Multi-Agent Reinforcement Learning

Solution Concepts for Games

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MARL Book

Multi-Agent Reinforcement Learning: Foundations and Modern Approaches

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To be published by MIT Press
(print version scheduled for fall 2024)

This lecture is based on *Multi-Agent Reinforcement Learning: Foundations and Modern Approaches* by Stefano V. Albrecht, Filippos Christianos and Lukas Schäfer

The book can be downloaded for free here.

Lecture Outline

- Joint policy and expected return
- Equilibrium solution concepts
- Additional solution criteria
- Complexity of computing equilibria

MARL Problem = **Game Model** *e.g. normal-form game, stochastic game, POSG* + **Solution Concept** *e.g. Nash equilibrium, social welfare, ...*

What does it mean to act optimally in a multi-agent system?

- Maximizing returns of one agent is no longer enough to determine a solution
- We must consider the joint policy of all agents
- This is less straightforward, and there are many different solution concepts

Joint Policy and Expected Return

Joint Policy

The joint policy is the combination of all agents' policies.

- $\pi = (\pi_1, ..., \pi_n)$ is the joint policy

The probability of a joint action under joint policy π can be defined as:

$$\pi(a^\tau | h^\tau) = \prod_{j \in I} \pi_j(a_j^\tau | h_j^\tau)$$

Note

This definition assumes probabilistic independence between agents' policies.

Additional Notation

In the multi-agent setting, we add the following notation:

- $\hat{h} = \{(s^\tau, o^\tau, a^\tau)_{\tau=0}^{t-1}, s^t, o^t\}$ is the **full history** containing:
  - s^τ, states up to $t-1$
  - o^τ, joint observations up to $t-1$
  - a^τ, joint actions up to $t-1$
  - s^t and o^t at current time step t
- $\sigma(\hat{h}^t) = (o^0, ..., o^t)$ is a function that returns the joint observation history from the **full history**
- $\mathcal{O}(o^t | a^{t-1}, s^t)$ is the joint observation probability

History Based Expected Return

Given a joint policy π, we can define the expected return for agent i under π as the probability-weighted sum of returns for agent i under all possible **full histories.**

- Let $\hat{H}$ be a set containing all full histories $\hat{h}^t$ for $t \to \infty$
- then the expected return for agent i under joint policy π is given by:

$$
\begin{aligned}
U_i(\pi) &= \lim_{t\to\infty} \mathbb{E}_{\hat{h}^t \sim (\mu, \mathcal{T}, \mathcal{O}, \pi)} \left[u_i(\hat{h}^t) \right] \\
&= \sum_{\hat{h}^t \in \hat{H}} \Pr(\hat{h}^t | \pi) u_i(\hat{h}^t)
\end{aligned}
$$

History Based Expected Return - Continued

The probability of a full history $\Pr(\hat{h}^t|\pi)$ is:

$$\Pr(\hat{h}^t|\pi) = \mu(s^0)\mathcal{O}(o^0|\emptyset, s^0) \prod_{\tau=0}^{t-1} \pi(a^\tau|\hat{h}^\tau)\mathcal{T}(s^{\tau+1}|s^\tau, a^\tau)\mathcal{O}(o^{\tau+1}|a^\tau, s^{\tau+1})$$

$u_i(\hat{h}^t)$ is the discounted return for agent i in the **full history**

$$u_i(\hat{h}^t) = \sum_{\tau=0}^{t-1} \gamma^\tau R_i(s^\tau, a^\tau, s^{\tau+1})$$

Recursive Expected Returns

Expected returns under a **joint policy** can also be defined recursively, analogously to the Bellman recursion.

$$V_i^\pi(\hat{h}) = \sum_{a \in A} \pi(a \mid \sigma(\hat{h})) Q_i^\pi(\hat{h}, a)$$

We can use this to define a Q function for the individual agent i as follows:

$$Q_i^\pi(\hat{h}, a) = \sum_{s' \in S} \mathcal{T}(s' \mid s(\hat{h}), a) \left[\mathcal{R}_i(s(\hat{h}), a, s') + \gamma \sum_{o' \in O} O(o' \mid a, s') V_i^\pi(\langle \hat{h}, a, s', o' \rangle) \right]$$

