Exercise 1. A&F 2.5.1. Evaluate $\oint_\gamma f(z)dz$ where γ is the unit circle centered at the origin for the follwoing functions f.

Solution 1.
(a) $f(z) = e^{iz}$. The function $f(z)$ is entire as it is the composition of two entire functions e^w and iz. Its derivative is $f'(z) = ie^{iz}$. By Cauchy's Theorem, this means that for the closed curve γ, we have

$$\oint_\gamma e^{iz} dz = 0. \tag{1}$$

(b) $f(z) = e^{z^2}$. Once again $f(z)$ is entire as it is the composition of two entire functions e^w and iz. By Cauchy's Theorem, this means

$$\oint_\gamma e^{iz} dz = 0. \tag{2}$$

(c) $f(z) = \frac{1}{z-1/2}$. The function $f(z)$ is analytic except at $z = \frac{1}{2}$ which is contained in γ, so we cannot use Cauchy's theorem. We can instead use the residue theorem. Writing f as its Taylor-Laurent series about $z_0 = \frac{1}{2}$,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - 1/2)^n = 1(z - 1/2)^{-1}. \tag{3}$$

Here we can see that $a_n = 0$ for all $n \neq -1$ and $a_{-1} = 1$. Therefore, by the Residue theorem, we have

$$\oint_C f(z)dz = 2\pi i a_{-1} = 2\pi i. \tag{4}$$

(d) $f(z) = \frac{1}{z^2-4}$. The function $f(z)$ is analytic except at $z = 2, -2$, neither of which are in the contour γ. Since $f(z)$ is analytic on and within γ, we can apply Cauchy's theorem, so that

$$\oint_\gamma \frac{1}{z^2 - 4} dz = 0. \tag{5}$$

(e) $f(z) = \frac{1}{2z^2+1}$. This function is analytic except at $z_\pm = i\frac{\sqrt{2}}{2}, -i\frac{\sqrt{2}}{2}$ which are contained in the contour γ. We can then write

$$f(z) = \frac{1}{2z^2 + 1} = \frac{1}{2(z - i\frac{\sqrt{2}}{2})(z + i\frac{\sqrt{2}}{2})}. \tag{6}$$

We'll now compute the residues at $z_\pm$ using the following formula for the residue of f at z_0

$$\text{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z). \tag{7}$$

For z_+., we can compute

$$\operatorname{Res}(f, z_+) = \lim_{z \to z_+} \left(z - i\frac{\sqrt{2}}{2} \right) f(z) \tag{8}$$
$$= \lim_{z \to z_+} \frac{1}{2(z + i\frac{\sqrt{2}}{2})} \tag{9}$$
$$= \frac{1}{2\left(i\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} \tag{10}$$
$$= \frac{1}{2i\sqrt{2}} = \frac{\sqrt{2}}{4i} \tag{11}$$

Similarly, we can compute the residue at z_-

$$\operatorname{Res}(f, z_-) = \lim_{z \to z_-} \left(z + i\frac{\sqrt{2}}{2} \right) f(z) \tag{12}$$
$$= \lim_{z \to z_-} \frac{1}{2(z - i\frac{\sqrt{2}}{2})} \tag{13}$$
$$= \frac{1}{2\left(-i\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)} \tag{14}$$
$$= -\frac{1}{2i\sqrt{2}} = -\frac{\sqrt{2}}{4i} \tag{15}$$
$$= -\operatorname{Res}(f, z_+) \tag{16}$$

We can then use the residue theorem to compute the integral of f over γ as follows

$$\oint_C \frac{1}{2z^2 + 1} dz = 2\pi i \left(\operatorname{Res}(f, z_+) + \operatorname{Res}(f, z_-) \right) = 0 \tag{17}$$

(f) $\sqrt{z - 4} 0 \leq \arg z < 2\pi$. With the specification on the argument, this function is analytic with derivative $\frac{1}{z-4}$ except on the real line for $x > 4$. Therefore, the integral of this function over the unit circle is 0 by Cauchy's Theorem since it is analytic within and on the unit circle

$$\oint_\gamma \sqrt{z - 4} dz = 0. \tag{18}$$

Exercise 2. A&F 2.5.5. Evaluate the integral

$$\int_0^\infty e^{ix^2} \tag{19}$$

using the contour $C(R)$ which is the closed circular section in the upper half plane with boundary points $(0,0), (0,R)$, and $Re^{i\frac{\pi}{4}}$.

