Exercise 1. Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X and function f such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but not $\{\varnothing, \Omega\}$. Give a function g such that $\sigma(g(X)) = \{\varnothing, \Omega\}$.

Solution 1. Consider the space $\Omega = \{-1, 0, 1\}$ with σ-algebra $\mathcal{F} = 2^{\Omega}$ and the uniform probability measure $\mathbb{P}$ on Ω, and $X(\omega) = \omega$. The σ-algebra $\sigma(X) = \mathcal{F}$. We can define $f(x) = |x|$. $f(X)$ takes two values $0, 1$. Therefore, our possible pre-images of $f(X)$ for a Borel set B are $\varnothing$ if $0, 1 \notin B$, Ω if $0, 1 \notin B$, $\{-1, 1\}$ if $1 \in B, 0 \notin B$ and $\{0\}$ if $0 \in B, 1 \notin B$. This means that

$$\sigma(f(X)) = \sigma(\{\varnothing, \Omega, \{-1, 1\}, \{0\}\}). \tag{1}$$

This σ-algebra does not contain the event $\{0, 1\}$ and therefore it is strictly smaller than $\mathcal{F}$. For g, we choose the function $g(x) = 0$. There are only two possible preimages of $g(X)$ in Ω. One of which is the entire space Ω when $0 \in B$ and the other is $\varnothing$ when $0 \notin B$. Therefore, the σ-algebra $\sigma(g(X))$ must be $\{\varnothing, \Omega\}$.

Exercise 2. Give examples of events A, B, C with probability strictly between 0 and 1 so that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \tag{2}$$
$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C) \tag{3}$$
$$\mathbb{P}(B \cap C) \neq \mathbb{P}(B)\mathbb{P}(C) \tag{4}$$
$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \tag{5}$$

Are A, B, and C independent?

Solution 2. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{F} = 2^{\Omega}$, and $\mathbb{P}$ be the uniform measure on Ω. Consider the events

$$A = \{3, 4, 5, 6\}, \quad B = \{1, 2, 3, 4\}, \quad C = \{4, 5, 7, 8\}. \tag{6}$$

Since $\mathbb{P}$ is uniform, we have that $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{4}{8} = \frac{1}{2}$. We can compute that

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{3, 4\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(C) \tag{7}$$
$$\mathbb{P}(A \cap C) = \mathbb{P}(\{4, 5\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(C) \tag{8}$$
$$\mathbb{P}(B \cap C) = \mathbb{P}(\{4\}) = \frac{1}{8} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(B)\mathbb{P}(C) \tag{9}$$
$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{4\}) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C). \tag{10}$$
$$\tag{11}$$

From the above equations, we can see that the events B and C are not independent. Therefore, the collection of events $\{A, B, C\}$ is not independent as an independent collection of events must at least be pairwise independent.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that Ω is countably infinite, and $\mathcal{F} = 2^{\Omega}$. Show that it is impossible for there to exist a countable collection of events $A_1, A_2, \ldots \in \mathcal{F}$ which are independent such that $\mathbb{P}(A_i) = \frac{1}{2}$ for each i.

Solution 3. Let $\omega \in \Omega$ and $n \in \mathbb{N}$. For every event A_i, $i \in \mathbb{N}$ in our countable collection, then we have that either $\omega \in A_i$ or $\omega \in A_i^c$. For each i with $1 \leq i \leq n$, define

$$E_i = \begin{cases} A_i, & \omega \in A_i \\ A_i^c, & \omega \in A_i^c. \end{cases} \tag{12}$$

