Exercise 1. Consider a diffusion $X = (X_t)_{t\geq 0}$ that lives on a finite interval (l, r), $0 < l < r < \infty$ and satisfies the SDE

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t.$$

One can easily check that the endpoints l and r are regular. Assume both endpoints are killing. Find the transition density $\Gamma(t, x; T, y)$ of X.

Solution 1. As l and r are regular killing endpoints, we have that

$$\Gamma(t, l; T, y)\mathrm{d}y = \Gamma(t, r; T, y)\mathrm{d}y = 0.$$

Therefore, we want to operator $\mathcal{A}$ to act on functions which are 0 on the boundary. In this case, we have that $\mu(t, X_t) = \mu x$ and $\sigma(t, x) = \sigma x$, so that

$$\mathcal{A} = \mu x \partial_x + \frac{\sigma^2}{2} x^2 \partial_{xx}.$$

This then gives the following PDE for the Kolomogorov Forward equation

$$\begin{aligned} \partial_t \Gamma = -\mu x \partial_x \Gamma - \frac{\sigma^2}{2} x^2 \partial_{xx} \Gamma \\ \Gamma(t, x, t, \cdot) = \delta_x. \end{aligned}$$

The right side of the above equation is an Euler ODE. We'll now find an eigenfunction expansion to the following problem. That is, we want eigenfunctions φ which satisfy

$$\begin{aligned} \mathcal{A}\varphi = \lambda\varphi \\ \varphi(l) = \varphi(r) = 0. \end{aligned}$$

To find the eigenfunctions, we write the eigenvalue problem.

$$\frac{\sigma^2}{2} x^2 \varphi'' + \mu x \varphi' - \lambda \varphi = 0, \varphi(l) = \varphi(r) = 0$$

which one can notice as an Euler equation. Writing the solution in terms of x^r, we see that r must satisfy

$$\begin{aligned} \frac{\sigma^2}{2} r(r-1) + \mu r - \lambda = 0 \\ s r^2 + (\mu - s) r - \lambda = 0, \end{aligned}$$

where $s = \sigma^2/2$. We can solve this quadratic as

$$r = \frac{s - \mu \pm \sqrt{(\mu - s)^2 + 4s\lambda}}{2s} = \frac{s - u}{2s} \pm \sqrt{\frac{(\mu - s)^2}{4s^2} + \frac{\lambda}{s}}.$$

In general, this is complex as long as

$$-\frac{(\mu - s)^2}{4s} > \lambda.$$

We'll proceed with this case in mind. Our solution is then of the form

$$\varphi(x) = c_1 x^{\left(\frac{s-\mu}{2s}\right)} \sin\left(\ln(x)\sqrt{\frac{(\mu - s)^2}{4s^2} + \frac{\lambda}{s}}\right) + c_2 x^{\left(\frac{s-u}{2s}\right)} \cos\left(\ln(x)\sqrt{\frac{(\mu - s)^2}{4s^2} + \frac{\lambda}{s}}\right).$$

First, we'll attempt to apply the boundary conditions, we require that $\varphi(r) = 0$, so that

$$\ln r \sqrt{\frac{(\mu - s)^2}{4s^2} + \frac{\lambda_n}{s}} = n\pi$$
$$\lambda_n = -\frac{(\mu - s)^2}{4s} + s\left(\frac{n\pi}{\ln r}\right)^2$$

This allows us to simplify

$$\varphi_n(x) = c_1 x^{(s-\mu)/2s} \sin(\ln x \frac{n\pi}{\ln r}) + c_2 x^{(s-\mu)/2s} \cos(\ln x \frac{n\pi}{\ln r}).$$

This can only satisfy the sin term at $x = r$ so we conclude that $c_2 = 0$. Frankly, I'm a bit unsure how to proceed from here, but I'll show how to finish.
Using this family of eigenfunctions, we can compute the solution as

$$\Gamma(t, x; T, y) = m(y) \sum_n \exp((T - t)\lambda_n)\varphi_n(y)\varphi_n(x),$$

by Corollary 9.5.5 where m is the speed density given by (pg 153) as

$$m(y) = \frac{2}{\sigma^2} \exp(\frac{2\mu}{\sigma^2} x).$$

Exercise 2. Consider a two-dimensional diffusion process $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ that satisfy the SDEs

