# 1 Background

## 1.1 Root Systems

Recall that a **(crystallographic) root system** is a finite spanning set of a real inner product space $V$ such that

(1) If $\alpha \in \Phi$ and $k\alpha \in \Phi$, then $k = \pm 1$.

(2) Let $s_\alpha$ be the reflection across the hyperplane orthogonal to $\alpha$. If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) \in \Phi$.

(3) If $\alpha, \beta \in \Phi$ then $\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \frac{1}{2}\mathbb{Z}$.

(#2 is the most important.) The elements of a root system $\Phi$ are called **roots**.

We say that a tuple of roots $\Delta = (\alpha_1, \ldots, \alpha_d)$ in $\Phi$ is a choice of **simple roots** if and only if $\Delta$ is a basis of $V$, and for every root $\beta = \sum b_i \alpha_i$ such that all $b_i$ have the same sign.

Every root system admits a (non-unique) choice of simple roots $\Delta$. The elements of the dual basis of $\Delta$ (that is, the tuple $(\check{\omega}_1, \ldots, \check{\omega}_d)$ such that $\langle \alpha_i, \check{\omega}_j\rangle = \delta_{i,j}$) are called the **fundamental coweights**.

The **dominant chambre** $\mathcal{C}$ is the set of vectors $\mathbf{z} \in V$ such that $\langle \mathbf{z}, \alpha_i\rangle \geq 0$. Equivalently,

$$\mathcal{C} = \left\{ \sum_{k=1}^{d} z_k \check{\omega}_k : z_k \geq 0 \text{ for all } k \right\}$$

In this formulation it is clear that the dominant chambre is precisely the "positive orthant" in the coweight basis.

The root systems are completely classified in the following sense: a root system is **reducible** if it can be written as $\Phi_1 \cup \Phi_2$ where $\Phi_1$ is a root system for a subspace $U \subsetneq V$ and $\Phi_2$ is a root system for $U^\perp$. There are four infinite families of root systems (the **classical types** $A_d, B_d, C_d$ and $D_d$) and five others (the **exceptional types** $E_6, E_7, E_8, F_4$ and $G_2$). The subscript in the name is the dimension of $V$.

## 1.2 Reflectable Walks

**Definition.** Let $G$ be a group acting on a real inner product space $V$ with a distinguished basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_d)$. We say that a nonempty set of vectors $\mathcal{S}$ is a $(G, \mathcal{B})$**-reflectable step set** if:

- for all $g \in G$ and $s \in \mathcal{S}$, we have $g \cdot s \in \mathcal{S}$; and
- for all $s \in \mathcal{S}$ and $1 \leq i \leq d$, there is an integer $c_i$ such that $\langle s, \mathbf{b}_i\rangle \in \{-c_i, 0, c_i\}$.

If additionally no proper subset of $\mathcal{S}$ is also $(G, \mathcal{B})$-reflectable, we say $\mathcal{S}$ is **transitive**.

Although usually formulated slightly differently, the following "Proposition" may be taken as the definition of a **cominiscule coweight**:

**Proposition 1.** *Let $\Phi$ be a root system of rank $d$ with associated Weyl group $W$. Let $\Delta = (\alpha_1, \ldots, \alpha_i)$ be a choice of simple roots for $\Phi$ and $(\breve{\omega}_1, \ldots, \breve{\omega}_d)$ be the corresponding fundamental coweights. Then $\mathcal{S}^{(i)} := W \cdot \breve{\omega}_i$ is a $(W, \Delta)$-reflectable step set if and only if $\breve{\omega}_i$ is cominiscule.*

Let us call these $\mathcal{S}^{(k)}$ **cominiscule models**. Note that by definition, cominiscule models are transitive. Since the $c_i$ in the definition of reflectability are all 1 (and in particular, are independent of $k$), the union of cominiscule models is also a valid $(W, \Delta)$-reflectable step set.

**Example 1.** *For a root system of type $A_d$, all fundamental coweights are cominiscule. Using the usual simple roots $\alpha_i = e_i - e_{i+1}$, the step set $\mathcal{S}^{(1)}$ in the coweight basis is*

$$\mathcal{S}^{(1)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \right\},$$

*so that the corresponding stepset inventory is $S^{(1)}(\mathbf{z}) = z_1 + \sum_{i=1}^{d-1} \frac{z_{i+1}}{z_i} + \frac{1}{z_d}$.*

**Remark 2.** *Type $A_d$ is somehow the most interesting case. For a root system of types $B_d$ or $C_d$, only $\breve{\omega}_1$ is cominiscule. For a root system of type $D_d$, only $\breve{\omega}_1$ and $\breve{\omega}_2$ are cominiscule. There are no cominiscule coweights of type $G_2$, and I don't know about the remaining types ($E_6, E_7, E_8$, and $F_4$).*

# 2 Well-Behaved Models

For a step set $\mathcal{S}$, let $Q_{\mathcal{S}}(\mathbf{z}, t) := \sum_{p_1, \ldots, p_d, n \geq 0} \mathrm{Walk}_{\mathcal{S}}(\mathbf{0} \to (p_1, \ldots, p_d)) z_1^{p_1} \ldots z_d^{p_d} t^n$ be the generating function for (unweighted) walks with steps in $\mathcal{S}$, staying in the positive orthant.

