Trail Equations

1 Foundational Functions

Any unspecified w, s, c, d are taken to be functions of x, e.g. $w \implies w(x), s \implies s(x)$.

1.1 Basis Functions

$$w(x) = \sqrt{8x+1}, \qquad s = \frac{(w-1)^2}{4(w+7)}, \qquad c = \frac{(w+3)^2}{16}, \qquad d = \frac{(w+3)^2}{8(w+1)}.$$

1.2 Indexed Basis Functions

$$s_0 = x, \qquad s_{-1} = \frac{1}{s(1/x)}, \qquad s_j = s(s_{j-1}), \qquad s_{-j} = s_{-1}(s_{-j+1}), \qquad c_j = c(s_j), \qquad d_j = d(s_j).$$

1.3 Basis Function Derivatives

$$w'(x) = \frac{4}{\sqrt{8x+1}}, \qquad s' = \frac{1}{w} - \frac{64}{w(w+7)^2},$$

$$s'_{-1} = \frac{s'(1/x)}{x^2 \cdot s^2(1/x)} = \frac{s'(1/x) \cdot s_{-1}^2}{x^2}, \qquad c' = \frac{w'(w+3)}{8}, \qquad d' = \frac{1}{2w} - \frac{2}{w(w+1)^2}.$$

1.4 Indexed Basis Function Derivatives

$$s'_0 = 1, \qquad s'_j = \prod_{k=0}^{j-1} s'(s_k), \qquad s'_{-j} = \prod_{k=-j+1}^{0} s'_{-1}(s_k), \qquad c'_j = c'(s_j) \cdot s'_j, \qquad d'_j = d'(s_j) \cdot s'_j.$$

1.5 Composite functions

$$S_k = 1 + \sum_{j=0}^{k-1} \prod_{n=0}^{j} (d_n - 1), \qquad T_\ell = \sum_{j=1}^{\ell-1} \prod_{n=1}^{j} (d_{-n} - 1)^{-1},$$

$$P_k = 1 + \sum_{j=0}^{k-1} \prod_{n=0}^{j} (c_n - 1), \qquad Q_\ell = \sum_{j=1}^{\ell-1} \prod_{n=1}^{j} (c_{-n} - 1)^{-1}.$$

$$A_i = \prod_{n=0}^{i} (d_n - 1), \qquad B_i = \prod_{n=0}^{i} (c_n - 1), \qquad C_i = \prod_{n=1}^{i} (d_{-n} - 1)^{-1}, \qquad D_i = \prod_{n=1}^{i} (c_{-n} - 1)^{-1}.$$

$$E_i = \sum_{j=0}^{i} \frac{d'_j}{d_j - 1}, \qquad F_i = \sum_{j=0}^{i} \frac{c'_j}{c_j - 1}, \qquad G_i = \sum_{j=1}^{i} \frac{-d'_{-j}}{d_{-j} - 1}, \qquad H_i = \sum_{j=1}^{i} \frac{-c'_{-j}}{c_{-j} - 1}.$$

2 Properties Of Basis Functions

(1) $s : (0, \infty) \to (0, \infty)$ are increasing.

(2) $s : (0, 3) \to (0, \frac{1}{3})$ are increasing.

(3) $s_{-1} : (\frac{1}{3}, \infty) \to (3, \infty)$ are increasing.

(4) $w : (0, 3) \to (1, 5)$ are increasing.

(5) $w, c, d : (0, \infty) \to (1, \infty)$ are increasing.

(6) $s(x) = \frac{x^2}{2} + O(x^3)$ as $x \to 0$.

(7) For $x \in (0, \infty), s(x) = \frac{\beta(x)^2}{\beta(x)+2}$ for $\beta(x) = \frac{w-1}{4}$.

(8) For $x \in (0, \infty), \beta(x) \leq x$.

(9) For $x \in (0, \infty), s(x) < x$.

2.1 $x \in [1/3, 3]$ and $j \in [-5, 5]$

(1) $s_j : (0, \infty) \to (0, \infty)$ are increasing.

(2) $c_j, d_j : (0, \infty) \to (1, \infty)$ are increasing.

(3) $S_k, P_k, : (0, \infty) \to (1, \infty)$ are increasing.

(4) $T_\ell, Q_\ell : (0, \infty) \to (0, \infty)$ are decreasing.

(5) $A_i, B_i : (0, \infty) \to (0, \infty)$ are increasing.

(6) $C_i, D_i : (0, \infty) \to (0, \infty)$ are decreasing.

(7) $E_i, F_i : (0, \infty) \to (0, \infty)$

(8) $G_i, H_i : (0, \infty) \to (-\infty, 0)$

(9) $w' : (0, \infty) \to (0, 4)$ is decreasing.

(10) $c' : (0, \infty) \to (\frac{1}{2}, 2)$ is decreasing.

(11) $s' : (0, \infty) \to (0, 0.124)$ is concave down, $\max s' \leq 0.124$ where $w \in [3.2, 3.3]$.