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}W_t^1 \\ \mathrm{d}Y_t &= \mathrm{d}W_t^2, \end{aligned}$$

where W_t^1 and W_t^2 are two independent Brownian motions. Define a function u as follows

$$\begin{aligned} u(x,y) &= \mathbb{E}[\varphi(X_\tau) \mid X_t = x, Y_t = y] \\ \tau &= \inf\{s \geq t \mid Y_s = a\}. \end{aligned}$$

1. State a PDE and boundary conditions satisfied by the function u.

2. Let us define the Fourier transform and the inverse Fourier transform, respectively, as follows

$$\begin{aligned} \text{FT:} \quad \hat{f}(\omega) &= \int e^{-i\omega x} f(x)\mathrm{d}x \\ \text{IFT:} \quad f(x) &= \frac{1}{2\pi}\int e^{i\omega x}\hat{f}(\omega)\mathrm{d}\omega \end{aligned}$$

   Use Fourier trasnforms and a conditioning argument to derive an expression for $u(x,y)$ as an inverse Fourier transform. Use this result to derive an explicit form for $\mathbb{P}(X_\tau \in \mathrm{d}z \mid X_t = x, Y_t = y)$ i.e. an expression involving no integrals.

3. Show the expression you derived in part (2) for $u(x,y)$ satisfies the PDE and BCs you stated in part (1).

Solution 2. (1.) Per section 9.6, we see that u must satisfy

$$\begin{aligned} \mathcal{A}u(x,y) &= 0, \quad (x,y) \in \mathbb{R}\times(-\infty, a) \\ u(x,a) &= \varphi(x), \end{aligned}$$

where

$$\mathcal{A} = \frac{1}{2}\partial_{xx} + \frac{1}{2}\partial_{xy} + \frac{1}{2}\partial_{yy}$$

This holds since the hitting time τ is almost surely finite. We can rewrite these equations as

$$\begin{aligned} \frac{1}{2}\partial_{xx}u + \frac{1}{2}\partial_{yy}u = 0, \\ u(x,a) = \varphi(x), \end{aligned}$$

using that $\sigma(t,x)$ is the identity matrix as the two Brownian motions are independent and μ is the 0 vector.

(2.) We begin by writing X_τ in terms of its Fourier transform using IFT

$$\varphi(X_\tau) = \frac{1}{2\pi}\int \exp(i\omega X_\tau)\hat{\varphi}(\omega)\mathrm{d}\omega.$$

Plugging this into the definition of u, we then have that

$$\begin{aligned}
u(x,y) &= \mathbb{E}\left[\frac{1}{2\pi}\int \exp(i\omega X_\tau)\hat{\varphi}(\omega)\mathrm{d}\omega \mid X_t = x, Y_t = y\right] \\
&= \frac{1}{2\pi}\int \hat{\varphi}(\omega)\mathbb{E}\left[\exp(i\omega X_\tau) \mid X_t = x, Y_t = y\right]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int \hat{\varphi}(\omega)\mathbb{E}\left[\mathbb{E}\left[\exp(i\omega X_\tau) \mid X_t = x, Y_t = y\right] \mid X_t = x, Y_t = y\right]\mathrm{d}\omega,
\end{aligned}$$

where we've used the tower property. Noticing that as X_t is a Brownian motion, we have that $X_\tau \sim \mathrm{Norm}(x, \tau - t)$ since $X_t - X_t$ has normal distribution with mean zero and variance $\tau - t$ and $X_t = x$. This allows us to write $\mathbb{E}[\exp(i\omega X_\tau)]$ as $\exp(i\omega x)\exp(-\frac{1}{2}\omega^2(\tau - t))$, so that

$$u(x,y) = \frac{1}{2\pi}\int \hat{\varphi}(\omega)\exp(i\omega x)\mathbb{E}\left[\exp\left(-\frac{1}{2}\omega^2(\tau - t)\right) \mid X_t = x, Y_t = y\right]\mathrm{d}\omega$$

Notice the conditional expectation now depends only on the hitting time τ starting from $Y_t = y$. Since Y_t is also a Brownian motion, we can shift the Brownian motion so that $Z_0 = Y_t - y$ and $Z_{\tau - t} = Y_\tau - y = a - y$. Under Z_t, we have that

