Exercise 1. (a) Let $\hat{f}(s)$ and $\hat{g}(s)$ be the Laplace transforms of one-sided function $f(t)$ and $g(t)$ respectively. Show that the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ is

$$\int_0^t f(t-\tau)g(\tau)d\tau.$$

(b) Use Laplace transform and the result in (a) to solve the following ordinary differential equation

$$\frac{d^2}{dt^2}y + 4y = f(t),$$

subject to the initial conditions: $y(0) = 0$, $\frac{dy}{dt}(0) = y'(0) = 0$.

Solution 1. (a) We write the definition of the inverse Laplace transform as

$$\mathcal{L}^{-1}([\hat{f}(s)\hat{g}(s)] = \frac{1}{2\pi i}\int_L e^{st}\hat{f}(s)\hat{g}(s)ds,$$

where L is chosen to be to the right of any singularities of $\hat{f}$ and $\hat{g}$ as discussed in class. Writing the definition of $\hat{g}$ we see that

$$\begin{aligned}\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)] &= \frac{1}{2\pi i}\int_L e^{st}\hat{f}(s)\hat{g}(s)ds \\ &= \frac{1}{2\pi i}\int_L e^{st}\hat{f}(s)\left(\int_0^\infty e^{-s\tau}g(\tau)d\tau\right)ds.\end{aligned}$$

Re-arranging the integrals and using the definition of the inverse Laplace transform, we have that

$$\begin{aligned}\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)] &= \frac{1}{2\pi i}\int_0^\infty g(\tau)\int_L e^{s(t-\tau)}\hat{f}(s)dsd\tau \\ &= \frac{1}{2\pi i}\int_0^\infty g(\tau)\int_L e^{s(t-\tau)}\hat{f}(s)dsd\tau \\ &= \int_0^\infty g(\tau)\mathcal{L}^{-1}[\hat{f}(s)](t-\tau) \\ &= \int_0^\infty g(\tau)f(t-\tau)d\tau\end{aligned}$$

Since the f and g are one sided, we have that $f(t-\tau) = 0$ when $\tau > t$ and $g(\tau) = 0$ for $\tau < 0$. Therefore,

$$\mathcal{L}^{-1}[\hat{f}(s)\hat{g}(s)] = \int_0^t g(\tau)f(t-\tau)d\tau$$

(b) We begin by taking the Laplace transform of both sides of the equation

$$\mathcal{L}[y'' + 4y] = \mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[f(t)].$$

Using the transform formulas for derivatives and initial conditions, we have that

$$\mathcal{L}[y''] = s^2\hat{y} - sy(0) - y'(0) = s^2\hat{y}.$$

Therefore, we have that

$$(s^2+4)\hat{y} = \hat{f}(s) \implies \hat{y} = \frac{\hat{f}(s)}{s^2+4}.$$

This means our desired solution $y(t)$ is given by

$$y(t) = \mathcal{L}^{-1}\left[\hat{f}(s)\cdot\frac{1}{s^2+4}\right]$$

We can solve this using the formula derived in part (a). Setting $\hat{g}(s) = \frac{1}{s^2+4}$, we solve first for this function's inverse transform. Looking at pg. 90 of Prof. Tung's book, we can see that

$$\mathcal{L}\left[\frac{\sin(2t)}{2}\right] = \frac{1}{s^2+4} \implies g(t) = \frac{\sin(2t)}{2}.$$

Using the formula derived in (a), we have that

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\hat{f}(s)\cdot\frac{1}{s^2+4}\right] \\ &= \frac{1}{2}\int_0^t f(t-\tau)\sin(2\tau)d\tau. \end{aligned}$$

Exercise 2. Solve the following Laplace equation

$$\frac{\partial^2}{\partial x^2}\varphi + \frac{\partial^2}{\partial y^2}\varphi = 0$$

in the upper half plane subject to the boundary conditions: $\varphi \to 0$ as $y \to \infty$ and $\varphi \to 0$ as $x \to \pm\infty$ and

$$\varphi(x, 0) = \frac{x}{x^2 + a^2}.$$

Solution 2. Assuming a solution $\varphi(x, y)$ to this equation exists, we take its Fourier transform in x, so that our differential equation satisfies

$$-\mathcal{F}\left[\frac{\partial^2}{\partial x^2}\varphi\right] = \mathcal{F}\left[\frac{\partial^2}{\partial y^2}\varphi\right]$$

By the differentiation rules of Fourier series, we have that

$$\mathcal{F}\left[\frac{\partial^2}{\partial x^2}\varphi\right] = -\lambda^2\mathcal{F}[\varphi].$$

