Exercise 1. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = 2^\Omega$. Define a probability measure $\mathbb{P}$ as follows:

$$\mathbb{P}(a) = 1/6, \mathbb{P}(b) = 1/3, \mathbb{P}(c) = 1/4, \mathbb{P}(d) = 1/4. \tag{1}$$

Next define three random variables X, Y, Z by

$$X(a) = 1, X(b) = 1, X(c) = -1, X(d) = -1 \tag{2}$$
$$Y(a) = 1, Y(b) = -1, Y(c) = 1, Y(d) = -1 \tag{3}$$

and $Z = X + Y$.
(a) List the sets in $\sigma(X)$. (b) Calculate $\mathbb{E}(Y \mid X)$. (c) Calculate $\mathbb{E}(Z \mid X)$.

Solution 1.
(a) We know that $\sigma(X) = \sigma(\{\{X = -1\}, \{X = 1\}\})$. Since $\{X = -1\} = \{c, d\}$ and $\{X = 1\} = \{a, b\}$, we can see that

$$\sigma(X) = \{\varnothing, \{a, b\}, \{c, d\}, \Omega\}. \tag{4}$$

(b) The conditional expectation of Y given X, $\mathbb{E}(Y \mid X = X(\omega))$ has two possible values $X(\omega) = 1$ and $X(\omega) = -1$. Using the definition of the random variable X, we know that if $x = 1$, that either $\omega = a, b$ and that if $x = -1$, $\omega = c, d$. Therefore, we have that

$$\mathbb{E}(Y \mid X) = \begin{cases} \mathbb{E}(Y \mid X = 1) & = \frac{\mathbb{P}(a)Y(a) + \mathbb{P}(b)Y(b)}{\mathbb{P}(\{a,b\})} = -1/3, \quad \omega \in \{a, b\} \\ \mathbb{E}(Y \mid X = -1) & = \frac{\mathbb{P}(c)Y(c) + \mathbb{P}(d)Y(d)}{\mathbb{P}(\{c,d\})} = 0, \quad \omega \in \{c, d\} \end{cases} \tag{5}$$

(c) We can repeat this procedure on $\mathbb{E}(Z \mid X)$ to see that

$$\mathbb{E}(Z \mid X) = \begin{cases} \mathbb{E}(Z \mid X = 1) & = \frac{\mathbb{P}(a)Z(a) + \mathbb{P}(b)Z(b)}{\mathbb{P}(\{a,b\})} = 2/3, \quad \omega \in \{a, b\} \\ \mathbb{E}(Z \mid X = -1) & = \frac{\mathbb{P}(c)Z(c) + \mathbb{P}(d)Z(d)}{\mathbb{P}(\{c,d\})} = -1, \quad \omega \in \{c, d\} \end{cases} \tag{6}$$

Exercise 2. (a) Prove that $\mathbb{E}\left(\mathbb{E}(X \mid \mathcal{F})\right) = \mathbb{E}X$. (b) Show that if $\mathcal{G} \subset \mathcal{F}$ and $\mathbb{E}X^2 < \infty$, then

$$\mathbb{E}([X - \mathbb{E}(X \mid \mathcal{F})]^2) + \mathbb{E}([\mathbb{E}(X \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{G})]^2) = \mathbb{E}([X - \mathbb{E}(X \mid \mathcal{G})]^2) \tag{7}$$

Solution 2.
(a) Consider the σ-algebra $\mathcal{G}_0 = \{\varnothing, \Omega\}$. Since $\mathcal{G}_0 \subset \mathcal{F}$, we have that

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}) \mid \mathcal{G}_0) = \mathbb{E}(X \mid \mathcal{G}_0). \tag{8}$$

We can see that for all $A \in \mathcal{G}_0$ then

$$\int_A \mathbb{E}(X \mid \mathcal{F}) d\mathbb{P} = \int_A X d\mathbb{P}. \tag{9}$$

Due to our choice of the trivial σ-algebra, we have that

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F})) = \int_\Omega \mathbb{E}(X \mid \mathcal{F}) d\mathbb{P} = \int_\Omega X d\mathbb{P} = \mathbb{E}(X). \tag{10}$$

