# AMATH 567A: Applied Complex Analysis

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# Introduction

This document is a collection of my personal notes on complex analysis during my time as a student in AMATH 567A in fall quarter 2020 at the University of Washington.

## Logistics

Homework is 70 % of the the grade and is due on Wed (at midnight?). Final is worth 30 % of the grade and will be take-home. There will be office hours with Tung on Tues. and one day to be determined. TA office hours are Mon. and Wed. from 4 to 5pm.

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# 1. Complex Numbers

We begin our discussion of complex analysis with their motivation. Consider the polynomial equation with integer coefficients

$$x^2 + 1 = 0. \tag{1.1}$$

In the real numbers $\mathbb{R}$, this has no solution. Since $x^2 \geq 0$ for any real number $x$, we know that

$$x^2 + 1 > 0 \text{ for all } x \in \mathbb{R}.$$

In an ideal world, we would want to have a number system in which every polynomial equation has a solution. This is where the complex numbers arise. Breaking from our traditions of real numbers, we can simply invent some other number $i$, so that

$$i^2 = -1 \Rightarrow i = \sqrt{-1}.$$

We call this number $i$ an *imaginary* or *complex* number. As we can see, the number $i$ is clearly **not** a real number due to it having a negative square. Using $i$, we can define the set of complex numbers as follows. The complex numbers are numbers of the form

**Definition 1.1.** We define the set of complex numbers as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}. \tag{1.2}$$

Notice that under this construction, $\mathbb{R}$ is the subset of $\mathbb{C}$ such that $b = 0$. In fact, we can define two binary operations *complex addition* $+_{\mathbb{C}}$ and *complex muliplication* $\cdot_{\mathbb{C}}$ on $\mathbb{C}$ which turn out to operate simuliarly to those in the real numbers.

**Definition 1.2.** Given complex numbers $z = a + ib$ and $w = c + id$, define

$$z +_{\mathbb{C}} w = (a + ib) + (c + id) = (a + c) + i(b + d) \tag{1.3}$$
$$z \cdot_{\mathbb{C}} w = = (a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc). \tag{1.4}$$

*Remark* 1.3. We often write $zw$ in place of $z \cdot_{\mathbb{C}}$ for simplicity.

You can derive these operations directly from the definition of the complex numbers yourself by using the substitution $i = \sqrt{-1}$. One can show that under these operations $\mathbb{C}$ is a field much like $\mathbb{R}$ and that in fact, real and complex addition and multiplication are exactly the same when working on $\mathbb{R} \subset \mathbb{C}$. In this way, we can think of $\mathbb{C}$ as an extension of $\mathbb{R}$. Because of this, we'll drop the subscript $_{\mathbb{C}}$ when dealing with complex operations and simply use the same notation as their real counterparts $+$ and $\cdot$.

**Proposition 1.4.** *The complex numbers $\mathbb{C}$ are a field when endowed with complex addition and multiplication as above.*

Since complex numebers are a field, we are also able to define divison for these numbers as follows

$$z \tag{1.5}$$

One significant benefit of extending the real numbers to the complex numbers is that it guarantees us that any polynomial with complex coefficients of degree $n$ has exactly $n$ complex roots. This statement is called the *Fundamental Theorem of Algebra* and we'll prove it for later. For now, let's return to our beginning example, but now using complex numbers. We can rely on the algebraic rules defined above to see that

$$z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i. \tag{1.6}$$

By extending our problem to the complex numbers, we can see that $z^2 + 1$ had two roots all along. They just lived in the complex numbers, outside of the real number line.

The reality is that complex numbers also behave like vectors in nice ways. Instead of thinking of the complex number $z = x + iy$ as being a sum, we can write it in terms of coordinates as a point in the plane $(x, y)$. This allows us to define the magnitude of complex numbers as the distance from the origin $0 \in \mathbb{C}$ as

$$|z| = \sqrt{x^2 + y^2}. \tag{1.7}$$

This is simply the Pythagorean theorem that you may have seen in earlier mathematics classes. We can use this similarity to show that this definition of magnitude defines a distance metric on $\mathbb{C}$.

