Problem Set 5 Solutions

This problem set is due **at 11:59pm** on **Thursday, October 20, 2016.**

EXERCISES (NOT TO BE TURNED IN)

LP

- Do Exercise 29.1-4 in CLRS on Page 856.
- Do Exercise 29.1-8 in CLRS on Page 857.
- Do Exercise 29.2-4 in CLRS on Page 864.
- Do Exercise 29.4-5 in CLRS on Page 885.

Problem 5-1. Considerate tours [50 points]

Tourists spawn in Kendall Square, s, at a rate of p. We would like to send them along MIT's network of hallways, which we model as a connected directed graph $G = (V, E)$ with no duplicate edges, to the student center, t. Each hallway $e = (i, j) \in E$ has a maximum rate of tourists it can sustain, u_{ij}, as well as a disturbance ratio, c_{ij}, such that the disturbance caused by sending tourists at a rate of k down this hallway is $k \cdot c_{ij}$. Our goal is to distribute the tourists–that is, to pick a rate of tourists for each hallway–such as to minimize the total (additive) disturbance across all hallways.

(a) [10 points] Formulate this problem as an LP.

Solution: Denote by x_{ij} the rate of tourists sent down hallway $e = (i, j)$. (Default to 0 if the hallway does not exist.) We want to minimize the total cost, subject to the capacity constraints and flow conservation.

$$
\begin{aligned}
\min \quad & \sum_{i,j} x_{ij} c_{ij} \\
\text{subject to} \quad & x_{ij} \leq u_{ij} && \forall e = (i, j) \in E \\
& \sum_{j \neq i} [x_{ji} - x_{ij}] = 0 && \forall i \in V \setminus \{s, t\} \\
& \sum_{j \neq s} [x_{sj} - x_{js}] = p \\
& \sum_{j \neq t} [x_{jt} - x_{tj}] = p \\
& x_{ij} \geq 0
\end{aligned}
$$

Note that the flow conservation constraints are not independent. That is, one could remove either the source or sink flow constraint with no effect on the feasible region.

(b) [5 points] Suppose all values are positive integers less than U. Give an upper bound on the runtime of the interior-point method cited in lecture 8 for this problem.

Solution: There are $|E|$ variables. In standard form, A is a $\Theta(|E|)$ by $|E|$ matrix. Plugging into the runtime given in the lecture notes gives $O((m + n)^{3.5} L^2) = O(\Theta(|E|)^{3.5}(\Theta(|E|^2) \log U)^2) = O(|E|^{7.5} \log^2 U)$.

(c) [20 points] Write the dual of the program you gave in part **(a)**.

Solution: In standard minimization form, we have

$$\begin{aligned}
\min \quad & \sum_{i,j} x_{ij} c_{ij} \\
\text{subject to} \quad & -x_{ij} \geq -u_{ij} && \forall e = (i,j) \in E \\
& \sum_{j \neq i} [x_{ji} - x_{ij}] \geq 0 && \forall i \in V \setminus \{s,t\} \\
& \sum_{j \neq i} [-x_{ji} + x_{ij}] \geq 0 && \forall i \in V \setminus \{s,t\} \\
& \sum_{j \neq s} [x_{sj} - x_{js}] \geq p \\
& \sum_{j \neq s} [-x_{sj} + x_{js}] \geq -p \\
& \sum_{j \neq t} [x_{jt} - x_{tj}] \geq p \\
& \sum_{j \neq t} [-x_{jt} + x_{tj}] \geq -p \\
& x_{ij} \geq 0
\end{aligned}$$

Let the first $|E|$ dual variables be w_e, the next $|V| - 2$ be d_i^+, the next $|V| - 2$ be d_i^-, and the last four $-d_s^+$,$-d_s^-$,d_t^+, and d_t^-. This gives the dual as

$$\begin{aligned}
\max \quad & (pd_t^+ - pd_t^-) - (pd_s^+ - pd_s^-) - \sum_e w_e u(e) \\
\text{subject to} \quad & -w_e + (d_j^+ - d_j^-) - (d_i^+ - d_i^-) \leq c_{ij} && \forall e = (i,j) \in E \\
& w_e, d_i^+, d_i^- \geq 0
\end{aligned}$$

which can be simplified using variables without nonnegativity constraints as

$$\begin{aligned}
\max \quad & pd_t - pd_s - \sum_e w_e u(e) \\
\text{subject to} \quad & -w_e + d_j - d_i \leq c_{ij} && \forall e = (i,j) \in E \\
& w_e \geq 0
\end{aligned}$$

.

