Exercise 1. Consider the non-homogeneous problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{g}(t).$$

(a) Let $\mathbf{x} = \mathbf{M}\mathbf{y}$ where the columns of $\mathbf{M}$ are the eigenvector of the above problems. (b) Write the equations in terms of $\mathbf{y}$ and multiply through by $\mathbf{M}^{-1}$, (c) Show that the resulting equation is

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y} + \mathbf{M}^{-1}\mathbf{g}(t).$$

where $\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered.
(d) Show that this system is now decoupled so that each component of $\mathbf{y}$ can be solved independently of the other components.

Solution 1. Letting $\mathbf{x} = \mathbf{M}\mathbf{y}$, we write that

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{M}\mathbf{y} + \mathbf{g}(t),$$

assuming that A is diagonalizable, we have that

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{D},$$

where $\mathbf{D}$ is the diagonal matrix with diagonal entries as the eigenvalues of $\mathbf{A}$. Then it follows

$$\frac{d\mathbf{x}}{dt} = \mathbf{M}\mathbf{D}\mathbf{y} + \mathbf{g}(t),$$

noting that $\frac{d\mathbf{x}}{dt} = \mathbf{M}\frac{d\mathbf{y}}{dt}$ and multiplying the above through by M, we see that

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y} + \mathbf{M}^{-1}\mathbf{g}(t).$$

We can write out each of the components of this equation as

$$\frac{dy_i}{dt} = \lambda_i y_i + (\mathbf{M}^{-1}\mathbf{g}(t))_i$$

where the subscript i denotes the i-th component and λ_i is the i-th eigenvalue of $\mathbf{A}$. This shows that each of the components are independent of one another as dy_i/dt is independent of any other y_j.

Exercise 2. Given $L = -d^2/dx^2$, find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -10e^x, \quad y(0) = 0, y'(1) = 0.$$

Solution 2. We rewrite the above equation as

$$-y'' = 10e^x + 2y.$$

Using this, we'll represent this as a Sturm Liouville problem which has solution

$$u(x) = \sum_{n=1}^{\infty} \frac{(f, u_n)}{\lambda_n - 2} u_n(x),$$

for eigenvalues λ_n and eigenfunction $u_n(x)$. Formulating this the following eigenvalue problem, we can solve for λ_n and u_n,

$$-\frac{d^2u_n}{dx^2} = \lambda_n u_n,$$

which has solutions of the form

$$u_n = c_1 \sin(\sqrt{\lambda_n}x) + c_2 \cos(\sqrt{\lambda_n}x).$$

In order for this to satisfy our desired boundary condition $u_n(0) = 0$, we have that $c_2 = 0$. The second boundary condition then gives that $y'(1) = 0$ or

$$\sqrt{\lambda_n} \cos \sqrt{\lambda_n} = 0$$

which has solutions when

$$\sqrt{\lambda_n} = \frac{\pi}{2}(2n+1)$$

so that

$$\lambda_n = \left(\frac{2n+1}{2}\pi\right)^2,$$

we then have eigenfunctions

$$u_n = c_n \sin\left(\frac{\pi(2n+1)}{2}x\right),$$

where c_n is a normalization constant. We can compute this normalization constant with

$$\begin{aligned}
(u_n, u_n) &= \int_0^1 \sin^2\left(\sqrt{\lambda_n}x\right) dx \\
&= \frac{1}{2}\int_0^1 1 - \cos(2\sqrt{\lambda_n}x) dx \\
&= \frac{1}{2} - \frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}} \\
&= \frac{1}{2} - \frac{\sin((2n+1)\pi)}{4\sqrt{\lambda_n}} = \frac{1}{2},
\end{aligned}$$

where we've used that $\sin x$ is zero for $x = k\pi, k \in \mathbb{Z}$. Now, we have normalization constant

$$\begin{aligned} c_n &= (u_n, u_n)^{-1/2} = \sqrt{2} \\ u_n(x) &= \sqrt{2}\sin(\sqrt{\lambda_n x}). \end{aligned}$$

We can then see that our u_n form an orthonormal set of functions with real valued eigenvalues. We can then compute the inner product of our in-homogenous term $f(x) = 10e^x$ and the eigenfunctions as

$$\begin{aligned} (10e^x, u_n) &= 10\sqrt{2}\int_0^1 e^x \sin\left(\sqrt{\lambda_n x}\right) dx \\ &= \frac{10\sqrt{2}}{\lambda_n + 1}\left(e^x \sin(\sqrt{\lambda_n x}) - e^x\sqrt{\lambda_n}\cos(\sqrt{\lambda_n x})\right)_{x=0}^{x=1} \\ &= \frac{10\sqrt{2}}{\lambda_n + 1}\left(\sqrt{\lambda_n} + e\sin\sqrt{\lambda_n}\right) \\ &= \frac{10\sqrt{2}}{\lambda_n + 1}\left(\sqrt{\lambda_n} + e(-1)^n\right), \end{aligned}$$

where we've solved the integral with integration by parts and noted that $\sin\sqrt{\lambda_n} = \pm 1$ in the last line. Finally, we can solve for our solution as

