\Rightarrow\; z = \frac{1}{2}\left(w + \frac{1}{w}\right) \quad \text{(derivation omitted)}$$

The branch cut for the $\mathrm{Ln}(w)$ function corresponds to w being a negative real number, i.e, $-\infty < w < 0$).

Note: $w \in (-\infty, 0) \;\Rightarrow\; z \in (-1, -\infty)$

$\uparrow$

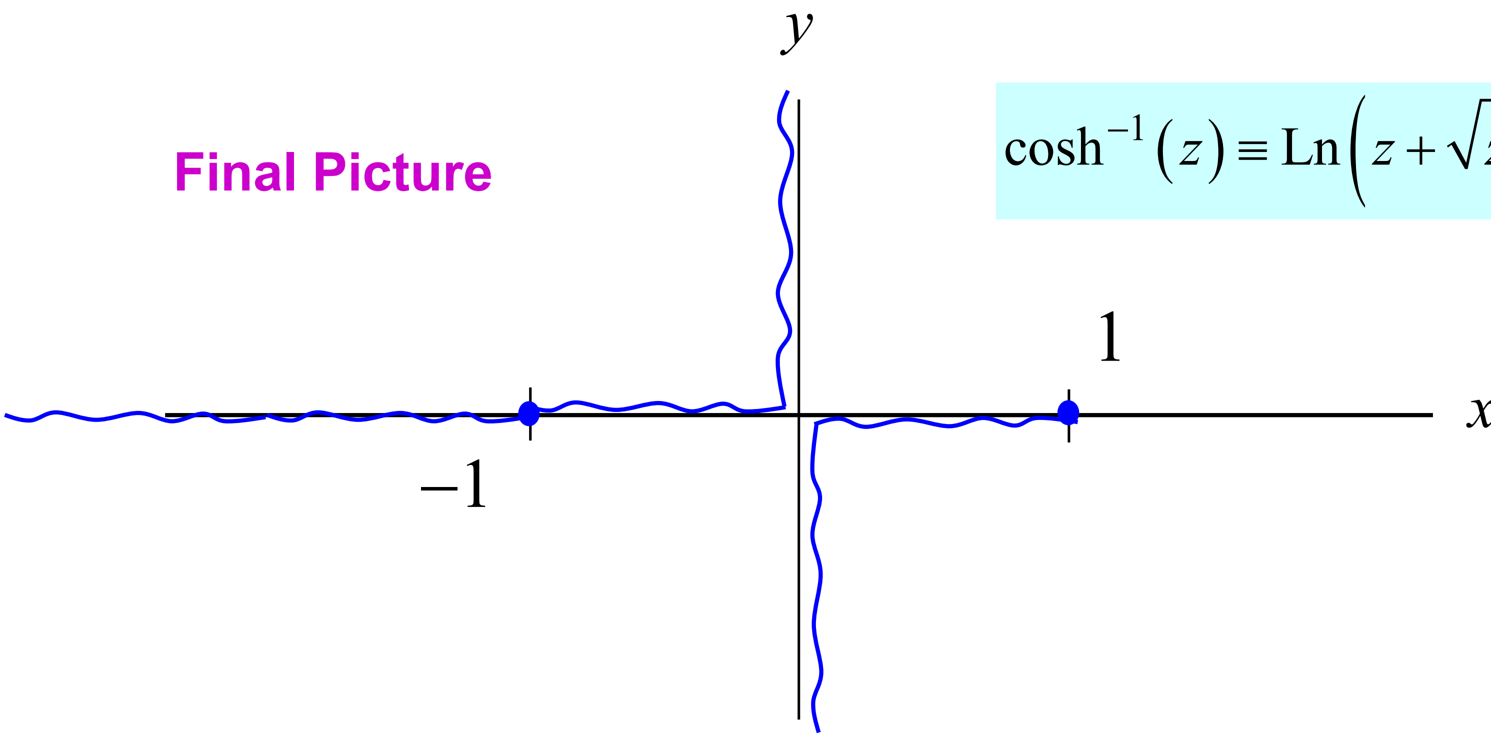
$w = -1$

Analytic Continuation Principle (cont.)

Example (cont.)

Final Picture

$$\cosh^{-1}(z) \equiv \mathrm{Ln}\left(z + \sqrt{z^2 - 1}\right)$$



$F(z)$ is an analytic continuation of $\cosh^{-1}(x)$ off of the real axis.

