Exercise 1. Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.

Solution 1.
1a. If we were to flip a biased coin 3 times, we would have 8 possible outcomes:

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}. \tag{1}$$

Since Ω is finite, we can simply take the σ-algebra of events to be all subsets of Ω i.e. $\mathcal{F} = 2^{\Omega}$. Since the coin is biased, we'll say the probability of heads for each toss is p with the probability of tails being $q = 1 - p$. For each outcome $\omega \in \Omega$, we define $|H|_{\omega}$ to be the number of heads in ω. Similarly, we define $|T|_{\omega}$ to be the number of tails in ω. Then assuming that each toss is independent, we define the probability mass function as

$$f(\omega) = p^{|H|_{\omega}} q^{|T|_{\omega}}. \tag{2}$$

We can then compute that

$$\sum_{\omega \in \Omega} f(\omega) = \sum_{\omega \in \Omega} p^{|H|_{\omega}} q^{|T|_{\omega}}. \tag{3}$$

Noticing that $|H|_{\omega} + |T|_{\omega} = 3$ for all ω, we can write this

$$\sum_{\omega \in \Omega} f(\omega) = \sum_{k=1}^{3} \binom{3}{k} p^k q^{3-k}, \tag{4}$$

since each head-tail count $(k, 3-k)$ appears exactly $\binom{3}{k}$ times. We can use the binomial theorem to compute that

$$\sum_{\omega \in \Omega} f(\omega) = \sum_{k=1}^{3} \binom{3}{k} p^k q^{3-k}, \tag{5}$$

$$= (p+q)^3 = (p+1-p)^3 = 1. \tag{6}$$

Therefore, f is indeed a probability mass function and induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

1b. If we were to draw from an urn which contains two blue and two red balls $U = \{B, B, R, R\}$ without replacements, then our possible samples are

$$\Omega = \{BB, BR, RB, RR\}. \tag{7}$$

Once again since this probability space is finite, we can just take the σ-algebra of events to be $\mathcal{F} = 2^{\Omega}$. This leaves us to find the probability mass function. Assuming that each ball is equally likely to be pulled and we do not replace the balls, then we have the probability of pulling blue is equal to the fraction of balls currently in the urn which are blue. This same logic applies for red balls. Therefore, we can compute the associated probability mass function p as:

$$p(BB) = \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6} \quad p(BR) = \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3} \tag{8}$$

$$p(RB) = \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3} \quad p(RR) = \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6} \tag{9}$$

This is certainly a probability mass function as

$$\sum_{\omega \in \Omega} p(\omega) = p(BB) + p(BR) + p(RB) + p(RR) = 1. \tag{10}$$

Therefore, we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as p induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

Exercise 2 (No translation-invariant random integer)**.** Show that there is no probability measure $\mathbb{P}$ on the integers $\mathbb{Z}$ with the discrete σ-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $\mathbb{P}(E+n) = \mathbb{P}(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer n. $E+n$ is obtained by adding n to every element of E.

Solution 2.

Exercise 3 (No translation-invariant random real)**.** Show that there is no probability measure $\mathbb{P}$ on the reals $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $\mathbb{P}(E + x) = \mathbb{P}(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real x. Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by intervals $(a, b] \subset \mathbb{R}$.

Solution 3.

Exercise 4. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = all subsets of $\mathbb{R}$ so that A or A^c is countable. $\mathbb{P}(A) = 0$ in the first case and $\mathbb{P}(A) = 1$ in the second. Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Solution 4.

Exercise 5. A collection $\mathcal{A}$ of subsets of Ω is called an **algebra** if $A, B \in \mathcal{A}$ implies A^c and $A \cup B$ are in $\mathcal{A}$. (a) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra. (b) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

Solution 5.