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \frac{(f, u_n)}{\lambda_n - 2} u_n(x) \\ &= 20\sum_{n=1}^{\infty}\left(\frac{\sqrt{\lambda_n} + e(-1)^n}{(\lambda_n + 1)(\lambda_n - 2)}\right)\sin(\sqrt{\lambda_n x}). \end{aligned}$$

Exercise 3. Given $L = -d^2/dx^2$, find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -x, \quad y(0) = 0, y(1) + y'(1) = 0.$$

Solution 3. We rewrite this as

$$-y'' = 2y + x.$$

Rewriting this as an eigenvalue problem, we have

$$-\frac{d^2u_n}{dx^2} = \lambda_n u_n,$$

where λ_n and u_n are the n-th eigenvalue and eigenfunction respectively. This eigenvalue problem has solutions of the form

$$u_n = c_1 \sin(\sqrt{\lambda_n}x) + c_2 \cos(\sqrt{\lambda_n}x).$$

In order for this to satisfy our desired boundary condition $u_n(0) = 0$, we have that $c_2 = 0$. The second boundary condition then gives that

$$\sqrt{\lambda_n} + \tan(\sqrt{\lambda_n}) = 0.$$

As this is a transcendental function, I will not solve it by hand. Our normalized eigenfunctions are then of the form

$$u_n = \left(\frac{2}{1 + \cos^2\sqrt{\lambda_n}}\right)^{\frac{1}{2}} \sin\sqrt{\lambda_n}x, n \in \mathbb{N}.$$

This is equation 60 in the lecture notes. Next, we compute inner product of our inhomogenous term x and our basis functions, so that

$$\begin{aligned}
(f, u_n) &= \left(\frac{2}{1 + \cos^2\sqrt{\lambda_n}}\right)^{\frac{1}{2}} \int_0^1 x \sin(\sqrt{\lambda_n}x)dx \\
&= \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n(1 + \cos^2(\sqrt{\lambda_n}))^{\frac{1}{2}}}
\end{aligned}$$

Writing f in terms of its eigenfunction expansion is then

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} (f, u_n) u_n(x) \\
&= 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x)}{\lambda_n(1 + \cos^2(\sqrt{\lambda_n}))},
\end{aligned}$$

which gives us a final solution to our differential equation

$$\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} \frac{(f, u_n)}{\lambda_n - 2} u_n(x) \\
&= 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x)}{\lambda_n(\lambda_n - 2)(1 + \cos^2(\sqrt{\lambda_n}))},
\end{aligned}$$

Exercise 4. Consider the Sturm-Liouville eigenvalue problem:

$$Lu = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda\rho(x)u, 0 < x < L,$$

with the boundary conditions

$$\alpha_1 u(0) + \beta_1 u'(0) = 0$$
$$\alpha_2 u(L) + \beta_2 u'(L) = 0$$

and with $p(x) > 0$, $\rho(x) > 0$, and $q(x) \geq 0$ and with $p(x), \rho(x), q(x)$ and $p'(x)$ continuous over $0 < x < L$. With the inner product,

$$(\varphi, \psi) = \int_0^L \rho(x)\varphi(x)\overline{\psi}(x)dx$$

show the following:

(a) L is a self-adjoint operator.

(b) Eigenfunctions corresponding to different eigenvalues are orthogonal

(c) Eigenvalues are real, non-negative, and eigenfunctions may be chosen to be real value.

(d) Each eigenvalue is simple i.e. it only has one eigenfunction. (Hint: recall that for each eigenvalue, there can be at most two linearly independent solutions - calculate the Wronskian of these two solutions adn see what it implies).

Solution 4. a) We'll show that

$$(Lu, v) - (u, Lv) = 0.$$

We write the difference as

$$(Lu, v) - (u, Lv) = \int_0^L Lu(x) \cdot \overline{v}(x) - u(x) \cdot \overline{Lv}(x)dx$$

We can simplify this as

$$(Lu, v) - (u, Lv) = \int_0^L u(x)\frac{d}{dx}\left[p(x)\overline{v'}(x)\right] - \overline{v}(x)\frac{d}{dx}\left[p(x)u'(x)\right]dx$$
$$+ \int_0^L q(x)u(x)\overline{v}(x) - q(x)\overline{v}(x)u(x)dx.$$