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\omega^2(\tau - t)\right) \mid X_t = x, Y_t = y\right] = \exp\left(-\left|a - y\right|\sqrt{\omega^2}\right) = \exp(-\left|y - a\right|\left|\omega\right|)$$

after applying theorem 7.5.2. MLN. This leaves us with the following formula for $u(x,y)$

$$\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int \hat{\varphi}(\omega)\exp\left(i\omega x - \left|y - a\right|\left|\omega\right|\right)\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int \hat{\varphi}(\omega)\exp\left(-\left|y - a\right|\left|\omega\right|\right)\exp\left(i\omega x\right)\mathrm{d}\omega.
\end{aligned}$$

We now have the inverse Fourier Transform of a product of functions, which will give us a convolution going backwards. This means that

$$\begin{aligned}
u(x,y) &= \frac{1}{2\pi}\int \mathcal{F}[\varphi(x)]\left[\exp\left(-\left|y - a\right|\left|\omega\right|\right)\right]\exp\left(i\omega x\right)\mathrm{d}\omega. \\
&= \mathcal{F}^{-1}\left(\mathcal{F}[\varphi(x)]\left[\exp\left(-\left|y - a\right|\left|\omega\right|\right)\right]\right) \\
&= \int_{-\infty}^{\infty} \varphi(u)\mathcal{F}^{-1}\left[\exp\left(-\left|y - a\right|\left|\omega\right|\right)\right](x - u)\mathrm{d}u,
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. All that remains is to compute the inverse transform of the inside term. Individually, these have inverse Fourier transform

$$\mathcal{F}^{-1}\left[\exp\left(-\left|y-a\right|\left|\omega\right|\right)\right](x) = \frac{|y-a|}{\left|y-a\right|^2 + x^2}$$

Plugging this into the previous equation, we get that

$$u(x,y) = \int_{-\infty}^{\infty} \varphi(u) \frac{|y-a|}{\left|y-a\right|^2 + (x-u)^2} \mathrm{d}u.$$

We can turn this into a conditional probability by picking φ to be an indicator function $\mathbf{1}_{X_\tau = z}$. Therefore, we have that

$$\begin{aligned}\mathbb{P}(X_\tau \in \mathrm{d}z \mid X_t = x, Y_t = y) &= \int_{-\infty}^{\infty} \varphi(u) \frac{|y-a|}{\left|y-a\right|^2 + (x-u)^2} \mathrm{d}u \\ &= \frac{|y-a|}{\left|y-a\right|^2 + (x-z)^2}.\end{aligned}$$

Differentiating this equation with respect to x and y shows that this indeed satisfies Lapalace's equation. Further, we see that this satisfies the boundary condition as the above probability is zero when $y = a$ unless $z = x$.

Exercise 3. Consider a continuous-time $(n+1)$-state Markov process $X(t)$, $X \in \mathcal{S} = \{0, 1, 2, \cdots, n\}$, with transition rates $g(i,j)$. Let state 0 be an absorbing state, e.g., all $g(0,j) = 0$, $1 \leq j \leq n$. Let τ_k be a hitting time:

$$\tau_k := \inf \big\{ t \geq 0 : X(t) = 0, X(0) = k \big\}.$$

(a) Show that

$$\sum_{1 \leq k \leq n} g(j,k)\mathbb{E}[\tau_k] = -1.$$

(b) Derive a system of equations relating $\mathbb{E}[\tau_k^2]$ to $\mathbb{E}[\tau_j]$, $1 \leq j, k \leq n$.
(c) Now if both states 0 and n are absorbing, let u_k be the probability of $X(t)$, starting with $X(0) = k$, being absorbed into state 0 and $1 - u_k$ be the probability being absorbed into state n. Derive a system of equations for u_k.

