M54 conjecture

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1 Abstract

2 Introduction

This paper proves the necessary and sufficient condition for the existence of time-invariant Nash Equilibrium for the stake-governed random turn game – Trail of Lost Pennies – introduced by [Hammond]. Trail of Lost Pennies is a variant of Tug-of-war, a class of games that has a history dates back to [some year] by [author name].

2.1 Game set up

The Trail of Lost Pennies plays on the infinite integer line with a counter initially placed at the origin. In the beginning of each turn, two players, namely Maxine (who plays to the right) and Mina (who plays to the left), wager a non-negative finite real amount, denoted a and b respectively. Then, the counter moves one unit to the right with probability $\frac{a}{a+b}$, else moves one unit to the left. Hence, the counter's location X is a discrete stochastic process $X(t) : \mathbb{N} \to \mathbb{Z}$, mapping from the game's time-step to counter's location (we take $\mathbb{N}$ to include zero).

If the counter's location tends to $+\infty$ as $t \to \infty$, Maxine and Mina receive a predefined payout, denoted m_∞ and n_∞ respectively, while receiving a predefined payout $m_{-\infty}$ and $n_{-\infty}$ respectively when the counter's location tends to $-\infty$, with the payouts constrained by $m_{-\infty} < m_\infty$ and $n_\infty < n_{-\infty}$. Therefore, we can specify our game set up entirely on the 4 parameters and denote our game by $\text{Trail}(m_{-\infty}, m_\infty, n_{-\infty}, n_\infty)$.

2.2 Motivations, definitions, and theorems

The time-invariant Nash Equilibrium of Trail of Lost Pennies is in fact characterized by a system of equations, in particular its *positive* solutions. Hence, we introduce the definition and related theorems motivate our result later.

Definition 1 (ABMN system). *Let $a_i, b_i, m_i, n_i \in \mathbb{R}$ be the non-negative finite wager of Maxine and Mina, mean payout of Maxine and Mina respectively when counter is located at $i \in \mathbb{Z}$. Then the ABMN system is the set of equations*

$$(a_i + b_i)(m_i + a_i) = a_i m_{i+1} + b_i m_{i-1} \tag{1}$$
$$(a_i + b_i)(n_i + b_i) = a_i n_{i+1} + b_i n_{i-1} \tag{2}$$
$$(a_i + b_i)^2 = b_i(m_{i+1} - m_{i+1}) \tag{3}$$
$$(a_i + b_i)^2 = a_i(n_{i-1} - n_{n+1}), \tag{4}$$

where i ranges over $\mathbb{Z}$.

Definition 2 (ABMN solution)**.** *A solution to this system of equations is said to have boundary data* $(m_{-\infty}, m_\infty, n_{-\infty}, n_\infty)$ *when*

$$\lim_{k\to\infty} m_{-k} = m_{-\infty}, \quad \lim_{k\to\infty} m_k = m_\infty, \quad \lim_{k\to\infty} n_{-k} = n_{-\infty}, \quad \lim_{k\to\infty} n_k = n_\infty.$$

For such a solution, the <u>*Mina margin*</u> *is set equal to* $\frac{n_{-\infty}-n_\infty}{m_\infty-m_{-\infty}}$. *A solution is called* <u>*positive*</u> *if* $a_i, b_i > 0$ *for all* $i \in \mathbb{Z}$. *It is called* <u>*strict*</u> *if* $m_{i+1} > m_i$ *and* $n_i > n_{i+1}$ *for* $i \in \mathbb{Z}$. *(include strict ?)*

Theorem 1 (Positive ABMN solution)**.** *(Include ?) Let* $\{(a_i, b_i, m_i, n_i) \in (0,\infty)^2 \times \mathbb{R}^2 : i \in \mathbb{Z}\}$ *be a positive ABMN solution. Then,*

1. *the solution is strict;*
2. *the solution has boundary conditions (data?)* $(m_{-\infty}, m_\infty, n_{-\infty}, n_\infty)$ *that satisfy* $m_{-\infty} < m_\infty$ *and* $n_\infty < n_{-\infty}$*;*
3. *the values* $m_{-\infty}, m_\infty, n_\infty$*, and* $n_{-\infty}$ *are real numbers. As such, the Mina margin* $\frac{n_{-\infty}-n_\infty}{m_\infty-m_{-\infty}}$ *exists and is a positive finite real number.*

