Exercise 1. Suppose that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and pick $A \in \mathcal{F}$. Define

$$Z(\omega) = \begin{cases} X(\omega), & \omega \in A \\ Y(\omega), & \omega \in A^c. \end{cases} \tag{1}$$

Show that Z is a random variable.

Solution 1. Let B be a Borel set and consider the set $\{Z \in B\}$. We can write $Z^{-1}B = (Z^{-1}B \cap A) \cup (Z^{-1}B \cap A^c)$. This allows us to see that

$$\{Z \in B\} = \{\omega \mid \omega \in (Z^{-1}B \cap A) \cup (Z^{-1}B \cap A^c)\} = \{\omega \mid \omega \in Z^{-1}B \cap A\} \cup \{\omega \mid \omega \in Z^{-1}B \cap A^c\}. \tag{2}$$

since the preimage of a set union is the union of the preimages of the sets composing it. Since $(Z^{-1}B \cap A) \subset A$ and $(Z^{-1}B \cap A^c) \subset A^c$, we know that $Z = X$ on $Z^{-1}B \cap A$ and $Z = Y$ on $Z^{-1}B \cap A^c$. This means that

$$\{Z \in B\} = \{\omega \mid \omega \in X^{-1}(B) \cap A\} \cup \{\omega \mid \omega \in Y^{-1}(B) \cap A^c\}. \tag{3}$$

Since both the events on the righthand side are in $\mathcal{F}$ because X and Y are random variables, their union $\{Z \in B\} \in \mathcal{F}$. This shows $\{Z \in B\}$ is measurable for any Borel set B, meaning Z is a random variable.

Exercise 2. Suppose that X is a continuous random variable with distribution F_X. Let g be a strictly increasing continuous function and define $Y = g(X)$.

a) What is the distribution function of Y F_Y?

b) What is f_Y the density function of Y?

Solution 2.
a) We can compute the distribution function of Y directy as

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(g(X) \leq x). \tag{4}$$

Since g is a strictly increasing continuous function it has a continuous inverse g^{-1}, therefore,

$$F_Y(x) = \mathbb{P}(g(X) \leq x) = \mathbb{P}(X \leq g^{-1}(x)) = (F_X \circ g^{-1})(x). \tag{5}$$

b) Here we assume that X has a density function $f_X = F_X'$. We can find the density of Y by taking the derivative of its distrubution function, so that

$$f_Y(x) = \frac{d}{dx}(F_Y(x)) = \frac{d}{dx}(F_X(g^{-1}(x))). \tag{6}$$

Assuming that g is additionally differentiable and using the chain rule, we can compute that

$$\begin{aligned} f_Y(x) &= \frac{d}{dx}\left[g^{-1}(x)\right] \cdot F_X'(g^{-1}(x)) && (7) \\ &= \frac{d}{dx}\left[g^{-1}(x)\right] \cdot f_X(g^{-1}(x)) && (8) \end{aligned}$$

The second equality follows from the definition of the density. In this case, the derivative of the inverse function g^{-1} should always be positive since g is strictly increasing, but in general, I believe we have to use $\left|\frac{d}{dx}g^{-1}(x)\right|$ to ensure the density is non-negative.

Exercise 3. Suppose that X is a continuous random variable with distribution function F_X. Find F_Y where Y is given by:

a) X^2

b) $\sqrt{|X|}$

c) $\sin X$

d) $F_X(X)$

Solution 3. a) We'll begin by trying to compute the distribution of Y directly,

$$\begin{aligned} F_Y(x) &= \mathbb{P}(Y \leq x) && (9) \\ &= \mathbb{P}(X^2 \leq x). && (10) \end{aligned}$$

When $x \geq 0$, we can see that

$$\begin{aligned} F_Y(x) &= \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) && (11) \\ &= F_X(\sqrt{x}) - F_X(-\sqrt{x}). && (12) \end{aligned}$$

If $x < 0$,

$$\begin{aligned} F_Y(x) &= \mathbb{P}(X^2 \leq x) && (13) \\ &= \mathbb{P}(X^2 \leq x < x) = 0 && (14) \end{aligned}$$

since the square of real-valued quantities must be non-negative. We can put this together to show that

$$F_Y(x) = \begin{cases} F_X(\sqrt{x}) - F_X(-\sqrt{x}), & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{15}$$

b) Once again computing directly

$$\begin{aligned} F_Y(x) &= \mathbb{P}(Y \leq x) && (16) \\ &= \mathbb{P}(\sqrt{|X|} \leq x) & = \mathbb{P}(|X| \leq x^2) && (17) \end{aligned}$$