Solution 2. We begin by considering the integral

$$I_R = \oint_{C(R)} e^{iz^2} dz. \tag{20}$$

Since the function $f(x) = e^{iz^2}$ is analytic in the entire complex plane, we have that $I_R = 0$ by Cauchy Theorem. Breaking the contour $C(R)$ into three parts, we see that

$$\oint_{C(R)} e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_{[Re^{i\frac{\pi}{4}},0]} e^{iz^2} dz + \int_{C_1(R)} e^{iz^2} dz. \tag{21}$$

Here, the first integral on the righthand side is the integral on the real line from $(0,R)$, the second is the integral on the line from $[Re^{i\frac{\pi}{4}}, 0]$, the third integral is the integral on the circular section between R and $Re^{i\frac{\pi}{4}}$. We'll begin by trying to estimate the third integral. We'll start by working with the contour $C_1(R,\theta_0)$ which is the circular sector between $Re^{i\theta_0}$ and $Re^{i\frac{\pi}{4}}$ which we'll parameterize by the function $\gamma_{R,\theta_0}(t) = Re^{it}$ for $t \in [\theta, \frac{\pi}{4}]$. Then, we can bound the third integral by

$$\begin{aligned}
\left|\int_{C_1(R,\theta_0)} e^{iz^2} dz\right| &\leq \text{length}(C_1(R,\theta_0)) \cdot \sup_{z\in C_1(R,\theta_0)} \left|e^{iz^2}\right| && (22)\\
&= \left(\frac{\pi}{4} - \theta_0\right) \sup_{z\in C_1(R,\theta_0)} \left|e^{iz^2}\right| && (23)\\
&\leq \frac{\pi}{4} \sup_{z\in C_1(R,\theta_0)} \left|e^{iz^2}\right|. && (24)
\end{aligned}$$

Now we'll compute $\sup_{z\in C_1(R,\theta_0)} \left|e^{iz^2}\right|$. Writing $z \in C_1(R,\theta_0)$ in polar exponential form as $z = Re^{i\theta}$ for some $\theta \in [\theta_0, \frac{\pi}{4}$, we can simplify the exponent of $f(z)$ as

$$\begin{aligned}
iz^2 = iR^2 e^{i2\theta} &= iR^2(\cos 2\theta + i \sin 2\theta) && (25)\\
&= -R^2(\sin 2\theta + i \cos 2\theta). && (26)
\end{aligned}$$

Exponentiating, we can compute $|f(z)|$ as

$$\begin{aligned}
\left|e^{iz^2}\right| = \left|e^{-R^2(\sin 2\theta + i\cos 2\theta)}\right| &= \left|e^{-R^2 \sin 2\theta}\right| \left|e^{-iR^2 \cos 2\theta}\right| && (27)\\
&= \left|e^{-R^2 \sin 2\theta}\right|, && (28)
\end{aligned}$$

where $\left|e^{-iR^2\cos 2\theta}\right| = 1$ since $R^2 \cos 2\theta$ is real. Since $\sin(x) > \frac{2x}{\pi}$ on $x \in [0, \frac{\pi}{2}]$, we have that

$$\sup_{z\in C_1(R,\theta_0)} \left|e^{iz^2}\right| \leq \sup_{\theta\in[\theta_0,\frac{\pi}{4}]} \left|e^{-R^2\frac{4\theta}{\pi}}\right| = e^{-R^2\frac{4\theta_0}{\pi}}. \tag{29}$$

We obtain this suprenum since the function $e^{-R^2\frac{4\theta}{\pi}}$ is positive and monotonically decreasing in θ.
Therefore, we can see that

$$\left|\int_{C_1(R,\theta_0)} e^{iz^2}dz\right| \leq \frac{\pi}{4}e^{-R^2\frac{4\theta_0}{\pi}}. \tag{30}$$

We can see that since $C_1(R, \theta_0)$ becomes similar to $C_1(R)$ as $\theta_0 \to 0$ then

$$\lim_{R\to\infty}\left|\int_{C_1(R)} f(z)dz\right| = \lim_{R\to\infty}\lim_{\theta_0\to 0}\left|\int_{C_1(R,\theta_0)} e^{iz^2}dz\right| \leq \lim_{R\to\infty}\lim_{\theta_0\to 0}\frac{\pi}{4}e^{-R^2\frac{4\theta_0}{\pi}}. \tag{31}$$

