Exercise 1.
(c) Evaluate
$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a^2, b^2 > 0.$$
(d) Evaluate
$$\int_0^\infty \frac{dx}{x^6+1}.$$

Solution 1.
(c) We'll begin by considering the curve $C(R)$ which is the upper semi-circle in $\mathbb{C}$ centered at 0 with radius R. We can divide this into two parts $[-R, R]$ and C_R which is the circular section of the semi-circle. We can then write
$$\int_{C(R)} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-R}^{R} \frac{dx}{(x^2+a^2)(x^2+b^2)} + \int_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)}. \tag{1}$$
Theorem 4.2.1. in A&F shows that that $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ has
$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 0$$
since $f(z)$ is rational and the degree of the denominator (4) exceeds the degree of the numerator (0) by more than two. Therefore, taking the limit as of $R \to \infty$ of equation (1), we see that
$$\lim_{R\to\infty} \int_{C(R)} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, \tag{2}$$
where the last equation follows from the fact the integrand is even. We can compute the leftmost integral using the Residue theorem. The function $\frac{1}{(z^2+a^2)(z^2+b^2)}$ has singularities at $\pm ia$ and $\pm ib$. Since we're dealing with the upper half circle, only two of this of these will be contained within our contour for large enough R. These are $z_1 = ia$ and $z_2 = ib$ assuming $a, b > 0$. Using the fact that the numerator of the integrand is analytic and that the zeros of the denominator are simple, we can write the residues as
$$\operatorname{Res}(f; z_0) = \frac{1}{2z_0(z_0^2+a^2) + 2z_0(z_0^2+b^2)}. \tag{3}$$
Evaluating at z_1 and z_2, we see that
$$\operatorname{Res}(f; ia) = \frac{1}{2ia(2a^2) + 2ia(a^2+b^2)} = \frac{1}{2ai(3a^2+b^2)} \tag{4}$$
$$\operatorname{Res}(f; ib) = \frac{1}{2ib(2b^2) + 2ib(a^2+b^2)} = \frac{1}{2bi(a^2+3b^2)} \tag{5}$$

Therefore, we can compute the final integral as

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \pi i(\mathrm{Res}(f;ia) + \mathrm{Res}(f;ib)) && (6)\\
&= \frac{\pi}{2}\left(\frac{1}{a(3a^2+b^2)} + \frac{1}{b(a^2+3b^2)}\right) && (7)\\
&= \frac{\pi}{2}\left(\frac{1}{3a^3+ab^2} + \frac{1}{a^2b+3b^3}\right). && (8)
\end{aligned}$$

(d) We consider the same general approach as in the previous part. We keep the definitions of $C(R)$ and C_R the same. Since $f(z) = \frac{1}{z^6+1}$ also satisfies the conditions of Theorem 4.2.1, we have that

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^6+1} = 0.$$

Additionally, our integrand is even again, so that

$$\lim_{R\to\infty} \int_{C(R)} \frac{dz}{z^6+1} = 2\int_0^\infty \frac{dz}{z^6+1}. \tag{9}$$

Once again, we can compute the integral on the left using the residue theorem. The integrand $\frac{1}{z^6+1}$ has singularities at $e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6}$ in the upper half plane. Therefore, the integral can be written as

$$\lim_{R\to\infty} \int_{C(R)} \frac{dz}{z^6+1} = 2\pi i(\mathrm{Res}(f; e^{i\pi/6}) + \mathrm{Res}(f; e^{i\pi/2}) + \mathrm{Res}(f; e^{i5\pi/6})). \tag{10}$$

Since the poles of $\frac{1}{z^6+1}$ are simple, we can compute the residues using the fact that the numerator is analytic and the denominator has only simple zero at z_0 as

$$\mathrm{Res}(f; z_0) = \frac{1}{6z_0^5} \tag{11}$$

as in section 5.4 of Prof. Tung's notes. Therefore,

$$\mathrm{Res}(f; e^{i\pi/6}) = \frac{1}{6e^{i5\pi/6}}, \quad \mathrm{Res}(f; e^{i\pi/2}) = \frac{1}{6e^{i5\pi/2}}, \quad \mathrm{Res}(f; e^{i5\pi/6}) = \frac{1}{6e^{i25\pi/6}}. \tag{12}$$

