1 Background

TODO: Not sure if this is the right formulation, but we'll roll with it

Definition. Let G be a group acting on an inner product space V with a distinguished basis $\mathcal{B} = (b_1, \dots, b_d)$. We say that a nonempty set of vectors $\mathcal{S}$ is a $(G, \mathcal{B})$**-reflectable step set** if:

- for all $g \in G$ and $s \in \mathcal{S}$, we have $g \cdot s \in S$; and
- for all $s \in \mathcal{S}$ and $1 \leq i \leq d$, there is an integer c_i such that $\langle s, b_i \rangle \in \{-c_i, 0, c_i\}$.

If additionally no proper subset of $\mathcal{S}$ is also $(G, \mathcal{B})$-reflectable, we say $\mathcal{S}$ is **irreducible**.

TODO: Exposition on root systems to be written.

Proposition 1. *Let Φ be a root system of rank d with associated Weyl group W. Let $\Delta = (\alpha_1, \dots, \alpha_i)$ be a choice of simple roots for Φ and $(\check{\omega}_1, \dots, \check{\omega}_d)$ be the corresponding fundamental coweights. Then $\mathcal{S}^{(i)} := W \cdot \check{\omega}_i$ is a (W, Δ)-reflectable step set if and only if $\check{\omega}_i$ is cominiscule.*

Let us call these $\mathcal{S}^{(i)}$ **cominiscule models**. Note that by definition, cominiscule models are irreducible.

Example 1. *For a root system of type A_d, all fundamental coweights are cominiscule. Using the usual simple roots $\alpha_i = e_i - e_{i+1}$, the step set $\mathcal{S}^{(1)}$ in the coweight basis is*

$$\mathcal{S}^{(1)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \right\},$$

so that the corresponding stepset inventory is $S^{(1)}(\mathbf{z}) = z_1 + \sum_{i=1}^{d-1} \frac{z_{i+1}}{z_i} + \frac{1}{z_d}$.

2 Well-Behaved Models

For a step set $\mathcal{S}$, let $Q_{\mathcal{S}}(\mathbf{z}, t) := \sum_{p_1, \dots, p_d, n \geq 0} \mathrm{Walk}_{\mathcal{S}}(\mathbf{0} \to (p_1, \dots, p_d)) z_1^{p_1} \dots z_d^{p_d} t^n$ be the generating function for (unweighted) walks with steps in $\mathcal{S}$, staying in the positive orthant.

Below, we say that a step set $\mathcal{S}$ is **well-behaved** if there is some Laurent polynomial G such that

$$Q_{\mathcal{S}}(\mathbf{a}, t) = \mathrm{Diag} \left(\frac{G(\mathbf{z})}{\prod_{i=1}^{d} (1 - z_i) \cdot \left(1 - t z_1 \dots z_d S(\frac{\mathbf{a}}{\mathbf{z}})\right)} \right),$$

where we write $\frac{\mathbf{a}}{\mathbf{z}}$ as as a shorthand for $(\frac{a_1}{z_1}, \cdots \frac{a_d}{z_d})$. In this case, we write $\mathcal{V} := \mathcal{V}(H)$ for the vanishing locus of the denominator $\prod_{i=1}^{d}(1-z_i) \cdot \big(1 - tz_1 \ldots z_d S(\frac{\mathbf{a}}{\mathbf{z}})\big)$.

TODO: Do we need to state some non-divisibility properties for G?

Proposition 2. *Given a well-behaved step set $\mathcal{S}$, the strata of $\mathcal{V}$ are (as sets),*

$$\Sigma_I = \left\{(\mathbf{z} \in \mathbb{C}^d, t) \ : z_i = 1 \text{ and } t = \big(z_1 \ldots z_d S(\tfrac{\mathbf{a}}{\mathbf{z}})\big)^{-1}\right\}; \qquad T_I = \left\{(\mathbf{z} \in \mathbb{C}^d, t) \ : z_i = 1\right\},$$

where I is a subset of $\{1, \ldots, d\}$. Moreover, all critical points of $\mathcal{V}$ in T_I are also in Σ_I.

