Exercise 1. Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz, \tag{1}$$

where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems by both

(i) enclosing the singular points in side C

(ii) enclosing the singular points outside C (by including the point at infinity)

Show that you obtain the same result in both cases.

(a) $\frac{z^2+1}{z^2-a^2}, \quad a^2 < 1.$

(b) $\frac{z^2+1}{z^3}.$

(c) $z^2 e^{-1/z}.$

Hint: the point at infinity is defined as $t = 1/z$ as $z \to 0$.

Solution 1.
(a) We'll begin by computing the integral of $f(z) = \frac{z^2+1}{z^2-a^2} = \frac{(z-i)(z-i)}{(z-a)(z+a)}$. Notice that both $\pm a$ are in the unit circle C since a^2 is less than 1 and are the only singularities in C. Therefore, the integral

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(f; a) + \text{Res}(f; -a). \tag{2}$$

Since both of these residues are taken at simple poles, we can compute them easily as

$$\text{Res}(f; a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{z^2+1}{z+a} = \frac{a^2+1}{2a} \tag{3}$$

$$\text{Res}(f; -a) = \lim_{z\to -a}(z+a)f(z) = \lim_{z\to -a}\frac{z^2+1}{z-a} = -\frac{a^2+1}{2a}. \tag{4}$$

We then conclude that

$$\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^2-a^2}dz = 0. \tag{5}$$

Since the function $f(z)$ has no singularities outside the unit circle (other than at ∞), we can alternatively compute that

$$\frac{1}{2\pi i}\oint_C f(z)dz = -\text{Res}(f; \infty). \tag{6}$$

Using that the point at infinity is defined by $t = 1/z$ as $z \to 0$, we have that

$$\text{Res}(f; \infty) = -\text{Res}(f(z^{-1})z^{-2}; 0) \tag{7}$$

as shown in class. Therefore, we compute the residue of

$$f(z^{-1})z^{-2} = \frac{z^{-2}+1}{z^{-2}+a^2} \cdot z^{-2} = \frac{z^{-2}}{1+a^2z^2} + \frac{1}{1+a^2z^2}. \tag{8}$$

This is the sum of a function which is analytic at 0 ($\frac{1}{1+a^2z^2}$) and another which has a double pole at 0. Since the coefficients of the Taylor-Laurent series are additive, we can ignore the contribution of the analytic function, so that

$$\begin{aligned}
\operatorname{Res}(f(z^{-1})z^{-2}; 0) = \operatorname{Res}\left(\frac{z^{-2}}{1+a^2z^2}; 0\right) &= \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{z^{-2}}{1+a^2z^2}\right) && (9)\\
&= \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{1+a^2z^2}\right) && (10)\\
&= \lim_{z\to 0} \frac{-2a^2z}{1+a^2z^2} = 0. && (11)
\end{aligned}$$

Once again, this allows us to conclude that

$$\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^2-a^2}dz = 0. \tag{12}$$

(b) We'll now repeat this procedure but with the function $f(z) = \frac{z^2+1}{z^3} = z^{-3} + z^{-1}$. The only singularities of this function in C occur at the point 0 and since this is already written in form of its Taylor-Laurent series about 0, we have that

$$\frac{1}{2\pi i} \oint_C \frac{z^2+1}{z^3} = \text{Res}\left(\frac{z^2+1}{z^3}; 0\right) = a_{-1} = 1. \tag{13}$$

We can now compute this same integral using the singularity at 0 by noting that

$$\frac{1}{2\pi i} \oint_C \frac{z^2+1}{z^3} dz = \text{Res}(f(z^{-1})z^{-2}; 0). \tag{14}$$

In this case,

$$f(z^{-1})z^{-2} = (z^3 + z^1)z^{-2} = z + z^{-1}. \tag{15}$$

Computing the residue of this function at 0 is simple since it is in terms of its Taylor-Laurent series, we see that

$$\frac{1}{2\pi i} \oint_C \frac{z^2+1}{z^3} dz = \text{Res}(f(z^{-1})z^{-2}; 0) = 1. \tag{16}$$

