

Exercise 1 (Exercise 5.1.). Patients arrive at an emergency room as a Poisson process with intensity λ . The time to treat each patient is an independent exponential random variable with parameter μ . Let $X = (X_t)_{t \geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X . Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π . What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Hint: You can use Little's law which states that the expected number of people in the hospital at steady-state is equal to the expected rate of arrivals times the expected rate of processing time.

Solution 1. If we have individuals arrive at rate λ and leave with rate μ , we can write that in the limit as $s \rightarrow 0$,

$$\mathbb{P}(X_{t+s} = j \mid X_t = i) = \delta_{ij} + g(i, j)s + O(s^2).$$

In this case, we have that $g(i, i+1) = \lambda$ since patients arrive at rate λ . Additionally, when there are patients to be treated, we have that the probability of someone being treated in a small time interval is μ , so $g(i, i-1) = \mu$ for all $i > 0$. We can use this to write the generator of X as

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\lambda + \mu) & \lambda & 0 & 0 & \dots \\ 0 & \mu & -(\lambda + \mu) & \lambda & 0 & 0 & \dots \\ 0 & 0 & \mu & -(\lambda + \mu) & \lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{pmatrix}$$

If there is a stationary distribution π , then it must satisfy $\pi \mathbf{G} = 0$, so that

$$0 = -\lambda\pi_0 + \mu\pi_1$$

$$0 = \lambda\pi_{n-1} - (\lambda + \mu)\pi_n + \mu\pi_{n+1}$$

for $n \geq 1$. This shows immediately that $\pi_0 = \frac{\lambda}{\mu}$ and by continued substitution, we see that

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0.$$

Due to the fact that π is a distribution, we must have that

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

but this can only hold in the regime where $\lambda < \mu$. In which case, we have a stationary distribution given by

$$\pi_0 = \left(\frac{1}{1 - \lambda/\mu}\right)^{-1} = 1 - \lambda/\mu$$

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0, \quad n \geq 1,$$

where we've used the geometric series to simplify π_0 . As we can see, this is geometrically distributed with $p = 1 - \lambda/\mu$. We can then compute the expected total waiting time for a patient in the invariant distribution using Little's law. As the expected number of people in the hospital at steady-state is equal to the expected rate of arrivals times the expected processing time. From this it follows,

$$\mathbb{E}[\pi] = \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda},$$

where we've computed this expectation using the fact that the distribution on π is geometric. Since λ is the rate of expected arrivals, we have that $1/(\mu - \lambda)$ is the expected processing time.

Exercise 2 (Exercise 5.3). Let $X = (X_t)_{t \geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, \dots\}$ and with a generator G whose i th row has entries

$$g_{i,i-1} = i\mu, \quad g_{i,i} = -i\mu - \lambda, \quad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = k$. Find $G_{X_t}(s)$. What is the distribution of X_t as $t \rightarrow \infty$?

Solution 2. We begin by writing the Kolmogorov forward equation for X which is

$$\begin{aligned} \frac{dp_t(k, 0)}{dt} &= \mu p_t(k, 1) - \lambda p_t(k, 0) \\ \frac{dp_t(k, j)}{dt} &= \mu(j+1)p_t(k, j+1) - (\lambda + j\mu)p_t(k, j) + \lambda p_t(k, j-1) \end{aligned}$$

Multiplying through by s^j and summing across j , we have that

$$\frac{\partial G_{X_t}}{\partial t}(s) = \mu \sum_{j=0}^{\infty} (j+1) s^j p_t(k, j+1) - \sum_{j=0}^{\infty} (\lambda + j\mu) s^j p_t(k, j) + \lambda \sum_{j=1}^{\infty} s^j p_t(k, j-1)$$

Using that $\partial_s G_X = \sum_{j=1}^{\infty} j s^{j-1} p_t(i, j)$, we have that

$$\begin{aligned} \frac{\partial G_{X_t}}{\partial t}(s) &= \mu \sum_{j=1}^{\infty} j s^{j-1} p_t(k, j) - \lambda \sum_{j=0}^{\infty} s^j p_t(k, j) - \mu s \sum_{j=1}^{\infty} j s^{j-1} p_t(k, j) + \lambda s \sum_{j=0}^{\infty} s^j p_t(k, j) \\ &= \mu \frac{\partial G_{X_t}}{\partial s} - \lambda G_{X_t}(s) - \mu s \frac{\partial G_{X_t}}{\partial s} + \lambda s G_{X_t}(s) \\ &= \mu(1-s) \frac{\partial G_{X_t}}{\partial s} - \lambda(1-s) G_{X_t}(s) \end{aligned}$$

I used Mathematica to solve this, which gives that

$$G_{X_t}(s) = c_1 e^{\frac{\lambda}{\mu} s} \left(t - \frac{\log |s-1|}{\mu} \right)$$

for some constant c_1 . This doesn't seem to be the correct solution to me because it's undefined in the limit as $t \rightarrow \infty$. But in case I got something wrong earlier, my approach from here would be to get the generating function $G_{X_t}(s)$ and take the limit as $t \rightarrow \infty$. From there, I would look at the resulting generating function and compare it to the PGF of a known random variable to figure out the distribution of X_t as $t \rightarrow \infty$.

