Assignment 4

Cayley Trees

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Consider Cayley trees $T(k, P)$ as defined in Problem 4 of Barabasi's book Chapter 3 on Random Networks, for $k \geq 3$.

1. Find a formula for the diameter $d_{\max}$ of $T(k, P)$ in terms of k and P.
2. Find a formula for the number of nodes N of $T(k, P)$ in terms of k and P.
3. Does $T(k, P)$ exhibit small world behavior?
   To answer that, compute the limit of $d_{\max} / \ln N$ for fixed $k \geq 3$ and P going to infinity. If this limit is a constant larger than 0 then, yes, $T(k, P)$ has the small world property.

1 Definition

A Cayley tree is a symmetric tree, constructed starting from a central node of degree k. Each node at distance d from the central node has degree k, until we reach the nodes at distance P that have degree one and are called leaves. From Albert-László Barabási. *Network Science*. Cambridge University Press, 2016. ISBN: 9781107076266. URL: http://networksciencebook.com/chapter/3#homework3.

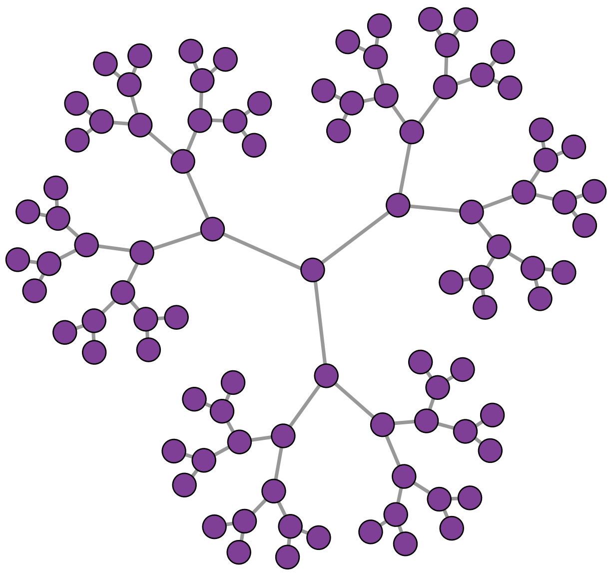

Figure 1: Cayley tree with $k = 3$ and $P = 5$.

2 Diameter

Considering the central node c of $T(k, P)$, we could denote each of the k subtrees from $T(k, P) - c$ as the *branches* of the Cayley tree. We can clearly see that the largest path on this tree must be from two leaves u and v on distinct branches. Since they are disconnected only in $T(k, P) - c$, we have $d(u, v) = d(u, c) + d(c, v) = P + P$. Therefore, $d_{\max} = 2P$.

3 Number of Nodes

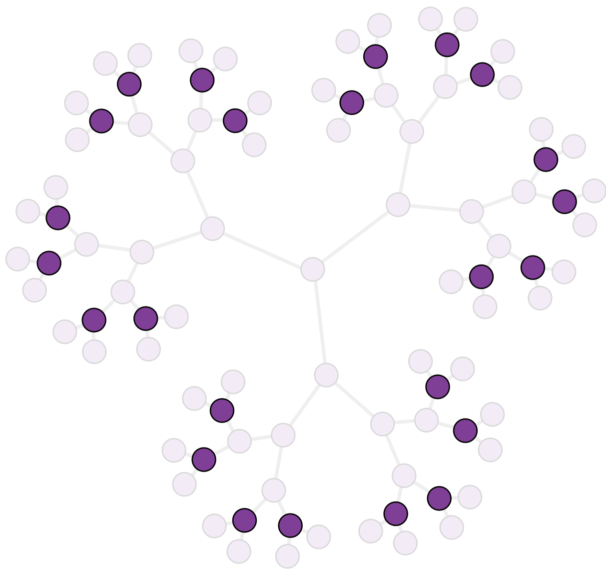

Figure 2: All 24 nodes with depth 4 of the $T(3,5)$ from figure 1.

We can detone the *depth* of a node u as the distance $d(u,c)$ from the central node c. Since each node has a unique depth, we can split the vertex set V into a partition of subsets $V_i \subseteq V$ with $0 \leq i \leq P$ such that:

$$V_i = \{v \in V \mid v \text{ has depth } i\}$$

For a layer V_i, $i \geq 2$, each node has k neighbors, one of which is the parent on V_{i-1}. Therefore, $|\,V_i\,| = (k-1) \cdot |\,V_{i-1}\,|$. The only exceptions are V_0 and V_1, with $|\,V_0\,| = 1$ and $|\,V_1\,| = k$, respectively. Expanding the recurrence we get:

$$\begin{aligned}
|\,V_i\,| &= (k-1)|\,V_{j-1}\,| \\
&= (k-1)(k-1)|\,V_{j-2}\,| \\
&= (k-1)^2|\,V_{j-2}\,| \\
&\cdots \\
&= (k-1)^{i-1}|\,V_1\,| \\
&= (k-1)^{i-1}k|\,V_0\,| \\
&= k(k-1)^{i-1}
\end{aligned}$$

Given that the maximum depth is P, we have:

$$\begin{aligned}
N = |\,V\,| = \sum_{i=0}^{P} |\,V_i\,| \\
&= |\,V_0\,| + \sum_{i=1}^{P} k(k-1)^{i-1} \\
&= 1 + k\frac{1-(k-1)^P}{1-(k-1)} \\
&= k\frac{(k-1)^P - 1}{k-2} + 1
\end{aligned}$$

4 Small World Behavior

As proposed, we need $\lim_{P\to\infty} \frac{d_{\max}}{\ln N}$. First note, however, that

$$\begin{aligned}
\lim_{P\to\infty} \frac{\ln N}{\ln(k-1)^P} &= \lim_{P\to\infty} \frac{N^{-1}\cdot \frac{dN}{dP}}{\ln(k-1)} = \frac{1}{\ln(k-1)} \lim_{P\to\infty} \frac{k\frac{(k-1)^P \ln(k-1)}{k-2}}{k\frac{(k-1)^P-1}{k-2}+1} \\
&= \lim_{P\to\infty} \frac{(k-1)^P}{(k-1)^P - 1 + \frac{k-2}{k}} = \lim_{P\to\infty} \frac{1}{1-\frac{-\frac{2}{k}}{(k-1)^P}} \\
&= 1
\end{aligned}$$

Therefore, the limit of $\frac{d_{\max}}{\ln N}$ is only positive when $\lim_{P\to\infty} \frac{d_{\max}}{\ln(k-1)^P}$ is also positive. Now,

$$\lim_{P\to\infty} \frac{d_{\max}}{\ln(k-1)^P} = \lim_{P\to\infty} \frac{2P}{P\ln(k-1)} = \frac{2}{\ln(k-1)} > 0$$

Thus,

$$\begin{aligned}
\lim_{P\to\infty} \frac{d_{\max}}{\ln N} &= \left(\lim_{P\to\infty} \frac{d_{\max}}{\ln N}\right) \times 1 \\
&= \lim_{P\to\infty} \frac{d_{\max}}{\ln N} \times \lim_{P\to\infty} \frac{\ln N}{\ln(k-1)^P} \\
&= \lim_{P\to\infty} \left(\frac{d_{\max}}{\ln N} \times \frac{\ln N}{\ln(k-1)^P}\right) \\
&= \lim_{P\to\infty} \frac{d_{\max}}{\ln(k-1)^P} \\
&= \frac{2}{\ln(k-1)}
\end{aligned}$$

For $k \geq 3$, this limit is positive, asserting that $T(k,P)$ exhibits small world behavior.