(12) $d' : (0, \infty) \to (0, 151)$ is concave down, $\max d' \leq 0.151$ where $w \in [2, 2.1]$.

(13) s'_j is increasing for $j \neq 1$ and $s'_1 = s'$ is concave down.

(14) c'_j is increasing for $j \neq 0, 1$, $c'_0 = c'$ is decreasing and c'_1 is concave down.

(15) d'_j is increasing for $j \neq -1, 0$, d'_{-1} is decreasing and $d'_0 = d'$ is concave down.

(16) S_k, P_k, T_ℓ, Q_ℓ are increasing.

2.2 Consequences On $\mathcal{M}'_{2,1}$

Numerical work suggests a decresing function.

3 Finite Mina Margin Map

3.1 $\mathcal{M}_{\ell,k}$

$$\mathcal{M}_{\ell,k}(x) = \frac{x(S_k + T_\ell)}{P_k + Q_\ell} = \frac{x\left[1 + \sum_{j=0}^{k-1}\prod_{n=0}^{j}(d_n - 1) + \sum_{j=1}^{\ell-1}\prod_{n=1}^{j}(d_{-n} - 1)^{-1}\right]}{1 + \sum_{j=0}^{k-1}\prod_{n=0}^{j}(c_n - 1) + \sum_{j=1}^{\ell-1}\prod_{n=1}^{j}(c_{-n} - 1)^{-1}}.$$

3.2 $\mathcal{M}_{\ell,k}$ Derivative

$$\mathcal{M}'_{\ell,k}(x) = \left\{\frac{1}{x} + \frac{\sum_{i=0}^{k-1}(E_i)(A_i) + \sum_{i=1}^{\ell-1}(G_i)(C_i)}{S_k + T_\ell} - \frac{\sum_{i=0}^{k-1}(F_i)(B_i) + \sum_{i=1}^{\ell-1}(H_i)(D_i)}{P_k + Q_\ell}\right\} \times \mathcal{M}_{\ell,k}(x)$$

$$= \left\{\frac{1}{x} + \frac{\sum_{i=0}^{k-1}\left[\left(\sum_{j=0}^{i}\frac{d'_j}{d_j - 1}\right)\left(\prod_{n=0}^{i}(d_n - 1)\right)\right] + \sum_{i=1}^{\ell-1}\left[\left(\sum_{j=1}^{i}\frac{-d'_{-j}}{d_{-j} - 1}\right)\left(\prod_{n=1}^{i}(d_{-n} - 1)^{-1}\right)\right]}{1 + \sum_{j=0}^{k-1}\prod_{n=0}^{j}(d_n - 1) + \sum_{j=1}^{\ell-1}\prod_{n=1}^{j}(d_{-n} - 1)^{-1}}\right.$$

$$\left. - \frac{\sum_{i=0}^{k-1}\left[\left(\sum_{j=0}^{i}\frac{c'_j}{c_j - 1}\right)\left(\prod_{n=0}^{i}(c_n - 1)\right)\right] + \sum_{i=1}^{\ell-1}\left[\left(\sum_{j=1}^{i}\frac{-c'_{-j}}{c_{-j} - 1}\right)\left(\prod_{n=1}^{i}(c_{-n} - 1)^{-1}\right)\right]}{1 + \sum_{j=0}^{k-1}\prod_{n=0}^{j}(c_n - 1) + \sum_{j=1}^{\ell-1}\prod_{n=1}^{j}(c_{-n} - 1)^{-1}}\right\} \times \mathcal{M}_{\ell,k}(x).$$

3.3 $\mathcal{M}_{2,1}$

$$
\begin{aligned}
\mathcal{M}_{2,1}(x) &= \frac{x\left[1+\sum\limits_{j=0}^{0}\prod\limits_{k=0}^{j}(d_k-1)+\sum\limits_{j=1}^{1}\prod\limits_{k=1}^{j}(d_{-k}-1)^{-1}\right]}{1+\sum\limits_{j=0}^{0}\prod\limits_{k=0}^{j}(c_k-1)+\sum\limits_{j=1}^{1}\prod\limits_{k=1}^{j}(c_{-k}-1)^{-1}} \\
&= \frac{x\left(1+(d_0-1)+(d_{-1}-1)^{-1}\right)}{1+(c_0-1)+(c_{-1}-1)^{-1}} = \frac{x\left(d+(d_{-1}-1)^{-1}\right)}{c+(c_{-1}-1)^{-1}}.
\end{aligned}
$$

