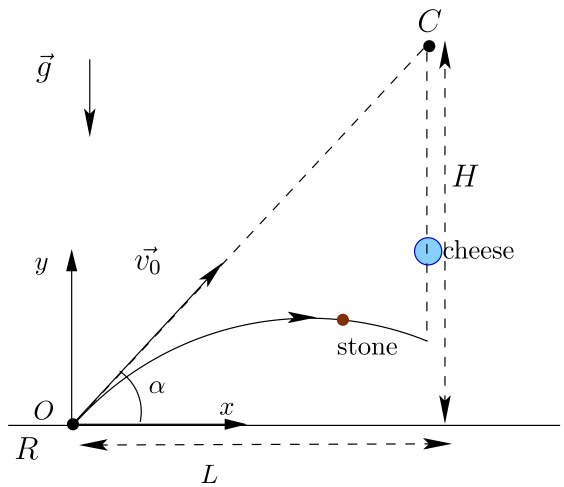
The crow and the fox

(Solution)

A diagram synthesising all available information is given in the problem. We add to that diagram a Oxy frame, where the origin O is the fox's position, and the angle α, which shows the direction of v_0 with respect to the x-axis.



a) The initial conditions are[1]:

- for the stone:

$$\text{position:} \begin{cases} x_0^S = 0 \\ y_0^S = 0 \end{cases} \qquad \text{velocity:} \begin{cases} v_{x,0}^S = v_0 \cos\alpha \\ v_{y,0}^S = v_0 \sin\alpha \end{cases}$$

- for the cheese:

$$\text{position:} \begin{cases} x_0^C = L \\ y_0^C = H \end{cases} \qquad \text{velocity:} \begin{cases} v_{x,0}^C = 0 \\ v_{y,0}^C = 0. \end{cases}$$

[1]For ease of writing, $||\vec{v}_0||$ is simply written v_0

1. Since both considered systems (system 1: the stone and system 2: the cheese) are only subject to gravity, their equations of motions are given by Newton's second law :

$$\ddot{\vec{r}}^S(t) = \vec{g} \qquad \text{et} \qquad \ddot{\vec{r}}^C(t) = \vec{g} \qquad \text{avec} \qquad \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Projecting both of these equations on the Oxy frame, we get four equations of motion:

- Equation for the stone, projected onto the Ox horizontal axis:

$$\ddot{x}^S(t) = 0.$$

Let's integrate this once:

$$\dot{x}^S(t) = cte.$$

The initial condition $v^S_{x,0} \equiv \dot{x}^S(t = t_0) = v_0 \cos\alpha$ allows to compute the constant

$$\dot{x}^S(t) = v_0 \cos\alpha.$$

To find the position, we integrate a second time:

$$x^S(t) = (v_0 \cos\alpha)t + cte'.$$

The initial condition $x^S_0 = 0$ gives us $cte' = 0$, and so

$$x^S(t) = (v_0 \cos\alpha)t.$$

- Equation for the stone, projected on Oy:

$$\ddot{y}^S(t) = -g.$$

We integrate once and use the initial condition $v^S_{y,0} = v_0 \sin\alpha$

$$\dot{y}^S(t) = -gt + v_0 \sin\alpha.$$

And we integrate and use the initial condition $y^S_0 = 0$ again to find

$$y^S(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t.$$

- Equations for the cheese projected onto Ox and Oy : Using the same method as above, we find

$$\begin{cases} \ddot{x}^C(t) = 0 \\ \ddot{y}^C(t) = -g \end{cases} \qquad \begin{cases} \dot{x}^C(t) = 0 \\ \dot{y}^C(t) = -gt \end{cases} \qquad \begin{cases} x^C(t) = L \\ y^C(t) = -\frac{1}{2}gt^2 + H. \end{cases}$$

2. The cheese and the stone do collide if there is a time $t = t_{coll}$ at which they have the same position, that is, if the following conditions are met:

$$x^S(t_{coll}) = x^C(t_{coll}) \qquad \text{ET} \qquad y^S(t_{coll}) = y^C(t_{coll}).$$

The condition $x^S(t_{coll}) = x^C(t_{coll})$ gives

$$(v_0 \cos\alpha)t_{coll} = L \Rightarrow t_{coll} = \frac{L}{v_0 \cos\alpha} = \frac{\sqrt{L^2 + H^2}}{v_0}. \tag{1}$$

While the condition $y^S(t_{coll}) = y^C(t_{coll})$ gives

$$-\frac{1}{2}gt_{coll}^2 + (v_0 \sin\alpha)t_{coll} = -\frac{1}{2}gt_{coll}^2 + H \Rightarrow t_{coll} = \frac{H}{v_0 \sin\alpha} = \frac{\sqrt{L^2 + H^2}}{v_0}. \tag{2}$$

Here, we used $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{H}{L}$. The solutions (**??**) and (**??**) are identical: there is, indeed, a cheese-stone collision at time

$$t_{coll} = \frac{\sqrt{L^2 + H^2}}{v_0}. \tag{3}$$

We find that the expression for t_{coll} does not depend on g: g accelerates both objects vertically and in the same fashion (mathematically, that's the term in $-\frac{1}{2}gt^2$), which has no influence on the horizontal motion of the objects, and so none on t_{coll} either.

b) We just proved that the the stone and the cheese will always collide, no matter the initial speed v_0. Isn't that surprising? However, if v_0 is too small, the cheese will hit the ground before the collision happens: we must restrict v_0 in that respect. This is not contradicting the previous statement: the collision would just "happen" at $y < 0$, that is, "in the ground". We would just have to dig a big enough hole to be able to see it.

In order for the collision to happen above ground, we must ensure $y^C(t_{coll}) > 0$, that is

$$
\begin{aligned}
y^C(t_{coll}) &= -\frac{1}{2}gt_{coll}^2 + H > 0 \\
\Rightarrow H &> \frac{1}{2}gt_{coll}^2 = \frac{1}{2}g\frac{L^2+H^2}{v_0^2} \\
\Rightarrow v_0 &> \sqrt{g\frac{L^2+H^2}{2H}}. \qquad (4)
\end{aligned}
$$

c) Let's check that the results we found are in the right dimensions.

- Equation **??**: $t_{coll} = \frac{\sqrt{L^2+H^2}}{v_0} : \frac{\left([m]^2\right)^{1/2}}{[m/s]} = \frac{[\cancel{m}]}{[\cancel{m}/s]} = [s]$. We find units of time.
- Equation **??**: $v_0 > \sqrt{g\frac{L^2+H^2}{2H}} : \left([m/s^2]\frac{[m]^2}{[m]}\right)^{1/2} = \frac{\cancel{[m]^{1/2}}[m]}{[s]\cancel{[m]^{1/2}}}$. We find units of speed.

Let's see a few limiting cases to check if we find what we expect.

- If v_0 tends towards infinity, t_{coll} must tend to 0 (Eq.**??**) and the collision will happen above ground (Eq.**??**)
- If v_0 tends to 0, t_{coll} must tend towards infinity (Eq.**??**) and the collision cannot happen above ground since Eq.**??** is never satisfied.
- If g tends to 0, the right-hand side of Eq.**??** tends to 0, so the collision will happen above ground, whatever the speed $v_0 > 0$.
- If H tends to 0, the right-hand side of Eq.**??** tends towards infinity, so the collision cannot happen above ground.

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