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Tableau sequences, open diagrams, and Baxter families

Sophie Burrill [a], Julien Courtiel [a], Eric Fusy [b,c], Stephen Melczer [d,e], Marni Mishna [a]

[a] *Department of Mathematics, Simon Fraser University, Burnaby, Canada*

[b] *PIMS (CNRS UMI 3069), University of British Columbia, Vancouver, Canada*

[c] *LIX, École Polytechnique, Palaiseau, France*

[d] *Cheriton School of Computer Science, University of Waterloo, Waterloo, ON, Canada*

[e] *U. Lyon, CNRS, ENS de Lyon, Inria, UCBL, Laboratoire LIP, France*

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ABSTRACT

Walks on Young's lattice of integer partitions encode many objects of algebraic and combinatorial interest. Chen et al. established connections between such walks and arc diagrams. We show that walks that start at $\varnothing$, end at a row shape, and only visit partitions of bounded height are in bijection with a new type of arc diagram — open diagrams. Remarkably, two subclasses of open diagrams are equinumerous with well known objects: standard Young tableaux of bounded height, and Baxter permutations. We give an explicit combinatorial bijection in the former case, and a generating function proof and new conjecture in the second case.

1. Introduction

The lattice of partition diagrams, where domination is given by inclusion of Ferrers diagrams, is known as Young's lattice. Walks on this lattice are important since they encode many objects of combinatorial and algebraic interest. A walk on Young's lattice can be listed as a sequence of Ferrers diagrams such that at most a single box is added or deleted at each step. A class of such sequences is

E-mail addresses: srb7@sfu.ca (S. Burrill), jcourtie@math.ubc.ca (J. Courtiel), fusy@lix.polytechnique.fr (E. Fusy), smelczer@uwaterloo.ca (S. Melczer), mmishna@sfu.ca (M. Mishna).

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also known as a *tableau family*. It is well known that there are several combinatorial classes in explicit bijection with tableau families ending in an empty shape, in particular when there are restrictions on the height of the tableaux which appear.

In this work we launch the study of tableau families that start at the empty partition and end with a partition composed of a single part: $\lambda = (m),\ m \geq 0$. Additionally, they are bounded, meaning that they only visit partitions that have at most k parts, for some fixed k. Remarkably, these have direct connections to both Young tableaux of bounded height and Baxter permutations. More precisely, we adapt results of Chen et al. [14] to the open diagrams of Burrill et al. [12], and use generating results of Bousquet-Mélou and Xin [9] to give proofs that these two classic combinatorial classes are in bijection with bounded height tableau families.

1.1. Part 1. Oscillating tableaux and Young tableaux of bounded height

The first tableau family that we consider is the set of *oscillating tableaux* with height bounded by k. These appear in the proofs of results on partitions avoiding certain nesting and crossing patterns [14], although they have a much longer history. They appear in the representation theory of the symplectic group, and elsewhere as up–down tableaux [6,35]. Our first main result is a new bijection connecting oscillating tableaux to the class of standard Young tableau of bounded height. Young tableaux are more commonly associated with oscillating tableau with no deletion step, but ours is a very different bijection. This result demonstrates a new facet of the ubiquity of Young tableaux.

Theorem 1. *The set of oscillating tableaux of size n with height bounded by k, which start at the empty partition and end in a row shape $\lambda = (m)$, is in bijection with the set of standard Young tableaux of size n with height bounded by $2k$, with m odd columns.*

The proof of Theorem 1 is by an explicit bijection between the two classes, an example of which is illustrated in Fig. 1. A slightly less refined version of this theorem was conjectured in an extended abstract version of this work [13]. Independently, but simultaneously to our own work, Krattenthaler [26] determined a different bijective map. Notably, he gave the interpretation of the m parameter as the number of odd columns.

One consequence of the bijective map is the symmetric joint distribution of two kinds of nesting patterns inside the class of involutions. Enumerative formulas for Young tableaux of bounded height have been known for almost half a century [20,21,5], but new generating function results can be derived from Theorem 1, notably an expression which can be written as a diagonal of a multivariate rational function. The analytic consequences of Theorem 1 are the subject of Section 4.3.

1.2. Part 2. Hesitating tableaux and Baxter permutations

In the second part, we consider the family of *hesitating tableaux*. These tableau sequences appear in studies of set partitions avoiding so-called *enhanced* nesting and crossing patterns. We make a generating function argument to connect hesitating tableaux that end in a row shape to Baxter permutations. A first computational proof of this identity was produced by Xin and Zhang [36]. Here we offer a slight variation on the computation and provide the intermediary details, using formulas of Bousquet-Mélou and Xin [9]. Specifically, the result is the following.

Theorem 2. *The number of hesitating tableaux of length $2n$ of height at most two and ending in a row is equal to the number B_{n+1} of Baxter permutations of length $n+1$, where*

$$B_n = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{1}$$

This theorem is a good candidate for a combinatorial proof. Baxter numbers have been described as the "big brother" of the well known Catalan numbers: they are the counting series for many

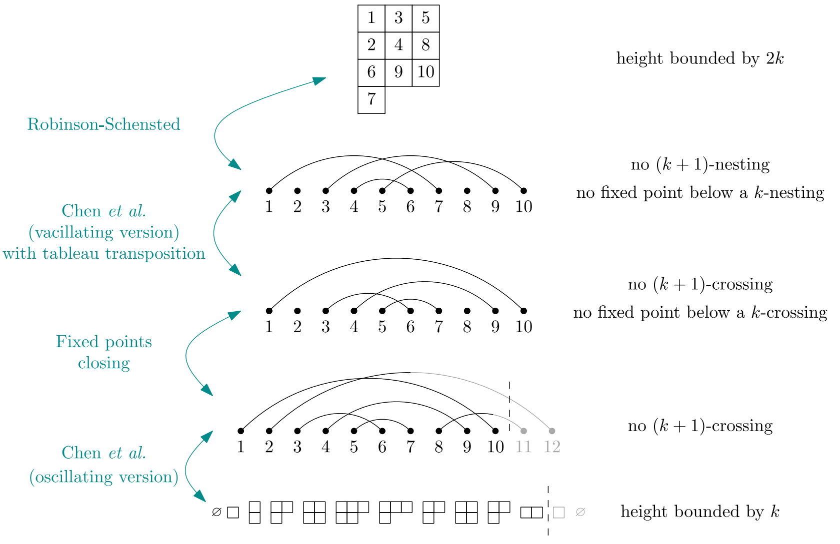


Fig. 1. The bijection behind Theorem 1. *From top to bottom:* standard Young tableau of size n with height bounded by k and m odd columns; involution diagram of size n with m fixed points and no *enhanced* $(k+1)$-nesting; involution diagram of size n with m fixed points, and no $(k+1)$-crossing, nor fixed point below a k-crossing; matching diagram of size $n+m$ with no $(k+1)$-crossing whose m last arcs are part of an m-nesting; oscillating tableau of size $n+m$ with height bounded by k, whose nth diagram is a row of length m. For this particular example, $n = 10$ and $k = m = 2$.

combinatorial classes, and these classes often contain natural subclasses which are counted by Catalan numbers. For example, doubly alternating Baxter permutations have a Catalan number counting sequence [24]. One consequence of Theorem 2 is a new two variable generating tree construction for Baxter numbers.

Unlike the results in Part 1, our proof of Theorem 2 is not a combinatorial bijection. One impediment to a bijective proof is a lack of a certain symmetry in the class of hesitating tableaux that is present in most known Baxter classes. A bijection would certainly be of interest, and in fact we conjecture a refinement of Theorem 2, in Conjecture 21, which could guide a combinatorial bijection.

We begin with definitions in Section 2, and some known bijections in Section 3. Then we focus on the standard Young tableaux of bounded height in Section 4, followed by our study of Baxter objects in Section 5.

2. The combinatorial classes

We begin with precise definitions for the combinatorial classes that are used in our results.

2.1. Tableaux families

As mentioned above, a common encoding of walks on Young's lattice is given by sequences of Ferrers diagrams. We consider three variants. Each sequence starts from the empty shape, and has a specified ending shape; the difference between the different families here is when one can add or remove a box. The *length* of a sequence is the number of elements, minus one. (It is the number of steps in the corresponding walk.)

A vacillating tableau is an *even* length sequence of Ferrers diagrams, written $(\lambda^{(0)}, \ldots, \lambda^{(2n)})$ where consecutive elements in the sequence are either the same or differ by one square, under the restriction[1] that $\lambda^{(2i)} \geq \lambda^{(2i+1)}$ and $\lambda^{(2i+1)} \leq \lambda^{(2i+2)}$.

[1] Recall $\lambda \leq \mu$ means that $\lambda_i \leq \mu_i$ for all i.

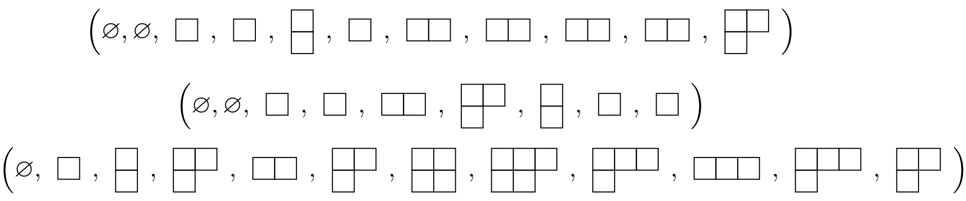


Fig. 2. *From top to bottom:* a vacillating tableau of length 10; a hesitating tableau of length 8; an oscillating tableau of length 11. In each case, the height is bounded by 2.

A hesitating tableau is an even length sequence of Ferrers diagrams, written $(\lambda^{(0)}, \dots, \lambda^{(2n)})$ where consecutive differences of elements in the sequence are either the same or differ by one square, under the following restrictions:

- if $\lambda^{(2i)} = \lambda^{(2i+1)}$, then $\lambda^{(2i+1)} < \lambda^{(2i+2)}$ (do nothing; add a box)
- if $\lambda^{(2i)} > \lambda^{(2i+1)}$, then $\lambda^{(2i+1)} = \lambda^{(2i+2)}$ (remove a box; do nothing)
- if $\lambda^{(2i)} < \lambda^{(2i+1)}$, then $\lambda^{(2i+1)} > \lambda^{(2i+2)}$ (add a box; remove a box).

