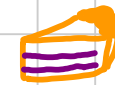


Actual pies
might look differently
(but will be delicious!)

①

Admin

- QUIZ I NEXT TUE → Carefully read the quiz handout
- Covers Lectures 1-9 (INCLUDING THIS LECTURE!)
- "META" Review session: Friday, 3/10, 7-9 PM, 34-101
In celebration of quiz being on π Day: Will have PIES!
- "Classic" Review session: Saturday, 3/11, 7-9 PM, 34-101



Same room as the lecture

Today: Linear programming (LP)

- Examples
- Standard form
- Geometric view
- Algorithms
- LP duality

Example: Optimal campaigning

How to campaign to win an election?

- Staff estimates votes obtained per \$ spent advertising in support of a particular issue

E.g.	<u>Policy/issue</u>	<u>Affected demographic</u>		
		<u>Urban</u>	<u>suburban</u>	<u>Rural</u>
y_1	Building roads	-2	5	3
y_2	Gun control	8	2	-5
y_3	Farm subsidies	0	0	10
y_4	Gasoline tax	10	0	-2

→ Each entry: How many votes will we win/lose per \$ spent on advertising on a given issue

(2)

→ Goal: Want to win majority in each demographic

	Urban	Suburban	Rural
Population:	100,000	200,000	50,000
Majority:	50,000	100,000	25,000

While spending as little dollars as needed

Let's phrase it in algebraic language!

Algebraic setup:

→ Let y_1, y_2, y_3, y_4 denote \$ spent on advertising on given issue

→ Goal: minimize $y_1 + y_2 + y_3 + y_4$ ← Total \$ spent
 subject to

$$\begin{aligned} -2y_1 + 8y_2 + 0 \cdot y_3 + 10y_4 &\geq 50,000 \leftarrow \\ 5y_1 + 2y_2 + 0 \cdot y_3 + 0 \cdot y_4 &\geq 100,000 \leftarrow \\ 3y_1 - 5y_2 + 10y_3 - 2y_4 &\geq 25,000 \leftarrow \end{aligned}$$

$y_1, y_2, y_3, y_4 \geq 0$ ← Ensure that we get all majorities
 can't "unadvertise"

→ Optimal solution:

$$\left. \begin{aligned} y_1^* &= \frac{2,050,000}{111} \approx 18,468 \\ y_2^* &= \frac{425,000}{111} \approx 3,829 \\ y_3^* &= 0 \\ y_4^* &= \frac{625,000}{111} \approx 5,631 \end{aligned} \right\} \Rightarrow \begin{aligned} \text{Total \$: } y_1^* + y_2^* + y_3^* + y_4^* &= \\ &= \frac{3,100,000}{111} \\ &\approx 27,928 \end{aligned}$$

→ Ok, but: Where is this coming from?

Linear programming (LP):

(3)

Minimize (or maximize) linear objective function

Subject to linear constraints (= inequalities & equalities)

→ variables: $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \Leftrightarrow \vec{x} \in \mathbb{R}^m$

→ objective function: $c_1 x_1 + c_2 x_2 + \dots + c_m x_m \Leftrightarrow \vec{c} \cdot \vec{x}$

→ constraints: $\sum_j A_{ij} x_j \leq b_j, \forall i \in \{1, \dots, n\}$

$$\Leftrightarrow A \cdot \vec{x} \leq \vec{b} \in \mathbb{R}^n$$

↳ $n \times m$ constraint matrix
with entries $A_{ij} \in \mathbb{R}$

Standard LP form:

$$\begin{aligned} \max \quad & \vec{c} \cdot \vec{x} \\ & A \cdot \vec{x} \leq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

What if the LP I wrote is not in this form?

No problem! Any LP can be transformed into this form!
(i.e., this formulation is universal)

To change: ① $\min \rightarrow \max$: $\vec{c} \rightarrow -\vec{c}$

② $\geq \rightarrow \leq$: multiply both sides by (-1)

③ $= \rightarrow \leq$: use both \leq & \geq + ②

④ $x_i \in \mathbb{R} \rightarrow x_i \geq 0$: introduce new variables

$$x_i^+ \geq 0 \text{ \& \> } x_i^- \geq 0$$

and substitute occurrences
of x_i with $(x_i^+ - x_i^-)$

(4)

Example:

$$\begin{aligned} \min \quad & -x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 - x_3 \leq 4 \\ & x_2 - x_3 = 2 \\ & x_1 + x_3 \geq 5 \\ & x_2, x_3 \geq 0 \end{aligned}$$

