Probabilities of Causation for Continuous and Vector Variables

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Abstract

Probabilities of causation (PoC) are valuable concepts for explainable artificial intelligence and practical decision-making. PoC are originally defined for scalar binary variables. In this paper, we extend the concept of PoC to continuous treatment and outcome variables, and further generalize PoC to capture causal effects between multiple treatments and multiple outcomes. In addition, we consider PoC for a sub-population and PoC with multi-hypothetical terms to capture more sophisticated counterfactual information useful for decision-making. We provide a nonparametric identification theorem for each type of PoC we introduce. Finally, we illustrate the application of our results on a real-world dataset about education.

1 INTRODUCTION

Probabilities of causation (PoC) are a family of probabilities quantifying whether one event was the real cause of another in a given scenario [Robins and Greenland, 1989, Pearl, 1999, Tian and Pearl, 2000, Pearl, 2009, Kuroki and Cai, 2011, Dawid et al., 2014, 2016, 2017, Murtas et al., 2017, Hannart and Naveau, 2018, Shingaki and Kuroki, 2021, Kawakami et al., 2023b]. PoC are valuable quantities for decision-making [Li and Pearl, 2019, 2022] and for explainable artificial intelligence (XAI) that aims to reduce the opaqueness of AI-based decision-making systems [Galhotra et al., 2021, Watson et al., 2021]. Pearl [1999] introduced three types of PoC over binary events, namely the probability of necessity and sufficiency (PNS), the probability of necessity (PN), and the probability of sufficiency (PS). They are defined based on the joint probability distribution of two potential outcomes. Tian and Pearl [2000] provided the bounds of PNS, PN, and PS in terms of observational and experimental data and showed that PNS, PN, and PS are identifiable under the assumptions of exogeneity and monotonicity. The problem of bounding PoC was further extended in [Li and Pearl, 2019, 2022, 2023, Mueller et al., 2022]. However, all these works are restricted to binary treatment and outcome. More recently, Li and Pearl [2024a,b] extended the problem of bounding PoC to multi-valued discrete treatment and outcome and provided bounds for various variants of PoC.

In this paper, we aim to extend the concept of PoC to continuous treatment and outcome. There is considerable interest in continuous treatment and outcome in causal inference [Hirano and Imbens, 2005, Kennedy et al., 2017, Bahadori et al., 2022], e.g., dose-response studies [Wong and Lachenbruch, 1996, Emilien et al., 2000, Ivanova et al., 2008] and policy evaluations with continuous actions [Kallus and Zhou, 2018, Krishnamurthy et al., 2019, Majzoubi et al., 2020]. For instance, doctors want to know the dose-response relationship between the amount of insulin and the blood sugar level.

We provide a nonparametric identification theorem for each type of PoC we introduced. The identification of binary PoC relies on a monotonicity assumption [Tian and Pearl, 2000]. We generalize the monotonicity assumption over binary treatment and outcome to continuous settings. We discuss the relationship of our proposed monotonicity assumption with another commonly used assumption in the causal inference literature - monotonicity over structural functions [Heckman and Vytlacil, 1999, Vytlacil, 2002, Heckman and Vytlacil, 2005, Chernozhukov and Hansen, 2005, Chernozhukov et al., 2007, Imbens and Newey, 2009].

We further extend the concept of PoC to capture causal effects between multiple treatments and multiple outcomes, which are drawing growing interests [Kang and Bates, 1990, Zhang, 1998, Sammel et al., 1999, Segal and Xiao, 2011, Lee et al., 2012, Kennedy et al., 2019, Rimal et al., 2019]. For instance, Hannart and Naveau [2018] investigated causal links between anthropogenic forcings, e.g., greenhouse gases (carbon dioxide, methane, nitrous oxide,

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halocarbons) emission and deforestation, and the observed climate changes, e.g., spatial–temporal vector of Earth surface temperature. They used a multivariate linear regression model with Gaussian noise to evaluate PoC.

We also introduce more complicated variants of PoC and provide identification theorems for them. They include PoC for a sub-population with specific covariates information considered by [Li and Pearl, 2022] and PoC with multi-hypothetical terms studied by Li and Pearl [2024a] for discrete treatment and outcome. These variants of PoC capture more sophisticated counterfactual information useful for decision-making.

Finally, we show an application of our results to a real-world dataset on education.

2 BACKGROUND AND NOTATION

We represent each variable with a capital letter (X) and its realized value with a small letter (x). Let $\mathbb{I}(x)$ be an indicator function that takes 1 if x is true; and 0 if x is false. Denote Ω_Y be the domain of Y, $\mathbb{E}[Y]$ be the expectation of Y, $\mathbb{P}(Y \leq y)$ be the cumulative distribution function (CDF) of continuous variable Y, and $\mathbb{P}(Y)$ be the probability of discrete variable Y. We denote $X \perp\!\!\!\perp Y|C$ if X and Y are conditionally independent given C.

Total order over vector space. We denote a total order on vectors of variables by $\prec$. For example, according to the lexicographical order [Harzheim, 2005], we order two dimensional vectors $(y_1, y_2) \prec_{\text{lexi}} (y_1', y_2')$ if "$y_1 < y_1'$", or "$y_1 = y_1'$ and $y_2 < y_2'$". A formal definition of the lexicographical order is given in Appendix A.

Structural Causal Models (SCM). We use the language of SCMs as our basic semantic and inferential framework [Pearl, 2009]. An SCM $\mathcal{M}$ is a tuple $\langle \boldsymbol{V}, \boldsymbol{U}, \mathcal{F}, \mathbb{P}_{\boldsymbol{U}} \rangle$, where $\boldsymbol{U}$ is a set of exogenous (unobserved) variables following a distribution $\mathbb{P}_{\boldsymbol{U}}$, and $\boldsymbol{V}$ is a set of endogenous (observable) variables whose values are determined by structural functions $\mathcal{F} = \{f_{V_i}\}_{V_i \in \boldsymbol{V}}$ such that $v_i := f_{V_i}(\mathbf{pa}_{V_i}, \boldsymbol{u}_{V_i})$ where $\mathbf{PA}_{V_i} \subseteq \boldsymbol{V}$ and $U_{V_i} \subseteq \boldsymbol{U}$. Each SCM $\mathcal{M}$ induces an observational distribution $\mathbb{P}_{\boldsymbol{V}}$ over $\boldsymbol{V}$, and a causal graph $G(\mathcal{M})$ over $\boldsymbol{V}$ in which there exists a directed edge from every variable in $\mathbf{PA}_{V_i}$ to V_i. An intervention of setting a set of endogenous variables $\boldsymbol{X}$ to constants $\boldsymbol{x}$, denoted by $do(\boldsymbol{x})$, replaces the original equations of $\boldsymbol{X}$ by the constants $\boldsymbol{x}$ and induces a *sub-model* $\mathcal{M}_{\boldsymbol{x}}$. We denote the potential outcome Y under intervention $do(\boldsymbol{x})$ by $Y_{\boldsymbol{x}}(\boldsymbol{u})$, which is the solution of Y in the sub-model $\mathcal{M}_{\boldsymbol{x}}$ given $\boldsymbol{U} = \boldsymbol{u}$.

Probabilities of Causation (PoC). Let X be a binary treatment taking values x_0 and x_1, and Y be a binary outcome taking values y_0 and y_1. PoC are defined as follows:

Definition 2.1 (PoC). *Probability of necessity and sufficiency (PNS), probability of necessity (PN), and probability of sufficiency (PS) are defined by [Pearl, 1999]:*

$$\begin{aligned} PNS &\triangleq \mathbb{P}(Y_{x_0} = y_0, Y_{x_1} = y_1), \\ PN &\triangleq \mathbb{P}(Y_{x_0} = y_0 | Y = y_1, X = x_1), \\ PS &\triangleq \mathbb{P}(Y_{x_1} = y_1 | Y = y_0, X = x_0). \end{aligned} \tag{1}$$

Tian and Pearl [2000] show that these PoC are identified under the following assumptions.

Assumption 2.1 (Exogeneity). $Y_{x_0} \perp\!\!\!\perp X$ *and* $Y_{x_1} \perp\!\!\!\perp X$.

Assumption 2.2 (Monotonicity). $\mathbb{P}(Y_{x_0} = y_1, Y_{x_1} = y_0) = 0$.

Under Assumptions 2.1 and 2.2, the PoC are identifiable by [Tian and Pearl, 2000]

$$\begin{aligned} \text{PNS} &= \mathbb{P}(Y = y_1 | X = x_1) - \mathbb{P}(Y = y_1 | X = x_0), \\ \text{PN} &= \frac{\mathbb{P}(Y = y_1 | X = x_1) - \mathbb{P}(Y = y_1 | X = x_0)}{\mathbb{P}(Y = y_1 | X = x_1)}, \\ \text{PS} &= \frac{\mathbb{P}(Y = y_1 | X = x_1) - \mathbb{P}(Y = y_1 | X = x_0)}{\mathbb{P}(Y = y_0 | X = x_0)}. \end{aligned} \tag{2}$$

3 POC FOR SCALAR CONTINUOUS VARIABLES

For ease of understanding, we will start with a single treatment variable X and a single outcome Y. We extend binary PoC for continuous variables, extend the monotonicity Assumption 2.2 to continuous settings, and provide an identification theorem.

3.1 POC DEFINITION

Let X be a continuous or discrete treatment variable, and Y be a continuous or discrete outcome variable. We assume the following SCM $\mathcal{M}_S$:

$$Y := f_Y(X, U), \; X := f_X(\epsilon_X), \tag{3}$$

where U and ϵ_X are latent exogenous variables.

We make the following assumption.

Assumption 3.1 (Exogeneity). $Y_x \perp\!\!\!\perp X$ *for all* $x \in \Omega_X$.

We note that if $\epsilon_X \perp\!\!\!\perp U$ then the exogeneity holds, and randomized controlled trials (RCT) on X ensure exogeneity. Exogeneity implies $\mathbb{P}(Y_x < y) = \mathbb{P}(Y < y | X = x)$.

We define PoC for continuous or discrete X and Y as a generalization of Definition 2.1.

Definition 3.1 (Probabilities of causation). *For any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$, we define three types of PoC as below:*[1]

$$\begin{aligned} &\textit{PNS}(y; x_0, x_1) \triangleq \mathbb{P}(Y_{x_0} < y \leq Y_{x_1}), \\ &\textit{PN}(y; x_0, x_1) \triangleq \mathbb{P}(Y_{x_0} < y | y \leq Y, X = x_1), \\ &\textit{PS}(y; x_0, x_1) \triangleq \mathbb{P}(y \leq Y_{x_1} | Y < y, X = x_0). \end{aligned} \tag{4}$$

PNS, PN, and PS are connected in the special case of binary X:

Lemma 3.1. *If X is binary, we have*

$$\begin{aligned} \textit{PNS}(y; x_0, x_1) = \textit{PN}(y; x_0, x_1)\mathbb{P}(y \leq Y, X = x_1) \\ + \textit{PS}(y; x_0, x_1)\mathbb{P}(Y < y, X = x_0). \end{aligned} \tag{5}$$

Remark on the connection of Def. 3.1 with the binary PoC in Def. 2.1: Suppose Y is binary with values $y_0 < y_1$, then Def. 3.1 with $y = y_1$ reduces to Def. 2.1. In general, the value of y in Def. 3.1 can be interpreted as an outcome threshold, such as the passing score for a test or a diagnostic threshold for blood pressure or blood glucose levels. Def. 3.1 focuses on the necessity/sufficiency of treatment x_1 w.r.t. x_0 to produce the event $Y \geq y$. We may introduce a binary outcome variable $O = \mathbb{I}(Y \geq y)$. Then $\text{PNS}(y; x_0, x_1) = \mathbb{P}(O_{x_0} = 0, O_{x_1} = 1)$, $\text{PN}(y; x_0, x_1) = \mathbb{P}(O_{x_0} = 0 | O = 1, X = x_1)$, and $\text{PS}(y; x_0, x_1) = \mathbb{P}(O_{x_1} = 1 | O = 0, X = x_0)$. Therefore, Def. 3.1 reduces to the standard definition of binary PoC over X and O. We note that this interpretation of PNS matches the use of PNS in [Hannart and Naveau, 2018].

