**Exercise 1.** Using Residue Calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}. \tag{1}$$

**Solution 1.** We see that the function $f(z) = \sin z / \sinh z$ has singularities at $\pi i k$ for $k \in \mathbb{Z}$. In order to avoid these singularities and evaluate the integral we will use a rectangular contour $\Gamma$ similar to A&F Fig. 4.3.5. Here, we let $C_{\epsilon_1}$ represent the top half of the circle centered at 0 with radius $\epsilon > 0$ and $C_{\epsilon_2}$ be centered at $\pi i$ with radius $\epsilon > 0$. Let $C_{SR}$ be contour going from $R$ to $R + i\pi$, $C_{SL}$ be the contour from $-R + i\pi$ to $-R$. Filling in the additional line segments, we have that

$$\oint_{\Gamma} f(z)dz = \left( \int_{C_{\epsilon_1}} + \int_{\epsilon}^{R} + \int_{C_{SR}} + \int_{R+i\pi}^{i\pi+\epsilon} + \int_{C_{\epsilon_2}} + \int_{i\pi-\epsilon}^{-R+i\pi} + \int_{C_{SL}} + \int_{-R}^{-\epsilon} \right) f(z)dz = 0 \tag{2}$$

by Cauchy's theorem since $\Gamma$ encloses no singularities. We begin by noting that

$$\int_{C_{\epsilon_1}} f(z) \to 0 \text{ as } \epsilon \to 0 \tag{3}$$

by A&F Theorem 4.3.1. since $f(z) \cdot z$ approaches 0 as $|z| \to 0$. This follows from the fact the limit of $f(z)$ approaches 1 and $z$ approaches 0 individually as $\epsilon \to 0$. Similarly, we can use Theorem 4.3.1. to compute that

$$\int_{C_{\epsilon_2}} f(z) \to -\pi i \text{Res}(f, i\pi) \text{ as } \epsilon \to 0 \tag{4}$$

There is an additional $-1$ in front due to the orientation of $C_{\epsilon_2}$. We can compute this residue using the fact that $i\pi$ is a simple pole, taking the limit

$$\text{Res}(f, i\pi) = \frac{\sin z_0}{\cosh z_0} = -\sin(i\pi) \tag{5}$$

Therefore, we have that

$$\int_{C_{\epsilon_2}} f(z) \to \pi i \sin(i\pi) \text{ as } \epsilon \to 0 \tag{6}$$

We next will simplify the integral over the parts of $\Gamma_\epsilon$ with $y = i\pi$ as

$$\left( \int_{R+i\pi}^{\epsilon+i\pi} + \int_{-\epsilon+i\pi}^{-R+i\pi} \right) f(z)dz = \left( \int_{R}^{\epsilon} + \int_{-\epsilon}^{-R} \right) \frac{\sin(x+i\pi)}{\sinh(x+i\pi)}dx. \tag{7}$$

We now use the sum of angles formula to show that

$$\begin{aligned}
\left( \int_{R+i\pi}^{\epsilon+i\pi} + \int_{-\epsilon+i\pi}^{-R+i\pi} \right) f(z)dz &= \left( \int_{R}^{\epsilon} + \int_{-\epsilon}^{-R} \right) \left( \frac{\sin x \cos i\pi}{\sinh(x+i\pi)} + \frac{\cos x \sin i\pi}{\sinh(x+i\pi)} \right) dx & (8)\\
&= -\left( \int_{R}^{\epsilon} + \int_{-\epsilon}^{-R} \right) \left( \frac{\sin x \cos i\pi}{\sinh(x)} + \frac{\cos x \sin i\pi}{\sinh(x)} \right) dx & (9)\\
&= -\left( \int_{R}^{\epsilon} + \int_{-\epsilon}^{-R} \right) \left( \frac{\sin x \cos i\pi}{\sinh(x)} \right) dx & (10)\\
&= \cos i\pi \left( \int_{\epsilon}^{R} + \int_{-R}^{-\epsilon} \right) \frac{\sin x}{\sinh(x)}dx & (11)
\end{aligned}$$