→

$$\begin{aligned} \max \quad & (x_1^+ - x_1^-) - x_2 - x_3 \\ \text{s.t.} \quad & (x_1^+ - x_1^-) - x_3 \leq 4 \\ & x_2 - x_3 \leq 2 \\ & -x_2 + x_3 \leq -2 \\ & -(x_1^+ - x_1^-) - x_3 \leq -5 \\ & x_1^+, x_1^-, x_2, x_3 \geq 0 \end{aligned}$$

Geometric view of LPs:

→ \vec{x} is a point in \mathbb{R}^m

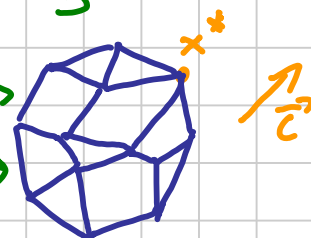
→ \vec{c} is a direction vector (length irrelevant for determining the optimal solution - gives only scaling of the objective)

⇒ $\max \vec{c} \cdot \vec{x}$ = want most extreme \vec{x} in direction of \vec{c} s.t. constraints

→ constraint $\vec{a} \cdot \vec{x} \leq b$ = halfspace bounded by the (hyper) plane

→ together constraints form a polytope

of constraints / # of rows of A with $\leq n$ polygonal facets



• Important: the polytope could be possibly no ← unbounded / empty ⇒ no solution (= LP is infeasible)

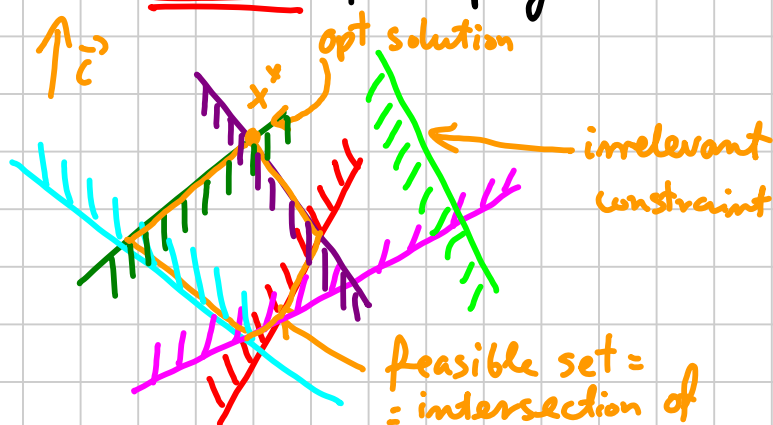
Best solution, i.e., optimal obj. value = too

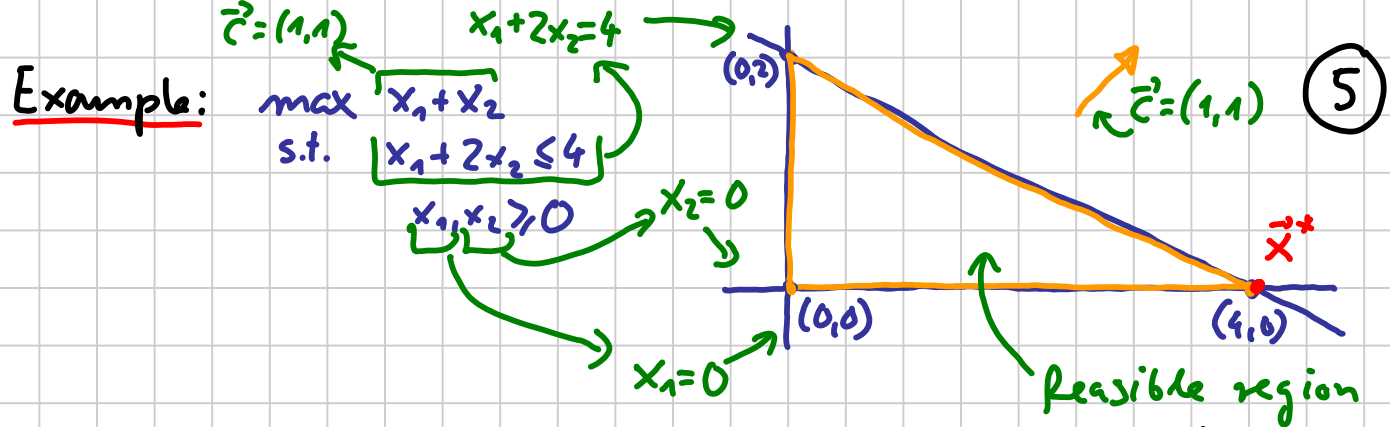
(Note: Polytope might be unbounded but still have a finite opt. obj. value)

→ Can show: if finite opt. obj. value then exists optimal solution x^* that is a vertex of the polytope

LP in 2D:

polytope → polygon
 halfspace → halfplane
 (hyper) plane → line





\Rightarrow Polygon bounded \Rightarrow need to just check all vertices

$\Rightarrow \vec{x} = (0,0) \rightarrow \vec{c}\vec{x} = 0, \vec{x} = (0,2) \rightarrow \vec{c}\vec{x} = 2, \vec{x} = (4,0) \rightarrow \vec{c}\vec{x} = 4,$

\Rightarrow opt. solution $\vec{x}^* = (4,0)$ &
opt. objective value $\vec{c}\vec{x}^* = 4$

How to solve an LP in general?

