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Asymptotic Lattice Path Enumeration Using Diagonals

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Abstract We consider d-dimensional lattice path models restricted to the first orthant whose defining step sets exhibit reflective symmetry across every axis. Given such a model, we provide explicit asymptotic enumerative formulas for the number of walks of a fixed length: the exponential growth is given by the number of distinct steps a model can take, while the sub-exponential growth depends only on the dimension of the underlying lattice and the number of steps moving forward in each coordinate. The generating function of each model is first expressed as the diagonal of a multivariate rational function, then asymptotic expressions are derived by analyzing the singular variety of this rational function. Additionally, we show how to compute subdominant growth, reflect on the difference between rational diagonals and differential equations as data structures for D-finite functions, and show how to determine first order asymptotics for the subset of walks that start and end at the origin.

Keywords Lattice path enumeration · D-finite · Diagonal · Analytic combinatorics in several variables · Weyl chambers

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1 Introduction

The reflection principle and its various incarnations have been indispensable in the study of lattice path models, particularly in the discovery of explicit enumerative formulas. Two examples include the study of walks in Weyl chambers initiated by Gessel and Zeilberger [17], and recent activity on walks restricted to the first quadrant using various kernel method approaches [8,10,11,21]. In these guises, the reflection principle is often a key element of the solution when dealing with D-finite[1] generating functions. This is no coincidence: the enumerative arguments in these works often contain an intermediate step where the generating function is expressed as a sub-series extraction of a rational function; this kind of expression is known to be D-finite by closure properties of the class. Such arguments typically continue by determining explicit representations of the generating functions which, unfortunately, can be cumbersome to manipulate. For example, recent work on walks in Weyl chambers has led to expressions which are determinants of large matrices with Bessel function entries [18,19,32]. Here we aim to determine asymptotics for a family of lattice path models arising naturally among those restricted to positive orthants—which correspond to walks in certain Weyl chambers—while avoiding such unwieldy representations. This is achieved by working directly with characterizations of generating functions as diagonals of multivariate rational functions, through the recently developed machinery for analytic combinatorics in several variables [27].

Coupling these two strategies—the systematic construction of diagonal representations and the application of analytic combinatorics in several variables—yields explicit, yet simple, asymptotic formulas for families of lattice path models. The focus of this article is d-dimensional walks whose set of allowable steps is symmetric with respect to every axis; we say such models are *highly symmetric*[2]. The techniques of analytic combinatorics in several variables apply in a rather straightforward way to derive dominant asymptotics for the number of walks ending anywhere in the non-negative orthant, and give an effective procedure to calculate higher order terms in the asymptotic expansions. Furthermore, we also consider the subfamily of walks that return to the origin (known as *excursions*). The highly symmetric models are good starting points in a wider study because the symmetries simplify both the process of constructing rational diagonal representations of the generating functions and the methods needed from analytic combinatorics in several variables to determine asymptotics. Once our equations are established, they are suitable input to existing computer algebra packages for asymptotics, such as that of Raichev [28].

The highly symmetric lattice path models we study fit well into the ongoing study of lattice path classes restricted to an orthant which take only "small" steps [3,7,10], and in particular are amenable to a kernel method treatment. This collection of models forms a little universe exhibiting many interesting phenomena, and recent work in two and three dimensions has used novel applications of algebra and analysis, along with new computational techniques, to determine exact and asymptotic enumeration

[1] A function is D-finite if it satisfies a linear differential equation with polynomial coefficients.

[2] The walks themselves are not required to possess any particular kind of symmetry.

formulas. One key predictor of the nature of a model's generating function (whether it is rational, algebraic, transcendental D-finite, or none of these) is the order of a group that is associated to each model. This group has its origins in the probabilistic study of random walks, namely [14], and when the group is finite it can sometimes be used to write generating functions as the positive part of an explicit multivariate rational Laurent series. The intimate relationship between combinatorial properties of a walk model and the nature of its generating function is explored in [7,25].

In 1992, Gessel and Zeilberger [17] outlined an extension of the reflection principle—originally used by André [1] in the nineteenth century to solve the two candidate ballot problem—to lattice walks on regions preserved under the actions of Coxeter–Weyl finite reflection groups; such regions are known as *Weyl chambers*. For highly symmetric models in two and three dimensions, the group defined by the kernel method coincides with that of the Coxeter–Weyl group for the Weyl chambers A_1^2 and A_1^3, respectively (a short survey of the basic results on Weyl chambers is included in Sect. 7 of this document). Indeed, one can use either viewpoint to generalize the study of two dimensional highly symmetric models to an arbitrary dimension. As these viewpoints are largely isomorphic, and the kernel method viewpoint is more self-contained, we begin this article by working through a straightforward application of the kernel method in order to write the generating function of a highly symmetric model in an arbitrary dimension as the diagonal of a multivariate rational function. We then present an asymptotic analysis of the coefficients of counting generating functions using techniques from the study of analytic combinatorics in several variables, and consequently link some of the combinatorial symmetries in a walk model to both analytic properties of the generating function, and geometric properties of an associated variety.

After this analysis is complete, we examine how these methods connect to walks in Weyl chambers, use results from their study to determine asymptotic results about excursions, and then discuss how the Weyl chamber viewpoint can be used in future work to examine larger classes of lattice path models using diagonal expressions. To start, we give a precise definition for highly symmetric models, and then state our main results.

1.1 Highly Symmetric Lattice Path Models

Concretely, the lattice path models we consider are characterized as follows. For a fixed dimension d, we define a model by its step set $\mathcal{S} \subseteq \{\pm 1, 0\}^d \setminus \{\mathbf{0}\}$ and say that $\mathcal{S}$ is *symmetric about the x_k axis* if $(i_1, \ldots, i_k, \ldots, i_d) \in \mathcal{S}$ implies $(i_1, \ldots, -i_k, \ldots, i_d) \in \mathcal{S}$. We further impose a nontriviality condition: for each coordinate there is at least one step in $\mathcal{S}$ which moves in the positive direction of that coordinate (this implies that for each coordinate there is a walk in the model which moves in that coordinate).

Definition 1.1 [Main Definition] Fix a dimension $d \in \mathbb{N}$ and let $\mathcal{S} \subseteq \{\pm 1, 0\}^d \setminus \{\mathbf{0}\}$. We say that $\mathcal{S}$ is a **nontrivial, highly symmetric lattice path model** if it satisfies the following properties:

(1) $\mathcal{S}$ is symmetric about the x_k axis for every $k = 1 \ldots d$; (highly symmetric)
(2) for each k, there is at least one element of $\mathcal{S}$ with a 1 in the kth coordinate. (nontrivial)

The main enumerative quantity of interest is the number s_n of walks taking steps from $\mathcal{S}$, starting at the origin and remaining in the positive orthant $\mathbb{N}^d = \mathbb{Z}^d_{\geq 0}$. Since the counting generating functions of the highly symmetric models can each be expressed as positive parts of multivariate rational Laurent series, (and consequently diagonals of rational functions in $(d+1)$ variables) they all have D-finite generating functions.[3]

The rational function obtained is then subjected to the asymptotic enumeration methods outlined in [27], in particular the cases which were developed by Pemantle, Raichev and Wilson in [26] and [29]. The singular variety defined by this rational function is central to this process and forcing a step set to have symmetry across each axis ensures that the variety is smooth, allowing us to calculate dominant asymptotics explicitly. This is not generally the case, in our experience, and thus we selected this particular kind of restriction as the starting point in our wider study. When the singular variety is not smooth, the analysis becomes more complicated due to an increased complexity in the integrals which yield asymptotic results. There are results in the text of Pemantle and Wilson [27] which apply to certain families of rational functions with non-smooth singular varieties, and applying these techniques to more general classes of lattice path models is ongoing work.