An oscillating tableau is simply a sequence of Ferrers diagrams such that at every stage a box is either added or deleted. Remark that the length of the sequence is not necessarily even.

In each case, if no diagram in the sequence is of height $k+1$, we say that the tableau has its *height bounded by* k. Fig. 2 shows examples of the different tableaux.

2.2. Lattice walks

Each integer partition represented as a Ferrers diagram in a tableau sequence can also be represented by a vector of its parts. If the tableau sequence is bounded by k, then a k-tuple is sufficient.

The sequence of vectors defines a lattice path. For example, each of the three tableau families above each directly corresponds to a lattice path family in the region

$$W_k = \{(x_1, x_2, \dots, x_k) : x_i \in \mathbb{Z}, x_1 \geq x_2 \geq \cdots \geq x_k \geq 0\}$$

starting at the origin $(0, \dots, 0)$. We can explicitly define three classes of lattice paths by translating the constraints on the tableau families.

Remark. Twice in this article, in order to relate previous results, we use a translation of this region and still identify it as W_k. The translated regions are identical to the original up to a small shift of coordinates. This change is detailed explicitly in the text (the allowed sets of steps are never changed).

Let e_i be the elementary basis vector with a 1 at position i and 0 elsewhere. The steps in our lattice model are all elementary vectors, with possibly one exception: the zero vector, also called *stay step*. The length of the walk increases with a stay step, but the position does not change.

W_k-**vacillating walk** is a walk of *even* length in W_k using (i) two consecutive stay steps; (ii) a stay step followed by an e_i step; (iii) a $-e_i$ step followed by a stay step; (iv) a $-e_i$ step followed by an e_j step.

W_k-**hesitating walk** has even length and steps occur in the following pairs: (i) a stay step followed by an e_i step; (ii) a $-e_i$ step followed by a stay step; (iii) an e_i step follow by a $-e_j$ step.

W_k-**oscillating walk** starts at the origin and takes steps of type e_i or $-e_i$, for $1 \leq i \leq k$. It does not permit stay steps.

Some examples are presented in Fig. 3.

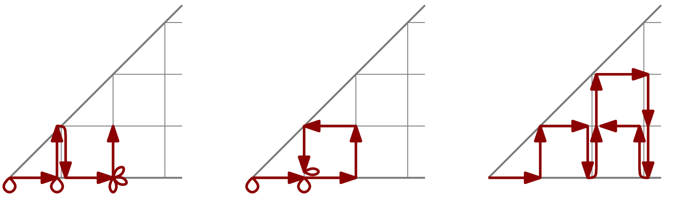


Fig. 3. *From left to right:* a W_2-vacillating walk; a W_2-hesitating walk; a W_2-oscillating walk. The stay steps are drawn as loops. (These walks correspond to the tableaux of Fig. 2.)

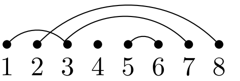


Fig. 4. The set partition $\pi = \{1, 3, 7\}, \{2, 8\}, \{4\}, \{5, 6\}$.

Fig. 5. Patterns in arc diagrams. *From left to right:* a 3-crossing; an enhanced 3-crossing; a 3-nesting; an enhanced 3-nesting.

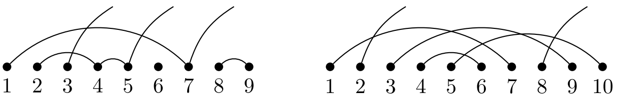


Fig. 6. An open partition and an open matching.

2.3. Open arc diagrams

Arc diagrams depict combinatorial objects as labelled graphs. They are a useful format for visualizing and detecting certain patterns. Matchings and set partitions are examples of classes that have natural representations using arc diagrams. In the arc diagram representation of a set partition of $\{1, 2, \dots, n\}$, a row of dots is labelled from 1 to n. A partition block $\{a_1, a_2, \dots, a_j\}$, ordered $a_1 < a_2 < \cdots < a_j$, is represented by the set of arcs

$$\{(a_1, a_2), (a_2, a_3), \dots, (a_{j-1}, a_j)\}$$

which are always drawn above the row of dots. We adopt the convention that a part of size one, say $\{i\}$, contributes a loop, that is a trivial arc (i, i). In this work, we do not draw the loops, although some authors do. The set partition $\pi = \{\{1, 3, 7\}, \{2, 8\}, \{4\}, \{5, 6\}\}$ is depicted as an arc diagram in Fig. 4. Matchings are represented similarly, with each pair contributing an arc.

A set of k distinct arcs $(i_1, j_1), \dots, (i_k, j_k)$ forms a *k-crossing* if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. They form an *enhanced k-crossing* if $i_1 < i_2 < \cdots < i_k \leq j_1 < j_2 < \cdots < j_k$. (By convention, an isolated dot of the partition forms an enhanced 1-crossing.) They form a *k-nesting* if $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. They form an *enhanced k-nesting* if $i_1 < i_2 < \cdots < i_k \leq j_k < \cdots < j_2 < j_1$ (As previously, $i_k = j_k$ means that i_k is an isolated element in the set partition.). Fig. 5 illustrates a 3-nesting, an enhanced 3-nesting, and a 3-crossing.

Recently, Burrill, Elizalde, Mishna and Yen [12] generalized arc diagrams by permitting *open arcs*: in these diagrams each arc has a left endpoint but not necessarily a right endpoint. The open arcs can be viewed as arcs "under construction". An *open partition* (resp. an *open matching*) is a set partition (resp. a matching) diagram with open arcs. In open matchings, the left endpoint of an open arc is never the right endpoint of another arc. Fig. 6 shows examples of such diagrams.

We are also interested in crossing and nesting patterns in open diagrams. Here we simplify the notation of [12]. A *k-crossing* in an open diagram is either a set of k mutually crossing arcs (as before), or the union of $k - 1$ mutually crossing arcs and an open arc whose left endpoint is to the right of the last left endpoint and to the left of the first right endpoint of the $k - 1$ crossing arcs. A *k-nesting*

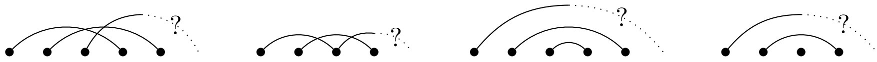


Fig. 7. Patterns in open diagrams. *From left to right:* a 3-crossing; an enhanced 3-crossing; a 3-nesting; an enhanced 3-nesting.

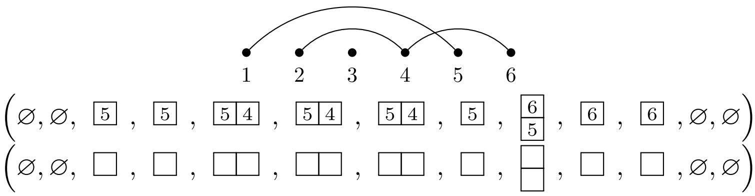


Fig. 8. *Top.* The set partition $\pi = \{1, 5\}, \{2, 4, 6\}, \{3\}$. *Middle.* The corresponding Young tableau sequence. *Bottom.* The vacillating tableau given by $\phi(\pi)$.

in an open diagram is either a set of k mutually nesting arcs, or a set of $k - 1$ mutually nesting arcs, and an open arc whose left endpoint is to the left of the $k - 1$ nesting arcs. We generalize enhanced k-crossings and enhanced k-nestings in an open diagram similarly. Examples are given in Fig. 7. If we want to point out that a crossing (or nesting) has no open arc, we say that it is a *plain* k-crossing (or k-nesting).

3. Bijections

3.1. Description of Chen, Deng, Du, Stanley, Yan's bijection

The work of Chen, Deng, Du, Stanley and Yan [14] describes nontrivial bijections between arc diagram families and tableau families. In this section we summarize a selection of their results, and adapt it to our needs. Their main bijection maps a set partition π to a sequence of Young tableaux,[2] the shapes of which form a vacillating tableau, denoted by $\phi(\pi)$. The exact same process is used by Chen et al. to treat hesitating tableaux and oscillating tableaux, and hence the main properties of the bijection are retained. We use their results for the proof of Propositions 6 and 7.

We describe here their bijection ϕ. Since we consider also partition diagrams with open arcs reaching to the right, we find it more convenient to formulate their mapping so as to read the partition from left to right, which makes us adopt mirror conventions. Let π be a set partition of size n. We are going to build from π a sequence of Young tableaux where the entries are decreasing in each row and each column — the fact that we use decreasing order instead of increasing order is a direct consequence of the change of the reading direction. The first entry is the empty Young tableau. We increment a counter i by one from 1 to n. A given step in the algorithm proceeds as follows. If i is the right-hand endpoint of an arc in π, then delete i from the previous tableau (it turns out that i must be in a corner). Otherwise, replicate the previous tableau. Then, after this move, if i is a left-hand endpoint of an arc (i, j) in π, insert j by the Robinson–Schensted insertion algorithm for the decreasing order into the previous tableau. If i is not a left-hand endpoint, replicate the previous tableau.

The output of this process is a sequence of Young tableaux starting from, and ending at, the empty Young tableau. The sequence of shapes is given by a vacillating tableau and is denoted $\phi(\pi)$.

Example. Consider the partition π from Fig. 8. The number 1 is the left-hand endpoint of the arc $(1, 5)$, but not the right-hand endpoint of any arc, so the first three Young tableaux are $\varnothing$, $\varnothing$, $\boxed{5}$. Similarly, 2 is the left-hand endpoint of $(2, 4)$ but not a right-hand endpoint, so the two following Young Tableaux

[2] A *Young tableau* is defined here as the filling of a Ferrers diagram with positive integers, such that the entries in each row and in each column are strictly decreasing (usually the entries are increasing; the reason for this change is explained later). The set of entries does not need to form an interval of the form $\{1, \dots, n\}$.

are $\boxed{5}$, $\boxed{5}\boxed{4}$. The number 3 is an isolated point, so the tableau $\boxed{5}\boxed{4}$ is repeated twice. The number 4 being the right-hand endpoint of $(2, 4)$ and the left-hand endpoint of $(4, 6)$, we delete 4, then we add 6: we obtain $\boxed{5}$, $\begin{array}{|c|}\hline 6\\ \hline 5\\ \hline\end{array}$. The rest of the sequence is given in Fig. 8.