Solution 3. Derive expectation for τ_k, We write that

$$\begin{aligned}\mathbb{P}(X_t = 0 \mid X(0) = k) = \mathbb{P}(\tau_k \leq t)\mathbb{P}(X_t = 0 \mid X(0) = k, \tau_k \leq t) \\ + \mathbb{P}(\tau_k > t)\mathbb{P}(X_t = 0 \mid X(0) = k, \tau_k > t).\end{aligned}$$

In the cases where $\mathbb{P}(\tau_k \leq t)$, we know the $X_t = 0$ since 0 is absorbing. We then have that

$$\mathbb{P}(X_t = \mid X(0) = k) = \mathbb{P}(\tau_k \leq t) + (1 - \mathbb{P}(\tau_k \leq t))\mathbb{P}(X_t = 0 \mid X(0) = k, \tau_k > t).$$

We can write the expectation as

$$\begin{aligned}\mathbb{E}[\tau_k] &= \int_0^\infty t \frac{d}{dt} p_t(k,0) \mathrm{d}t \\ &= \int_0^\infty t \sum_i p_t(k,i) g(i,0) \mathrm{d}t.\end{aligned}$$

We then have that

$$\begin{aligned}g(j,k)\mathbb{E}[\tau_k] &= \int_0^\infty t \sum_i p_t(k,i) g(j,k) g(i,0) \mathrm{d}t \\ \sum_{1 \leq k \leq n} g(j,k)\mathbb{E}[\tau_k] &= \int_0^\infty t \sum_k \sum_i g(j,k) p_t(k,i) g(i,0) \mathrm{d}t \\ &= \sum_i g(i,0) \int_0^\infty t \frac{\partial}{\partial t} p_t(j,i) \mathrm{d}t \\ &= -\int_0^\infty \sum_i p_t(j,i) g(i,0) \mathrm{d}t,\end{aligned}$$

where in last line we've used integration by parts to eliminate t. We can write this using the KFE, so that

$$\begin{aligned}\sum_{1\leq k\leq n} g(j,k)\mathbb{E}[\tau_k] &= -\int_0^\infty \sum_i p_t(j,i)g(i,0)\mathrm{d}t\\ &= -\int_0^\infty \frac{\partial}{\partial t}p_t(j,0)\mathrm{d}t\\ &= \lim_{t\to\infty} p_0(j,0) - p_t(j,0)\\ &= -1,\end{aligned}$$

since 0 is absorbing.
(b) We can write the expectation as

$$\begin{aligned}\sum_{1\leq k\leq n} g(j,k)\mathbb{E}[\tau_k^2] &= \int_0^\infty t^2\frac{d}{dt}p_t(k,0)\mathrm{d}t\\ &= \cdots\\ &= -2\int_0^\infty t\frac{\partial}{\partial t}p_t(j,0)\mathrm{d}t\\ &= -2\mathbb{E}[\tau_j].\end{aligned}$$

Essentially, we've repeated exactly what we did in part (a) with t^2 instead of t.
(c) We'll define

$$u_k = \lim_{t\to\infty} p_t(k,0).$$

As 0 and n are the only two absorbing states, we have that

$$1 - u_k = \lim_{t\to\infty} p_t(k,n).$$

We'll now consider the sum

$$\begin{aligned}\sum_{1\leq k\leq n} g(j,k)u_k &= \lim_{t\to\infty}\sum_{1\leq k<n} g(j,k)p_t(k,0)\\ &= \lim_{t\to\infty}\frac{\partial}{\partial t}p_t(j,0)\\ &= 0,\end{aligned}$$

since 0 is absorbing.

Exercise 4. This problem is set up in the language of Theorem 9.4.1. and its Corollary 9.4.2, but really is about solving a first-order linear ordinary differential equation (ODE) and carrying out asymptotic evaluation of an integral by Laplace's method.
Consider an Ito process $X(t)$ with boundaries, $X \in (0,1)$:

$$\mathrm{d}X(t) = \mu(X)\mathrm{d}t + \epsilon \mathrm{d}W(t),$$

where ϵ is a small constant, and $\mu(x)$ has a potential function $U(x)$: $\mu(x) = -\mathrm{d}U(x)/\mathrm{d}x$. The drift $\mu(x)$ has two roots $x_1, x_2 \in [0,1]$, $x_1 < x_2$, $\mu'_x(x_1) < 0$ and $\mu'_x(x_2) > 0$; they correspond to a local minimum, at x_1, and a local maximum, at x_2, of $U(x)$. The backward equation for the expected value of the hitting time, $T(x)$ is

$$\frac{\epsilon^2}{2}\frac{\mathrm{d}^2T(x)}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x} = -1, \ \frac{\mathrm{d}T(0)}{\mathrm{d}x} = 0, \ T(1) = 0.$$

