Exercise 1. Compute $\mathrm{d}(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution 1. Using Thm 8.2.1. (MLN) with $f(x) = x^4$,

$$W_T^4 = \int_0^T 4W_t^3 \mathrm{d}W_t + \frac{1}{2}\int_0^T 12W_t^2 \mathrm{d}t.$$

We can write this in differential form as

$$\mathrm{d}(W_t^4) = 6W_t^2 \mathrm{d}t + 4W_t^3 \mathrm{d}W_t.$$

We can solve for the expectation of W_T^4 as

$$\mathbb{E}[W_T^4] = 4\mathbb{E}\left[\int_0^T W_t^3 \mathrm{d}W_t\right] + 6\mathbb{E}\left[\int_0^T W_t^2 \mathrm{d}t\right].$$

The left most integral defines a martigale so that its expection is given by $I_0 = 0$, we then have

$$\begin{aligned}
\mathbb{E}[W_T^4] &= 6\mathbb{E}\left[\int_0^T W_t^2 \mathrm{d}t\right] \\
&= 6\int_0^T \mathbb{E}[W_t^2]\mathrm{d}t \\
&= 6\int_0^T t\mathrm{d}t = 3T^2,
\end{aligned}$$

where we've interchanged the order of integration using Fubini and used that the variance of the standard Brownian motion is $\mathbb{E}[W_t^2] = t$. We'll now repeat this method to solve for $\mathbb{E}[W_T^6]$. We have that

$$\mathbb{E}[W_T^6] = \mathbb{E}\left[\int_0^T 6W_t^5 \mathrm{d}W_t\right] + \mathbb{E}\left[\frac{1}{2}\int_0^T 30W_t^4 \mathrm{d}t\right].$$

Once again, the first integral is a martingale and has expectation 0. We can also switch the order of integral using Fubini's theorem so that

$$\begin{aligned}
\mathbb{E}[W_T^6] &= 15\int_0^T \mathbb{E}\left[W_t^4\right] \mathrm{d}t \\
&= 15\int_0^T 3t^2 \mathrm{d}t \\
&= 15T^3.
\end{aligned}$$

Exercise 2. Find an explicit expression for Y_T where

$$\mathrm{d}Y_t = r\mathrm{d}t + \alpha Y_t \mathrm{d}W_t$$

Hint: compute $\mathrm{d}(Y_t Z_t)$ where $Z_t \coloneqq \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$.

Solution 2. Using the product rule, we have that

$$\mathrm{d}(Y_t Z_t) = Z_t \mathrm{d}Y_t + Y_t \mathrm{d}Z_t + \mathrm{d}[Y, Z]_t.$$

With the choice of Z_t, we have that

$$\mathrm{d}Z_t = \alpha^2 Z_t \mathrm{d}t - \alpha Z_t \mathrm{d}W_t.$$

Putting this together, we compute

$$\begin{aligned} Z_t \mathrm{d}Y_t &= r Z_t \mathrm{d}t + \alpha Y_t Z_t \mathrm{d}W_t \\ Y_t \mathrm{d}Z_t &= \alpha^2 Z_t Y_t \mathrm{d}t - \alpha Z_t Y_t \mathrm{d}W_t. \end{aligned}$$

We can additionally compute

$$\mathrm{d}[Y, Z]_t = (\alpha Y_t \mathrm{d}W_t)(-\alpha Z_t \mathrm{d}W_t) = -\alpha^2 Y_t Z_t \mathrm{d}t.$$

We can now write

$$\begin{aligned} \mathrm{d}(Y_t Z_t) &= r Z_t \mathrm{d}t \\ Y_T Z_T &= Y_0 Z_0 + \int_0^T r Z_t \mathrm{d}t \end{aligned}$$

Noting that $Z_0 = 1$, we have that

$$\begin{aligned} Y_T &= \frac{1}{Z_T}\left(Y_0 + \int_0^T r \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)\mathrm{d}t\right) \\ &= \exp(\alpha W_T - \frac{1}{2}\alpha^2 T)\left(Y_0 + \int_0^T r \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)\mathrm{d}t\right) \end{aligned}$$

Exercise 3. Suppose X, Δ and Π are given by

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t, \quad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \quad \Pi_t = X_t \Delta_t,$$

where f is some smooth function. Show that if f satisfies

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) f(t, x) = 0.$$

for all (t, x), then Π is a martingale with respect to a filtration $\mathcal{F}$ for W.

Solution 3. Let $g(t, x) = \frac{\partial f}{\partial x}(t, x)$, we then have that

$$\begin{aligned} \mathrm{d}\Delta_t = \mathrm{d}g(t, X_t) &= \partial_t g(t, X_t)\mathrm{d}t + \partial_x g(t, X_t) dX_t + \frac{1}{2}\partial_{xx} g(t, X_t)\mathrm{d}[X, X]_t \\ &= \left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2}{\partial x^2} \right) g(t, x)\mathrm{d}t + \sigma X_t \frac{\partial}{\partial x} g(t, X_t)\mathrm{d}W_t. \end{aligned}$$

Using product rule, we write that

$$\mathrm{d}\Pi_t = \mathrm{d}(X_t \Delta_t) = \Delta_t \mathrm{d}X_t + X_t \mathrm{d}\Delta_t + \mathrm{d}[\Delta, X]_t.$$