**Proposition 1.5.** *We can use the magnitude of complex numbers to define a distance metric on $\mathbb{C}$. The distance between any two points $z$ and $w$ in $\mathbb{C}$ is given by the magnitude of their difference.*

$$d(z, w) = |z - w|. \tag{1.8}$$

Writing $z = a + ib$ and $w = c + id$, we can compute this by looking at the real and imaginary parts of $z$ and $w$ individually, so that

$$|z - w| = \sqrt{(a - c)^2 + (b - d)^2}. \tag{1.9}$$

*Remark* 1.6. Since $|\cdot|$ is a distance metric on $\mathbb{C}$, it satisfies the triangle inequality i.e. that

$$|z + w| \leq |z| + |w|. \tag{1.10}$$

Notice that this is equivalent to the distance between these points if we were to consider them as $z = (a, b)$ and $w = (c, d)$ in $\mathbb{R}^2$. With this representation in mind, we'll define some more operations for dealing with geometry of the complex numbers. To start, we'll break our complex number's down into two seperate components.

**Definition 1.7.** Given a complex number $z = a + bi$, we define the real and imaginary parts of $z$ to be

$$\mathrm{Re}[z] = a \text{ and } \mathrm{Im}[z] = b. \tag{1.11}$$

We also define the complex conjugate of $z$.

**Definition 1.8.** For any complex number $z = a + bi$, we define the *complex conjugate* of $z$ to be

$$\bar{z} = a - bi \tag{1.12}$$

The complex conjugate is extremely useful in discussion of the geometry of the complex plane. One direct consequence of our definition of it is that complex conjugate of a product of two complex numbers is the product of the complex conjugates of the individual numbers.

**Proposition 1.9.** *For any complex numbers $z$ and $w$, $\bar{zw} = \bar{z}\bar{w}$*

With a quick computation, we can see that magnitude of a complex number can also be found using its complex conjugate.

$$|z| = \sqrt{z\bar{z}}. \tag{1.13}$$

The complex conjugate also gives us an easy formula for computing the multiplicative inverse of any non-zero complex number.

**Proposition 1.10.** *If $z \neq 0$, then $z \cdot \left(\frac{\bar{z}}{|z|^2}\right) = 1$. Therefore, we can compute the multplicative inverse of $z$ as*

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{1.14}$$

## 1.1 Polar Coordinates

As we've seen, we can represent the complex numbers using Cartesian coordinates as ordered pairs in $\mathbb{R}^2$. We can also represent a complex number $z \in \mathbb{C}$ with polar coordinates.

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.15}$$

Euler was able to prove that the right had side of this equation was equivalent to

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.16}$$

This allows us to define sin and cos in terms of the complex exponential as

$$\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}), \tag{1.17}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{1.18}$$

*Proof of Euler's formula.*
Let $f(\theta) = \cos\theta + i\sin\theta$. Since both sin and cos satisfy the second order equation,

$$\frac{d^2}{d\theta^2}g + g = 0 \tag{1.19}$$

then we know that $f$ does as well with initial conditions $f(0) = 1$ and $f'(0) = i$. The exponential function should have the property that

$$\frac{d}{d\theta}e^{\lambda\theta} = \lambda e^{\lambda\theta}. \tag{1.20}$$

By taking another derivative, we can see that

$$\frac{d^2}{d\theta^2}e^{\lambda\theta} + e^{\lambda\theta} = (\lambda^2 + 1)(e^{\lambda\theta}). \tag{1.21}$$

We see that for $\lambda = i$, this function satisfies the same differential equation as $f$. Likewise, $e^{\lambda 0} = 1$ and $\frac{d}{d\theta}e^{\lambda\theta} \mid_{\theta=0} = \lambda$. Therefore, by the uniqueness of this differential equation, we have

