Exercise 1. Consider the singular equation:

$$\epsilon \frac{d^2 u}{dx^2} + (1+x)^2 \frac{du}{dx} + u = 0$$

with $u(0) = u(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) Obtain the leading order uniform solution using the WKB method.

(b) Plot the uniform solution for $\epsilon = 0.01$, 0.05, 0.1, 0.2.

Solution 1. We'll write $u(x)$ using a WKB expansion as follows

$$u(x) = \exp\left(\frac{S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots}{\epsilon} \right).$$

Plugging this expansion into the differential equation of interest, we have

$$\epsilon \left(\frac{S_{0x}^2(x)}{\epsilon^2} + \frac{2S_{0x}(x)S_{1x}}{\epsilon} + \cdots \frac{S_{0xx}}{\epsilon} + \cdots \right) u(x) + (1+x)^2 \left(\frac{S_{0x}}{\epsilon} + S_{1x}(x) + \cdots \right) u(x) + u(x) = 0.$$

using formula (360) from notes to compute the derivatives of u. Collecting powers of ϵ and removing $u(x)$ since all terms contain u we have the following hierarchy of equations.

$$\begin{aligned} O(\epsilon^{-1}): &\quad S_{0x}^2(x) + (1+x)^2 S_{0x}(x) = 0 \\ O(1): &\quad S_{0xx} + 2S_{0x}(x)S_{1x}(x) + (1+x)^2 S_{1x}(x) + 1 = 0. \end{aligned}$$

This leaves two possible solutions for the $O(\epsilon^{-1})$ term

$$S_{0x} = 0 \implies S_0 = C$$
$$S_{0x} = -(1+x)^2 \implies S_0 = -\frac{(1+x)^3}{3} + C.$$

The first case $S_{0x} = 0$ reduces the $O(1)$ equation to

$$(1+x)^2 S_{1x}(x) + 1 = 0,$$

so that

$$S_{1x} = -(1+x)^{-2} \implies S_1(x) = \frac{1}{1+x} + C.$$

Therefore, our first solution is given by

$$u(x) = C_1 \exp\left(\frac{1}{1+x} \right).$$

We can now consider the second case which has $O(1)$ equation

$$2(1+x) - (1+x)^2 S_{1x} + 1 = 0$$

This has solution

$$S_1(x) = -(1+x)^{-1} - \ln(1+x)^2,$$

which when plugged into the expansion gives

$$u(x) = \frac{C_2}{(1+x)^2} \exp\left(-\frac{1}{1+x} - \frac{1}{3\epsilon}(1+x)^3\right)$$

Our combined uniform solution is then

$$u(x) = C_1 \exp\left(\frac{1}{1+x}\right) + \frac{C_2}{(1+x)^2} \exp\left(-\frac{1}{1+x} - \frac{1}{3\epsilon}(1+x)^3\right).$$

We can use the boundary conditions to solve for the constants of interest. We require that

$$\begin{aligned} u(0) &= C_1 e + C_2 e^{-1} \exp\left(-\frac{1}{3\epsilon}\right) = 1 \\ u(1) &= C_1 e^{\frac{1}{2}} + \frac{C_2}{4} e^{-\frac{1}{2}} \exp\left(-\frac{8}{3\epsilon}\right) = 1. \end{aligned}$$

Starting with the first equation, we write that

$$C_2 = F e^{1+\frac{1}{3\epsilon}},$$

which allows us to simplfy so that

$$\begin{aligned} 1 &= C_1 e + F \\ 1 &= C_1 e^{-\frac{1}{2}} + \frac{F}{4} \exp\left(\frac{1}{2} - \frac{7}{3}\epsilon\right). \end{aligned}$$

Noticing that $\exp\left(\frac{1}{2} - \frac{7}{3}\epsilon\right)$ is appoximately 0 in the small ϵ limit tells us that

$$\begin{aligned} C_1 &= e^{-\frac{1}{2}} \\ C_2 &= (1 - e^{\frac{1}{2}}) e^{1+\frac{1}{3\epsilon}}. \end{aligned}$$

This gives solution

$$u(x) = \exp\left(\frac{1}{1+x} - \frac{1}{2}\right) + \frac{1-e^{1/2}}{(1+x)^2} \exp\left(-\frac{1}{1+x} + 1 - \frac{1}{3\epsilon}\left[(1+x)^3 - 1\right]\right).$$

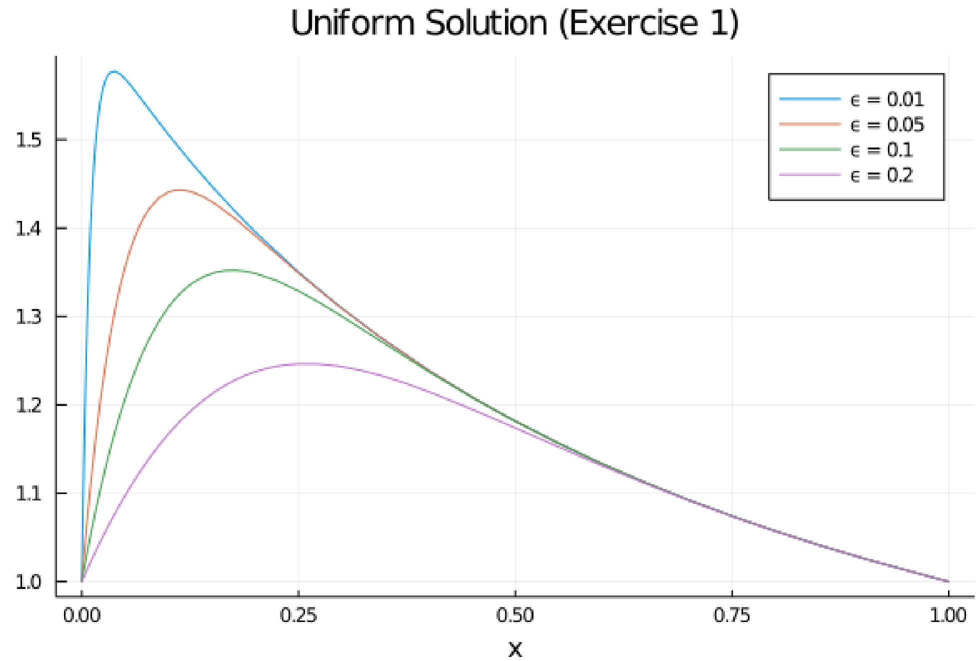

Figure 1: Uniform solution for various ϵ