Given a vacillating tableau $(\varnothing, \lambda_1, \ldots, \lambda_{2n-1}, \varnothing)$, there exists a unique way to fill the entries of the Ferrers diagrams into Young tableaux so that it corresponds to an image of a set partition. This has been proved in [14], and implies that ϕ is a bijection.

In an arc diagram, we say that the *segment* $[i, i+1]$ *is below* a k-crossing if the arc diagram contains k arcs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < i_2 < \cdots < i_k \le i$ and $i + 1 \le j_1 < j_2 < \cdots < j_k$. Similarly, the *segment* $[i, i + 1]$ *is below* a k-nesting if there exist k arcs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < i_2 < \cdots < i_k \le i$ and $i+1 \le j_k < \cdots < j_2 < j_1$. For instance, in Fig. 8, the segment $[3, 4]$ is below a 2-nesting but not below a 2-crossing, while the segment $[4, 5]$ is below a 2-crossing but not below a 2-nesting. With this definition we can formulate and prove a stronger version of [14, Theorem 3.2] (this property can also easily be seen in the growth diagram formulation of the bijection — see [25]).

Proposition 3. *Let π be a partition of size n and $\phi(\pi) = (\lambda_0, \ldots, \lambda_{2n})$. For every $i \in \{1, \ldots, n\}$, the segment $[i, i + 1]$ of π is below a k-crossing (resp. k-nesting) if and only if λ_{2i} in $\phi(\pi)$ has at least k rows (resp. k columns).*

Example. We continue our example and verify that $\lambda^{(6)} = \square\square$ has 2 columns but not 2 rows, and accordingly $[3, 4]$ is below a 2-nesting, but not a 2-crossing.

Proof. Let $(T_0, \ldots, T_{2n})$ be the sequence of Young tableaux corresponding to the partition π. We use some ingredients from the proof of Theorem 3.2 of [14, p. 1562][3]:

1. A pair (i, j) is an arc in the representation of π if and only if j is an entry in $T_{2i}, T_{2i+1}, \ldots, T_{2(j-1)}$;
2. Let $\sigma_i = w_1 w_2 \ldots w_r$ denote the permutation of the entries of T_i such that $w_1, w_2, \ldots, w_r$ have been inserted in $(T_0, \ldots, T_{2n})$ in this order;
3. The permutation σ_i has an increasing subsequence of length k if and only if the partition λ_i has at least k rows.

The following statements are then equivalent:

- The segment $[i, i + 1]$ is below a k-crossing.
- There exist k arcs $(i_1, j_1), \ldots, (i_k, j_k)$ in π such that
$$i_1 < i_2 < \cdots < i_k \le i \quad \text{and} \quad i + 1 \le j_1 < j_2 < \cdots < j_k.$$
- There exist k numbers $j_1 < j_2 < \cdots < j_k$ that are entries of T_{2i} such that $j_1, j_2, \ldots, j_k$ have been inserted in this order in $(T_0, \ldots, T_{2n})$.
- There exist k numbers $j_1 < j_2 < \cdots < j_k$ such that $j_1 j_2 \ldots j_k$ is a subsequence of σ_{2i}.
- The diagram λ_{2i} has at least k rows.

The proof for k-crossings is similar. $\square$

Considering all intervals $[i, i+1]$ for $1 \le i \le n$, we recover the statement of Theorem 3.2 from [14].

Corollary 4 (*Theorem 3.2 From [14]*)**.** *A set partition π has no $(k + 1)$-crossing (resp. no $(k + 1)$-nesting) if and only if no Ferrers diagram in the sequence $\phi(\pi)$ has $k + 1$ rows (resp. columns).*

Remark. The *crossing level* of a set partition π, denoted $cr(\pi)$, is the maximal k such that π has a k-crossing. Similarly, the *nesting level* of a set partition π, denoted $ne(\pi)$, is the maximal k such that π has a k-nesting. Chen et al. conclude from the previous corollary that the joint distribution of cr and ne over all the set partition diagrams of fixed size is symmetric. That is,

$$\sum_{\substack{\pi \text{ set partition diagram} \\ \text{of size } n}} x^{cr(\pi)} y^{ne(\pi)} = \sum_{\substack{\pi \text{ set partition diagram} \\ \text{of size } n}} y^{cr(\pi)} x^{ne(\pi)}.$$

3 Recall that one bijection is the mirror image of the other. So the indices differ between [14] and here.

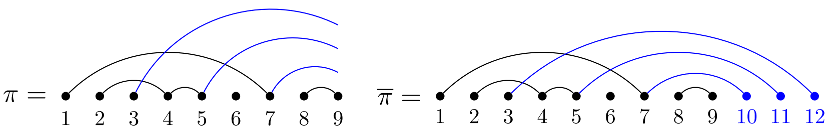


Fig. 9. *Left.* An open partition diagram, π, with 3 open arcs. *Right.* The corresponding closed partition diagram $\overline{\pi}$ ending with a 3-nesting, obtained by closing the 3 open arcs of π in reverse order.

Let τ denote transposition, the operation that transposes every Ferrers diagram inside a vacillating tableau. Then $\phi^{-1} \circ \tau \circ \phi$ swaps the crossing level and the nesting level of a set partition. Moreover, note that $\phi^{-1} \circ \tau \circ \phi$ preserves the opener/closer sequence, i.e., if the number i is an isolated point (resp. a left endpoint, a right endpoint, a left and right endpoint at the same time) in a partition π, then i is an isolated point (resp. a left endpoint, a right endpoint, a left and right endpoint at the same time) in $\phi^{-1} \circ \tau \circ \phi(\pi)$.

3.2. Bijections with open partitions

Next we describe a generalization of the bijection of Chen et al. to the class of tableaux ending at a row shape. We thereby link their bijection to the classes of Section 2.

Proposition 5. *A bijection can be constructed between any two of the following classes:*

1. *the set of open partition diagrams of length n with no $(k+1)$-crossing, with m open arcs;*
2. *the set of open partition diagrams of length n with no $(k+1)$-nesting, with m open arcs;*
3. *the set of vacillating tableaux of length $2n$, with maximum height bounded by k, ending in a row of length m;*
4. *the set of W_k-vacillating walks of length $2n$ ending at $(m, 0, \ldots, 0)$.*

Proof. *Bijection* $(1) \Leftrightarrow (3)$. We close the open diagrams in a canonical way, and then apply ϕ. More precisely, let π be an open partition diagram of length n with m open arcs and no $(k+1)$-crossing. We build a new partition diagram $\overline{\pi}$ of length $n+m$ without open arcs by closing the m open arcs of π in decreasing order. That is, if $i_1 < i_2 < \cdots < i_m$ denote the positions of the m open arcs of π, the partition $\overline{\pi}$ is the closure, obtained by replacing the m open arcs with the arcs $(i_1, n+m), (i_2, n+m-1), \ldots, (i_m, n+1)$, as shown in Fig. 9. Note that the open arcs are closed in such a way that no new crossing is created.

The m last elements of $\overline{\pi}$ form the end of an m-nesting. Consequently, each crossing of $\overline{\pi}$ has at most one element inside $\{n+1, \ldots, n+m\}$; so the preimage of any ℓ-crossing of $\overline{\pi}$ is also an ℓ-crossing. As π has no $(k+1)$-crossing, the diagram $\overline{\pi}$ has no $(k+1)$-crossing.

Let $\phi(\overline{\pi}) = (\lambda_0, \ldots, \lambda_{2(n+m)})$ be the image of $\overline{\pi}$ under ϕ. By Corollary 4, the height of this vacillating tableau is bounded by k. Moreover, the segment $[n, n+1]$ in $\overline{\pi}$ is below an m-nesting but not below a 2-crossing. By Proposition 3, it means that λ_{2n} is a column with at least m rows. Since $\phi(\overline{\pi})$ ends with an empty diagram and one can delete at most one cell every two steps, λ_{2n} has exactly m rows. Thus, $(\lambda_0, \ldots, \lambda_{2n})$ is a vacillating tableau of length $2n$, with maximum height bounded by k, ending in a column of length m.

The transformation is bijective: a vacillating tableau $(\lambda_0, \ldots, \lambda_{2n})$ from the set (3) can be concatenated with $((m-2), (m-2), \ldots, (1), \varnothing, \varnothing)$, where (j) denotes the partition of j only composed of a single part of size j. If we change its preimage under ϕ by opening the arcs ending in $\{n+1, \ldots, n+m\}$ into m open arcs, we recover the initial open diagram π.

Bijection $(2) \Leftrightarrow (3)$. The previous bijection is adapted with an additional application of the transposition operator τ.

Bijection $(3) \Leftrightarrow (4)$. This is a straightforward consequence of the encoding. As the vacillating tableaux end at a row of length m, the endpoints of the walks must be the point $(m, 0, \ldots, 0)$. $\square$

The open diagram case inherits many properties from the closed diagram case. For example, the statistics of crossing level and nesting level are equidistributed. Also, the problem of finding a direct bijection between open partitions with no k-crossing and open partitions with no k-nesting without going through the vacillating tableaux seems to be as difficult as the closed case.

However, the nesting level and the crossing level do not have symmetric joint distribution for open partitions. This constitutes a difference with the (closed) partition diagrams.

Furthermore, the other generalizations of Chen et al.— specifically the ones that concern the hesitating and oscillating tableaux — are similarly adapted by the same canonical closing strategy. Consequently, the proofs of the following results are almost identical to the vacillating case.

Proposition 6. *The following classes are in bijection:*

1. *the set of open matching diagrams of length n with no $(k+1)$-crossing, with m open arcs;*
2. *the set of open matching diagrams of length n with no $(k+1)$-nesting, with m open arcs;*
3. *the set of oscillating tableaux of length n, with height bounded by k, ending in a row of length m;*
4. *the set of W_k-oscillating walks of length n ending at $(m, 0, \ldots, 0)$.*

Proposition 7. *The following classes are in bijection:*

1. *the set of open partition diagrams of length n with no enhanced $(k+1)$-crossing, with m open arcs;*
2. *the set of open partition diagrams of length n with no enhanced $(k+1)$-nesting, with m open arcs;*
3. *the set of hesitating tableaux of length 2n, with height bounded by k, ending in a row of length m;*
4. *the set of W_k-hesitating walks of length $2n$ ending at $(m, 0, \ldots, 0)$.*

Fig. 1 depicts an example of application of Proposition 6: the fourth object is an open matching, and the last object is the tableau image of this open matching.