The boundary condition at $x = 0$ is understood as "reflecting the process", the boundary at $x = 1$ is understood as "killing the process".
(a) Show that the $T(x;\epsilon)$, the solution to the ODE,

$$T(x;\epsilon) = \frac{2}{\epsilon^2}\int_x^1 \mathrm{d}u \int_0^u \exp\left\{\frac{2}{\epsilon^2}\Big[U(u) - U(v)\Big]\right\}\mathrm{d}v.$$

(b) Using the result in (a) show that $T(x;\epsilon)$, as $\epsilon \to 0$, has an asymptotic expression that is independent of x,

$$T(x;\epsilon) \simeq \frac{2\pi}{\sqrt{U''(x_1)|U''(x_2)|}}\exp\left\{\frac{2}{\epsilon^2}\Big[U(x_2) - U(x_1)\Big]\right\}.$$

It only has to do with the "barrier height" $U(x_2) - U(x_1)$ and the curvatures at the x_1 and x_2.

Solution 4. (a) We begin by taking the first derivative of the function above

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x} &= -\frac{2}{\epsilon^2}\int_0^x \exp\left\{\frac{2}{\epsilon^2}\left[U(x) - U(v)\right]\right\}\mathrm{d}v \\ &= -\frac{2}{\epsilon^2}\exp\left\{\frac{2}{\epsilon^2}U(x)\right\}\int_0^x \exp\left\{-\frac{2}{\epsilon^2}U(v)\right\}\mathrm{d}v\end{aligned}$$

Next, taking the second derivative using product rule, we see

$$\frac{\mathrm{d}^2T}{\mathrm{d}^2x} = -\frac{2}{\epsilon^2} - \frac{2}{\epsilon^2}\left(\frac{2}{\epsilon^2}U'(x)\exp\left\{\frac{2}{\epsilon^2}U(x)\right\}\right)\int_0^x \exp\left\{-\frac{2}{\epsilon^2}U(v)\right\}\mathrm{d}v$$

.

Plugging this into the ODE, we have

$$-1 + \left(-U'(x)\frac{2}{\epsilon^2}\exp\left\{\frac{2}{\epsilon^2}U(x)\right\} - \mu(x)\frac{2}{\epsilon^2}\exp\left\{\frac{2}{\epsilon^2}U(x)\right\}\right)\int_0^x \exp\left\{-\frac{2}{\epsilon^2}U(v)\right\}\mathrm{d}v = -1,$$

where we've used that $\mu = -U'$. $T(x;\epsilon)$ is a solution to the ODE.
(b) We begin by rewriting $T(x;\epsilon)$ as nested integrals

$$T(x;\epsilon) = -\frac{2}{\epsilon^2}\int_0^x \exp\left(\frac{2}{\epsilon^2}U(u)\right)\int_0^u \exp\left(-\frac{2}{\epsilon^2}U(v)\right)\mathrm{d}v\mathrm{d}u.$$

We first will asymptotically evaluate the inner integral in the $\epsilon \to 0$ limit around x_1 which maximizes $-U$ as $(-U)''(x_1) = \mu'(x_1) < 0$ and $\mu(x_1) = 0$, so that

$$T(x;\epsilon) \simeq \frac{2}{\epsilon^2}\left(\sqrt{\frac{2\pi}{\frac{2}{\epsilon^2}\left|U''(x_1)\right|}}\exp\left(-\frac{2}{\epsilon^2}U(x_1)\right)\right)\int_0^x \exp\left(\frac{2}{\epsilon^2}U(u)\right)\mathrm{d}u.$$

We'll repeat this method to asymptotically evaluate the inner integral using that U is maximized at x_2 by similar argument to what we used for x_1. Using Laplace's method allows us to conclude

$$\begin{aligned}
T(x;\epsilon) &\simeq \frac{2}{\epsilon^2}\left(\sqrt{\frac{2\pi}{\frac{2}{\epsilon^2}\left|U''(x_1)\right|}}\exp\left(-\frac{2}{\epsilon^2}U(x_1)\right)\right)\left(\sqrt{\frac{2\pi}{\frac{2}{\epsilon^2}U''(x_2)}}\exp\left(\frac{2}{\epsilon^2}U(x_2)\right)\right)\\
&\simeq \frac{2\pi}{\sqrt{\left|U''(x_2)\right|U''(x_1)}}\exp\left(\frac{2}{\epsilon^2}[U(x_2) - U(x_1)]\right).
\end{aligned}$$

I skipped most of the algebra in the last line, but it's mostly manipulating square roots and combining the exponentials.