We can compute the three terms as

$$\begin{aligned} \Delta_t \mathrm{d}X_t &= (\sigma X_t) g(t, X_t)\mathrm{d}W_t \\ X_t \mathrm{d}\Delta_t &= \left(X_t \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X_t^3 \frac{\partial^2}{\partial x^2} \right) g(t, x)\mathrm{d}t + \sigma X_t^2 \frac{\partial}{\partial x} g(t, X_t)\mathrm{d}W_t \\ \mathrm{d}[\Delta, X]_t &= \sigma^2 X_t^2 \frac{\partial}{\partial x} g(t, X)\mathrm{d}t \end{aligned}$$

Therfore, we can write Π_t as

$$\begin{aligned} \Pi_T - \Pi_0 &= \int_0^T \left(X_t \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X_t^3 \frac{\partial^2}{\partial x^2} + \sigma^2 X_t^2 \frac{\partial}{\partial x} \right) g(t, X_t)\mathrm{d}t + \int_0^T \sigma X_t^2 \frac{\partial}{\partial x} g(t, X_t)\mathrm{d}W_t \\ &= \int_0^T X_t \left(\frac{\partial^2}{\partial t \partial x} + \frac{1}{2}\sigma X_t^2 \frac{\partial^3}{\partial x^3} + \sigma^2 X_t \frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t + \int_0^T \sigma X_t^2 \frac{\partial}{\partial x} g(t, X_t)\mathrm{d}W_t. \end{aligned}$$

Differentiating the condition to us with respect to f shows that

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) = \left(\frac{\partial^2}{\partial t \partial x} + \frac{1}{2}\sigma x^2 \frac{\partial^3}{\partial x^3} + \sigma^2 x \frac{\partial^2}{\partial x^2} \right) f(t, x) = 0$$

Therefore,

$$\Pi_T - \Pi_0 = \int_0^T \sigma X_t^2 \frac{\partial}{\partial x} g(t, X_t)\mathrm{d}W_t.$$

As this is an Ito integral with respect to W_t it is a martingale with respect to $\mathcal{F}_t$

Exercise 4. Suppose X is given by

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t.$$

For any smooth function f define

$$M_t^f \coloneqq f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2} \right) f(s, X_s)ds.$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}$ for W.

Solution 4. We begin by working with

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\mathrm{d}[X, X]_t \\
&= \left(\frac{\partial}{\partial t} + \mu(t, X_t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2}{\partial x^2} \right) f(t, X_t)\mathrm{d}t \\
&+ \sigma(t, X_t)\frac{\partial}{\partial x}f(t, X_t)\mathrm{d}W_t.
\end{aligned}$$

We can rewrite this in integral form as

$$\begin{aligned}
f(t, X_t) - f(0, X_0) = &\int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial x^2} \right) f(s, X_s)ds \\
&+ \int_0^t \sigma(t, X_t)\frac{\partial}{\partial x}f(t, X_t)\mathrm{d}W_t.
\end{aligned}$$

This allows us to rewrite M_t^f as

$$M_t^f = \int_0^t \sigma(t, X_t)\frac{\partial}{\partial x}f(t, X_t)\mathrm{d}W_t.$$

As this is an Ito integral with respect to W_t it is a martingale with respect to $\mathcal{F}_t$

Exercise 5. Let $X = (X_t)_{0 \le t \le T}$ be an OU process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathrm{d}X_t = K(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where $\{W_t\}_{0 \le t \le T}$ is a Brownian motion under probability $\mathbb{P}$. Then we can define a new probability measure $\tilde{\mathbb{P}}$ such that the process $\tilde{W} = (\tilde{W}_t)_{0 \le t \le T}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Then the OU process $X = (X_t)_{0 \le t \le T}$ on the new probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ will be

$$\mathrm{d}X_t = K(\theta^* - X_t)\mathrm{d}t + \sigma \mathrm{d}\tilde{W}_t.$$

Solution 5. We'll seek to write $\tilde{W}_t$ in temrs of W_t. Writing both representations of X_t in integral form, we have that

$$\begin{aligned} X_T - X_0 &= \int_0^T K(\theta - X_t)\mathrm{d}t + \int_0^T \sigma \mathrm{d}W_t \\ &= \int_0^T K(\theta^* - X_t)\mathrm{d}t + \int_0^T \sigma \mathrm{d}\tilde{W}_t. \end{aligned}$$

We can rewrite this so that

$$\begin{aligned} \int_0^T \mathrm{d}\tilde{W}_t &= \frac{1}{\sigma}\int_0^T K(\theta - X_t) - K(\theta^* - X_t)\mathrm{d}t + \int_0^T \mathrm{d}W_t \\ &= \frac{1}{\sigma}\int_0^T K(\theta - \theta^*)\mathrm{d}t + \int_0^T \mathrm{d}W_t. \end{aligned}$$

Alternatively, in differential form this is

$$\mathrm{d}\tilde{W}_t = \frac{1}{\sigma}K(\theta - \theta^*)\mathrm{d}t + \mathrm{d}W_t.$$

Using that $\tilde{W}_t$ is a Brownian motion under $\tilde{\mathbb{P}}$ and W_t is a Brownian motion under $\mathbb{P}$, we can then use the Girsanov theorem (Thm 8.5.5 in MLN) to see that

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(- \int_0^T \frac{1}{2}\left(\frac{K}{\sigma}(\theta - \theta^*)\right)^2 \mathrm{d}t - \int_0^T \frac{K}{\sigma}(\theta - \theta^*)\mathrm{d}W_t \right)$$