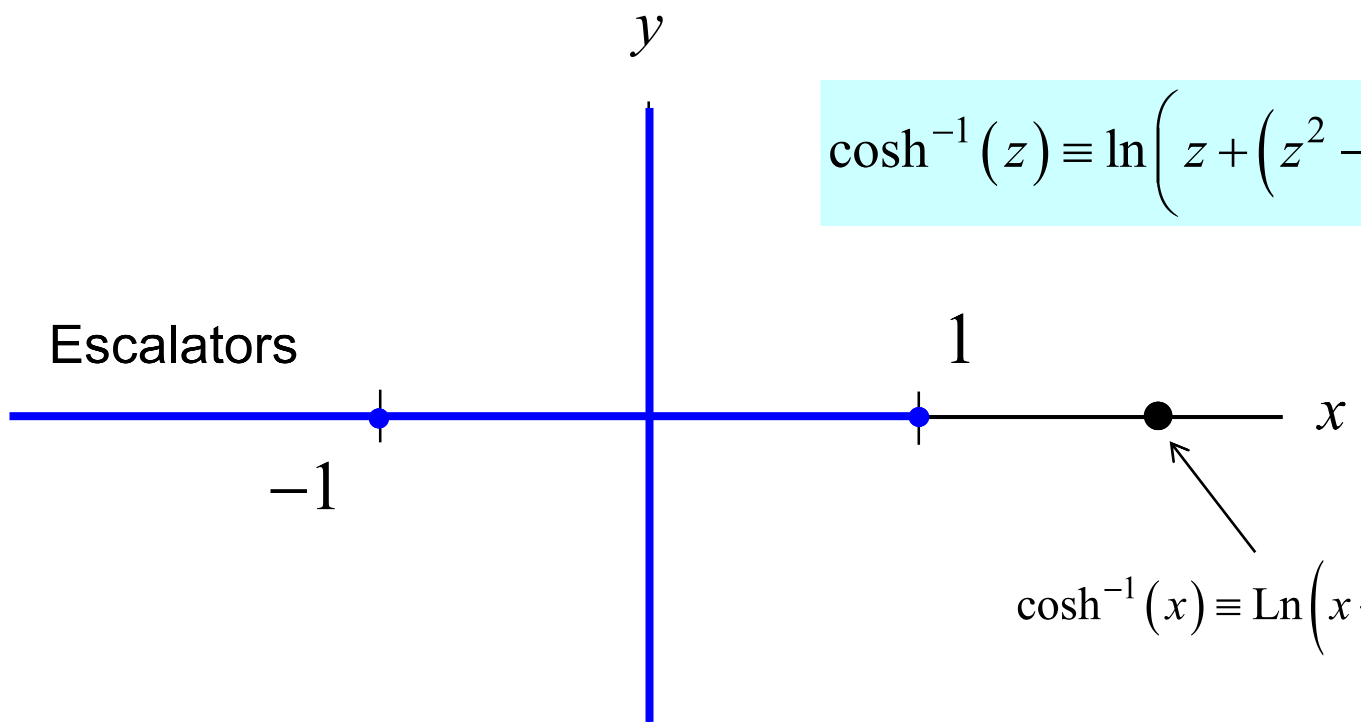
$F(z)$ is analytic in any <u>one</u> quadrant.

Analytic Continuation Principle (cont.)

Example (cont.)

Analytic continuation onto Riemann surface:

$$\cosh^{-1}(z) \equiv \ln\left(z + \left(z^2 - 1\right)^{1/2}\right)$$



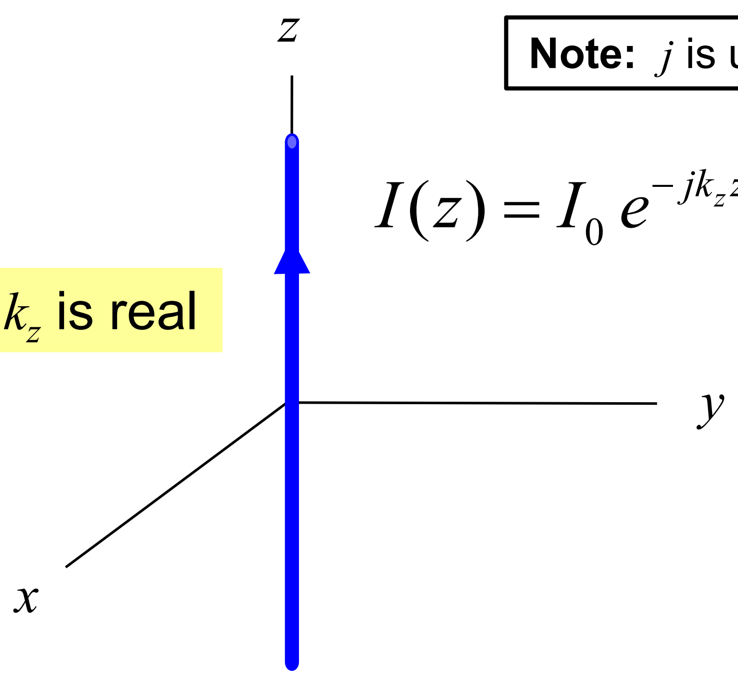
$$\cosh^{-1}(x) \equiv \mathrm{Ln}\left(x + \sqrt{x^2 - 1}\right)$$

There are an infinite number of sheets!