Exercise 5. As a special example of a Lévy process, let $Y(t)$ be the standard Poisson process with probability mass function

$$p_{Y(t)}(n) = \mathbb{P}\{Y(t) = n\} = \frac{t^n e^{-t}}{n!};$$

all jumps in the Poisson process have $\Delta Y = 1$. If one denotes the random times at which the jumps occur sequentially as $T_1, T_2, \cdots$, then $\{T_k\}_{k\geq 1}$ is a positive real-valued, discrete-time stochastic process with independent and stationary increments. This is in contrast to $Y(t)$ which is an integer-valued continuous-time stochastic process with independent and stationary increments. Y_t and T_k are widely called Poisson counting process and Poisson point process, respectively.

(a) Show that for any $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$, $(Y_{t_4} - Y_{t_3}) \perp\!\!\!\perp (Y_{t_2} - Y_{t_1})$ according to the definition of a Poisson process in Chapter 5; show also that for any $0 \leq t_1 < t_2 < \infty$, $(Y_{t_2} - Y_{t_1}) \sim Y_{t_2 - t_1}$.

(b) A standard Brownian motion $W(t)$ has independent and stationary increments, between t and $t + \tau$, that are normally distributed:

$$W(t+\tau) - W(t) \sim \mathcal{N}(0, \tau), \quad t, \tau \geq 0.$$

What is the distribution for the stationary increment $Y(t+\tau) - Y(t)$? What is the stationary increment $T_{k+\ell} - T_k$, where ℓ is a positive integer?

(c) Introducing time-changed Poisson process with rate function $\lambda(t) \geq 0$. Assuming that $\lambda(t)$ is uniformly bounded for all time t:

$$\tilde{Y}(t) := Y\left(\int_0^t \lambda(s)\mathrm{d}s\right).$$

Show that in the limit of $t \to \infty$,

$$\lim_{t\to\infty} \mathbb{P}\left(\left|\frac{\tilde{Y}(t)}{t} - \lambda(t)\right| > \epsilon\right) = 0, \ \forall \epsilon > 0.$$

(d) Show that for a continuous time two-state Markov process $X(t)$, $X \in \{-1, +1\}$, with transition rates $g(-1, +1) = g_+$ and $g(+1, -1) = g_-$, can be represented by an integral equation in terms of two independent Poisson processes $Y_1(t)$ and $Y_2(t)$ with time changes:

$$X(t) = X(0) + 2Y_1\left(g_+ \int_0^t \mathbf{1}_{-1}\big(X(s)\big)\mathrm{d}s\right) - 2Y_2\left(g_- \int_0^t \mathbf{1}_{1}\big(X(s)\big)\mathrm{d}s\right).$$

(e) Applying the result in (d), show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X(t)] = 2g_+ P_{-1}(t) - 2g_- P_1(t),$$

where $P_k(t) = \mathbb{P}(X(t) = k)$

Solution 5. (a) We write that

$$Y_t = \sum_{k=1}^{\infty} \mathbf{1}_{T_k \leq t},$$

so that Y_t counts the events which have occurred up to time t. We then have that for $t_1 < t_2 < t_3 < t_3$,

$$Y_{t_4} - Y_{t_3} = \sum_{k=1}^{\infty} \mathbf{1}_{T_k \leq t_4} - \mathbf{1}_{T_k \leq t_3} = \sum_{k=1}^{\infty} \mathbf{1}_{t_3 < T_k \leq t_4}.$$

As the underlying T_k are independent for the differences $Y_{t_4} - Y_{t_3}$ and $Y_{t_2} - Y_{t_1}$ and do not overlap, this representation shows that

$$Y_{t_4} - Y_{t_3} \text{ is independent of } Y_{t_2} - Y_{t_1}$$

Additionally, the derived equation for $Y_{t_4} - Y_{t_3}$ shows that the increments is the number of events which have occurred between the two time points as these increments are i.i.d. they must share the same distribution with $Y_{t_4 - t_3}$, so that

$$Y_{t_4} - Y_{t_3} \sim Y_{t_4 - t_3}.$$

(b) The increments of the counting process are Poisson distributed with

$$\begin{aligned} Y_{t+\tau} - Y_t &\sim \text{Pois}(\tau) \\ T_{t+l} - T_t &\sim \text{Gamma}(l, 1). \end{aligned}$$

