Exercise 1. Consider the singular equation:

$$\epsilon \frac{d^2u}{dx^2} + (1+x)^2 \frac{du}{dx} + u = 0$$

with $u(0) = u(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) Obtain a uniform approximation which is valid to $O(\epsilon)$ i.e. determine the leading order behavior and first correction.

(b) Show that assuming the boundary layer to be at $x = 1$ is inconsistent. (Hint: Use the stretched inner varible $\xi = (1-x)/\epsilon$).

(c) Plot the uniform solution for $\epsilon = 0.01$, 0.05, 0.1, 0.2.

Solution 1. (a) (Assuming this has a boundary layer at 0) We attempt the perturbation expansion which gives us terms

$$\begin{aligned} O(1): &\quad (1+x)^2 u_{0x} + u_0 = 0 \\ O(\epsilon): &\quad (1+x)^2 u_{1x} + u_1 = -u_{0xx}, \end{aligned}$$

for the outer problem this system only satisfies the right boundary condition i.e. $u(1) = 1$. This gives us solution for the leading order as

$$u_0(x) = e^{\frac{1}{x+1} - \frac{1}{2}}.$$

Computing the second derivative of this, we see that our $O(\epsilon)$ equation becomes

$$\begin{aligned} (1+x)^2 u_{1x} + u_1 &= -\frac{2x+3}{(x+1)^4} e^{\frac{1}{x+1} - \frac{1}{2}} \\ u_1(1) &= 0. \end{aligned}$$

This has solution

$$\begin{aligned} u_1(x) &= ce^{\frac{1}{x+1}} + \left(\frac{x}{2}\right)\left(\frac{e^{\frac{1}{x+1} - \frac{1}{2}}}{(x+1)^5}\right) + \left(\frac{7}{10}\right)\left(\frac{e^{\frac{1}{x+1} - \frac{1}{2}}}{(x+1)^5}\right) \\ c &= -\frac{1}{2^6}\left(\frac{12}{5}\right) e^{-\frac{1}{2}}. \end{aligned}$$

We'll now tackle the inner problem using the coordinate transformation $\xi = \frac{x}{\epsilon}$. In these coordinates, our equation becomes

$$\begin{aligned} u_{\xi\xi} + (1+\epsilon\xi)^2 u_\xi + \epsilon u = 0, \\ u_{\xi\xi} + u_\xi + 2\epsilon\xi u_\xi + \epsilon^2\xi^2 u_\xi + \epsilon u = 0 \end{aligned}$$

with bondary condition $u(\epsilon\xi = 0) = 1$. Now doing a pertubation expansion, we see that

$$\begin{aligned} O(1): &\quad u_{0\xi\xi} + u_{0\xi} = 0 \\ O(\epsilon): &\quad u_{1\xi\xi} + u_{1\xi} = -u_0 - 2\xi u_{0\xi} \end{aligned}$$

with boundary conditions $u_0(\xi = 0) = 1$ and $u_1(\xi = 0) = 0$. The leading order solution in this case is then given by

$$u_0(\xi) = A\exp(-\xi) + (1 - A).$$

We can additionally simplify and now solve the differential equation for $O(\epsilon)$, so that

$$u_{1\xi\xi} + u_{1\xi} = -A\exp(-\xi) - (1 - A) + 2A\xi\exp(-\xi).$$

This has solution

$$u_1(\xi) = -A\xi^2 e^{-\xi} - A\xi e^{-\xi} + A\xi - \xi.$$

We can now match these equations, so that

$$\begin{aligned}\lim_{x\to 0} u_{\text{out}}(x) &= \lim_{x\to 0}\exp\left(\frac{1}{x+1} - \frac{1}{2}\right) = e^{\frac{1}{2}}\\ \lim_{\xi\to\infty} u_{\text{in}}(\xi) &= \lim_{\xi\to\infty} A\exp(-\xi) + (1 - A) = 1 - A\end{aligned}$$

We then see

$$A = 1 - e^{\frac{1}{2}}.$$

We can then write our uniform solution as

$$\begin{aligned}u_{\text{unif}} &= u_{\text{out}} + u_{\text{in}} - e^{\frac{1}{2}}\\ &= \exp\left(\frac{1}{x+1} - \frac{1}{2}\right) + A\exp\left(-\frac{x}{\epsilon}\right) + 1 - A - e^{\frac{1}{2}}.\end{aligned}$$