1.2 Main Results

We present two main results in this work. The first appears as Theorem 3.4.

Theorem Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ define a d-dimensional, nontrivial, highly symmetric lattice path model. Then the number s_n of walks of length n taking steps in $\mathcal{S}$, beginning at the origin, and never leaving the positive orthant has asymptotic expansion

$$s_n = \left[\left(s^{(1)} \cdots s^{(d)} \right)^{-1/2} \pi^{-d/2} |\mathcal{S}|^{d/2} \right] \cdot n^{-d/2} \cdot |\mathcal{S}|^n + O\left(n^{-(d+1)/2} \cdot |\mathcal{S}|^n \right),$$

where $s^{(k)}$ denotes the number of steps in $\mathcal{S}$ which have kth coordinate 1.

This formula is easy to apply to any given model, and for certain infinite families of models as well.

Example 1.2 When $d = 2$ there are four nonisomorphic highly symmetric models in the quarter plane, listed in Table 1. Applying Theorem 3.4 verifies the asymptotic results guessed previously by [4]. $\triangleleft$

Example 1.3 Let $\mathcal{S} = \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$, the largest possible step set. This is symmetric across each axis. We compute that $|\mathcal{S}| = 3^d - 1$, and $s^{(j)} = 3^{d-1}$ for all j, so

$$s_n \sim \left(\frac{(3^d - 1)^{d/2}}{3^{d(d-1)/2} \cdot \pi^{d/2}} \right) \cdot n^{-d/2} \cdot (3^d - 1)^n.$$

$\triangleleft$

[3] The class of D-finite functions is closed under the diagonal operation [24].

Table 1 The four highly symmetric models with unit steps in the quarter plane and the asymptotic enumeration formulas

$\mathcal{S}$	Asymptotics	$\mathcal{S}$	Asymptotics
	$\frac{4}{\pi\sqrt{1\cdot 1}} \cdot n^{-1} \cdot 4^n = \frac{4}{\pi} \cdot \frac{4^n}{n}$		$\frac{4}{\pi\sqrt{2\cdot 2}} \cdot n^{-1} \cdot 4^n = \frac{2}{\pi} \cdot \frac{4^n}{n}$
	$\frac{6}{\pi\sqrt{3\cdot 2}} \cdot n^{-1} \cdot 6^n = \frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$		$\frac{8}{\pi\sqrt{3\cdot 3}} \cdot n^{-1} \cdot 8^n = \frac{8}{3\pi} \cdot \frac{8^n}{n}$

Example 1.4 Let $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the kth standard basis vector in $\mathbb{R}^d$, and consider the set of steps $\mathcal{S} = \{e_1, -e_1, \ldots, e_d, -e_d\}$. Then the number of walks of length n taking steps from $\mathcal{S}$ and never leaving the positive orthant has asymptotic expansion

$$s_n \sim \left(\frac{2d}{\pi}\right)^{d/2} n^{-d/2} \, (2d)^n.$$

$\triangleleft$

The second main result is a comparable statement for excursions, Theorem 7.4.

Theorem Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ define a nontrivial, highly symmetric lattice path model in dimension d. Then the number of excursions e_n of length n taking steps in $\mathcal{S}$, beginning and ending at the origin, and never leaving the positive orthant satisfies

$$e_n = O\left(\frac{|\mathcal{S}|^n}{n^{3d/2}}\right).$$

The work of Denisov and Wachtel [13], which investigates the exit times of random walks in a very general class of cones, gives a framework to determine the dominant asymptotics for such excursions. To be precise, Equation (12) of [13] implies that the number of excursions e_n satisfies the asymptotic estimate $e_n \sim C \cdot |\mathcal{S}|^n \cdot n^{-(d+p/2)}$ for some constant C, where p is defined in terms of the dimension d and a positive eigenvalue of the Laplace–Beltrami operator on the $d-1$ sphere (we note that this result holds for all walk models in an orthant, not just the highly symmetric models). Based on these results, Bostan et al. [7] gave an effective method for calculating the exponent $d + p/2$ in the two dimensional case; for the dimension two highly symmetric models their method shows that $e_n \sim C \cdot |\mathcal{S}|^n \cdot n^{-3}$, which matches the upper bound given by the above theorem.

1.3 Organization of This Paper

This article is organized as follows. Section 2 describes how to express the generating function using an orbit sum by applying the kernel method, following the strategy of [10]. We then derive Eq. (9), which describes the generating function as the diagonal

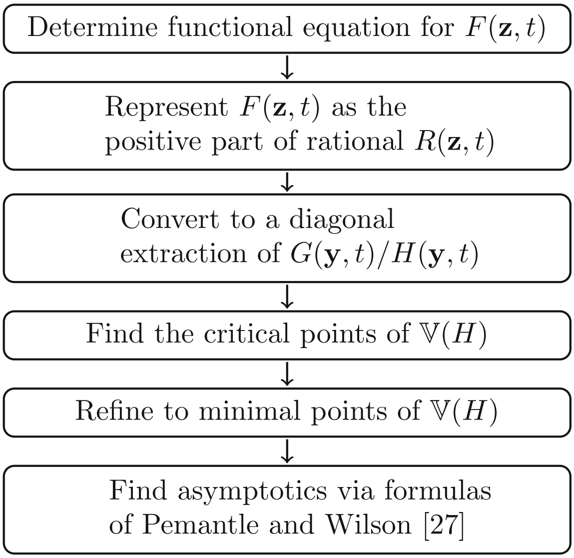

Fig. 1 The stratagem of determining asymptotics via the generalized kernel method for symmetric models

of a rational multi-variable series. Section 3 justifies why the work of Pemantle and Wilson [27] is applicable, with the asymptotic results computed in Sect. 3.3. We discuss the sub-dominant factor, and compute an example in Sect. 5. Section 6 discusses the differential equations satisfied by these generating functions, and how to use creative telescoping techniques to find them. We apply the results to some small examples, then conclude with a discussion of how these models fit into the context of walks in Weyl chambers, allowing us to obtain results on the asymptotics of walk excursions and consider other generalizations.

2 Deriving a Diagonal Expression for the Generating Function

Fix a dimension d and a highly symmetric set of steps $\mathcal{S} \subseteq \{\pm 1, 0\}^d \setminus \{\mathbf{0}\}$. Recall this means that $(i_1, \ldots, i_k, \ldots, i_d) \in \mathcal{S}$ implies $(i_1, \ldots, -i_k, \ldots, i_d) \in \mathcal{S}$. In this section we derive a functional equation for a multivariate generating function, apply the orbit sum method to derive a closed expression related to this generating function, and conclude by writing the univariate counting generating function for the number of walks as the complete diagonal of a rational function. The process is summarized in Fig. 1.

The following notation is used throughout:

$$\overline{z}_i = z_i^{-1}; \qquad \mathbf{z} = (z_1, \ldots, z_d); \qquad \mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d;$$
$$\mathbf{z}^{\mathbf{i}} = z_1^{i_1} \cdots z_d^{i_d}; \qquad \mathbf{z}_{\hat{k}} := (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d),$$

and we write $\mathbb{Q}[z_k, \overline{z_k}]$ to refer to the ring of Laurent polynomials in the variable z_k.

