Exercise 1. Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz, \tag{1}$$

where C is the unit circle centered at the origin with $f(z)$ given below.

(a) $\frac{z+1}{2z^3-3z^2-2z}$.

(b) $\frac{\cosh(z^{-1})}{z}$.

(c) $\frac{e^{-\cosh(z)}}{4z^2+\pi^2}$.

(d) $\frac{\ln(z+2)}{2z+1}$, $\quad -\pi < \arg(z+2) \leq \pi$

(e) $\frac{z+z^{-1}}{z(2z-(2z)^{-1})}$

Solution 1.
(a) We begin by factoring the denominator of the integrand

$$f(z) = \frac{z+1}{2z^3-3z^2-2z} = \frac{z+1}{z(2z+1)(z-2)}.$$

This function has singularities in C at $z = -1/2, 0$, but is otherwise analytic in C. By the Residue theorem, computing the integral is as simple as noting that

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}(f; -1/2) + \text{Res}(f; 0).$$

All of the poles of this function are simple, so we can compute the residues as follows

$$\begin{aligned}\text{Res}(f; -1/2) &= \lim_{z\to -1/2}(z+\frac{1}{2})\left(\frac{z+1}{z(2z+1)(z-2)}\right)\\ &= \lim_{z\to -1/2}\frac{z+1}{2z(z-2)} = -1/5\end{aligned}$$

We can compute the other residue as

$$\begin{aligned}\text{Res}(f; 0) &= \lim_{z\to 0}(z)\left(\frac{z+1}{z(2z+1)(z-2)}\right)\\ &= \lim_{z\to 0}\frac{z+1}{(2z+1)(z-2)} = -1/2\end{aligned}$$

We can now conclude that

$$\frac{1}{2\pi i}\oint_C \frac{z+1}{2z^3-3z^2-2z}dz = -7/10$$

(b) We note that the only singularity of $f(z) = \frac{\cosh(z^{-1})}{z}$ in C occurs at 0, so our desired integral is equal to $\text{Res}(f, 0)$. Writing the numerator in terms of its Taylor-Laurent series around 0, we see that

$$\cosh(z^{-1}) = \sum_{n=0}^{\infty} \frac{z^{-2n}}{(2n)!},$$

we then have that

$$f(z) = \frac{\cosh(z^{-1})}{z} = \sum_{n=0}^{\infty} \frac{z^{-2n-1}}{(2n)!}.$$

From this, we can see that the residue is given by the coefficient of this series when $n = 0$, so that

$$\frac{1}{2\pi i} \oint_C \frac{\cosh(z^{-1})}{z} dz = \text{Res}(f; 0) = 1$$

(c) The only singularities of

$$f(z) = \frac{e^{-\cosh(z)}}{4z^2 + \pi^2} = \frac{e^{-\cosh(z)}}{4(z - i\frac{\pi}{2})(z + i\frac{\pi}{2})},$$

are given by $\pm i\frac{\pi}{2}, 0$. None of these occur in C. Therefore, the function is analytic within and on C and we have

$$\frac{1}{2\pi i}\oint_C \frac{e^{-\cosh(z)}}{4z^2 + \pi^2} dz = 0$$

(d) Since we've restricted to $-\pi < \arg(z+2) \leq \pi$, the function $f(z) = \frac{\ln(z+2)}{2z+1}$ only has a singularity at $z = -\frac{1}{2}$. Therefore, the integral over C can be computed using only the residue of this function at $-\frac{1}{2}$. We compute this as

$$\begin{aligned}
\operatorname{Res}(f; -1/2) &= \lim_{z \to -1/2} (z + 1/2) \left(\frac{\ln(z+2)}{2z+1} \right) \\
&= \lim_{z \to -1/2} \frac{\ln(z+2)}{2} = \frac{\ln(3/2)}{2}.
\end{aligned}$$

Therefore, our integral is simply

$$\frac{1}{2\pi i} \oint_C \frac{\ln(z+2)}{2z+1} dz = \operatorname{Res}(f; -1/2) = \frac{\ln(3/2)}{2}.$$