Theorem 2 (Conditions for positive ABMN solution)**.** *Let* $I \subset (0,\infty)$ *equal to the set of values of the Mina margin* $\frac{n_{-\infty}-n_\infty}{m_\infty-m_{-\infty}}$*, where* $\{(a_i, b_i, m_i, n_i \in (0,\infty)^2 \times \mathbb{R}^2 : i \in \mathbb{Z}\}$ *ranges over the set of positive ABMN solutions. Then,*

1. *there exists a value* $\lambda \in (0,1]$ *such that* $I = [\lambda, \lambda^{-1}]$*;*
2. *a positive ABMN solution exists with boundary data* $(m_{-\infty}, m_\infty, n_{-\infty}, n_\infty) \in \mathbb{R}^4$ *if and only if* $m_{-\infty} < m_\infty$ *and* $n_\infty < n_{-\infty}$ *and the Mina margin* $\frac{n_{-\infty}-n_\infty}{m_\infty-m_{-\infty}} \in [\lambda, \lambda^{-1}]$*;*
3. *the value of* λ *is at most 0.999904.*

3 Main Result

[Hammond] conjectured that λ is at least 0.999902, and we will provide a computer-assisted proof that indeed $\lambda \geq 0.999902$. We first introduce some of the tools developed in [Hammond] that will be useful in our proof.

3.1 Some tools

Definition 3. *(Finite mina margin map)*

Definition 4. *Set* $w : (0,\infty) \to (1,\infty), w(x) = \sqrt{8x+1}$. *Writing* $w = w(x)$*, we further set*

$$s = \frac{(w-1)^2}{4(w+7)}, \qquad c = \frac{(w+3)^2}{16}, \qquad d = \frac{(w+3)^2}{8(w+1)} \quad \text{for } x \in (0,\infty)$$

Definition 5. *Let* $s_{-1} : (0,\infty) \to (0,\infty)$ *be given by* $s_{-1}(x) = \frac{1}{s(1/x)}$. *We now define a collection of functions* $s_i : (0,\infty) \to (0,\infty)$ *indexed by* $i \in \mathbb{Z}$. *We begin by setting* $s_0(x) = x$ *for* $x \in (0,\infty)$. *We then iteratively specify that for* $i \in \mathbb{N}_+$ *and* $x \in (0,\infty), s_i(x) = s(s_{i-1}(x))$ *and* $s_{-1}(x) = s_{-1}(s_{-i+1}(x))$. *Then, for* $j \in \mathbb{Z}$*, set* $c_j, d_j : (0,\infty) \to (0,\infty)$ *to be* $c_j(x) = c(s_j(x))$ *and* $d_j(x) = d(s_j(x))$.

Definition 6. *Set $P_0 = S_0 = 1$. For $k \in \mathbb{N}$, we iteratively specify*

$$P_{k+1}(x) = \prod_{i=0}^{k}(c_i(x) - 1) + P_k(x) \quad \text{and} \quad S_{k+1}(x) = \prod_{i=0}^{k}(d_i(x) - 1) + S_k(x).$$

Set $Q_1 = T_1 = 0$. For $k \in \mathbb{N}_+$, we then set

$$Q_{k+1}(x) = \prod_{i=1}^{k}(c_{-i}(x) - 1)^{-1} + Q_k(x) \quad \text{and} \quad T_{k+1}(x) = \prod_{i=1}^{k}(d_{-i}(x) - 1)^{-1} + T_k(x).$$

Lemma 1 (Another form of mina margin map)**.** *For $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$, the finite mina margin map takes the form*

$$\mathcal{M}_{\ell,k}(x) = \frac{x(S_k + T_\ell)}{P_k + Q_\ell}.$$

Lemma 2. *For $x \in [1/3, 3]$, $|\mathcal{M}(x) - \mathcal{M}_{5,4}(x)| \leq 6.3 \times 10^{-7}$.*

Theorem 3. $\lambda \in [0.999902, 0.999904]$.

Proof. The upper bound has been proved in [Hammond], and we will provide a computer-assisted proof for the lower bound of λ.

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