

Exercise 1. A&F 2.5.1. Evaluate $\oint_{\gamma} f(z)dz$ where γ is the unit circle centered at the origin for the following functions f .

Solution 1.

(a) $f(z) = e^{iz}$. The function $f(z)$ is entire as it is the composition of two entire functions e^w and iz . Its derivative is $f'(z) = ie^{iz}$. By Cauchy's Theorem, this means that for the closed curve γ , we have

$$\oint_{\gamma} e^{iz} dz = 0. \quad (1)$$

(b) $f(z) = e^{z^2}$. Once again $f(z)$ is entire as it is the composition of two entire functions e^w and z^2 . By Cauchy's Theorem, this means

$$\oint_{\gamma} e^{z^2} dz = 0. \quad (2)$$

(c) $f(z) = \frac{1}{z-1/2}$. The function $f(z)$ is analytic except at $z = \frac{1}{2}$ which is contained in γ , so we cannot use Cauchy's theorem. We can instead use the residue theorem. Writing f as its Taylor-Laurent series about $z_0 = \frac{1}{2}$,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - 1/2)^n = 1(z - 1/2)^{-1}. \quad (3)$$

Here we can see that $a_n = 0$ for all $n \neq -1$ and $a_{-1} = 1$. Therefore, by the Residue theorem, we have

$$\oint_{\gamma} f(z) dz = 2\pi i a_{-1} = 2\pi i. \quad (4)$$

(d) $f(z) = \frac{1}{z^2-4}$. The function $f(z)$ is analytic except at $z = 2, -2$, neither of which are in the contour γ . Since $f(z)$ is analytic on and within γ , we can apply Cauchy's theorem, so that

$$\oint_{\gamma} \frac{1}{z^2-4} dz = 0. \quad (5)$$

(e) $f(z) = \frac{1}{2z^2+1}$. This function is analytic except at $z_{\pm} = i\frac{\sqrt{2}}{2}, -i\frac{\sqrt{2}}{2}$ which are contained in the contour γ . We can then write

$$f(z) = \frac{1}{2z^2+1} = \frac{1}{2(z - i\frac{\sqrt{2}}{2})(z + i\frac{\sqrt{2}}{2})}. \quad (6)$$

We'll now compute the residues at z_{\pm} using the following formula for the residue of f at z_0

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z). \quad (7)$$

For z_+ , we can compute

$$\text{Res}(f, z_+) = \lim_{z \rightarrow z_+} \left(z - i\frac{\sqrt{2}}{2} \right) f(z) \quad (8)$$

$$= \lim_{z \rightarrow z_+} \frac{1}{2(z + i\frac{\sqrt{2}}{2})} \quad (9)$$

$$= \frac{1}{2\left(i\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} \quad (10)$$

$$= \frac{1}{2i\sqrt{2}} = \frac{\sqrt{2}}{4i} \quad (11)$$

Similarly, we can compute the residue at z_-

$$\text{Res}(f, z_-) = \lim_{z \rightarrow z_-} \left(z + i\frac{\sqrt{2}}{2} \right) f(z) \quad (12)$$

$$= \lim_{z \rightarrow z_-} \frac{1}{2(z - i\frac{\sqrt{2}}{2})} \quad (13)$$

$$= \frac{1}{2\left(-i\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)} \quad (14)$$

$$= -\frac{1}{2i\sqrt{2}} = -\frac{\sqrt{2}}{4i} \quad (15)$$

$$= -\text{Res}(f, z_+) \quad (16)$$

We can then use the residue theorem to compute the integral of f over γ as follows

$$\oint_{\gamma} \frac{1}{2z^2 + 1} dz = 2\pi i (\text{Res}(f, z_+) + \text{Res}(f, z_-)) = 0 \quad (17)$$

(f) $\sqrt{z-4}$, $0 \leq \arg z < 2\pi$. With this specification on the argument, this function is analytic with derivative $\frac{1}{z-4}$ except on the real line for $x > 4$. Therefore, the integral of this function over the unit circle is 0 by Cauchy's Theorem since it is analytic within and on the unit circle

$$\oint_{\gamma} \sqrt{z-4} dz = 0. \quad (18)$$

Exercise 2. A&F 2.5.5. Evaluate the integral

$$\int_0^\infty e^{ix^2} dx \quad (19)$$

using the contour $C(R)$ which is the closed circular section in the upper half plane with boundary points $(0, 0)$, $(R, 0)$, and $Re^{i\frac{\pi}{4}}$.