LP solving algorithms:

① Simplex: [Dantzig 1947]

\rightarrow current feasible solution \vec{x} walks from vertex to vertex of the feasible polytope, in direction of \vec{c}

\rightarrow Runtime: Practical but exponential in the worst-case

② Ellipsoid method: [Khachiyan 1979]

\rightarrow Maintains ellipsoid that is guaranteed to contain \vec{x}^*

\rightarrow In each step, shrinks the ellipsoid

\rightarrow Runtime: Polynomial in the worst-case, useful in theory, but impractical

of bits to describe A, \vec{b}, \vec{c}

③ Interior-point method: [Karmarkar 1984]

\rightarrow Current feasible solution \vec{x} moves inside the feasible polytope, vaguely in direction of \vec{c}

\rightarrow Runtime: Polynomial in the worst case & quite practical

$O(n^{3.5}L)$
 \uparrow
of constraints

Simplex

VS.

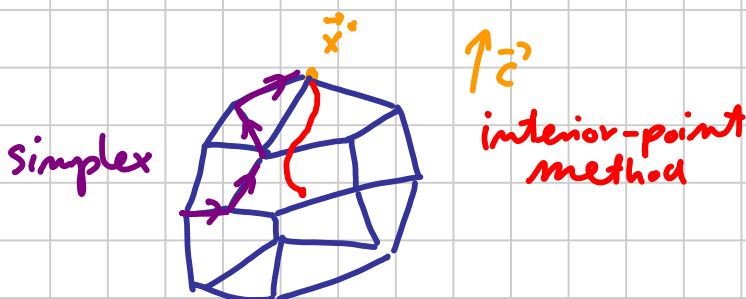
Interior-point method

(6)

→ moves on the edges of the polytope

→ highly attuned to the combinatorial structure of the constraints

→ moves inside the polytope
→ ignores much of the combinatorial structure of the constraints



CAUTION: A "close" relative to linear programming, called integer linear programming (ILP) is NP-complete ← means: (probably) **REALLY HARD** (in the worst case)

$$\begin{aligned} \text{ILP: } \max \quad & \vec{c} \cdot \vec{x} \\ \text{s.t. } & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq 0 \\ & x_1, x_2, \dots, x_n \text{ integral} \end{aligned}$$

← Very difficult to deal with algorithmically

LP Duality:

Recall the example LP we considered earlier:

$$\begin{aligned} \min \quad & y_1 + y_2 + y_3 + y_4 \\ \text{s.t. } \quad & -2y_1 + 8y_2 + 0 \cdot y_3 + 10y_4 \geq 50,000 \quad (1) \\ & 5y_1 + 2y_2 + 0 \cdot y_3 + 0 \cdot y_4 \geq 100,000 \quad (2) \\ & 3y_1 - 5y_2 + 10y_3 - 2y_4 \geq 25,000 \quad (3) \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

Earlier: we exhibited a feasible solution with obj. value $\frac{3,100,000}{111}$ and claimed this is best possible.

Can we somehow substantiate this claim?

Yes! Note: Any feasible solution to that LP has to satisfy constraints (inequalities) (1), (2), (3)

(7)

⇒ Consider taking the following linear combination of them:

$$\left. \begin{array}{l} \frac{25}{222} \cdot (1) \\ \frac{46}{222} \cdot (2) \\ \frac{14}{222} \cdot (3) \end{array} \right\} + \Rightarrow \underbrace{y_1 + y_2 + \frac{140}{222} y_3 + y_4}_{(A)} \geq \underbrace{\frac{3,100,000}{111}}_{(B)}$$

⇒ (A) has to be satisfied by only feasible solution

⇒ But: $\vec{c} \cdot \vec{y} = y_1 + y_2 + y_3 + y_4 \geq (B) \geq \frac{3,100,000}{111}$

⇒ Every feasible solution has to have obj. value $\geq \frac{3,100,000}{111}$

⇒ This proves that $\frac{3,100,000}{111}$ is indeed best possible here!

This is great, but where did the "magic" coefficients $(\frac{25}{222}, \frac{46}{222}, \frac{14}{222})$ come from ???

(in fact, "best" such coeff.)

