Exercise 1. Let X and Y_0, Y_1, $Y_2, \ldots$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that $\mathbb{E}\,|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, \ldots, Y_n)$ and $X_n = \mathbb{E}(X \mid \mathcal{F}_n)$. Show that the sequence $X_0, X_1, \ldots$ is a martingale with respect to the filtration $(\mathcal{F})_{n\geq 0}$

Solution 1. The random variables X_n are satisfy $\mathbb{E}\,|X_n| < \infty$ as they are the conditional expectation of X with respect to $\mathcal{F}_n$ and $\mathbb{E}\,|X| < \infty$. We can compute the conditional expectation of X_{n+1} with respect to $\mathcal{F}_n$ as

$$\begin{aligned}
\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) && (1)\\
&= \mathbb{E}(X \mid \mathcal{F}_n) && (2)\\
&= X_n, && (3)
\end{aligned}$$

where we have used the tower property and the definition of X_n.

Exercise 2. Let $X_0, X_1, \ldots$ be i.i.d. Bernoulli random variables with parameter p. Define $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n = 0, 1, 2, \ldots \tag{4}$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. Show that Z_n is a martingale with respect to this filtration.

Solution 2. We begin by noting that

$$Z_{n+1} = \left(\frac{1-p}{p}\right)^{2S_{n+1}-(n+1)} = \left(\frac{1-p}{p}\right)^{(2S_n - n)+(2X_{n+1}-1)} \tag{5}$$

$$= \left(\frac{1-p}{p}\right)^{2X_{n+1}-1} Z_n. \tag{6}$$

We can then compute the conditional expectation of Z_{n+1} with respect to $\mathcal{F}_n$ as

$$\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}-1} Z_n \mid \mathcal{F}_n\right) \tag{7}$$

$$= Z_n \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}-1} \mid \mathcal{F}_n\right) \tag{8}$$

$$= Z_n \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}-1}\right), \tag{9}$$

where we have used that $Z_n \in \mathcal{F}_n$ and that any function of only X_{n+1} is independent from $\mathcal{F}_n$. All that remains is to compute that final expectation. We do this by noting that $2X_{n+1} - 1$ is 1 with probability p and -1 with probability $1 - p$. Therefore,

$$\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}-1}\right) = p\left(\frac{1-p}{p}\right) + (1-p)\left(\frac{p}{1-p}\right) = p + 1 - p = 1. \tag{10}$$

Therefore, we conclude that

$$\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) = Z_n. \tag{11}$$

What remains to be shown is that $|Z_n| < \infty$. This follows from the fact that $|Z_n| < \max\{\left(\frac{1-p}{p}\right)^n, \left(\frac{1-p}{p}\right)^{-n}\}$. Either $\frac{1-p}{p}$ or $\frac{p}{1-p}$ is greater than one. In the case the first is greater than one, the random variable is bounded by $\left(\frac{1-p}{p}\right)^n$ since the maximum and minimum values that $2S_n - n$ can obtain are n and $-n$. Otherwise, if $\frac{p}{1-p} > 1$, then the random variable is bounded by $\left(\frac{1-p}{p}\right)^{-n}$. Therefore, Z_n is a martingale.

Exercise 3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + \cdots + \xi_n, \quad n \geq 1, \tag{12}$$

determine a martingale. Show that the summands are mutually uncorrelated i.e. that $\mathbb{E}(\xi_i \xi_j) = \mathbb{E}(\xi_i)\mathbb{E}(\xi_j)$ for $i \neq j$.

Solution 3. For any j, we must have $\mathbb{E}[\xi_j] = 0$. Otherwise, for $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$,

$$\mathbb{E}[\xi_j] = \mathbb{E}[X_j - X_{j-1}] = \mathbb{E}[\mathbb{E}(X_j - X_{j-1} \mid \mathcal{F}_{j-1})] = 0. \tag{13}$$