Although Def. 3.1 can be interpreted in terms of a binarized outcome $O = \mathbb{I}(Y \geq y)$. It is more natural and consistent to have a formulation in terms of the original variable Y rather than in terms of O. A major benefit of the proposed formulation is that it can be naturally extended to study more complex variants of PoC in Section 5 that are difficult to formulate in terms of a binarized outcome.

When X and Y are discrete variables taking values $\{x_1, \ldots, x_P\}$ and $\{y_1, \ldots, y_Q\}$, Li and Pearl [2024a] defined PNS by $\mathbb{P}(Y_{x_{i_1}} = y_{j_1}, Y_{x_{i_2}} = y_{j_2})$ ($1 \leq i_1, i_2 \leq P$, $1 \leq j_1, j_2 \leq Q$, $i_1 \neq i_2$ and $j_1 \neq j_2$). However, their definition is not suitable for a continuous outcome Y.

Example 3.1. *Consider the dose-response relationship between the blood sugar level (Y) and the amount of insulin (X). Let y be a blood sugar threshold, and x_0, x_1 be two insulin amount ($x_0 > x_1$). A doctor may want to know the probability (PNS) that the patient's blood sugar level would be greater than or equal to the threshold y had they taken x_1 amount of insulin, and would be less than y had they taken x_0 insulin. PN represents the probability that the patient's blood sugar level would be less than y had they taken x_0 insulin when the patient took x_1 insulin with sugar level greater than or equal to y. PS represents the probability that the patient's blood sugar level would be greater than or equal to y had they taken x_1 insulin when the patient took x_0 insulin with sugar level less than y.*

3.2 IDENTIFICATION ASSUMPTIONS

Tian and Pearl [2000] used montonicity Assumption 2.2 for binary treatment and outcome to identify binary PoC. Here we generalize this assumption to continuous and discrete cases, and discuss connections with several commonly used assumptions in the literature.

(**I**). **Monotonicity over Y_x.** We first propose the following assumption:

Assumption 3.2 (Strong Monotonicity over Y_x). *The potential outcomes Y_x satisfy, for any $x_0, x_1 \in \Omega_X$, either "$Y_{x_0}(u) \leq Y_{x_1}(u)$ $\mathbb{P}_U$-almost surely for every $u \in \Omega_U$" or "$Y_{x_1}(u) \leq Y_{x_0}(u)$ $\mathbb{P}_U$-almost surely for every $u \in \Omega_U$".*

Note that we allow both monotonic increasing and decreasing cases. It turns out that the PoC in Def. 3.1 can be identified under a weaker assumption:[2]

Assumption 3.3 (Monotonicity over Y_x). *The potential outcomes Y_x satisfy, for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$, "either $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = 0$ or $\mathbb{P}(Y_{x_1} < y \leq Y_{x_0}) = 0$".*

Introducing a binarized outcome $O = \mathbb{I}(Y \geq y)$, Assumption 3.3 becomes "$\mathbb{P}(O_{x_0} = 0, O_{x_1} = 1) = 0$ or $\mathbb{P}(O_{x_0} = 1, O_{x_1} = 0) = 0$". Assumption 3.3 is weaker than 3.2 since $\mathbb{P}(Y_{x_0} < Y_{x_1}) = 0$ implies $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = 0$ but not vice versa.

Next, we discuss several related assumptions used in the literature for various identification purposes.

(**II**). **Monotonicity over f_Y.** Monotonicity on U over structural function $f_Y(x, U)$ has appeared in the instrumental variable (IV) literature, e.g. [Vytlacil, 2002, Heckman and Vytlacil, 1999, 2005].

Assumption 3.4 (Monotonicity over f_Y). *The function $f_Y(x, U)$ is either monotonic increasing on U for all $x \in \Omega_X$ or monotonic decreasing on U for all $x \in \Omega_X$ almost surely w.r.t. $\mathbb{P}_U$.*

For example, Heckman and Vytlacil [2005] introduced the latent index model $Y := \mathbb{I}[f_Y(X) \geq U]$ for a binary outcome, which satisfies the above assumption.

(**III**). **Strict monotonicity over f_Y.** The following stronger monotonicity assumption have also been used [Chesher,

[1] We can equally define PNS as $\text{PNS}(y; x_0, x_1) \triangleq \mathbb{P}(Y_{x_0} \leq y < Y_{x_1})$. We will stay with Definition 3.1 in this paper.

[2] This assumption is not "$\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = 0$ for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$" or "$\mathbb{P}(Y_{x_1} < y \leq Y_{x_0}) = 0$ for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$."

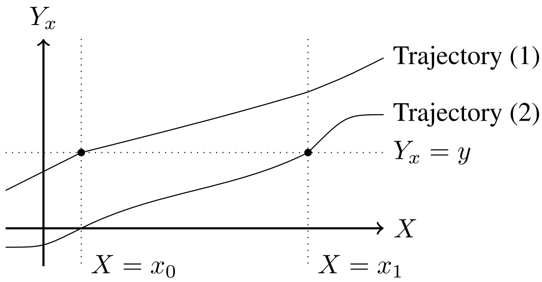

Figure 1: Trajectories for (1) $Y_x(u_{\rho(y;x_0)})$ and (2) $Y_x(u_{\rho(y;x_1)})$.

2003, Chernozhukov and Hansen, 2005, Chernozhukov et al., 2007, Imbens and Newey, 2009].

Assumption 3.5 (Strict monotonicity over f_Y)**.** *The function $f_Y(x, U)$ is either strictly monotonic increasing on U for all $x \in \Omega_X$ or strictly monotonic decreasing on U for all $x \in \Omega_X$ almost surely w.r.t. $\mathbb{P}_U$ with $\sup_{u \in \Omega_U} \mathfrak{p}(u) < \infty$.*

The condition $\sup_{u \in \Omega_U} \mathfrak{p}(u) < \infty$ means U is continuous distribution. For example, the widely used additive noise model $Y = f_Y(X) + U$ [Newey and Powell, 2003, Singh et al., 2019, Hartford et al., 2017, Xu et al., 2021, Kawakami et al., 2023a] satisfies this assumption.

Relationship between the three assumptions. We obtain that our proposed monotonicity Assumption 3.3 for continuous and discrete cases is equivalent to the monotonicity Assumption 3.4 over structural function $f_Y(x, U)$ under the following assumption:

Assumption 3.6. *Potential outcome Y_x has PDF p_{Y_x} for each $x \in \Omega_X$, and its support $\{y \in \Omega_Y : p_{Y_x}(y) \neq 0\}$ is the same for each $x \in \Omega_X$.*

This assumption is reasonable for continuous variables. For example, the linear regression model with Gaussian noise in [Hannart and Naveau, 2018] satisfies this assumption.

Theorem 3.1. *Under SCM $\mathcal{M}_S$ and Assumption 3.6, Assumptions 3.3 and 3.4 are equivalent, and Assumption 3.5 is a strictly stronger requirement than 3.4.*

3.3 IDENTIFICATION THEOREM

Next, we present an identification theorem. We denote the conditional CDF

$$\rho(y; x) \triangleq \mathbb{P}(Y < y | X = x). \tag{6}$$

Theorem 3.2 (Identification of PoC)**.** *Under SCM $\mathcal{M}_S$ and Assumptions 3.1, 3.3 (or 3.4, 3.5), and 3.6, PNS, PN, and PS are identifiable by*

$$\begin{aligned} \textit{PNS}(y; x_0, x_1) &= \max\{\rho(y; x_0) - \rho(y; x_1), 0\}, \\ \textit{PN}(y; x_0, x_1) &= \max\left\{\frac{\rho(y; x_0) - \rho(y; x_1)}{1 - \rho(y; x_1)}, 0\right\}, \\ \textit{PS}(y; x_0, x_1) &= \max\left\{\frac{\rho(y; x_0) - \rho(y; x_1)}{\rho(y; x_0)}, 0\right\} \end{aligned} \tag{7}$$

for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$ such that $\rho(y; x_1) < 1$ and $\rho(y; x_0) > 0$.

We can use the trajectories of potential outcomes to visualize and explain the above identification result for PNS. The trajectory $\{(x, Y_x(u)) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ represents potential outcome $Y_x(u)$ vs. X for the subject $U = u$. *Under Assumptions 3.3 (or 3.4, 3.5), the subjects' trajectories do not cross over each other* (they may overlap). We denote $u_{\rho(y;x)} = \sup\{u : f_Y(x, u) < y\}$ for any $x \in \Omega_X$ and $y \in \Omega_Y$, and $Y_x(u_{\rho(y;x)})$ is the potential outcome for subject $u_{\rho(y;x)}$. Consider the two trajectories shown in Figure 1. Trajectory (1) $\{(x, Y_x(u_{\rho(y;x_0)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ goes through the point (x_0, y), and Trajectory (2) $\{(x, Y_x(u_{\rho(y;x_1)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ goes through the point (x_1, y). The trajectory of subject u lies in the region between Trajectories (1) and (2) if and only if $Y_{x_0}(u) < y \leq Y_{x_1}(u)$. Thus, we have $\text{PNS}(y; x_0, x_1) = \mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = \mathbb{P}(Y_{x_0} < y) - \mathbb{P}(Y_{x_1} < y)$, where $\mathbb{P}(Y_{x_0} < y)$ represents the probability of a subject's trajectory being below Trajectory (1) and $\mathbb{P}(Y_{x_1} < y)$ represents the probability of a subject's trajectory being below Trajectory (2).

4 POC FOR VECTOR CONTINUOUS VARIABLES

In this section, we extend PoC to vectors of continuous or discrete variables $\boldsymbol{Y}$ and $\boldsymbol{X}$, and we consider PoC for a sub-population with specific covariates information. The benefits of considering the subject's covariates include (i) they reveal the heterogeneity of causal effects; and (ii) they weaken identification assumptions.

4.1 CONDITIONAL POC DEFINITION

Let $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{C}$ be a set of continuous or discrete treatment variables, outcome variables, and covariates, respectively. We assume the following SCM $\mathcal{M}_T$:

$$\boldsymbol{Y} := f_{\boldsymbol{Y}}(\boldsymbol{X}, \boldsymbol{C}, \boldsymbol{U}), \boldsymbol{X} := f_{\boldsymbol{X}}(\boldsymbol{C}, \boldsymbol{\epsilon_X}), \boldsymbol{C} := f_{\boldsymbol{C}}(\boldsymbol{\epsilon_C}) \tag{8}$$

The functions $f_{\boldsymbol{Y}}$, $f_{\boldsymbol{X}}$, and $f_{\boldsymbol{C}}$ are vector-valued functions. $\boldsymbol{\epsilon_X}$, $\boldsymbol{\epsilon_C}$, and $\boldsymbol{U}$ are latent exogenous variables. We assume that the domains $\Omega_{\boldsymbol{Y}}$ and $\Omega_{\boldsymbol{U}}$ are totally ordered sets with $\preceq$. Let the dimensions of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{C}, \boldsymbol{U}$ be d_X, d_Y, d_C, d_U.

We make the following assumption.

Assumption 4.1 (Conditional exogeneity). *$\boldsymbol{Y_x} \perp\!\!\!\perp \boldsymbol{X}|\boldsymbol{C}$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$.*

Conditional exogeneity implies $\mathbb{P}(\boldsymbol{Y_x} \prec \boldsymbol{y}|\boldsymbol{C} = \boldsymbol{c}) = \mathbb{P}(\boldsymbol{Y} \prec \boldsymbol{y}|\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{C} = \boldsymbol{c})$ for any $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

We define the multivariate conditional PoC as below:

Definition 4.1 (Conditional PoC). *For any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, we define conditional PoC by*

$$
\begin{aligned}
&\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) \triangleq \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}|\boldsymbol{C} = \boldsymbol{c}),\\
&\mathit{PN}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) \triangleq \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y}|\boldsymbol{y} \preceq \boldsymbol{Y}, \boldsymbol{X} = \boldsymbol{x}_1, \boldsymbol{C} = \boldsymbol{c}),\\
&\mathit{PS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) \triangleq \mathbb{P}(\boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}|\boldsymbol{Y} \prec \boldsymbol{y}, \boldsymbol{X} = \boldsymbol{x}_0, \boldsymbol{C} = \boldsymbol{c}).
\end{aligned} \tag{9}
$$

$\mathrm{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c})$ provides a measure of the sufficiency and necessity of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0$ to produce $\boldsymbol{Y} \succeq \boldsymbol{y}$ given $\boldsymbol{C} = \boldsymbol{c}$. $\mathrm{PN}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c})$ provides a measure of the necessity of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0$ to produce $\boldsymbol{Y} \succeq \boldsymbol{y}$ given $\boldsymbol{C} = \boldsymbol{c}$. $\mathrm{PS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c})$ provides a measure of the sufficiency of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0$ to produce $\boldsymbol{Y} \succeq \boldsymbol{y}$ given $\boldsymbol{C} = \boldsymbol{c}$.