Notice that each E_i has probability $\mathbb{P}(E_i) = \frac{1}{2}$ since $\mathbb{P}(A_i) = \frac{1}{2} = 1 - \frac{1}{2} = \mathbb{P}(A_i^c)$. Further, the collection of events E_i for $1 \leq i \leq n$ is independent since any collection consisting of events A_i^c and A_j must be independent for $i \neq j$. Therefore, since $\omega \in E_i$ for every i, we have that

$$\mathbb{P}(\{\omega\}) \leq \mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \frac{1}{2^n} \tag{13}$$

for every $n \in \mathbb{N}$. Since this holds for arbitrary n, it follows that $\mathbb{P}(\{\omega\}) = 0$. Since the space Ω is countable and $\mathbb{P}$ is countably additive, we have that

$$\mathbb{P}(\Omega) = 1 = \mathbb{P}(\bigcup_{\omega \in \Omega} \{\omega\}) = \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}), \tag{14}$$

but this is a contradiction since

$$\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = \sum_{\omega \in \Omega} 0 = 0 \neq 1. \tag{15}$$

Exercise 4. (a) Let $X \geq 0$ and $Y \geq 0$ be independent random variables with distribution functions F and G. Find the distribution function of XY.
(b) If $X \geq 0$ and $Y \geq 0$ are independent continuous random variables with density functions f and g, find the density function of XY.
(c) If X and Y are independent exponentially distributed random variables with parameter λ, find the density function of XY.

Solution 4.
(a) Consider the function $h(x,y) = 1_{\{xy \leq z\}}$ for some $z \geq 0$. This is a measurable non-negative function, we then have that

$$\mathbb{E}[h(X,Y)] = \int_{[0,\infty)} \int_{[0,\infty)} h(x,y)\mu(dx)\nu(dy), \tag{16}$$

where μ is the distribution of X and ν is the distribution Y. Note that the integrals above are taken across the non-negative reals since we know that both X and Y are non-negative random variables. First using our choice of an indicator function to simplify the expectation above, we compute that

$$\mathbb{E}[h(X,Y)] = \mathbb{E}[1_{\{XY \leq z\}}] = \mathbb{P}(XY \leq z). \tag{17}$$

Secondly, we focus on simplifying using our choice of distributions, so that

$$\begin{aligned}
\int_{[0,\infty)} h(x,y)\mu(dx) &= \int_{[0,\infty)} h(x,y)\mu(dx) && (18)\\
&= \int_{[0,\infty)} 1_{\{xy \leq z\}}\mu(dx) && (19)\\
&= \int_{[0,\infty)} 1_{\{x \leq \frac{z}{y}\}}\mu(dx) && (20)\\
&= F\left(\frac{z}{y}\right). && (21)
\end{aligned}$$

Therefore, we can see that

$$\mathbb{P}(XY \leq z) = \int_{[0,\infty)} F\left(\frac{z}{y}\right)\nu(dy) = \int_{[0,\infty)} F\left(\frac{z}{y}\right) dG(y). \tag{22}$$

(b) Starting from the previous part, we can simplify in terms of our density functions, so that

$$\mathbb{P}(XY \leq z) = \int_{[0,\infty)} F\left(\frac{z}{y}\right) dG(y) \tag{23}$$

$$= \int_{[0,\infty)} \left(\int_0^{\frac{z}{y}} f(x)dx\right) dG(y) \tag{24}$$

$$= \int_0^{\infty} \left(\int_0^{\frac{z}{y}} f(x)dx\right) g(y)dy \tag{25}$$

$$= \int_0^{\frac{z}{y}} \int_0^{\infty} f(x)g(y)dy\ dx \tag{26}$$

where we have substituted the distribution functions for their respective densities and used Fubini's theorem in the last step. We can use compute the density of XY by taking the derivative of this with respect to z. Using the first fundamental theorem of calculus, we see that

$$f_{XY}(z) = \int_0^{\infty} \frac{1}{y} f(z/y)g(y)dy. \tag{27}$$

(c) In the case both X and Y are exponentially distributed with parameter λ, we have that $f(x) = g(x) = \lambda e^{-\lambda x}$. Plugging this into the above equation, we have

$$f_{XY}(z) = \int_0^{\infty} \frac{1}{y} \left(\lambda e^{-\lambda \frac{z}{y}}\right) \left(\lambda e^{-\lambda y}\right) dy \tag{28}$$

$$= \lambda^2 \int_0^{\infty} e^{-\lambda(y+\frac{z}{y})} \frac{1}{y} dy. \tag{29}$$