Exercise 1 (Exercise 4.1.)**.** A six-sided die is rolled repeatedly. Which of the following a Markov chains? For those that are, find the one-step transition matrix. (a) X_n is the largest number rolled up to the nth roll. (b) X_n is the number of sixes rolled in the first n rolls. (c) At time n, X_n is the time since the last six was rolled. (d) At time n, X_n is the time until the next six is rolled.

Solution 1. (a) Since X_n is the largest number rolled up to the n-th roll, we have that

$$X_{n+1} = \max\{X_n, Y_{n+1}\},$$

where Y_{n+1} is the $(n+1)$-th roll. As this only depends on the most recent value of X, this is a Markov chain. Since each roll of the six-sided die is independent, we have that Y_{n+1} has uniform distribution on $\{1,2,3,4,5,6\}$. It follows that the probability that $X_{n+1} = X_n$ is given by

$$\mathbb{P}(X_{n+1} = X_n \mid X_n = i) = \mathbb{P}(Y_{n+1} \geq i) = \frac{i}{6}.$$

and for each $6 \geq j > i \geq 1$, we have

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \frac{1}{6}.$$

For any $i < X_n$, we have additionally that $\mathbb{P}(X_{n+1} = i \mid X_n) = 0$ as the maximum is non-decreasing. We can write these probabilities in the one step transition matrix as

$$\mathbf{P} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

or more compactly as

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} 1/6 \text{ if } j > i \\ i/6 \text{ if } j = i \\ 0 \text{ if } j < i. \end{cases}$$

(b) Setting X_n to be the number of sixes in the first n rolls, we have that

$$X_{n+1} = X_n + 1 + Y_{n+1},$$

where Y_{n+1} is the indicator variable for whether the $n+1$-th roll was a six or not. This follows because

$$X_{n+1} = X_n + Y_{n+1}$$

For $i, j \in \mathbb{N}_0$, we have that

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} 5/6 \text{ if } j = i \\ 1/6 \text{ if } j = i + 1 \\ 0 \text{ otherwise }, \end{cases}$$

since Y_{n+1} is 1 with probability 1/6 and 0 otherwise. This shows that X is a Markov chain with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$

(c) If we define X_n to be the number of rolls since the last six, we see that

$$X_{n+1} = X_n(1 - Y_{n+1}) + 1,$$

where Y_{n+1} is 1 if the $(n+1)$th roll is a six and 0 otherwise. It follows that for $i, j \in \mathbb{N}$, we have

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} 5/6 \text{ if } j = i + 1 \\ 1/6 \text{ if } j = 1 \\ 0 \text{ otherwise } , \end{cases}$$

since the count will reset to 1 if Y_{n+1} is a six which occurs with probability 1 and will increase by one otherwise.

(d) If the time to the next roll of a six is known as X_n, then there are two options either $X_n > 1$, in which case $X_{n+1} = X_n - 1$ as the countdown simply ticks down once. Otherwise, we have $X_n = 1$, in which case the next roll is certainly a six and the time to the next six is not deterministic. Assuming that each roll is independent, we can calculate the probability of the time to the next six for X_{n+1} given $X_n = 1$ as being geometrically distributed with rate 1/6 since this is the probability that one rolls a six, so that

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} 1 \text{ if } j = i - 1 \text{ and } i \geq 2 \\ \frac{1}{6}\left(1 - \frac{1}{6}\right)^{j-1} \text{ if } i = 1 \text{ and } j \geq 1 \\ 0 \text{ otherwise .} \end{cases}$$

Exercise 2 (Exercise 4.2)**.** Let $Y_n = X_{2n}$. Compute the transition matrix for Y when (a) X is a simple random walk and (b) X is a branching process with generating function G for its offspring distribution.

Solution 2. (a) Notice that each step of our random walk X is given by $X_{n+1} = X_n + \xi_{n+1}$, where

$$\xi_n = \begin{cases} 1 \text{ with prob } p \\ -1 \text{ with prob } q \end{cases}$$

This shows that for

$$Y_{n+1} - Y_n = X_{2n+2} - X_{2n} = \xi_{2n+1} + \xi_{2n+2}$$

From this, we see that $Y_{n+1} = Y_n + \xi_{2n+1} + \xi_{2n+2}$ where

$$\xi_{2n+1} + \xi_{2n+2} = \begin{cases} 2 \text{ with prob } p^2 \\ 0 \text{ with prob } 2pq \\ -2 \text{ with prob } q^2 \end{cases}$$

This means that for integers i and j

$$p_{ij} = \mathbb{P}(Y_{n+1} = j \mid Y_n = i) = \begin{cases} p^2 \text{ if } j = i + 2 \\ 2pq \text{ if } j = i \\ q^2 \text{ if } j = i - 2 \\ 0 \text{ otherwise.} \end{cases}$$