2.1 A Functional Equation

To begin, we define the generating function:

$$F(\mathbf{z},t)=\sum_{\substack{n\geq 0\\ \mathbf{i}\in\mathbb{N}^d}} s_{\mathbf{i}}(n)\mathbf{z}^{\mathbf{i}}t^n=\sum_{n\geq 0}\left(\sum_{\mathbf{i}\in\mathbb{N}^d} s_{\mathbf{i}}(n)z_1^{i_1}\cdots z_d^{i_d}\right)t^n\in\mathbb{Q}[z_1,\ldots,z_d][\![t]\!], \tag{1}$$

where $s_{\mathbf{i}}(n)$ counts the number of walks of length n taking steps from $\mathcal{S}$ which stay in the positive orthant and end at lattice point $\mathbf{i}\in\mathbb{N}^d$. Note that the series $F(\mathbf{1},t)$ is the generating function for the total number of walks in the orthant, and we can recover the series for walks ending on the hyperplane $z_k=0$ by setting $z_k=0$ in the series $F(\mathbf{z},t)$ (the variables $z_1,\ldots,z_d$ are referred to as *catalytic* variables in the literature, as they are present during the analysis and removed at the end of the 'reaction' via specialization to 1). We also define the function (known as either the *characteristic polynomial* or the *inventory* of $\mathcal{S}$) by

$$S(\mathbf{z})=\sum_{\mathbf{i}\in\mathcal{S}}\mathbf{z}^{\mathbf{i}}\in\mathbb{Q}\left[z_1,\overline{z}_1,\ldots,z_d,\overline{z}_d\right]. \tag{2}$$

In many recent analyses of lattice walks, functional equations are derived by translating the following description of a walk into a generating function equation: a walk is either an empty walk, or a shorter walk followed by a single step. To ensure the condition that the walks remain in the positive orthant, we must not count walks that add a step with a negative kth component to a walk ending on the hyperplane $z_k=0$. To account for this, it is sufficient to subtract an appropriate multiple of F from the functional equation: $t\overline{z}_kF(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_d,t)$. However, if a given step has several negative components we must use the principle of inclusion and exclusion to prevent over compensation. To be explicit, the generating function $F(\mathbf{z},t)$ satisfies the functional equation

$$\begin{aligned}(z_1\cdots z_d)F(\mathbf{z},t)&=(z_1\cdots z_d)+t(z_1\cdots z_d)S(\mathbf{z})F(\mathbf{z},t)\\&\quad-t\sum_{V'\subseteq[d]}(-1)^{|V'|}\left[(z_1\cdots z_d)S(\mathbf{z})F(\mathbf{z},t)\right]_{\{z_j=0:j\in V'\}},\end{aligned} \tag{3}$$

where $[d]=\{1,\ldots,d\}$. Basic manipulations then give the following result.

Lemma 2.1 *Let $F(\mathbf{z},t)$ be the multivariate generating function described above. Then*

$$(z_1\cdots z_d)\left(1-tS(\mathbf{z})\right)F(\mathbf{z},t)=(z_1\cdots z_d)+\sum_{k=1}^{d}A_k\left(\mathbf{z}_{\hat{k}},t\right), \tag{4}$$

for some $A_k\in\mathbb{Q}\left[\mathbf{z}_{\hat{k}}\right][\![t]\!]$.

Example 2.2 Set $\mathcal{S} = \{e_1, -e_1, \ldots, e_d, -e_d\}$. In this case $S(\mathbf{z}) = \sum_{j=1}^{d}(z_j + \overline{z}_j)$, so $(z_1 \cdots z_d)S(\mathbf{z})$ vanishes when at least two of the z_j are zero, and the generating function satisfies

$$(z_1 \cdots z_d)\,(1 - tS(\mathbf{z}))\,F(\mathbf{z}, t) = (z_1 \cdots z_d) + \sum_{k=1}^{d} t(z_1 \ldots z_{k-1}z_{k+1} \ldots z_d)F\left(\mathbf{z}_{\hat{k}}\right).$$

◁

2.2 The Orbit Sum Method

The orbit sum method, also known as the obstinate kernel method, has three main steps: find a suitable group $\mathcal{G}$ of rational maps; apply the elements of the group to the functional equation and form a telescoping sum; and (ultimately) represent the generating function of a model as the positive series extraction of an explicit rational function. Bousquet-Mélou and Mishna [10] illustrate the applicability in the case of lattice walks, and it has been adapted to several dimensions by Bostan et al. [3].

2.2.1 The Group $\mathcal{G}$

For any d-dimensional model, we define the group $\mathcal{G}$ of 2^d rational maps by

$$\mathcal{G} := \left\{(z_1, \ldots, z_d) \mapsto (z_1^{i_1}, \ldots, z_d^{i_d}) : (i_1, \ldots, i_d) \in \{-1, 1\}^d\right\}. \tag{5}$$

Given $\sigma \in \mathcal{G}$, we can consider σ as a map on $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][\![t]\!]$ through the group action defined by $\sigma\,(A(\mathbf{z}, t)) := A\,(\sigma(\mathbf{z}), t)$. Due to the symmetry of the step set across each axis, one can verify that $\sigma(S(\mathbf{z})) = S(\sigma(\mathbf{z})) = S(\mathbf{z})$ always holds. The fact that this group does not depend on the step set of the model—only on the dimension d—is crucial to obtaining the general results here[4]. When d equals 2, the group $\mathcal{G}$ matches the groups used by [10,14]. As we will see in Sect. 7, $\mathcal{G}$ corresponds to the Weyl group of the Weyl chamber A_1^d, where the step set $\mathcal{S}$ can be studied in the context of Gessel and Zeilberger [17].

2.2.2 A Telescoping Sum

Next we apply each of the 2^d elements of $\mathcal{G}$ to Eq. (4), and take a weighted sum. Define $\operatorname{sgn}(\sigma) = (-1)^r$, where $r = \#\{k : \sigma(\mathbf{z})_k = \overline{z}_k\}$ is the number of components of $(z_1, \ldots, z_d)$ which σ maps to their reciprocals, and let $\sigma_k \in \mathcal{G}$ be the map

[4] In general, the kernel method associates to each set of steps $\mathcal{S} \subset \{\pm 1, 0\}^d$ some (possibly unique) group of rational maps (which groups arise, and what properties they possess, is still being investigated). When this group is finite an analysis similar to the one presented here often, but not always, yields an explicit expression for the corresponding generating function, and proves that this generating function is D-finite. See [10] for the two dimensional case and [3] for some results in three dimensions.

$$\sigma_k : (z_1, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_d) \mapsto (z_1, \ldots, z_{k-1}, \overline{z}_k, z_{k+1}, \ldots, z_d).$$

Lemma 2.3 *Let $F(\mathbf{z}, t)$ be the generating function counting the number of walks of length n with endpoint marked. Then, as elements of the ring $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][\![t]\!]$,*

$$\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d)\sigma(F(\mathbf{z}, t)) = \frac{\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d)}{1 - tS(\mathbf{z})}. \tag{6}$$

Proof For each $\sigma \in \mathcal{G}$ we have $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma_k \sigma)$ and, for the A_k in Equation (4),

$$\sigma\left(A_k\left(\mathbf{z}_{\hat{k}}, t\right)\right) = \left(\sigma_k \sigma\right)\left(A_k(\mathbf{z}_{\hat{k}}, t)\right).$$

Thus, we can apply each $\sigma \in \mathcal{G}$ to Eq. (4) and sum the results, weighted by $\operatorname{sgn}(\sigma)$, to cancel each A_k term on the right hand side. Minor algebraic manipulations, along with the fact that the group elements fix $S(\mathbf{z})$, then give Eq. (6). □

2.2.3 Positive Series Extraction

Next, we note that each term in the expansion of

$$\begin{aligned} \operatorname{sgn}(\sigma_1) \cdot \sigma_1(z_1 \cdots z_d)\sigma_1(F(\mathbf{z}, t)) &= -(\overline{z}_1 z_2 \cdots z_d) F(\overline{z}_1, z_2, \ldots, z_d, t) \\ &\in \mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][\![t]\!] \end{aligned}$$

has a negative power of z_1. In fact, unless σ is the identity any summand $\sigma(z_1 \cdots z_d)\sigma(F(\mathbf{z}, t))$ on the left hand side of Eq. (6) contains a negative power of at least one variable in any term of its expansion.