The first statement we've proved in 561. The second uses that the between event times are independent exponentials with rate λ as from Thm 5.1.5. Their sum is then Gamma distributed with parameters l and 1.
(c) We start with the transformed Poisson process and write

$$\left| \frac{\tilde{Y}(t)}{t} - \lambda(t) \right| = \left| \frac{1}{t} Y \left(\int_0^t \lambda(s) \mathrm{d}s \right) - \lambda(t) \right|.$$

By mean value theorem for integrals, we have that for $s_\star(t) \in (0, t)$, we have that

$$\frac{1}{t} \int_0^t \lambda(s) \mathrm{d}s = \lambda(s_\star).$$

By uniform boundness, we then have that

$$\left| \frac{1}{t} \int_0^t \lambda(s) \mathrm{d}s \right| = |\lambda(s_\star)| \leq M,$$

for some fixed $M > 0$. We can then write out write out the probability distribution of the time changed Poisson as

$$\mathbb{P}(\tilde{Y}(t) = kt) = (\lambda(s_\star))^{kt} \frac{1}{(kt)!} \exp(-\lambda(s_\star)).$$

We then subtract $\lambda(t)t$ from both sides so that

$$\mathbb{P}([\tilde{Y}(t) - \lambda(t)t] \approx [k - \lambda(t)]t) \leq \frac{M^{kt}}{(kt)!} \exp(-\lambda(s_\star)) \leq \frac{M^{kt}}{(kt)!},$$

where we've used that $\lambda(t)$ is non-negative. Yes, we know the distribution is integer valued but we're playing fast and loose with it. For any fixed $k \in \mathbb{N}$ as $t \to \infty$, we have

$$\frac{M^{kt}}{(kt)!} \xrightarrow{t\to\infty} 0,$$

which shows that

$$\mathbb{P}\left(\left|\tilde{Y}(t)/t - \lambda(t)\right| > \epsilon\right) \xrightarrow{t\to\infty} 0.$$

(d)Assuming that Δt is sufficiently, small only one of the integrals can be non-zero, in the case that $X(t) = 1$

$$X(t + \Delta t) - X(t) = -2Y_2 \left(g_- \int_t^{t+\Delta t} \mathrm{d}s\right) = -2Y_2 \left(g_- \Delta t\right)$$

This will match up when Y_1 returns 1, so the conditional probability is then given by

$$\begin{aligned}\mathbb{P}(X(t + \Delta t) = -1 \mid X(t) = 1) &= f_{\text{Pois}}(g_- \Delta t, 1)\\ &= -g_- \Delta t \exp(-g_- \Delta t)\\ &= 1 - g_- \Delta t + O(\Delta t^2),\end{aligned}$$

where last line we've used $\exp(-x) \approx 1 - x + x^2$. This then shows the g_- is the generator element corresponding to $g(1, -1)$ using the definition of the generator. We can repeat this argument assuming $X(t) = -1$ to show that $g(-1, 1) = g_+$

$$\mathbb{P}(X(t + \Delta t) = 1 \mid X(t) = -1) = 1 - g_+ \Delta t + O(\Delta t^2).$$

This shows that this representation is equivalent to the original Markov chain.
(e) Taking the expectation of the result in (d), we have that

$$\begin{aligned}\mathbb{E}[X(t)] &= X(0) + 2\mathbb{E}\left[Y_1\left(g_+ \int_0^t \mathbf{1}_{-1}\big(X(s)\big)\mathrm{d}s\right)\right] - 2\mathbb{E}\left[Y_2\left(g_- \int_0^t \mathbf{1}_1\big(X(s)\big)\mathrm{d}s\right)\right]\\ &= X(0) + 2g_+ \int_0^t \mathbb{E}\mathbf{1}_{-1}\big(X(s)\big)\mathrm{d}s - 2g_- \int_0^t \mathbb{E}\mathbf{1}_1\big(X(s)\big)\mathrm{d}s,\end{aligned}$$

where we've interchanged the integral and expectation and then used that the expectation of an indicator event is its probability. Taking the derivative of this equation, gives the result

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X(t)] = 2g_+ P_{-1}(t) - 2g_- P_1(t).$$

Exercise 6. Let $P = (P_t)_{t\geq 0}$ be a Poisson process with intensity λ.