Now, we fix $i < j$ consider

$$\mathbb{E}[\xi_i \xi_j] = \mathbb{E}[\mathbb{E}((X_i - X_{i-1})(X_j - X_{j-1}) \mid \mathcal{F}_i)] \tag{14}$$

$$\tag{15}$$

We can continue by taking the conditional expectation within the expectation

$$\mathbb{E}[\xi_i \xi_j] = \mathbb{E}[\mathbb{E}((X_i - X_{i-1})(X_j - X_{j-1}) \mid \mathcal{F}_i)] \tag{16}$$

$$\mathbb{E}[(X_i - X_{i-1})\mathbb{E}(X_j - X_{j-1}) \mid \mathcal{F}_i], \tag{17}$$

where we have used that $(X_i - X_{i-1}) \in \mathcal{F}_i$. Next, we can show that

$$\mathbb{E}(\xi_j \mid \mathcal{F}_i) = \mathbb{E}(X_j - X_{j-1} \mid \mathcal{F}_i) = \mathbb{E}(X_j \mid \mathcal{F}_i) - \mathbb{E}(X_{j-1} \mid \mathcal{F}_i) = X_i - X_i = 0 \tag{18}$$

where we have used the martigale property X_n. Therefore,

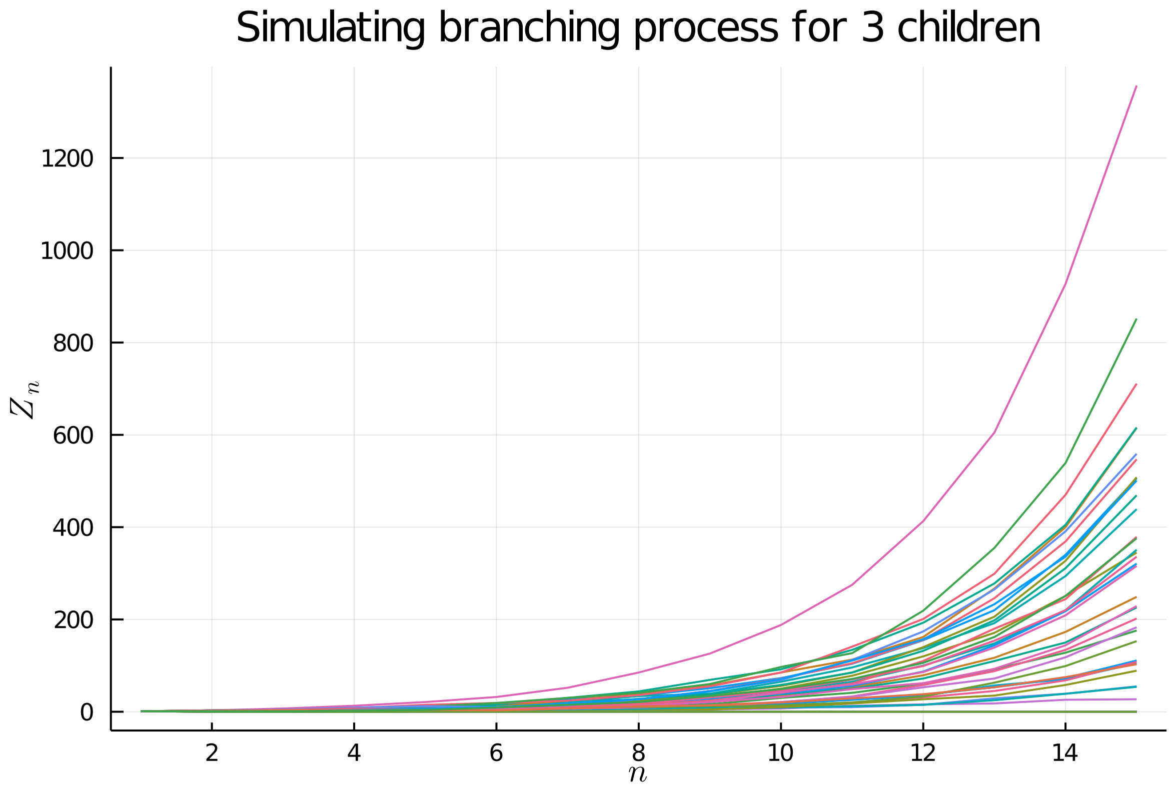
$$\mathbb{E}[\xi_i \xi_j] = \mathbb{E}[(X_i - X_{i-1}) \cdot 0] = 0. \tag{19}$$

Exercise 4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8$, $p_1 = 3/8$, $p2 = 3/8$, $p3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?