With this in mind, for an element $A(\mathbf{z}, t) \in \mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][\![t]\!]$ we let $[z_k^{\geq}]A(\mathbf{z}, t)$ denote the sum of all terms of $A(\mathbf{z}, t)$ which contain only non-negative powers of z_k. Lemma 2.4 then follows from the identity

$$\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d) = (z_1 - \overline{z}_1) \cdots (z_d - \overline{z}_d),$$

which can be proved by induction.

Lemma 2.4 *Let $F(\mathbf{z}, t)$ be the generating function counting the number of walks of length n with endpoint marked. Then*

$$F(\mathbf{z}, t) = [z_1^{\geq}] \cdots [z_d^{\geq}] R(\mathbf{z}, t), \tag{7}$$

where

$$R(\mathbf{z}, t) = \frac{(z_1 - \overline{z}_1) \cdots (z_d - \overline{z}_d)}{(z_1 \cdots z_d)(1 - tS(\mathbf{z}))}.$$

Since the class of D-finite functions is closed under positive series extraction—as shown in [24]—an immediate consequence is the following.

Corollary 2.5 *For any nontrivial highly symmetric model $\mathcal{S}$ of walks both the complete generating function $F(\mathbf{z}, t)$ and the counting generating function $F(\mathbf{1}, t)$ are D-finite functions.*

2.3 The Generating Function as a Diagonal

Given an element

$$B(\mathbf{z}, t) = \sum_{n \geq 0} \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} b_{\mathbf{i}}(n) z_1^{i_1} \cdots z_d^{i_d} \right) t^n \in \mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][[t]],$$

we let Δ denote the (complete) diagonal operator

$$\Delta B(\mathbf{z}, t) := \sum_{n \geq 0} b_{n,\ldots,n}(n) t^n.$$

There is a natural correspondence between the diagonal operator and extracting the positive part of a multivariate power series, as in Eq. (7).

Proposition 2.6 *Let $B(\mathbf{z}, t)$ be an element of $\mathbb{Q}[z_1, \overline{z}_1, \ldots, z_d, \overline{z}_d][[t]]$. Then*

$$[z_1^{\geq}] \cdots [z_d^{\geq}] B(\mathbf{z}, t) \bigg|_{z_1=1,\ldots,z_d=1} = \Delta \left(\frac{B(\overline{z}_1, \ldots, \overline{z}_d, z_1 \cdots z_d \cdot t)}{(1 - z_1) \cdots (1 - z_d)} \right). \tag{8}$$

Proof The right hand side of Eq. (8) is given by

$$\Delta \left(\sum_{k \geq 0} z_1^k \right) \cdots \left(\sum_{k \geq 0} z_d^k \right) \left(\sum_{n \geq 0} \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} b_{\mathbf{i}}(n) z_1^{n-i_1} \cdots z_d^{n-i_d} \right) t^n \right)$$

so that the coefficient of t^n in the diagonal is the sum of all terms $b_{\mathbf{i}}(n)$ with $i_1, \ldots, i_d \geq 0$ (by assumption there are only finitely many which are nonzero). But this is exactly the coefficient of t^n on the left hand side. □

We note also that in the context of lattice path models with step set $\mathcal{S} \subseteq \{\pm 1, 0\}^d \setminus \{\mathbf{0}\}$, the modified generating function $F(\overline{z}_1, \ldots, \overline{z}_d, z_1 \cdots z_d \cdot t)$ is actually a power series in the variables $z_1, \ldots, z_d, t$ (as a walk cannot move farther in any direction than its number of steps). Combining Lemma 2.4 and Proposition 2.6 implies that the generating function for the number of walks can be represented as $F(\mathbf{1}, t) = \Delta \left(\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \right)$ where, as $S(\mathbf{z}) = S(\overline{z}_1, \ldots, \overline{z}_d)$ by our symmetry condition,

$$\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} = \frac{(1 - z_1^2) \cdots (1 - z_d^2)}{1 - t(z_1 \cdots z_d) S(\mathbf{z})} \cdot \frac{1}{(1 - z_1) \cdots (1 - z_d)}$$

$$= \frac{(1+z_1)\cdots(1+z_d)}{1 - t(z_1\cdots z_d)S(\mathbf{z})}. \tag{9}$$

To be precise, $G(\mathbf{z}, t)$ and $H(\mathbf{z}, t)$ are defined as the numerator and denominator of Eq. (9).

Example 2.7 For the walks defined by $\mathcal{S} = \{e_1, -e_1, \ldots, e_d, -e_d\}$, we have

$$\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} = \frac{(1+z_1)\cdots(1+z_d)}{1 - t\sum_{k=1}^{d}(1+z_k^2)(z_1\cdots z_{k-1}z_{k+1}\cdots z_d)}.$$

Note that this rational function is not unique, in the sense that there are other rational functions whose diagonals yield the same counting sequence. ◁

2.4 The Singular Variety Associated to the Kernel

Here, we pause to note that the *combinatorial* symmetries of the step sets that we consider affect the *geometry* of the variety of $H(\mathbf{z}, t)$—called the *singular variety*. This has a direct impact on both the asymptotics of the counting sequence under consideration and the ease with which asymptotics are computed. In particular, factors of the form $(1-z_k)$ present in the denominator of this rational function can contribute to the existence of non-simple poles, and thus a non-smooth singular variety. Although non-smooth varieties can be handled in many cases—see [27]—having a smooth singular variety is the easiest situation in which one can work in the multivariate setting. Understanding the interplay between the step set symmetry and the singular variety geometry, and in the process dealing with the non-smooth cases, is a promising direction for future work.

3 Analytic Combinatorics in Several Variables

Following work of Pemantle and Wilson [26] and Raichev and Wilson [29], we determine the dominant asymptotics for the diagonal of the multivariate power series $\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}$ by studying the variety (complex set of zeroes) $\mathcal{V} \subseteq \mathbb{C}^{d+1}$ of the denominator

$$H(\mathbf{z}, t) = 1 - t\cdot(z_1\cdots z_d)S(\mathbf{z}).$$

To begin, we compute a set of singular points known as the *critical points* which contain all singular points which *could potentially* affect the asymptotics of $\Delta(G/H)$; this is done in Sect. 3.1. The set of critical points is then refined to a set of *minimal points*, which determine the dominant asymptotics up to an exponential decay; this is computed in Sect. 3.2. The minimal points are the critical points which are ‘closest’ to the origin in a sense made precise below.

In fact, for any highly symmetric model there is only one singular point which determines dominant asymptotics: the point $\boldsymbol{\rho} = (\mathbf{1}, 1/|\mathcal{S}|)$. This uniformity aids greatly in computing the quantities required in the analysis of a general step set, in order to obtain Theorem 3.4.

The enumerative results come from calculating a Cauchy residue type integral, and after characterizing the minimal points we determine asymptotics in Sect. 3.3 using pre-computed formulas for such integrals found in [27].