- $s(\hat{h})$ denotes the last state in $\hat{h}$ such that $s(\hat{h}) = s^t$

Recursive Expected Returns - Continued

- $V_i^\pi(\hat{h})$ is the **value** or **expected return** for agent i when agents follow **joint policy** π
- $Q_i^\pi(\hat{h}, a)$ is the **expected return** for agent i when agents execute **joint action** a after $\hat{h}$ and follow π thereafter
- Given the definition for $V_i^\pi(\hat{h})$ and $Q_i^\pi(\hat{h}, a)$, we can define the expected return for agent i at the initial state s^0 as:

$$U_i(\pi) = \mathbb{E}_{s_0 \sim \mu, o_0 \sim \mathcal{O}(\cdot|\emptyset, s_0)} \left[V_i^\pi(\langle s_0, o_0 \rangle) \right]$$

Best Response

Best Response

The **best-response** policy is the policy that maximizes the expected return for agent i against a given set of policies for all other agents $\pi_{-i} = (\pi_1, ..., \pi_{i-1}, \pi_{i+1}, ..., \pi_n)$.

- A best response for agent i to π_{-i} is a policy π_i that maximizes the expected return for agent i when facing π_{-i}

$$\mathrm{BR}_i(\pi_{-i}) = \arg\max_{\pi_i} U_i(\langle \pi_i, \pi_{-1} \rangle)$$

- Where $\langle \pi_i, \pi_{-1} \rangle$ is the entire **joint policy**

Equilibrium Solution Concepts

Minimax

Minimax is a solution concept defined for **two-agent zero-sum** games. Recall that one agent's reward is the negative of the other agent's reward.

- The existence of minimax solution in **normal-form** games was first proven by John von Neumann (1928)
- Minimax solutions also exist in **two-agent zero-sum stochastic games** with finite episode lengths like chess and Go.

Minimax Definition

In a two-agent, zero-sum game, a joint policy $\pi = (\pi_i, \pi_j)$ is a minimax solution if

$$\begin{aligned} U_i(\pi) &= \max_{\pi'_i} \min_{\pi'_j} U_i(\pi'_i, \pi'_j) && (1) \\ &= \min_{\pi'_j} \max_{\pi'_i} U_i(\pi'_i, \pi'_j) && (2) \\ &= -U_j(\pi). && (3) \end{aligned}$$

- Equation 1 is the minimum expected return agent i can guarantee against any opponent
- Equation 2 is the minimum expected return agent j can **force** on agent i
- A minimax solution is the **best response** to the **worst-case** opponent
- (π_i, π_j) is a minimax solution if $\pi_i \in \mathrm{BR}_i(\pi_j)$ <u>and</u> $\pi_j \in \mathrm{BR}_j(\pi_i)$

Minimax via Linear Programming

We can obtain a minimax solution for non-repeated zero-sum normal-form games by solving two linear programs, one for each agent.

$$
\begin{aligned}
\text{minimize} \quad & U_j^* \\
\text{subject to} \quad & \sum_{a_i \in A_i} \mathcal{R}_j(a_i, a_j) x_{a_i} \leq U_j^* && \forall a_j \in A_j \\
& x_{a_i} \geq 0 && \forall a_i \in A_i \\
& \sum_{a_i \in A_i} x_{a_i} = 1
\end{aligned}
$$

- Minimizing agent j's return U_j^*
- Such that no single action of agent j can receive a greater return than U_j^* when agent i follows $\pi_i(a_i) = x_{a_i}$

Nash Equilibrium

The Nash equilibrium solution concept applies the idea of a **mutual best response** to general-sum games with two or more agents.

- John Nash (1950) proved the existence of such a solution for **general-sum non-repeated normal-from games**
- No agent i can improve its expected returns by changing its policy π_i assuming other agents policies remain fixed

$$\forall i, \pi_i' : U_i(\pi_i', \pi_{-i}) \leq U_i(\pi)$$

- Each agent's policy in the Nash equilibrium is a **best response** to all other agent's policies

Nash Equilibrium in Matrix Games

	C	D
C	-1,-1	-5,0
D	0,-5	-3,-3

Figure: Prisoners Dilemma

	A	B
A	10	0
B	0	10

Figure: Coordination Game

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Figure: Rock Paper Scissors

Can you identify the Nash equilibria?

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NE at D, D (-3, -3)

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Two NE's at A, A (10) and B, B (10)

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Figure: Rock Paper Scissors

NE is to choose actions uniformly at random with expected return 0

Folk Theorem in Repeated Normal Form Games

Folk theorems provide solutions for **repeated normal-form games** showing that any **set** of **feasible** and **enforceable** expected returns $\hat{U}$ can be achieved by an equilibrium solution if agents are far-sighted (γ close to 1).