4. Young tableaux, involutions and open matchings

4.1. Bijections

We can now prove our first main result, namely Theorem 1. Our strategy is to use Proposition 6, and prove the following result, from which Theorem 1 is a straightforward consequence.

Proposition 8. *The set of standard Young tableaux of size n with height bounded by $2k$ and m odd columns is in bijection with the set of open matching diagrams of length n, with m open arcs and with no $(k+1)$-crossing.*

As far as we can tell, the essential connections behind this theorem were first conjectured by Burrill [11]. First, she experimentally observed the equivalent connection between standard Young tableaux of bounded height and open matchings with no $(k+1)$-nesting [11, Conjecture 6.2.1], and then she proved the correspondence between open matchings with no $(k+1)$-nesting and oscillating tableau ending in a row [11, Section 7.3].

The following lemma presents a classic property of the Robinson–Schensted correspondence (see for example [33]).

Lemma 9 (*Robinson–Schensted Correspondence*)**.** *The set of standard Young tableaux of size n with height bounded by k and m odd columns is in bijection with involutions of size n with m fixed points and no decreasing subsequence of length $k+1$.*

As a first step, Lemma 9 yields combinatorial objects that are close to open matchings. Indeed, involutions have a very natural arc diagram representation: cycles $(i\,j)$ are represented by an arc (i, j), and fixed points are isolated dots.

Example. Let Y be the standard Young tableau from Fig. 1. The image of (Y, Y) under the Robinson–Schensted correspondence is the involution $(1\,7)(3\,9)(4\,6)(5\,10)$. Its diagram is the second object of Fig. 1.

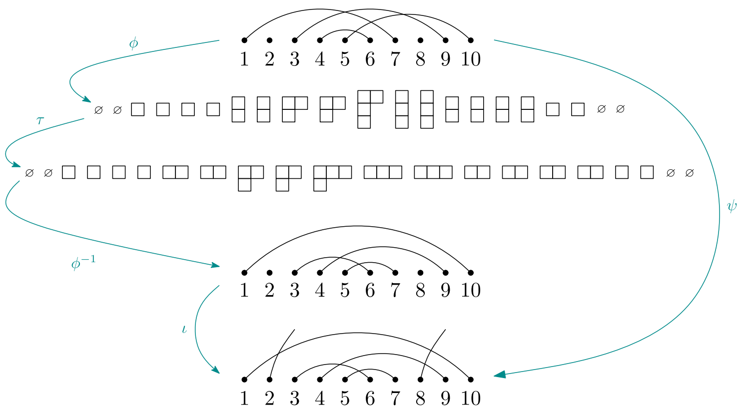


Fig. 10. Image of the involution (1 7)(3 9)(4 6)(5 10) under ψ.

There is a simple correspondence between decreasing sequences in an involution and enhanced nestings in its arc diagram representation.

Lemma 10. *Let $k \in \mathbb{Z}_{\geq 1}$. An involution has no decreasing subsequence of length $2k-1$ if and only if there is no enhanced k-nesting in its arc diagram representation.*

Proof. Let α be an involution. If its arc diagram has an enhanced k-nesting then α contains k cycles $(i_1 j_1), \ldots, (i_k j_k)$ that satisfy $i_1 < i_2 < \cdots < i_k \leq j_k < \cdots < j_1$, which clearly induces a decreasing subsequence of length $2k-1$.

Conversely, assume that there exist $2k-1$ numbers $i_1 < i_2 < \cdots < i_{2k-1}$ such that $\alpha(i_{2k-1}) < \cdots < \alpha(i_1)$. If $\alpha(i_k) - i_k \geq 0$, then $i_1 < \cdots < i_k \leq \alpha(i_k) < \cdots < \alpha(i_1)$: this means that $(i_1, \alpha(i_1)), \ldots, (i_k, \alpha(i_k))$ form an enhanced k-nesting. Otherwise, $\alpha(i_k) - i_k \leq 0$. Thus $\alpha(i_{2k-1}) < \cdots < \alpha(i_k) \leq i_k < \cdots < i_{2k-1}$: the arcs $(\alpha(i_{2k-1}), i_{2k-1}), \ldots, (\alpha(i_k), i_k)$ form an enhanced k-nesting. $\square$

By the two preceding lemmas, the proof of Proposition 8 is reduced to the proof that involution diagrams of length n with m fixed points and no enhanced $(k+1)$-nesting are in bijection with open matching diagrams of length n with m open arcs and no $(k+1)$-crossing. This is established by the following lemma.

Lemma 11. *There is a bijection ψ between involution diagrams and open matching diagrams, such that for α an involution diagram and $\beta = \psi(\alpha)$, the diagrams α and β have same length, the number of fixed points in α is the number of open arcs in β, and for any $\ell \geq 1$ there is an enhanced ℓ-nesting in α if and only if there is an ℓ-crossing in β. In addition the opener/closer sequence of α (seeing fixed points as openers) is the same as the opener/closer sequence of β.*

Proof. We describe ψ, a bijective map between involutions and open matchings. It is formed as a composition of other maps. We have already defined ϕ, the bijection from set partition diagrams to vacillating tableaux from Section 3, and τ, the transpose action which can be applied to any tableau sequence. We add ι, the operation that changes every isolated dot in an involution diagram into an open arc. Let ψ be the composition $\iota \circ \phi^{-1} \circ \tau \circ \phi$. Fig. 10 shows an example of the action of ψ.

Since ϕ, τ and ι can all be reversed, the mapping ψ is bijective. Moreover, recall from the remark at the end of Section 3.1, $\phi^{-1} \circ \tau \circ \phi$ preserves the opener–closer sequence. Therefore, every involution of size n with m fixed points is mapped under ψ to an open matching diagram of size n with m open arcs, such that the opener/closer sequence is preserved (seeing fixed points as openers).

Assume that an involution α has an enhanced ℓ-nesting $(i_1, j_1) \cdots (i_\ell, j_\ell)$. If $i_\ell \neq j_\ell$, this enhanced nesting is also a plain ℓ-nesting. By the remark at the end of Section 3.1, we know that $\phi^{-1} \circ \tau \circ \phi(\alpha)$ has an ℓ-crossing, so the same holds for $\psi(\alpha)$.

If $i_\ell = j_\ell$, then i_ℓ is a fixed point of α and hence an open arc in $\psi(\alpha)$. Moreover, the segment $[i_\ell, i_\ell+1]$ is below the $(\ell-1)$-nesting $(i_1, j_1)\cdots(i_{\ell-1}, j_{\ell-1})$. So, by Proposition 3, the $2i_\ell$-th diagram of $\phi(\alpha)$ has at least $\ell-1$ columns. The $2i_\ell$-th diagram of $\tau\circ\phi(\alpha)$ has then at least $\ell-1$ rows, and so $[i_\ell, i_\ell+1]$ is below a $(\ell-1)$-crossing in $\psi(\alpha)$. Thus, i_ℓ is in $\psi(\alpha)$ an open arc below a $(\ell-1)$-crossing. Hence the open matching $\psi(\alpha)$ has an ℓ-crossing. The converse is proved similarly.

In summary, ψ is a bijection between involution diagrams of size n with m fixed points and no enhanced k-nesting and open matchings diagrams of size n with m open arcs and no k-crossing. □

We direct the reader to Fig. 1 if they want a complete illustration of Theorem 1 or Proposition 8.

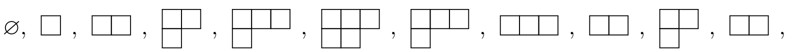
Krattenthaler's bijection

As mentioned in the introduction, Krattenthaler has also described [26] an explicit bijective map to proves Theorem 1. This map, like ours, relies on the Robinson–Schensted correspondence, but in addition it relies on *jeu de taquin*[4] moves. These two maps share some similarities, although they do not appear to be linked by any simple map. In fact, both bijections can be described in terms of either arc diagrams or growth diagrams, since growth diagrams are an alternative encoding of Chen et al.'s bijection. Krattenthaler uses growth diagrams [26].

The difference between the two bijections concerns the treatment of fixed points which appear when the standard Young tableaux are viewed as involutions. More precisely, the two bijections put into correspondence involutions with m fixed points whose diagrams contain no enhanced $(k+1)$-nesting (second object of Fig. 1), and matchings whose diagrams have no $(k+1)$-nesting and the m last arcs form the end of an m-crossing (second-last object of Fig. 1). Roughly, both bijections reorganize the set of the m fixed points, and close them to form an m-crossing at the end — and all this is done without creating any $(k+1)$-nesting. This is however realized in very different ways.

Unlike ours, Krattenthaler's bijection "*preprocesses*" standard Young tableaux with *jeu de taquin* to complete odd columns. Already at this stage, the arc diagrams are quite different. Notably, if we translate Krattenthaler's preprocessing to the world of arc diagrams, we observe that every fixed point has been closed, and it is not the last, but the first m arcs of this diagram that are part of an m-crossing. The required form is obtained by reversing this image.

Since the mechanisms of *jeu de taquin* and the ones of Chen et al.'s bijection are quite different, we think there is no obvious connection between Krattenthaler's bijection and ours. This is observable in practice: the bijections significantly differ on numerous examples. For example, Krattenthaler's bijection applied to the standard Young tableau of Fig. 1 gives the sequence



which differs in several positions.

Changing "even" by "odd"

Remark that standard Young tableaux with height bounded by an odd number are also characterized in terms of open matching diagrams (but instead constrained by the plain nestings or crossings):

Proposition 12. *The following classes are in bijection:*

(i) *the set of standard Young tableaux of size n with m odd columns and height bounded by* $2k-1$*;*
(ii) *the set of involutions of size n with m fixed points and no decreasing subsequence of length* $2k$*;*
(iii) *the set of open matching diagrams of length n with no plain k-crossing and with m open arcs;*
(iv) *the set of open matching diagrams of length n with no plain k-nesting and with m open arcs.*

Proof. The Robinson–Schensted correspondence (specifically, the property described in Lemma 9) gives a straightforward bijection between (i) and (ii). Then, seeing isolated points as open arcs, it is easy to adapt Lemma 10 in order to show the correspondence between (ii) and (iv). Finally the bijection between (iii) and (iv) is given by $\phi^{-1}\circ\tau\circ\phi$, where ϕ and τ are defined in Section 3. □

4 *Jeu de taquin* is an operation on Young tableaux invented by Schützenberger [31].

