Quiz 1

- Do not open this quiz booklet until you are directed to do so. Read all the instructions first.
- The quiz contains 5 problems, several with multiple parts. You have 80 minutes to earn 80 points.
- This quiz booklet contains 10 pages, including this one, and a sheet of scratch paper.
- Write your solutions in the space provided. If you run out of space, continue your answer on the back of the same sheet and make a notation on the front of the sheet.
- Do not waste time deriving facts that we have studied. Just cite results from class.
- When we ask you to give an algorithm in this quiz, describe your algorithm in English or pseudocode, and provide a short argument for correctness and running time. You do not need to provide a diagram or example unless it helps make your explanation clearer.
- Do not spend too much time on any one problem. Generally, a problem's point value is an indication of how many minutes to spend on it.
- **Please write your name on every single page of this exam.**
- Good luck!

Problem	Title	Points	Parts	Grade	Initials
0	Name	1	11		
1	True/False	18	6		
2	Resizing van Emde Boas Trees	12	2		
3	Another Algorithm for MST	20	3		
4	Parity Problems	14	3		
5	K-Sum Subset	15	1		
Total		80			

Name: ____________________

Circle your recitation:

F10	F11	F11	F12	F1	F2	F3	F11	F12	F1
Kelly	Shalom	Victor	Angus	Min	Devin	Paul	Brando	Nirvan	Daniel
R01	R02	R03	R04	R05	R06	R07	R08	R09	F10

Quiz 1-1: [18 points] T/F Questions

Mark each statement as either true or false. You have to **provide a short explanation for each**.

(a) Given an array A with n integers, we can output the set of elements of A with rank between $n/4$ and $3n/4$ in $O(n)$ time.

(b) It is possible to multiply two $n \times n$ matrices in time $o(n^3)$.

(c) Given a graph $G = (V, E)$ where $|E| = \Theta(|V|^2)$, Johnson's algorithm can be used to find all pairs shortest paths in time $o(V^3)$.

(d) Let $G = (V, E)$ and let H be the subgraph of G induced by some set of vertices $V' \subset V$. That is, $H = (V', E')$ where E' consists of all edges both of whose endpoints are in V'. Then every MST of H is a subgraph of some MST of G.

(e) Suppose a Las Vegas probabilistic algorithm is run to termination on a set of inputs. You later find out that the psuedo-random number generator used by the algorithm to generate its randomness was corrupted, and instead the algorithm was using fixed numbers for randomness. Then the output of the Las Vegas algorithm is still guaranteed to be correct.

(f) If an algorithm runs in time $\Theta(n)$ with probability 0.9999 and in time $\Theta(n^2)$ with the remaining probability, then its expected run-time is $\Theta(n)$.

Quiz 1-2: [12 points] Resizing van Emde Boas Trees

We modify the van Emde Boas data structure for storing elements in the range $\{1, \ldots, u\}$ so instead of having $\sqrt{u}$ blocks of size $\sqrt{u}$ at each node, we have 1000 blocks of size $u/1000$ each.

(a) **[6 points]** Write the recurrence for the run-time complexity of inserting an element to the new data structure. Solve the recurrence.

(b) **[6 points]** Write the recurrence for the space complexity of the new data structure and solve it.

Quiz 1-3: [20 points] Another Algorithm for MST

In this problem, you will develop another algorithm for finding the minimum spanning tree of a graph G.

- For each vertex v, choose the minimum weight edge adjacent to v and mark the edge.
- For each marked edge (u, v), contract u and v into one new vertex w. That is, every edge that was connected to u is now connected to w (with the same weight), and similarly for edges connected to v.
- Repeat until there is only one vertex left.
- Return all marked edges.

(a) [5 points] Find the worst-case run time of the algorithm.

(b) [5 points] Prove that at each iteration, all the marked edges belong to the MST of the input graph of that iteration. With this, briefly argue (in one sentence) that the algorithm returns the minimum spanning tree of G.

(c) **[10 points]** A planar graph is a graph that can be drawn on a 2D plane (e.g., on paper) without any edges crossing. Edges are parallel if they connect the same two vertices. From graph theory, we know that for a planar graph $G = (V, E)$ with no parallel edges $|E| \leq 3|V|$.

Find an $O(|V|)$ time algorithm for finding the minimum spanning tree of a planar graph with no parallel edges. Modify the algorithm and analysis from previous parts.

Hint: If you contract two vertices in a planar graph, the resulting graph is still planar.

Quiz 1-4: Parity Problems

Your friend has discovered a black magic algorithm to solve the all-pairs shortest paths problem on a weighted undirected graph $G = (V, E)$ in $O(V^2)$ time. However, due to some technical details, it only works on graphs with odd edge weights. You are tasked with designing a way to make the algorithm work on more general graphs. You may assume that all edge weights are natural numbers.

(a) **[5 points]** Describe an algorithm to produce vertex weights $h : V \to \{0, 1\}$ with the property that $h(u) - h(v) + w(u, v)$ is odd for every edge $(u, v) \in E$, where $w(u, v)$ is the weight of the edge (u, v). You may assume that there exists at least one function h satisfying the above property.

(b) **[6 points]** Prove that a such a weighting $h : V \to \{0, 1\}$ exists if G contains no cycles with an odd number of even-weight edges.

(c) **[3 points]** Design an algorithm to solve the all-pairs shortest paths problem on an undirected graph that does not contain any cycles with an odd number of even-weight edges in $O(V^2)$ time. You may assume that your friend's algorithm works as described.

Quiz 1-5: [15 points] K-Sum Subset

Suppose we have an array A with n elements, each of which is an integer in the range of $[0...50n]$. Give an $O(n \log n)$ algorithm that when given an integer $0 \leq t \leq 5 \cdot 50n$, determines whether there exist at most 5 (not necessarily distinct) elements of A which sum to t. That is, output "YES" if there exist $i_1, \ldots, i_k$ with $k \leq 5$ such that $A[i_1] + \ldots + A[i_k] = t$ and output "NO" otherwise.

Scratch page