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.22}$$

□

# 2. Elementary Functions

## 2.1 Complex Exponental

**Definition 2.1** (Complex exponential)**.** Define

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{i=1}^{\infty} \frac{z^n}{n!} \tag{2.1}$$

**Proposition 2.2** (Properties of the complex exponential)**.** *The complex exponential has the following properties:*

*(i) First*

*(ii) Second*

## 2.2 Complex logarithm

Can the familiar logarithm function in the real numbers also be generalized easily in the complex numbers? Let's start from the definition of the logarithm. Suppose $w = f(z)$ is a function such that $z = e^w$ with $e$ defined as above. With $z = re^{i\theta}$ and $w = u + iv$. We see that

$$r = e^u \text{ and } \theta = v.$$

Therefore, $u = \ln r$ is real and singular-valued, but $\theta$ can only be determined up to a shift by $2n\pi$ since the complex exponential $e^{i\theta}$ is equivalent for

$$\theta = \theta_p + 2n\pi \text{ with } -\pi < \theta_p \leq \pi. \tag{2.2}$$

We can somewhat solve this issue by restricting $\theta$ to the interval $[0, 2\pi]$ or $[-\pi, \pi]$.

## 2.3 Analytic functions

**Definition 2.3.** A function $f$ of a complex variable $z$ is *analytic* at a point $z_0$ if it is single-valued and its derivative exists not only at $z_0$ but at each point $z$ in some neighborhood of $z_0$. We call the function analytic in a region $R$ if it is analytic at every point in $R$.

This definition is a bit terse. Let's explore what this means through a couple of examples and non-examples of analytic functions.

**Example 2.4** (Differentiating $z^2$)**.** As an example, let's differentiation the function $f(z) = z^2$.

$$\frac{d}{dz} z^2 = ... = 2z \tag{2.3}$$

**Example 2.5** (Non-differentiability of $\bar{z}$)**.** Consider now $f(z) = \bar{z}$.

The reality is that functions like $f(z) = |z|^2 = z\bar{z}$ are really functions of two variables $z$ and $\bar{z}$ since

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2}(z - \bar{z}). \tag{2.4}$$

**Informal perspective on path independence.** Consider a complex function of two variables $f(z, \bar{z})$. Supposing we expand $f$ in a Taylor series, we see that

$$f(z, \bar{z}) - f(z_0, \bar{z}_0) = a(z - z_0) + b(\bar{z} - \bar{z}_0) + \text{higher order terms.} \tag{2.5}$$

Taking the derivative allows the higher order terms to fall out giving:

$$\lim_{z \to z_0} \frac{f(z, z_0) - f(z_0, \bar{z}_0)}{z - z_0} = a + b\frac{\bar{z} - \bar{z}_0}{z - z_0}. \tag{2.6}$$

**Multivalued Functions** Our definition of analyticity required that the function in question be single-valued. As we've seen, this requirement automatically disqualifies complex functions like

$$z^{1/2} \text{ and } \ln z$$

from being analytic despite the fact they are at least somewhere differentiable in the real case. These functions are nowhere analytic due to the fact they are multi-valued, but we can make these singular valued by restricting the argument of $z$ to an inteval of length $2\pi$ e.g. we might define that $\arg z \in (0, 2\pi]$. This restriction creates a line of discontinuity, so it is certainly not differentiable along that line. In hopes to find a region where it is analytic we must first cut away.