EM Example

Assume a radiating phased line source on the z axis.

Note: j is used here instead of i.



$$I(z) = I_0 \, e^{-jk_z z}$$

k_z is real

The magnetic vector potential is (ECE 6341):

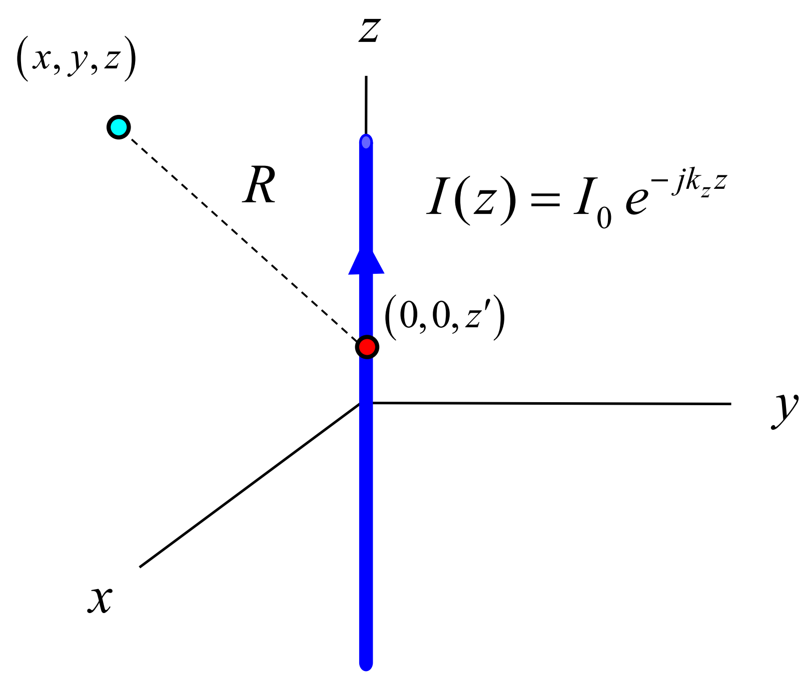
$$A_z = \left(\frac{\mu I_0}{4j}\right) H_0^{(2)}(k_\rho \rho) \, e^{-jk_z z}$$

$$k_\rho = \left(k_0^2 - k_z^2\right)^{1/2} = \begin{cases} \sqrt{k_0^2 - k_z^2}, & |k_z| < k_0 \\ -j\sqrt{k_z^2 - k_0^2}, & |k_z| > k_0 \end{cases}$$

EM Example (cont.)

We can also write (from ECE 6340):

$$A_z = \int_{-\infty}^{\infty} I_0\, e^{-jk_z z'} \left(\mu_0 \frac{e^{-jk_0 R}}{4\pi R} \right) dz'$$



Note: The integral converges for <u>real</u> k_z.

Hence

$$A_z = \int_{-\infty}^{\infty} I_0\, e^{-jk_z z'} \left(\mu_0 \frac{e^{-jk_0 R}}{4\pi R} \right) dz' = \left(\frac{\mu_0 I_0}{4j} \right) H_0^{(2)}(k_\rho \rho)\, e^{-jk_z z}$$

Exists only for real k_z Exists for complex k_z

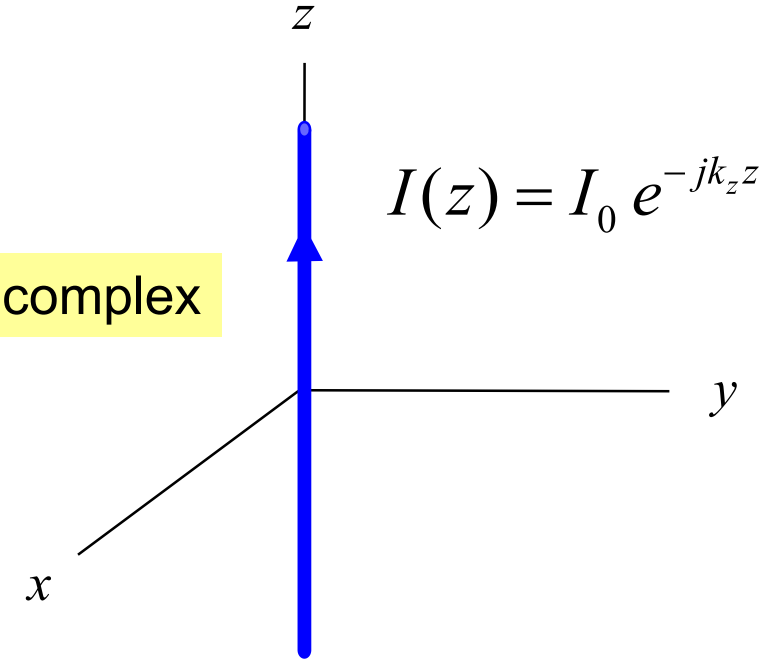
The second form is the <u>analytic continuation</u> of the first form off of the real axis.