where we have used that $\sinh(x) = -\sinh(x + i\pi)$ and the fact that $\cos x/\sinh x$ is even and that the bounds of our integral are symmetric. Next, we show that $\int_{C_{SL}}$ and $\int_{C_{SR}} \to 0$ as $R \to \infty$. This follows from the fact that

$$\left|\int_0^\pi f(R + i\theta)d\theta\right| \le \pi \cdot \sup_{\theta\in[0,\pi]} (f(R + i\theta)) \tag{12}$$

$$= \pi \sup_{\theta\in[0,\pi]} \frac{|\sin(R + i\theta)|}{|\sinh(R + i\theta)|} \to 0 \text{ as } R \to \infty, \tag{13}$$

since $|\sin(R + i\theta)| = \left|(e^{-\theta+iR} - e^{\theta+iR})/2i\right|$ which is bounded by a constant not depending on $R$ and $|\sinh(R + i\theta)| = \left|(e^{R+i\theta} - e^{-R-i\theta})/2\right|$ which goes to infinity as $R \to \infty$. This same argument holds for the integral over $C_{SL}$. Reducing what remains of our integral, we see that

$$(1 + \cos i\pi)\left(\int_\epsilon^R + \int_{-R}^{-\epsilon}\right) f(x)dx + \int_{C_{\epsilon_1}} f(z)dz + \int_{C_{\epsilon_2}} f(z)dz = 0 \tag{14}$$

Taking the limits as $R \to \infty$ and $\epsilon \to 0$, we see that

$$(1 + \cos i\pi) \int_{-\infty}^{\infty} f(x)dx = -\pi i \sin i\pi. \tag{15}$$

We can then simplify this to solve for the desired integral

$$\int_{-\infty}^{\infty} f(x)dx = -\pi i \frac{\sin(i\pi)}{1 + \cos i\pi} \tag{16}$$

$$= -\pi i \tan\left(\frac{i\pi}{2}\right) \tag{17}$$

$$= \pi \tanh\left(\frac{\pi}{2}\right), \tag{18}$$

where we have used the half angle formula for $\tan x$ and the fact that $\tanh x = -i \tan ix$.

**Exercise 2.** Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{1+\cos x}{(x-\pi)^2}. \tag{19}$$

**Solution 2.** We'll begin by noting

$$I = \int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2} dx = \int_{-\infty}^{\infty} \frac{1+\cos(x+\pi)}{((x+\pi)-\pi)^2} dx = \int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx, \tag{20}$$

where we have just shifted the integral and noted that we are integrating over the entire real line. We'll now shift our focus to the function $f(z) = \frac{1-e^{iz}}{z^2}$ since

$$\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{1-e^{iz}}{z^2} dz\right) = \int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx \tag{21}$$

We'll now consider the integral of $f(z)$ over the contour $\Gamma$ which is comprised of the following segments

$$\oint_{\Gamma} = \int_{-R}^{-\epsilon} - \int_{C_\epsilon} + \int_{\epsilon}^{R} + \int_{C_R} = 0, \tag{22}$$

here $C_\epsilon$ is the circle centered at 0 with radius $\epsilon$ and oriented counter-clockwise, and $C_R$ is the circle centered at 0 with radius $R$ and also oriented counter-clockwise. We have also used Cauchy's Theorem, since the function $f(z)$ is analytic within and on the contour $\Gamma$. We can see that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$ since

$$\left|\int_{C_R} f(z)dz\right| \le \left|\frac{1-e^{iz}}{z^2}\right| \cdot \pi R \le \frac{2\pi R}{R^2}. \tag{23}$$

We can then write that in the limit as $R \to \infty$

$$\int_{-\infty}^{-\epsilon} f(z)dz + \int_{\epsilon}^{\infty} f(z)dz = \int_{C_\epsilon} f(z)dz \tag{24}$$