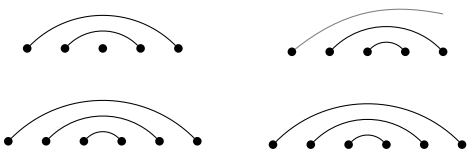


Fig. 11. *Left.* An enhanced 3-nesting with respectively 5 and 6 dots. *Right.* A 3-nesting with respectively 5 and 6 dots (in an open matching).

4.2. A new symmetric joint distribution for involutions

While looking for the previous bijection we found a surprising symmetry property for involutions, which is now presented. Section 3 contained the definition of *nesting level*. In the context of involution diagrams, this notion can be refined in two different ways, depending on whether we regard involution diagrams as enhanced set partition diagrams or as open matchings.

The *enhanced nesting level* of an involution α, denoted $\mathrm{ne}_{\downarrow}(\alpha)$, is the maximal number of *dots* in an enhanced nesting of α (note that k marks the number of dots not number of arcs). Pursuant to Lemma 10, the number $\mathrm{ne}_{\downarrow}(\alpha)$ is also the length of the longest decreasing subsequence of α. Similarly, we define the *open nesting level* of an involution α, denoted $\mathrm{ne}_{\leftarrow}(\alpha)$: after transforming the diagram of α into an open matching by changing every isolated point into an open arc, the open nesting level of α is the maximal number of dots inside a nesting.

Remark that an enhanced nesting and a nesting in an open diagram are identical if both have an even number of dots; these are then plain nestings. The difference is made when the number of dots is odd, say $2k+1$. In this case, an enhanced nesting is made of a dot *below* a plain k-nesting, while a nesting in an open matching is made of an open arc to the *left* of a plain k-nesting. This justifies the notation $\mathrm{ne}_{\downarrow}$ and $\mathrm{ne}_{\leftarrow}$. Fig. 11 compares the two patterns.

Example 13. The open nesting level of the involution $(1\ 7)(3\ 9)(4\ 6)(5\ 10)$, depicted in Fig. 11, is 5: the numbers 2, 3, 4, 6, 9 form a nesting if we transform the dot 2 into an open arc. However the enhanced nesting level of the same involution is 4: there is no dot below any 2-nesting.

A (weak) link between the two statistics can be easily derived from the preceding study, as stated in the following proposition.

Proposition 14. *There is a bijection θ from involution diagrams to involution diagrams such that for α any involution diagram, $\beta = \theta(\alpha)$, and $\ell \geq 1$, there is an enhanced ℓ-nesting in α if and only if there is an ℓ-nesting in the open matching obtained by changing every fixed point in β by an open arc. In other words, there exists a bijection θ between involutions α such that $\mathrm{ne}_{\downarrow}(\alpha) = 2k-1$ or $2k$ and involutions β such that $\mathrm{ne}_{\leftarrow}(\beta) = 2k-1$ or $2k$.*

In addition, θ preserves the length, the number of fixed points and the opener–closer sequence (viewing fixed points as openers).

Proof. We define θ as the composition of the mapping ψ described by Lemma 11 (from involution diagrams to open matching diagrams) and the correspondence between (1) and (2) in Proposition 6. All the stated properties of θ are direct consequences of Lemma 11 and Proposition 6. $\square$

Note that an enhanced nesting is preserved when the diagram is reflected. This is not true for an odd nesting in an open diagram, because the isolated point must be to the left of the nesting. Despite the fact they do not share this property, the enhanced nesting level and the open nesting level have symmetric distribution, as stated in the following theorem.

Theorem 15. *The statistics $\mathrm{ne}_{\leftarrow}$ and $\mathrm{ne}_{\downarrow}$ have a symmetric joint distribution over all the involutions of size n with m fixed points, i.e.,*

$$\sum_{\substack{\alpha \text{ involution} \\ \text{of size } n \\ \text{with } m \text{ fixed points}}} x^{\mathrm{ne}_{\leftarrow}(\alpha)} y^{\mathrm{ne}_{\downarrow}(\alpha)} = \sum_{\substack{\alpha \text{ involution} \\ \text{of size } n \\ \text{with } m \text{ fixed points}}} y^{\mathrm{ne}_{\leftarrow}(\alpha)} x^{\mathrm{ne}_{\downarrow}(\alpha)}.$$

Remark. The bijection θ from Proposition 14 does not swap the statistics $\mathrm{ne}_\downarrow$ and $\mathrm{ne}_\leftarrow$. For instance, the involution $\alpha = (1\ 5)(2\ 3)$ of size 5 is mapped to the involution $\beta = (2\ 3)(4\ 5)$: we have $\mathrm{ne}_\downarrow(\alpha) = \mathrm{ne}_\leftarrow(\alpha) = 4$ but $\mathrm{ne}_\leftarrow(\beta) = 3$ and (even worse!) $\mathrm{ne}_\downarrow(\beta) = 2$. Nonetheless, the existence of the function θ (and more particularly the fact that an involution with enhanced nesting level $2k-1$ or $2k$ is mapped under θ to an involution with open nesting level $2k-1$ or $2k$) is sufficient to prove Theorem 15.

Proof. Consider all involutions of fixed size, with a fixed number of fixed points. Let $a_{i,j}$ be the number of involutions α in this class such that $\mathrm{ne}_\downarrow(\alpha) = i$ and $\mathrm{ne}_\leftarrow(\alpha) = j$. By Proposition 14, the bijection θ maps involutions α such that $\mathrm{ne}_\downarrow(\alpha) = 2k-1$ or $2k$ to involutions β such that $\mathrm{ne}_\leftarrow(\beta) = 2k-1$ or $2k$; hence

$$\sum_{j\geq 0} \left(a_{2k-1,j} + a_{2k,j}\right) = \sum_{i\geq 0} \left(a_{i,2k-1} + a_{i,2k}\right). \tag{2}$$

We can simplify the expression in Eq. (2) as the values $\mathrm{ne}_\downarrow(\alpha)$ and $\mathrm{ne}_\leftarrow(\alpha)$ can only differ by at most one for a given involution α. Indeed, if ℓ denotes the maximal number of arcs inside a nesting of an involution, the open nesting level and the enhanced nesting level must equal either 2ℓ or $2\ell+1$. Therefore, $a_{i,j} = 0$ except for pairs (i, j) of the form $(2\ell, 2\ell)$, $(2\ell, 2\ell+1)$, $(2\ell+1, 2\ell)$ or $(2\ell+1, 2\ell+1)$. Eq. (2) can be thus rewritten as:

$$a_{2k-1,2k-1} + a_{2k-1,2k-2} + a_{2k,2k} + a_{2k,2k+1} = a_{2k-2,2k-1} + a_{2k-1,2k-1} + a_{2k,2k} + a_{2k+1,2k},$$

or after simplification

$$a_{2k,2k+1} - a_{2k+1,2k} = a_{2k-2,2k-1} - a_{2k-1,2k-2}.$$

In other words, the sequence $\left(a_{2k,2k+1} - a_{2k+1,2k}\right)$ is constant over all $k \geq 0$. But since it equals 0 for $k = 0$, we have for every $k \geq 0$,

$$a_{2k,2k+1} = a_{2k+1,2k}.$$

The other terms $a_{i,j}$ such that $i \neq j$ vanish, so the last equality is sufficient to conclude the proof. $\square$

The previous proof is simple but not constructive: can we describe an involution (on involutions) that swaps the statistics $\mathrm{ne}_\leftarrow$ and $\mathrm{ne}_\downarrow$? The answer is yes, and a description can be given in terms of iterations of θ, where θ is the mapping defined by Proposition 14.

Lemma 16. *Let $\theta^{(\ell)}$ be the ℓth iteration of θ and $A_{i,j}$ be the set of involutions α such that $\mathrm{ne}_\downarrow(\alpha) = i$ and $\mathrm{ne}_\leftarrow(\alpha) = j$.*

For every α in $A_{2k,2k+1}$ with $k \geq 0$, there exists $m \geq 1$ such that

$$\theta^{(\ell)}(\alpha) \notin A_{2k,2k+1} \cup A_{2k+1,2k} \quad \text{for } \ell \in \{1, \ldots, m-1\},$$

and $\theta^{(m)}(\alpha) \in A_{2k+1,2k}$. Moreover, for every α' in $A_{2k+1,2k}$, there exists $m' \geq 1$ such that $\theta^{(\ell')}(\alpha') \notin A_{2k,2k+1} \cup A_{2k+1,2k}$ for $\ell' \in \{1, \ldots, m'-1\}$, and $\theta^{(m')}(\alpha') \in A_{2k,2k+1}$.

In other words, in the orbit of any involution under θ (this orbit is cyclic since θ is bijective and the set of involutions of fixed size is finite), the elements of $A_{2k,2k+1} \cup A_{2k+1,2k}$ alternate between $A_{2k,2k+1}$ and $A_{2k+1,2k}$.

An example of this correspondence is illustrated in Fig. 12.

Proof. Consider $i > 0$ such that $\theta^{(i)}(\alpha) = \alpha$ (such an i exists as θ acts bijectively on the finite set of involutions of a fixed length). Since $\mathrm{ne}_\leftarrow(\alpha) = 2k+1$, we have $\mathrm{ne}_\downarrow(\theta^{(i-1)}(\alpha)) > 2k$.