(e) We begin by simplifying the integrand

$$f(z) = \frac{z + z^{-1}}{z(2z - (2z)^{-1})} = \frac{z + z^{-1}}{2z^2 - \frac{1}{2}} = \frac{z + z^{-1}}{2(z - \frac{1}{2})(z + \frac{1}{2})}.$$

We can then see that this function has singularities at $0, \pm\frac{1}{2}$, all of which are contained in C. We can then compute the residues at these various points using that they are all simple poles. Beginning with $\frac{1}{2}$,

$$\begin{aligned}
\operatorname{Res}(f; 1/2) &= \lim_{z \to 1/2} (z - 1/2) \left(\frac{z + z^{-1}}{2(z - \frac{1}{2})(z + \frac{1}{2})} \right) \\
&= \lim_{z \to 1/2} \frac{z + z^{-1}}{2(z + \frac{1}{2})} \\
&= 5/4
\end{aligned}$$

Next,

$$\begin{aligned}
\operatorname{Res}(f; -1/2) &= \lim_{z \to -1/2} (z + 1/2) \left(\frac{z + z^{-1}}{2(z - \frac{1}{2})(z + \frac{1}{2})} \right) \\
&= \lim_{z \to -1/2} \frac{z + z^{-1}}{2(z - \frac{1}{2})} \\
&= 5/4
\end{aligned}$$

We can compute the final residue as

$$\begin{aligned}
\operatorname{Res}(f; 0) &= \lim_{z \to 0} z \left(\frac{z}{2z^2 - \frac{1}{2}} + \frac{z^{-1}}{2z^2 - \frac{1}{2}} \right) \\
&= \lim_{z \to 0} \left(\frac{z^2}{2z^2 - \frac{1}{2}} + \frac{1}{2z^2 - \frac{1}{2}} \right) \\
&= 0 + 2.
\end{aligned}$$

We can then compute the integral as

$$\frac{1}{2\pi i} \oint_C f(z)dz = \operatorname{Res}(f; 1/2) + \operatorname{Res}(f; -1/2) + \operatorname{Res}(f; 0) = 9/2.$$

Exercise 2. Show that

$$I = \int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{\pi}{2}(1-e^{-1})$$

Solution 2. We instead will work with the integrand $f(z) = \frac{e^{iz}}{z(z^2+1)}$ and compute the integral

$$J = \int_{-\infty}^{\infty} \frac{e^{iz}}{z(z^2+1)}dz$$

since $\mathrm{Im}(J) = I$. We will compute the integral J with the contour $\Gamma = C_\epsilon + C_R + [-R, -\epsilon] + [\epsilon, R]$, where C_ϵ is the upper half circle centered at 0 with radius ϵ oriented clockwise and C_R the upper half circle centered at 0 with radius R and oriented counter-clockwise. We then have that

$$\int_\Gamma f(z)dz = \int_{C_\epsilon} f(z)dz + \int_\epsilon^R f(z)dz + \int_{C_R} f(z)dz + \int_{-R}^{-\epsilon} f(z)dz.$$

We can compute the integral of the left hand side by Residue theorem since the contour Γ contains the singularity of $f(z)$ at i, so that

$$\int_\Gamma f(z)dz = 2\pi i \mathrm{Res}(f, i).$$

We can compute this residue as

$$\begin{aligned}
\mathrm{Res}(f; i) &= \lim_{z\to i}(z-i)\left(\frac{e^{iz}}{z(z+i)(z-i)}\right) \\
&= \lim_{z\to i} \frac{e^{iz}}{z(z+i)} \\
&= -\frac{e^{-1}}{2}.
\end{aligned}$$

Therefore,

$$-\pi i e^{-1} = \int_{C_\epsilon} f(z)dz + \int_\epsilon^R f(z)dz + \int_{C_R} f(z)dz + \int_{-R}^{-\epsilon} f(z)dz.$$

As $\epsilon \to 0$, we can compute the integral over C_ϵ as

$$\int_{C_\epsilon} f(z)dz = -\pi i \mathrm{Res}(f, 0).$$

We can compute this residue as

$$\begin{aligned}
\mathrm{Res}(f; 0) &= \lim_{z\to 0} z \cdot \frac{e^{iz}}{z(z^2+1)} \\
&= \lim_{z\to 0} \frac{e^{iz}}{z^2+1} = 1.
\end{aligned}$$