(d) [5 points] Restrict your attention to the case $p = 1$, $u(e) = \infty$. How would this affect your answer to the previous part?

Solution: With $u(e) = \infty$, we no longer have the variables w_e.

$$\begin{aligned} \max \quad & d_t - d_s \\ \text{subject to} \quad & d_j - d_i \le c_{ij} \qquad\qquad \forall e = (i, j) \in E \end{aligned}$$

(e) [10 points] What common problem is this new LP a formulation of? Why is it not surprising that solving the considerate tours problem can solve this other problem?

Solution: This is the shortest path problem from s to t, with c_{ij}'s representing the length of edges. d_v represents the shortest distance from s to a vertex v.

It is not surprising, because if we just want to send 1 unit of flow, we should send it along the shortest/minimum cost path.

Problem 5-2. Lost in the supermarket [50 points]

You are the manager of a very poorly laid out supermarket. Your supermarket has many aisles that are individually straight but can point in all sorts of directions and intersect in arbitrary fashions. Unsurprisingly, kids keep getting lost. In order to quickly find lost children, you decide to station employees at some of the intersections, so that they can collectively look down every aisle. Of course, being a thrifty manager, you want to devote as few employees as you can to this.

Denote the set of intersections by V and the set of aisles by E. Assume that an aisle can be seen from either of the two adjacent intersections but cannot be seen from any other intersection.

(a) [10 points] Formulate this problem as an integer program. That is, you may restrict some of your optimization variables to be integral.

Solution: Let x_v denote whether an employee is at v. We want to minimize the total number of employees subject to the constraint that each edge has at least one employee on an endpoint.

$$\begin{aligned} \min \quad & \sum x_v & \\ \text{subject to} \quad & x_u + x_v \geq 1 & \forall e = \{u, v\} \in E \\ & x_v \in \{0, 1\} & \forall v \in V \end{aligned}$$

(b) [5 points] Is this problem solvable using the algorithms cited in lecture?

Solution: No, it's an ILP.

(c) [15 points] Relax the program you gave in part **(a)**. That is, drop any integrality constraints. Now, write the dual of the relaxed program.

Solution: The relaxed program is

$$\begin{aligned} \min \quad & \sum x_v & \\ \text{subject to} \quad & x_u + x_v \geq 1 & \forall e = \{u, v\} \in E \\ & - x_v \geq -1 & \forall v \in V \\ & x_v \geq 0 & \forall v \in V \end{aligned}$$

Let y_e be the variable corresponding to the constraint involving edge e in the primal. z_v is defined similarly.

$$
\begin{aligned}
\max \quad & \sum y_e - \sum z_v \\
\text{subject to} \quad & \sum_{e=\{u,v\} \in E} y_e - z_v \leq 1 && \forall v \in V \\
& y_e, z_v \geq 0 && \forall e \in E, v \in V
\end{aligned}
$$

(d) [15 points] Finally, constrain the variables in the dual program to be integers. What problem is this final program a formulation of?

Solution: This is maximum matching.

First, note that the ≤ 1 constraint in the primal is not necessary (i.e. the optimal solutions are the same without the constraint), and removing them means that you don't get the z_v's in the dual, making it more immediately recognizable as maximum matching.

Otherwise, you could argue that the z_v's could all be set to zero in an optimal solution. (Take an optimal solution with some nonzero z_v. Reduce that z_v and some adjacent y_e by z_v. Then all constraints are satisfied and the objective function value does not change.) Then, it is clear that this is the maximum matching problem.