- $\hat{U}$ is **feasible** if it can be achieved by a joint policy π
- $\hat{U}$ is **enforceable** if each $\hat{U}_i$ is at least as great as the agent's minmax value v_i

$$v_i = \min_{\pi_{-i}^m m} \max_{\pi_i^m m} U_i(\pi_i^m m, \pi_{-i}^m m)$$

- When $\hat{U}$ is enforceable no agent has an incentive to deviate from π, deviating results in other agents **enforcing** the minmax value $v_i \leq \hat{U}_i$

ϵ-Nash Equilibrium

Exact Nash equilibria are difficult to compute:

- In settings with more than two players, action probabilities may be irrational numbers
- Exact Nash equilibria are often computationally too expensive (more on slide 31)
- We can relax the conditions by requiring that no agent can improve its expected return by more than some amount $\epsilon > 0$
- In a general-sum game with n agent, a joint policy π is an ϵ-Nash equilibrium for $\epsilon > 0$ if:

$$\forall i, \pi'_i : U_i(\pi'_i, \pi_{-i}) - \epsilon \leq U_i(\pi)$$

ϵ-Nash Equilibrium can be far from Nash Equilibrium

	C	D
A	100,100	0,0
B	1,2	1,1

- Unique Nash equilibrium at A, C
- ϵ-Nash equilibrium when $\epsilon = 1$ at B, D and A, C
- ϵ-Nash equilibrium may not be a good approximation for the true Nash equilibrium
- Returns under $\epsilon-$Nash equilibrium can differ greatly from returns under the Nash equilibrium

Correlated Equilibrium

We have previously assumed that the agent's policies are probabilistically independent, which can limit expected returns.

- **Correlated equilibria**, allow for correlated policies
- π_c is a central policy that provides a probability distribution across all agents' actions
- Agents can follow action 'recommendations' $\pi_c(a)$ or deviate by choosing another action, represented by action modifier ξ_i
- then a **correlated equilibrium** can be defined as:

$$\sum_{a \in A} \pi_c(a) \mathcal{R}_i(\langle \xi_i(a_i), a_{-i} \rangle) \leq \sum_{a \in A} \pi_c(a) \mathcal{R}_i(a)$$

- Nash equilibria are a special case of correlated equilibria

Correlated Equilibrium Chicken Game

	S	L
S	0,0	7,2
L	2,7	6,6

Non-Correlated Nash Equilibrium:

- Deterministic: $\pi_i(S) = 1, \pi_j(S) = 0 \rightarrow (7, 2)$ and $\pi_i(S) = 0, \pi_j(S) = 1 \rightarrow (2, 7)$
- Probabilistic: $\pi_i(S) = \frac{1}{3}, \pi_j(S) = \frac{1}{3} \rightarrow \approx (4.66, 4.66)$

Correlated Equilibrium Chicken Game

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Correlated Nash Equilibrium:

- $\pi_c(L, L) = \pi_c(S, L) = \pi_c(L, S) = \frac{1}{3}$ and $\pi_c(S, S) = 0$
- Expected return $= 7 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = 5$
- Assuming knowledge of π_c, if agent i receives recommendation L they know agent j will choose either S or L with probability 0.5
- Thus expected return is $0.5 \cdot 6 + 0.5 \cdot 2 = 4$, which is greater than deviating from the action where the agent i has an expected return $0.5 \cdot 0 + 0.5 \cdot 7 = 3.5$

Coarse Correlated Equilibrium

Coarse correlated equilibria are more general than correlated equilibria where the action modifier ξ_i is not conditioned on the recommended action given by π_C.

- In correlated equilibrium $\xi_i : A_i \rightarrow A_i$, such that it takes the recommended action and provides an alternative action
- Coarse correlated equilibrium ξ_i is a constant action
- In other words the agent needs to choose to deviate from the recommended action before seeing it
- The coarse correlated equilibrium plays an important role in no-regret learning discussed in later slides

Correlated Equilibrium via Linear Programming

Similar to a minimax we can solve for correlated equilibria using a linear program for each agent i:

$$
\begin{aligned}
\text{maximise} \quad & \sum_{a \in A} \sum_{i \in I} x_a \mathcal{R}_i(a) \\
\text{subject to} \quad & \sum_{\substack{a \in A \\ a_i = a'_i}} x_a \mathcal{R}_i(a) \geq \sum_{\substack{a \in A \\ a_i = a''_i}} x_a \mathcal{R}_i(a'', a_{-i}) & \forall i \in I, a'_i, a''_i \in A_i \\
& x_a \geq 0 & \forall a \in A \\
& \sum_{a \in A} x_a = 1
\end{aligned}
$$

- The constraint ensures that no agent can increase their return by deviating from the action a'_i sampled under the joint policy $\pi(a) = x_a$, to some other action a'_i

Shortcomings of Equilibrium Solutions

Equilibrium solutions have been adopted as standard solution concepts in MARL but have limitations.