Therefore in the limit as $\epsilon \to 0$, we have

$$\pi i(1 - e^{-1}) = -\pi i e^{-1} + \pi i = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz.$$

By Jordan's lemma, the integral over C_R goes to 0 in the limit as $R \to \infty$, which leaves us with

$$\pi i(1 - e^{-1}) = \int_{-\infty}^{\infty} f(z)dz.$$

We now note that this implies

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx = \pi(1 - e^{-1})$$

due to our choice of $f(z)$. This integrand of the above integral is even, so we can conclude that

$$\int_{0}^{\infty} \frac{\sin x}{x(x^2+1)} dx = \frac{\pi}{2}(1 - e^{-1}).$$

Exercise 3. Consider the function

$$f(z) = \ln(z^2 - 1),$$

made single-valued by restricting the angles in the following ways, with

$$z_1 = z - 1 = r_1 e^{i\theta_1}, \quad z_2 = z + 1 = r_2 e^{i\theta_2}$$

(a) $-\frac{3\pi}{2} < \theta_1 \leq \frac{\pi}{2}, \quad -\frac{3\pi}{2} < \theta_2 \leq \frac{\pi}{2}$

(b) $0 < \theta_1 \leq 2\pi, \quad 0 < \theta_2 \leq 2\pi$

(c) $-\pi < \theta_1 \leq \pi, \quad 0 < \theta_2 \leq 2\pi$

Find where the branch cuts are for each case by locating where the function is discontinuous. Use the AB tests and show your results.

Solution 3. We begin by simplifying $f(z)$ in terms of z_1 and z_2, so that

$$\begin{aligned} f(z) &= \ln(z^2 - 1) = \ln(z-1) + \ln(z+1) \\ &= \ln(z_1) + \ln(z_2) \\ &\ln(r_1 r_2) + i(\theta_1 + \theta_2). \end{aligned}$$

(a) Let's pick a point $A = iy - \epsilon$ with $y > 0$ and ϵ small which is slightly to the left of a point on the positive imaginary axis. We can then compute that

$$z_1(A) = A - 1 = -(1+\epsilon) + iy \quad z_2(A) = A + 1 = 1 - \epsilon + iy.$$

We can additionally find the angle using the arctangent, so that

$$\theta_1(A) = \arctan(-\frac{y}{1+\epsilon}) - \pi, \quad \theta_2(A) = \arctan(\frac{y}{1-\epsilon}), \tag{2}$$

where we've added a constant to shift the angle into the proper range. Then for $B = iy + \epsilon$ which is slightly to the right of the positive imaginary axis. We can compute that

$$\begin{aligned} z_1(B) = B - 1 = -(1-\epsilon) + iy \quad z_2(B) = B + 1 = 1 + \epsilon + iy \\ \theta_1(B) = \arctan(-\frac{y}{1-\epsilon}) - \pi, \quad \theta_2(B) = \arctan(\frac{y}{1+\epsilon}). \end{aligned}$$

In the limit as $\epsilon \to 0$, the various magnitudes r converge, and we can see that

$$\lim_{\epsilon \to 0} \theta_1(A) + \theta_2(A) = \lim_{\epsilon \to 0} \theta_1(B) + \theta_2(B),$$

so the function is continuous along the positive imaginary axis.

(b)

Case 1: $x \geq 1$ Now picking $A = x + i\epsilon$ for $x \geq 1$ and $\epsilon > 0$ which is slightly above the real axis. We can compute that

$$z_1(A) = A - 1 = x - 1 + i\epsilon, \quad z_2(A) = A + 1 = x + 1 + i\epsilon.$$

We can compute the angles $\theta_1(A)$ and $\theta_2(A)$ as being slightly above 0 due to the side of the branch cut they are on. Similarly, for a point $B = x - i\epsilon$ for $x \geq 1$ and $\epsilon > 0$ which is slightly below the real axis. We see that the angle $\theta_1(B)$ and $\theta_2(B)$ are slightly below 2π. Taking the limit as $\epsilon \to 0$ from above, we have that

$$\lim_{\epsilon \to 0^+} \theta_1(A) + \theta_2(A) = 0 + 0 \neq 2\pi + 2\pi = \lim_{\epsilon \to 0^+} \theta_1(B) + \theta_2(B),$$

so $f(z)$ is discontinuous along the real axis where $x > 1$.