Hannart and Naveau [2018] studied multivariate PNS where the outcomes are the space-time vectorial random variables of the Earth's surface temperatures. Li and Pearl [2019, 2024a, 2022] considered conditional PNS over discrete variables in their benefit function and called it z-specific PNS, but their definition of PNS is different from ours and is not suitable for continuous variables.

4.2 IDENTIFICATION ASSUMPTIONS AND THEOREM

We generalize Assumptions 3.3, 3.4, and 3.5 to multivariate outcomes and treatments with covariates as below, respectively.

Assumption 4.2 (Conditional monotonicity over $\boldsymbol{Y_x}$). *The potential outcomes $\boldsymbol{Y_x}$ satisfy: for any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, either $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}|\boldsymbol{C} = \boldsymbol{c}) = 0$ or $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_0}|\boldsymbol{C} = \boldsymbol{c}) = 0$.*

This assumption extends Assumptions 3.3 to totally ordered vector variables.

Assumption 4.3 (Conditional monotonicity over $f_{\boldsymbol{Y}}$). *The function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is either (i) monotonic increasing on $\boldsymbol{U}$ with $\preceq$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$, or (ii) monotonic decreasing on $\boldsymbol{U}$ with $\preceq$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$.*

This assumption says that the function $f_{\boldsymbol{Y}}$ preserves the total order from $\Omega_{\boldsymbol{U}}$ to $\Omega_{\boldsymbol{Y}}$ given $\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{C} = \boldsymbol{c}$.

Assumption 4.4 (Strict conditional monotonicity over $f_{\boldsymbol{Y}}$). *The function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is either (i) strictly monotonic increasing on $\boldsymbol{U}$ with $\preceq$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$ with $\sup_{\boldsymbol{u} \in \Omega_{\boldsymbol{U}}} \mathfrak{p}(\boldsymbol{u}|\boldsymbol{C} = \boldsymbol{c}) < \infty$ for all $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, or (ii) strictly monotonic decreasing on $\boldsymbol{U}$ with $\preceq$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$ with $\sup_{\boldsymbol{u} \in \Omega_{\boldsymbol{U}}} \mathfrak{p}(\boldsymbol{u}|\boldsymbol{C} = \boldsymbol{c}) < \infty$ for all $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.*

This assumption implies that there exists a one-to-one mapping from $\Omega_{\boldsymbol{U}}$ to $\Omega_{\boldsymbol{Y}}$ given $\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{C} = \boldsymbol{c}$.

Assumptions 4.2, 4.3, and 4.4 reduce to Assumptions 3.3, 3.4, and 3.5 under SCM $\mathcal{M}_S$, respectively. We establish the relationships between Assumptions 4.2, 4.3, and 4.4 under the following assumption:

Assumption 4.5. *Potential outcome $\boldsymbol{Y_x}$ has conditional PDF $p_{\boldsymbol{Y_x}|\boldsymbol{C}=\boldsymbol{c}}$ given $\boldsymbol{C} = \boldsymbol{c}$ for each $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, and its support $\{\boldsymbol{y} \in \Omega_{\boldsymbol{Y}} : p_{\boldsymbol{Y_x}|\boldsymbol{C}=\boldsymbol{c}}(\boldsymbol{y}) \neq 0\}$ is the same for each $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.*

This assumption is similar to Assumption 3.6 and reasonable for continuous variables. For example, the multivariate linear regression model with Gaussian noise in [Hannart and Naveau, 2018] satisfies this assumption.

Theorem 4.1. *Under SCM $\mathcal{M}_T$ and Assumption 4.5, Assumptions 4.2 and 4.3 are equivalent, and Assumption 4.4 is a strictly stronger requirement than 4.3.*

For example, the additive noise model $\boldsymbol{Y} := f_{\boldsymbol{Y}}(\boldsymbol{X}, \boldsymbol{C}) + \boldsymbol{U}$ satisfies all Assumptions 4.2, 4.3, and 4.4.

We denote conditional CDF

$$\rho(\boldsymbol{y}; \boldsymbol{x}, \boldsymbol{c}) \triangleq \mathbb{P}(\boldsymbol{Y} \prec \boldsymbol{y}|\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{C} = \boldsymbol{c}) \tag{10}$$

for all $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$, $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$. Then. we have the following theorem:

Theorem 4.2 (Identification of conditional PoC). *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, PNS, PN, and PS are identifiable by*

$$
\begin{aligned}
&\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) = \max\{\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c}) - \rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c}), 0\},\\
&\mathit{PN}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) = \max\left\{\frac{\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c}) - \rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c})}{1 - \rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c})}, 0\right\},\\
&\mathit{PS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) = \max\left\{\frac{\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c}) - \rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c})}{\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c})}, 0\right\}
\end{aligned} \tag{11}
$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$ such that $\rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c}) < 1$ and $\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c}) > 0$.

Remark. PoC, like $\mathrm{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1)$, can be computed through conditional PoC:

$$\mathrm{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1) = \int_{\boldsymbol{c} \in \Omega_{\boldsymbol{C}}} \mathrm{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{c}) \mathfrak{p}(\boldsymbol{c}) d\boldsymbol{c} \tag{12}$$

where $\mathfrak{p}(\boldsymbol{c})$ is PDF of $\boldsymbol{C}$. Then, we can estimate it under weaker conditions than required by Theorem 3.2 since the conditional version of the assumptions required by Theorem 4.2 are weaker.

5 VARIANTS OF PROBABILITIES OF CAUSATION

In this section, we study several more complicated variants of PoC.

5.1 PNS WITH EVIDENCE

We consider PNS with evidence $(\boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c})$ denoted by $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$.[3] Evidence provides the situation-specific information and restricts the attention to PNS for a sub-population.

For instance, revisiting Example 3.1, for a patient with a certain age and body weight, a doctor may want to know the probability that the patient's blood sugar level would be greater than or equal to the threshold y had they taken x_1 amount of insulin, and would be less than y had they taken x_0 insulin, when the patient took x' amount of insulin and had blood sugar level y'. This probability is given by $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1} | Y = y', X = x', \boldsymbol{C} = \boldsymbol{c})$ where $\boldsymbol{c}$ stands for the patient's age and body weight.

Note that for a binary treatment and outcome, PNS with evidence $(X = x_1, Y = y_1)$ coincides with PN, and PNS with evidence $(X = x_0, Y = y_0)$ coincides with PS. However, for continuous treatment and outcome, we could have PNS with different evidence.

Definition 5.1 (Conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$)**.** *We define conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$ as*

$$\begin{aligned} &\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) \\ &\triangleq \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} | \boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c}) \end{aligned} \tag{13}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, and $\boldsymbol{y}, \boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$.

PNS$(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$ provides a measure of the sufficiency and necessity of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0$ to produce $\boldsymbol{Y} \succeq \boldsymbol{y}$ given the evidence $(\boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c})$.

We denote the conditional CDF

$$\rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) \triangleq \mathbb{P}(\boldsymbol{Y} \preceq \boldsymbol{y}' | \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c}) \tag{14}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, which differs from $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$ in that it includes the point $\boldsymbol{Y} = \boldsymbol{y}'$.

We obtain the following theorem:

Theorem 5.1 (Identification of conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$)**.** *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, we have*

(A). *If $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) \neq \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$, then we have*

$$\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) = \max\{\alpha/\beta, 0\}, \tag{15}$$

where

$$\begin{aligned} \alpha = &\min\{\rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c}), \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})\} \\ &- \max\{\rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c}), \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})\}, \\ \beta = &\rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) - \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) \end{aligned} \tag{16}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$.

(B). *If $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) = \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$, then we have*

$$\begin{aligned} &\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) \\ &= \mathbb{I}(\rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c}) \leq \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) < \rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c})) \end{aligned} \tag{17}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$.

We provide an explanation of this result based on analyzing the trajectories of potential outcomes in Appendix B.

Assumption 4.4 implies $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) = \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$. Then, we have the following corollary:

Corollary 5.1. *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.4, and 4.5, we have*

$$\begin{aligned} &\mathit{PNS}(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) \\ &= \mathbb{I}(\rho(\boldsymbol{y}; \boldsymbol{x}_1, \boldsymbol{c}) \leq \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) < \rho(\boldsymbol{y}; \boldsymbol{x}_0, \boldsymbol{c})) \end{aligned} \tag{18}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$.

5.2 CONDITIONAL PNS WITH MULTI-HYPOTHETICAL TERMS

To address questions involving multiple counterfactual statements jointly, Li and Pearl [2024a,b] considered (conditional) PNS with multi-hypothetical terms $\mathbb{P}(Y_{x_{i_1}} = y_{j_1}, Y_{x_{i_2}} = y_{j_2}, \ldots, Y_{x_{i_P}} = y_{j_P} | \boldsymbol{C} = \boldsymbol{c})$ when X and Y are discrete scalar variables taking values $\{x_1, \ldots, x_P\}$ and $\{y_1, \ldots, y_Q\}$. However, their definition is not applicable to continuous outcome Y. Here, we define conditional PNS with multi-hypothetical terms that are applicable to both discrete and continuous cases.

Example 5.1. *Extending Example 3.1, the overdose of insulin may cause low blood sugar, which is also harmful to patients. Then, the blood sugar level of a patient should be between a lower bound y_1 and an upper bound y_2. Let x_0, x_1, x_2 be three insulin amount ($x_0 > x_1 > x_2$). A doctor may conclude that the x_1 amount of insulin is better than x_0, x_2 if the following counterfactual situations are simultaneously true: the patient's blood sugar level (i) would be less than the lower bound y_1 had they taken x_0 amount of insulin, (ii) would be greater than or equal to the lower bound y_1 and less than the upper bound y_2 had they taken x_1 amount, and (iii) would be greater than or equal to the upper bound y_2 had they taken x_2 amount. The doctor wants to know the probability of the above counterfactual situations, which is given by $\mathbb{P}(Y_{x_0} < y_1 \leq Y_{x_1} < y_2 \leq Y_{x_2})$.*

[3]Note that PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$ include PN and PS with evidence as special cases.

Definition 5.2 (Conditional PNS with multi-hypothetical terms)**.** *Conditional PNS with multi-hypothetical terms PNS($\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{c}$) is defined by* $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y}_1 \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}, \boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y}_2 \preceq \boldsymbol{Y}_{\boldsymbol{x}_2}, \dots, \boldsymbol{Y}_{\boldsymbol{x}_{P-1}} \prec \boldsymbol{y}_P \preceq \boldsymbol{Y}_{\boldsymbol{x}_P} | \boldsymbol{C} = \boldsymbol{c})$ *for any sets of values* $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P)$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P)$, *and any* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, *where* $\overline{\boldsymbol{y}}$ *is a set of thresholds of outcome, and* $\overline{\boldsymbol{x}}$ *is a set of treatments.*

For instance, when $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \boldsymbol{y}_2)$, PNS$(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{c}) = \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y}_1 \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y}_2 \preceq \boldsymbol{Y}_{\boldsymbol{x}_2} | \boldsymbol{C} = \boldsymbol{c})$ measures the sufficiency and necessity of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0, \boldsymbol{x}_2$ to produce $\boldsymbol{y}_1 \preceq \boldsymbol{Y} \prec \boldsymbol{y}_2$ given $\boldsymbol{C} = \boldsymbol{c}$.

We have the following theorem:

Theorem 5.2 (Identification of conditional PNS with multi-hypothetical terms)**.** *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, PNS($\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{c}$) is identifiable by*

$$\begin{aligned}\textit{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{c}) = \max\Big\{ &\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\} \\ &- \max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\}, 0\Big\}\end{aligned} \tag{19}$$

for any $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^{P}$, *and* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

We provide an explanation of this result based on analyzing the trajectories of potential outcomes in Appendix B.