We first verify our claim in the previous section that the variety is smooth (that is, at every point on $\mathcal{V}$ at least one of the partial derivatives H_{z_k} or H_t does not vanish). Indeed, any non-smooth point on $\mathcal{V}$ would have to satisfy both

$$\begin{aligned} 1 - t(z_1 \cdots z_d)S(\mathbf{z}) &= H = 0 \\ \text{and} \quad -(z_1 \cdots z_d)S(\mathbf{z}) &= H_t = 0, \end{aligned}$$

which can never occur. Equivalently, this shows that at each point in $\mathcal{V}$ there exists a neighbourhood $N \subseteq \mathbb{C}^{d+1}$ such that $\mathcal{V} \cap N$ is a complex submanifold of N.

3.1 Critical Points

The next step is to find the critical points, determined through an appeal to stratified Morse theory. The critical points for a smooth variety are precisely the solutions to the following *critical point equations*:

$$H = 0, \quad tH_t = z_1 H_{z_1}, \quad tH_t = z_2 H_{z_2}, \quad \ldots, \quad tH_t = z_d H_{z_d},$$

which we now solve. As each step in $\mathcal{S}$ has coordinates taking values in $\{-1, 0, 1\}$, we may collect the coefficients of z_k and use symmetry to write, for unique Laurent polynomials $S_1^{(k)}(\mathbf{z}_{\hat{k}})$ and $S_0^{(k)}(\mathbf{z}_{\hat{k}})$,

$$S(\mathbf{z}) = (\overline{z}_k + z_k)S_1^{(k)}(\mathbf{z}_{\hat{k}}) + S_0^{(k)}(\mathbf{z}_{\hat{k}}). \tag{10}$$

The equation $tH_t = z_k H_{z_k}$ states

$$t(z_1 \cdots z_d)S(\mathbf{z}) = t(z_1 \cdots z_d)S(\mathbf{z}) + t(z_1 \cdots z_d)(z_k S_{z_k}(\mathbf{z})),$$

which implies

$$0 = t(z_1 \cdots z_d) \cdot z_k S_{z_k}(\mathbf{z}) = t\left(z_k^2 - 1\right)(z_1 \cdots z_{k-1} z_{k+1} \cdots z_d)S_1^{(k)}(\mathbf{z}_{\hat{k}}). \tag{11}$$

Note that while $(z_1 \cdots z_{k-1} z_{k+1} \cdots z_d)S_1^{(k)}(\mathbf{z}_{\hat{k}})$ is a polynomial, $S_1^{(k)}(\mathbf{z}_{\hat{k}})$ itself is a Laurent polynomial, so one must be careful when specializing variables to 0 in the expression. This calculation characterizes the critical points of $\mathcal{V}$.

Proposition 3.1 *The point $(\mathbf{z}, t) = (z_1, \ldots, z_d, t) \in \mathcal{V}$ is a critical point of $\mathcal{V}$ if and only if for each $1 \leq k \leq d$ either:*

(1) $z_k = \pm 1$ *or,*

(2) *the polynomial* $(y_1 \cdots y_{k-1}y_{k+1} \cdots y_d)S_1^{(k)}(\mathbf{y}_{\hat{k}})$ *has a root at* $\mathbf{z}$.

Proof We have shown above that the critical point equations reduce to Eq. (11). Furthermore, if t were zero at a point on $\mathcal{V}$ then $0 = H(z_1, \ldots, z_n, 0) = 1$, a contradiction. □

The Laurent polynomial $S_1^{(k)}(\mathbf{y}_{\hat{k}})$ is the sum of the terms in the Laurent polynomial $S(\mathbf{z})$ which correspond to the steps in $\mathcal{S}$ with kth coordinate 1.

3.2 Minimal Points

Roughly speaking, among the critical points, only those which are 'closest' to the origin contribute to the asymptotics, up to an exponentially decaying error. This is analogous to the single variable case, where the singularities of minimum modulus are those which contribute to the dominant asymptotic term. To be precise, for any point $(\mathbf{z}, t) \in \mathbb{C}^{d+1}$ we define the closed polydisk

$$D(\mathbf{z}, t) := \{(\mathbf{w}, t') \in \mathbb{C}^{d+1} : |t'| \leq |t| \text{ and } |w_j| \leq |z_j| \quad \text{for } j = 1, \ldots, d\}.$$

The critical point $(\mathbf{z}, t)$ is called *strictly minimal* if $D(\mathbf{z}, t) \cap \mathcal{V} = \{(\mathbf{z}, t)\}$, and *finitely minimal* if this intersection contains only a finite number of points, all of which are on the boundary of $D(\mathbf{z}, t)$. Finally, we call a critical point *isolated* if there exists a neighbourhood of $\mathbb{C}^{d+1}$ where it is the only critical point. In our case, we need only be concerned with isolated finitely minimal points. In addition as we deal with rational functions all singularities are poles.

Proposition 3.2 *The point* $\boldsymbol{\rho} = (\mathbf{1}, 1/|\mathcal{S}|)$ *is a finitely minimal point of the variety* $\mathcal{V}$. *Furthermore, any point in* $D(\boldsymbol{\rho}) \cap \mathcal{V}$ *is an isolated critical point.*

Proof The point $\boldsymbol{\rho}$ is critical as it lies on $\mathcal{V}$ and its first d coordinates are all 1. Suppose $(\mathbf{w}, t_{\mathbf{w}})$ lies in $\mathcal{V} \cap D(\boldsymbol{\rho})$, where we note that any choice of $\mathbf{w}$ uniquely determines $t_{\mathbf{w}}$ on $\mathcal{V}$. Then, as $t_{\mathbf{w}} \neq 0$,

$$\left| \sum_{(i_1,\ldots,i_d)\in\mathcal{S}} w_1^{i_1+1} \cdots w_d^{i_d+1} \right| = \left| (w_1 \cdots w_d) S(\mathbf{w}) \right| = \left| \frac{1}{t_{\mathbf{w}}} \right| \geq |\mathcal{S}|.$$

But $(\mathbf{w}, t_{\mathbf{w}}) \in D(\boldsymbol{\rho})$ implies $|w_j| \leq 1$ for each $1 \leq j \leq d$. Thus, the above inequality states that the sum of $|\mathcal{S}|$ complex numbers of modulus at most 1 has modulus $|\mathcal{S}|$. The only way this can occur is if each term in the sum has modulus 1, and all terms point in the same direction in the complex plane. By symmetry, and the assumption that we take a positive step in each direction, there are two terms of the form $w_2^{i_2+1} \cdots w_d^{i_d+1}$ and $w_1^2 w_2^{i_2+1} \cdots w_d^{i_d+1}$ in the sum, so that w_1^2 must be 1 in order for them to point in the same direction. This shows $w_1 = \pm 1$, and the same argument applies to each w_k, so there are at most 2^d points in $\mathcal{V} \cap D(\boldsymbol{\rho})$.