Case 2: $0 < x < 1$ Now when $A = x + i\epsilon$, we have the same $z_1(A)$, $z_2(A)$, $z_1(B)$, and $z_2(B)$ but now $z_1(A)$ and $z_1(B)$ have negative real part so that in the limit

$$\lim_{\epsilon \to 0^+} \theta_1(A) + \theta_2(A) = \pi + 0 \neq \pi + 2\pi = \lim_{\epsilon \to 0^+} \theta_1(B) + \theta_2(B).$$

The case where $x = 1$ is similar, but instead $z_1(A)$ and $z_1(B)$ have 0 real part and $\theta_1(A)$ and $\theta_1(B)$ converge to $\pi/2$ and $3\pi/2$ as $z_1(A)$ and $z_1(B)$ live on the positive and negative imaginary axes respectively , so the function $f(z)$ is discontinuous along the positive real axis.

(c) In this case, we have one branch cut along the negative real axis and another along the positive axis. We'll split into a couple of cases.
Picking $A = x + i\epsilon$ for $\epsilon > 0$ and $x \in \mathbb{R}$ gives a point which is slightly above the real axis. We have that

$$z_1(A) = A - 1 = x - 1 + i\epsilon, \quad z_2(A) = A + 1 = x + 1 + i\epsilon. \tag{3}$$

For $B = x - i\epsilon$, we have a point slightly below the real axis, so that

$$z_1(B) = B - 1 = x - 1 - i\epsilon, \quad z_2(B) = B + 1 = x + 1 - i\epsilon.$$

Case 1: $x > 1$ We have that in the limit as $\epsilon \to 0$, $\theta_1(A) = 0$ since $z_1(A)$ is slightly above the positive real axis. Also, we have that in the same limit, $\theta_2(A) = 0$ since $z_2(A)$ is slightly above the real axis as well. For similar reasons, we have that $\theta_1(B) = 0$ and $\theta_2(B) = 2\pi$. This shows that the function is discontinuous along $x \geq 1$

Case 2: $x < -1$ In this case, the four points of interest z_1, z_2 for A and B all have negative real part, so that in the limit as $\epsilon \to 0$,

$$\begin{aligned} \theta_1(A) = \pi \quad \theta_2(A) = \pi \\ \theta_1(B) = -\pi \quad \theta_2(B) = \pi. \end{aligned}$$

This shows that the function is also discontinuous along $x \leq -1$.

Case 3: $x \in (-1, 1)$ In this case, we have that $z_1(A)$ and $z_1(B)$ have negative real part and $z_2(A)$ and $z_2(B)$ have positive real part, so that in the limit

$$\begin{aligned} \theta_1(A) = \pi \quad \theta_2(A) = 0 \\ \theta_1(B) = -\pi \quad \theta_2(B) = 2\pi. \end{aligned}$$

This shows that the function is actually continuous on $(-1, 1)$.

Case 4: $x = 1$ In this case, $z_1(A)$ and $z_1(B)$ have zero real part and $z_2(A)$ and $z_2(B)$ have positive real part, so that in the limit

$$\begin{aligned} \theta_1(A) = \pi/2 \quad \theta_2(A) = 0 \\ \theta_1(B) = -\pi/2 \quad \theta_2(B) = 2\pi. \end{aligned}$$

So the function is discontinuous here.

Case 5: $x = -1$ In this case, we have that $z_1(A)$ and $z_1(B)$ have negative real part and $z_2(A)$ and $z_2(B)$ have zero real part, so that in the limit

$$\theta_1(A) = \pi \quad \theta_2(A) = \pi/2$$
$$\theta_1(B) = -\pi \quad \theta_2(B) = 3\pi/2.$$

So the function is discontinuous here.
This shows that the function is continuous along $(-1, 1)$ and that $f(z)$ is discontinuous along the real axis where $|x| \geq 1$.