Additional Practice Problems for the Final Exam

- The following set of practice problems has been provided to supplement the earlier set of practice problems that were released. Please see the instructions provided in that document.

Problem-1: T/F Questions. [3 points each] Answer *true* or *false* to each of the following. *No justification is required for your answer.*

(a) [T/F] Consider a function, $f(x)$, that is *strictly convex* on the interval $[-2017, 2017]$. Then $\frac{d^2}{dx^2} f(x) > 0$ for all $x \in [-2017, 2017]$.

(b) [T/F] The gradient descent algorithm either diverges to $-\infty$ or converges to a local minimum or maximum.

(c) [T/F] Given a function f that is has α-strong convexity and β-smoothness. Suppose we can show that f has $\frac{\alpha}{2}$-strong convexity. Then we also expect for the gradient descent algorithm to converge faster.

(d) [T/F] Suppose we would like to identify a local minima for a function f using the gradient descent algorithm. Picking the step size to equal the inverse Hessian will get to the local minima in a single step.

Problem-2: Distributed Coloring [20 points]

Consider an undirected graph $G = (V, E)$ in which every vertex has degree at most Δ. Define a new graph $G' = (V', E')$, the ***Cartesian product*** of G with a clique of size $\Delta + 1$.

Specifically, V' is the set of pairs (v, i) for all vertices $v \in V$ and integers i with $0 \le i \le \Delta$, and E' consists of two types of edges:

(a) For each edge $\{u, v\} \in E$, there is an edge between (u, i) and (v, i) in E', for all $0 \le i \le \Delta$. (Thus, each index i forms a copy of G.)

(b) For each vertex $v \in V$, there is an edge between (v, i) and (v, j) in E', for all $i \ne j$ with $0 \le i, j \le \Delta$. (Thus each v forms a $(\Delta + 1)$-clique.)

Here is an example of this transformation with $\Delta = 3$:

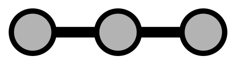

Figure 1: Graph G.

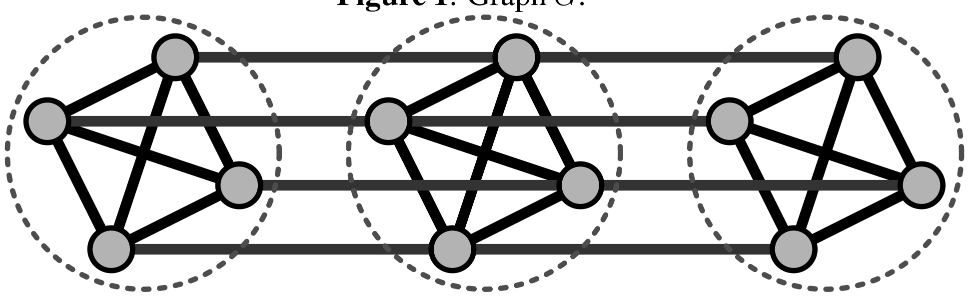

Figure 2: The Cartesian product G' of G and a clique of size 4.

(a) Let S be any *maximal* independent set of G' (i.e., adding any other vertex to S would violate independence). Prove that, for each vertex $v \in V$, S contains exactly one of the $\Delta + 1$ vertices in V' of the form (v, i).

Hint: Use the Pigeonhole Principle.

(b) Now consider a synchronous network of processes based on the graph G, where every vertex knows an upper bound Δ on the degree. Give a distributed algorithm to find a vertex $(\Delta + 1)$-coloring of G, i.e., a mapping from vertices in V to colors in $\{0, 1, \dots, \Delta\}$ such that adjacent vertices have distinct colors. The process associated with each vertex should output its color. Argue correctness.

Hint: Combine part (a) with Luby's algorithm.

(c) Analyze the expected time and communication costs for solving the coloring problem in this way, including the cost of Luby's algorithm.