5.3 CONDITIONAL PNS WITH MULTI-HYPOTHETICAL TERMS AND EVIDENCE

We consider PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$, combining the settings in Definitions 5.1 and 5.2. Evidence provides the situation-specific information and restricts the attention to a sub-population.

For instance, revisiting Example 3.1, for a patient with a certain age and body weight, a doctor may want to know the probability that the patient's blood sugar level (i) would be less than the lower bound y_1 had they taken x_0 amount of insulin, (ii) would be greater than or equal to the lower bound y_1 and less than the upper bound y_2 had they taken x_1 amount, and (iii) would be greater than or equal to the upper bound y_2 had they taken x_2 amount, when the patient took x' amount of insulin and had blood sugar level y'. This probability is given by $\mathbb{P}(Y_{x_0} < y_1 \leq Y_{x_1} < y_2 \leq Y_{x_2} | Y = y', X = x', \boldsymbol{C} = \boldsymbol{c})$ where $\boldsymbol{c}$ stands for the patient's age and body weight.

Definition 5.3 (Conditional PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$)**.** *Conditional PNS with multi-hypothetical terms and evidence* $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$ *PNS($\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c}$) is defined by* $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y}_1 \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}, \boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y}_2 \preceq \boldsymbol{Y}_{\boldsymbol{x}_2}, \dots, \boldsymbol{Y}_{\boldsymbol{x}_{P-1}} \prec \boldsymbol{y}_P \preceq \boldsymbol{Y}_{\boldsymbol{x}_P} | \boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c})$ *for any* $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^{P}$, *and* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

When $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \boldsymbol{y}_2)$, PNS$(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})$ measures the sufficiency and necessity of $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0, \boldsymbol{x}_2$ to produce $\boldsymbol{y}_1 \preceq \boldsymbol{Y} \prec \boldsymbol{y}_2$ given the evidence $(\boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c})$.

We have the following theorem.

Theorem 5.3 (Identification of conditional PNS with multi--hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$)**.** *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, we have*

(A). *If* $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) \neq \rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, *then we have*

$$\textit{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c}) = \max\{\gamma/\delta, 0\}, \tag{20}$$

where

$$\begin{aligned}\gamma = \min\Big\{&\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\}, \rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\Big\} \\ - \max\Big\{&\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\}, \rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\Big\}, \\ \delta = \rho^o&(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) - \rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\end{aligned} \tag{21}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^{P}$, *and* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

(B). *If* $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) = \rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, *then we have*

$$\begin{aligned}&\textit{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c}) \\ &= \mathbb{I}\Big(\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\} \leq \rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) \\ &\qquad < \min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\}\Big)\end{aligned} \tag{22}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^{P}$, *and* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

In addition, we have the following corollary:

Corollary 5.2. *Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.4, and 4.5, we have*

$$\begin{aligned}&\textit{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c}) \\ &= \mathbb{I}\Big(\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\} \leq \rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) \\ &\qquad < \min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\}\Big)\end{aligned} \tag{23}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^{P}$, *and* $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

6 APPLICATION TO A REAL-WORLD DATASET

Dataset. We take up an open dataset in the UC Irvine Machine Learning Repository `https://archive.ics.uci.edu/dataset/320/student+performance` about student performance in mathematics in secondary education of two Portuguese schools. Secondary education lasts three years, and students are tested once a year, three times in total. The data attributes include demographic, social, and school-related features and student grades. The sample size is 649 with no missing values. Prior research using this data aimed to predict the students' performance based on their attributes [Cortez and Silva, 2008, Helwig, 2017]. We assess the causal relationship between the students' performance, study time, and extra paid classes via estimating PoC introduced in this paper.

Variables. We take the scores of mathematics in the final period (Y^1), in the second period (Y^2), and in the first period (Y^3) as the outcome variables $\boldsymbol{Y} = (Y^1, Y^2, Y^3)$. Y^1, Y^2, Y^3 take values from $\{0, 1, \dots, 20\}$. We assume a lexicographical order $\succ_{\text{lexi}}$ on $\boldsymbol{Y}$. For example, $(Y^1, Y^2, Y^3) \succ_{\text{lexi}} (6, 6, 6)$ means "$Y^1 > 6$" or "$Y^1 = 6 \wedge Y^2 > 6$" or "$Y^1 = 6 \wedge Y^2 = 6 \wedge Y^3 > 6$". We consider "*study time in a week*" (X^1) and "*extra paid classes within the course subject*" (X^2) (yes: $X^2 = 2$, no: $X^2 = 1$) as treatment variables $\boldsymbol{X} = (X^1, X^2)$. We select "sex", "failures", "schoolsup", "famsup", and "goout" as the covariates ($\boldsymbol{C}$), which were chosen in [Helwig, 2017] in a previous study.

We assume Assumption 4.3 which means that latent exogenous variables, such as the student's mental and physical conditions during the test day, have monotonic impacts on the test scores.

Estimation Methods. All identification theorems in the paper compute PoC through conditional CDFs, e.g. $\rho(\boldsymbol{y}; \boldsymbol{x}, \boldsymbol{c}) = \mathbb{P}(\boldsymbol{Y} \prec \boldsymbol{y} | \boldsymbol{X} = \boldsymbol{x}, \boldsymbol{C} = \boldsymbol{c})$. We estimate the conditional CDFs by logistic regression using the "glm" function in R. We conduct the bootstrapping [Efron, 1979] to reveal the distribution of the estimator.

Results. We consider the subject whose ID number is 1. Let the values of her covariates be $\boldsymbol{c}_1$. In reality, she studied for 2 hours a week and took no extra paid classes ($\boldsymbol{x}' = (2, 1)$), and got 6, 6, and 5 scores in the final, second, and first grades, respectively ($\boldsymbol{y}' = (6, 6, 5)$). The other attributes of her are shown in Appendix D.

In the first study, we evaluate conditional PNS, PN, and PS by setting $\boldsymbol{y} = (6, 6, 6)$, $\boldsymbol{x}_0 = (2, 1)$, $\boldsymbol{x}_1 = (4, 2)$, and $\boldsymbol{C} = \boldsymbol{c}_1$ in Def. 4.1 to reveal the necessity/sufficiency of setting $\boldsymbol{x}_1$ w.r.t. $\boldsymbol{x}_0$ to produce $\boldsymbol{Y} \succeq_{\text{lexi}} \boldsymbol{y}$ in the subpopulation characterized by $\boldsymbol{C} = \boldsymbol{c}_1$. The estimated values of conditional PNS, PN, and PS are

$$\begin{aligned} &\text{PNS:} \quad 8.862\% (\text{CI} : [1.122\%, 19.510\%]), \\ &\text{PN:} \quad 9.212\% (\text{CI} : [1.133\%, 20.647\%]), \\ &\text{PS:} \quad 72.331\% (\text{CI} : [27.975\%, 93.022\%]), \end{aligned} \tag{24}$$

where CI represents 95% confidence intervals. The PNS value above represents the probability of the following statement:

"*A student with attributes value $\boldsymbol{c}_1$ would get scores $\boldsymbol{Y} \succeq_{lexi} \boldsymbol{y}$ had she studied 4 hours a week and taken extra classes and would get scores $\boldsymbol{Y} \prec_{lexi} \boldsymbol{y}$ had she studied 2 hours a week and taken no extra class.*"

PN means the probability of the following statement:

"*A student with attributes value $\boldsymbol{c}_1$ would get scores $\boldsymbol{Y} \prec_{lexi} \boldsymbol{y}$ had she studied 2 hours a week and taken no extra class when, in reality, she scored $\boldsymbol{Y} \succeq_{lexi} \boldsymbol{y}$, studied 4 hours a week, and took extra classes.*"

And PS means the probability of the following statement:

"*A student with attributes value $\boldsymbol{c}_1$ would get scores $\boldsymbol{Y} \succeq_{lexi} \boldsymbol{y}$ had she studied 4 hours a week and taken extra classes when, in reality, she scored $\boldsymbol{Y} \prec_{lexi} \boldsymbol{y}$, studied 2 hours a week, and took no extra class.*"

The results reveal that PNS and PN are relatively low, and PS is relatively high. In other words, studying 4 hours and taking extra classes for students with attributes value $\boldsymbol{c}_1$ are unlikely "necessary and sufficient" or "necessary" to achieve $\boldsymbol{Y} \succeq_{\text{lexi}} \boldsymbol{y}$ compared to studying 2 hours and taking no extra class; however, they are highly "sufficient".

In the second study, we consider more detailed evidence than the first study and evaluate conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$, letting $\boldsymbol{y} = (6, 6, 6)$, $\boldsymbol{y}' = (6, 6, 5)$, $\boldsymbol{x}_0 = (2, 1)$, $\boldsymbol{x}_1 = (4, 2)$, $\boldsymbol{x}' = (2, 1)$, and $\boldsymbol{C} = \boldsymbol{c}_1$ in Def. 5.1. The estimated value is

$$\text{PNS:} \ 0.024\% \quad (\text{CI} : [0.000\%, 0.243\%]), \tag{25}$$

which means the probability of the following statement:

"*A student with attributes value $\boldsymbol{c}_1$ would get scores $\boldsymbol{Y} \succeq_{lexi} \boldsymbol{y}$ had she studied 4 hours a week and taken extra classes and would get scores $\boldsymbol{Y} \prec_{lexi} \boldsymbol{y}$ had she studied 2 hours a week and taken no extra class when she scored $\boldsymbol{Y} = \boldsymbol{y}'$, studied 2 hours a week, and took no extra class in reality.*"

We reveal that this probability is very low, that is, studying 4 hours and taking extra classes for students with $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}_1)$ are probably not "necessary and sufficient" to achieve $\boldsymbol{Y} \succeq_{\text{lexi}} \boldsymbol{y}$ compared to studying 2 hours and taking no extra class.

In the third study, we evaluate conditional PNS with multi-hypothetical terms, letting $\boldsymbol{y}_1 = (5, 5, 5)$, $\boldsymbol{y}_2 = (6, 6, 6)$,

$\boldsymbol{x}_0 = (1,1)$, $\boldsymbol{x}_1 = (2,1)$, $\boldsymbol{x}_2 = (4,2)$, and $\boldsymbol{C} = \boldsymbol{c}_1$ in Def. 5.2. The estimated value is

$$\text{PNS: } 0.000\% \quad (\text{CI} : [0.000\%, 0.000\%]), \tag{26}$$

which means the joint probability of the following three counterfactual statements:

"*(i) A student with attributes value $\boldsymbol{c}_1$ would get scores $\boldsymbol{Y} \succeq_{lexi} \boldsymbol{y}_2$ had she studied 4 hours a week and taken extra classes,*
(ii) she would get scores $\boldsymbol{y}_1 \preceq_{lexi} \boldsymbol{Y} \prec_{lexi} \boldsymbol{y}_2$ had she studied 2 hours a week and taken no extra classes, and
(iii) she would get scores $\boldsymbol{Y} \prec_{lexi} \boldsymbol{y}_1$ had she studied 1 hour a week and taken no extra classes."

We reveal that this probability is close to zero, that is, studying 2 hours and taking no extra class for students with attributes value $\boldsymbol{c}_1$ are not "necessary and sufficient" to achieve $\boldsymbol{y}_1 \preceq_{\text{lexi}} \boldsymbol{Y} \prec_{\text{lexi}} \boldsymbol{y}_2$ compared to "studying 1 hour and taking no extra class" or "studying 4 hours and taking extra classes".

Finally, we consider more detailed evidence than the third study and evaluate conditional PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$, letting $\boldsymbol{y}_1 = (5,5,5)$, $\boldsymbol{y}_2 = (6,6,6)$, $\boldsymbol{y}' = (6,6,5)$, $\boldsymbol{x}_0 = (1,1)$, $\boldsymbol{x}_1 = (2,1)$, $\boldsymbol{x}_2 = (4,2)$, $\boldsymbol{x}' = (2,1)$, and $\boldsymbol{C} = \boldsymbol{c}_1$ in Def. 5.3. The estimated value is

$$\text{PNS: } 96.711\% \quad (\text{CI} : [59.059\%, 100.000\%]), \tag{27}$$

which represents the probability of the above three counterfactual statements in the third study given additional information $\boldsymbol{x}'$ and $\boldsymbol{y}'$. Unlike PNS with multi-hypothetical terms in the third study, PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}_1)$ is relatively high. That is, studying 2 hours and taking no extra class with $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}_1)$ are highly "necessary and sufficient" to achieve $\boldsymbol{y}_1 \preceq_{\text{lexi}} \boldsymbol{Y} \prec_{\text{lexi}} \boldsymbol{y}_2$ compared to "studying 1 hour and taking no extra class" and "studying 4 hours and taking extra classes".