Key insight: Finding such "magic" coefficients can be phrased by itself as an LP!

Specifically: Given an LP (in standard form)

"Primal" LP: $\max \sum_j c_j x_j$
 (P) s.t. $\sum_j A_{ij} x_j \leq b_i \quad \forall_i$
 $x_j \geq 0 \quad \forall_j$

[Compactly:
 $\max \vec{c} \cdot \vec{x}$
 s.t. $A \vec{x} \leq \vec{b}$
 $\vec{x} \geq 0$]

We consider a "dual" LP

"Dual" LP: $\min \sum_i b_i y_i$
 (D) s.t. $\sum_i A_{ij} y_i \geq c_j \quad \forall_j$
 $y_i \geq 0 \quad \forall_i$

[$\min \vec{b} \cdot \vec{y}$
 s.t. $A^T \vec{y} \geq \vec{c}$
 $\vec{y} \geq 0$]

Intuition: $\vec{x} \in \mathbb{R}^m \rightarrow x_j$ corresponds to j-th constraint in (D)
 $\vec{y} \in \mathbb{R}^n \rightarrow y_i$ ——— i-th constraint in (P)

(8)

Weak LP Duality: If \vec{x} is a feasible solution to (P)
& \vec{y} is a feasible solution to (D)

Then: $\vec{c} \cdot \vec{x} \leq \vec{b} \cdot \vec{y}$

In other words: Obj. value of any feasible \vec{y} provides an upper bound
on obj. value of any feasible \vec{x} & vice versa

\Rightarrow If \vec{x}^* is opt. solution to (P) & \vec{y}^* an opt. solution to (D)

then

$$\max_{\substack{A\vec{x} \leq \vec{b} \\ \vec{x} \geq 0}} \vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{x}^* \leq \vec{b} \cdot \vec{y}^* = \min_{\substack{A^T \vec{y} \geq \vec{c} \\ \vec{y} \geq 0}} \vec{b} \cdot \vec{y}$$

Proof of Weak LP Duality:

(Mirrors our reasoning from the example.)

Now, \vec{y} gives the "magic" coefficients for (P) & vice versa)

\rightarrow Taking a linear combination of constraints of (P) with \vec{y} as coeff.:

$$\Rightarrow \sum_i y_i \left(\sum_j A_{ij} x_j \right) \leq \sum_i y_i b_i$$

All $y_i \geq 0$, so no \leq flips!

$$\Rightarrow \vec{y} A \vec{x} = \sum_{i,j} A_{ij} y_i x_j \leq \sum_i y_i b_i = \vec{b} \cdot \vec{y} \quad (*)$$

\rightarrow Taking a linear combination of constraints of (D) with \vec{x} as coeff.

Again: $\vec{x} \geq 0$

$$\Rightarrow \sum_j x_j \sum_i A_{ij} y_i \geq \sum_j x_j c_j = \vec{c} \cdot \vec{x}$$

$$\Rightarrow \vec{x} A^T \vec{y} = \sum_{i,j} A_{ij} x_j y_i \geq \vec{c} \cdot \vec{x} \quad (**)$$

identity $v^T M u = u^T M^T v$

$$\Rightarrow \vec{c} \cdot \vec{x} \leq \vec{x} A^T \vec{y} \stackrel{(**)}{=} \vec{y} A \vec{x} \leq \vec{b} \cdot \vec{y}$$

\square

So, we know that if $OPT(P)$ is the best obj. value for (P) & $OPT(D)$ is the best obj. value for (D) then (9)

$$OPT(P) \leq OPT(D)$$

(Note: We assume here that both $OPT(P)$ and $OPT(D)$ are defined & finite, i.e., that neither (P) nor (D) are infeasible or unbounded)

Question: Is this inequality ever strict? That is, are the "magic coefficients arguments" not exhaustive?
Surprisingly (?): NO!

Strong LP duality: If \vec{x}^* & \vec{y}^* are optimal feasible solutions to (P) & (D), respectively, then:

[We skip the proof of this theorem]

$$\vec{c} \cdot \vec{x}^* = \vec{b} \cdot \vec{y}^*$$

Moreover: Only one of the following four possibilities exists:

- Both (P) & (D) have optimal solutions (as above)
- (P) is unbounded & (D) is infeasible
- (D) is unbounded & (P) is infeasible
- Both (P) & (D) are infeasible (not too interesting but possible!)

Note: → \vec{y}^* delivers best "magic" coefficients to prove optimality of \vec{x}^* & vice versa

→ Roles of (P) & (D) are completely symmetric

$$\Rightarrow \text{"Dual of (D)" = (P)}$$

→ Max Flow Min Cut Thm is a special case of the above strong duality → See Recitation notes