Exercise 1. Let X_1, X_2, $X_3, \dots$ be a sequence of random variables such that

$$X_n \sim \text{Geo}(\lambda/n) \text{ for } n \in \mathbb{N},$$

where $\lambda > 0$ is a constant. Define a new sequence as

$$Y_n = \frac{X_n}{n}.$$

Show that Y_n converges in distribution to Exponential(λ).

Solution 1. Our goal is to show that

$$|\mathbb{P}(Y_n < x) - \mathbb{P}(Y < x)| \to_{n\to\infty} 0,$$

where Y has Exponential(λ) distribution. We begin by computing

$$\begin{aligned}\mathbb{P}(Y_n > x) &= \mathbb{P}(X_n > nx)\\ &= \mathbb{P}(X_n > \lfloor nx \rfloor)\end{aligned}$$

since the geometric distribution only has mass at integers. Using that the X_n has Geo(λ/n) distribution, we see that

$$\begin{aligned}\mathbb{P}(Y_n > x) &= \left(1 - \frac{\lambda}{n}\right)^{\lfloor nx \rfloor}\\ &= \left(1 - \frac{\lambda}{n}\right)^{nx}\left(1 - \frac{\lambda}{n}\right)^{\lfloor nx \rfloor - nt}.\end{aligned}$$

Notice that $0 \geq \lfloor nx \rfloor - nt \geq 1$, so that in the limit the right hand term converges to 1. The lefthand term converges to $e^{-\lambda x}$. Therefore, we have that

$$\lim_{n\to\infty} |\mathbb{P}(Y_n > x) - \mathbb{P}(Y > x)| = e^{-\lambda x} - e^{-\lambda x} = 0.$$

Therefore Y_n converges in distribution to $Y \sim$ Exponential(λ).

Exercise 2. Consider the sample space $\Omega = [0, 1]$ with uniform probability. Define the sequence X_n, $n \in \mathbb{N}$ as

$$X_n(\omega) = \frac{n}{n+1}\omega + (1-\omega)^n.$$

Also define $X(\omega) = \omega$. Show that X_n converges almost surely to X.

Solution 2. We begin by writing

$$|X_n(\omega) - X(\omega)| = \left|(1-\omega)^n - \frac{1}{n+1}\omega\right|.$$

Taking the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} |X_n(\omega) - X(\omega)| = \begin{cases} 1 \text{ if } \omega = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Therefore,

$$\mathbb{P}(\lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0) = \mathbb{P}(\omega \in (0, 1]) = 1.$$

Therefore X_n converges almost surely to X.

Exercise 3. Show that if X_n is any sequence of random variables, then there are constants $c_n \to \infty$ so that $X_n/c_n \to^{\text{a.s.}} 0$.

Solution 3. For each X_n pick $a_n > 0$ so that

$$\mathbb{P}(|X_n| \geq a_n) = \mathbb{P}(|X_n/a_n| \geq 1) < \frac{1}{2^n}.$$

Now defining $c_n = \max\{a_n^2, c_{n-1}+1\}$ with $c_0 = 1$. This sequence is c_n is a positive increasing sequence with $c_n \to \infty$. Now we'll consider the events $\{|X_n/c_n| \geq \epsilon\}$ for any $\epsilon > 0$. We have that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n/c_n| \geq \epsilon) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq |c_n|\,\epsilon) \leq \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq |a_n|^2\,\epsilon),$$

where we've used that $c_n \geq a_n^2$. From this it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n/c_n| \geq \epsilon) \leq \sum_{n=1}^{\infty} \mathbb{P}(|X_n/a_n| \geq a_n\epsilon).$$

For any $\epsilon > 0$, there will exist N so that for $n > N$, $a_n\epsilon > 1$. This means that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n/c_n| \geq \epsilon) \leq \sum_{n=1}^{N} \mathbb{P}(|X_n/a_n| \geq a_n\epsilon) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \infty.$$

Therefore by Borel-Cantelli, we have that

$$\mathbb{P}(|X_n/c_n| \geq \epsilon \text{ i.o.}) = 0$$

for all $\epsilon > 0$. This tells us that

$$\mathbb{P}(\lim_{n\to\infty} |X_n/c_n| = 0) = 1,$$

so X_n/c_n converges almost surely to 0.

Exercise 4. Let X_n be independent with $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$. Show that

a) $X_n \to^{\mathbb{P}} 0$ if and only if $p_n \to 0$.

b) $X_n \to^{\text{a.s.}} 0$ if and only if $\sum_n p_n < \infty$.

Solution 4. a) Suppose that for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) = 0.$$

Then for $\epsilon \in (0, 1)$, we have that

$$\mathbb{P}(|X_n| > \epsilon) = p_n.$$

Therefore, we have that

$$\lim_{n\to\infty} p_n = 0.$$

The reverse direction follows since once again

$$p_n = \mathbb{P}(|X_n| > \epsilon), \text{ when } \epsilon \in (0, 1).$$

b) We'll consider the event

$$\mathbb{P}(\lim_{n\to\infty} |X_n| = 0).$$

In the case this event has probability one, then $\mathbb{P}(|X_n| > \epsilon \text{ i.o}) = 0$ for any $\epsilon \in (0, 1)$. As these events are independent, we can use to the contrapositive of Borel-Cantelli (MLN 6.6.1) to see that

$$\mathbb{P}(|X_n| > \epsilon \text{ i.o}) < 1 \implies \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \epsilon) < \infty.$$

Due to our choice of $\epsilon < 1$, we have that

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \mathbb{P}(X_n = 1) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \epsilon) < \infty.$$

In the case that

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \mathbb{P}(X_n = 1) < \infty$$

Borel-Cantell tells us that

$$\mathbb{P}(X_n = 1 \text{ i.o.}) = 0.$$

Therefore,

$$\mathbb{P}(\lim_{n\to\infty} |X_n| = 0) = 1,$$

so X_n converges to 0 almost surely.

Exercise 5. Show that a sequence of random variables $X_1, X_2, \dots$ for which

$$\mathbb{P}(X_n = 1) = \frac{1}{n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$$

has limit $X_n \to^{\mathbb{P}} 0$ but the convergence is not almost surely. While your proof needs not to be perfectly rigorous, you are not allowed to use theorems from this class. In other words, show all the steps in your proof.

Solution 5. For simplicity, we assume that the X_n are independent.We can see that the X_n converge to 0 in probability because for $\epsilon \in (0, 1)$,

$$\mathbb{P}(|X_n| > \epsilon) = \mathbb{P}(X_n = 1) = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

In what follows, we'll show that work with the events $\{X_n = 0\}$. In order for this to not converge almost surely, we would need to show that $\mathbb{P}(X_n = 1 \text{ i.o.}) > 0$ i.e. there is a some set of non-zero measure on which $X_n = 1$ infinitely often. That is, for a fixed m, we'll begin by looking at the probability that $X_n = 0$ for all $n \geq m$

$$\mathbb{P}(X_n = 0 \text{ for all } n \geq m) = \prod_{n=m}^{\infty} \mathbb{P}(X_n = 0) = \prod_{n=m}^{\infty} \left(1 - \frac{1}{n}\right) = 0,$$

where we've used that the X_n are independent. This implies

$$\mathbb{P}(X_n = 1 \text{ for some } n \geq m) = 1 - \mathbb{P}(X_n = 0 \text{ for all } n \geq m) = 1.$$

This means there is zero probability that any sequence can converge after a fixed m since there will eventually be $X_n = 1$. In short, this means that

$$\mathbb{P}(X_n = 0 \text{ i.o}) = 0,$$

so that X_n does not converge almost surely to 0.