In the limit as $\epsilon \to 0$, we can evaluate the right hand side using Theorem 4.3.1., so that

$$\int_{C_\epsilon} f(z)dz \to i\pi \mathrm{Res}(f,0) \tag{25}$$

We can compute this residue by looking directly at the Taylor-Laurent series of $f(z)$

$$\frac{1}{z^2} \cdot \left(1 - e^{iz}\right) = \frac{1}{z^2}\left(-iz + \frac{(iz)^2}{2!} - \cdots\right) \tag{26}$$

$$= \frac{-i}{z} - \frac{1}{2} - \cdots \tag{27}$$

Looking at this series, we see that $\mathrm{Res}(f,0) = -i$. This means that taking the limit as $\epsilon = 0$, we see

$$\int_{-\infty}^{\infty} f(z)dz = \pi. \tag{28}$$

Since this is real and the integral exists, we see that our desired integral is $I = \pi$

**Exercise 3.** Evaluate the following integral using residue calculus

$$I = \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2} dx \tag{29}$$

where $-1 < a < 1$,$a \neq 0$ and $-\pi < b < \pi$, $b \neq 0$. Justify all key steps. Do not use the general formula for this integral

**Solution 3.** We'll consider the same contour as in page 64 of Prof. Tung's notes. We begin by showing that $\left|\int_{C_R} f(z)\right| \to 0$ as $R \to \infty$ where $C_R$ is the circle centered at 0 with radius $R$ for angles $\theta \in [0, 2\pi)$. This follows from the fact that

$$\left|\int_{C_R} f(z)\right| \leq 2\pi R \cdot \left|\frac{R^a e^{ia\theta}}{R^2 e^{i\theta} + 2Re^{i\theta} + 1}\right| = O\left(\frac{R^{1+a}}{R^2}\right) = O(R^{a-1}). \tag{30}$$

Since $a \in (-1, 1)$, this means the power $a - 1 < 0$ and the integral goes to 0 in the limit as $R \to \infty$. Now looking at $C_2$ which is the circle centered at 0 and with radius $\rho$ and oriented counter-clockwise, we see that

$$\begin{aligned}
\left|\int_{C_2} f(z)dz\right| &= \left|\int_{2\pi}^0 f(\rho e^{i\theta})\rho i e^{i\theta} d\theta\right| && (31)\\
&= \left|\int_{2\pi}^0 \frac{\rho^{a+1} e^{i(a+1)\theta}}{1 + 2\cos(b)\rho e^{i\theta} + \rho^2 e^{2i\theta}} i d\theta\right| && (32)\\
&\leq \int_{2\pi}^0 \left|\frac{\rho^{a+1} e^{i(a+1)\theta}}{1 + 2\cos(b)\rho e^{i\theta} + \rho^2 e^{2i\theta}} i\right| d\theta && (33)\\
&\leq \int_{2\pi}^0 \frac{|\rho^{a+1}|}{|1| + |2\cos(b)\rho| + |\rho^2|} d\theta \to 0 \text{ as } \rho \to 0. && (34)
\end{aligned}$$

We have used that the $a+1 > 0$ to make the final limit argument and the triangle inequality in the computations above. Next, we consider the integral over $C_1$ where $z = re^{2\pi i}$ along the real axis from $R$ to 0, so that

$$\begin{aligned}
\int_{C_1} f(z)dz &= \int_R^0 f(re^{2\pi i})e^{2\pi i} dr && (35)\\
&= \int_R^0 \frac{r^a e^{2\pi i(a+1)}}{1 + 2r\cos(b)e^{2\pi i} + r^2 e^{4\pi i}} dr && (36)\\
&= e^{2\pi i(a+1)} \int_R^0 \frac{r^a}{1 + 2r\cos(b) + r^2} dr && (37)\\
&= -e^{2\pi i(a+1)} \int_0^R \frac{r^a}{1 + 2r\cos(b) + r^2} dr && (38)\\
&= -e^{2\pi i(a+1)} \int_{C_0} f(z)dz, && (39)
\end{aligned}$$

where $C_0$ is the curve where $z = r$ from 0 to $R$. Due to our choice of contour $C = C_0 + C_1 + C_2 + C_R$, we have that in the limit as $R \to \infty$ and $\rho \to 0$

$$\oint_C f(z) = (1 - e^{2\pi i(a+1)}) \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2} dx. \tag{40}$$