1. What is the Levy measure ν of P?
2. Let $\mathrm{d}X_t = \mathrm{d}P_t$. Define $u(t,x) = \mathbb{E}[\varphi(X_T) \mid X_t = x]$. Find $u(t,x)$ and verify that it solves the Kolmogorov Backward Equation.

Solution 6. (1.) Using definition 10.1.6 and 10.17, we can write the Levy measure of P as

$$\nu(U) = \mathbb{E}N(1,U) = \mathbb{E}\left[\sum_{s:0<s\leq 1} \mathbf{1}_{\Delta P_s \in U}\right].$$

We'll now compute the probability distribution of ΔP_s, we have that

$$N_{t+\mathrm{dt}} - N_t \sim \mathrm{Pois}(\lambda \mathrm{d}t),$$

so that jumps can only of size one since this depends on powers of $\lambda\mathrm{d}t$. We then have that the number of jumps of size one in U depends on whether or not U contains 1. We can express this as:

$$\nu(U) = \mathbb{E}\left[\sum_{n=1}^{P_1} \mathbf{1}_{1\in U}\right] = \begin{cases} \lambda = \mathbb{E}\left[P_1\right], & 1 \quad \in U \\ 0, & 1 \quad \notin U. \end{cases}$$

We can write the full measure as

$$\nu(U) = \lambda \mathbf{1}_{1\in U} \text{ or } \nu(x) = \lambda\delta(x-1).$$

(2.) As a pure jump process, we can write P_t in the form

$$\begin{aligned} P_t &= \int_{|z|<1/2} z\tilde{N}(t,\mathrm{d}z) + \int_{|z|\geq 1/2} zN(t,\mathrm{d}z) \\ &= \int_{|z|\geq 1/2} zN(t,\mathrm{d}z) = N(t,\mathrm{d}z) \end{aligned}$$

where we've used that the Poisson process has jumps of size 1. In short, this allows us to write

$$\mathrm{d}P_t = \lambda t + \int_{\mathbb{R}} z\tilde{N}(t,\mathrm{d}z).$$

Following the notes, we have that

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i\xi x + (T-t)\psi(\xi))\hat{\varphi}(\xi).$$

We can find the characteristic exponent ψ as

$$\psi(\xi) = \lambda(\exp(i\xi) - 1),$$

following the derivation on pg 175 (MLN). Plugging this in, we have that

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i\xi x + (T-t)\lambda(\exp(i\xi) - 1))\hat{\varphi}(\xi)\mathrm{d}\xi.$$

page 186 We'll show that this satisfies the KBE with generator

$$\begin{aligned}\mathcal{A} &= \lambda\partial_x + \int_{\mathbb{R}} \lambda\delta_1(z)(\theta_z - 1 - z\partial_x)\\ &= \lambda\partial_x + \lambda\theta_1 - \lambda - \lambda\partial_x\\ &= \lambda(\theta_1 - 1).\end{aligned}$$

We then have that

$$\begin{aligned}\mathcal{A}u &= \lambda[u(t,x+1) - u(t,x)]\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \lambda(\exp(i\xi) - 1)\exp(i\xi x + (T-t)\lambda(\exp(i\xi) - 1))\hat{\varphi}(\xi)\mathrm{d}\xi.\end{aligned}$$

Also notice that after interchanging the integral and derivative with respect to t, we have that

$$\partial_t u = -\frac{1}{2\pi}\int_{\mathbb{R}} \lambda(\exp(i\xi) - 1)\exp(i\xi x + (T-t)\lambda(\exp(i\xi) - 1))\hat{\varphi}(\xi)\mathrm{d}\xi.$$

Therefore, u satisfies the KBE

$$\mathcal{A}u + \partial_t u = 0.$$

We can also check the boundary condition is satisfied as

$$u(T,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i\xi x)\hat{\varphi}(\xi)\mathrm{d}\xi = \varphi(x).$$