**Exercise 1.** The concept of *change of measure* in terms of a Radon-Nikodym derivative can be summarized as in the following diagram:

$$\begin{tikzcd}
(\Omega, \mathcal{F}, \mathbb{P}) \arrow[r, "X(\omega)"] \arrow[d, "\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)"'] & f_X(x) \arrow[d] \\
(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \arrow[r, "X(\omega)"] & \tilde{f}_X(x)
\end{tikzcd}$$

(a) Assuming that in the diagram, both probability density functions $f_X(x)$ and $\tilde{f}_X(x)$ for a random variable $X(\omega)$ are given. Find the RND $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)$ in terms of the $X(\omega)$.
(b) In the diagram below, for a given random variable $X : \Omega \to \mathbb{R}$ and a smooth function $g(x) : \mathbb{R} \to \mathbb{R}$, let us assume random variable $Y(\omega) = h^{-1}\big(X(\omega)\big)$ under the new measure $\tilde{\mathbb{P}}$ has a probability density function

$$\tilde{f}_Y(x) = f_X(x).$$

$$\begin{tikzcd}
(\Omega, \mathcal{F}, \mathbb{P}) \arrow[r, "X(\omega)"] \arrow[d, "\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega) = g[X(\omega)]"'] & f_X(x) \arrow[d] \\
(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \arrow[r, "X(\omega)"] \arrow[ur, "Y(\omega)"] & \tilde{f}_X(x)
\end{tikzcd}$$

Find the function $h(y)$.
(c) Now consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $X(\omega) = \big(X_1, X_2, \cdots, X_n\big)(\omega)$ is a $n$-dimensional random variables, whose sccessive differences $X_j - X_{j-1}$ are all conditionally, normally distributed independent random variables:

$$X_{j+1} - X_j \sim \mathcal{N}\Big(\mu_{j+1}(X_j), \sigma^2_{j+1}(X_j)\Big).$$

Find the change of measures $Z(\omega) = \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)$ such that under the new measure $\tilde{\mathbb{P}}$,

$$X_{j+1} - X_j \sim \mathcal{N}\big(0, \sigma^2_{j+1}(X_j)\big).$$

What is the conditional expectation

$$\mathbb{E}\big[Z|X_1, \cdots, X_k\big]$$

for $k < n$?
(d) Applying the result from (c), show the following expression

$$Z_T = \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{b^2(X_s)}{A^2(X_s)}\right) \mathrm{d}s - \int_0^T \left(\frac{b(X_s)}{A(X_s)}\right) \mathrm{d}W(t)\right\}$$

represents a change of measure, from $\mathbb{P}$ to $\tilde{\mathbb{P}}$. The Ito process on $[0, T]$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + A(X_t)\mathrm{d}W(t),$$

under measure $\mathbb{P}$ then becomes

$$\mathrm{d}X_t = A(X_t)\mathrm{d}\widetilde{W}(t)$$

under measure $\tilde{\mathbb{P}}$.

**Solution 1.** (a) Just to be clear I'm assuming that $\tilde{f}_X$ and $f_X$ are densities with respect to Lebesgue which I just write as $\mathrm{d}x$, so that we have

$$\mathbb{P}(X \in A) = \int_A f_X(x)\mathrm{d}x$$
$$\tilde{\mathbb{P}}(X \in A) = \int_A \tilde{f}_X(\omega)\mathrm{d}x.$$

We then have that if $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}$ is the Radon-Nikodym derivative

$$\begin{aligned}
\tilde{\mathbb{P}}(X \in A) &= \int_A \tilde{f}_X(x)\mathrm{d}x \\
&= \int_A \mathrm{d}\tilde{\mathbb{P}} \\
&= \int_A \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\mathrm{d}\mathbb{P} \\
&= \int_A \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} f_X(x)\mathrm{d}x,
\end{aligned}$$

where we've written $\mathrm{d}\mathbb{P} = f_X(x)\mathrm{d}x$ and $\mathrm{d}\tilde{\mathbb{P}} = \tilde{f}_X(x)\mathrm{d}x$. The above equalities hold when