Since the righthand most function is monotonically decreasing and bounded by 0 in both R and θ_0, we can re-arrange the limits and show that

$$\lim_{R\to\infty}\left|\int_{C_1(R)} e^{iz^2}dz\right| = \lim_{R\to\infty}\lim_{\theta_0\to 0}\left|\int_{C_1(R,\theta_0)} e^{iz^2}dz\right| = 0. \tag{32}$$

This implies that $\lim\limits_{R\to\infty}\int_{C_1(R)} e^{iz^2}dz = 0$ which we'll use a bit later.
We'll now work on simplifying the integral $\int_{[Re^{i\frac{\pi}{4}},0]} e^{iz^2}dz$. We can immediately see that we can reverse the orientation of this integral and instead parameterize this integral using the curve $\gamma_2(r) = re^{i\frac{\pi}{4}}$ for $r \in [0, R]$. We can then compute

$$\int_{[Re^{i\frac{\pi}{4}},0]} e^{iz^2}dz = -\int_{\gamma_2} e^{iz^2}dz. \tag{33}$$

We can simplify the righthand side by computing that

$$\int_{\gamma_2} e^{iz^2}dz = \int_0^R e^{i\gamma(r)^2}\gamma'(r)dr \tag{34}$$

$$= \int_0^R e^{i(re^{i\pi/4})^2}(e^{i\pi/4})dr \tag{35}$$

$$= e^{i\pi/4}\int_0^R e^{i(r^2e^{i\pi/2}}dr \tag{36}$$

$$= e^{i\pi/4}\int_0^R e^{i(ir^2)}dr \tag{37}$$

$$= e^{i\pi/4}\int_0^R e^{-r^2}dr. \tag{38}$$

Therefore, taking the limit of 21 as R approaches ∞, we see that $\lim_{R\to\infty} I_R = 0$, and therefore

$$0 = \lim_{R\to\infty} \Big(\int_{-R}^{R} e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr + \int_{C_1(R)} e^{iz^2} dz \Big) \tag{39}$$

As shown before, the third integral on the righthand side approaches 0 in the limit, so that

$$0 = \lim_{R\to\infty} \Big(\int_{0}^{R} e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr \Big). \tag{40}$$

To conclude, we'll use the additive property of limits and the fact that $\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$. This means that

$$\begin{aligned} \int_0^\infty e^{ix^2} dx = \lim_{R\to\infty} \int_0^R e^{ix^2} dx &= \lim_{R\to\infty} e^{i\pi/4} \int_0^R e^{-r^2} dr && (41) \\ &= e^{i\pi/4} \int_0^\infty e^{-r^2} dr && (42) \\ &= e^{i\pi/4} \frac{\sqrt{\pi}}{2}. && (43) \end{aligned}$$

This allows us to conclude that

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \frac{\sqrt{\pi}}{2}. \tag{44}$$

Exercise 3. A&F 2.5.6. Evaluate the integral

$$I = \int_{\mathbb{R}} \frac{dx}{x^2+1} \tag{45}$$

using the contour $C(R)$ which is the closed semicircle in the upper half plane with endopoints $(-R, 0)$ and $(0, R)$.

Solution 3. Throughout, we assume that $R > 1$ for simplicity. Following the outliine of the previous, we can decompose the integral of $f(z) = 1/(z^2+1)$ over $C(R)$ as follows

$$\oint_{C(R)} \frac{dz}{z^2+1} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_1(R)} \frac{dz}{z^2+1}. \tag{46}$$

Here, $C_1(R)$ represents the rounded section of the upper half circle which we parameter as $\gamma(t) = Re^{i\theta}$ for $\theta \in [0, \pi]$. We can then bound the integral as in the the previous problem by

$$\left| \int_{C_1(R)} \frac{dz}{z^2+1} \right| \leq \pi R \cdot \left(\sup_{z \in C_1(R)} \left| \frac{1}{z^2+1} \right| \right) \tag{47}$$

since the curve $C_1(R)$ has length πR. We can compute the suprenum that noting that each $z = Re^{i\theta}$, so that

$$\left| \frac{1}{z^2+1} \right| \leq \frac{1}{||z^2| - |1||} \tag{48}$$

$$= \frac{1}{||R^2| - 1|} \tag{49}$$

$$= \frac{1}{|R^2 - 1|} \tag{50}$$

by the reverse triagle inequality and the fact that $|z| = \left| Re^{i\theta} \right| = |R|$. This shows that

$$\left| \int_{C_1(R)} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{|R^2 - 1|}. \tag{51}$$

