AMATH 567
Applied Complex Variables
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Problem 1. From A&F 2.5.1. Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

(a) e^{iz} (b) e^{z^2} (c) $\dfrac{1}{z-1/2}$

(d) $\dfrac{1}{z^2-4}$ (e) $\dfrac{1}{2z^2+1}$ (f) $\sqrt{z-4}$, 0<=arg(z-4)<2π

Problem 2. From A&F 2.5.5. We wish to evaluate the integral

$$\int_0^\infty e^{ix^2}\mathrm{d}x.$$

Consider the contour

$$I_R = \oint_{C_{(R)}} e^{iz^2}\mathrm{d}z,$$

where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points $0, R$, and $Re^{i\pi/4}$. Show that $I_R = 0$ and that

$$\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2}\mathrm{d}z = 0,$$

where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$. Hint: Use $\sin(x) \geq \frac{2x}{\pi}$ on $0 \leq x \leq \pi/2$.

Then, breaking up the contour $C_{(R)}$ into three component parts, deduce

$$\lim_{R\to\infty}\left(\int_0^R e^{ix^2}\,dx - e^{i\pi/4}\int_0^R e^{-r^2}\mathrm{d}r\right) = 0,$$

and from the well-known result of real integration:

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$.

Problem 3. From A&F 2.5.6. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + 1}.$$

Show how to evaluate this integral by considering

$$\oint_{C(R)} \frac{\mathrm{d}z}{z^2 + 1},$$

where $C_{(R)}$ is the closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x axis. Hint: use

$$\frac{1}{z^2 + 1} = \frac{-1}{2i} \left(\frac{1}{z + i} - \frac{1}{z - i} \right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables.

Problem 4. From A&F 3.3.5. Let

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} \mathrm{d}\theta \\ &= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin\theta) \mathrm{d}\theta. \end{aligned}$$

The functions $J_n(t)$ are called Bessel functions, which are well-known special functions in mathematics and physics.