(c) We now switch to the function

$$f(z) = z^2 e^{-1/z} = z^2 \left(1 - z^{-1} + \frac{z^{-2}}{2!} - \frac{z^{-3}}{3!} + \cdots \right) = z^2 - z + \frac{1}{2} - \frac{z^{-1}}{3!} + \cdots , \tag{17}$$

where we have expanded $e^{-1/z}$ in terms of its Taylor-Laurent series about 0. Using the fact the only singularity of $f(z)$ in C is 0, we can compute that

$$\frac{1}{2\pi i} \oint_C z^2 e^{-1/z} dz = \text{Res}(z^2 e^{-1/z}; 0) = -\frac{1}{3!} = -1/6. \tag{18}$$

We can now solve this problem by instead using the singularity at ∞. We see that

$$f(1/z)1/z^2 = (z^{-2}e^{-z}) \cdot z^{-2} = z^{-4}e^{-z} \tag{19}$$

$$= z^{-4} \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots \right) \tag{20}$$

$$= z^{-4} - z^{-3} + \frac{z^{-2}}{2!} - \frac{z^{-1}}{3!} + \cdots , \tag{21}$$

where we have expanded e^{-z} in terms of its Taylor-Laurent series about 0. With this expansion, we can clearly see $\text{Res}(f(z)z^{-2}; 0) = -1/6$ and conclude

$$\frac{1}{2\pi i} \oint_C z^2 e^{-1/z} dz = \text{Res}(f(z^{-1})z^{-2}; 0) = -1/6. \tag{22}$$

Exercise 2. Find the Fourier transform of

$$f(t) = \begin{cases} 1, & t \in (-a, a) \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Then, do the inverse transform using techniques of contour integration, e.g. Jordan's lemma, principal values, etc.

Solution 2. We begin by doing the Fourier transform of f, so that

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} f(t) dt = \int_{-a}^{a} e^{-i\lambda t} dt, \tag{24}$$

since the integrand is 0 for t such that $|t| \geq a$ and just $e^{i\lambda t}$ for $|t| < a$. We can now evaluate this integral using the anti-derivative of an exponential, so that

$$F(\lambda) = \int_{-a}^{a} e^{-i\lambda t} dt \tag{25}$$

$$= -\frac{e^{-i\lambda t}}{i\lambda}|_{-a}^{a} \tag{26}$$

$$= -\frac{1}{i\lambda}\left(e^{-i\lambda a} - e^{i\lambda a}\right) \tag{27}$$

$$= \frac{2}{\lambda}\sin(\lambda a). \tag{28}$$

We'll now take the inverse Fourier transform of this function as

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda t} F(\lambda) d\lambda. \tag{29}$$

We'll start by writing this integral in terms of exponentials

$$\hat{f}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i\lambda t}\frac{\left(e^{i\lambda a} - e^{-i\lambda a}\right)}{\lambda} d\lambda \tag{30}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda(t+a)} - e^{i\lambda(t-a)}}{\lambda} d\lambda \tag{31}$$

$$= \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} \frac{e^{iz(t+a)}}{z} dz - \int_{-\infty}^{\infty} \frac{e^{iz(t-a)}}{z} dz\right) \tag{32}$$

$$\tag{33}$$

Case 1: $t \in (-a, a)$ We'll begin by integrating the first integral. Since $t > -a$, we have that the first integral disappears on C_R the upper semicircle centered at 0 with radius $R \to \infty$. That is,

$$\int_{\Gamma} \frac{e^{iz(t+a)}}{z} dz = \left(\int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^{R} + \int_{C_R}\right)\frac{e^{iz(t+a)}}{z} dz, \tag{34}$$

where C_ϵ is the upper half circle centered at 0 with radius ϵ and oriented clockwise and Γ the contour consisting of the contours on the right hand side of the integral. By Cauchy's theorem, we have that $\int_\Gamma = 0$. We can compute the integral over C_ϵ using

$$\int_{C_\epsilon} \frac{e^{iz(t+a)}}{z} dz = -\pi i \text{Res}\left(\frac{e^{iz(t+a)}}{z}, 0\right) = -\pi i \text{ as } \epsilon \to 0. \tag{35}$$