Let m denote the smallest $j > 0$ such $\mathrm{ne}_\downarrow(\theta^{(j)}(\alpha)) > 2k$. We have then $\mathrm{ne}_\downarrow(\theta^{(m-1)}(\alpha)) \leq 2k$. Using the properties of θ we know that $\mathrm{ne}_\leftarrow(\theta^{(m)}(\alpha)) \leq 2k$, hence $\mathrm{ne}_\downarrow(\theta^{(m)}(\alpha)) \leq 2k+1$. As $\mathrm{ne}_\downarrow(\theta^{(m)}(\alpha)) > 2k$, we must have

$$\mathrm{ne}_\downarrow(\theta^{(m)}(\alpha)) = 2k+1 \quad \text{and} \quad \mathrm{ne}_\leftarrow(\theta^{(m)}(\alpha)) = 2k.$$

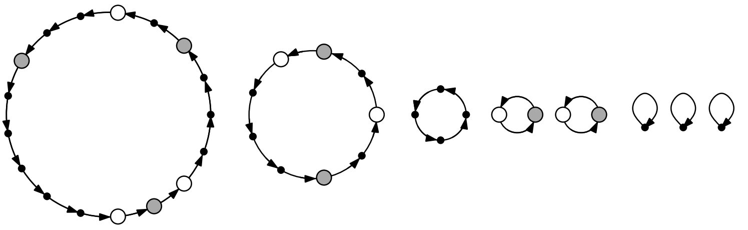


Fig. 12. Schematic representation of typical orbits under θ. The white circles represent the elements of $A_{2k,2k+1}$, the gray circles the elements of $A_{2k+1,2k}$ and the small points are the remaining elements.

Moreover, by minimality of m, there is no $\ell \in \{1, \ldots, m-1\}$ such that $\theta^{(\ell)}(\alpha) \in A_{2k+1,2k}$. For the same reason, there is no $\ell \in \{1, \ldots, m-1\}$ such that $\theta^{(\ell)}(\alpha) \in A_{2k,2k+1}$, since by the property of preservation of θ it would imply $\mathrm{ne}_{\downarrow}(\theta^{(\ell-1)}(\alpha)) > 2k$.

The property on α' can be proved by symmetry. □

The previous lemma sets out how to build the desired involution. Essentially, from an involution of $A_{2k,2k+1}$, we iterate θ until obtaining an involution of $A_{2k+1,2k}$. If, on the other hand, the involution belongs to $A_{2k+1,2k}$ we want to go backward, so we iterate θ^{-1} until obtaining an involution of $A_{2k,2k+1}$. If an involution does not fall under one of the previous forms, it necessarily belongs to a set of the form $A_{\ell,\ell}$, and we can then set this involution as a fixed point.

Proposition 17. *Let α be an involution and $A_{i,j}$ be the set of involutions α such that $\mathrm{ne}_{\downarrow}(\alpha) = i$ and $\mathrm{ne}_{\leftarrow}(\alpha) = j$. If $\alpha \in A_{2k,2k+1}$, set m_α as the smallest integer m such that $\theta^{(m)}(\alpha) \in A_{2k+1,2k}$. If $\alpha \in A_{2k+1,2k}$, set m_α as the opposite of the smallest integer m such that $\theta^{(-m)}(\alpha) \in A_{2k,2k+1}$. Otherwise, set m_α as 0.*

The mapping $\alpha \mapsto \theta^{(m_\alpha)}(\alpha)$ is an involution on the class of involutions that exchanges the statistics $\mathrm{ne}_{\downarrow}$ and $\mathrm{ne}_{\leftarrow}$. It preserves the size of the involutions, the number of fixed points and the opener/closer sequence (considering fixed points as openers).

Remark. What about the *open crossing level* of an involution, that is to say the maximum number of dots contained in a k-crossing, when this involution is transformed into an open matching? It is easy to see that the open crossing level shares the same distribution as the open nesting level or the enhanced nesting level (in particular via the bijection $\phi^{-1} \circ \tau \circ \phi$). However, this statistic does not have a symmetric distribution, whether it is with the open nesting level or with the enhanced nesting level.

4.3. *Consequences of Theorem 1*

There are two immediate generating function consequences to Theorem 1. The first is a non-trivial determinant identity that can be deduced from the equivalence of these two classes, and the second is a diagonal expression for the generating function of Young tableaux of bounded height. Both use enumeration results of Weyl chamber walks [19,23].

4.3.1. *Consequence: The combinatorics of a determinant identity*

The exponential generating functions of both Young tableau of bounded height and of W_k-oscillating walks have both already appeared in the literature, and both in a surprisingly similar form. Theorem 1 thus yields an interesting determinant identity. Both involve hyperbolic Bessel function of the first kind of order j that we modify them slightly. Denote by $b_j(t)$ the sum

$$b_j(t) = I_j(2t) = \sum_n \frac{t^{2n+j}}{n!(n+j)!}.$$

Using the generating function expression for Young tableau of bounded (even) height developed by works of Gordon, Houten, Bender and Knuth [20,21,5], which depends on the parity of k, and Grabiner–Magyar's [23] formula for the exponential generating function of the W_k-oscillating walks of length n between two given points we deduce

$$\det[b_{i-j}+b_{i+j-1}]_{1\le i,j\le k}=\sum_{u=0}^{k-1}(-1)^u\sum_{\ell=u}^{2k-1-2u}b_\ell\det[b_{i-j}-b_{kd-i-j}]_{0\le i\le k-1, i\ne u, 1\le j\le k-1}.$$

The expression on the right hand side follows from the formula of Grabiner and Magyar's formula, which starts as an infinite sum but simplifies to a finite sum after applying the identity $b_{-k}=b_k$ and a co-factor expansion of the determinants.

In fact, Krattenthaler [26] notes that one can be deduced from the other by application of the symmetric function analysis of Goulden [22]. The bijective proof here may give more insight to the intermediary technicalities, and should be of a different flavour than those of Stembridge [34], which involves sets of non-intersecting paths.

4.3.2. Consequence: A diagonal expression

The second consequence is an expression of the generating function for standard Young tableaux as a diagonal of a rational function. A diagonal Δf of a formal power series f is the univariate subseries defined by the sum over $(i_0, i_1, \ldots, i_k)\in\mathbb{N}^{k+1}$

$$\Delta\sum_{i_0,i_1,\ldots,i_k}a(i_0,i_1,\ldots,i_k)z_0^{i_0}z_1^{i_1}\ldots z_k^{i_k}=\sum_n a(n,n,\ldots,n)z^n.$$

These expressions are of interest for several reasons.

First, it can be used to determine the nature of the generating function. Lipshitz [27] proved that the diagonal of a D-finite function[5] is D-finite. Since rational functions are D-finite, diagonals of rational functions are also D-finite.

Gessel and Zeilberger [19] gave a formula for the generating function of certain families of walks in Weyl chambers. The oscillating walks in W_k fit into their formalism, and the resulting expression can be manipulated into a diagonal of rational function. This gives a new proof of the D-finiteness of the class of standard Young tableaux of bounded height. The problem was first posed by Stanley [32], and proved for generic k by Gessel in 1990 [18]. Gessel's proof requires symmetric functions, and also relies on the same closure properties of D-finite functions. The subsequent proof by Goulden which uses the above relations might be insightful.

Furthermore, there has been much recent activity on different computer algebra approaches to diagonal expressions. For example, we can use the work of [8] to determine bounds on the shape of the annihilating differential equation, and potentially the methods of Pemantle and Wilson [30] could be useful to give more precise asymptotic estimates for Young tableaux of bounded height. Their work has been already applied to lattice path problems [28].

Theorem 18. *The ordinary generating function, $Y_k(z)$ for Young tableaux of height bounded by $2k$ satisfies the formula*

$$z^{2k-1}Y_k(z)=-\Delta\left[\frac{z_0^{2k-1}(z_3z_4^2\cdots z_k^{k-2})(z_1+1)\prod_{1\le j<i\le k}(z_i-z_j)(z_iz_j-1)\cdot\prod_{2\le i\le k}(z_i^2-1)}{1-z_0(z_1\cdots z_k)(z_1+\bar{z}_1+\cdots z_k+\bar{z}_k)}\right].$$

The proof of Theorem 18 is a direct application of Gessel and Zeilberger's formula for reflectable walks in Weyl chambers [19,23] for the class of oscillating walks in W_k.

5 A function is D-finite if the set of all its partial derivatives spans a vector space of finite dimension.

5. Tableau sequences as Baxter classes

The combinatorial class that came to be known as Baxter permutations was introduced in 1967 in a paper of Baxter [2] studying compositions of commuting functions. A Baxter permutation of size n is a permutation $\sigma \in \mathfrak{S}_n$ such that there are no indices $i < j < k$ satisfying $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$. We shall denote by B_n the number of Baxter permutations of size n. They constitute entry A001181 of the Online Encyclopedia of Integer Sequences (OEIS) [29]. The authors Chung, Graham, Hoggatt and Kleiman [15] found the explicit formula of Eq. (1).

Many combinatorial classes have subsequently been discovered to have the same counting sequence — for example triples of lattice paths [16] and plane bipolar orientations [3]. A recent comprehensive survey of Felsner, Fusy, Noy and Orden [17] finds many structural commonalities among these seemingly diverse families of objects. Remarkably, there are intuitive bijections connecting these classes, see for instance [7].

The generating function of hesitating tableaux in Proposition 7 (i) was determined by Xin and Zhang [36]. Baxter numbers appear in their Table 3, and they state that the equivalence between the two series is proved by verifying that the B_n satisfy the recurrence they deduce.

We give a slightly different proof and describe some of the consequences of the result. For example, the classes of Proposition 7 have combinatorial bijections between them, but they *do not* share many of the properties of the other known Baxter classes. However, each of them does have a natural subclass of objects enumerated by Catalan numbers, as many Baxter families also do. (For example, non-crossing partitions are counted by Catalan numbers.)

Proposition 19. *The following classes are in bijection:*

(i) *the set of hesitating tableaux of length* $2n$ *with height bounded by* 2*, starting with empty diagram, ending in a row;*
(ii) *the set of open partition diagrams of length n with no enhanced* 3*-crossing;*
(iii) W_2*-hesitating walks of length n ending on the x-axis;*
(iv) *Baxter permutations of size* $n+1$.

Remark that Theorem 2 is simply the implication that (i) and (iv) from Proposition 19 are in bijection. We prove the bijection between (iii) and (iv) with a generating function argument, and deduce the other bijections using Proposition 7.

5.1. *Proof of Theorem 2*

Theorem 2 has already been proved by Xin and Zhang [36] and in a previous version of this work [13]. For the sake of completeness and pedagogical purpose we provide here an alternative proof relying on explicit coefficient expressions given by the Lagrange inversion formula, as computed in [9]. Moreover we conjecture a stronger result that could be useful as a guide to prove the bijection combinatorially.