We can then compute the desired integral as

$$\int_0^\infty \frac{dz}{z^6+1} = \frac{\pi i}{6}\left(\frac{1}{e^{i5\pi/6}} + \frac{1}{e^{i5\pi/2}} + \frac{1}{e^{i25\pi/6}}\right) \tag{13}$$

$$= \frac{\pi i}{6}\left(e^{-5\pi/6} + e^{-\pi/2} + e^{-i\pi/5}\right) \tag{14}$$

$$= \frac{\pi i}{6}(-2i) = \frac{\pi}{3}. \tag{15}$$

Above, we used that $e^{-5\pi i/2} = e^{-i\pi/2} = -i$, $e^{-i25\pi/6} = e^{-i\pi/6} = -\frac{\sqrt{3}}{2} - i/2$ and $e^{-i5\pi/6} = \frac{\sqrt{3}}{2} - i/2$.

Exercise 2. Evaluate the following integrals
(a)
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx, \quad a^2 > 0.$$
(b)
$$\int_{-\infty}^{\infty} \frac{\cos(kx) dx}{(x^2 + a^2)(x^2 + b^2)}, \quad a^2, b^2, k > 0.$$
(h)
$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}.$$

Solution 2. (a) Once again, we'll work in the upper half plane and use the contours $C(R), [-R, R]$, and C_R. Instead of working directly with the integral at hand, we'll instead using the integrand $f(z) = \frac{ze^{-z}}{z^2+a^2}$ since

$$\operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz\right) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx. \tag{16}$$

Here we can use Jordan's Lemma since $g(z) = \frac{z}{z^2+a^2}$ converges to 0 uniformly as $z \to \infty$, so that we see that $\int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz \to 0$ as $R \to \infty$. It follows that

$$\lim_{R\to\infty} \int_{C(R)} \frac{ze^{iz}}{z^2 + a^2} dz = \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \sum_j \operatorname{Res}(f; z_j). \tag{17}$$

We can then compute the residues in the upper half plane at the singularity $z = ia$ for $a > 0$ since it is the sole isolated singularity of f in the upper half plane. Since the numerator is analytic and the denominator has only simple poles, we can compute the residue as

$$\operatorname{Res}(f; ia) = \left(\frac{z_0 e^{iz_0}}{2z_0}\right)_{z_0 = ia} = \frac{iae^{i(ia)}}{2ia} = \frac{e^{-a}}{2}. \tag{18}$$

Plugging this in,

$$\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \frac{e^{-a}}{2} = \pi i e^{-a}. \tag{19}$$

Taking the imaginary part of this, we conclude

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}. \tag{20}$$

(b) We use the integrand $\frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)}$ and same contours as above. Since the norm of this function is the same as the integrand in (1c) because $\left|e^{ikz}\right| = 1$, we know that

$$\int_{C_R} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} \to 0 \tag{21}$$

as $R \to \infty$. We could alternatively use Jordan's Lemma. Next, using the residue theorem and the fact that $\int_{C(R)} = \int_{C_R} + \int_{-R}^{R}$, we can compute that for $f(z) = \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)}$

$$\int_{-\infty}^{\infty} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i \left(\operatorname{Res}(f; ia) + \operatorname{Res}(f; ib)\right). \tag{22}$$

Since the numerator of this is analytic and the denominator has only simple poles, we can compute the residues here using the formula

$$\operatorname{Res}(f, z_0) = \frac{e^{ikz}}{2z_0(z_0^2+a^2) + 2z_0(z_0^2+b^2)} \tag{23}$$

as in Section 5.4 of Prof. Tung's notes. Since the only two singularities in the upper half circle are ia, ib for $a, b > 0$, we can then compute that the integral is

$$\int_{-\infty}^{\infty} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = \pi \left(\frac{e^{-ka}}{3a^3+ab^2} + \frac{e^{ikb}}{a^2b+3b^3} \right). \tag{24}$$

The real part of this integral is our desired integral since the above quantity is entirely real, we just have that

$$\int_{-\infty}^{\infty} \frac{e^{ikz}dz}{(z^2+a^2)(z^2+b^2)} = \int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)} \tag{25}$$

Therefore, our final answer

$$\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)} = \pi \left(\frac{e^{-ka}}{3a^3+ab^2} + \frac{e^{-kb}}{a^2b+3b^3} \right). \tag{26}$$

(h) To solve this integral, we'll write $\frac{1}{(5-3\sin\theta)^2}$ in terms of complex exponentials, so that

$$U(\theta) = \frac{1}{(5-3\sin\theta)^2} = \frac{1}{(5-\frac{3}{2i}(z-z^{-1}))^2} = U(z). \tag{27}$$