EM Example (cont.)

$$A_z = \left(\frac{\mu_0 I_0}{4j}\right) H_0^{(2)}(k_\rho \rho)\, e^{-jk_z z}$$

$$k_\rho = \left(k_0^2 - k_z^2\right)^{1/2}$$
$$= -j\left(k_z^2 - k_0^2\right)^{1/2}$$

k_z is complex



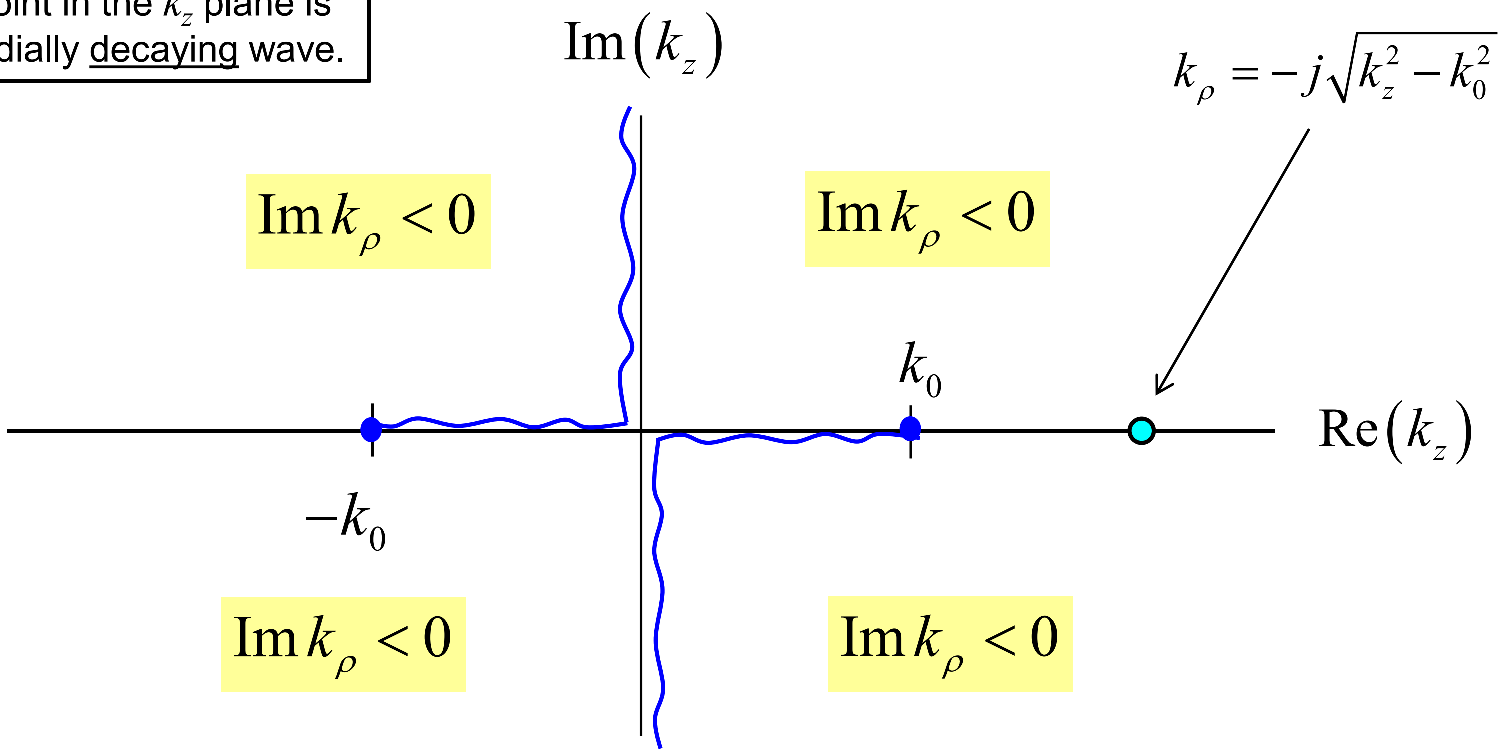
In order for this to be the analytic continuation off of the real axis of the integral form, we must chose the branch of the square root function <u>correctly</u> so that it changes smoothly and it is correct when k_z = real > k_0.