3.4 $\mathcal{M}_{2,1}$ Derivative

$$
\begin{aligned}
\mathcal{M}'_{2,1}(x) = &\left\{\frac{1}{x}+\frac{\sum\limits_{i=0}^{0}\left[\left(\sum\limits_{j=0}^{i}\frac{d'_j}{d_j-1}\right)\left(\prod\limits_{k=0}^{i}(d_k-1)\right)\right]+\sum\limits_{i=1}^{1}\left[\left(\sum\limits_{j=1}^{i}\frac{-d'_{-j}}{d_{-j}-1}\right)\left(\prod\limits_{k=1}^{i}(d_{-k}-1)^{-1}\right)\right]}{1+\sum\limits_{j=0}^{0}\prod\limits_{k=0}^{j}(d_k-1)+\sum\limits_{j=1}^{1}\prod\limits_{k=1}^{j}(d_{-k}-1)^{-1}}\right. \\
&\left.-\frac{\sum\limits_{i=0}^{0}\left[\left(\sum\limits_{j=0}^{i}\frac{c'_j}{c_j-1}\right)\left(\prod\limits_{k=0}^{i}(c_k-1)\right)\right]+\sum\limits_{i=1}^{1}\left[\left(\sum\limits_{j=1}^{i}\frac{-c'_{-j}}{c_{-j}-1}\right)\left(\prod\limits_{k=1}^{i}(c_{-k}-1)^{-1}\right)\right]}{1+\sum\limits_{j=0}^{0}\prod\limits_{k=0}^{j}(c_k-1)+\sum\limits_{j=1}^{1}\prod\limits_{k=1}^{j}(c_{-k}-1)^{-1}}\right\}\times\mathcal{M}_{2,1}(x)
\end{aligned}
$$

$$
\begin{aligned}
= &\left\{\frac{1}{x}+\frac{\left(\frac{d'_0}{d_0-1}\right)(d_0-1)+\left(\frac{-d'_{-1}}{d_{-1}-1}\right)(d_{-1}-1)^{-1}}{1+(d_0-1)+(d_{-1}-1)^{-1}}-\frac{\left(\frac{c'_0}{c_0-1}\right)(c_0-1)+\left(\frac{-c'_{-1}}{c_{-1}-1}\right)(c_{-1}-1)^{-1}}{1+(c_0-1)+(c_{-1}-1)^{-1}}\right\} \\
&\times\frac{x\left[1+(d_0-1)+(d_{-1}-1)^{-1}\right]}{1+(c_0-1)+(c_{-1}-1)^{-1}}.
\end{aligned}
$$

3.5 $\mathcal{M}_{5,4}$

$$\mathcal{M}_{5,4}(x) = \frac{x\left[1 + \sum\limits_{j=0}^{3}\prod\limits_{k=0}^{j}(d_k - 1) + \sum\limits_{j=1}^{4}\prod\limits_{k=1}^{j}(d_{-k} - 1)^{-1}\right]}{1 + \sum\limits_{j=0}^{3}\prod\limits_{k=0}^{j}(c_k - 1) + \sum\limits_{j=1}^{4}\prod\limits_{k=1}^{j}(c_{-k} - 1)^{-1}}.$$

3.6 $\mathcal{M}_{5,4}$ Derivative

$$\mathcal{M}'_{5,4}(x) = \left\{\frac{1}{x} + \frac{\sum\limits_{i=0}^{3}(E_i)(A_i) + \sum\limits_{i=1}^{4}(G_i)(C_i)}{S_3 + T_4} - \frac{\sum\limits_{i=0}^{3}(F_i)(B_i) + \sum\limits_{i=1}^{4}(H_i)(D_i)}{P_3 + Q_4}\right\} \times \mathcal{M}_{5,4}(x)$$

$$= \left\{\frac{1}{x} + \frac{\sum\limits_{i=0}^{3}\left[\left(\sum\limits_{j=0}^{i}\frac{d'_j}{d_j - 1}\right)\left(\prod\limits_{k=0}^{i}(d_k - 1)\right)\right] + \sum\limits_{i=1}^{4}\left[\left(\sum\limits_{j=1}^{i}\frac{-d'_{-j}}{d_{-j} - 1}\right)\left(\prod\limits_{k=1}^{i}(d_{-k} - 1)^{-1}\right)\right]}{1 + \sum\limits_{j=0}^{3}\prod\limits_{k=0}^{j}(d_k - 1) + \sum\limits_{j=1}^{4}\prod\limits_{k=1}^{j}(d_{-k} - 1)^{-1}}\right.$$
$$\left. - \frac{\sum\limits_{i=0}^{3}\left[\left(\sum\limits_{j=0}^{i}\frac{c'_j}{c_j - 1}\right)\left(\prod\limits_{k=0}^{i}(c_k - 1)\right)\right] + \sum\limits_{i=1}^{4}\left[\left(\sum\limits_{j=1}^{i}\frac{-c'_{-j}}{c_{-j} - 1}\right)\left(\prod\limits_{k=1}^{i}(c_{-k} - 1)^{-1}\right)\right]}{1 + \sum\limits_{j=0}^{3}\prod\limits_{k=0}^{j}(c_k - 1) + \sum\limits_{j=1}^{4}\prod\limits_{k=1}^{j}(c_{-k} - 1)^{-1}}\right\} \times \mathcal{M}_{5,4}(x)$$