We first set up some notation. Let $\bar{x} = \frac{1}{x}$, and consider the ring of formal series $\mathbb{Q}[x, \bar{x}][[t]]$. The operator CT_x extracts the constant term in x of series of $\mathbb{Q}[x, \bar{x}][[t]]$. We recall the work of Bousquet-Mélou and Xin [9]. Here, we only require the $k = 2$ case from their work, and have consequently eliminated some of the subscripts from the statements of their results. Also, note that their definition of W_2 is shifted one unit to the right, hence in the statement of their results, walks start at $(1, 0)$ rather than $(0, 0)$.

Let Q denote the first quadrant in the plane, $Q = \{(x, y) : x, y \geq 0\}$, and let W_2 denote the region $W_2 = \{(x, y) : x > y \geq 0\}$. Walks taking n steps that start at λ and end at μ and remain in Q and W_2 are, respectively, denoted by $q(\lambda, \mu, n)$ and $w(\lambda, \mu, n)$.

Bousquet-Mélou and Xin's Proposition 12 in [9], based on a classic reflection argument, implies the following. For any starting and ending points λ and μ in W_2, the number of W_2-hesitating walks going from λ to μ can be expressed in terms of the number of Q-hesitating walks:

$$w(\lambda, \mu, n) = q(\lambda, \mu, n) - q(\lambda, \overline{\mu}, n)$$

where $\overline{(x,y)} = (y,x)$. They define a simple sign reversing involution between pairs of walks; the walks restricted to W_2 appear as fixed points.

We consider the following two generating functions for Q-hesitating walks that start at $(1,0)$ and end on an axis:

$$H(x;t) = \sum_{i\geq 1, n\geq 0} q((1,0),(i,0),2n)x^i t^n$$
$$\text{and} \quad V(y;t) = \sum_{i\geq 1, n\geq 0} q((1,0),(0,i),2n)y^i t^n.$$

By applying the proposition we see immediately that the bivariate generating function $W(x;t)$ for W_2-hesitating walks that start at $(0,1)$ and end on the x- axis satisfies the formula

$$W(x;t) = \sum_{i\geq 1, n\geq 0} w_2((1,0),(i,0),2n)\, x^i t^n = H(x;t) - V(x;t). \tag{3}$$

Theorem 2 is equivalent to the statement

$$W(1;t) = \sum B_{n+1} t^n. \tag{4}$$

Proof of Theorem 2. For $i \geq 0$, Bousquet-Mélou and Xin [9] show the following

$$[x^{i+1}]H(x;t) = \mathrm{CT}_x \frac{Y}{t(1+x)} \bar{x}^{2+i}(x^2 - \bar{x}^2Y^2 + \bar{x}^3 Y) \tag{5}$$

$$[x^{i+1}]V(x;t) = \mathrm{CT}_x \frac{Y}{t(1+x)} x^{3+i}(x^2 - \bar{x}^2Y^2 + \bar{x}^3 Y). \tag{6}$$

Thus, we deduce

$$\begin{aligned}[x^{i+1}]W(x;t) &= [x^{i+1}]H(x;t) - V(x;t)\\ &= \mathrm{CT}_x \frac{Y}{t(1+x)} (\bar{x}^{2+i} - x^{3+i})(x^2 - \bar{x}^2Y^2 + \bar{x}^3 Y).\end{aligned}$$

Hence we have

$$W(1;t) = T(t) + U(t) + V(t),$$

where

$$\begin{aligned}
T(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y}{t(1+x)} (\bar{x}^i - x^{5+i}) = \mathrm{CT}_x \frac{Y}{t(1+x)}(1 + x + x^2 + x^3 + x^4),\\
U(t) &= -\mathrm{CT}_x \sum_{i\geq 0} \frac{Y^3}{t(1+x)} (\bar{x}^{4+i} - x^{1+i}) = \mathrm{CT}_x \frac{Y^3}{t(1+x)} x,\\
V(t) &= \mathrm{CT}_x \sum_{i\geq 0} \frac{Y^2}{t(1+x)} (\bar{x}^{5+i} - x^i) = \mathrm{CT}_x \frac{Y^2}{t(1+x)}(-1 - x - x^2 - x^3).
\end{aligned}$$

Here, we have applied the following identity from [9] (valid for $k \geq 1$ and $\ell \in \mathbb{Z}$)

$$\mathrm{CT}_x \frac{Y^k}{t(1+x)} \bar{x}^\ell = \mathrm{CT}_x \frac{Y^k}{t(1+x)} x^{\ell-k+1}.$$

Hence, defining $A_{\ell,k}(t) = \mathrm{CT}_x \frac{Y^k}{t(1+x)} x^\ell$, we can collect terms to obtain

$$W(1;t) = \sum_{r=0}^{4} A_{r,1}(t) + A_{1,3}(t) - \sum_{r=0}^{3} A_{r,2}(t). \tag{7}$$

It is shown in [9] that the Lagrange inversion formula yields, for $n \in \mathbb{N}$,

$$[t^n]A_{\ell,k}(t) = \sum_{j\in\mathbb{Z}} a_n(\ell, k, j),$$

with

$$a_n(\ell, k, j) = \frac{k}{n+1}\binom{n+1}{j}\binom{n+1}{j+k}\binom{n}{j-\ell}.$$

Here we apply the convention $\binom{n}{j} = 0$ for $j < 0$ or $j > n$.

Next, it is straightforward to detect and check the linear relations (valid for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$)

$$a_n(4, 1, n-j+2) + a_n(1, 3, j-1) - a_n(2, 2, n-j+1) - a_n(3, 2, j) = 0,$$
$$a_n(1, 1, n-j) + a_n(2, 1, j+1) - a_n(0, 2, j) = 0,$$

which respectively give $A_{4,1}(t) + A_{1,3}(t) - A_{2,2}(t) - A_{3,2}(t) = 0$ and $A_{1,1}(t) + A_{2,1}(t) - A_{0,2}(t) = 0$. Remarkably, expression (7) for $W(1; t)$ simplifies to

$$W(t) = A_{0,1}(t) + A_{3,1}(t) - A_{1,2}(t). \tag{8}$$

For $n \geq 1$, the Baxter number B_n is given by $B_n = \sum_{j\in\mathbb{Z}} b_{n,j}$, with $b_{n,j} = \frac{\binom{n+1}{j-1}\binom{n+1}{j}\binom{n+1}{j+1}}{\binom{n+1}{1}\binom{n+1}{2}}$, and again it is easy to detect and check that (for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$)

$$a_n(0, 1, j) + a_n(3, 1, j+1) - a_n(1, 2, j) = b_{n+1,j+1},$$

so that $A_{0,1}(t) + A_{3,1}(t) - A_{1,2}(t) = \sum_{n\geq 0} B_{n+1}t^n$, and thus $[t^n]W(1; t) = B_{n+1}$. $\square$

5.2. Consequence: a new generating tree

A *generating tree* for a combinatorial class expresses recursive structure in a rooted plane tree with labelled nodes. The objects of size n are each uniquely generated, and the set of objects of size n comprise the nth level of the tree. They are useful for enumeration, and for showing that two classes are in bijection. Theorem 2 yields a new generating tree construction for Baxter objects.

Several different formalisms exist for generating trees, notably [1]. The central properties are as follows. Every object γ in a combinatorial class $\mathcal{C}$ is assigned a label $\ell(\gamma) \in \mathbb{Z}^k$, for some fixed k. There is a rewriting rule on these labels with the property that if two nodes have the same label then the ordered list of labels of their children is also the same. We consider labels that are pairs of positive integers, specified by $\{\ell_{\mathrm{Root}} : [i, j] \to \mathrm{Succ}([i, j])\}$, where ℓ_{Root} is the label of the root.

Two generating trees for Baxter objects are known in the literature, and one consequence of Theorem 2 is a third, using the generating tree for open partitions given by Burrill et al. [12]. This tree differs from the other two already at the third level, illustrating a very different decomposition of the objects. For the three different systems we give the succession rules, and the first 5 levels of the tree (unlabelled), in Fig. 13.

5.3. Conjecture: a refined identity of Baxter numbers

We have proved that the coefficients $a(n, m)$ counting W_2-hesitating walks of length $2n$ from $(0, 0)$ to $(m, 0)$ satisfy $\sum_m a(n, m) = B_{n+1}$, with B_n the nth Baxter number. A bijective proof is yet to be found, and in that perspective a natural question is whether the parameter m corresponds to a simple parameter on another Baxter family. The family of Q-*hesitating excursions*, i.e. hesitating walks in the lattice $Q = \{(x, y) : x, y \geq 0\}$ starting and ending at the origin, forms a good candidate, since we have strong evidence (though no proof) that m is distributed as a certain parameter on that family. But let us show first that it is indeed a Baxter family.

Proposition 20. *The number of Q-hesitating excursions of length $2n$ is equal to B_{n+1}.*

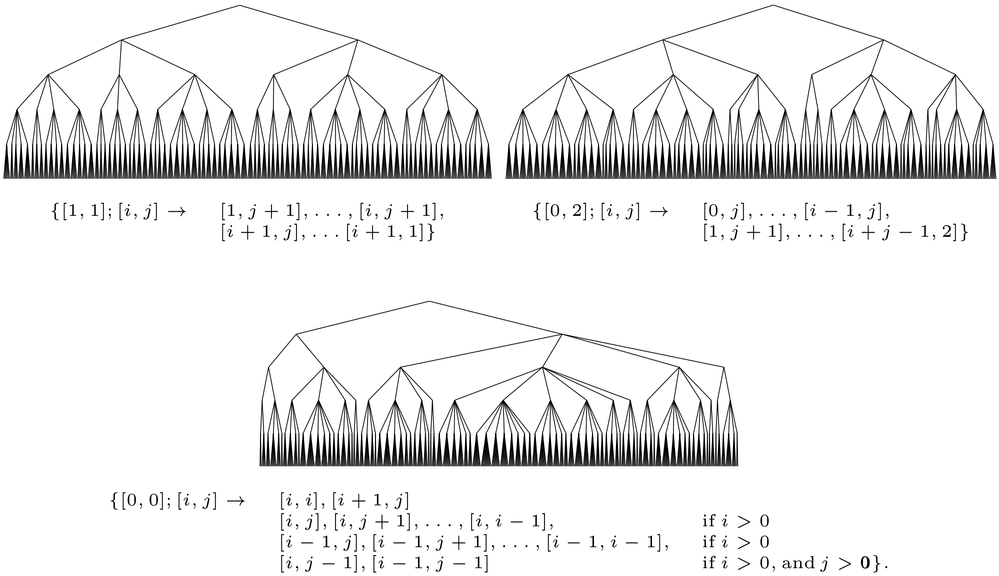


Fig. 13. The first five levels of each of the Baxter generating trees. They are respectively from [7,10,12].