Solution 2. We begin by considering the integral

$$I_R = \oint_{C(R)} e^{iz^2} dz. \quad (20)$$

Since the function $f(x) = e^{iz^2}$ is analytic in the entire complex plane, we have that $I_R = 0$ by Cauchy Theorem. Breaking the contour $C(R)$ into three parts, we see that

$$\oint_{C(R)} e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_{[Re^{i\frac{\pi}{4}}, 0]} e^{iz^2} dz + \int_{C_1(R)} e^{iz^2} dz. \quad (21)$$

Here, the first integral on the righthand side is the integral on the real line from $(0, R)$, the second is the integral on the straight line from $[Re^{i\frac{\pi}{4}}, 0]$, the third integral is over the circular section between $(R, 0)$ and $Re^{i\frac{\pi}{4}}$. We'll begin by trying to estimate the third integral.

We'll start by working with the contour $C_1(R, \theta_0)$ which is the circular sector between $Re^{i\theta_0}$ and $Re^{i\frac{\pi}{4}}$ which we'll parameterize by the function $\gamma_{R, \theta_0}(t) = Re^{it}$ for $t \in [\theta_0, \frac{\pi}{4}]$. Then, we can bound the integral over $C_1(R, \theta_0)$ by

$$\left| \int_{C_1(R, \theta_0)} e^{iz^2} dz \right| \leq \text{length}(C_1(R, \theta_0)) \cdot \sup_{z \in C_1(R, \theta_0)} |e^{iz^2}| \quad (22)$$

$$= \left(\frac{\pi}{4} - \theta_0 \right) \sup_{z \in C_1(R, \theta_0)} |e^{iz^2}| \quad (23)$$

$$\leq \frac{\pi}{4} \sup_{z \in C_1(R, \theta_0)} |e^{iz^2}|. \quad (24)$$

Now we'll compute $\sup_{z \in C_1(R, \theta_0)} |e^{iz^2}|$. Writing $z \in C_1(R, \theta_0)$ in polar exponential form as $z = Re^{i\theta}$ for some $\theta \in [\theta_0, \frac{\pi}{4}]$, we can simplify the exponent of $f(z)$ as

$$iz^2 = iR^2 e^{i2\theta} = iR^2 (\cos 2\theta + i \sin 2\theta) \quad (25)$$

$$= -R^2 (\sin 2\theta + i \cos 2\theta). \quad (26)$$

Exponentiating, we can compute $|f(z)|$ as

$$|e^{iz^2}| = |e^{-R^2 (\sin 2\theta + i \cos 2\theta)}| = |e^{-R^2 \sin 2\theta}| |e^{-iR^2 \cos 2\theta}| \quad (27)$$

$$= |e^{-R^2 \sin 2\theta}|, \quad (28)$$

where $\left| e^{-iR^2 \cos 2\theta} \right| = 1$ since $R^2 \cos 2\theta$ is real. Since $\sin(x) > \frac{2x}{\pi}$ on $x \in [0, \frac{\pi}{2}]$, we have that

$$\sup_{z \in C_1(R, \theta_0)} \left| e^{iz^2} \right| \leq \sup_{\theta \in [\theta_0, \frac{\pi}{4}]} \left| e^{-R^2 \frac{4\theta}{\pi}} \right| = e^{-R^2 \frac{4\theta_0}{\pi}}. \quad (29)$$

We obtain this supremum since the function $e^{-R^2 \frac{4\theta}{\pi}}$ is positive and monotonically decreasing in θ . Therefore, we can see that

$$\left| \int_{C_1(R, \theta_0)} e^{iz^2} dz \right| \leq \frac{\pi}{4} e^{-R^2 \frac{4\theta_0}{\pi}}. \quad (30)$$

We can see that since $C_1(R, \theta_0)$ becomes similar to $C_1(R)$ as $\theta_0 \rightarrow 0$ then

$$\lim_{R \rightarrow \infty} \left| \int_{C_1(R)} f(z) dz \right| = \lim_{R \rightarrow \infty} \lim_{\theta_0 \rightarrow 0} \left| \int_{C_1(R, \theta_0)} e^{iz^2} dz \right| \leq \lim_{R \rightarrow \infty} \lim_{\theta_0 \rightarrow 0} \frac{\pi}{4} e^{-R^2 \frac{4\theta_0}{\pi}}. \quad (31)$$

Since the righthand most function is monotonically decreasing and bounded by 0 in both R and θ_0 , we can re-arrange the limits and show that

$$\lim_{R \rightarrow \infty} \left| \int_{C_1(R)} e^{iz^2} dz \right| = \lim_{R \rightarrow \infty} \lim_{\theta_0 \rightarrow 0} \left| \int_{C_1(R, \theta_0)} e^{iz^2} dz \right| = 0. \quad (32)$$

This implies that $\lim_{R \rightarrow \infty} \int_{C_1(R)} e^{iz^2} dz = 0$ which we'll use a bit later.

