Exercise 1. (a) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Determine its eigenvalues and eigenvectors and the algebraic and geometric multiplicity of each.
(b) Consider the matrix

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Determine its eigenvalues and eigenvectors and the algebraic and geometric multiplicity of each.

Solution 1.
(a) We can compute that the characteristic polynomial is given by $p(\lambda) = (2-\lambda)^3$ since $\mathbf{A}-\lambda\mathbf{I}$ is a diagonal matrix and its determinant is the product of its diagonal entries. Therefore, its eigenvalues are simply $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and the algebraic multiplicity of this eigenvalue is 3. We can see that the corresponding eigenvectors are given by the euclidean basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ since the matrix $\mathbf{A} = 2\mathbf{I}$. Using the fact that $\mathbf{A} - 2\mathbf{I} = 0$, we see that the nullspace has dimension 3, so the eigenvalue $\lambda = 2$ has geometric multiplicity 3.
(b) Similarly, we can compute the characteristic polynomial as $p(\lambda) = (2-\lambda)^3$ since $\mathbf{B} - \lambda\mathbf{I}$ is an upper triangular matrix and its determinant is the product of its diagonal entries. Therefore, its eigenvalues are simply $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and the algebraic multiplicity is 3. We can compute the eigenvector as $\mathbf{v}_1 = \mathbf{e}_1$. Further, we can compute the geometric multiplicity by looking at

$$\mathbf{B} - 2\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can see that this matrix has rank 2, so the geometric multiplicity of the eigenvalue $\lambda = 2$ is $3 - 2 = 1$.

Exercise 2. For each of the following statements, prove that it is true or give a counter example to show it is false. Here each matrix $A \in \mathbb{C}^{m\times m}$ unless otherwise indicated.

(a) If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.

(b) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is $-\lambda$.

(c) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is $\overline{\lambda}$.

(d) If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is nonsingular, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.

(e) If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.

(f) If $\mathbf{A}$ is Hermitian and λ is an eigenvalue of $\mathbf{A}$, then $|\lambda|$ is a singular value of $\mathbf{A}$.

(g) If $\mathbf{A}$ is diagonalizable and all its eigenvalues are equal, then $\mathbf{A}$ is diagonal.

Solution 2.
(a) Suppose that $\mathbf{v}$ is the eigenvector corresponding to λ, then we have that

$$\begin{aligned}(\mathbf{A} - \mu\mathbf{I})\mathbf{v} &= \mathbf{A}\mathbf{v} - \mu\mathbf{v} \\ &= \lambda\mathbf{v} - \mu\mathbf{v} \\ &= (\lambda - \mu)\mathbf{v}.\end{aligned}$$

Therefore, $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.
(b) This is false. Consider the matrix $\mathbf{A} = 2\mathbf{I}$. This matrix's eigenvalue is 2 as can be seen by its diagonal entries, but -2 is not an eigenvalue.
(c) Suppose that $\mathbf{v}$ is the eigenvector corresponding to λ, then we have that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Taking the complex conjugate of this equation, we see that

$$\overline{\mathbf{A}\mathbf{v}} = \mathbf{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}},$$

where we have used that $\mathbf{A}$ is real. This shows that $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$.
(d) Suppose that $\mathbf{v}$ is the eigenvector corresponding to λ, then we have that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Multiplying by $\mathbf{A}^{-1}$, we see that

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{v} = \lambda\mathbf{A}^{-1}\mathbf{v} \implies \mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$

Notice, we assume that the eigenvalue $\lambda \neq 0$. This is because the matrix $\mathbf{A}$ is non-singular and has non-zero determinant and therefore, cannot have 0 eigenvalues.
(e) This is not true. Consider the matrix

$$\mathbf{W} = \begin{pmatrix} 0 & 27 \\ 0 & 0 \end{pmatrix}.$$

This matrix is upper triangular and therefore, its eigenvalues are its diagonals which are 0.
(f) We know that the non-zero singular values are given by the square roots of the eigenvalues of $\mathbf{A}^*\mathbf{A} = \mathbf{A}^2$ where we've used that $\mathbf{A}$ is Hermitian. Therefore, the singular values are

simply the square roots of the eigenvalues of $\mathbf{A}^2$ as shown in HW 2. Using the fact that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for eigenvalue λ and corresponding eigenvector $\mathbf{v}$, we have that

$$\mathbf{A}^2\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}, \tag{1}$$

so λ^2 is an eigenvalue of $\mathbf{A}^2$. We see then that the singular values of $\mathbf{A}$ are just $\sigma = \sqrt{\lambda^2} = |\lambda|$.
(g) Suppose that the matrix is diagonalizable so that $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$. In the case that the all the eigenvalues are the same, we have that $\mathbf{\Lambda} = \lambda\mathbf{I}$. Therefore, we have that

$$\mathbf{A} = \lambda\mathbf{P}\mathbf{I}\mathbf{P}^{-1} = \lambda\mathbf{I},$$

so that $\mathbf{A}$ is diagonal.

Exercise 3. Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ be tridiagonal and Hermitian, with all its sub and super diagonal entries non-zero. Prove that the eigenvalues of $\mathbf{A}$ are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $\mathbf{A} - \lambda\mathbf{I}$ has rank at least $m - 1$)

Solution 3. We begin by writing the matrix $\mathbf{A}$ in terms of a block diagonal matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{a} & 0 \\ \mathbf{C} & \mathbf{d}, \end{pmatrix}$$

where $\mathbf{a}, \mathbf{d}$ are vectors in $\mathbb{C}^{m-1}$ and $\mathbf{C}$ is an upper triangular matrix of size $(m-1) \times (m-1)$. Notice that the matrix $\mathbf{C}$ must be upper triangular and its diagonal entries are all non-zero under our assumption that all sub-diagonal entries of $\mathbf{A}$ must be non-zero. It follows then that the matrix $\mathbf{C}$ is non-singular. From this, we know that $\operatorname{rank}(\mathbf{A} - \lambda\mathbf{I}) \geq m - 1$ for all $\lambda \in \mathbb{C}$. In the case that λ is an eigenvalue of $\mathbf{A}$, this means that the geometric multiplicity of λ must be exactly one. Since $\mathbf{A}$ is Hermitian, we know that it is diagonalizable, and therefore, non-defective i.e. the geometric multiplicity of its eigenvalues must be equal to their algebraic multiplicity. Since these multiplicities are 1, each of the eigenvalues of $\mathbf{A}$ must be distinct.