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(x) = \frac{\tilde{f}_X(x)}{f_X(x)}.$$

This appears to be just the likelihood ratio of the densities.
(b) By the definition $Y = h^{-1}(X)$, we have that

$$\begin{aligned}
\tilde{\mathbb{P}}(Y \in h^{-1}(A)) &= \int_{h^{-1}(A)} \tilde{f}_Y(y)\mathrm{d}y \\
&= \int_{h^{-1}(A)} \tilde{f}_X(h(y)) \left|\frac{\mathrm{d}h}{\mathrm{d}y}(y)\right| \mathrm{d}y \\
&= \int_A \tilde{f}_X(u)\mathrm{d}u = \int_A \mathrm{d}\tilde{\mathbb{P}}
\end{aligned}$$

using change of variables and a change of measure from $\tilde{\mathbb{P}}$ to $\mathbb{P}$ in the last line using (a). Additionally, we have that $\tilde{f}_Y(x) = f_X(x)$, so that

$$\mathbb{P}(X \in h^{-1}(A)) = \int_{h^{-1}(A)} \mathrm{d}\mathbb{P} = \int_A \mathrm{d}\tilde{\mathbb{P}} = \tilde{\mathbb{P}}(X \in A)$$

Therefore, for any $y$, $h$ satisfies

$$f_X(y) = \tilde{f}_X(h(y)).$$

(c) Using a bit of intuition from (a), we see that

$$\begin{aligned}\tilde{P}(A) &= \int_A Z_{j+1}(\omega)\mathrm{d}\mathbb{P} \\ &= \int_A Z_{j+1} C_{j+1} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_{j+1}(X_j)}{\sigma_{j+1}(X_j)}\right)^2\right)\mathrm{d}x,\end{aligned}$$

where $C_{j+1}$ is the normalization constant for the distribution of $X_{j+1} - X_j$ under $\mathbb{P}$. Our desired mean zero normal distribution satisfies

$$\tilde{P}(A) = \int_A C_{j+1} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma_{j+1}(X_j)}\right)^2\right)\mathrm{d}x,$$

where $C_{j+1}$ is the same normalization constant as before. Therefore, we can write $Z$ as

$$Z_{j+1} = \exp\left(-\frac{x\mu_{j+1}(X_j)}{\sigma^2_{j+1}(X_j)} + \frac{1}{2}\frac{\mu^2_{j+1}(X_j)}{\sigma^2_{j+1}(X_j)}\right),$$

replacing $Z_{j+1}$ with the above will give us the desired integral. For each change of measure $Z_{j+1}$. we can now compute the conditional expectation as

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[Z_{j+1} \mid X_1, X_2, \dots X_j] &= \int_\Omega Z_{j+1}(\omega)\mathrm{d}\mathbb{P} \\ &= \int_\Omega C_{j+1} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma_{j+1}(X_j)}\right)^2\right)\mathrm{d}x \\ &= \tilde{\mathbb{P}}(\Omega) = 1,\end{aligned}$$

which makes sense as $Z$ is the Radon Nikodym derivative $\mathrm{d}\tilde{\mathbb{P}}/\mathrm{d}\mathbb{P}$.

(d) Using Girsanov theorem (Thm 8.6.1 in MLN) allows us to define a change of measure $\mathrm{d}\tilde{W} = \left(\frac{b(X_t)}{A(X_t)}\right)\mathrm{d}t + \mathrm{d}W_t$. Re-arranging this, allows us to write

$$\mathrm{d}W_t = \mathrm{d}\tilde{W}_t - \left(\frac{b(X_t)}{A(X_t)}\right)\mathrm{d}t$$

We can then apply this change of measure to $X$ to show that

$$\begin{aligned}
\mathrm{d}X_t &= b(X_t)\mathrm{d}t + A(X_t)\mathrm{d}W_t \\
\mathrm{d}X_t &= b(X_t)\mathrm{d}t + A(X_t)\left(\mathrm{d}\tilde{W}_t - \left(\frac{b(X_t)}{A(X_t)}\right)\mathrm{d}t\right) \\
&= A(X_t)\mathrm{d}\tilde{W}_t,
\end{aligned}$$

under the new measure $\tilde{\mathbb{P}}$.

