Exercise 1. A&F 2.5.1. Evaluate $\oint_\gamma f(z)dz$ where γ is the unit circle centered at the origin for the follwoing functions f.

Solution 1.
(a) $f(z) = e^{iz}$. The function $f(z)$ is entire as it is the composition of two entire functions e^w and iz. Its derivative is $f'(z) = ie^{iz}$. By Cauchy's Theorem, this means that for the closed curve γ, we have

$$\oint_\gamma e^{iz} dz = 0. \tag{1}$$

(b) $f(z) = e^{z^2}$. Once again $f(z)$ is entire as it is the composition of two entire functions e^w and iz. By Cauchy's Theorem, this means

$$\oint_\gamma e^{iz} dz = 0. \tag{2}$$

(c) $f(z) = \frac{1}{z-1/2}$. The function $f(z)$ is analytic except at $z = \frac{1}{2}$ which is contained in γ, so we cannot use Cauchy's theorem. We can instead use the residue theorem. Writing f as its Taylor-Laurent series about $z_0 = \frac{1}{2}$,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - 1/2)^n = 1(z - 1/2)^{-1}. \tag{3}$$

Here we can see that $a_n = 0$ for all $n \neq -1$ and $a_{-1} = 1$. Therefore, by the Residue theorem, we have

$$\oint_C f(z)dz = 2\pi i a_{-1} = 2\pi i. \tag{4}$$

(d) $f(z) = \frac{1}{z^2-4}$. The function $f(z)$ is analytic except at $z = 2, -2$, neither of which are in the contour γ. Since $f(z)$ is analytic on and within γ, we can apply Cauchy's theorem, so that

$$\oint_\gamma \frac{1}{z^2 - 4} dz = 0. \tag{5}$$

(e) $f(z) = \frac{1}{2z^2+1}$. This function is analytic except at $z_\pm = i\frac{\sqrt{2}}{2}, -i\frac{\sqrt{2}}{2}$ which are contained in the contour γ. We can then write

$$f(z) = \frac{1}{2z^2 + 1} = \frac{1}{2(z - i\frac{\sqrt{2}}{2})(z + i\frac{\sqrt{2}}{2})}. \tag{6}$$

We'll now compute the residues at $z_\pm$ using the following formula for the residue of f at z_0

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)f(z). \tag{7}$$

For z_+., we can compute

$$\operatorname{Res}(f, z_+) = \lim_{z \to z_+} \left(z - i\frac{\sqrt{2}}{2} \right) f(z) \tag{8}$$

$$= \lim_{z \to z_+} \frac{1}{2(z + i\frac{\sqrt{2}}{2})} \tag{9}$$

$$= \frac{1}{2\left(i\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} \tag{10}$$

$$= \frac{1}{2i\sqrt{2}} = \frac{\sqrt{2}}{4i} \tag{11}$$

Similarly, we can compute the residue at z_-

$$\operatorname{Res}(f, z_-) = \lim_{z \to z_-} \left(z + i\frac{\sqrt{2}}{2} \right) f(z) \tag{12}$$

$$= \lim_{z \to z_-} \frac{1}{2(z - i\frac{\sqrt{2}}{2})} \tag{13}$$

$$= \frac{1}{2\left(-i\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)} \tag{14}$$

$$= -\frac{1}{2i\sqrt{2}} = -\frac{\sqrt{2}}{4i} \tag{15}$$

$$= -\operatorname{Res}(f, z_+) \tag{16}$$

We can then use the residue theorem to compute the integral of f over γ as follows

$$\oint_C \frac{1}{2z^2 + 1} dz = 2\pi i \left(\operatorname{Res}(f, z_+) + \operatorname{Res}(f, z_-) \right) = 0 \tag{17}$$

(f)

Exercise 2. A&F 2.5.5.

Solution 2.

Exercise 3. A&F 2.5.6.

Solution 3.

Exercise 4. A&F 3.3.5.

Solution 4. In order to find the coeffiecients of the Taylor-Laurent Series about 0 of $f(z) = e^{\frac{t}{2}(z-z^{-1})} = \sum_{n\in\mathbb{Z}} a_n z^n$, we use the formula

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}}dz \tag{18}$$

where γ is the unit circle parameterized as $\gamma(\theta) = e^{i\theta}$ for $\theta \in [-\pi, \pi]$. We can then simplify the integral as

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_\gamma \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}}dz & (19)\\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}}\cdot ie^{i\theta}d\theta. & (20)\end{aligned}$$

We can simplify $e^{i\theta} - e^{-i\theta}$ as $2i\sin\theta$ and combine the terms $e^{i\theta}$ and $e^{i(n+1)\theta}$, so that

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}}\cdot ie^{i\theta}d\theta = \frac{i}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{\frac{t}{2}(2i\sin\theta)}}{e^{in\theta}}d\theta \tag{21}$$

Combining the top and bottom halves of the integrad and canceling the i in front, we get that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)}d\theta. \tag{22}$$

We can further simplify this using $e^{-ix} = \cos x - i\sin x$, which gives us

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta - t\sin\theta)d\theta + \frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(n\theta - t\sin\theta)d\theta. \tag{23}$$

Show that $g(\theta) = n\theta - t\sin\theta$ is odd.
Therefore, $\cos(g(\theta))$ is even and $\sin(g(\theta))$ is odd. This means that

$$\begin{aligned} \int_{-\pi}^{\pi}\cos(n\theta - t\sin\theta)d\theta &= 2\int_0^{\pi}\cos(n\theta - t\sin\theta)d\theta & (24)\\ \int_{-\pi}^{\pi}\sin(n\theta - t\sin\theta)d\theta &= 0 & (25)\end{aligned}$$

Therefore,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)}d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)d\theta. \tag{26}$$