Therefore, we have that

$$\lim_{R \to \infty} \int_{C_1(R)} \frac{dz}{z^2+1} = 0. \tag{52}$$

Next, we'll compute the integral $\oint_{C(R)} \frac{dz}{z^2+1}$ with the residue theorem. Since the contour $C(R)$ only contains the singularity i, we can write the integral as

$$\oint_{C(R)} \frac{dz}{z^2+1} = 2\pi i \text{Res}(f, i). \tag{53}$$

We can easily compute this residue as

$$\text{Res}(f, i) = \lim_{z \to i} (z - i)\left(\frac{1}{z^2+1}\right) = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}. \tag{54}$$

Therefore, we have that

$$\oint_{C(R)} \frac{dz}{z^2+1} = 2\pi i \mathrm{Res}(f,i) = \pi. \tag{55}$$

Now taking the limit as $R \to \infty$, we see that

$$\pi = \lim_{R\to\infty} \Big(\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_1(R)} \frac{dz}{z^2+1} \Big). \tag{56}$$

Since the rightmost integral vanishes in the limit, we have that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi. \tag{57}$$

Verification by real integration We can verify this using real integration since

$$\int_{-R}^{R} \frac{dx}{x^2+1} = \arctan(R) - \arctan(-R). \tag{58}$$

Taking the limit as $R \to \infty$, we see that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \lim_{R\to\infty} \Big[\arctan(R) - \arctan(-R) \Big] = \pi/2 - (-\pi/2) = \pi. \tag{59}$$

Exercise 4. A&F 3.3.5.

Solution 4. In order to find the coeffiecients of the Taylor-Laurent Series about 0 of $f(z) = e^{\frac{t}{2}(z-z^{-1})} = \sum_{n\in\mathbb{Z}} a_n z^n$, we use the formula

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}} dz \tag{60}$$

where γ is the unit circle parameterized as $\gamma(\theta) = e^{i\theta}$ for $\theta \in [-\pi, \pi]$. We can then simplify the integral as

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_\gamma \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}} dz &(61)\\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} \cdot ie^{i\theta} d\theta. &(62)\end{aligned}$$

We can simplify $e^{i\theta} - e^{-i\theta}$ as $2i\sin\theta$ and combine the terms $e^{i\theta}$ and $e^{i(n+1)\theta}$, so that

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} \cdot ie^{i\theta} d\theta = \frac{i}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(2i\sin\theta)}}{e^{in\theta}} d\theta \tag{63}$$

Combining the top and bottom halves of the integrand and canceling the i in front, we get that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta. \tag{64}$$

We can further simplify this using $e^{-ix} = \cos x - i\sin x$, which gives us

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta - t\sin\theta) d\theta + \frac{i}{2\pi}\int_{-\pi}^{\pi} \sin(n\theta - t\sin\theta) d\theta. \tag{65}$$

The key to the next equality is that$g(\theta) = n\theta - t\sin\theta$ is odd. We show this directly as

$$\begin{aligned} g(-\theta) = -n\theta - t\sin -\theta &= -(n\theta + t\sin -\theta) &(66)\\ &= -(n\theta - t\sin\theta) \quad (\sin\theta \text{ is odd }) &(67)\\ &= -g(\theta). &(68)\end{aligned}$$

Therefore, $\cos(g(\theta))$ is even and $\sin(g(\theta))$ is odd due to composition rules for odd and even functions. This means that

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos(n\theta - t\sin\theta) d\theta = 2\int_0^{\pi} \cos(n\theta - t\sin\theta) d\theta &(69)\\ &\int_{-\pi}^{\pi} \sin(n\theta - t\sin\theta) d\theta = 0 &(70)\end{aligned}$$

by integral theorems for even and odd functions. Therefore,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - t\sin\theta) d\theta. \tag{71}$$