We can then re-write the desired integral in terms of a contour integral on the unit circle C

$$\int_0^{2\pi} U(\theta)d\theta = \oint_C U(z)\frac{dz}{iz} = 2\pi i \sum_j \text{Res}\left(\frac{U}{iz}; z_j\right), \tag{28}$$

where z_j are the isolated singularities of $\frac{U}{iz}$ in the unit circle C. Therefore,

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \oint_C \frac{1}{(5-\frac{3}{2i}(z-z^{-1}))^2}\frac{dz}{iz}. \tag{29}$$

We can simplify the integrand of the right hand side as

$$\begin{aligned}
\frac{U(z)}{iz} &= \frac{1}{iz}\cdot\frac{1}{(5-\frac{3}{2i}(z-z^{-1}))^2} = \frac{-iz}{z^2(5-\frac{3}{2i}(z-z^{-1}))^2} && (30)\\
&= \frac{-iz}{(5z-\frac{3}{2i}(z^2-1))^2} && (31)\\
&= \frac{-iz}{(-\frac{3}{2i}z^2+5z+\frac{3}{2i})^2} && (32)\\
&= \frac{-iz}{(\frac{1}{2}(1+3iz)(z-3i))^2} && (33)\\
&= \frac{-4iz}{(1+3iz)^2(z-3i)^2} && (34)\\
&= \frac{-4iz}{(3i)^2(z-i/3)^2(z-3i)^2} && (35)\\
&= \frac{4i}{9(z-i/3)^2(z-3i)^2} && (36)
\end{aligned}$$

We can compute the zeroes of the denominator as $\frac{i}{3}$ and $3i$. Since only $\frac{i}{3}$ is in the unit circle, we can compute the integral as

$$\int_0^{2\pi} U(\theta)d\theta = 2\pi i\text{Res}\left(\frac{U}{iz}; \frac{i}{3}\right) \tag{37}$$

In this case, the residue can be computed using the double pole formula

$$\operatorname{Res}\left(\frac{U}{iz};\frac{i}{3}\right) = \lim_{z\to i/3}\frac{d}{dz}\left[\frac{U}{iz}\cdot\left(z-\frac{i}{3}\right)^2\right] \tag{38}$$

$$= \lim_{z\to i/3}\frac{d}{dz}\left[\frac{4i}{9(z-i/3)^2(z-3i)^2}\cdot\left(z-\frac{i}{3}\right)^2\right] \tag{39}$$

$$= \frac{4i}{9}\lim_{z\to i/3}\frac{d}{dz}\left[\frac{z}{(z-3i)^2}\right] \tag{40}$$

$$= \frac{4i}{9}\lim_{z\to i/3} -\frac{(z+3i)}{(z-3i)^3} \tag{41}$$

$$= -\frac{4i}{9}\frac{10i/3}{512i/27} \tag{42}$$

$$= -\frac{40i}{512} = -i\frac{5}{64} \tag{43}$$

We now finish computing the integral as

$$\int_0^{2\pi} U(\theta)d\theta = 2\pi i\left(-i\frac{5}{64}\right) = \frac{5\pi}{32}. \tag{44}$$

Exercise 3. Use a sector contour C with radius R centered at the origin with angles $0 \leq \theta \leq \frac{2\pi}{5}$ to find for $a > 0$,

$$I = \int_0^\infty \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}.$$

Solution 3. We'll break down the contour C mentioned above in three parts, so that $\oint_C = \int_{C_x} + \int_{C_R} + \int_{C_L}$. Let C_L is the straight line between $Re^{i2\pi/5}$ and 0, C_R be the circular section, and C_x be the line segment from $[0, R]$ on the real axis. We then have that

$$\int_0^R \frac{dx}{x^5 + a^5} + \int_{C_R} \frac{dz}{z^5 + a^5} + \int_{C_L} \frac{dz}{z^5 + a^5} = 2\pi i \sum_j \text{Res}\left(\frac{1}{z^5 + a^5}; z_j\right) \tag{45}$$

by the residue theorem. By theorem 4.2.1 of A&F,

$$\int_{C_R} \frac{dz}{z^5 + a^5} \to 0, \tag{46}$$

as $R \to \infty$. We can also make the substitution on C_L that $z = xe^{2\pi i/5}$ for $x \in [0, R]$, so that

$$\int_{C_L} \frac{dz}{z^5 + a^5} = \int_R^0 \frac{e^{2\pi i/5}}{x^5 + a^5} dx = -e^{2\pi i/5} \int_0^R \frac{dx}{x^5 + a^5}. \tag{47}$$