Like in the previous problem, this is 0 is zero for $x < 0$, but for $x \geq 0$, we can compute

$$\begin{aligned} F_Y(x) &= \mathbb{P}(Y \leq x) && (18) \\ &= \mathbb{P}(-x^2 \leq X \leq x^2) && (19) \\ &= F_X(x^2) - F_X(-x^2). && (20) \end{aligned}$$

We can write this compactly as

$$F_Y(x) = \begin{cases} F_X(x^2) - F_X(-x^2), & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{21}$$

c) Computing directly,

$$F_Y(x) = \mathbb{P}(Y \leq x) \tag{22}$$
$$= \mathbb{P}(\sin X \leq x). \tag{23}$$

Since $\sin X$ is bounded between $[0, 1]$, we have that $F_Y(x) = 1$ for $x > 1$ and $F_Y(x) = 0$ for $x < 0$. Otherwise, if $x \in [0, 1]$, we can take the inverse of $\sin X$, so that $\mathbb{P}(\sin X \leq x) = \mathbb{P}(X \leq \arcsin x)$

$$F_Y(x) = \begin{cases} 1, \quad x > 1 \\ F_X(\arcsin x), \quad x \in [0, 1] \\ 0, \quad x < 1 \end{cases} \tag{24}$$

d) Computing directly,

$$F_Y(x) = \mathbb{P}(Y \leq x) \tag{25}$$
$$= \mathbb{P}(F_X(X) \leq x) \tag{26}$$
$$= \mathbb{P}(X \leq \tilde{F}_X^{-1}(x)), \tag{27}$$

where $\tilde{F}_X^{-1}$ is the pseudo-inverse of the distribution F defined by

$$\tilde{F}_X^{-1} = \inf\{x \mid F(x) \geq u\}. \tag{28}$$

Exercise 4. Define $X\colon [0,1] \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} 1, & \omega \in [0,1] \cap \mathbb{Q}^c \\ 0, & \omega \in [0,1] \cap \mathbb{Q}. \end{cases} \tag{29}$$

As defined, X is the function which takes rational numbers in $[0,1]$ to 0 and irrational numbers in $[0,1]$ to 1. Assuming that X is defined on $([0,1], \mathcal{B}[0,1], \mathbb{P})$ where $\mathbb{P}$ is the Lebesgue measure. Show that X is a random variable. If it is, what are its distribution function and expectation? Does X has a density function? Is X discrete?

Solution 4.
X is a random variable. Let $A = \mathbb{Q}^c \cap [0,1]$. We will show that $A \in \mathcal{B}[0,1]$. By definition of $\mathcal{B}[0,1]$, we know that every rational number $q \in \mathbb{Q} \cap [0,1]$ has $\{q\} \in \mathcal{B}[0,1]$ since $[0,q)$ and $(q,1]$ are Borel sets and

$$[0,1] \setminus ([0,q) \cup (q,1]) = \{q\}. \tag{30}$$

We also know that $\mathbb{P}(\{q\}) = 0$ under the Lebesgue measure since

$$\mathbb{P}([0,q) \cup (q,1]) = 1 \text{ and } \mathbb{P}([0,q) \cup (q,1] \cup \{q\}) = \mathbb{P}([0,1]) = 1. \tag{31}$$

Since there are a countable number of rationals, we have that

$$\mathbb{P}(\mathbb{Q} \cap [0,1]) = \sum_{q \in \mathbb{Q}} \mathbb{P}(\{q\}) = 0 \tag{32}$$

by countable additivity. This means that A is a measurable set with $\mathbb{P}(A) = 1$ since $A^c = \mathbb{Q} \cap [0,1]$ is measurable with $\mathbb{P}(A^c) = 0$. Notice that with this choice of A, we can see that X is simply the indicator function of A and as it is an indicator function of a measurable event it is a random variable.
Distribution of X. We can compute the distribution of X using the fact that it is either 1 or 0. If $\omega \in \mathbb{Q}$, then $X = 0$ which occurs with probability 0. Similarly, if $\omega \notin \mathbb{Q}$, then $X = 1$ which occurs with probability 1. Therefore, our distribution is described by

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 1, & x = 1 \\ 0, & x < 1. \end{cases} \tag{33}$$

Expectation of X. Since X is an indicator function, we can compute its expectation quite simply as

$$\mathbb{E}\left[X\right] = \mathbb{P}(A) = 1. \tag{34}$$

Density of X. The random variable has no density since it is simply a point mass at $x = 1$.
X is discrete. The random variable X is discrete since it can only take the values 0 and 1.