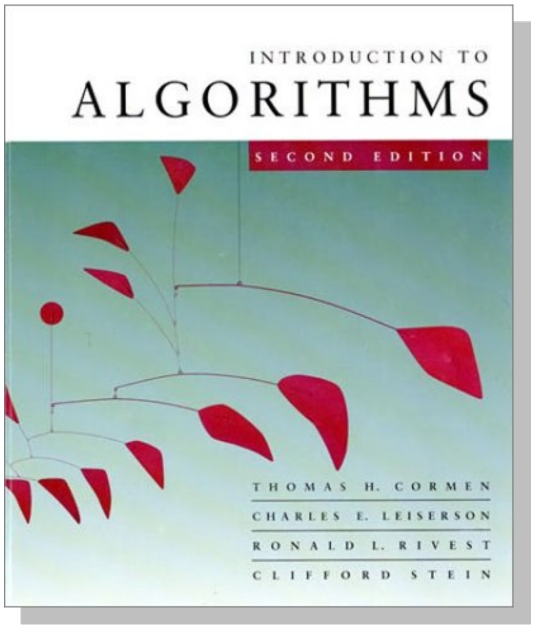
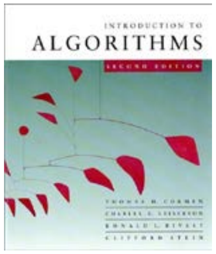
Introduction to Algorithms
6.046J/18.401J



LECTURE 15
Dynamic Programming

- Longest common subsequence
- Optimal substructure
- Overlapping subproblems

Prof. Charles E. Leiserson



Dynamic programming

Design technique, like divide-and-conquer.

Example: ***Longest Common Subsequence (LCS)***

- Given two sequences $x[1 \mathinner{.\,.} m]$ and $y[1 \mathinner{.\,.} n]$, find a longest subsequence common to them both.

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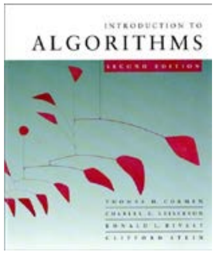

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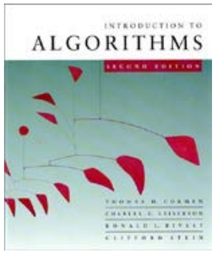

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x: A B C B D A B

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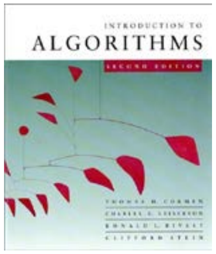
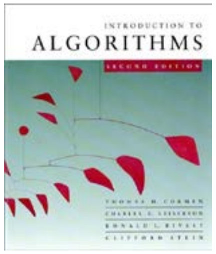
$\left.\right\}$ BCBA = LCS(x, y)

functional notation, but not a function



Brute-force LCS algorithm

Check every subsequence of $x[1 \,.\,.\, m]$ to see if it is also a subsequence of $y[1 \,.\,.\, n]$.



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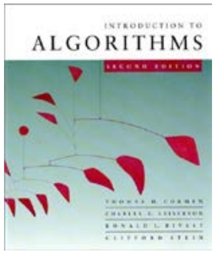
Analysis

- Checking $= O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time $= O(n2^m)$
$= \text{exponential time.}$



Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
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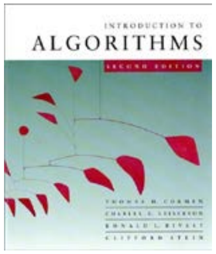
Notation: Denote the length of a sequence s by $|s|$.



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Simplification:

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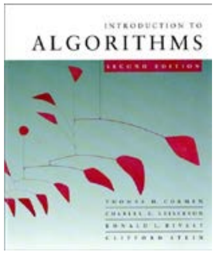
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|\,s\,|$.

Strategy: Consider ***prefixes*** of x and y.