**Exercise 2.** The Ornstein-Uhlenbeck process, defined by linear SDE

$$\mathrm{d}X(t) = -\mu X(t)\mathrm{d}t + \sigma \mathrm{d}W(t),\ X(0) = x_0,$$

in which $\sigma$ and $\mu > 0$ are two constants, has its Kolmogorov forward equation

$$\frac{\partial}{\partial t}\Gamma(x_0; t, x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\Gamma(x_0; t, x) + \frac{\partial}{\partial x}\Big(\mu x \Gamma(x_0; t, x)\Big), \tag{1}$$

with the initial condition $\Gamma(x_0; 0, x) = \delta(x - x_0)$.
(a) Show that the solution to the linear PDE (1) has a Gaussian form and find the solution.
(b) What is the limit of

$$\lim_{t\to\infty} \Gamma(x_0; t, x)?$$

(c) Find $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$.
(d) You note that $\mathbb{E}[X(t)]$ is the same as the solution to the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu x$, which is obtained when $\sigma = 0$. Is this result true for a nonlinear SDE?

**Solution 2.** (a) We begin by writing

$$X_t = x_0 e^{-\mu t} + \sigma \int_0^t \exp(-k(t-s))\mathrm{d}W_s.$$

This follows from Example 8.3.6. (MLN), but can be derived by using Ito's formula on $Z_t = e^{\mu t} X_t$. This can be written as a normal distribution as it is a constant plus an Ito integral. We can write this normal distribution using its expectation and variance so that

$$\begin{aligned}
\mathbb{E}[X_t] &= x_0 e^{-\mu t} + \mathbb{E}\left[\sigma \int_0^t \exp(-k(t-s))\mathrm{d}W_s\right] \\
&= x_0 e^{-\mu t} \\
\mathrm{Var}[X_t] &= \mathrm{Var}\left[\sigma \int_0^t \exp(-\mu(t-s))\mathrm{d}W_s\right] \\
&= \frac{\sigma^2}{2\mu}\left(1 - \exp(-2\mu t)\right).
\end{aligned}$$

We can then write the transition density as a Gausian

$$\begin{aligned}
\Gamma(x_0; t, x) &\sim \text{Normal}\left(x_0 e^{-\mu t}, \frac{\sigma^2}{2\mu}\left(1 - \exp(-2\mu t)\right)\right) \\
\Gamma(x_0; t, x) &= \sqrt{\frac{\mu}{\pi\sigma^2(1 - \exp(-2\mu t))}} \exp\left(-\mu \frac{(x - x_0 e^{-\mu t})^2}{\sigma^2(1 - \exp(-2\mu t))}\right).
\end{aligned}$$

(b) Taking $t \to \infty$, we see that this distribution approaches

$$\begin{aligned}
\lim_{t\to\infty} \Gamma(x_0, t, x) &\sim \text{Normal}\left(0, \frac{\sigma^2}{2\mu}\right) \\
\Gamma(x_0; t, x) &= \sqrt{\frac{\mu}{\pi\sigma^2}} \exp\left(-\mu \frac{x^2}{\sigma^2}\right).
\end{aligned}$$

(c) This was solved above.
(d) Is what result clear? That the expectation of an SDE corresponds to the ODE that represents its drift? I believe this property can failure in cases where the martingale property fails for the $\mathrm{d}W_t$ integral. Basically, any SDE of the form $\mathrm{d}X_t = a(X_t, t)\mathrm{d}t + b(X_t, t)\mathrm{d}W_t$ with

$$\mathbb{E}\left[\int_0^t b(X_s, s)\mathrm{d}W_s\right] \neq 0$$

will not have the desired result.