Below, we say that a step set $\mathcal{S}$ is **well-behaved** if there is some Laurent polynomial $G$ such that

$$Q_{\mathcal{S}}(\mathbf{a}, t) = \mathrm{Diag}\left( \frac{G(\mathbf{z})}{\prod_{i=1}^{d} (1 - z_i) \cdot \left(1 - t z_1 \ldots z_d S(\frac{\mathbf{a}}{\mathbf{z}})\right)} \right),$$

where we write $\frac{\mathbf{a}}{\mathbf{z}}$ as as a shorthand for $(\frac{a_1}{z_1}, \cdots \frac{a_d}{z_d})$. In this case, we write $\mathcal{V} := \mathcal{V}(H)$ for the vanishing locus of the denominator $\prod_{i=1}^{d} (1 - z_i) \cdot \left(1 - t z_1 \ldots z_d S(\frac{\mathbf{a}}{\mathbf{z}})\right)$.

TODO: Do we need to state some non-divisibility properties for $G$?

**Proposition 3.** *Given a well-behaved step set $\mathcal{S}$, the strata of $\mathcal{V}$ are (as sets),*

$$\Sigma_I = \left\{(\mathbf{z} \in \mathbb{C}^d, t) \ : z_i = 1 \text{ and } t = \left(z_1 \dots z_d S(\tfrac{\mathbf{a}}{\mathbf{z}})\right)^{-1}\right\}; \qquad T_I = \left\{(\mathbf{z} \in \mathbb{C}^d, t) \ : z_i = 1\right\},$$

*where $I$ is a subset of $\{1, \dots, d\}$. Moreover, all critical points of $\mathcal{V}$ in $T_I$ are also in $\Sigma_I$.*

*Proof.* TODO: Prove this. I don't know how to argue the geometry. The second statement is easy: the relevant matrix is lower-triangular on $T_I^\circ$. $\square$

Let $\Sigma_I^\circ$ denote the points of $\Sigma_I$ which are not in $\Sigma_{I'}$ for any larger $I \subset I'$; that is, for which $z_j \neq 1$ for all $j \notin I$.

**Theorem 4.** *Given a well-behaved step set $\mathcal{S}$, the critical points contained in the open stratum $\Sigma_I^\circ$ are the solutions $\mathbf{p} = (p_1, \dots, p_d)$ to the equations $p_j \partial_j S(\frac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}} = 0$ for all $j \in \{1, \dots, n\} \setminus I$ and $p_i = 1$ for $i \in I$.*

TODO: Do the critical point equations assume some additional geometry? I'm nervous because I have not said "transverse" at any point...

*Proof.* Begin by defining the notation $I = \{i_1, \dots, i_k\}$, and $\tilde{S}(\mathbf{z}) := 1 - t z_1 \cdots z_d S(\frac{\mathbf{a}}{\mathbf{z}})$, as well as $t_0 = \left(p_1 \cdots p_d S(\frac{\mathbf{a}}{\mathbf{p}})\right)^{-1}$.

The critical point equations of Pemantle–Wilson [CITE: Equations (8.3.1–2)] show that $\mathbf{p} \in \Sigma_I^\circ$ is a critical point of $\mathcal{V}$ precisely when the following determinants vanish for all $j \in \{1, \dots, d\} \setminus I$:

$$0 = \det \begin{bmatrix} -z_{i_1} & & 0 & 0 & 0 \\ & \ddots & & & \\ 0 & & -z_{i_k} & 0 & 0 \\ z_{i_1}\partial_{i_1}\tilde{S}(\mathbf{z}) & & z_{i_k}\partial_{i_k}\tilde{S}(\mathbf{z}) & z_j \partial_j \tilde{S}(\mathbf{z}) & -t z_1 \cdots z_d S(\frac{\mathbf{a}}{\mathbf{z}}) \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}_{\mathbf{z}=\mathbf{p},\ t=t_0}. \tag{1}$$

Performing a cofactor expansion along the last row shows that Equations (1) are equivalent to

$$\begin{aligned} 0 &= \left[(z_{i_1} \cdots z_{i_k})\big(t(z_1 \cdots z_d)S(\tfrac{\mathbf{a}}{\mathbf{z}}) + z_j \partial_j \tilde{S}(\mathbf{z})\big)\right]_{\mathbf{z}=\mathbf{p},\ t=t_0} \\ 0 &= t_0 S(\tfrac{\mathbf{a}}{\mathbf{p}}) - t_0 S(\tfrac{\mathbf{a}}{\mathbf{p}}) - t_0\left[(z_1 \cdots z_j^2 \cdots z_d)\partial_j S(\tfrac{\mathbf{a}}{\mathbf{z}})\right]_{\mathbf{z}=\mathbf{p}} \\ 0 &= p_j \partial_j \log S(\tfrac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}} \end{aligned}$$

Since log is a monotonic function, $\partial_j \log S(\frac{\mathbf{a}}{\mathbf{z}}) = 0$ if and only if $\partial_j S(\frac{\mathbf{a}}{\mathbf{z}}) = 0$. $\square$

TODO: The last steps feel a little bit illegal. I think that one could be more careful and get to the same conclusion, but I'm not sure what to shore up. I'm also constantly worried about $p_j = 0$. It seems that some casework needs to be done, and it's very possible that other solutions exist, but I am going to pretend as if not.