(b) Assuming the boundary layer to instead be at $x = 1$, we first work on the outer problem (near $x = 0$) using equations

$$\begin{aligned}O(1):&\quad (1+x)^2 u_{0x} + u_0 = 0\\ O(\epsilon):&\quad (1+x)^2 u_{1x} + u_1 = -u_{0xx},\end{aligned}$$

with boundary condition $u_0(0) = 1$ and $u_1(0) = 0$. This gives leading order solution

$$u_0(x) = e^{\frac{1}{x+1} - 1}.$$

Next, for the inner problem (near $x = 1$), we again do a change of variables $\xi = (1 - x)/\epsilon$, so that the equation becomes

$$\begin{aligned}\frac{\epsilon}{\epsilon^2}u_{\xi\xi} + \frac{1}{\epsilon}(2 - \epsilon\xi)^2 u_\xi + u &= 0\\ u_{\xi\xi} + 4u_\xi - 4\epsilon\xi u_\xi + \epsilon^2\xi^2 u_\xi + \epsilon u &= 0\\ u(1) &= 1,\end{aligned}$$

This gives leading order equation

$$\begin{aligned} u_{0\xi\xi} + 4u_{0\xi} &= 0 \\ u_0(1) &= 1. \end{aligned}$$

This has solution

$$u_0(\xi) = Ae^{-4\xi} + (1 - A).$$

To form a uniform solution, we require that

$$e^{-\frac{1}{2}} = \lim_{x \to 1} u_{\text{out}}(x) = \lim_{\xi \to -\infty} u_{\text{inner}}(\xi) = -\infty.$$

This is inconsistent as the right limit is infinite.
(c) Plots attached below.

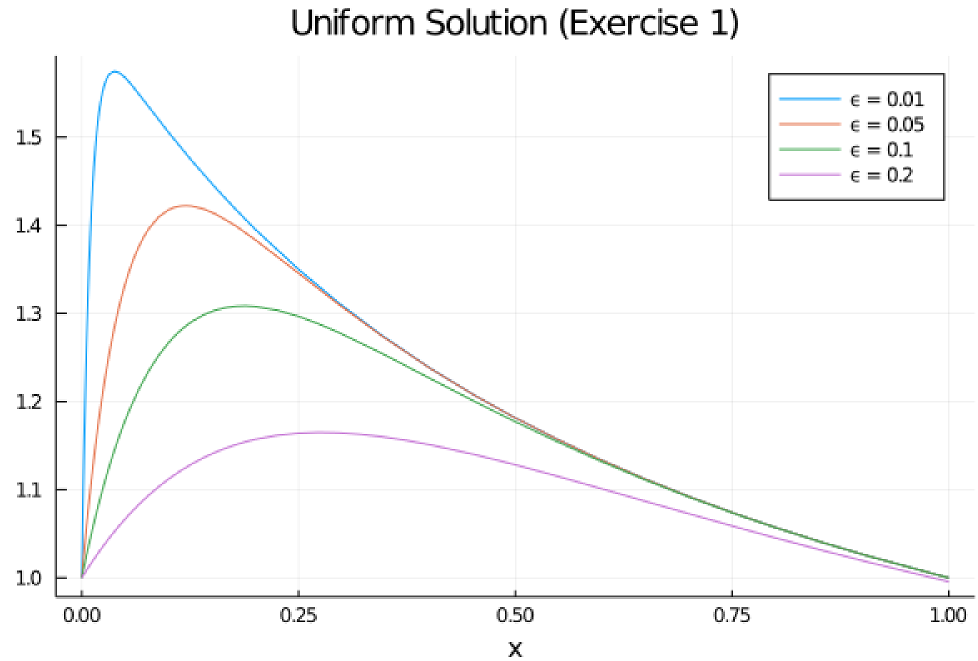

Figure 1: Plot of the uniform solution for exercise 1.

Exercise 2. Consider the singular equation:

$$\epsilon \frac{d^2u}{dx^2} - x^2 \frac{du}{dx} - u = 0$$

with $u(0) = u(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) With the method of dominant balance, show that there are three distinguished limits: $\delta = \epsilon^{\frac{1}{2}}$, $\delta = \epsilon$, and $\delta = 1$ (the outer problem). Write down each of the problems inthe various distinguished limits.

(b) Obtain the leading order uniform approximation (Hint: there are boundary layers ar $x = 0$ and $x = 1$).

(c) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution 2. (a) We begin by doing a perturbation expansion of the equations, so that we have equations

$$\begin{aligned} O(1): &\quad -x^2 u_{0x} - u_0 = 0 \\ O(\epsilon): &\quad -x^2 u_{1x} - u_1 = -u_{0xx}. \end{aligned}$$

As we do not know the location of the bounday layer, we will knot impose a specifc boundary condition. Rather, we show that our solution will be of the form

$$u_0(x) = Ce^{\frac{1}{x}},$$

for some constant C dependent on the boundary condition we impose. In order to analyze the inner problem, we'll introduce a stretching as before $\xi = x/\delta$ near $x = 0$. This transforms our equations to

$$\begin{aligned} \frac{\epsilon}{\delta^2} u_{\xi\xi} - \delta\xi^2 u_\xi - u = 0 \\ \epsilon u_{\xi\xi} - \delta^3 \xi^2 u_\xi - \delta^2 u = 0, \end{aligned}$$