We can compute this residue by simply using the simple pole formula from Prof. Tung's notes. Therefore, in the limit, we have

$$\int_{-\infty}^{\infty} \frac{e^{iz(t+a)}}{z} dz = \pi i \tag{36}$$

For the second integral, we cannot use Jordan's lemma since $t - a < 0$. We can instead do the same kind of integration after flipping the contour we use to integrate about the real axis. Therefore, using the mirrored contours, we have that

$$0 = \int_\Gamma \frac{e^{iz(t-a)}}{z} dz = \left(\int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_\epsilon^R + \int_{C_R}\right) \frac{e^{iz(t-a)}}{z} dz, \tag{37}$$

where once again the integral over C_R disappears since we now use the lower half semi-circle. This leaves us to compute the integral as

$$\int_{-\infty}^{\infty} \frac{e^{iz(t-a)}}{z} dz = -\pi i, \tag{38}$$

where we have used the same methods for calculating the residue. The only difference is the sign of the integral over C_ϵ which is reversed due to the reversal of orientation of C_ϵ we get from reflecting about the real axis. That is, C_ϵ has become the lower half circle about 0 with radius ϵ, but is now oriented counterclockwise, which gives us the flipped sign of our integral. The other integrals are left unchanged by this reflection. Therefore, we can compute that for $t \in (-a, a)$

$$\hat{f}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i\lambda t} \frac{(e^{i\lambda a} - e^{-i\lambda a})}{\lambda} d\lambda = \frac{1}{2\pi i}(\pi i - (-\pi i)) = 1. \tag{39}$$

Case 2: $|t| > a$ In this case, there are two sub-cases $t < -a < a$ and $-a < a < t$. In either cases, we can repeat the analysis done above but with both integrals having the same contour Γ. Due to both integrands having the same residue at 0, this then means that

$$I = \int_{-\infty}^{\infty} \frac{e^{iz(t-a)}}{z} dz = \int_{-\infty}^{\infty} \frac{e^{iz(t-a)}}{z} dz. \tag{40}$$

Therefore, for t with $|t| > a$,

$$\hat{f}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i\lambda t} \frac{(e^{i\lambda a} - e^{-i\lambda a})}{\lambda} d\lambda = \frac{1}{2\pi i}(I - I) = 0. \tag{41}$$

Case 3: $t = \pm a$ In the case that $t = a$, we have to compute the integral

$$\hat{f}(a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i2az} - 1}{z} dz \tag{42}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{aiz} \frac{e^{iaz} - e^{-iaz}}{2iz} dz \tag{43}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{aiz} \frac{\sin(ax)}{x} \tag{44}$$
$$= \frac{1}{\pi} \left(\int_{-\infty}^{\infty} \frac{\cos(ax)\sin(ax)}{x} dx + i \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x} dx \right) \tag{45}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(ax)\sin(ax)}{x} dx \tag{46}$$

The function $\frac{\sin^2(ax)}{x} dx$ is odd, so its integral goes to 0. Translating back to complex exponentials, we see that $\cos(ax)\sin(ax) = (1/4i)(e^{i2ax} - e^{-i2ax}) = \sin(2ax)/2$. Therefore, we have that

$$\hat{f}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin 2ax}{x} dx = 1/2, \tag{47}$$

where we have used that $\int_{-\infty}^{\infty} \sin tx/x dx = \pi$ for $t > 0$ as shown in Prof. Tung's book. Now we let $t = -a$, we solve the integral

$$\hat{f}(-a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 - e^{-i2az}}{z} dz. \tag{48}$$

We can follow essentially the same arithmetic to see that

$$\hat{f}(-a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iaz} \frac{e^{iaz} - e^{-iaz}}{z} dz \tag{49}$$
$$= \frac{1}{\pi} \left(\int_{-\infty}^{\infty} \frac{\cos(ax)\sin(ax)}{x} dx - i \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x} dx \right) \tag{50}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(ax)\sin(ax)}{x} dx. \tag{51}$$

Therefore, $\hat{f}(a) = \hat{f}(-a) = 1/2$. We can see that this is the same as our original function $f(t)$ for all $t \neq \pm a$.