We have performed additional analyses. To evaluate the effect of study time (X^1) only, we let $\boldsymbol{x}_1 = (4,1)$ in the first and second analyses, and $\boldsymbol{x}_2 = (4,1)$ in the third and fourth analyses. The results are shown in Appendix D, and all estimated PoC are lower than that obtained with joint effect of study time and extra paid classes. To evaluate the effect of extra paid classes (X^2) only, we let $\boldsymbol{x}_1 = (2,2)$ in the first and second analyses. The results are shown in Appendix D, and all estimated PoC are also lower than the results with joint effect.

7 CONCLUSION

We introduce new types of PoC to capture the causal effects between multiple continuous treatments and outcomes and provide identification theorems. The results greatly expand the range of causal questions that researchers can tackle going beyond binary treatment and outcome. In this paper, we focus on the form of PoC where all treatments are intervened. The scenario of just intervening only a subset of all treatment variables is also useful in real life [Lu et al., 2022, Li et al., 2023], which will be future research. In settings where the monotonicity assumptions do not hold, we may explore methods for bounding PoC. However, for continuous variables, we can not straightforwardly apply linear programming formulation used for bounding binary PoC in [Tian and Pearl, 2000, Li and Pearl, 2024a]. Bounding PoC introduced in this paper will be an interesting future work.

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Appendix to "Probabilities of Causation for Continuous and Vector Variables"

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A ADDITIONAL INFORMATION ON BACKGROUND AND NOTATION

Orders. We explain the orders used in this paper. The definition of the total order is as below [Harzheim, 2005]:

Definition A.1 (Total order)**.** *A total order on a set Ω is a relation "$\preceq$" on Ω satisfying the following four conditions for all $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \Omega$:*

1. $\boldsymbol{a}_1 \preceq \boldsymbol{a}_1$*;*
2. *if* $\boldsymbol{a}_1 \preceq \boldsymbol{a}_2$ *and* $\boldsymbol{a}_2 \preceq \boldsymbol{a}_3$ *then* $\boldsymbol{a}_1 \preceq \boldsymbol{a}_3$*;*
3. *if* $\boldsymbol{a}_1 \preceq \boldsymbol{a}_2$ *and* $\boldsymbol{a}_2 \preceq \boldsymbol{a}_1$ *then* $\boldsymbol{a}_1 = \boldsymbol{a}_2$*;*
4. *at least one of* $\boldsymbol{a}_1 \preceq \boldsymbol{a}_2$ *and* $\boldsymbol{a}_2 \preceq \boldsymbol{a}_1$ *holds.*

In this case we say that the ordered pair $(\Omega, \preceq)$ is a totally ordered set. The inequality $\boldsymbol{a} \preceq \boldsymbol{b}$ of total order means $\boldsymbol{a} \prec \boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{b}$, and the relationship $\neg(\boldsymbol{a} \preceq \boldsymbol{b}) \Leftrightarrow \boldsymbol{a} \succ \boldsymbol{b}$ holds for a totally ordered set, where $\neg$ means the negation.

Definition A.2 (Lexicographical order for the Cartesian product)**.** *A lexicographic order $\prec$ on the Cartesian product of two sets Ω_A and Ω_B with order relations $\preceq_A$ and $\preceq_B$ satisfies: for all $(a_1, b_1) \in \Omega_A \times \Omega_B$ and $(a_2, b_2) \in \Omega_A \times \Omega_B$, $(a_1, b_1) \prec (a_2, b_2)$ if and only if either*

1. $a_1 \prec_A a_2$*, or*
2. $a_1 = a_2$ *and* $b_1 \prec_B b_2$*.*

The lexicographic order can be readily extended to Cartesian products of arbitrary length by recursively applying this definition, i.e., by observing that $\Omega_A \times \Omega_B \times \Omega_C = \Omega_A \times (\Omega_B \times \Omega_C)$. This is widely used as one example of the total order for vector space. The order defined by Mahalanobis' distance and Gaussian distribution [Hannart and Naveau, 2018] is one example of the total orders. Briefly, they consider mapping ϕ from $\Omega = \mathbb{R}^d$ to $\mathbb{R}$, and define $\boldsymbol{a}_0 \preceq \boldsymbol{a}_1$ by the relationship $\phi(\boldsymbol{a}_0) \leq \phi(\boldsymbol{a}_1)$.

B ADDITIONAL INFORMATION ON ANALYZING TRAJECTORIES

In this section, we give the analyzing the trajectories of potential outcomes of Theorems 5.1 and 5.2.

Analyzing Trajectories of Theorem 5.1. We denote $\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x},\boldsymbol{c})} = \sup\{\boldsymbol{u} : f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y}\}$ and $\boldsymbol{u}_{\rho^o(\boldsymbol{y};\boldsymbol{x},\boldsymbol{c})} = \sup\{\boldsymbol{u} : f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{u}) \preceq \boldsymbol{y}\}$. Given $\boldsymbol{C} = \boldsymbol{c}$, the trajectory $\{(\boldsymbol{x}, \boldsymbol{Y}_{\boldsymbol{x}}(\boldsymbol{u})) \in \Omega_{\boldsymbol{X}} \times \Omega_{\boldsymbol{Y}}; \forall \boldsymbol{x} \in \Omega_{\boldsymbol{X}}\}$ represents potential outcome $\boldsymbol{Y}_{\boldsymbol{x}}(\boldsymbol{u})$ vs. $\boldsymbol{X}$ for the subject $\boldsymbol{U} = \boldsymbol{u}$. *Given $\boldsymbol{C} = \boldsymbol{c}$, under Assumptions 4.1 and 4.2 (or 4.3, 4.4), the subjects' trajectories do not cross over each other* (they may overlap).

Consider the trajectories shown in Figure 2. Given $\boldsymbol{C} = \boldsymbol{c}$, Trajectory (1) $\{(\boldsymbol{x}, \boldsymbol{Y}_{\boldsymbol{x}}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})})) \in \Omega_{\boldsymbol{X}} \times \Omega_{\boldsymbol{Y}}; \forall \boldsymbol{x} \in \Omega_{\boldsymbol{X}}\}$ goes through the point $(\boldsymbol{x}_0, \boldsymbol{y})$, Trajectory (2) $\{(\boldsymbol{x}, \boldsymbol{Y}_{\boldsymbol{x}}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})})) \in \Omega_{\boldsymbol{X}} \times \Omega_{\boldsymbol{Y}}; \forall \boldsymbol{x} \in \Omega_{\boldsymbol{X}}\}$ goes through the point

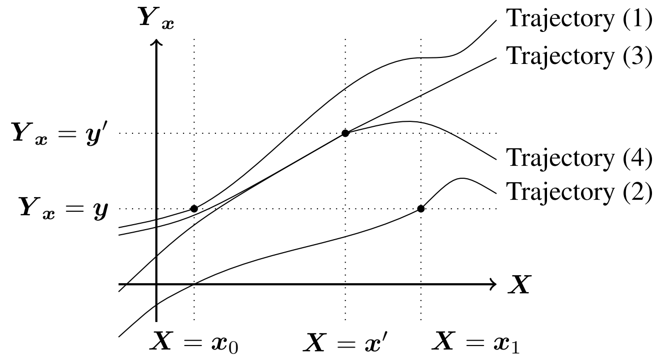

Figure 2: Trajectories for (1) $Y_x(u_{\rho(y;x_0,c)})$, (2) $Y_x(u_{\rho(y;x_1,c)})$, (3) $Y_x(u_{\rho^o(y';x',c)})$ and (4) $Y_x(u_{\rho(y';x',c)})$.

(x_1, y), Trajectory (3) $\{(x, Y_x(u_{\rho^o(y';x',c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ and Trajectory (4) $\{(x, Y_x(u_{\rho(y';x',c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ go through the point (x', y'). Given $C = c$, the trajectory of subject u lies between in the region between Trajectories (1) and (2) if and only if they satisfy $Y_{x_0} \prec y \preceq Y_{x_1}$ given $C = c$. Given $C = c$, the trajectory of subject u lies in the region between Trajectories (3) and (4) if and only if they satisfy $(Y = y', X = x')$ given $C = c$. Thus, we have $\mathbb{P}(Y_{x_0} \prec y \preceq Y_{x_1} | Y = y', X = x', C = c)$ is

$$\max\left\{\frac{\min\{\mathbb{P}(Y_{x_0} \prec y | C = c), \mathbb{P}(Y_{x'} \preceq y' | C = c)\} - \max\{\mathbb{P}(Y_{x_1} \prec y | C = c), \mathbb{P}(Y_{x'} \prec y' | C = c)\}}{\mathbb{P}(Y_{x'} \preceq y' | C = c) - \mathbb{P}(Y_{x'} \prec y' | C = c)}, 0\right\}, \quad (28)$$

where $\mathbb{P}(Y_{x_0} \prec y | C = c)$ represents the probability of a subject's trajectory being below Trajectory (1), $\mathbb{P}(Y_{x_1} \prec y | C = c)$ represents the probability of a subject's trajectory being below Trajectory (2), $\mathbb{P}(Y_{x'} \prec y' | C = c)$ represents the probability of a subject's trajectory being below Trajectory (3) and $\mathbb{P}(Y_{x'} \preceq y' | C = c)$ represents the probability of a subject's trajectory being below Trajectory (4). When $\rho(y'; x', c) = \rho^o(y'; x', c)$, Trajectory (3) coincides with Trajectory (4). $\mathrm{PNS}(y; x_0, x_1, y', x', c)$ represents whether Trajectory (3) or (4) lies in the region between Trajectories (1) and (2), and takes value either 0 or 1.

Analyzing Trajectories of Theorem 5.2. We provide trajectories-based explanation on Theorem 5.2 when $P = 2$. Consider the trajectories shown in Figure 3. Given $C = c$, Trajectory (1) $\{(x, Y_x(u_{\rho(y_1;x_0,c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ goes through the point (x_0, y), Trajectory (2) $\{(x, Y_x(u_{\rho(y_1;x_1,c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ goes through the point (x_1, y), Trajectory (3) $\{(x, Y_x(u_{\rho(y_2;x_1,c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ and Trajectory (4) $\{(x, Y_x(u_{\rho(y_2;x_2,c)})) \in \Omega_X \times \Omega_Y; \forall x \in \Omega_X\}$ go through the point (x', y'). Given $C = c$, the trajectories of subject u lies in the region between Trajectories (1) and (2) if and only if they satisfy $Y_{x_0} \prec y_1 \preceq Y_{x_1}$. Given $C = c$, the trajectories of subject u lies in the region between Trajectories (3) and (4) if and only if they satisfy $Y_{x_1} \prec y_2 \preceq Y_{x_2}$. Thus, we have $\mathbb{P}(Y_{x_0} \prec y_1 \preceq Y_{x_1} \prec y_2 \preceq Y_{x_2} | C = c)$ is

$$\max\{\min\{\mathbb{P}(Y_{x_0} \prec y_1 | C = c), \mathbb{P}(Y_{x_1} \preceq y_2 | C = c)\} - \max\{\mathbb{P}(Y_{x_1} \prec y_1 | C = c), \mathbb{P}(Y_{x_2} \prec y_2 | C = c)\}, 0\}, \quad (29)$$

where $\mathbb{P}(Y_{x_0} \prec y_1 | C = c)$ represents the probability of a subject's trajectory being below Trajectory (1), $\mathbb{P}(Y_{x_1} \prec y_2 | C = c)$ represents the probability of a subject's trajectory being below Trajectory (2), $\mathbb{P}(Y_{x_1} \prec y_2 | C = c)$ represents the probability of a subject's trajectory being below Trajectory (3) and $\mathbb{P}(Y_{x_2} \preceq y_2 | C = c)$ represents the probability of a subject's trajectory being below Trajectory (4).

C PROOFS

We give the proof of lemmas, theorems, and corollary in the body of the paper.

