Exercise 1. Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.

Solution 1.

Exercise 2 (No translation-invariant random integer)**.** Show that there is no probability measure $\mathbb{P}$ on the integers $\mathbb{Z}$ with the discrete σ-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $\mathbb{P}(E+n) = \mathbb{P}(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer n. $E+n$ is obtained by adding n to every element of E.

Solution 2.

Exercise 3 (No translation-invariant random real)**.** Show that there is no probability measure $\mathbb{P}$ on the reals $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $\mathbb{P}(E+x) = \mathbb{P}(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real x. Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by intervals $(a, b] \subset \mathbb{R}$.

Solution 3.

Exercise 4. Let $\Omega = \mathbb{R}$, $\mathcal{F} =$ all subsets of $\mathbb{R}$ so that A or A^c is countable. $\mathbb{P}(A) = 0$ in the first case and $\mathbb{P}(A) = 1$ in the second. Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Solution 4.

Exercise 5. A collection $\mathcal{A}$ of subsets of Ω is called an **algebra** if $A, B \in \mathcal{A}$ implies A^c and $A \cup B$ are in $\mathcal{A}$. (a) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra. (b) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution 5.