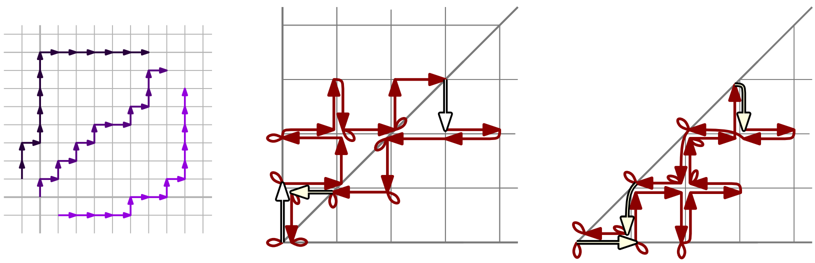


Fig. 14. *Left.* A non-intersecting triple of lattice paths. *Middle.* A Q-hesitating excursion. The stay steps are drawn as loops. The switch-multiplicity of the walk is 3 (the white arrows indicate the marked steps). *Right.* A W_2-hesitating excursion with 3 marked steps each leaving the diagonal. These three objects are in correspondence.

Proof. We show an easy bijection with the set $\mathcal{T}_n$ of non-intersecting triples of lattice paths each of length n with steps either $N = (0, 1)$ (north steps) or $E = (1, 0)$ (east steps), with respective starting points $(-1, 1)$, $(0, 0)$, $(1, -1)$ and respective ending points $(k-1, n-k+1)$, $(k, n-k)$, $(k+1, n-k-1)$ for some $k \in \{0, \dots, n\}$. For 3 distinct points p_1, p_2, p_3 in $\mathbb{Z}^2$ on a same line of slope -1, ordered from top-left to bottom-right, define the *distance-pair* for (p_1, p_2, p_3) as the pair (i, j) of nonnegative integers such that $x(p_1) = x(p_2) - i - 1$ and $x(p_3) = x(p_2) + j + 1$. Let $(P_1, P_2, P_3) \in \mathcal{T}_n$. For $r \in \{0, \dots, n\}$ and $i \in \{1, 2, 3\}$, let $p_i^{(r)}$ be the point on P_i after r steps, and let $d(r)$ be the distance-pair for $(p_1^{(r)}, p_2^{(r)}, p_3^{(r)})$; note that $d(0) = (0, 0)$ and $d(n) = (0, 0)$ and that $d(r) \in q$ for $0 \le r \le n$. Moreover, for $0 < r \le n$, the vector $\delta(r) := d(r) - d(r-1)$ is in the set $\{(\pm 1, 0), (0, \pm 1), (1, -1), (-1, 1), (0, 0)\}$, with two possibilities for being $(0, 0)$ (whether the rth steps in P_1, P_2, P_3 are all north or all east). Hence the situation for the successive distance-pairs $d_0, \dots, d_n$ is exactly the same as for the successive points of even rank in a Q-excursion of length $2n$. Fig. 14 (left and middle) illustrates this bijection. □

We now define a secondary parameter m for Q-excursions. Let w be a Q-excursion of length $2n$ where e_r denotes the rth step, for $1 \le r \le 2n$. Consider, if any, the first step e_{i_1} that visits the region $x < y$. Then consider, if any, the first step e_{i_2} after e_{i_1} that visits the region $x > y$, and so on (switching between $x < y$ and $x > y$ each time). We have here a stopping iterative process yielding, for some $m \geq 0$, m marked steps $e_{i_1}, \ldots, e_{i_m}$ with $i_1 < \cdots < i_m$; m is called the *switch-multiplicity* of the excursion. For instance, the switch-multiplicity of the excursion at the middle of Fig. 14 is 3. Note also that $m \leq n$ since two marked steps cannot be consecutive (the case $m = n$ is reached by the unique excursion where $i_1 = 1, i_2 = 3, i_3 = 5, \ldots$, i.e., the excursion that alternates pairs of steps $(0, 1), (0, -1)$ with pairs of steps $(1, 0), (-1, 0)$). Denote by $q(n, m)$ the number of Q-hesitating excursions of length $2n$ and switch-multiplicity m, and $a(n, m)$ the number of W_2-hesitating walks of length $2n$ from $(0, 0)$ to $(m, 0)$.

Conjecture 21. *For $n, m \geq 0$, we have $q(n, m) = a(n, m)$.*

We have thought of the switch-multiplicity as a natural candidate because of the analogy with a well-known bijection between excursions of length $2n$ on the line $\mathbb{Z}$ and walks of length $2n$ starting at 0 on the half-line $\mathbb{Z}_{\geq 0}$ (with steps in ± 1 for both types of walks), where a similar switch-multiplicity parameter for excursions (this time switching between $\mathbb{Z}_{<0}$ and $\mathbb{Z}_{>0}$) corresponds to half the ending abscissa of positive walks. However, what surprises us is that, while we have strong evidence the conjecture is true, we do not even have a proof for $m = 1$ (the case $m = 0$ is trivial).

Let us now slightly reformulate the conjecture so that we have W_2-hesitating walks on both sides. Consider a Q-hesitating excursion w of length $2n$, with $e_{i_1}, \ldots, e_{i_m}$ the steps given by the stopping iterative process (switching between $x < y$ and $x > y$). Accordingly w splits into a concatenated sequence of $m+1$ parts $\pi_0, \ldots, \pi_m$, where π_0 is the part before e_{i_1} ($\pi_0 = w$ if $m = 0$), for $1 \leq h < m$, the part π_h is between e_{i_h} (included) and $e_{i_{h+1}}$ (excluded), and for $m \geq 1$, the part π_m is the ending part of W_2 starting from e_{i_m}. Each walk π_i starts and ends on the diagonal $x = y$ and stays in $x \geq y$ for i even and in $x \leq y$ for i odd. Hence, if we reflect each odd walk π_{2i+1} according to the diagonal $x = y$, we obtain a W_2-hesitating walk from $(0, 0)$ to $(0, 0)$ with m marked steps (the steps at positions $i_1, \ldots, i_m$) each entering the diagonal $x = y$. In addition, due to recording the marked steps, there is no loss of information (the original excursion can be recovered). This correspondence is depicted by Fig. 14 (middle and right). Hence, if we denote by $a(n; i, j, m)$ the number of W_2-hesitating walks of length $2n$ from $(0, 0)$ to (i, j) with $2n$ steps and m marked steps each leaving the diagonal $x = y$, Conjecture 21 can be reformulated as:

Conjecture 22 (*Reformulation*)**.** *For $n, m \geq 0$, we have*

$$a(n; m, 0, 0) = a(n; 0, 0, m).$$

Actually, an even stronger symmetry seems to hold:

Conjecture 23. *For $n, i, j \geq 0$, we have*

$$a(n; i, 0, j) = a(n; j, 0, i).$$

Note that there is clearly a one-to-one correspondence between steps leaving the diagonal $x = y$ and steps ending at the diagonal $x = y$. Hence $a(n; i, j, m)$ is also the number of W_2-hesitating walks of length $2n$ from $(0, 0)$ to (i, j) with $2n$ steps and m marked steps each ending at the diagonal $x = y$. In that form it is easy to obtain a recurrence for the coefficients $a(n; i, j, m)$ by considering the effect of adding the last two steps (note that each of the two last steps has to be unmarked if empty or not ending at $x = y$, and might be either unmarked or marked if non-empty and ending at $x = 1$). Denote by $\mathcal{S}$ the set of steps $\{(\pm 1, 0), (0, \pm 1), (-1, 1), (1, -1)\}$ together with the two stationary steps s_1 and s_2, where s_1 simulates taking the pair $(1, 0)$ and $(-1, 0)$ as a single step, and s_2 simulates taking the pair $(0, 1)$ and $(0, -1)$ as a single step. We have the following recurrence (with δ the Kronecker symbol), from which we have been able to check that Conjecture 23 holds for all n, i, j with $n \leq 56$:

- for $n = 0$,

$$a(n; i, j, m) = 1 \quad \text{if } i = j = m = 0,$$
$$a(n; i, j, m) = 0 \quad \text{otherwise},$$

- for $n > 0$,

$$
\begin{aligned}
a(n; i, j, m) &= 0 \quad \text{for } (i, j, m) \notin \mathcal{D} := \{0 \le j \le i,\ 0 \le m \le n\},\\
a(n; i, j, m) &= \delta_{i=j} \cdot \sum_{s \in \mathcal{S} \setminus s_1} a(n-1; i - x(s), j - y(s), m)\\
&\quad + \delta_{i>j} \cdot \sum_{s \in \mathcal{S}} a(n-1; i - x(s), j - y(s), m)\\
&\quad + \delta_{i=j} \cdot \sum_{s \in \mathcal{S} \setminus s_1} a(n-1; i - x(s), j - y(s), m-1)\\
&\quad + \delta_{i=j+1} \cdot a(n-1; i, j, m-1) \quad \text{for } (i, j, m) \in \mathcal{D}.
\end{aligned}
$$

6. Conclusion

We conclude with a few thoughts on future directions. We are led to wonder if the new interpretation of oscillating tableaux is significant in representation theory, or if it is simply a form that facilitates enumeration. Furthermore, other generalizations, such as osculating walkers [4], could be interpreted in this context.

Baxter numbers generalize, in some sense, Catalan numbers. Both are ubiquitous combinatorial sequences, and both are related to hesitating walk families. Perhaps a bijection between the Catalan sub-classes of one of the classes in Proposition 19 and one of the previously known Baxter classes could be extended to a bijection for the full class. This might also permit new interpretations of hesitating walks in higher dimensions.

Finally let us mention that in very recent work with Mathias Lepoutre we have been able to find a combinatorial derivation of Theorem 2.

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