By Proposition 3.1 every such point $(\mathbf{w}, t_{\mathbf{w}}) \in \mathcal{V} \cap D(\boldsymbol{\rho})$ is critical, and to show it is isolated it is sufficient to prove $S_1^{(k)}(\mathbf{w}_{\hat{k}}) \neq 0$ for all $1 \leq k \leq d$. Indeed, if $S_1^{(k)}(\mathbf{w}_{\hat{k}}) = 0$ then $\mathbf{w} \in \mathcal{V}$ implies

$$|t_{\mathbf{w}}| = \frac{1}{\left|w_1 \cdots w_d S_0^{(k)}(\mathbf{w}_{\hat{k}})\right|} \geq \frac{1}{\left|S_0^{(k)}(\mathbf{w}_{\hat{k}})\right|} \geq \frac{1}{S_0^{(k)}(\mathbf{1})} > \frac{1}{|\mathcal{S}|},$$

by our assumption that $\mathcal{S}$ contains a step which moves forward in the kth coordinate. This contradicts $(\mathbf{w}, t_{\mathbf{w}}) \in D(\boldsymbol{\rho})$. □

3.3 Asymptotics Results

To apply the formulas of [26] we need to define a few quantities. To start, we note that on all of $\mathcal{V}$ we may parametrize the coordinate t as

$$t(\mathbf{z}) := \frac{1}{z_1 \cdots z_d S(\mathbf{z})}.$$

For each point $(\mathbf{w}, t_{\mathbf{w}}) \in \mathcal{V} \cap D(\boldsymbol{\rho})$ the analysis of [26] shows that the asymptotics of a parametrized integral, which in turn determines the asymptotics of a given walk model depends on the function

$$\begin{aligned}
\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta}) &= \log\left(\frac{t(w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d})}{t_{\mathbf{w}}}\right) + i\sum_{k=1}^{d} \theta_k \\
&= \log\left(\frac{S(\mathbf{w})}{e^{i(\theta_1+\cdots+\theta_d)} S(w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d})}\right) + i(\theta_1 + \cdots + \theta_d) \\
&= \log S(\mathbf{w}) - \log S(w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d}) + K, \qquad (12)
\end{aligned}$$

where K is a constant multiple of $2\pi i$ coming from the branch cut which defines the logarithm; the asymptotic results we derive depend only on the Hessian of $\tilde{f}^{(\mathbf{w})}$, meaning the precise value of this constant does not affect the analysis. Next we let $\mathcal{H}_{\mathbf{w}}$ denote the determinant of the Hessian of $\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta})$ at $\mathbf{0}$:

$$\mathcal{H}_{\mathbf{w}} := \det \tilde{f''}^{(\mathbf{w})}(\mathbf{0}) = \begin{vmatrix} \tilde{f}^{(\mathbf{w})}_{\theta_1\theta_1}(\mathbf{0}) & \tilde{f}^{(\mathbf{w})}_{\theta_1\theta_2}(\mathbf{0}) & \cdots & \tilde{f}^{(\mathbf{w})}_{\theta_1\theta_d}(\mathbf{0}) \\ \tilde{f}^{(\mathbf{w})}_{\theta_2\theta_1}(\mathbf{0}) & \tilde{f}^{(\mathbf{w})}_{\theta_2\theta_2}(\mathbf{0}) & \cdots & \tilde{f}^{(\mathbf{w})}_{\theta_2\theta_d}(\mathbf{0}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}^{(\mathbf{w})}_{\theta_d\theta_1}(\mathbf{0}) & \tilde{f}^{(\mathbf{w})}_{\theta_d\theta_2}(\mathbf{0}) & \cdots & \tilde{f}^{(\mathbf{w})}_{\theta_d\theta_d}(\mathbf{0}) \end{vmatrix}.$$

If $\mathcal{H}_{\mathbf{w}} \neq 0$ then we say $(\mathbf{w}, t_{\mathbf{w}})$ is *non-degenerate*. The main asymptotic result of smooth multivariate analytic combinatorics, in this restricted context, is the following[5].

[5] Theorem 3.5 of [26] allows for asymptotic expansions of coefficient sequences more generally defined from multivariate functions than the diagonal sequence. The formula listed in Theorem 3.3 is stated only for diagonal sequences, as are our definitions of critical and minimal points.

Theorem 3.3 (Adapted from Theorem 3.5 of [26]) *Suppose that the meromorphic function $F(\mathbf{z}, t) = G(\mathbf{z}, t)/H(\mathbf{z}, t)$ non-degenerate, isolated, strictly minimal pole at $(\mathbf{w}, t_{\mathbf{w}})$, and tH_t does not vanish at this point. Then there is an asymptotic expansion*

$$[t^n]\Delta F(\mathbf{z}, t) \sim (w_1 \cdots w_d \cdot t_{\mathbf{w}})^{-n} \sum_{l \geq l_0} C_l n^{-(d+l)/2} \tag{13}$$

for constants C_l, where l_0 is the degree to which G vanishes near $(\mathbf{w}, t_{\mathbf{w}})$. When G does not vanish at $(\mathbf{w}, t_{\mathbf{w}})$ then $l_0 = 0$ and the leading term of this expansion is

$$C_0 = (2\pi)^{-d/2} \mathcal{H}_{\mathbf{w}}^{-1/2} \cdot \frac{G(\mathbf{w}, t_{\mathbf{w}})}{tH_t(\mathbf{w}, t_{\mathbf{w}})}. \tag{14}$$

In fact, Corollary 3.7 of [26] shows that in the case of a finitely minimal point one can simply sum the contributions of each point. Combining this with the above calculations gives our main result.

Theorem 3.4 *Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ define a d-dimensional, nontrivial, highly symmetric lattice path model. Then the number s_n of walks of length n taking steps in $\mathcal{S}$, beginning at the origin, and never leaving the positive orthant has asymptotic expansion*

$$s_n = \left[\left(s^{(1)} \cdots s^{(d)} \right)^{-1/2} \pi^{-d/2} |\mathcal{S}|^{d/2} \right] \cdot n^{-d/2} \cdot |\mathcal{S}|^n + O\left(n^{-(d+1)/2} \cdot |\mathcal{S}|^n \right), \tag{15}$$

where $s^{(k)}$ denotes the number of steps in $\mathcal{S}$ which have kth coordinate 1.

Proof Recall that the point $\boldsymbol{\rho}$ is minimal by Proposition 3.2. We begin by verifying that each point $(\mathbf{w}, t_{\mathbf{w}}) \in \mathcal{V} \cap D(\boldsymbol{\rho})$ satisfies the following conditions, meaning Corollary 3.7 of Pemantle and Wilson [26] can be applied to derive asymptotics:

1. $(\mathbf{w}, t_{\mathbf{w}})$ **is a simple pole**. As $\mathcal{V}$ is smooth, the point $(\mathbf{w}, t_{\mathbf{w}})$ is a simple pole.
2. $(\mathbf{w}, t_{\mathbf{w}})$ **is isolated**. This is proven in Proposition 3.2.
3. tH_t **does not vanish at** $(\mathbf{w}, t_{\mathbf{w}})$. This follows from $t_{\mathbf{w}} H_t(\mathbf{w}, t_{\mathbf{w}}) = 1/(w_1 \cdots w_d) \neq 0$.
4. $(\mathbf{w}, t_{\mathbf{w}})$ **is non-degenerate**. Directly taking partial derivatives in Eq. (12) implies

$$\tilde{f}^{(\mathbf{w})}_{\theta_j \theta_k}(\mathbf{0}) = \begin{cases} w_j w_k \dfrac{S_{y_j y_k}(\mathbf{w}) S(\mathbf{w}) - S_{y_j}(\mathbf{w}) S_{y_k}(\mathbf{w})}{S(\mathbf{w})^2} & : j \neq k \\ \dfrac{S_{y_j y_j}(\mathbf{w}) S(\mathbf{w}) + w_j S_{y_j}(\mathbf{w}) S(\mathbf{w}) - S_{y_j}(\mathbf{w})^2}{S(\mathbf{w})^2} & : j = k \end{cases}.$$

Since $S_{y_j}(\mathbf{y}) = (1 - y_j^{-2}) S_1^{(j)}(\mathbf{y}_{\hat{j}})$ we see that $S_{y_j}(\mathbf{w}) = 0$. Similarly, one can calculate that $S_{y_j y_j}(\mathbf{w}) = 2S_1^{(j)}(\mathbf{w})$ and $S_{y_j y_k}(\mathbf{w}) = 0$ for $j \neq k$, so that the

Hessian of $\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta})$ at $\mathbf{0}$ is a diagonal matrix and

$$\mathcal{H}_{\mathbf{w}} = \frac{2^d}{S(\mathbf{w})^d} S_1^{(1)}(\mathbf{w}) \cdots S_1^{(d)}(\mathbf{w}). \tag{16}$$

The proof of Proposition 3.2 implies that $S_1^{(k)}(\mathbf{w}) \neq 0$ for any $1 \leq k \leq d$, so each $(\mathbf{w}, t_{\mathbf{w}})$ is non-degenerate.