C.1 PROOFS IN SECTION 3

Lemma C.1. *Under SCM $\mathcal{M}_S$ and Assumption 3.6, Assumption 3.3 implies Assumption 3.4.*

Proof. Suppose the negation of Assumption 3.4:

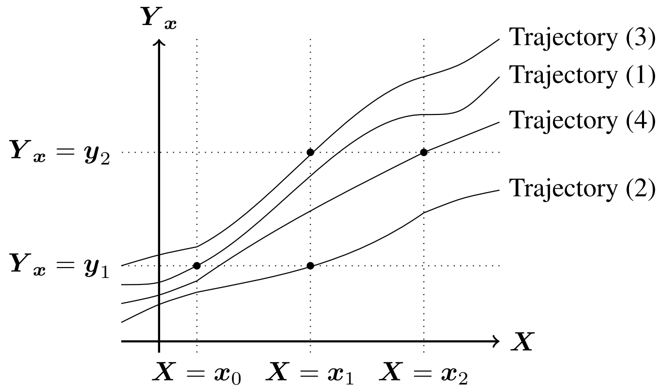

Figure 3: Trajectories for (1) $\boldsymbol{Y_x}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_0,\boldsymbol{c})})$, (2) $\boldsymbol{Y_x}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_1,\boldsymbol{c})})$, (3) $\boldsymbol{Y_x}(\boldsymbol{u}_{\rho(\boldsymbol{y}_2;\boldsymbol{x}_1,\boldsymbol{c})})$ and (4) $\boldsymbol{Y_x}(\boldsymbol{u}_{\rho(\boldsymbol{y}_2;\boldsymbol{x}_2,\boldsymbol{c})})$.

there exists a set $\mathcal{U}$ such that $0 < \mathbb{P}(\mathcal{U}) < 1$, and

$$f_Y(x_0, u_0) \geq y > f_Y(x_0, u_1) \wedge f_Y(x_1, u_0) < y \leq f_Y(x_1, u_1)$$

for some $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$ and for any $u_0, u_1 \in \mathcal{U}$ such that $u_0 < u_1$.

Assumption 3.6 guarantees the existence of overlapping y values in the above since "no overlap" situation $\{f_Y(x_0, u) : u \in \Omega_U\} \cap \{f_Y(x_1, u) : u \in \Omega_U\} = \emptyset$ means the intersection of the support of Y_{x_0} and the support of Y_{x_1} is empty, which violates Assumption 3.6.

Then we have

$$f_Y(x_0, u_0) \geq y > f_Y(x_1, u_0) \text{ and } f_Y(x_0, u_1) < y \leq f_Y(x_1, u_1) \text{ for some } x_0, x_1 \in \Omega_X \text{ and } y \in \Omega_Y \text{ and}$$
$$\text{for any } u_0, u_1 \in \mathcal{U} \text{ such that } u_0 < u_1,$$

and it implies

$$f_Y(x_0, u) \geq y > f_Y(x_1, u) \text{ and } f_Y(x_0, u) < y \leq f_Y(x_1, u) \text{ for some } x_0, x_1 \in \Omega_X \text{ and } y \in \Omega_Y \text{ and for any } u \in \mathcal{U}.$$

This implies the negation of Assumption 3.3 $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) \neq 0$ and $\mathbb{P}(Y_{x_1} < y \leq Y_{x_0}) \neq 0$ for some $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$ since $f_Y(x_0, u) \geq y > f_Y(x_1, u) \Leftrightarrow Y_{x_0}(u) > y \geq Y_{x_1}(u)$ and $f_Y(x_0, u) < y \leq f_Y(x_1, u) \Leftrightarrow Y_{x_1}(u) \geq y > Y_{x_0}(u)$. In conclusion, we have lemma C.1 by taking a contraposition of the above statements. $\square$

Lemma C.2. *Under SCM $\mathcal{M}_S$ and Assumption 3.6, Assumption 3.4 implies Assumption 3.3.*

Proof. First, we denote $u_{sup} = \sup\{u : f_Y(x_0, u) < y\}$. We consider the situations "the function $f_Y(x, U)$ is monotonic increasing on U" and "the function $f_Y(x, U)$ is monotonic decreasing on U", separately.

(1). If the function $f_Y(x, U)$ is **monotonic increasing** on U for all $x \in \Omega_X$ almost surely w.r.t. $\mathbb{P}_U$, we have

$$f_Y(x_0, u_{sup}) \leq f_Y(x_0, u) \text{ and } f_Y(x_1, u_{sup}) \leq f_Y(x_1, u) \tag{30}$$

for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $u \geq u_{sup}$. We have the following statements:

1. Supposed $f_Y(x_0, u_{sup}) > f_Y(x_1, u_{sup})$, we have $y = f_Y(x_0, u_{sup}) > f_Y(x_1, u_{sup}) \geq f_Y(x_1, u) = Y_{x_1}(u)$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $f_Y(x_0, u) < y$. It means $Y_{x_0}(u) < y \Rightarrow Y_{x_1}(u) < y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ and $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = 0$.
2. Supposed $f_Y(x_0, u_{sup}) \leq f_Y(x_1, u_{sup})$, we have $f_Y(x_1, u) \geq f_Y(x_1, u_{sup}) \geq f_Y(x_0, u_{sup}) = y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $f_Y(x_0, u) \geq y$. It means $Y_{x_0}(u) \geq y \Rightarrow Y_{x_1}(u) \geq y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ and $\mathbb{P}(Y_{x_1} < y \leq Y_{x_0}) = 0$.

Then, Assumption 3.4 holds.

(2). If the function $f_Y(x, U)$ is **monotonic decreasing** on U for all $x \in \Omega_X$ almost surely w.r.t. $\mathbb{P}_U$, we have

$$f_Y(x_0, u_{sup}) \geq f_Y(x_0, u) \text{ and } f_Y(x_1, u_{sup}) \geq f_Y(x_1, u) \tag{31}$$

for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $u \geq u_{sup}$. We have the following statements:

1. Supposed $f_Y(x_0, u_{sup}) \leq f_Y(x_1, u_{sup})$, we have $y = f_Y(x_0, u_{sup}) \leq f_Y(x_1, u_{sup}) \leq f_Y(x_1, u) = Y_{x_1}(u)$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $f_Y(x_0, u) \geq y$. It means $Y_{x_0}(u) \geq y \Rightarrow Y_{x_1}(u) \geq y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ and $\mathbb{P}(Y_{x_1} < y \leq Y_{x_0}) = 0$.
2. Supposed $f_Y(x_0, u_{sup}) > f_Y(x_1, u_{sup})$, we have $f_Y(x_1, u) \leq f_Y(x_1, u_{sup}) < f_Y(x_0, u_{sup}) = y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ such that $f_Y(x_0, u) < y$. It means $Y_{x_0}(u) < y \Rightarrow Y_{x_1}(u) < y$ for $\mathbb{P}_U$-almost every $u \in \Omega_U$ and $\mathbb{P}(Y_{x_0} < y \leq Y_{x_1}) = 0$.

Thus, Assumption 3.4 holds. In conclusion, Assumption 3.4 implies Assumption 3.3 □

Theorem 3.1. *Under SCM $\mathcal{M}_S$ and Assumption 3.6, Assumptions 3.3 and 3.4 are equivalent, and Assumptions 3.5 is a strictly stronger requirement than 3.4.*

Proof. We have Theorem 3.1 from Lemma C.1 and C.2. □

Theorem 3.2. *(Identification of PoC) Under SCM $\mathcal{M}_S$ and Assumptions 3.1, 3.3 (or 3.4, 3.5), and 3.6, PNS, PN, and PS are identifiable by*

$$\begin{aligned}
\textit{PNS}(y; x_0, x_1) &= \max\{\rho(y; x_0) - \rho(y; x_1), 0\},\\
\textit{PN}(y; x_0, x_1) &= \max\left\{\frac{\rho(y; x_0) - \rho(y; x_1)}{1 - \rho(y; x_1)}, 0\right\},\\
\textit{PS}(y; x_0, x_1) &= \max\left\{\frac{\rho(y; x_0) - \rho(y; x_1)}{\rho(y; x_0)}, 0\right\}
\end{aligned} \tag{32}$$

for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$ such that $\rho(y; x_1) < 1$ and $\rho(y; x_0) > 0$.

Proof. Under Assumptions 3.1 and 3.4,

$$\begin{aligned}
\text{PNS}(y; x_0, x_1) &= \mathbb{P}(Y_{x_0} < y \leq Y_{x_1})\\
&= \mathbb{P}(u_{\rho(y;x_1)} \leq u < u_{\rho(y;x_0)})\\
&= \max\{\rho(y; x_0) - \rho(y; x_1), 0\}
\end{aligned} \tag{33}$$

for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$, where $u_{\rho(y;x_0)} = \sup\{u : f_Y(x_0, u) < y\}$ and $u_{\rho(y;x_1)} = \sup\{u : f_Y(x_1, u) \leq y\}$. Note that all u such that $u \leq u_{\rho(y;x)}$ satisfy $f_Y(x, u) < y$ from Assumption 3.4. In addition, $\text{PN}(y; x_0, x_1)$ and $\text{PS}(y; x_0, x_1)$ are given from the following relationship:

$$\text{PN}(y; x_0, x_1) = \frac{\text{PSN}(y; x_0, x_1)}{\mathbb{P}(y \leq Y | X = x_1)}, \quad \text{PS}(y; x_0, x_1) = \frac{\text{PSN}(y; x_0, x_1)}{\mathbb{P}(Y < y | X = x_0)}, \tag{34}$$

$\mathbb{P}(y \leq Y | X = x_1) = 1 - \rho(y; x_1)$ and $\mathbb{P}(Y < y | X = x_0) = \rho(y; x_0)$ for any $x_0, x_1 \in \Omega_X$ and $y \in \Omega_Y$. □

C.2 PROOFS IN SECTION 4

Theorem 4.1. *Under SCM $\mathcal{M}_T$ and Assumption 4.5, Assumptions 4.2 and 4.3 are equivalent, and Assumptions 4.4 is a strictly stronger requirement than 4.3.*

Proof. We show the proof of equivalence of assumptions.

(Assumption 4.2 ⇒ Assumption 4.3.) For any $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, from Assumption 4.5, if we have the negation of Assumption 4.3

$$\text{there exists a set } \mathcal{U} \text{ such that } 0 < \mathbb{P}(\mathcal{U}) < 1, \text{ and}$$

$$f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_0) \succeq \boldsymbol{y} \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_1) \wedge f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_0) \prec y \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_1)$$

$$\text{for some } \boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}} \text{ and } \boldsymbol{y} \in \Omega_{\boldsymbol{Y}} \text{ and for any } \boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{U} \text{ such that } \boldsymbol{u}_0 \preceq \boldsymbol{u}_1,$$

then we have

$$f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_0) \succeq \boldsymbol{y} \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{u}_0) \text{ and } f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_1) \prec \boldsymbol{y} \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_1) \text{ for some } \boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}} \text{ and } \boldsymbol{y} \in \Omega_{\boldsymbol{Y}} \text{ and for any } \boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathcal{U} \text{ such that } \boldsymbol{u}_0 \preceq \boldsymbol{u}_1,$$

and we also have

$$f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \text{ and } f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y} \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \text{ for some } \boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{X}} \text{ and } \boldsymbol{y} \in \Omega_{\boldsymbol{Y}} \text{ and for any } \boldsymbol{u} \in \mathcal{U}.$$

This implies the negation of Assumption 3.3 $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} | \boldsymbol{C} = \boldsymbol{c}) \neq 0$ and $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_0} | \boldsymbol{C} = \boldsymbol{c}) \neq 0$ for some $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Omega_{\boldsymbol{Y}}$ and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$ since $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \Leftrightarrow \boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \succ \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u})$ and $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y} \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \Leftrightarrow \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \succ \boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u})$.

(Assumption 4.3 $\Rightarrow$ Assumption 4.2.) For any $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, we denote $\boldsymbol{u}_{sup} = \sup\{\boldsymbol{u} : f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \preceq \boldsymbol{y}\}$. We consider the situations "the function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is monotonic increasing on $\boldsymbol{U}$" and "the function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is monotonic decreasing on $\boldsymbol{U}$", separately.