$k_\rho = -j\left(k_z^2 - k_0^2\right)^{1/2}$ = negative imaginary number for k_z = real > k_0

EM Example (cont.)

$$k_\rho = -j\left(k_z^2 - k_0^2\right)^{1/2}$$

Every point in the k_z plane is now a radially decaying wave.



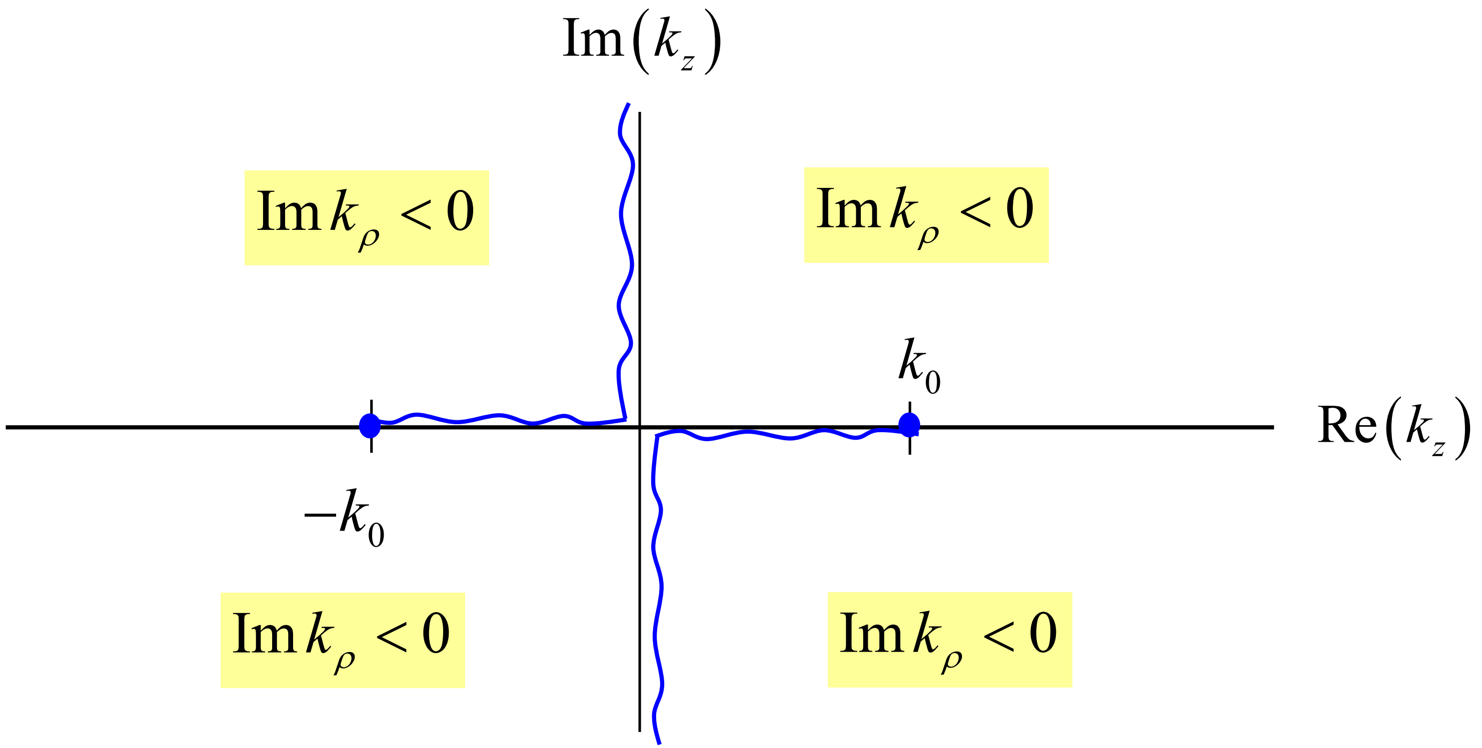
The Sommerfeld branch cuts are a convenient choice.

(but not necessary)

EM Example (cont.)