Thus, we can apply Corollary 3.7 of [26] and sum the expansions (13) at each point in $\mathcal{V} \cap D(\boldsymbol{\rho})$ to obtain the asymptotic expansion

$$s_n \sim |\mathcal{S}|^n \sum_{\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})} \left(\sum_{l \geq l_{\mathbf{w}}} C_l^{\mathbf{w}} n^{-(d+l)/2} \right) \tag{17}$$

for constants $C_l^{\mathbf{w}}$, where $l_{\mathbf{w}}$ is the degree to which $G(\mathbf{y}, t)$ vanishes near $(\mathbf{w}, t_{\mathbf{w}})$. Since the numerator $G(\mathbf{y}, t) = (1 + y_1) \cdots (1 + y_d)$ vanishes at all points of $\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})$ except for $\boldsymbol{\rho} = (\mathbf{1}, 1/|\mathcal{S}|)$, the dominant term of (17) is determined only by the contribution of $\mathbf{w} = \boldsymbol{\rho}$. Substituting the value for $\mathcal{H}_{\boldsymbol{\rho}}$ given by Eqs. (16) into (14) gives the desired asymptotic result. □

4 Examples

We now give two examples, both of which determine critical points by directly solving the critical point equations. The first example has only a finite number of critical points, all of which are minimal points. In contrast, the second example contains a *curve* of critical points (however, as guaranteed by Proposition 3.2, no point on this curve is a minimal point).

Example 4.1 Consider the three dimensional highly symmetric model defined by the step set

$$\mathcal{S} = \{(-1, 0, \pm 1), (1, 0, \pm 1), (0, 1, \pm 1), (0, -1, \pm 1)\},$$

with characteristic polynomial

$$S(x, y, z) = (x + y + \overline{x} + \overline{y})(z + \overline{z}).$$

The kernel equation for the complete generating function is

$$\begin{aligned} xyz(1-tS(x, y, z))F(x, y, z, t) = xyz &- ty(z^2+1)F(0, y, z) - tx(z^2+1)F(x, 0, z) \\ &- t(x^2y + y^2x + y + x)F(x, y, 0) \\ &+ txF(x, 0, 0) + tyF(0, y, 0). \end{aligned}$$

By Lemma 2.4 we deduce $F(1,1,1,t) = \Delta B(x,y,z,t)$ where

$$B(x,y,z,t) = \frac{(\overline{x}-x)(\overline{y}-y)(\overline{z}-z)}{\overline{x}\,\overline{y}\,\overline{z}(1-txyzP(\overline{x},\overline{y},\overline{z}))} \cdot \frac{1}{(1-x)(1-y)(1-z)}$$
$$= \frac{(1+x)(1+y)(1+z)}{1-t(z^2+1)(x+y)(xy+1)}.$$

Next, we verify that the denominator $H(x,z,y,t)$ of $B(x,y,z,t)$ is smooth—i.e., that H and its partial derivatives don't vanish together at any point. This can be checked automatically by computing a Gröbner Basis of the ideal generated by H and its partial derivatives.

In pseudo-code:[6]

$> H := 1 - t(z^2+1)(x+y)(xy+1):$

$> \texttt{GroebnerBasis}([H, H_x, H_y, H_z, H_t], \texttt{plex}(t,x,y,z));$

$$[1]$$

The critical points can be computed:

$> \texttt{GroebnerBasis}([H, tH_t - xH_x, tH_t - yH_y, tH_t - zH_z], \texttt{plex}(t,x,y,z));$

$$[z^2-1, y^2-1, x-y, 8t-y]$$

This implies that there is a finitely minimal critical point $\boldsymbol{\rho} = (1,1,1,1/8)$, where

$$T(\boldsymbol{\rho}) \cap \mathcal{V} = \{(1,1,1,1/8), (1,1,-1,1/8), (-1,-1,1,-1/8), (-1,-1,-1,-1/8)\}.$$

The value of $\mathcal{H}_{\mathbf{w}}$ can be calculated at each point to be 1/4. For instance:

$> f := \log S(\mathbf{1}) - \log S\left(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\right):$

$> \texttt{subs}\,(\theta_1 = 0, \theta_2 = 0, \theta_3 = 0, \texttt{det(Hessian}\,(f, [\theta_1, \theta_2, \theta_3])));$

$$1/4$$

Equations (13) and (14) then give the asymptotic result

$$s_n \sim 4\sqrt{2} \cdot \pi^{-3/2} \cdot n^{-3/2} \cdot 8^n.$$

◁

Example 4.2 Consider the model in three dimensions restricted to the positive octant taking the twelve steps

$$\mathcal{S} = \{(-1,0,\pm 1), (1,0,\pm 1), (0,1,\pm 1), (0,-1,\pm 1), (\pm 1,1,0), (\pm 1,-1,0)\}.$$

[6] The input is formatted for Maple version 18.

Now, by our previous analysis, $F(1, 1, 1, t) = \Delta B(x, y, z, t)$ where

$$B(x, y, z, t) = \frac{(1+x)(1+y)(1+z)}{1 - t(z^2+1)(x+y)(xy+1) - tz(y^2+1)(x^2+1)}. \tag{18}$$

The denominator $H(x, z, y, t)$ of $B(x, y, z, t)$ can again be verified to be smooth, but the ideal encoding the critical point equations is no longer zero dimensional; i.e., there are an infinite number of critical points. For instance, the following calculation shows that any point $(1, -1, z, 1/4z)$ with $z \neq 0$ is a non-isolated critical point:

$> H := 1 - t(z^2+1)(x+y)(xy+1) - tz(y^2+1)(x^2+1):$

$> I := \mathtt{subs}\left(x = 1, y = -1, [H, tH_t - xH_x, tH_t - yH_y, tH_t - zH_z]\right):$

$> \mathtt{GroebnerBasis}(I, \mathtt{plex}(t, x, y, z));$

$$[1 - 4tz]$$

Note that none of these points are minimal—so Proposition 3.2 is not contradicted—since

$$|(1)\cdot(-1)\cdot(z)\cdot(1/4z)| = 1/4 > \frac{1}{|\mathcal{S}|}.$$

In fact, here we have the finitely minimal point $\boldsymbol{\rho} = (1, 1, 1, 1/12)$ with

$$T(\boldsymbol{\rho}) \cap \mathcal{V} = \{(1, 1, 1, 1/12), (-1, -1, -1, -1/12)\}.$$

