AMATH 584A: Applied Linear Algebra

Marlin Figgins

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Introduction

Table of Contents

1. Lecture 1: Overview

This course will be entirely about the problem of $Ax = b$. That is, we're concerning with linear systems. In fact, many problems are of this form. In the age of data science, these variables A and x can get huge quickly. In your typical linear algebra classes, you learn to solve this with Gaussian elimination, but the reality is that this is one of the slowest ways you can solve this problem.

1.1 Matrix Decompositions

Matrix decompositions allow us to solve the problem $Ax = b$ much faster. Let's start with the case of complex square matrices $A \in \mathbb{C}^{n\times n}$.

To solve this problem with Gaussian Elimination, the cost would be on the order of $O(n^3)$. This is fine for small matrices, but immagine you're dealing with large matrices and this begins to blow up in computation time rather quickly.

1.1.1 LU decomposiiton

The LU decomposition allows us to represent our matrix A as

$$A = LU \tag{1.1}$$

where L is lower triangular and U is upper triangular. Our problem becomes

$$Ax = b \tag{1.2}$$
$$LUx = b \tag{1.3}$$
$$Ux = y \tag{1.4}$$
$$Ly = b \tag{1.5}$$

This allows us to use forward and back substituion individually which are of order $O(n^2)$ to solve this probelm. This LU decompoisition already gives a saving of order of n. This is all well and good, but what does it take to get an LU decomposition?

1.1.2 QR decomposition

We want to express our matrix A in the form

$$A = QR \tag{1.6}$$

where Q is a unitary matrix and R is upper triangular. Solving $Ax = b$ with this decomposiiton gives us,

$$QRx = b \tag{1.7}$$
$$Rx = y \tag{1.8}$$
$$Qy = b \tag{1.9}$$
$$Q^T[Qy = b] \tag{1.10}$$
$$y = Q^T b \tag{1.11}$$

1.1.3 Eigenvalue Decomposition

We can write the eigenvale decomposition as

$$A = V\Lambda V^{-1} \tag{1.12}$$

Using this to solve $Ax = b$, we get that

$$V^{-1}[V\Lambda V^{-1}x = b] \tag{1.13}$$
$$\Lambda y = V^{-1}b \tag{1.14}$$

Since Λ is diagonal, the answer is very clear here.

1.1.4 Singular Value Decomposition

The singular value decomposition is one of the most important decomposition algorithms. We decompose A as

$$A = U\Sigma V^* \tag{1.15}$$

Solving $Ax = b$,

$$U\Sigma V^* x = b \tag{1.16}$$
$$\Sigma V^* x = U^* b \tag{1.17}$$
$$\Sigma \hat{x} = \hat{b} \tag{1.18}$$