**Exercise 3.** The time-independent solution to a Kolmogorov forward equation gives a stationary probability density function for the Ito process $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t)$:

$$-\frac{\partial}{\partial x}\Big(\mu(x)f(x)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)f(x)\Big) = 0.$$

This is a linear, second-order ODE. We assume that both $\mu(x)$ and $\sigma(x)$ satisfy the conditions required to have a solution $f(x)$ on the entire $\mathbb{R}$. Find the expression for the general solution. There are two constants of integration, which should be determined according to appropriate probabilistic reasoning.

**Solution 3.** We write the ODE as

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)f(x)\Big) = \frac{\partial}{\partial x}\Big(\mu(x)f(x)\Big).$$

Integrating, we see that

$$\frac{\partial}{\partial x}(\sigma^2(x)f(x)) = 2\mu(x)f(x) + C$$

Using the product rule on the right side, we have that

$$f'(x)\sigma^2(x) = (2\mu(x) - 2\sigma(x)\sigma'(x))f(x) + C$$

We then have that

$$f'(x) = \left(\frac{2\mu(x) - 2\sigma(x)\sigma'(x)}{\sigma^2(x)}\right) f(x) + \frac{C}{\sigma^2(x)}$$

To ensure that $f'(x)$ approach zero in its tails, we set $C = 0$. Integrating once more from $-\infty$ to $x$, we see that

$$\begin{aligned}
f(x) &= C_2 \exp\left(\int_{-\infty}^{x} \frac{2\mu(\xi) - 2\sigma(\xi)\sigma'(\xi)}{\sigma^2(\xi)}\mathrm{d}\xi\right)\\
&= C_2 \exp\left(\int_{-\infty}^{x} \frac{2\mu(\xi)}{\sigma^2(\xi)}\mathrm{d}\xi - \int_{-\infty}^{x} \frac{2\sigma'(\xi)}{\sigma(\xi)}\mathrm{d}\xi\right)\\
&= C_2 \exp\left(\int_{-\infty}^{x} \frac{2\mu(\xi)}{\sigma^2(\xi)}\mathrm{d}\xi - \log(\sigma^2(x)) + \log(\sigma^2(-\infty))\right)\\
&= \frac{C_2}{\sigma^2(x)} \exp\left(\int_{-\infty}^{x} \frac{2\mu(\xi)}{\sigma^2(\xi)}\mathrm{d}\xi\right),
\end{aligned}$$

where we've wrapped $\sigma^2(-\infty)$ into the constant $C_2$. Here, $C_2$ will serve as a normalizing constant so that $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$ i.e.

$$C_2^{-1} = \int_{\mathbb{R}} \frac{1}{\sigma^2(x)} \exp\left(\int_{-\infty}^{x} \frac{2\mu(\xi)}{\sigma^2(\xi)}\mathrm{d}\xi\right)\mathrm{d}x.$$

**Exercise 4.** Let $X$ be a solution to the following SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \delta\sqrt{X_t}\mathrm{d}W_t.$$

Define

$$u(t,x) = \mathbb{E}\left[\exp\left(-\int_t^t X_s \mathrm{d}s\right) \mid X_t = x_T\right]$$

Derive a PDE for the function $u$. To solve the PDE for $u$, try a solution of the form

$$u(t,x) = \exp(-xA(t) - B(t)),$$

where $A$ and $B$ are determinisitc functions of $t$, Show that $A$ and $B$ must satisfy a coupled pairs of ODEs with appropriate terminal conidtions at time $T$, Bonus question: solve the ODEs it may be helpful to note that one of the ODEs is a Riccati equation).