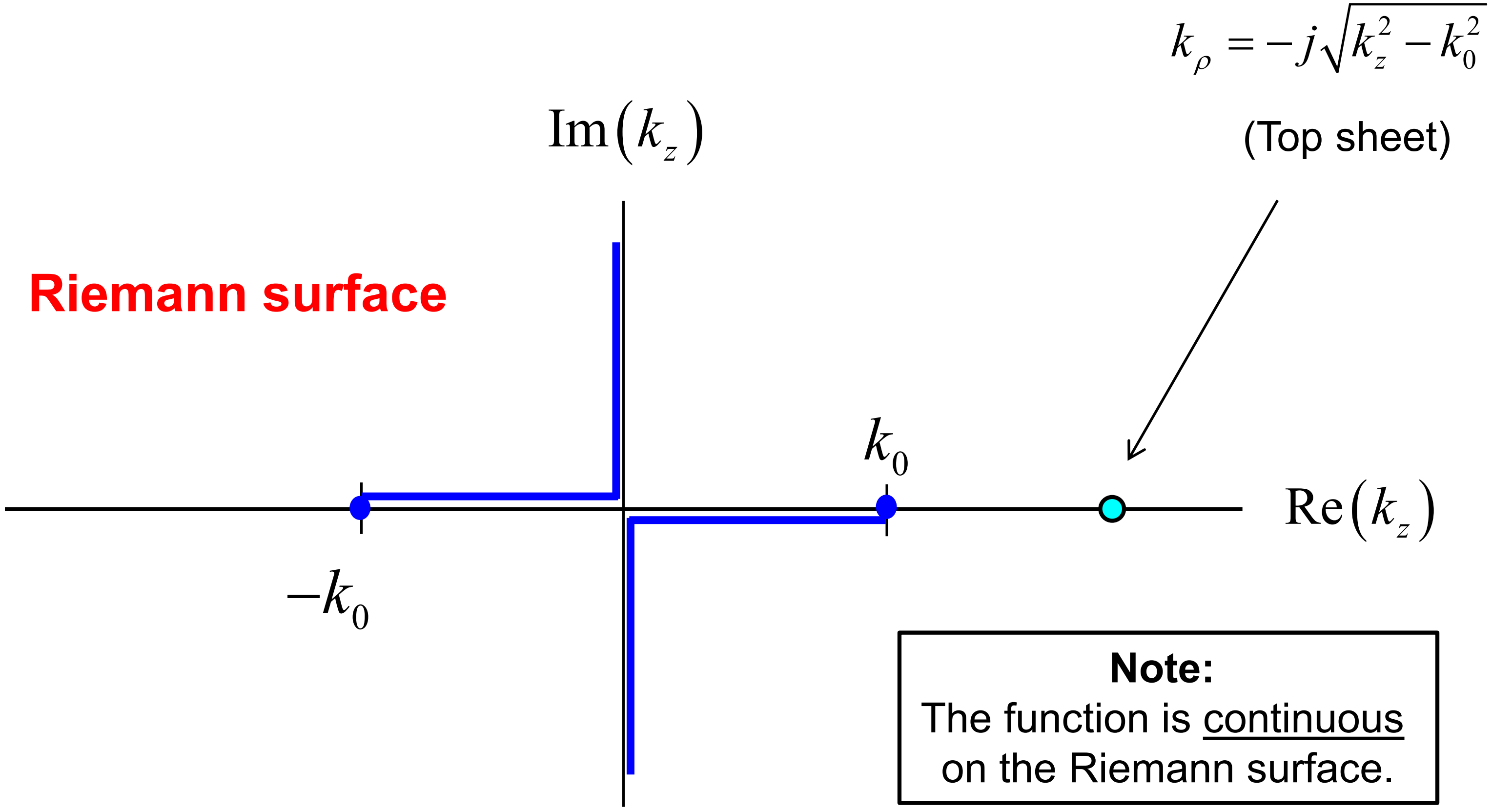
$$k_\rho = -j\sqrt{k_z^2 - k_0^2}$$

The use of the radical sign is equivalent to having Sommerfeld branch cuts.



EM Example (cont.)

$$k_\rho = -j\left(k_z^2 - k_0^2\right)^{1/2}$$



Note:
The function is <u>continuous</u> on the Riemann surface.