(e) [5 points] Give a counterexample to show that strong duality does **not** necessarily hold for the integer programs given in part **(a)** and part **(d)**.

Solution: There are many trivial examples, such as the 3-cycle (need at least 2 vertices to cover, but can only put one edge in matching).

Problem 5-3. LP Oracles [50 points]

Your homework this week is to find, to within an error of $\pm\frac{1}{2}$, the optimal objective function value for a very large LP, given to you written down on a sheet of paper. Specifically, you have something of the form

$$\max_x \ c^T x$$

$$\begin{aligned} \text{subject to } Ax &\le b \\ x_i &\ge 0 \end{aligned}$$

where c and b are n by 1 vectors and A is an n by n matrix. We will treat each entry in A, b, and c as being of constant size.

Because you skipped lecture, you have no knowledge of algorithms for solving LPs. However, you happen to have a friend who can look at any LP and instantly give a feasible solution, if one exists. You realize that you can take advantage of this talent by repeatedly modifying your LP using eraser and pencil and asking your friend to give a feasible solution.[1]

Assume the LP is feasible and not unbounded.

(a) [25 points] Let the optimal objective function value be X. Give an algorithm to complete your homework assignment making only $O(\log X)$ calls to your friend and $O(n)$ edits before the first call and between each consecutive pair of calls.

Solution: Add to your LP a constraint on the objective value. First, show your friend

$$\max_x \ c^T x$$

$$\begin{aligned} \text{subject to } Ax &\le b \\ -c^T x &\le -z \end{aligned}$$

with $z = 0$.

There are two cases:

- If he reports that there is a solution, then we know that $X \ge 0$. Now try with $z = 1, 2, 4, ...$ until your friend reports that there is no solution. At this point you know $2^{i-1} \le X < 2^i$ for some constant i, and you can now binary search in that range for X.

[1]That is, your answer to this problem should not involve using any LP solver other than your friend.

- Otherwise, we know that $X < 0$. Now try with $z = -1, -2, -4, ...$ until your friend reports that there is a solution. At this point you know $-2^i \leq X < -2^{i-1}$, and you can now binary search in that range for X.

(b) [25 points] Before asking your friend to help, you realize that X could be very large, and you don't want to take so much of his time. Give an algorithm to complete your homework assignment making $O(n^2)$ edits to the LP and a constant number of calls to your friend.

Solution: Using strong duality, we can ask our friend for a solution to

$$\begin{aligned} Ax &\leq b \\ A^T y &\geq c \\ c^T x &= b^T y \\ x_i, y_i &\geq 0 \end{aligned}$$

Any feasible solution to this gives an optimal solution to the primal (by taking only the variables x). We can then evaluate the objective function on this to get X.

Problem 5-4. Constraints [50 points]

Consider an arbitrary LP P in m dimensions with dual D. Suppose we are given a particular optimal solution x to P.

(a) [25 points] Suppose the variable x_i is not equal to zero. What does this imply about the constraint corresponding to x_i in D? Prove your claim. Is the converse of your claim true?

Solution:

By strong duality, there exists an optimal solution y to the dual such that

$$c^T x = y^T Ax$$
$$(c - y^T A)x = 0$$

. Thus for each primal variable, either it is zero or the corresponding dual constraint is tight. So $x_i \neq 0$ implies the corresponding dual constraint is tight.

The converse is not true; it is possible for a dual constraint to be tight and the corresponding primal variable to be zero.

(b) [25 points] Evaluting the objective function of P on x gives you the optimal objective function value of both P and D, but what if you want to actually find an optimal solution to D? Without relying on any LP solvers, give an algorithm that can take an optimal solution to P and produce an optimal solution to D.

Assume that the number of constraints is equal to the number of variables. Assume that your given primal solution x has no zero entries. Finally, assume that the rows of A are linearly independent.

Solution: All x_i are nonzero so all dual constraints are tight. This gives n linearly independent equations for n variables, so there exists a unique solution, y. The optimal solution to the dual exists by strong duality, so y must be optimal.