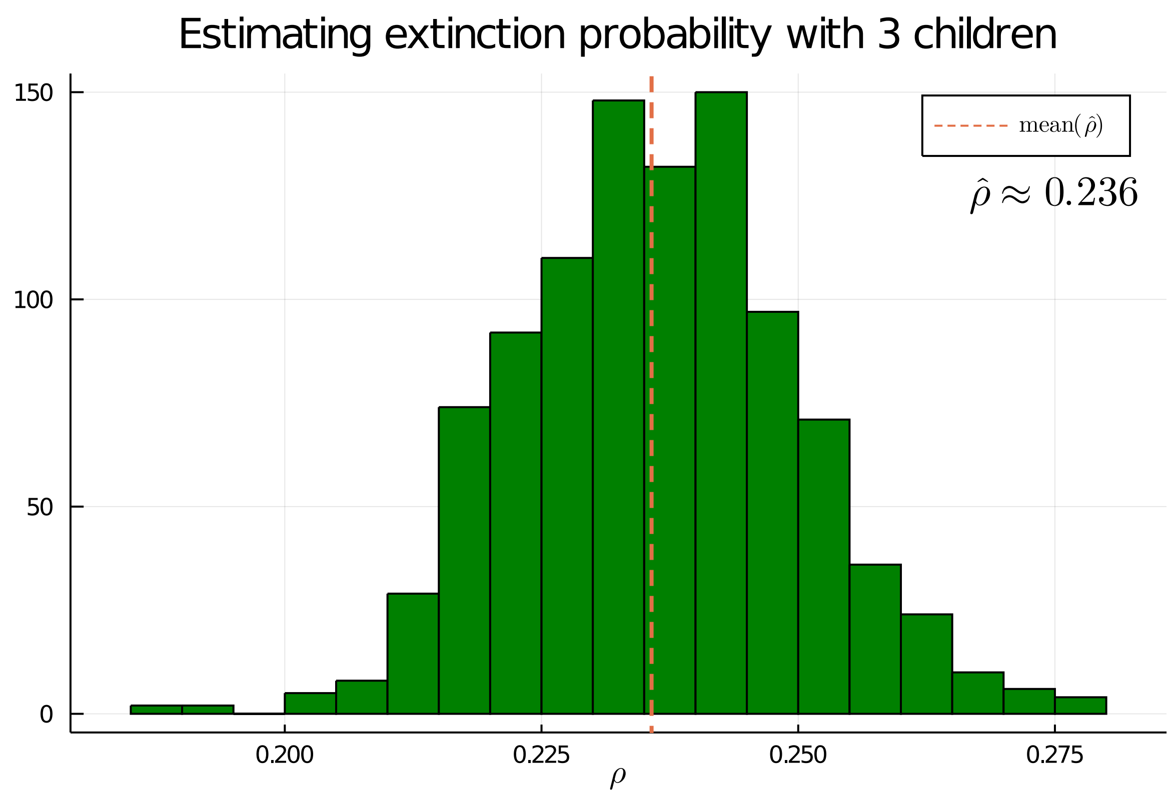
Solution 4. Above, we can see that the offspring distribution is given by $\xi \sim \text{Binom}(3,\, 1/2)$ with mean $\mathbb{E}[\xi] = \mu = 1.5$. Since $\mu > 1$ and $Z_0 = 0$, we can use theorem 4.3.12 from Durrett to compute the extinction probably as the solution to $\varphi(\rho) = \rho$ in $[0, 1)$, where φ is the generating function for the offspring distribution. We look for solutions to the equation

$$\begin{aligned} \varphi(s) - s &= 1/8 + 3/8s + 3/8s^2 + 1/8s^3 - s && (20) \\ &= 1/8 - 5/8s + 3/8s^2 + 1/8s^3 = 0. && (21) \end{aligned}$$

Plotting the zeroes of this polynomial should give us our answer as 0.236. I also implemented this in Julia to get a rough approximation of this probability with repeated sampling.
In the case that each family has only two children, we have $\xi \sim \text{Binom}(2,\, 1/2)$ with mean $\mathbb{E}[\xi] = \mu = 1$. This means that we have $p_0 = 1/4, p_1 = 1/2, p_2 = 1/4$. In this case, we can use theorem 4.3.11, so we see that extinction is inevitable, so $\rho = 1$. I also approximated this probability in Julia as well.



Simulating branching process for 3 children
Z_n
n
0
200
400
600
800
1000
1200
2
4
6
8
10
12
14



Estimating extinction probability with 3 children
mean($\hat{\rho}$)
$\hat{\rho} \approx 0.236$
ρ
0
50
100
150
0.200
0.225
0.250
0.275

Each family has 2 children