(b) Fixing $Y_n = X_{2n} = i$, we see that for

$$X_{2n+1} = \sum_{k=1}^{i} \xi_k,$$

where ξ_k are i.i.d draws with common generating function G. Since i is fixed, we can compute the generating function at the next time step as

$$G_{X_{2n+1}}(s) = G(s)^i. \tag{1}$$

For the next time step, we have that

$$Y_{n+1} = X_{2n+2} = \sum_{k=0}^{X_{2n+1}} \xi_k.$$

Since X_{2n+1} is independent of the ξ_k, we have that

$$G_{Y_{n+1}}(s) = G_{X_{2n+1}}(G(s)) = G(G(s))^i.$$

With this generating function, we can compute the probability of having j offspring given $Y_n = i$ as

$$p_{ij} = \mathbb{P}(Y_{n+1} = j \mid Y_n = i) = \frac{1}{j!}\left[\frac{d^j}{dt^j} G_{Y_{n+1}}(t)\right]_{t=0} = \frac{1}{j!}\left[\frac{d^j}{dt^j} G(G(t))^i\right]_{t=0},$$

where we have used that for a random variable X with generating function $G_X(s)$

$$\mathbb{P}(X = j) = \frac{1}{j!}\left[\frac{d^j}{dt^j} G_X(t)\right]_{t=0}$$

Exercise 3 (Exercise 4.3)**.** Let X be a Markov chain with state space S and absorbing state k. Suppose that $j \to k$ for all $k \in S$. Show that all states other than k are transient.

Solution 3. Fix a state $j \neq k$. Since $j \to k$, we know there exists an integer n so that $p_n(j,k) > 0$. As k is an absorbing state, we have that $p(k,i) = 0$ for any state i. This means that for all $m \geq 1$, $p_m(k,j) = 0$. Therefore, if $X_0 = j$ and $X_n = k$ with probability $p_n(j,k) > 0$, then X cannot return to state j as $p_m(k,j) = 0$ for all $m \geq 1$. This means that the state $j \neq k$ cannot be recurrent and is therefore transient.

Exercise 4 (Exercise 4.4)**.** Suppose that two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i \mid X_0 = i) = \mathbb{P}(\tau_i < \tau_j \mid X_0 = j),$$

where $\tau_j = \inf\{n \geq 1 \mid X_n = j\}$. Show that if $X_0 = i$, the expected number of visits to j prior to re-visiting i is one.

Solution 4. Let Z be the random variable describing the number of visits to state j before re-visiting i. To start, we define

$$\begin{aligned} p &= \mathbb{P}(\tau_j < \tau_i \mid X_0 = i) = \mathbb{P}(\tau_i < \tau_j \mid X_0 = j) \\ 1 - p &= \mathbb{P}(\tau_i \leq \tau_j \mid X_0 = i) = \mathbb{P}(\tau_j \leq \tau_i \mid X_0 = j) \end{aligned}$$

In this way, we can define the probability that we start at $X_0 = i$ and reach state j without first returning to i as p. Once we are at j, the probability of returning to j without first reaching state i is then $1 - p$. Starting at $X_0 = i$, we see that the probability of returning to i only after k visits is

$$\mathbb{P}(Z = 0) = 1 - p, \quad \mathbb{P}(Z = k) = p^2(1 - p)^{k-1}, k > 0.$$

We can compute this expectation directly as

$$\begin{aligned} \mathbb{E}[Z] &= \sum_{k=0}^{\infty} k\mathbb{P}(Z = k) \\ &= p \sum_{k=1}^{\infty} kp(1 - p)^{k-1} \\ &= p\mathbb{E}[\text{Geom}(p)] \\ &= 1 \end{aligned}$$

Exercise 5 (Exercise 4.5)**.** Let X be a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}$$

Find the invariant distribution π, and the mean-recurrence times $\overline{\tau}_j$ for $j = 1, 2, 3$.

Solution 5. Supposing the invariant distribution exists, we have that

$$\begin{aligned} \pi_1 &= (1-2p)\pi_1 + p\pi_2 \\ \pi_2 &= 2p\pi_1 + (1-2p)\pi_2 + 2p\pi_3 \\ \pi_3 &= p\pi_2 + (1-2p)\pi_3 \end{aligned}$$

The first equation implies that $2\pi_1 = \pi_2$. The second implies $2\pi_3 = \pi_2$. Using that $1 = \pi_1 + \pi_2 + \pi_3$, this shows that

$$\pi = (1/4, 1/2, 1/4), \quad \overline{\tau} = (4, 2, 4)$$

since $\mathbf{P}$ is irreducible and $\pi_i = 1/\overline{\tau}_i$.