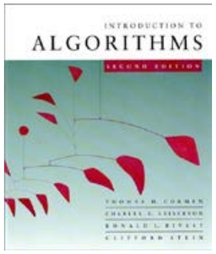
- Define $c[i, j] = |\,\mathrm{LCS}(x[1\,.\,.\,i], y[1\,.\,.\,j])\,|$.
- Then, $c[m, n] = |\,\mathrm{LCS}(x, y)\,|$.



Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$



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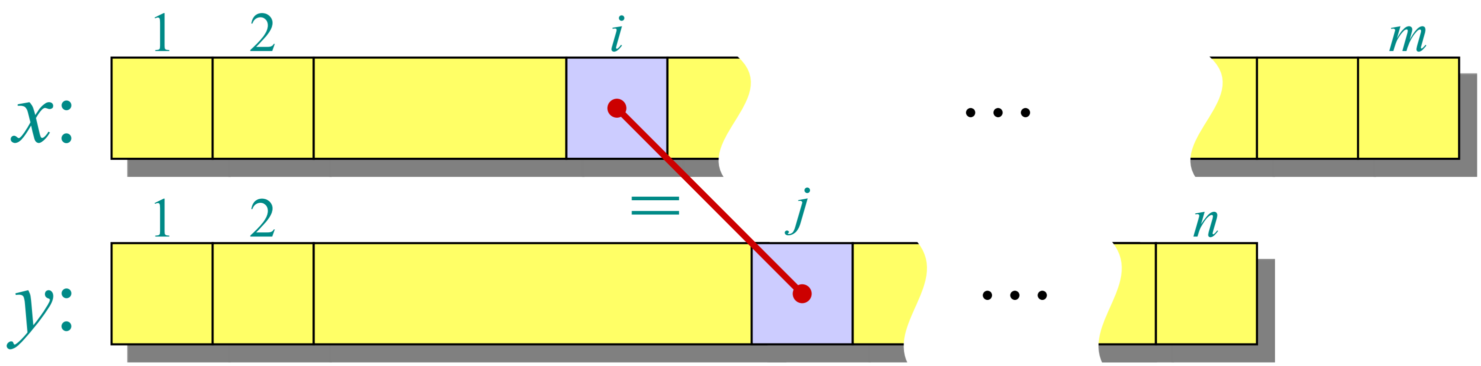
Proof. Case $x[i] = y[j]$:



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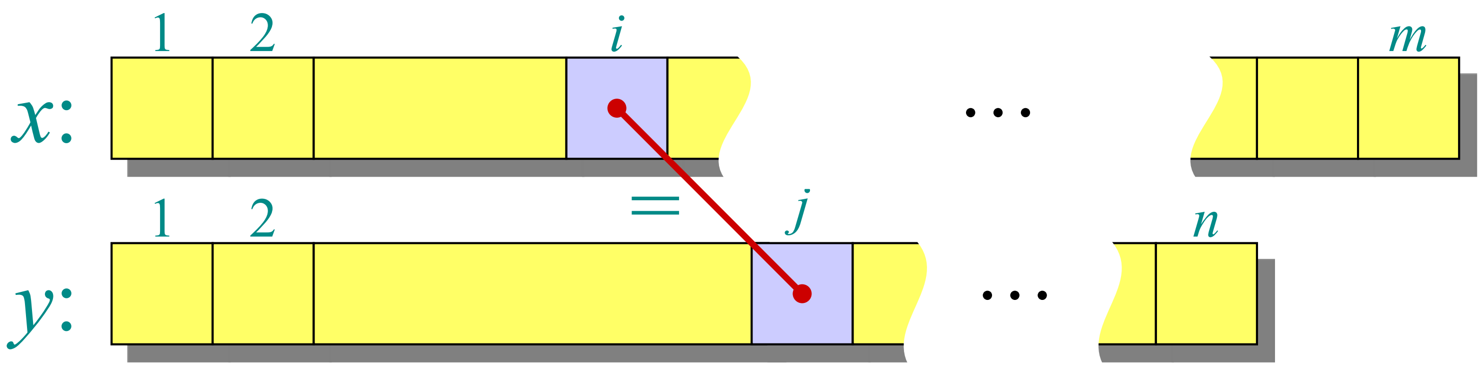
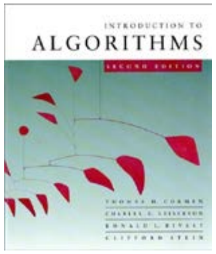
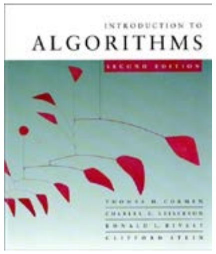