Proof. TODO: Prove this. I don't know how to argue the geometry. The second statement is easy: the relevant matrix is lower-triangular on T_I°. □

Let Σ_I° denote the points of Σ_I which are not in $\Sigma_{I'}$ for any larger $I \subset I'$; that is, for which $z_j \neq 1$ for all $j \notin I$.

Theorem 3. *Given a well-behaved step set $\mathcal{S}$, the critical points contained in the open stratum Σ_I° are the solutions $\mathbf{p} = (p_1, \ldots, p_d)$ to the equations $p_j \partial_j S(\frac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}} = 0$ for all $j \in \{1, \ldots, n\} \setminus I$ and $p_i = 1$ for $i \in I$.*

TODO: Do the critical point equations assume some additional geometry? I'm nervous because I have not said "transverse" at any point...

Proof. Begin by defining the notation $I = \{i_1, \ldots, i_k\}$, and $\tilde{S}(\mathbf{z}) := 1 - tz_1 \cdots z_d S(\frac{\mathbf{a}}{\mathbf{z}})$, as well as $t_0 = \Big(p_1 \cdots p_d S(\frac{\mathbf{a}}{\mathbf{p}})\Big)^{-1}$.

The critical point equations of Pemantle–Wilson [CITE: Equations (8.3.1–2)] show that $\mathbf{p} \in \Sigma_I^\circ$ is a critical point of $\mathcal{V}$ precisely when the following determinants vanish for all $j \in \{1, \ldots, d\} \setminus I$:

$$0 = \det \begin{bmatrix} -z_{i_1} & & 0 & 0 & 0 \\ & \ddots & & & \\ 0 & & -z_{i_k} & 0 & 0 \\ z_{i_1}\partial_{i_1}\tilde{S}(\mathbf{z}) & & z_{i_k}\partial_{i_k}\tilde{S}(\mathbf{z}) & z_j\partial_j\tilde{S}(\mathbf{z}) & -tz_1 \cdots z_d S(\frac{\mathbf{a}}{\mathbf{z}}) \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}_{\mathbf{z}=\mathbf{p},\ t=t_0}. \tag{1}$$

Performing a cofactor expansion along the last row shows that Equations (1) are equivalent to

$$0 = \Big[(z_{i_1} \cdots z_{i_k})\big(t(z_1 \cdots z_d)S(\tfrac{\mathbf{a}}{\mathbf{z}}) + z_j\partial_j\tilde{S}(\mathbf{z})\big)\Big]_{\mathbf{z}=\mathbf{p},\ t=t_0}$$

$$0 = t_0 S(\tfrac{\mathbf{a}}{\mathbf{p}}) - t_0 S(\tfrac{\mathbf{a}}{\mathbf{p}}) - t_0\Big[(z_1 \cdots z_j^2 \cdots z_d)\partial_j S(\tfrac{\mathbf{a}}{\mathbf{z}})\Big]_{\mathbf{z}=\mathbf{p}}$$

$$0 = p_j \partial_j \log S(\tfrac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}}$$

Since log is a monotonic function, $\partial_j \log S(\frac{\mathbf{a}}{\mathbf{z}}) = 0$ if and only if $\partial_j S(\frac{\mathbf{a}}{\mathbf{z}}) = 0$. □

TODO: The last steps feel a little bit illegal. I think that one could be more careful and get to the same conclusion, but I'm not sure what to shore up. I'm also constantly worried about $p_j = 0$. It seems that some casework needs to be done, and it's very possible that other solutions exist, but I am going to pretend as if not.