We'll now work on simplifying the integral $\int_{[Re^{i\frac{\pi}{4}}, 0]} e^{iz^2} dz$. We can immediately see that we can reverse the orientation of this integral and instead parameterize this integral using the curve $\gamma_2(r) = re^{i\frac{\pi}{4}}$ for $r \in [0, R]$, so that

$$\int_{[Re^{i\frac{\pi}{4}}, 0]} e^{iz^2} dz = - \int_{\gamma_2} e^{iz^2} dz. \quad (33)$$

We can simplify the righthand side by computing that

$$\int_{\gamma_2} e^{iz^2} dz = \int_0^R e^{i\gamma(r)^2} \gamma'(r) dr \quad (34)$$

$$= \int_0^R e^{i(re^{i\pi/4})^2} (e^{i\pi/4}) dr \quad (35)$$

$$= e^{i\pi/4} \int_0^R e^{i(r^2 e^{i\pi/2})} dr \quad (36)$$

$$= e^{i\pi/4} \int_0^R e^{i(ir^2)} dr \quad (37)$$

$$= e^{i\pi/4} \int_0^R e^{-r^2} dr. \quad (38)$$

Taking the limit of 21 as R approaches ∞ , we then see $\lim_{R \rightarrow \infty} I_R = 0$, and therefore

$$0 = \lim_{R \rightarrow \infty} \left(\int_{-R}^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr + \int_{C_1(R)} e^{iz^2} dz \right) \quad (39)$$

As shown before, the third integral on the righthand side approaches 0 in the limit, so that

$$0 = \lim_{R \rightarrow \infty} \left(\int_0^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr \right). \quad (40)$$

To conclude, we'll use the additive property of limits and the fact that $\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$ in order to show

$$\int_0^\infty e^{ix^2} dx = \lim_{R \rightarrow \infty} \int_0^R e^{ix^2} dx = \lim_{R \rightarrow \infty} e^{i\pi/4} \int_0^R e^{-r^2} dr \quad (41)$$

$$= e^{i\pi/4} \int_0^\infty e^{-r^2} dr \quad (42)$$

$$= e^{i\pi/4} \frac{\sqrt{\pi}}{2}. \quad (43)$$

This allows us to conclude that

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \frac{\sqrt{\pi}}{2}. \quad (44)$$

Exercise 3. A&F 2.5.6. Evaluate the integral

$$I = \int_{\mathbb{R}} \frac{dx}{x^2 + 1} \quad (45)$$

using the contour $C(R)$ which is the closed semicircle in the upper half plane with endpoints $(-R, 0)$ and $(0, R)$.

Solution 3. Throughout, we assume that $R > 1$ for simplicity. Following the outline of the previous problem, we can decompose the integral of $f(z) = 1/(z^2 + 1)$ over $C(R)$ as follows

$$\oint_{C(R)} \frac{dz}{z^2 + 1} = \int_{-R}^R \frac{dx}{x^2 + 1} + \int_{C_1(R)} \frac{dz}{z^2 + 1}. \quad (46)$$

Here, $C_1(R)$ represents the rounded section of the upper half circle which we parameter as $\gamma(t) = Re^{i\theta}$ for $\theta \in [0, \pi]$. We can then bound the integral as in the the previous problem by

$$\left| \int_{C_1(R)} \frac{dz}{z^2 + 1} \right| \leq \pi R \cdot \left(\sup_{z \in C_1(R)} \left| \frac{1}{z^2 + 1} \right| \right) \quad (47)$$

since the curve $C_1(R)$ has length πR . We can compute the supremum that noting that each $z = Re^{i\theta}$, so that

$$\left| \frac{1}{z^2 + 1} \right| \leq \frac{1}{||z^2| - |1||} \quad (48)$$

$$= \frac{1}{||R^2| - 1|} \quad (49)$$

$$= \frac{1}{|R^2 - 1|} \quad (50)$$

by the reverse triangle inequality and the fact that $|z| = |Re^{i\theta}| = |R|$. This shows that

$$\left| \int_{C_1(R)} \frac{dz}{z^2 + 1} \right| \leq \frac{\pi R}{|R^2 - 1|}. \quad (51)$$