Let $z[1 \,..\, k] = \mathrm{LCS}(x[1 \,..\, i], y[1 \,..\, j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \,..\, k-1]$ is CS of $x[1 \,..\, i-1]$ and $y[1 \,..\, j-1]$.



Proof (continued)

Claim: $z[1\,.\,.\,k-1] = \mathrm{LCS}(x[1\,.\,.\,i-1], y[1\,.\,.\,j-1])$.
Suppose w is a longer CS of $x[1\,.\,.\,i-1]$ and $y[1\,.\,.\,j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \,\|\, z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1\,.\,.\,i]$ and $y[1\,.\,.\,j]$ with $|w \,\|\, z[k]| > k$. Contradiction, proving the claim.



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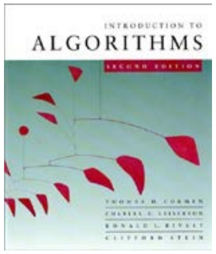
Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. $\square$

Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

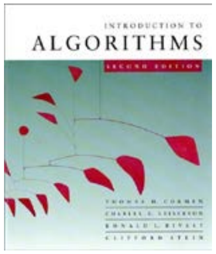


Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.



Recursive algorithm for LCS

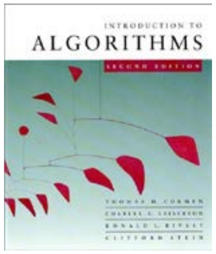
LCS(x, y, i, j) // ignoring base cases

    if $x[i] = y[j]$

        then $c[i, j] \leftarrow$ LCS$(x, y, i–1, j–1) + 1$

        else $c[i, j] \leftarrow \max\{LCS(x, y, i–1, j)$, LCS$(x, y, i, j–1)\}$

    return $c[i, j]$



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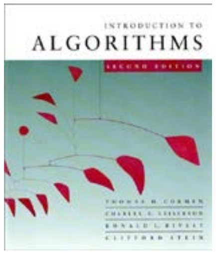
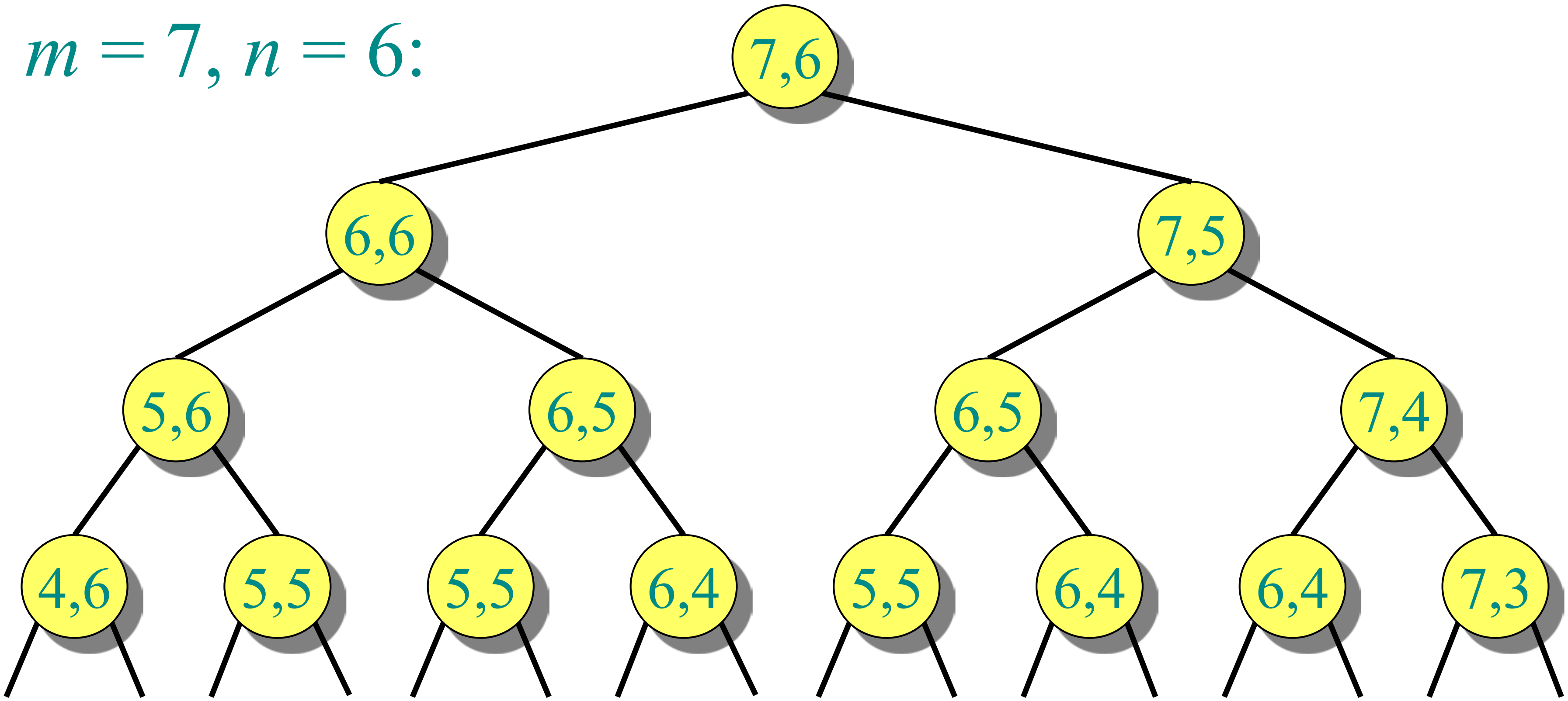
Worse case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.



Recursion tree

$m = 7, n = 6$:



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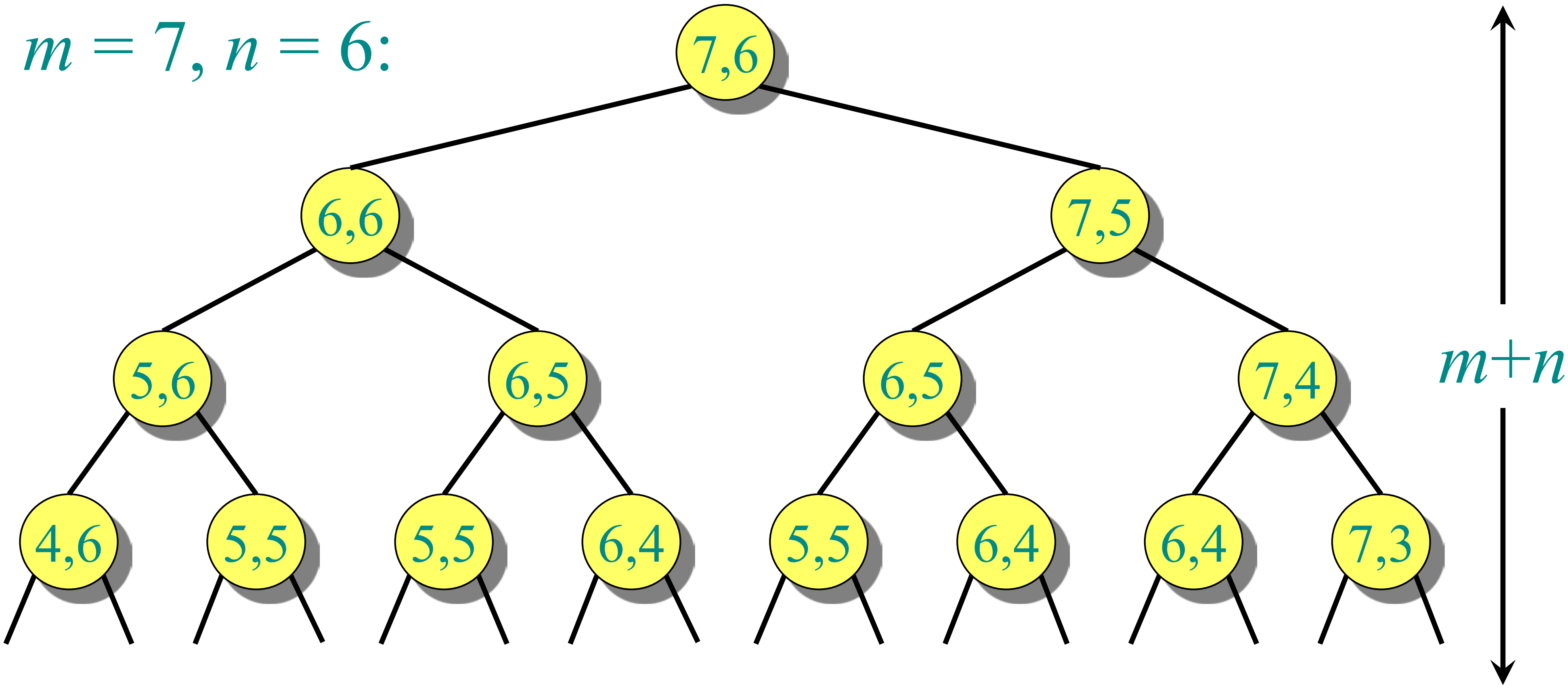

Height $= m + n \Rightarrow$ work potentially exponential.