This is great as we can now use the residue theorem to address the integral

$$\oint_C f(z)dz = 2\pi i \sum \mathrm{Res}(f, z_j). \tag{41}$$

Therefore, all that remains is to compute the residues of $f$ at its singularities. We begin by factoring the denominator of the integrand of $I$ using the quadratic formula

$$z_\pm = \frac{-2\cos b \pm \sqrt{4\cos^2 b - 4}}{2} = -\cos b \pm \sqrt{\cos^2 b - 1} = -\cos b \pm i\left|\sin b\right|, \tag{42}$$

where we have used that $1 - \cos^2 x = |\sin x|$. In the case that $b > 0$, we have that $\sin b = |\sin b|$, so

$$z_+ = -e^{-ib} \quad z_- = -e^{ib}. \tag{43}$$

Otherwise, we have that these are switched since $b < 0$, $-\sin b = |\sin b|$. In what follows, we just assume that $b \in (0, \pi)$. Now that we have shown our singularities are given by $z_+$ and $z_-$. We can then factor the integrand as

$$f(z) = \frac{z^a}{1 + 2z\cos(b) + z^2} = \frac{z^a}{(z - z_+)(z - z_-)}. \tag{44}$$

We can compute the residues as

$$\mathrm{Res}(f, z_+) = \lim_{z \to z_+} (z - z_+) \left( \frac{z^a}{(z - z_+)(z - z_-)} \right) = \frac{z_+^a}{z_+ - z_-} = \frac{(-1)^a e^{-iab}}{2i\sin b} \tag{45}$$

$$\mathrm{Res}(f, z_-) = \lim_{z \to z_-} (z - z_-) \left( \frac{z^a}{(z - z_+)(z - z_-)} \right) = \frac{z_-^a}{z_- - z_+} = -\frac{(-1)^a e^{iab}}{2i\sin b}. \tag{46}$$

To be thorough, we can also compute residues and integral for the case $b < 0$, so that

$$\mathrm{Res}(f, z_+) = \lim_{z \to z_+} (z - z_+) \left( \frac{z^a}{(z - z_+)(z - z_-)} \right) = \frac{z_+^a}{z_+ - z_-} = -\frac{(-1)^a e^{iab}}{2i\sin b} \tag{47}$$

$$\mathrm{Res}(f, z_-) = \lim_{z \to z_-} (z - z_-) \left( \frac{z^a}{(z - z_+)(z - z_-)} \right) = \frac{z_-^a}{z_- - z_+} = \frac{(-1)^a e^{-iab}}{2i\sin b}, \tag{48}$$

which shows that our integral will be unchanged as the sum of the residues is unchanged. We can then compute that

$$\oint_C f(z)dz = \frac{(-1)^a \pi}{\sin b} \left( e^{-iab} - e^{iab} \right) \tag{49}$$

$$\tag{50}$$

We can take this to compute our desired integral as

$$\begin{aligned}
\int_0^\infty \frac{x^a}{1+2x\cos(b)+x^2}dx &= (-1)^a \frac{\pi}{\sin b}\left(\frac{e^{-iab}-e^{iab}}{1-e^{2\pi i(a+1)}}\right) && (51)\\
&= (-1)^a \frac{\pi}{\sin b}\left(\frac{e^{-iab}-e^{iab}}{1-e^{2\pi i(a+1)}}\right)\left(\frac{e^{-\pi i a}}{e^{-\pi i a}}\right) && (52)\\
&= (-1)^a \frac{\pi e^{-\pi i a}}{\sin b}\left(\frac{e^{-iab}-e^{iab}}{e^{-\pi i a}-e^{\pi i a}}\right) && (53)\\
&= (-1)^a \frac{\pi e^{-\pi a i}}{\sin b}\left(\frac{\sin(ab)}{\sin(a\pi)}\right) && (54)\\
&= e^{\pi \alpha i} \frac{\pi e^{-\pi a i}}{\sin b}\left(\frac{\sin(ab)}{\sin(a\pi)}\right) && (55)\\
&= \frac{\pi}{\sin b}\left(\frac{\sin(ab)}{\sin(a\pi)}\right), && (56)
\end{aligned}$$

where we have used the definition of sine in terms of complex exponentials and in the last line we have used that $(-1)^\alpha = e^{\pi\alpha i}$.