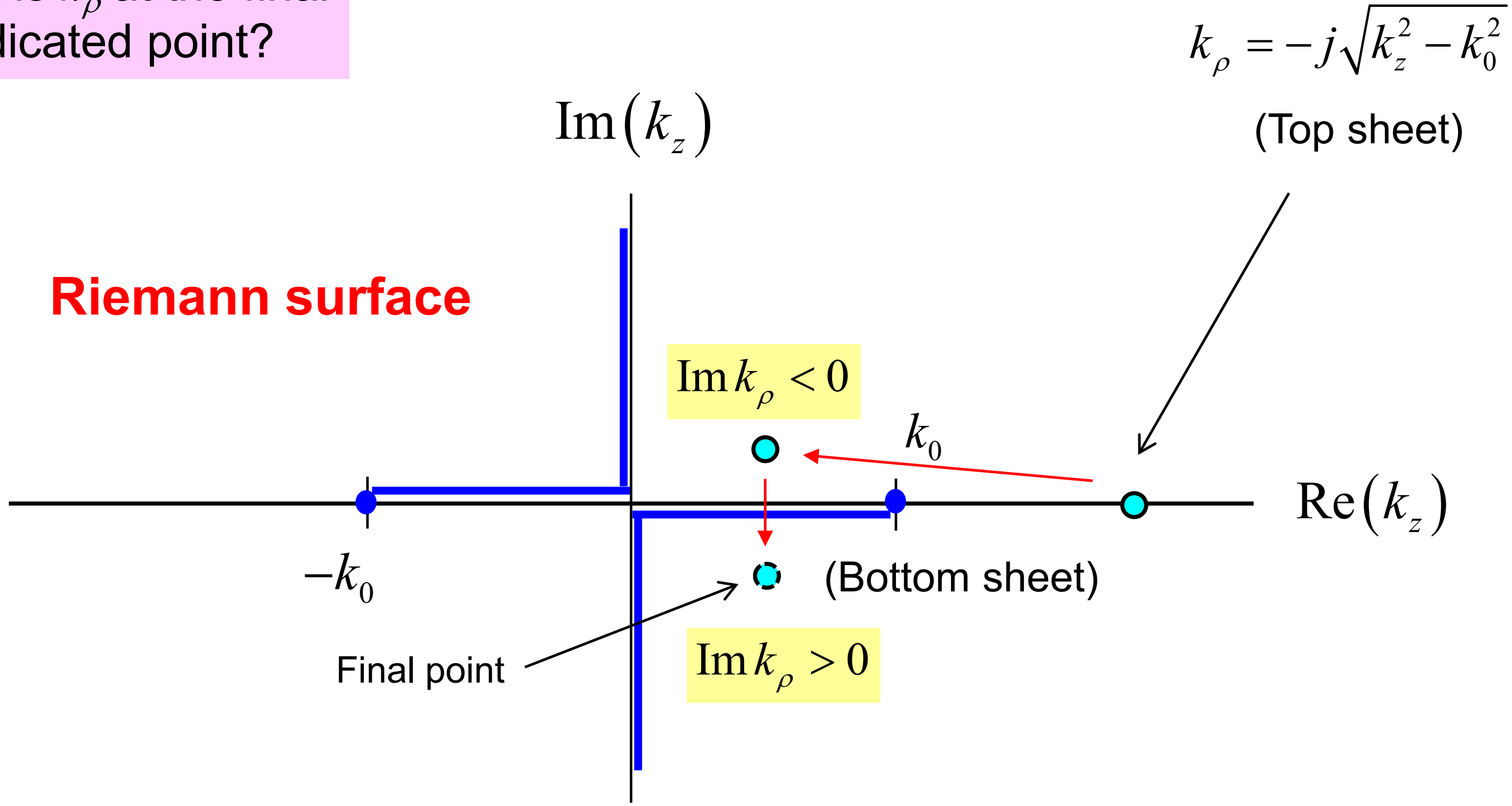
We can now let k_z wander anywhere we wish on the Riemann surface, and we know how to calculate the square root. (We analytically continue to the entire Riemann surface.)

EM Example (cont.)

Example

$$k_\rho = -j\left(k_z^2 - k_0^2\right)^{1/2}$$

What is k_ρ at the final indicated point?



At the indicated final point, the imaginary part of k_ρ is chosen to be positive.