The way to balance the first and last terms (given $O(\delta^3)$ is much smaller than $O(\delta^2)$) which gives a potential boundary layer of width $O(\epsilon^{1/2})$.
We now consider the distinguished limit near $x = 1$, for which we define stetching $\xi = (1 - x)/\delta$. This transforms our equation as

$$\epsilon u_{\xi\xi} + \delta(1 - \delta\xi)^2 u_\xi - \delta^2 u = 0.$$

In this case, the term $(1 - \delta\xi)^2$ is approximately one and the δ^2 is much smaller than the rest, so we have

$$\epsilon u_{\xi\xi} + \delta u_\xi = 0$$

and distinguished limit $\delta = \epsilon$. To construct the inner solution, we first begin with $x = 0$ and $\xi = x/\epsilon^{1/2}$ as derived above which has leading order solution

$$O(1): \quad u_{0\xi\xi} - u_0 = 0, u_0(0) = 1.$$

This gives solution $u_{\text{in}0} = u_0(\xi) = e^{-\xi}$ as we require that u_0 is bounded in the $\xi \to \infty$ limit. Next for the second inner solution near $x = 1$ with stetching $\xi = (1 - x)/\epsilon$ as derived above, we have leading order equation

$$\begin{aligned} u_{0\xi\xi} + u_{0\xi} &= 0, \\ u_0(0) &= 1, \end{aligned}$$

which is solved as

$$u_{\text{in}1}(\xi) = u_0(\xi) = Ae^{-\xi} + (1 - A).$$

We'll now proceed to match our solutions

$$\begin{aligned} u_{\text{out}} &= Ce^{\frac{1}{x}} \\ u_{\text{in}0} &= e^{-\xi} \\ u_{\text{in}1} &= Ae^{-\xi} + (1 - A). \end{aligned}$$

Matching our equations results in the equations

$$\begin{aligned} \lim_{x \to 0} u_{\text{out}} &= \lim_{\xi \to \infty} u_{\text{in}0} \\ \lim_{x \to 1} u_{\text{out}} &= \lim_{\xi \to \infty} u_{\text{in}1}. \end{aligned}$$

Solving this shows that

$$\begin{aligned} C \lim_{x \to 0} e^{\frac{1}{x}} &= \lim_{\xi \to \infty} e^{-\xi} = 0 \\ 0 = \lim_{x \to 1} Ce^{\frac{1}{x}} &= \lim_{\xi \to \infty} Ae^{-\xi} + (1 - A) = 1 - A, \end{aligned}$$

so $C = 0$ and $A = 1$. We then write the uniform solution as

$$u_{\text{unif}} = \exp\left(-\frac{x}{\epsilon^{\frac{1}{2}}}\right) + \exp\left(-\frac{1 - x}{\epsilon}\right),$$

where we've included the proper definitions for various ξ.
(c) Solutions plotted below.

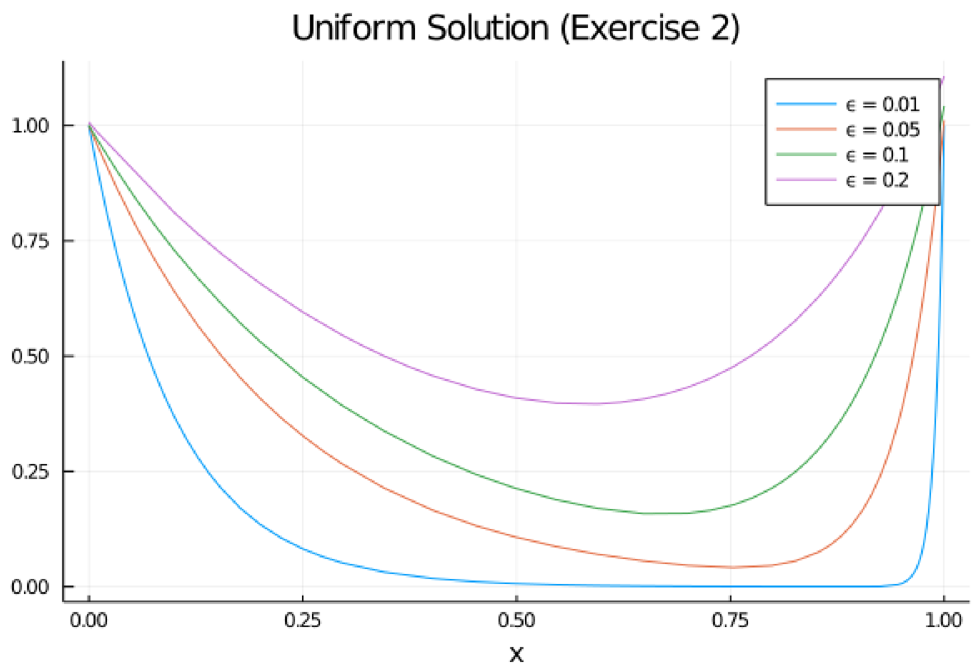

Figure 2: Plot of the uniform solution for exercise 2.