**Solution 4.** We use theorem 9.2.1 (MLN) with $\mu(t, X_t) = \kappa(\theta - X_t)$ and $\sigma(t, X_t) = \delta\sqrt{X_t}$. W This gives a PDE

$$\partial_t u = -\kappa(\theta - x)\partial_x u - \frac{\delta^2}{2}x\partial_{xx}u.$$

Supposing that our solution is of the form above, we can write out its derivative as

$$\begin{aligned}
\partial_t u &= (-xA'(t) - B'(t))u \\
\partial_x u &= -A(t)u \\
\partial_{xx} u &= A(t)^2 u
\end{aligned}$$

Putting these together with the derived PDE , we have that

$$(-xA'(t) - B'(t))u = A(t)\kappa(\theta - x)u - A(t)^2\frac{\delta^2}{2}u.$$

This gives ODES

$$\begin{aligned}
-xA'(t) - A(t)\kappa(\theta - x) + \frac{\delta^2}{2}A(t)^2 &= B'(t) \\
u(T, x) &= x_T
\end{aligned}$$

**Exercise 5.** For $i = 1, 2, \ldots, d$, let $X^{(i)}$ satisfy

$$\mathrm{d}X_t^{(i)} = -\frac{b}{2} X_t^{(i)} \mathrm{d}t + \frac{1}{2} \sigma \mathrm{d}W_t^{(i)},$$

where the $(W^{(i)})_{i=1}^d$ are independent Brownian motions. Define

$$R_t = \sum_{i=1}^d (X_t^{(i)})^2, \quad B_t = \sum_{i=1}^d \int_0^t \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)}.$$

Show that $B$ is a Brownian motion. Derive an SDE for $R$ that involves only $\mathrm{d}t$ and $\mathrm{d}B_t$ term i.e. no $\mathrm{d}W_t^{(i)}$ should appear.

**Solution 5.** We'll use Levy's characterization of the Brownian motion to show that $B$ is a Brownian motion. We begin by noting that $B_0 = 0$ since

$$B_0 = \sum_{i=1}^d \int_0^0 \frac{1}{\sqrt{R_s}} X_s^{(i)} \mathrm{d}W_s^{(i)} = 0.$$

We write $B_t$ in differential form

$$\mathrm{d}B_t = \sum_{i=1}^d \frac{1}{\sqrt{R_t}} X_t^{(i)} \mathrm{d}W_t^{(i)}.$$

Next, we compute the quadratic variation as

$$\mathrm{d}[B, B]_t = \frac{1}{R_t} \sum_{i=1}^d (X_t^{(i)})^2 \mathrm{d}t,$$

where we've used that $\mathrm{d}W_{t^{(i)}}$ are independent to cancel the cross terms. Using the definition of $R_t$, we have that

$$\mathrm{d}[B, B] = \mathrm{d}t \implies [B, B]_t = t.$$

What remains is to show that $B_t$ is a martingale. We begin by considering the integrability condition

$$\mathbb{E} \int_0^t \frac{1}{R_s} (X_s^{(i)})^2 \mathrm{d}W_s^{(i)} < \infty,$$

which is satisfied for all $i = 1, \ldots, d$. Therefore, $B_t$ is an Ito integral and a martingale. From thos, we see $B_t$ satisfies Levy's characterization of a Brownian motion. To derive an SDE for $R$, we begin by differentiating $R_t$ so that

$$\begin{aligned}
\mathrm{d}R_t &= \sum_{i=1}^{d}\left[2X_t\mathrm{d}X_t^{(i)} + (\mathrm{d}X_t^{(i)})^2\right] \\
&= \sum_{i=1}^{d}\left(-b(X_t^{(i)})^2 + \frac{1}{4}\sigma^2\right)\mathrm{d}t + \sigma X_t\mathrm{d}W_t^{(i)} \\
&= \left(\frac{d}{4}\sigma^2 - b\sum_{i=1}^{d}(X_t^{(i)})^2\right)\mathrm{d}t + \sigma\sum_{i=1}^{d}X_t^{(i)}\mathrm{d}W_t^{(i)} \\
&= \frac{d}{4}\sigma^2 - bR_t\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}B_t
\end{aligned}$$