$$\left(k_\rho = +j\sqrt{k_z^2 - k_0^2}\right)$$

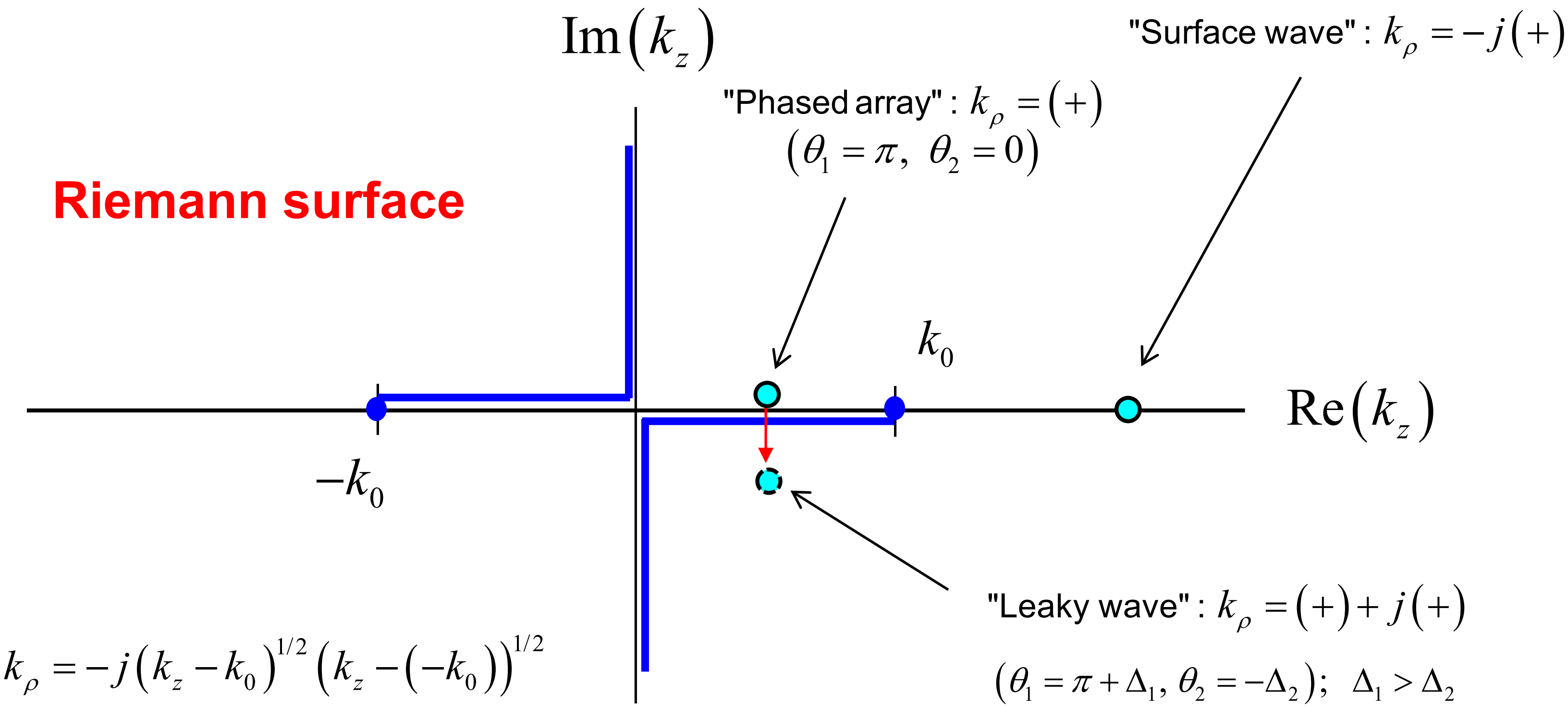
EM Example (cont.)

Example

What type of wave is at the indicated points?

$$k_\rho = -j\left(k_z^2 - k_0^2\right)^{1/2}$$

Riemann surface



Note : $k_\rho = -j\left(k_z - k_0\right)^{1/2}\left(k_z - (-k_0)\right)^{1/2}$

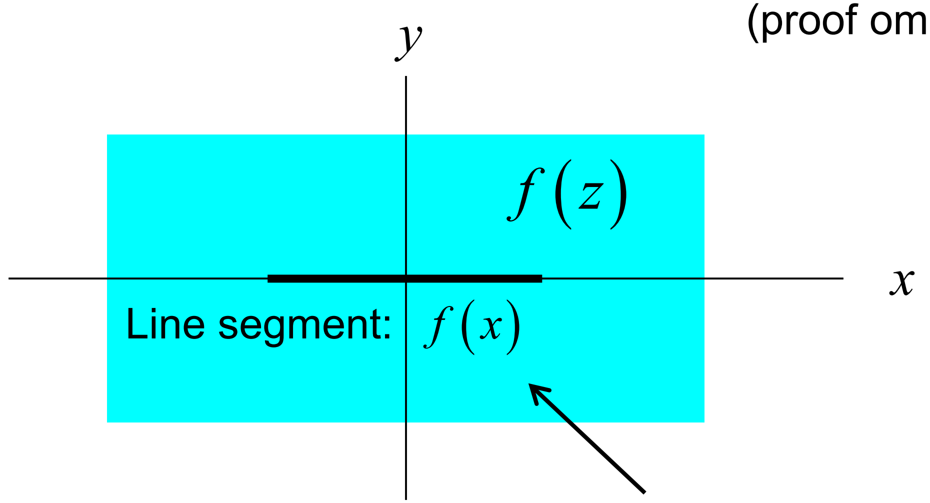
Note:
The leaky wave field is the analytic continuation of the phased array field when k_z becomes complex.

Schwarz Reflection Principle

Assume that $f(z)$ is the analytic continuation of a real function $f(x)$ off the real axis (or a segment of the real axis).

Then within the analytic region, we have $\boxed{f\left(z^*\right) = \left[f\left(z\right)\right]^*}$

(proof omitted)



$f(z)$ is assumed analytic in this region.

Examples:

$$\sin(z),\ e^z,\ J_n(z)$$

$$\ln(z),\ \ \mathrm{Re}(z) > 0$$ (assuming branch cut on negative real axis)