Therefore, we have that

$$\lim_{R \rightarrow \infty} \int_{C_1(R)} \frac{dz}{z^2 + 1} = 0. \quad (52)$$

Next, we'll compute the integral $\oint_{C(R)} \frac{dz}{z^2 + 1}$ using the residue theorem. Since the contour $C(R)$ only contains the singularity i , we can write the integral as

$$\oint_{C(R)} \frac{dz}{z^2 + 1} = 2\pi i \text{Res}(f, i). \quad (53)$$

We can easily compute this residue as

$$\text{Res}(f, i) = \lim_{z \rightarrow i} (z - i) \left(\frac{1}{z^2 + 1} \right) = \lim_{z \rightarrow i} \frac{1}{z + i} = \frac{1}{2i}. \quad (54)$$

Therefore, we have that

$$\oint_{C(R)} \frac{dz}{z^2 + 1} = 2\pi i \operatorname{Res}(f, i) = \pi. \quad (55)$$

Now taking the limit as $R \rightarrow \infty$, we see that

$$\pi = \lim_{R \rightarrow \infty} \left(\int_{-R}^R \frac{dx}{x^2 + 1} + \int_{C_1(R)} \frac{dz}{z^2 + 1} \right). \quad (56)$$

Since the rightmost integral vanishes in the limit, we have that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi. \quad (57)$$

Verification by real integration We can verify this using real integration since

$$\int_{-R}^R \frac{dx}{x^2 + 1} = \arctan(R) - \arctan(-R). \quad (58)$$

Taking the limit as $R \rightarrow \infty$, we see that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + 1} = \lim_{R \rightarrow \infty} \left[\arctan(R) - \arctan(-R) \right] = \pi/2 - (-\pi/2) = \pi. \quad (59)$$

Exercise 4. A&F 3.3.5.

Solution 4. In order to find the coefficients of the Taylor-Laurent Series about 0 of $f(z) = e^{\frac{t}{2}(z-z^{-1})} = \sum_{n \in \mathbb{Z}} a_n z^n$, we use the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}} dz \quad (60)$$

where γ is the unit circle parameterized as $\gamma(\theta) = e^{i\theta}$ for $\theta \in [-\pi, \pi]$. We can then simplify the integral as

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{\frac{t}{2}(z-z^{-1})}}{z^{n+1}} dz \quad (61)$$

$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} \cdot ie^{i\theta} d\theta. \quad (62)$$

We can simplify $e^{i\theta} - e^{-i\theta}$ as $2i \sin \theta$ and combine the terms $e^{i\theta}$ and $e^{i(n+1)\theta}$, so that

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} \cdot ie^{i\theta} d\theta = \frac{i}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(2i \sin \theta)}}{e^{in\theta}} d\theta \quad (63)$$

Combining the top and bottom halves of the integrand and canceling the i in front, we get that

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t \sin \theta)} d\theta. \quad (64)$$

We can further simplify this using $e^{-ix} = \cos x - i \sin x$, which gives us

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta - t \sin \theta) d\theta + \frac{i}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta - t \sin \theta) d\theta. \quad (65)$$

The key to the next equality is that $g(\theta) = n\theta - t \sin \theta$ is odd. We show this directly as

$$g(-\theta) = -n\theta - t \sin -\theta = -(n\theta + t \sin -\theta) \quad (66)$$

$$= -(n\theta - t \sin \theta) \quad (\sin \theta \text{ is odd}) \quad (67)$$

$$= -g(\theta). \quad (68)$$

It follows that $\cos(g(\theta))$ is even and $\sin(g(\theta))$ is odd due to composition rules for odd and even functions. This means that

$$\int_{-\pi}^{\pi} \cos(n\theta - t \sin \theta) d\theta = 2 \int_0^{\pi} \cos(n\theta - t \sin \theta) d\theta \quad (69)$$

$$\int_{-\pi}^{\pi} \sin(n\theta - t \sin \theta) d\theta = 0 \quad (70)$$

by integral theorems for even and odd functions. Therefore,

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t \sin \theta)} d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - t \sin \theta) d\theta. \quad (71)$$