Exercise 3 (Exercise 5.4). Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t + dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $O(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let N_0 and let τ_1 be the time of the first jump of N , If $\lambda(t) = c/(1 + t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

Solution 3. Since N is a poisson process with intensity function $\lambda(t)$, we have that

$$\mathbb{P}(X_{t+s} = i + 1 \mid X_t = i) = \lambda(t)s + O(s^2).$$

This allows us to write the generator of N as

$$G_t = \begin{pmatrix} -\lambda(t) & \lambda(t) & 0 & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & 0 & -\lambda(t) & \lambda(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With this generator, we can write the forward and backward Kolmogorov equations.

Solving the forward equation. We begin with the forward so that

$$\frac{d\mathbf{P}_t}{dt} = \mathbf{P}_t G_t.$$

Writing out the matrix multiplication, we have the following forward equations

$$\begin{aligned} \frac{dp_t}{dt}(0, 0) &= -\lambda(t)p_t(0, 0) \\ \frac{dp_t}{dt}(0, j) &= \lambda(t)p_t(0, j-1) - \lambda(t)p_t(0, j), \end{aligned}$$

where we have assumed that $N_0 = 0$ and $j > 0$. Using the generating function of N_t , we can transform this equation since

$$G_{N_t}(s) = \sum_{j=0}^{\infty} s^j p_t(0, j), \quad \frac{\partial G_{N_t}}{\partial t}(s) = \sum_{j=0}^{\infty} s^j \frac{dp_t(0, j)}{dt}.$$

This way, we can write that

$$\begin{aligned} \frac{\partial G_{N_t}}{\partial t}(s) &= \sum_{j=0}^{\infty} s^j \frac{dp_t(0, j)}{dt} \\ &= \lambda(t) \left(\sum_{j=1}^{\infty} s^j p_t(0, j-1) - \sum_{j=0}^{\infty} s^j p_t(0, j) \right) \\ &= \lambda(t) \left(s \sum_{j=0}^{\infty} s^j p_t(0, j) - \sum_{j=0}^{\infty} s^j p_t(0, j) \right) \\ &= \lambda(t)(s-1) \sum_{j=0}^{\infty} s^j p_t(0, j) \\ &= \lambda(t)(s-1)G_{N_t}(s), \end{aligned}$$

where we've used the Kolomogorov forward equation and the definition of the generating function. We can see that this differential equation is solved pretty simply as an exponential function, so that

$$G_{N_t}(s) = \exp \left((s-1) \int_0^t \lambda(\tau) d\tau \right).$$

We can double check this solution is sufficient by taking the partial derivative with respect to t . Notice that this is the generating function for a Poisson random variable which shows that

$$N_t \sim \text{Poisson} \left(\int_0^t \lambda(\tau) d\tau \right).$$

Writing the Backward equation We write the backward equation as

$$\frac{d\mathbf{P}_t}{dt} = G_t \mathbf{P}_t.$$

Completing the matrix multiplication, we write

$$\begin{aligned} \frac{dp_t}{dt}(i, 0) &= -\lambda(t)p_t(i, 0) \\ \frac{dp_t}{dt}(i, j) &= \lambda(t)p_t(i, j+1) - \lambda(t)p_t(i, j) \end{aligned}$$

First jump time τ_1 If we define the first jump time of N_t to be τ_1 , we have that if $\tau_1 < t$, then $N_t > 0$ since N is non-decreasing. Using this fact, we can write the CDF of τ_1 as

$$\begin{aligned} \mathbb{P}(\tau_1 < t) &= \mathbb{P}(N_t > 0) \\ &= 1 - \mathbb{P}(N_t = 0) \\ &= 1 - \exp \left(- \int_0^t \lambda(\tau) d\tau \right) \end{aligned}$$

