

Exercise 1. Suppose that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and pick $A \in \mathcal{F}$. Define

$$Z(\omega) = \begin{cases} X(\omega), & \omega \in A \\ Y(\omega), & \omega \in A^c. \end{cases} \quad (1)$$

Show that Z is a random variable.

Solution 1. Let B be a Borel set and consider the set $\{Z \in B\}$. We can write $B = (B \cap A) \cup (B \cap A^c)$. This allows us to see that

$$\{Z \in B\} = \{Z \in (B \cap A) \cup (B \cap A^c)\} = \{Z \in B \cap A\} \cup \{Z \in B \cap A^c\}. \quad (2)$$

since the preimage of a set union is the union of the preimages of the sets composing it. Since $(B \cap A) \subset A$ and $(B \cap A^c) \subset A^c$, we know that $Z = X$ on $B \cap A$ and $Z = Y$ on $B \cap A^c$. This means that

$$\{Z \in B\} = \{X \in B \cap A\} \cup \{Y \in B \cap A^c\}. \quad (3)$$

Since both the events on the righthand side are in \mathcal{F} because X and Y are random variables, their union $\{Z \in B\} \in \mathcal{F}$. This shows $\{Z \in B\}$ is measurable for any Borel set B , meaning Z is a random variable.

Exercise 2. Suppose that X is a continuous random variable with distribution F_X . Let g be a strictly increasing continuous function and define $Y = g(X)$.

a) What is the distribution function of Y F_Y ?

b) What is f_Y the density function of Y ?

Solution 2.

a) We can compute the distribution function of Y directly as

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(g(X) \leq x). \quad (4)$$

Since g is a strictly increasing continuous function it has a continuous inverse g^{-1} , therefore,

$$F_Y(x) = \mathbb{P}(g(X) \leq x) = \mathbb{P}(X \leq g^{-1}(x)) = (F_X \circ g^{-1})(x). \quad (5)$$

b) Here we assume that X has a density function $f_X = F'_X$. We can find the density of Y by taking the derivative of its distribution function, so that

$$f_Y(x) = \frac{d}{dx}(F_Y(x)) = \frac{d}{dx}(F_X(g^{-1}(x))). \quad (6)$$

Assuming that g is additionally differentiable and using the chain rule, we can compute that

$$f_Y(x) = \frac{d}{dx} [g^{-1}(x)] \cdot F'_X(g^{-1}(x)) \quad (7)$$

$$= \frac{d}{dx} [g^{-1}(x)] \cdot f_X(g^{-1}(x)) \quad (8)$$

The second equality follows from the definition of the density. In this case, the derivative of the inverse function g^{-1} should always be positive since g is strictly increasing, but in general, I believe we have to use $\left| \frac{d}{dx} g^{-1}(x) \right|$ to ensure the density is non-negative.

Exercise 3. Suppose that X is a continuous random variable with distribution function F_X . Find F_Y where Y is given by:

- a) X^2
- b) $\sqrt{|X|}$
- c) $\sin X$
- d) $F_X(X)$

Solution 3. a) We'll begin by trying to compute the distribution of Y directly,

$$F_Y(x) = \mathbb{P}(Y \leq x) \quad (9)$$

$$= \mathbb{P}(X^2 \leq x). \quad (10)$$

When $x \geq 0$, we can see that

$$F_Y(x) = \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) \quad (11)$$

$$= F_X(\sqrt{x}) - F_X(-\sqrt{x}). \quad (12)$$

If $x < 0$,

$$F_Y(x) = \mathbb{P}(X^2 \leq x) \quad (13)$$

$$= \mathbb{P}(X^2 \leq x < x) = 0 \quad (14)$$

since the square of real-valued quantities must be non-negative. We can put this together to show that

$$F_Y(x) = \begin{cases} F_X(\sqrt{x}) - F_X(-\sqrt{x}), & x \geq 0 \\ 0, & x < 0. \end{cases} \quad (15)$$

b) Once again computing directly

$$F_Y(x) = \mathbb{P}(Y \leq x) \quad (16)$$

$$= \mathbb{P}(\sqrt{|X|} \leq x) = \mathbb{P}(|X| \leq x^2) \quad (17)$$

Like in the previous problem, this is 0 for $x < 0$, but for $x \geq 0$, we can compute