(b) Since $\mathbb{E}(X^2) < \infty$ and $\mathcal{G} \subset \mathcal{F}$, we have $\mathbb{E}(X \mid \mathcal{G}) \in L^2(\mathcal{F})$. Therefore, the random variable $Z = \mathbb{E}(X \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{G}) \in L^2(\mathcal{F})$ as well. From here, we can essentially follow the minimization proof from the lecture notes to see that

$$\begin{aligned}
\mathbb{E}(X - \mathbb{E}(X \mid \mathcal{G}))^2 &= \mathbb{E}(X - \mathbb{E}(X \mid \mathcal{F}) + \mathbb{E}(X \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{G}))^2 && (11)\\
&= \mathbb{E}(X - \mathbb{E}(X \mid \mathcal{F}))^2 + \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{G}))^2, && (12)
\end{aligned}$$

where the second to last equality follows from the fact that the cross term $\mathbb{E}((X - \mathbb{E}(X \mid \mathcal{F}))Z) = 0$. The full computation is as follows

$$\begin{aligned}
\mathbb{E}((X - \mathbb{E}(X \mid \mathcal{F}))Z) &= \mathbb{E}(ZX - \mathbb{E}(ZX \mid \mathcal{F})) && (13)\\
&= \mathbb{E}(ZX) - \mathbb{E}(\mathbb{E}(ZX \mid \mathcal{F})) && (14)\\
&= 0. && (15)
\end{aligned}$$

Here, we have used that $Z \in \mathcal{F}$ to place it within the conditional expectation and problem (2a).

Exercise 3. An important special case of the previous result (2b) occurs when $\mathcal{G} = \{\varnothing, \Omega\}$. Let $\text{Var}(X \mid \mathcal{F}) = \mathbb{E}(X^2 \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{F})^2$. Show that

$$\text{Var}(X) = \mathbb{E}(\text{Var}(X \mid \mathcal{F})) + \text{Var}(\mathbb{E}(X \mid \mathcal{F})). \tag{16}$$

Solution 3.
Taking $\mathcal{G} = \{\varnothing, \Omega\}$ in (2b), we see that

$$\begin{aligned} \text{Var}(X) = \mathbb{E}([X - \mathbb{E}(X)]^2) &= \mathbb{E}([X - \mathbb{E}(X \mid \mathcal{F})]^2) + \mathbb{E}([\mathbb{E}(X \mid \mathcal{F}) - \mathbb{E}(X)]^2) & (17) \\ &= \mathbb{E}([X - \mathbb{E}(X \mid \mathcal{F})]^2) + \text{Var}(\mathbb{E}(X \mid \mathcal{F})), & (18) \end{aligned}$$

by the definition of variance and the fact that $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{G}))$. This leaves us with

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}([X - \mathbb{E}(X \mid \mathcal{F})]^2) + \text{Var}(\mathbb{E}(X \mid \mathcal{F})) & (19) \\ &= \mathbb{E}(\mathbb{E}([X - \mathbb{E}(X \mid \mathcal{F})]^2 \mid \mathcal{F})) + \text{Var}(\mathbb{E}(X \mid \mathcal{F})) & (20) \\ &= \mathbb{E}(\text{Var}(X \mid \mathcal{F})) + \text{Var}(\mathbb{E}(X \mid \mathcal{F})), & (21) \end{aligned}$$

where in the last line we have used problem (2a) and the fact that $\text{Var}(X \mid \mathcal{F}) = \mathbb{E}[(X - \mathbb{E}(X \mid \mathcal{F}))^2]$

Exercise 4. Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with mean μ and variance σ^2 and let N be an independent positive integer value random variable with $\mathbb{E}N^2 < \infty$ and $X = Y_1 + \cdots + Y_N$. Show that

$$\mathrm{Var}(X) = \sigma^2 \mathbb{E}N + \mu^2 \mathrm{Var}(N). \tag{22}$$

Solution 4. Using the result of Exercise 3, we have that

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X \mid N)) + \mathrm{Var}(\mathbb{E}(X \mid N)). \tag{23}$$

Conditioning on N, we have that

$$\mathbb{E}(X \mid N) = \mathbb{E}\left(\sum_{i=1}^{N} Y_i \mid N\right) = \sum_{i=1}^{N} \mathbb{E}\left(Y_i \mid N\right) = \sum_{i=1}^{N} \mathbb{E}(Y_i) = \mu N, \tag{24}$$

by linearity and the fact that each Y_i has mean μ. Similarly, we can compute that

$$\mathrm{Var}(X \mid N) = \mathrm{Var}\left(\sum_{i=1}^{N} Y_i \mid N\right) = \sum_{i=1}^{N} \mathrm{Var}\left(Y_i \mid N\right) = \sum_{i=1}^{N} \mathrm{Var}(Y_i) = \sigma^2 N, \tag{25}$$

where we are able to take the sum of the variances since all the random variables Y_i are independent and therefore uncorrelated. This then shows that

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}(\sigma^2 N) + \mathrm{Var}(\mu N) && (26) \\ &= \sigma^2 \mathbb{E}N + \mu^2 \mathrm{Var}N. && (27) \end{aligned}$$