Therefore, our transformed differential equation satisfies

$$\lambda^2\mathcal{F}[\varphi] = \mathcal{F}\left[\frac{\partial^2}{\partial y^2}\varphi\right] = \frac{\partial^2}{\partial y^2}\mathcal{F}\left[\varphi\right],$$

where we've interchanged the y derivative and our integration in the Fourier transform. For simplicity, we'll set $\mathcal{F}[\varphi] = \hat{\varphi}$, so that we see

$$\lambda^2\hat{\varphi} = \frac{\partial^2}{\partial y^2}\hat{\varphi} \implies \hat{\varphi}(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}$$

for functions A and B which depend on λ. Since we require that $\varphi \to 0$ as $y \to \infty$ and φ is bounded for $y > 0$, this means that $A = 0$ for $\lambda > 0$ and $B = 0$ for $\lambda < 0$, so we write

$$\hat{\varphi}(\lambda, y) = C(\lambda)e^{-|\lambda|y},$$

for some function $C(\lambda)$. Using that $\varphi(x, 0) = \frac{x}{x^2+a^2} = f(x)$, we have that

$$\hat{\varphi}(\lambda, 0) = \hat{f}(\lambda) = C(\lambda)e^{-|\lambda|\cdot 0}.$$

This shows that our transformed solution satisfies $C(\lambda) = \hat{f}(\lambda)$, so that

$$\hat{\varphi}(\lambda, y) = \hat{f}(\lambda)e^{-|\lambda|y}.$$

To return our solution to the desired (x, y) coordinates, we'll use the convolution theorem for Fourier transforms. As shown on page 286 of A& F, we have that the inverse transform of $e^{-|\lambda|y}$ is given by $g(x, y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$. Therefore, our solution is given by the convolution theorem for Fourier transforms as

$$\begin{aligned}
\varphi(x,y) &= \int_{-\infty}^{\infty} f(t)g(x-t,y)dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\frac{y}{(x-t)^2+y^2}dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+a^2}\cdot\frac{y}{(x-t)^2+y^2}dt \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{t}{(t^2+a^2)((x-t)^2+y^2)}dt.
\end{aligned}$$

We can solve this integral with contour integration. Let Γ be the contour for the upper half circle, and for $h(t) = \frac{t}{(t^2+a^2)((x-t)^2+y^2)}$ we have that

$$\int_\Gamma h(t)dt = \int_{C_R} h(t)dt + \int_{-R}^{R} h(t)dt.$$

As $h(t)$ is a rational function with denominator degree at least 2 greater than the numerator degree, we have that the integral over C_R goes to 0 in the limit as $R \to \infty$. This means that

$$\int_\Gamma h(t)dt = \int_{-\infty}^{\infty} h(t) \text{ as } R \to \infty.$$

We can solve for this integral using the residue theorem. In the upper half plane where Γ is defined, $h(t)$ has singularities at ia and $iy + x$ in the upper half plane as $a > 0$ and $y > 0$. Therefore, this shows that

$$\int_{-\infty}^{\infty} h(t)dt = 2\pi i(\mathrm{Res}(h; ia) + \mathrm{Res}(h; iy + x)).$$

Since each of the singularities are simple poles, we can write a formula for the Residues using the derivative of the denominator, so that

$$\mathrm{Res}(h; t_0) = \frac{t}{2t((x-t)^2+y^2) - 2(t^2+a^2)(x-t)}.$$

Plugging in the singularity $t_0 = ia$, we have that

$$\begin{aligned}
\mathrm{Res}(h; ia) &= \frac{ia}{2ia((x-ia)^2+y^2) - 2((ia)^2+a^2)(x-ia)} \\
&= \frac{ia}{2ia((x-ia)^2+y^2)} \\
&= \frac{1}{2((x-ia)^2+y^2)}.
\end{aligned}$$

Next, for $t_0 = iy + x$, we have that

$$\begin{aligned}
\operatorname{Res}(h; iy+x) &= \frac{iy+x}{2(iy+x)([x-(iy+x)]^2+y^2) - 2((iy+x)^2+a^2)(x-(iy+x))} \\
&= -\frac{iy+x}{2((iy+x)^2+a^2)(-iy)} \\
&= \frac{x}{2iy((iy+x)^2+a^2)} + \frac{1}{2((iy+x)^2+a^2)}
\end{aligned}$$

We have that

$$\begin{aligned}
\int_{-\infty}^{\infty} h(t)dt &= 2\pi i\left(\frac{1}{2((x-ia)^2+y^2)} + \frac{x}{2iy((iy+x)^2+a^2)} + \frac{1}{2((iy+x)^2+a^2)}\right) \\
&= \pi\left(\frac{i}{(x-ia)^2+y^2} + \frac{x}{y((iy+x)^2+a^2)} + \frac{i}{(iy+x)^2+a^2}\right)
\end{aligned}$$

Plugging this into the integral solution for φ, we have

$$\varphi(x,y) = \frac{yi}{(x-ia)^2+y^2} + \frac{x}{(iy+x)^2+a^2} + \frac{yi}{(iy+x)^2+a^2}.$$

As a simple check, we can see that this solution satisfies our initial condition when $y = 0$.

Happy Thanksgiving! Thank you for your thorough grading.