(1). If the function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is **monotonic increasing** on $\boldsymbol{U}$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$, we have

$$f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \text{ and } f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \tag{35}$$

for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $\boldsymbol{u} \succeq \boldsymbol{u}_{sup}$. We have the following statements:

1. Supposed $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup})$, we have $\boldsymbol{y} = f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succeq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) = Y_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u})$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y}$. It means $\boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y} \Rightarrow \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ and $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} | \boldsymbol{C} = \boldsymbol{c}) = 0$.
2. Supposed $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup})$, we have $f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \succeq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succeq f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) = \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y}$. It means $\boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \Rightarrow \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ and $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_0} | \boldsymbol{C} = \boldsymbol{c}) = 0$.

Then, these imply Assumption 3.4.

(2). If the function $f_{\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{c}, \boldsymbol{U})$ is **monotonic decreasing** on $\boldsymbol{U}$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}}$ almost surely w.r.t. $\mathbb{P}_{\boldsymbol{U}}$, we have

$$f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succeq f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \text{ and } f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succeq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \tag{36}$$

for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $\boldsymbol{u} \succeq \boldsymbol{u}_{sup}$. We have the following statements:

1. Supposed $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup})$, we have $\boldsymbol{y} = f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) = \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u})$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y}$. It means $\boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u}) \succeq \boldsymbol{y} \Rightarrow \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{u}) \succeq \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ and $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_0} | \boldsymbol{C} = \boldsymbol{c}) = 0$.
2. Supposed $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) \succ f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup})$, we have $f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}) \prec f_{\boldsymbol{Y}}(\boldsymbol{x}_1, \boldsymbol{c}, \boldsymbol{u}_{sup}) \preceq f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}_{sup}) = \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ such that $f_{\boldsymbol{Y}}(\boldsymbol{x}_0, \boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y}$. It means $\boldsymbol{Y}_{\boldsymbol{x}_0}(\boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y} \Rightarrow \boldsymbol{Y}_{\boldsymbol{x}_1}(\boldsymbol{c}, \boldsymbol{u}) \prec \boldsymbol{y}$ for $\mathbb{P}_{\boldsymbol{U}}$-almost every $\boldsymbol{u} \in \Omega_{\boldsymbol{U}}$ and $\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} | \boldsymbol{C} = \boldsymbol{c}) = 0$.

Then, Assumption 4.3 holds. In conclusion, Assumption 4.3 implies Assumption 4.2.

$\square$

Theorem 4.2. *(Identification of conditional PoC) Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, PNS, PN, and PS are identifiable by*

$$\begin{aligned}
PNS(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c}) &= \max\{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}),0\},\\
PN(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c}) &= \max\left\{\frac{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})}{1-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})},0\right\},\\
PS(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c}) &= \max\left\{\frac{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})}{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})},0\right\}
\end{aligned} \tag{37}$$

for any $\boldsymbol{x}_0,\boldsymbol{x}_1\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$, and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$ such that $\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})<1$ and $\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})>0$.

Proof. Under Assumptions 4.1 and 4.2 (or 4.3),

$$\begin{aligned}
\mathrm{PNS}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c}) &= \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}\preceq\boldsymbol{Y}_{\boldsymbol{x}_1}|\boldsymbol{C}=\boldsymbol{c})\\
&= \mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})})\\
&= \max\{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}),0\}
\end{aligned} \tag{38}$$

for any $\boldsymbol{x}_0,\boldsymbol{x}_1\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$ and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$, where $\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})}=\sup\{\boldsymbol{u}: f_{\boldsymbol{Y}}(\boldsymbol{x}_0,\boldsymbol{c},\boldsymbol{u})\prec\boldsymbol{y}\}$ and $\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})}=\sup\{\boldsymbol{u}: f_{\boldsymbol{Y}}(\boldsymbol{x}_1,\boldsymbol{c},\boldsymbol{u})\prec\boldsymbol{y}\}$.

$\mathrm{PN}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})$ and $\mathrm{PS}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})$ are given from the following relationship:

$$\mathrm{PN}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})=\frac{\mathrm{PNS}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})}{\mathbb{P}(\boldsymbol{y}\preceq\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}_1,\boldsymbol{C}=\boldsymbol{c})},\quad \mathrm{PS}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})=\frac{\mathrm{PNS}(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{c})}{\mathbb{P}(\boldsymbol{Y}\prec\boldsymbol{y}|\boldsymbol{X}=\boldsymbol{x}_0,\boldsymbol{C}=\boldsymbol{c})}, \tag{39}$$

and $\mathbb{P}(\boldsymbol{y}\preceq\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}_1,\boldsymbol{C}=\boldsymbol{c})=1-\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})$ and $\mathbb{P}(\boldsymbol{y}\prec\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}_0,\boldsymbol{C}=\boldsymbol{c})=\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})$ for any $\boldsymbol{x}_0,\boldsymbol{x}_1\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$ and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$. □

C.3 PROOFS IN SECTION 5

Theorem 5.1. *(Identification of conditional PNS with evidence $(\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})$) Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, we have*

(A). *If $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\neq\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, then we have*

$$PNS(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})=\max\{\alpha/\beta,0\}, \tag{40}$$

where

$$\begin{aligned}
\alpha &= \min\{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c}),\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}-\max\{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}),\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\},\\
\beta &= \rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})-\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})
\end{aligned} \tag{41}$$

for any $\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}'\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$, $\boldsymbol{y}'\in\Omega_{\boldsymbol{Y}}$, and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$.

(B). *If $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})=\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, then we have*

$$PNS(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})=\mathbb{I}(\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})\le\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})<\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})) \tag{42}$$

for any $\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}'\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$, $\boldsymbol{y}'\in\Omega_{\boldsymbol{Y}}$, and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$.

Proof. Under Assumptions 4.1 and 4.3, if $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\neq\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, we have

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}\preceq\boldsymbol{Y}_{\boldsymbol{x}_1}|\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}',\boldsymbol{C}=\boldsymbol{c})\\
&=\frac{\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}\preceq\boldsymbol{Y}_{\boldsymbol{x}_1},\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}'|\boldsymbol{C}=\boldsymbol{c})}{\mathbb{P}(\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}'|\boldsymbol{C}=\boldsymbol{c})}\\
&=\frac{\mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})},\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})}{\mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\prec\boldsymbol{u}\preceq\boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})}\\
&=\frac{\max\{\min\{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c}),\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}-\max\{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}),\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\},0\}}{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})-\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\\
&=\max\left\{\frac{\min\{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c}),\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}-\max\{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}),\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}}{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})-\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})},0\right\}.
\end{aligned} \tag{43}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$ and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$. This represents the statement (A). Otherwise, since $\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})} = \boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}$, we have

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y} \preceq \boldsymbol{Y}_{\boldsymbol{x}_1} | \boldsymbol{Y} = \boldsymbol{y}', \boldsymbol{X} = \boldsymbol{x}', \boldsymbol{C} = \boldsymbol{c}) \\
&= \mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})} \preceq \boldsymbol{u} \prec \boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})} | \boldsymbol{u} = \boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}) \\
&= \mathbb{I}(\boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c})} \preceq \boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})} \prec \boldsymbol{u}_{\rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c})}) \\
&= \mathbb{I}(\rho(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{c}) \leq \rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) < \rho(\boldsymbol{y};\boldsymbol{x}_0,\boldsymbol{c}))
\end{aligned} \tag{44}$$

for any $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$ and $\boldsymbol{y} \in \Omega_{\boldsymbol{Y}}$. This represents the statement (B). □

Theorem 5.2. *(Identification of conditional PNS with multi-hypothetical terms) Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, PNS$(\overline{\boldsymbol{y}}; \overline{\boldsymbol{x}}, \boldsymbol{c})$ is identifiable by*

$$\textit{PNS}(\overline{\boldsymbol{y}}; \overline{\boldsymbol{x}}, \boldsymbol{c}) = \max\left\{ \min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_{p-1}, \boldsymbol{c})\} - \max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_p, \boldsymbol{c})\}, 0\right\} \tag{45}$$

for any $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^P$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

Proof. Under Assumptions 4.1 and 4.3, we have

$$\begin{aligned}
&\text{PNS}(\overline{\boldsymbol{y}}; \overline{\boldsymbol{x}}, \boldsymbol{c}) \\
&= \mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0} \prec \boldsymbol{y}_1 \preceq \boldsymbol{Y}_{\boldsymbol{x}_1}, \boldsymbol{Y}_{\boldsymbol{x}_1} \prec \boldsymbol{y}_2 \preceq \boldsymbol{Y}_{\boldsymbol{x}_2}, \dots, \boldsymbol{Y}_{\boldsymbol{x}_{P-1}} \prec \boldsymbol{y}_P \preceq \boldsymbol{Y}_{\boldsymbol{x}_P} | \boldsymbol{C} = \boldsymbol{c}) \\
&= \mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_0,\boldsymbol{c})} \preceq \boldsymbol{u} \prec \boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_1,\boldsymbol{c})}, \boldsymbol{u}_{\rho(\boldsymbol{y}_2;\boldsymbol{x}_1,\boldsymbol{c})} \preceq \boldsymbol{u} \prec \boldsymbol{u}_{\rho(\boldsymbol{y}_2;\boldsymbol{x}_2,\boldsymbol{c})}, \dots, \boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_P,\boldsymbol{c})} \preceq \boldsymbol{u} \prec \boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_{P-1},\boldsymbol{c})}) \\
&= \mathbb{P}(\boldsymbol{u}_{\max_p\{\rho(\boldsymbol{y}_P;\boldsymbol{x}_P,\boldsymbol{c})\}} \prec \boldsymbol{u} \preceq \boldsymbol{u}_{\min_p\{\rho(\boldsymbol{y}_P;\boldsymbol{x}_{P-1},\boldsymbol{c})\}}) \\
&= \max\left\{ \min_p\{\rho(\boldsymbol{y}_P; \boldsymbol{x}_{P-1}, \boldsymbol{c})\} - \max_p\{\rho(\boldsymbol{y}_P; \boldsymbol{x}_P, \boldsymbol{c})\}, 0\right\}
\end{aligned} \tag{46}$$

for any $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^P$ and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$. □

Theorem 5.3. *(Identification of conditional PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$) Under SCM $\mathcal{M}_T$ and Assumptions 4.1, 4.2 (or 4.3, 4.4), and 4.5, we have*

(A). *If $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) \neq \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$, then we have*

$$\textit{PNS}(\overline{\boldsymbol{y}}; \overline{\boldsymbol{x}}, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) = \max\{\gamma/\delta, 0\}, \tag{47}$$

where

$$\begin{aligned}
&\gamma = \min\left\{ \min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_{p-1}, \boldsymbol{c})\}, \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})\right\} - \max\left\{ \max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_p, \boldsymbol{c})\}, \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})\right\}, \\
&\delta = \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) - \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})
\end{aligned} \tag{48}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^P$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

(B). *If $\rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) = \rho^o(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c})$, then we have*

$$\textit{PNS}(\overline{\boldsymbol{y}}; \overline{\boldsymbol{x}}, \boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c}) = \mathbb{I}\Big(\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_p, \boldsymbol{c})\} \leq \rho(\boldsymbol{y}'; \boldsymbol{x}', \boldsymbol{c}) < \min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p; \boldsymbol{x}_{p-1}, \boldsymbol{c})\}\Big) \tag{49}$$

for any $\boldsymbol{x}' \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{y}' \in \Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_P) \in \Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_P) \in \Omega_{\boldsymbol{Y}}^P$, and $\boldsymbol{c} \in \Omega_{\boldsymbol{C}}$.