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  - Nash equilibria do not always maximize expected returns
  - E.g., In Prisoner's Dilemma, (D,D) is Nash but (C,C) yields higher returns

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- Non-uniqueness:
  - There can be multiple (even infinite) equilibria, each with different expected returns
- Incompleteness:
  - Equilibria for sequential games don't specify actions for off-equilibrium paths i.e. paths not specified by equilibrium policy $\Pr(\hat{h}|\pi) = 0$
  - If there is a temporary disturbance that leads to an **off-equilibrium** path, the equilibrium policy π does not specify actions to return to a **on-equilibrium** path

Refinement Concepts

Pareto Optimality

To address some of these limitations, we can add additional solution requirements such as **Pareto optimality**. A joint policy π is **Pareto-optimal** if it is not **Pareto-dominated** by any other joint policy.

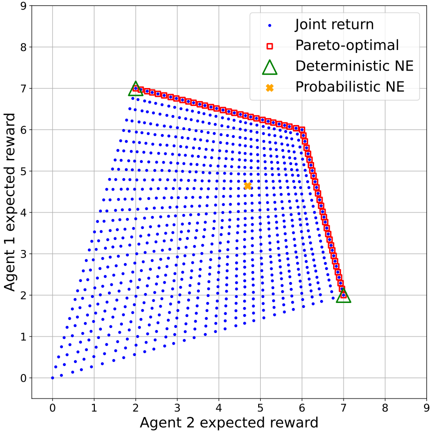
A joint policy π is **Pareto-dominated** by another policy π' if

$$\forall i : U_i(\pi') \geq U_i(\pi) \quad \text{and} \quad \exists i : U_i(\pi') > U_i(\pi).$$

Intuition

A joint policy is **Pareto-optimal** if there is no other joint policy that improves the expected return for at least one agent without reducing the expected return for any other agent.

Pareto-Optimal Solution in the Chicken Game



	S	L
S	0,0	7,2
L	2,7	6,6

- The figure shows the discretized space of joint policies for the chicken matrix game
- Each blue dot represents the expected joint return obtained by a joint policy

Social Welfare and Fairness

To further constrain the space of desirable solutions, we can consider social welfare and fairness concepts.

Welfare optimality:

$$W(\pi) = \sum_{i \in I} U_i(\pi)$$

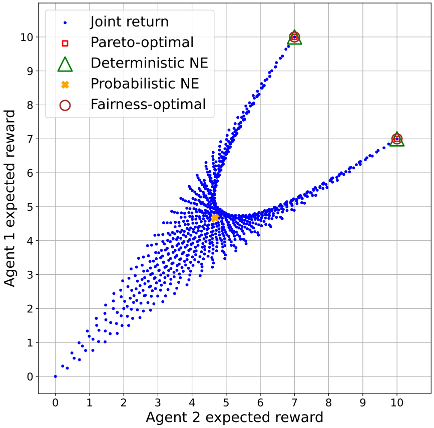
- A joint policy π is **welfare-optimal** if $\pi \in \arg\max_{\pi'} W(\pi')$

Fairness optimality:

$$F(\pi) = \prod_{i \in I} U_i(\pi)$$

- A joint policy π is **fairness-optimal** if $\pi \in \arg\max_{\pi'} F(\pi')$

Fairness in Battle of the Sexes



	A	B
A	10,7	2,2
B	0,0	7,10

- 2 agents agreeing to meet at either location A or B, with each agent having a preference for one or the other location
- A, A and B, B are both Nash equilibria and fairness optimal
- In the chicken game, there is only 1 Pareto optimal and fairness optimal solution

No Regret

Regret measures the difference between the rewards an agent received versus the rewards it would have received by choosing a different action in past episodes.

