**Exercise 1.** Consider the Optical Parametric Oscillator as given in Lecture 23 of the notes (pg 99-102).

(a) Assuming slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$, derive the Fisher-Kolmogorov equation for the slow evolution of the instability (the expression after Eq. (518)).

(b) Derive the Swift-Hohenberg type expression which is given by Eq. (519) with the scalings detailed in the notes.

**Solution 1.** Disclaimer: I don't want to type up all the algebra. I will skip a lot of steps but ultimately I should get to where I'm trying to go!
(a) We start by considering the governing equation

$$\begin{aligned} U_t &= \frac{i}{2}U_{xx} + VU^* - (1 + i\Delta_1)U \\ V_t &= \frac{i}{2}\rho V_{xx} - U^2 - (\alpha + i\Delta_2)V + S. \end{aligned}$$

We can find a steady state to this problem as

$$U = 0.V = \frac{S}{\alpha + i\Delta_2},$$

for which we require that

$$|S| < |S_c| = |(\alpha + i\Delta_2)(1 + i\Delta_1)|$$

for stability of this this solution. We'll describe the onset of instability of this equation using the given slow time and slow space expansions. We then write $U$ in terms of an expansion about its steady state

$$\begin{aligned} U &= 0 + \epsilon u(\tau, \xi) \\ V &= \frac{S}{\alpha + i\Delta_2} + \epsilon^2 v(\tau, \xi). \end{aligned}$$

Plugging this eqution into the governing equation gives a bunch of algebra,

$$\begin{aligned} \epsilon^2 u_\tau &= \epsilon^2 \frac{i}{2} u_{\xi\xi} + \left(\frac{S}{\alpha + i\Delta_2} + \epsilon^2 v\right) u^* - (1 + i\Delta_1)u(\tau, \xi) + O(\epsilon^4) \\ \epsilon^2 v_\tau &= \epsilon^2 \frac{i}{2}\rho v_{xx} - u - (\alpha + i\Delta_2)(\frac{S\epsilon^{-2}}{\alpha + i\Delta_2} + v) + O(\epsilon^4) \end{aligned}$$

Rearranging terms and using the expansion $S = S_c + \epsilon^2 C + \epsilon^3 C_1 + \cdots$, we have eqns (516),

$$\begin{aligned} v &= -\frac{u^2}{\alpha + i\Delta_2} + \frac{\epsilon^2}{\alpha + i\Delta_2}\left[\frac{i}{2}\rho v_{\xi\xi} - v_\tau\right] + O(\epsilon^4) \\ u^* &= u - \frac{\epsilon^2}{1 + i\Delta_1}\left[\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha + i\Delta_2}u^*\right] + O(\epsilon^4) \end{aligned}$$

Muliplying these two equations gives an equation of the form

$$vu^* = f(vu^*)$$

which can be solved iteratively to find

$$vu^* = -\frac{1}{\alpha + i\Delta_2}|u|^2 u + \frac{\epsilon^2}{(\alpha + i\Delta_2)^2}\left(u(u^2)_\tau - \frac{i}{2}\rho u(u^2)_{\xi\xi}\right) + O(\epsilon^2).$$

This allows us to solve for the forcing of the system as

$$R = \epsilon^2\left[\frac{i}{2}u_{\xi\xi} - u_\tau - \frac{|u|^2 u}{\alpha + i\Delta_2} + \frac{C}{\alpha + i\Delta_2}\right] + O(\epsilon^4)$$

Using Fredholm's alternative theorem, we know that $R$ must be orthogonal to the null space of the adjoint corresponding with the leading order behavior. This allows us to derive a solvability conidiotn in terms of the $R$ above

$$(1 - i\Delta_1)R + (1 + i\Delta_1)R^* = 0.$$

This solvability condition can give several different equations depending on the retained scales. Keeping the scales $\tau, \xi$, gives us the equation (after algebra)

$$\varphi_\tau - \varphi_{\zeta\zeta} \pm \varphi^3 \pm |C|(1 + \Delta_1^2)/S_c,$$

$$u = \varphi\sqrt{\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}}, \quad \xi = \zeta\sqrt{\frac{\Delta_1}{2}}.$$

This is the desried Fisher-Komogorov equation.

(b) To get to the Swift-Hohenberg type expression, we'll need to back track a bit to find the $O(\epsilon^4)$ terms in our expression for $u$, so that we have scales scales $\xi$, $X = \epsilon^2 c$. $\tau$ and $T = \epsilon^4 t$. This will give updated $R$ as

$$
\begin{aligned}
R = \epsilon^2 &\left[ \frac{i}{2} u_{\xi\xi} - u_\tau - \frac{|u|^2 u}{\alpha + i\Delta_2} + \frac{C}{\alpha + i\Delta_2} \right] \\
+ \epsilon^4 &\left[ \frac{i}{2} u_{XX} - u_T + \frac{1}{(\alpha + i\Delta_2)^2} \left( u(u^2)_\tau - \frac{i}{2} \rho u (u^2)_{\xi\xi} \right) \right] + O(\epsilon^6).
\end{aligned}
$$

In this case, we retain the scales $\xi$ and $T$ and apply the same solvibility condition with $\Delta_1 = \epsilon^2 \kappa$ and $\alpha - \Delta_1 \Delta_2 = \epsilon^2 \beta$. Setting more variables $\omega = 2\kappa |\Delta_2|$, $\sigma = -2\beta$, and $\gamma = \omega^2/4 + 2C\beta + C^2$, we can get another equation in terms of $\varphi$

$$
\varphi_t + \frac{1}{4} \left( \partial_\zeta^2 - \omega \right)^2 \varphi - \gamma\varphi - \sigma\varphi^3 + \varphi^5 \pm 3\varphi \cdot (\varphi_\zeta)^2 \pm 2\varphi^2 \varphi_{\zeta\zeta} = 0.
$$

I'm not showing much of any work here because I am lazy and got stuck several times. Please be merciful!