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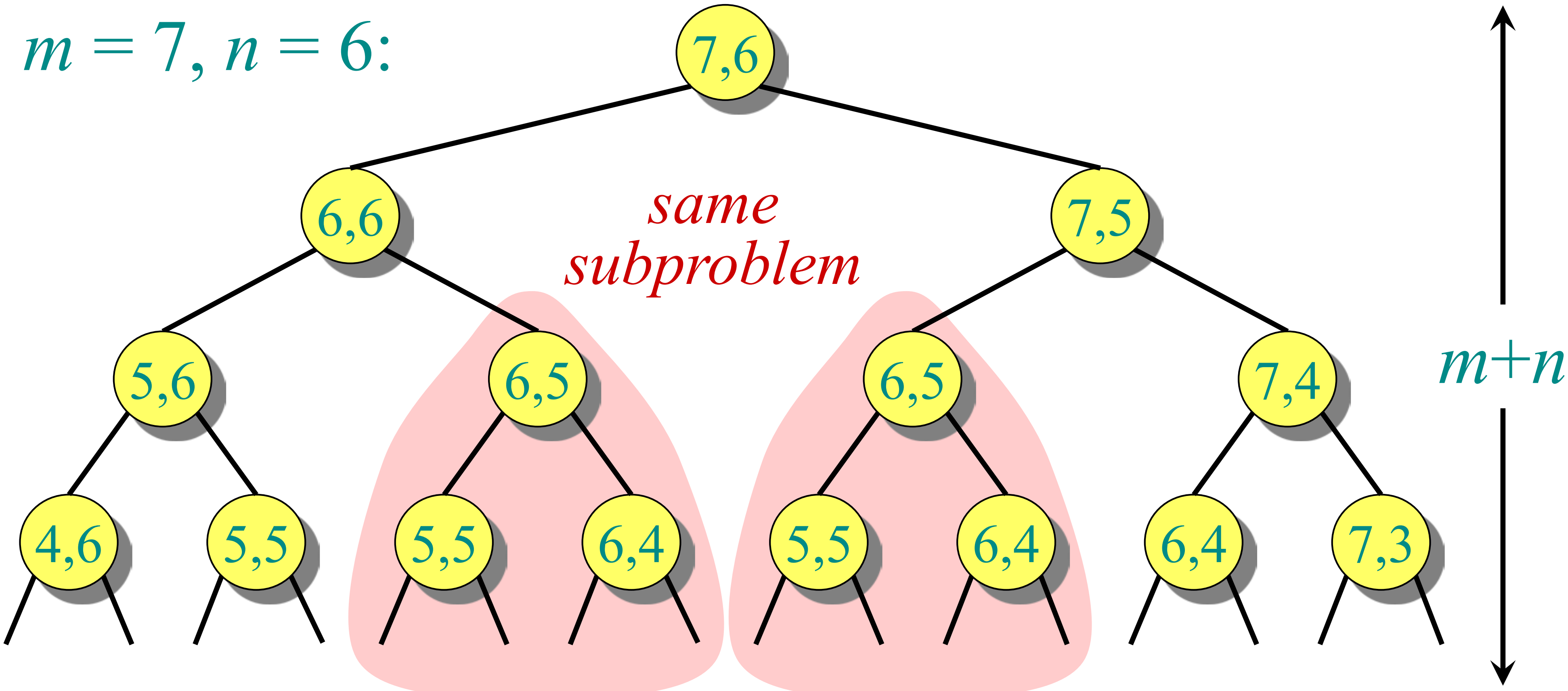
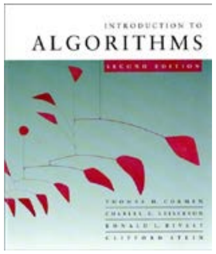


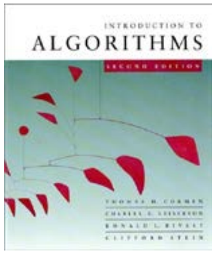
Height $= m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!



Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.



Dynamic-programming hallmark #2

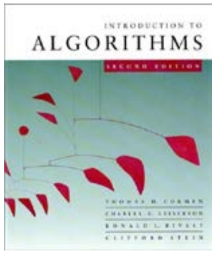
Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.



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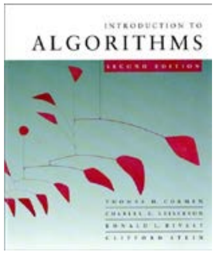
same as before



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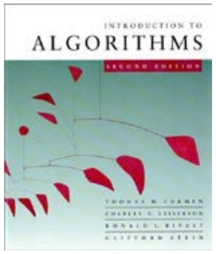
same as before

Time $= \Theta(mn) =$ constant work per table entry.
Space $= \Theta(mn)$.

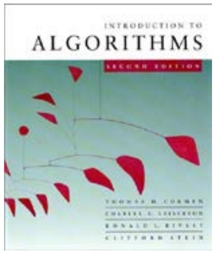


Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4



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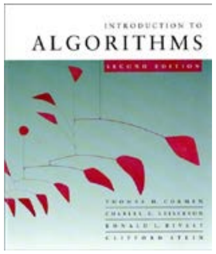
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4



Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

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D	0	0	1	1	1	2	2	2
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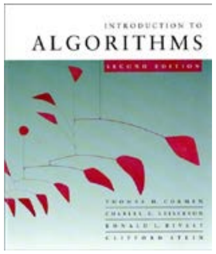

Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

Space = $\Theta(mn)$.

Exercise:

$O(\min\{m, n\})$.

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
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