Remark 4. *Compared to the notation of Pemantle–Wilson, we have permuted the factors f_i so that the coordinates corresponding to I would be placed in the leftmost columns of the relevant minor. We have also written k where they would write $d-k$, or perhaps $(d+1)-(k+1)$.*

3 The Cominiscule Model $\mathcal{S}^{(1)}$ in Type A_d

Recall that $S^{(1)}$, the stepset generating function for the cominiscule Model

Proposition 5. *The cominiscule model $\mathcal{S}^{(1)}$ is well-behaved.*

Proof. TODO: Prove this. I think this is pretty reasonable to show using Theorem 7.1 of Melczer–Mishna, but haven't worked through the details. $\square$

As described in Example 1, $S^{(1)} = z_1 + \sum_{i=1}^{d-1} \frac{z_{i+1}}{z_i} + \frac{1}{z_d}$, which gives an explicit form to the solutions in Theorem 3:

$$p_j \partial_j S(\tfrac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}} = \begin{cases} -a_1 \cdot \frac{1}{p_1} + \frac{a_2}{a_1} \cdot \frac{p_1}{p_2} & \text{if } j = 1 \\ -\frac{a_d}{a_{d-1}} \cdot \frac{p_{d-1}}{p_d} + \frac{1}{a_d} \cdot p_d & \text{if } j = d \\ -\frac{a_j}{a_{j-1}} \cdot \frac{p_{j-1}}{p_j} + \frac{a_{j+1}}{a_j} \cdot \frac{p_j}{p_{j+1}} & \text{otherwise} \end{cases}$$

Clearing denominators, the equations become

$$\begin{cases} a_1^2 p_2 & = a_2 \cdot p_1^2 & \text{if } 1 \notin I \\ a_d^2 \cdot p_{d-1} & = a_{d-1} \cdot p_d^2 & \text{if } d \notin I \\ a_j^2 \cdot p_{j+1} p_{j-1} & = a_{j-1} a_{j+1} \cdot p_j^2 & \text{for } j \in \{2, \dots, d-1\} \setminus I \end{cases}$$

Conjecture 6. *The smooth critical points of the model $\mathcal{S}^{(1)}$ in Σ_I° are $(\zeta a_1, \dots, \zeta^d a_d)$ when $I = \{1, \dots, d\}$ where ζ is a $(d-1)^{th}$ root of unity.*

For each subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $\{1, \dots, d\}$, denote its complement by $J = \{1, \dots, d\} \setminus I$. Define also $i_0 = 0, i_{k+1} = d+1$, and $a_0 = a_{k+1} = 1$.

Then for each proper I, there is a unique critical point $\mathbf{p} = (p_1, \dots, p_d)$ of the model $\mathcal{S}^{(1)}$ in the open stratum Σ_I°. It has coordinates $p_i = 1$ for all $i \in I$, and $p_j = a_j(a_{\ell-1}a_\ell)^{-\frac{d-1}{d}(i_\ell - j+1)}$ for all $j \in J$ such that $i_{\ell-1} < j < i_\ell$.

Proof. TODO: Prove or correct the above conjecture.

The system is "tridiagonal" (nonlinear, but same idea) and so one can probably solve it by induction; the observations below are simply what I got after computing out from p_1 to p_4 (and back again). It seems to be rather difficult to keep track of the roots of unity carefully so I just went back at the end and did a backwards "engineer's induction" to determine that the root of unity had to have order dividing $d+1$.

Some calculations suggest that when $I = \{1, \dots, d\}$, the solutions are $p_i = \zeta^i a_i$, where ζ is a $(d{+}1)^{\text{th}}$ root of unity. When I is smaller, things are more annoying because the "endpoints" of the intervals $[i_\ell, i_{\ell+1}]$ will generally contribute to the weights; for instance if $I = \{1, \dots, d-1\}$, the same calculations suggest that $p_i = \zeta^i a_i a_d^{(i-d)\frac{d-1}{d}}$.

However, it does seem like these extra weights will guarantee that the $\zeta = 1$ whenever a_d is not a root of unity; that is, whenever $a_d \neq 1$. (But $a_i \neq 1$ is probably a foundational assumption because otherwise there would be cancellation between G and the denominator!) This does agree with the July 12 calculations for $d = 2$. $\square$

References