The term in the last line is clearly 0, we can additionally expand the integrand in the first line with the product rule

$$\int_0^L u(x)\frac{d}{dx}\left[p(x)\overline{v'}(x)\right] - \overline{v}(x)\frac{d}{dx}\left[p(x)u'(x)\right]dx$$
$$= \int_0^L \frac{d}{dx}\left[p(x)\left(u(x)\overline{v'}(x) - u'(x)\overline{v}(x)\right)\right]$$
$$= \left[p(x)\left(u(x)\overline{v'}(x) - u'(x)\overline{v}(x)\right)\right]_{x=0}^{x=L}$$

In the case that u and v share boundary conditions, this must be 0. We'll show this as follows. Under our assumed boundary conditions, both u and v satisfy

$$g(0) = -\frac{\beta_1}{\alpha_1} g'(0)$$
$$g(L) = -\frac{\beta_2}{\alpha_2} g'(L).$$

We then have that

$$\begin{aligned} &u(L)\overline{v'}(L) - u'(L)\overline{v}(L) \\ &= -\frac{\alpha_2}{\beta_2} u(L)\overline{v}(L) + \frac{\alpha_2}{\beta_2} u(L)\overline{v}(L) = 0 \end{aligned}$$

The same follows for the other boundary at 0. This shows that

$$(Lu, v) - (u, Lv) = 0,$$

i.e. that L is self-adjoint.

b) Suppose that we have eigenfunctions u_n and u_m corresponding to distinct eigenvalues $\lambda_n \neq \lambda_m$. We have that

$$(Lu_m, u_n) = (u_m, Lu_n)$$

since L is self-adjoint. Further, using that u_m and u_n are eigenfunctions, we have

$$(Lu_m, u_n) = \lambda_m (u_m, u_n) \quad (u_m, Lu_n) = \overline{\lambda}_n = (u_m, u_n).$$

As $\lambda_n \neq \lambda_m$, we see that

$$(\lambda_m - \overline{\lambda}_n)(u_m, u_n) = (Lu_m, u_n) - (u_m, Lu_n) = 0$$

which implies that $(u_m, u_n) = 0$.

c) Once again assuming that u_m is an eigenfunction, we have that

$$\begin{aligned}(Lu_m, u_m) &= (u_m, Lu_m)\\ (Lu_m, u_m) &= \lambda_m (u_m, u_m)\\ (u_m, Lu_m) &= \overline{\lambda}_m (u_m, u_m).\end{aligned}$$

This implies that

$$\lambda_m (u_m, u_m) = \overline{\lambda}_m (u_m, u_m),$$

which implies that $\lambda_m = \overline{\lambda}_m$ i.e. the eigenvalue is real. To show the eigenvalues are non-negative, we begin with the eigenvalue problem and multiply by $u(x)$

$$\int_0^L \frac{d}{dx}\left[p(x)u'(x)\right]u(x) + q(x)u^2(x) + \lambda\rho(x)u^2(x)dx = 0.$$

Integrating the first by parts, we see

$$\int_0^L \frac{d}{dx}\left[p(x)u'(x)\right]u(x) = \left[p(x)u'(x)u(x)\right]_{x=0}^{x=L} - \int_0^L p(x)u'(x)^2 dx$$

Putting this together, we have

$$\left[p(x)u'(x)u(x)\right]_{x=0}^{x=L} + \int_0^L q(x)u^2(x) + \lambda\rho(x)u^2(x) - p(x)u'(x)^2 dx = 0$$

Plugging in boundary conditions, we have that

$$\lambda\int_0^L \rho(x)u^2(x)dx + \int_0^L q(x)u^2(x)dx = \int_0^L p(x)u'(x)^2 dx + \frac{\alpha_2}{\beta_2}p(L)u(L)^2 - \frac{\alpha_1}{\beta_1}p(0)u(0)^2.$$

I want to make a claim here on the right hand side but the fact that I don't have conditions on α_1, α_2, β_1 and β_2 seems to be preventing this.

d) Suppose that y_m, y_n are eigenfunctions corresponding to the eigenvalue λ. Writing the Wronksian of these two functions at 0, we have

$$y_m(0)y_n'(0) - y_m'(0)y_n(0) = 0$$

because y_m and y_n have the same boundary conditions at 0. Now writing

$$y_m(x)Ly_n(x) - y_n(x)Ly_m(x) = 0$$

since y_m and y_n correspond to the same eigenvalue, we can expand the definition of the operator, so that

$$-y_m(x)\frac{d}{dx}\left[p(x)y_n'(x)\right] + y_n(x)\frac{d}{dx}\left[p(x)y_m'(x)\right] = 0$$

We can recognize this as the product rule of another function which shows

$$\frac{d}{dx}\left[p(x)(y_n y_m' - y_n' y_m)\right] = 0,$$

which implies that

$$p(x)W(y_n, y_m)(x) = 0$$

for all x i.e. the Wronskian is identically 0 and y_n and y_m are linearly dependent.