$$F_Y(x) = \mathbb{P}(Y \leq x) \quad (18)$$

$$= \mathbb{P}(-x^2 \leq X \leq x^2) \quad (19)$$

$$= F_X(x^2) - F_X(-x^2). \quad (20)$$

We can write this compactly as

$$F_Y(x) = \begin{cases} F_X(x^2) - F_X(-x^2), & x \geq 0 \\ 0, & x < 0. \end{cases} \quad (21)$$

c) Computing directly,

$$F_Y(x) = \mathbb{P}(Y \leq x) \quad (22)$$

$$= \mathbb{P}(\sin X \leq x). \quad (23)$$

Since $\sin X$ is bounded between $[0, 1]$, we have that $F_Y(x) = 1$ for $x > 1$ and $F_Y(x) = 0$ for $x < 0$. Otherwise, if $x \in [0, 1]$, we can take the inverse of $\sin X$, so that $\mathbb{P}(\sin X \leq x) = \mathbb{P}(X \leq \arcsin x)$

$$F_Y(x) = \begin{cases} 1, & x > 1 \\ F_X(\arcsin x), & x \in [0, 1] \\ 0, & x < 0 \end{cases} \quad (24)$$

d) Computing directly,

$$F_Y(x) = \mathbb{P}(Y \leq x) \quad (25)$$

$$= \mathbb{P}(F_X(X) \leq x) \quad (26)$$

$$= \mathbb{P}(X \leq \tilde{F}_X^{-1}(x)), \quad (27)$$

where \tilde{F}_X^{-1} is the pseudo-inverse of the distribution F defined by

$$\tilde{F}_X^{-1} = \inf\{x \mid F(x) \geq u\}. \quad (28)$$

Exercise 4. Define $X: [0, 1] \rightarrow \mathbb{R}$ by

$$X(\omega) = \begin{cases} 1, & \omega \in [0, 1] \cap \mathbb{Q}^c \\ 0, & \omega \in [0, 1] \cap \mathbb{Q}. \end{cases} \quad (29)$$

As defined, X is the function which takes rational numbers in $[0, 1]$ to 0 and irrational numbers in $[0, 1]$ to 1. Assuming that X is defined on $([0, 1], \mathcal{B}[0, 1], \mathbb{P})$ where \mathbb{P} is the Lebesgue measure. Show that X is a random variable. If it is, what are its distribution function and expectation? Does X has a density function? Is X discrete?

Solution 4.

X is a random variable. Let $A = \mathbb{Q}^c \cap [0, 1]$. We will show that $A \in \mathcal{B}[0, 1]$. By definition of $\mathcal{B}[0, 1]$, we know that every rational number $q \in \mathbb{Q} \cap [0, 1]$ has $\{q\} \in \mathcal{B}[0, 1]$ since $[0, q]$ and $(q, 1]$ are Borel sets and

$$[0, 1] \setminus ([0, q] \cup (q, 1]) = \{q\}. \quad (30)$$

We also know that $\mathbb{P}(\{q\}) = 0$ under the Lebesgue measure since

$$\mathbb{P}([0, q] \cup (q, 1]) = 1 \text{ and } \mathbb{P}([0, q] \cup (q, 1] \cup \{q\}) = \mathbb{P}([0, 1]) = 1. \quad (31)$$

Since there are a countable number of rationals, we have that

$$\mathbb{P}(\mathbb{Q} \cap [0, 1]) = \sum_{q \in \mathbb{Q}} \mathbb{P}(\{q\}) = 0 \quad (32)$$

by countable additivity. This means that A is a measurable set with $\mathbb{P}(A) = 1$ since $A^c = \mathbb{Q} \cap [0, 1]$ is measurable with $\mathbb{P}(A^c) = 0$. Notice that with this choice of A , we can see that X is simply the indicator function of A and as it is an indicator function of a measurable event it is a random variable.

Distribution of X. We can compute the distribution of X using the fact that it is either 1 or 0. If $\omega \in \mathbb{Q}$, then $X = 0$ which occurs with probability 0. Similarly, if $\omega \notin \mathbb{Q}$, then $X = 1$ which occurs with probability 1. Therefore, our distribution is described by

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 1, & x = 1 \\ 0, & x < 1. \end{cases} \quad (33)$$

Expectation of X. Since X is an indicator function, we can compute its expectation quite simply as

$$\mathbb{E}[X] = \mathbb{P}(A) = 1. \quad (34)$$

Density of X. The random variable has no density since it is simply a point mass at $x = 1$.

X is discrete. The random variable X is discrete since it can only take the values 0 and 1.