With this in mind, we can compute that τ_1 has density

$$f_{\tau_1}(t) = \lambda(t) \exp \left(- \int_0^t \lambda(\tau) d\tau \right)$$

In the case that $\lambda(t) = c/(1+t)$, we can compute the exponent as

$$\int_0^t \lambda(\tau) d\tau = \int_0^t \frac{c}{1+x} dx = c \ln(1+t).$$

Therefore, our simplified density is

$$f_{\tau_1}(t) = \frac{c}{1+t} \exp(-c \ln(1+t)) = \frac{c}{(1+t)^{c+1}}.$$

This allows us to compute the expectation of τ_1 as

$$\mathbb{E}[\tau_1] = \int_0^\infty \frac{ct}{(1+t)^{c+1}}.$$

Using integration by parts and assuming that $c \neq 0, c \neq 1$, we have that

$$\begin{aligned}\mathbb{E}[\tau_1] &= \int_0^\infty \frac{ct}{(1+t)^{c+1}} \\ &= \frac{t}{(1+t)^c} \Big|_{t=0}^\infty + \int_0^\infty \frac{1}{(1+t)^c} dt \\ &= \left[\frac{t}{(1+t)^c} - \frac{1}{(c+1)} \frac{1}{(1+t)^{c+1}} \right] \Big|_{t=0}^\infty\end{aligned}$$

notice that the first term of this only exists if and only if $c > 1$, elsewhere this expectation is not defined.

Exercise 4 (Exercise 5.5). Let N be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1 - p$. Find $G_{N_t}(s) = \mathbb{E}s^{N_t}$. What is the mean and variance of N_t ?

Solution 4. To solve this, we begin by noting that

$$\begin{aligned} p_t(i, j) &= \mathbb{P}(N_t = j \mid N_0 = i) \\ &= \mathbb{P}(N_t = j \mid N_0 = i, \Lambda = \lambda_1)\mathbb{P}(\Lambda = \lambda_1) + \mathbb{P}(N_t = j \mid N_0 = i, \Lambda = \lambda_2)\mathbb{P}(\Lambda = \lambda_2) \\ &= p \cdot p_{1,t}(i, j) + (1 - p) \cdot p_{2,t}(i, j) \end{aligned}$$

Writing the generating function of N_t , we have that

$$G_{N_t}(s) = \sum_{j=0}^{\infty} p_t(i, j)s^j = p \sum_{j=0}^{\infty} p_{1,t}(i, j)s^j + (1 - p) \sum_{j=0}^{\infty} p_{2,t}(i, j)s^j.$$

Therefore,

$$G_{N_t}(s) = pG_{N_{1,t}}(s) + (1 - p)G_{N_{2,t}}(s),$$

where $G_{N_{1,t}}(s)$ is the generating function of a Poisson process with intensity λ_1 and $G_{N_{2,t}}(s)$ is the generating function of a Poisson process with intensity λ_2 . We can compute the expectation of N_t as

$$\begin{aligned} \mathbb{E}[N_t] &= G_{N_t}'(1) = pG_{N_{1,t}}'(1) + (1 - p)G_{N_{2,t}}'(1) \\ &= p\mathbb{E}[N_{1,t}] + (1 - p)\mathbb{E}[N_{2,t}] \\ &= p\lambda_1 t + (1 - p)\lambda_2 t \\ &= (p\lambda_1 + (1 - p)\lambda_2)t \end{aligned}$$

We can use a similar technique to compute the variance as

$$\text{Var}[N_t] = G_{N_t}''(1) + G_{N_t}'(1) - (G_{N_t}'(1))^2$$

We begin by computing

$$G_{N_t}''(1) = pG_{N_{1,t}}''(1) + (1 - p)G_{N_{2,t}}''(1).$$

Using that $G_{N_{1,t}}(s) = \exp(\lambda_1 t(s - 1))$, $G_{N_{2,t}}(s) = \exp(\lambda_2 t(s - 1))$, we have that

$$G_{N_t}''(1) = p(\lambda_1 t)^2 + (1 - p)(\lambda_2 t)^2.$$

Putting this all together, we have that

$$\begin{aligned} \text{Var}[N_t] &= p(\lambda_1 t)^2 + (1 - p)(\lambda_2 t)^2 + p\lambda_1 t + (1 - p)\lambda_2 t - (p\lambda_1 t + (1 - p)\lambda_2 t)^2 \\ &= p\lambda_1^2 t^2 + (1 - p)\lambda_2^2 t^2 \\ &\quad + p\lambda_1 t + (1 - p)\lambda_2 t \\ &\quad - p^2\lambda_1^2 t^2 - 2p(1 - p)\lambda_1\lambda_2 t^2 - (1 - p)^2\lambda_2^2 t^2. \end{aligned}$$

This is probably able to be simplified, but I don't really want to do more algebra.