**Example 2.6** (Re-defining $z^{1/2}$)**.** Let's redefine a single-valued $z^{1/2}$ by

$$f(z) = z^{1/2}, \quad 0 \leq \arg z < 2\pi. \tag{2.7}$$

Consider points $A = re^{i0}$ and $B = re^{i2\pi}$. Then wihtout restricting our argument, we would have

$$f(A) = r^{1/2} \text{ and } f(B) = (re^{i2\pi})^{1/2} = r^{1/2}e^{i\pi} = -r^{1/2}. \tag{2.8}$$

Therefore, without restricting our argument we would run into issues rather quickly. Due to our actual definition which includes $A$ in its domain and not $B$, we would observe that $f(A) = r^{1/2}$. Regardless of our choice of whether to include $A$ or $B$, this introduces a line of discontuity along the positive real axis. This is called the *branch cut.*

**Branch cuts** The above example shows that a branch cut goes from the origin to the point at infinity $z_\infty = \lim_{z\to 0} \frac{1}{z}$. We call these two points the *branch points.*

## 2.4 Cauchy-Riemann equations.

Here is the first interesting conseequence of our requirement that the derivative of $f$ be path-independent. As we showed before, if $f$ is analytic at $z$, then

$$f'(z) = \lim_{\Delta z \to 0} = \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2.9}$$

We can take the derivative along two seperate directions. First, along the real axis.
Now along the imaginary axis.
This gives us the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x}. \tag{2.10}$$

**Example 2.7.** Let's consider an example

$$f(z) = z \cdot z^* = x^2 + y^2 \tag{2.11}$$

**Laplace's Equation.** One application of this is Laplace's equation

$$\nabla^2 \varphi = 0, \tag{2.12}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The solution is a harmoninc function.

**Theorem 2.8.** *The real and imaginary parts of an analytic function are both harmonic functions.*

*Proof.* Since $f$ is analytic, it satisifies the Cauchy-Riemann equations. Therefore, □

**Example 2.9** (Equipotentials of $f(z) = z^2$)**.** We'll now plot the level curves of the function

$$f(z) = z^2 = (x^2 - y^2) + i2xy. \tag{2.13}$$

# 3. Complex Integration

We can define this with the Riemann sum so that

$$\int_C f(z)dz = \lim_{L\to 0} \sum_{j=1}^{n} f(\zeta_j)\Delta z_j, \tag{3.1}$$

where $\zeta_j = \frac{1}{2}(z_j + z_{j-1})$, $\Delta z_j = z_j - z_{j-1}$, and $L = \max_j |\Delta z_j|$. When it comes to actually evaluating this integral, we typically write it in terms of its real and imaginary parts, so that $f(z) = u(x,y) + iv(x,y)$. Then

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx). \tag{3.2}$$

Here, we see that integrals in the complex plane can be reduced to real integrals in the $x - y$ plane. It's our hope that these integrals are path independent to simplify these 2D real integrals.

### Cauchy's Integral Theorem

**Theorem 3.1.** *If $f(z)$ is analytic inside and on a simple closed curve $C$, then*

$$\oint_C f(z)dz = 0. \tag{3.3}$$

We can prove this statment using Green's theorem in real variables

**Theorem 3.2** (Green's Theorem)**.**

$$\oint_C (V_1(x,y)dx + V_2(x,y)dy) = \iint (\frac{d}{dx}V_2 - \frac{d}{dy}V_1)dxdy \tag{3.4}$$

Applying this to prove Cauchy's Integral Theroem, we see taht

$$\oint_C (udx - vdy) = -\iint_S (\frac{dv}{dz} + \frac{du}{dy})dxdy \tag{3.5}$$
$$\oint_C (vdx - udy) = -\iint_S (\frac{du}{dz} + \frac{dv}{dy})dxdy \tag{3.6}$$

**Path Independence**

**Proposition 3.3.** *For an analytic function $f$, we have*

$$\int_{z_0}^{z} f(z)dz = F(z) - F(z_0) \tag{3.7}$$

*if $f$ is analytic for all paths $C$ from $z_0$ to $z$ which be continuously deformed without leaving the region of analyticity of $f$.*

*Proof.* Using Cauchy's Theorem and taking difference between integrals on curves $C_1$ and $C_2$. □