- In non-repeated normal-form games (assuming actions of other agents are fixed) **regret**, is:

$$Regret_i^z = \max_{a_i \in A_i} \sum_{e=1}^{z} \left[\mathcal{R}(\langle a_i, a_{-i}^e \rangle) - \mathcal{R}_i(a^e) \right]$$

- Let a^e denote the joint action in episode $e = 1, ..., z$
- There is no regret if regret is at most 0 as $z \to \infty$
- In general-sum games with n agents, the agents have no regret if:

$$\forall i : \lim_{z \to \infty} \frac{1}{z} Regret_i^z \leq 0$$

No Regret in Prisoners Dilemma

Episode e	1	2	3	4	5	6	7	8	9	10
Action a_1^e	C	C	D	C	D	D	C	D	D	D
Action a_2^e	C	D	C	D	D	D	C	C	D	C
Reward $\mathcal{R}_1(a^e)$	-1	-5	0	-5	-3	-3	-1	0	-3	0
Reward $\mathcal{R}(\langle C, a_2^e \rangle)$	-1	-5	-1	-5	-5	-5	-1	-1	-5	-1
Reward $\mathcal{R}(\langle D, a_2^e \rangle)$	0	-3	0	-3	-3	-3	0	0	-3	0

- Agent 1 receives total reward -21, always playing D would have resulted -15
- Thus, $\text{Regret}_1^{10} = -15 + 21 = 6$

Generalizing No-Regret to Stochastic Games and POSGs

For each agent i we introduce:

- A finite space of policies Π_i from which agent i can select a policy
- Let π^e denote the joint policy from episode $e = 1, ..., z$ with $\pi_i^e \in \Pi_i$ for all $i \in I$
- Agent i's regret for not having chosen the best policy across episodes is then defined as

$$Regret_i^z = \max_{\pi_i \in \Pi_i} \sum_{e=1}^{z} \left[U_i(\langle \pi_i, \pi_{-i}^e \rangle) - U_i(\pi^e) \right]$$

Note

This equation is equivalent to the previous equation for normal-form games if each Π_i is a set of **deterministic** policies corresponding to an action $a_i \in A_i$

Complexity of Computing Equilibria

Complexity of computing equilibria

Algorithmic game theory evaluates the computational difficulty of finding certain game solutions.

- Normal-form games provide a complexity lower bound for more complex games.
- Two-agent zero-sum games have **polynomial-time** minimax solutions via linear programming
- Correlated equilibria in non-repeated normal-form general-sum games can also be computed in **polynomial time**
- Nash equilibria computation is more complex due to the independence assumption and cannot be done using linear programming
- Finding Nash equilibria (NASH problem) is a **total search** problem and has been proven to be **PPAD complete**, one other PPAD complete problem is the END-OF-LINE Problem

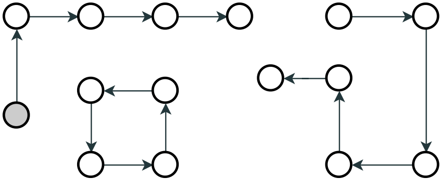
END-OF-LINE PPAD Complexity

The END-OF-LINE problem is PPAD complete, meaning all other problems in PPAD, including the NASH problem, can be reduced to it

END-OF-LINE Definition: Let $G(k) = (V, E)$ be a directed graph consisting of

- a finite set V containing 2^k nodes (each node is represented as a bit-string of length k)
- a finite set $E = \{(a, b) \mid a, b \in V\}$ of directed edges (from node a to node b, for $a, b \in V$) such that:
  - if $(a, b) \in E$ then $\exists a' \neq a : (a', b) \in E$ and $\nexists b' \neq b : (a, b') \in E$
- The goal is to find a node $e \neq s$ in this graph using two functions:
  - Parent(v) and Child(v), which return the parent or child node of v, respectively

END-OF-LINE - Continued



- The PPAD ”parity argument” ensures the existence of a sink node corresponding to a given source node (in grey)
- If a source node is given we know node e must exist
- To find e we can start at source and repeatedly call Child(v) until we find e
- As the graph scales 2^k this means finding e may require **exponential time** in the worst case.

Complexity Considerations for MARL

- **Reduction to NASH:** Computing Brouwer fixed points and other PPAD problems are reducible to the NASH problem, indicating there are no known efficient (polynomial time) algorithms for solving NASH
- **Approximate ϵ-Nash Equilibrium:** PPAD-completeness holds for both approximate solutions ($\epsilon > 0$) and exact solutions ($\epsilon = 0$), with approximations often necessary due to potentially irrational equilibria
- **Implications for MARL:** MARL algorithms are unlikely to be a silver bullet for finding Nash equilibria efficiently
- **Research Focus in MARL:** Research often targets identifying exploitable structures in certain game types, but MARL may still require **exponential** time when such structures are unavailable.

Summary

We covered:

- Best Response and minimax
- Equilibrium solutions
- Additional solution criteria
- Complexity of finding Nash equilibria

Next we'll cover:

- MARL in games