Proof. Under Assumptions 4.1 and 4.3, if $\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c}) \neq \rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})$, Eq. (20) holds since we have

$$\begin{aligned}
&\text{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})\\
&=\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}_1\preceq\boldsymbol{Y}_{\boldsymbol{x}_1},\boldsymbol{Y}_{\boldsymbol{x}_1}\prec\boldsymbol{y}_2\preceq\boldsymbol{Y}_{\boldsymbol{x}_2},\dots,\boldsymbol{Y}_{\boldsymbol{x}_{P-1}}\prec\boldsymbol{y}_P\preceq\boldsymbol{Y}_{\boldsymbol{x}_P}|\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}',\boldsymbol{C}=\boldsymbol{c})\\
&=\frac{\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}_1\preceq\boldsymbol{Y}_{\boldsymbol{x}_1},\boldsymbol{Y}_{\boldsymbol{x}_1}\prec\boldsymbol{y}_2\preceq\boldsymbol{Y}_{\boldsymbol{x}_2},\dots,\boldsymbol{Y}_{\boldsymbol{x}_{P-1}}\prec\boldsymbol{y}_P\preceq\boldsymbol{Y}_{\boldsymbol{x}_P},\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}'|\boldsymbol{C}=\boldsymbol{c})}{\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}'}=\boldsymbol{y}'|\boldsymbol{C}=\boldsymbol{c})}\\
&=\frac{\mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_0,\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_1,\boldsymbol{c})},\dots,\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_{P-1},\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_P,\boldsymbol{c})},\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})}{\mathbb{P}(\boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})}\\
&=\frac{\max\{\min\{\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\},\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}-\max\{\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\},\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\},0\}}{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})-\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\\
&=\max\left\{\frac{\min\{\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\},\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}-\max\{\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\},\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})\}}{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})-\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})},0\right\}
\end{aligned}\tag{50}$$

for any $\boldsymbol{x}'\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{y}'\in\Omega_{\boldsymbol{Y}}$, $\overline{\boldsymbol{x}}=(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_P)\in\Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}}=(\boldsymbol{y}_1,\dots,\boldsymbol{y}_P)\in\Omega_{\boldsymbol{Y}}^{P}$ and $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$. This represents the statement (A). Otherwise, since $Y_x(\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})=Y_x(\boldsymbol{u}_{\rho^o(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})$, we have

$$\begin{aligned}
&\text{PNS}(\overline{\boldsymbol{y}};\overline{\boldsymbol{x}},\boldsymbol{y}',\boldsymbol{x}',\boldsymbol{c})\\
&=\mathbb{P}(\boldsymbol{Y}_{\boldsymbol{x}_0}\prec\boldsymbol{y}_1\preceq\boldsymbol{Y}_{\boldsymbol{x}_1},\boldsymbol{Y}_{\boldsymbol{x}_1}\prec\boldsymbol{y}_2\preceq\boldsymbol{Y}_{\boldsymbol{x}_2},\dots,\boldsymbol{Y}_{\boldsymbol{x}_{P-1}}\prec\boldsymbol{y}_P\preceq\boldsymbol{Y}_{\boldsymbol{x}_P}|\boldsymbol{Y}=\boldsymbol{y}',\boldsymbol{X}=\boldsymbol{x}',\boldsymbol{C}=\boldsymbol{c})\\
&=\mathbb{P}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_0,\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_1,\boldsymbol{c})},\dots,\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_{P-1},\boldsymbol{c})}\preceq\boldsymbol{u}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_P,\boldsymbol{c})}|\boldsymbol{u}=\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})})\\
&=\mathbb{I}(\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_0,\boldsymbol{c})}\preceq\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_1;\boldsymbol{x}_1,\boldsymbol{c})},\dots,\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_{P-1},\boldsymbol{c})}\preceq\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\prec\boldsymbol{u}_{\rho(\boldsymbol{y}_P;\boldsymbol{x}_P,\boldsymbol{c})})\\
&=\mathbb{I}(\boldsymbol{u}_{\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\}}\preceq\boldsymbol{u}_{\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})}\prec\boldsymbol{u}_{\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\}})\\
&=\mathbb{I}\Big(\max_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_p,\boldsymbol{c})\}\leq\rho(\boldsymbol{y}';\boldsymbol{x}',\boldsymbol{c})<\min_{p=1,\dots,P}\{\rho(\boldsymbol{y}_p;\boldsymbol{x}_{p-1},\boldsymbol{c})\}\Big)
\end{aligned}\tag{51}$$

for any $\overline{\boldsymbol{x}}=(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_P)\in\Omega_{\boldsymbol{X}}^{P+1}$, $\overline{\boldsymbol{y}}=(\boldsymbol{y}_1,\dots,\boldsymbol{y}_P)\in\Omega_{\boldsymbol{Y}}^{P}$, $\boldsymbol{x}'\in\Omega_{\boldsymbol{X}}$, $\boldsymbol{c}\in\Omega_{\boldsymbol{C}}$, $\boldsymbol{y}'\in\Omega_{\boldsymbol{Y}}$ and $\boldsymbol{y}\in\Omega_{\boldsymbol{Y}}$. This represents the statement (B). □

D ADDITIONAL INFORMATION ON APPLICATION

In this section, we provide additional information on the application.

D.1 DETAILS OF DATASET

First, we explain all variables in the application. We pick up the following variables as **outcomes***.*

1. *G1 - first period grade (numeric: from 0 to 20)*
2. *G2 - second period grade (numeric: from 0 to 20)*
3. *G3 - final grade (numeric: from 0 to 20, output target)*

We pick up the following variables as **treatments***.*

1. *studytime - weekly study time (numeric: 1 - < 2 hours, 2 - 2 to 5 hours, 3 - 5 to 10 hours, or 4 - >10 hours)*
2. *paid - extra paid classes within the course subject (Math or Portuguese) (binary: yes or no)*

We show the other variables as potential **covariates***.*

1. *school - student's school (binary: 'GP' - Gabriel Pereira or 'MS' - Mousinho da Silveira)*
2. *sex - student's sex (binary: 'F' - female or 'M' - male)*
3. *age - student's age (numeric: from 15 to 22)*
4. *address - student's home address type (binary: 'U' - urban or 'R' - rural)*
5. *famsize - family size (binary: 'LE3' - less or equal to 3 or 'GT3' - greater than 3)*

Table 1: Attributes of the ID number 1 subject.

school	sex	age	address	famsize	Pstatus	Medu	Fedu	Mjob	Fjob	reason
GP	F	18	U	GT3	A	4	4	at_home	teacher	course
guardian	traveltime	studytime	failures	schoolsup	famsup	paid	activities	nursery	higher	internet
mother	2	2	0	yes	no	no	no	yes	yes	no
romantic	famrel	freetime	goout	Dalc	Walc	health	absences	G1	G2	G3
no	4	3	4	1	1	3	6	5	6	6

6. *Pstatus - parent’s cohabitation status (binary: ’T’ - living together or ’A’ - apart)*
7. *Medu - mother’s education (numeric: 0 - none, 1 - primary education (4th grade), 2 “ 5th to 9th grade, 3 “ secondary education or 4 “ higher education)*
8. *Fedu - father’s education (numeric: 0 - none, 1 - primary education (4th grade), 2 “ 5th to 9th grade, 3 “ secondary education or 4 “ higher education)*
9. *Mjob - mother’s job (nominal: ’teacher’, ’health’ care related, civil ’services’ (e.g. administrative or police), ’at home’ or ’other’)*
10. *Fjob - father’s job (nominal: ’teacher’, ’health’ care related, civil ’services’ (e.g. administrative or police), ’at home’ or ’other’)*
11. *reason - reason to choose this school (nominal: close to ’home’, school ’reputation’, ’course’ preference or ’other’)*
12. *guardian - student’s guardian (nominal: ’mother’, ’father’ or ’other’)*
13. *traveltime - home to school travel time (numeric: 1 - <15 min., 2 - 15 to 30 min., 3 - 30 min. to 1 hour, or 4 - >1 hour)*
14. *failures - number of past class failures (numeric: n if 1<=n<3, else 4)*
15. *schoolsup - extra educational support (binary: yes or no)*
16. *famsup - family educational support (binary: yes or no)*
17. *activities - extra-curricular activities (binary: yes or no)*
18. *nursery - attended nursery school (binary: yes or no)*
19. *higher - wants to take higher education (binary: yes or no)*
20. *internet - Internet access at home (binary: yes or no)*
21. *romantic - with a romantic relationship (binary: yes or no)*
22. *famrel - quality of family relationships (numeric: from 1 - very bad to 5 - excellent)*
23. *freetime - free time after school (numeric: from 1 - very low to 5 - very high)*
24. *goout - going out with friends (numeric: from 1 - very low to 5 - very high)*
25. *Dalc - workday alcohol consumption (numeric: from 1 - very low to 5 - very high)*
26. *Walc - weekend alcohol consumption (numeric: from 1 - very low to 5 - very high)*
27. *health - current health status (numeric: from 1 - very bad to 5 - very good)*
28. *absences - number of school absences (numeric: from 0 to 93)*

We show the attributes of ID number 1 in Table 1.

D.2 ADDITIONAL ANALYSES OF APPLICATION

We give three additional analyses of the four applications in the body of the paper.

Effect of study time only. *First, we evaluate conditional PNS, PN, and PS, letting $\boldsymbol{y} = (6, 6, 6)$, $\boldsymbol{x}_0 = (2, 1)$, $\boldsymbol{x}_1 = (4, 1)$, and $\boldsymbol{c}_1$ in Def. 4.1. The estimated values of conditional PNS, PN, and PS are*

$$\begin{aligned} \textit{PNS:} \quad & 2.491\%(\textit{CI}: [0.000\%, 7.395\%]), \\ \textit{PN:} \quad & 2.709\%(\textit{CI}: [0.000\%, 8.060\%]), \\ \textit{PS:} \quad & 25.864\%(\textit{CI}: [0.000\%, 73.544\%]), \end{aligned} \tag{52}$$

respectively. Second, we evaluate conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$*, letting* $\boldsymbol{y} = (6,6,6)$*,* $\boldsymbol{y}' = (6,6,5)$*,* $\boldsymbol{x}_0 = (2,1)$*,* $\boldsymbol{x}_1 = (4,1)$*,* $\boldsymbol{x}' = (2,1)$*, and* $\boldsymbol{c}_1$ *in Def. 5.1. Then, the estimated value of it is*

$$\textit{PNS:}\ 0.000\% \quad (\textit{CI}: [0.000\%, 0.000\%]). \tag{53}$$

Third, we evaluate conditional PNS with multi-hypothetical terms, letting $\boldsymbol{y}_1 = (5,5,5)$*,* $\boldsymbol{y}_2 = (6,6,6)$*,* $\boldsymbol{x}_0 = (1,1)$*,* $\boldsymbol{x}_1 = (2,1)$*,* $\boldsymbol{x}_2 = (4,1)$*, and* $\boldsymbol{c}_1$ *in Def. 5.2. The estimated value of it is*

$$\textit{PNS:}\ 0.000\% \quad (\textit{CI}: [0.000\%, 0.000\%]). \tag{54}$$

Finally, we evaluate conditional PNS with multi-hypothetical terms and evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$*, letting* $\boldsymbol{y}_1 = (5,5,5)$*,* $\boldsymbol{y}_2 = (6,6,6)$*,* $\boldsymbol{y}' = (6,6,5)$*,* $\boldsymbol{x}_0 = (1,1)$*,* $\boldsymbol{x}_1 = (2,1)$*,* $\boldsymbol{x}_2 = (4,1)$*,* $\boldsymbol{x}' = (2,1)$*, and* $\boldsymbol{c}_1$ *in Def. 5.3 . We eliminate the results of NA, then the estimated value of it is*

$$\textit{PNS:}\ 42.489\% \quad (\textit{CI}: [0.000\%, 100.000\%]). \tag{55}$$

Effect of extra paid classes only. *First, we evaluate conditional PNS, PN, and PS, letting* $\boldsymbol{y} = (6,6,6)$*,* $\boldsymbol{x}_0 = (1,1)$*,* $\boldsymbol{x}_1 = (2,2)$*, and* $\boldsymbol{c}_1$ *in Def. 4.1. The estimated values of conditional PNS, PN, and PS are*

$$\begin{aligned} &\textit{PNS:} && 7.700\%(\textit{CI}: [1.072\%, 16.614\%]), \\ &\textit{PN:} && 8.132\%(\textit{CI}: [1.090\%, 18.139\%]), \\ &\textit{PS:} && 65.398\%(\textit{CI}: [37.015\%, 89.795\%]), \end{aligned} \tag{56}$$

respectively. Second, we evaluate conditional PNS with evidence $(\boldsymbol{y}', \boldsymbol{x}', \boldsymbol{c})$*, letting* $\boldsymbol{y} = (6,6,6)$*,* $\boldsymbol{y}' = (6,6,5)$*,* $\boldsymbol{x}_0 = (1,1)$*,* $\boldsymbol{x}_1 = (2,2)$*,* $\boldsymbol{x}' = (2,1)$*, and* $\boldsymbol{c}_1$ *in Def. 5.1. Then, the estimated value of it is*

$$\textit{PNS:}\ 0.009\% \quad (\textit{CI}: [0.000\%, 0.139\%]). \tag{57}$$