Since the only pole of $\frac{1}{z^5+a^5}$ inside C is $z_0 = ae^{i\pi/5}$ for R large enough, we have that in the limit as $R \to \infty$

$$\int_0^\infty \frac{dx}{x^5 + a^5} = \frac{2\pi i}{1 - e^{2\pi i/5}} \text{Res}\left(\frac{1}{z^5 + a^5}; ae^{i\pi/5}\right). \tag{48}$$

All that remains is to compute the residue, which is given by

$$\text{Res}\left(\frac{1}{z^5 + a^5}; ae^{i\pi/5}\right) = \left(\frac{1}{5z^4}\right)_{z_0} = \frac{1}{5a^4} e^{-4\pi i/5}. \tag{49}$$

Combining the last two equations allows us to conclude that

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^5 + a^5} &= \frac{2\pi i}{1 - e^{2\pi i/5}} \frac{1}{5a^4} e^{-4\pi i/5} && (50)\\
&= \frac{\pi}{5a^4} \left(\frac{2ie^{-4\pi i/5}}{1 - e^{2\pi i/5}}\right) && (51)\\
&= \frac{\pi}{5a^4} \left(\frac{2i}{e^{-\pi i/5} - e^{\pi i/5}}\right) e^{-\pi i} && (52)\\
&= \frac{\pi}{5a^4} \left(\frac{2i}{e^{\pi i/5} - e^{-\pi i/5}}\right) && (53)\\
&= \frac{\pi}{5a^4 \sin \pi/5} && (54)
\end{aligned}$$

where we have multiplied by $e^{-\pi i/5}/e^{-\pi i/5} = 1$ in the third line and simplified $e^{-i\pi} = -1$ in the fourth and used that $\sin x = (e^{ix} - e^{-iz})/2i$ in the last.

Exercise 4. A function that is analytic for all $z \in \mathbb{C}$ is called entire.

(a) Show that any bounded entire function is necessarily constant.

(b) Suppose $f(z)$ is an entire function, not necessarily, bounded, but such that $\mathrm{Im}(f(z)) \leq 0$. Show that $f(z)$ is necessarily constant.

Solution 4. (a) We'll show that the derivative of a bounded entire function must be 0 on $\mathbb{C}$. Suppose that the $|f(z)| \leq M$ for all $z \in \mathbb{C}$. For fixed $a \in \mathbb{C}$, $f(z)$ is analytic in and on the circle of radius R centered at a since $f(z)$ is entire. Therefore, we can use Cauchy's bound for derivatives (page 30 in Prof. Tung's book), to see that

$$|f'(a)| \leq \frac{M}{R}. \tag{55}$$

Since this holds for any circle of radius R, we see that $|f'(a)| = 0$. Since this holds for arbitrary $a \in \mathbb{C}$, we know that $f'(z) = 0$ for all $z \in \mathbb{C}$. This means the function $f(z)$ must be constant.

(b) Since the function $f(z)$ has $\text{Im}(f(z)) \leq 0$, the function $f(z) - i$ has $\text{Im}(f(z) - i) \leq -1$ for all $z \in \mathbb{C}$ and therefore it is non-zero on the entire complex plane. It follows that the function

$$h(z) = \frac{1}{f(z) - i} \tag{56}$$

is entire since it is reciprocal of a non-zero entire function. This function $h(z)$ is bounded since

$$|h(z)| = \frac{1}{|f(z) - i|} \tag{57}$$

$$= \frac{1}{\text{Re}(f(z) - i)^2 + \text{Im}(f(z) - i)^2} \tag{58}$$

$$\leq \frac{1}{\text{Re}(f(z) - i)^2 + 1} \leq 1, \tag{59}$$

where we've used that $|z| = \text{Re}(z)^2 + \text{Im}(z)^2$ and $\text{Im}(f(z) - i)^2 \geq 1$. By part a, it follows that $h(z)$ is constant over $\mathbb{C}$ i.e. $h(z) = c$ for some constant $c \in \mathbb{C}$. We can compute that $f(z) = \frac{1}{c} + i$, so $f(z)$ is constant over $\mathbb{C}$.