**Remark 5.** *Compared to the notation of Pemantle–Wilson, we have permuted the factors $f_i$ so that the coordinates corresponding to $I$ would be placed in the leftmost columns of the relevant minor. We have also written $k$ where they would write $d-k$, or perhaps $(d+1)-(k+1)$.*

# 3 The Cominiscule Model $\mathcal{S}^{(1)}$ in Type $\mathrm{A}_d$

**Proposition 6.** *The step set $\mathcal{S}^{(1)}$ for the root system $A_1$ is well-behaved.*

*Proof.* TODO: Prove this. I think this is pretty reasonable to show using Theorem 7.1 of Melczer–Mishna, but haven't worked through the details. □

As described in Example 2, $S^{(1)} = z_1 + \sum_{i=1}^{d-1} \frac{z_{i+1}}{z_i} + \frac{1}{z_d}$, which gives an explicit form to the solutions in Theorem 4:

$$p_j \partial_j S(\tfrac{\mathbf{a}}{\mathbf{z}})\Big|_{\mathbf{z}=\mathbf{p}} = \begin{cases} -a_1 \cdot \frac{1}{p_1} + \frac{a_2}{a_1} \cdot \frac{p_1}{p_2} & \text{if } j = 1 \\ -\frac{a_d}{a_{d-1}} \cdot \frac{p_{d-1}}{p_d} + \frac{1}{a_d} \cdot p_d & \text{if } j = d \\ -\frac{a_j}{a_{j-1}} \cdot \frac{p_{j-1}}{p_j} + \frac{a_{j+1}}{a_j} \cdot \frac{p_j}{p_{j+1}} & \text{otherwise} \end{cases}$$

Clearing denominators, the equations become

$$\begin{cases} a_1^2 p_2 & = a_2 \cdot p_1^2 & \text{if } 1 \notin I \\ a_d^2 \cdot p_{d-1} & = a_{d-1} \cdot p_d^2 & \text{if } d \notin I \\ a_j^2 \cdot p_{j+1} p_{j-1} & = a_{j-1} a_{j+1} \cdot p_j^2 & \text{for } j \in \{2, \ldots, d-1\} \setminus I \end{cases}$$

**Conjecture 7.** *The smooth critical points of the model $\mathcal{S}^{(1)}$ in $\Sigma_I^\circ$ are $(\zeta a_1, \ldots, \zeta^d a_d)$ when $I = \{1, \ldots, d\}$ where $\zeta$ is a $(d-1)^{th}$ root of unity.*

*For each subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $\{1, \ldots, d\}$, denote its complement by $J = \{1, \ldots, d\} \setminus I$. Define also $i_0 = 0, i_{k+1} = d+1$, and $a_0 = a_{k+1} = 1$.*

*Then for each proper $I$, there is a unique critical point $\mathbf{p} = (p_1, \ldots, p_d)$ of the model $\mathcal{S}^{(1)}$ in the open stratum $\Sigma_I^\circ$. It has coordinates $p_i = 1$ for all $i \in I$, and $p_j = a_j (a_{\ell-1} a_\ell)^{-\frac{d-1}{d}(i_\ell - j + 1)}$ for all $j \in J$ such that $i_{\ell-1} < j < i_\ell$.*

*Proof.* TODO: Prove or correct the above conjecture.

The system is "tridiagonal" (nonlinear, but same idea) and so one can probably solve it by induction; the observations below are simply what I got after computing out from $p_1$ to $p_4$ (and back again). It seems to be rather difficult to keep track of the roots of unity carefully so I just went back at the end and did a backwards "engineer's induction" to determine that the root of unity had to have order dividing $d+1$.

Some calculations suggest that when $I = \{1, \ldots, d\}$, the solutions are $p_i = \zeta^i a_i$, where $\zeta$ is a $(d+1)^{\text{th}}$ root of unity. When $I$ is smaller, things are more annoying because the "endpoints" of the intervals $[i_\ell, i_{\ell+1}]$ will generally contribute to the weights; for instance if $I = \{1, \ldots, d-1\}$, the same calculations suggest that $p_i = \zeta^i a_i a_d^{(i-d)\frac{d-1}{d}}$.

However, it does seem like these extra weights will guarantee that the $\zeta = 1$ whenever $a_d$ is not a root of unity; that is, whenever $a_d \neq 1$. (But $a_i \neq 1$ is probably a foundational assumption because otherwise there would be cancellation between $G$ and the denominator!) This does agree with the July 12 calculations for $d = 2$. $\square$

# References