As above, the value of $\mathcal{H}_{\mathbf{w}}$ can be calculated at both points $\mathbf{w} \in T(\boldsymbol{\rho}) \cap \mathcal{V}$ and Eqs. (13) and (14) can be applied to give the asymptotic result

$$s_n \sim 3\sqrt{3}\cdot \pi^{-3/2}\cdot n^{-3/2}\cdot 12^n,$$

which matches Theorem 3.4. $\triangleleft$

5 Higher Order Terms

Building upon the work of Pemantle and Wilson, Raichev and Wilson [29] refined the asymptotics of Eq. (13) and found expressions for the higher order constants $C_1, C_2, \ldots$, theoretically allowing one to calculate the contribution of each minimal point $\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})$. To be explicit, Theorem 3.8 of [29] gives the asymptotic contribution of the minimal point $\mathbf{w}$ as

$$\begin{aligned} s_n^{(\mathbf{w})} = |\mathcal{S}|^n \cdot \left[2^{-d}\pi^{-d/2} S(\mathbf{w})^{d/2} \cdot \left(S_1^{(1)}(\mathbf{w}) \cdots S_1^{(d)}(\mathbf{w})\right)^{-1/2}\right] \cdot n^{-d/2} \\ \cdot \sum_{k=0}^{N-1} n^{-k} L_k(\tilde{u}^{(\mathbf{w})}, \tilde{f}^{(\mathbf{w})}) + O\left(|\mathcal{S}|^n \cdot n^{-(d-1)/2-N}\right). \end{aligned} \tag{19}$$

With $\star$ denoting the Hadamard product

$$(a_1, \dots, a_d) \star (b_1, \dots, b_d) = (a_1 b_1, \dots, a_d b_d),$$

we have

$$\begin{aligned}
\tilde{u}^{(\mathbf{w})}(\boldsymbol{\theta}) &:= -\frac{1}{t_{\mathbf{w}}} \cdot \frac{G(\mathbf{w} \star e^{i\boldsymbol{\theta}}, t_{\mathbf{w}})}{H_t(\mathbf{w} \star e^{i\boldsymbol{\theta}}, t_{\mathbf{w}})} \\
g_{\mathbf{w}}(\boldsymbol{\theta}) &:= \log S(\mathbf{w}) - \log S(\mathbf{w} \star e^{i\boldsymbol{\theta}}) - \frac{1}{2}\boldsymbol{\theta} \cdot \tilde{f''}^{(\mathbf{w})}(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}^T \\
L_k(\tilde{u}^{(\mathbf{w})}, \tilde{f}^{(\mathbf{w})}) &:= \sum_{r=0}^{2k} \frac{\mathcal{D}^{r+k}\left(\tilde{u}^{(\mathbf{w})} \cdot g_{\mathbf{w}}^r\right)(\mathbf{0})}{(-1)^k 2^{r+k} r!(r+k)!},
\end{aligned}$$

and $\mathcal{D}$ is the differential operator

$$\mathcal{D} = -\sum_{0 \le r,s \le d} \left(\operatorname{Inv} \tilde{f''}^{(\mathbf{w})}\right)_{r,s} \partial_{\theta_r} \partial_{\theta_s} = -\frac{S(\mathbf{w})}{2} \sum_{r=0}^{d} \frac{1}{S_1^{(1)}(\mathbf{w})} \partial_{\theta_r}^2.$$

This expression is quite involved, making it hard to derive a *general* asymptotic theorem with higher order terms. However, it is explicit and effective for any given step set. The principal difficulty in the smooth case is the identification of points which actually contribute to the asymptotic growth. In the case of highly symmetric models this is accomplished through the characterization of minimal points given in Proposition 3.2.

Example 5.1 Consider the two dimensional model with step set $\{N, S, NE, SE, NW, SW\}$ = . A direct application of Theorem 3.4 implies

$$s_n \sim \frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}.$$

By Proposition 3.2, to find the minimal points we simply solve the equation

$$H(x, y, t) = 1 - t(1 + y^2 + x + xy^2 + x^2 + x^2y^2) = 0,$$

in t for all $(x, y) \in \{\pm 1\}^2$ and check whether the corresponding solution $t_{x,y}$ satisfies $|t_{x,y}| = 1/|\mathcal{S}| = 1/6$. Of the four possible points, we get only two minimal points: the expected point $\boldsymbol{\rho} = (1, 1, 1/6)$ along with the point $\boldsymbol{\sigma} = (1, -1, 1/6)$.

Computing the terms in expansion (19) at these two minimal points—aided by the Sage implementation of [28]—gives the asymptotic contributions:

$$\begin{aligned}
s_n^{(\boldsymbol{\rho})} &= 6^n \left(\frac{\sqrt{6}}{\pi n} - \frac{17\sqrt{6}}{16\pi n^2} + \frac{605\sqrt{6}}{512\pi n^3} + O(1/n^4) \right), \text{ and} \\
s_n^{(\boldsymbol{\sigma})} &= (-6)^n \left(\frac{\sqrt{6}}{4\pi n^2} - \frac{33\sqrt{6}}{64\pi n^3} + O(1/n^4) \right).
\end{aligned}$$

Thus, the counting sequence for the number of walks of length n has asymptotic expansion

$$s_n = 6^n\left(\frac{\sqrt{6}}{\pi n} - \frac{\sqrt{6}(17-4(-1)^n)}{16\pi n^2} + \frac{\sqrt{6}(38720-16896(-1)^n)}{32768\pi n^3} + O(1/n^4)\right).$$

We verified this expression to the second order term through numerical computations (due to the fast growth of the sequence, extremely high precision arithmetic would be needed to verify further terms in this expansion). ◁

6 From Diagonals to Differential Equations

We have remarked in Corollary 2.5 that for any highly symmetric model $\mathcal{S}$ the corresponding generating function $F(\mathbf{1}, t)$ is D-finite. Indeed, from the expression $F(\mathbf{1}, t) = \Delta G(\mathbf{z}, t)/H(\mathbf{z}, t)$ it is possible in principle to compute an annihilating linear differential equation of $F(\mathbf{1}, t)$ through the use of algorithms for creative telescoping. These algorithms are typically grouped into those that perform elimination in an Ore algebra (such as Zeilberger [33]), and those which use an ansatz of undetermined coefficients. The main strategy is to compute differential operators annihilating multivariate integrals, and connect them to diagonals of rational functions via the following relations:

$$\frac{1}{2\pi i}\int_{\Omega}\frac{B(z_1, z_2/z_1, z_3, \ldots, z_d, t)}{z_2}dz_2 = \Delta_{1,2}B(\mathbf{z}, t) \tag{20}$$

$$\left(\frac{1}{2\pi i}\right)^d \int_T \frac{B(z_1, z_2/z_1, z_3/z_2, \ldots, z_d/z_{d-1}, t/z_d)}{z_1 z_2 \cdots z_d}d\mathbf{z} = \Delta B(\mathbf{z}, t). \tag{21}$$

Here Ω is an appropriate contour in $\mathbb{C}$ containing the origin, T is an appropriate torus in $\mathbb{C}^d$ containing the containing the origin, the function

$$B(\mathbf{z}, t) = \sum_{n\geq 0}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} b_{\mathbf{i}}(n) z_1^{i_1}\cdots z_d^{i_d}\right)t^n$$

is analytic in a neighbourhood of the origin, and $\Delta_{1,2}$ denotes the partial diagonal operator

$$\Delta_{1,2}B(\mathbf{z}, t) = \sum_{n\geq 0}\left(\sum_{\substack{\mathbf{i}\in\mathbb{Z}^d\\ i_1=i_2}} b_{\mathbf{i}}(n) z_2^{i_2}\cdots z_d^{i_d}\right)t^n.$$

The reader is directed to [6,23] for details on how these methods work and are implemented in modern computer algebra systems. In Table 2 we have computed annihilators for the four highly symmetric models in two dimensions using an ansatz method

Table 2 Annihilating differential equations for the highly symmetric quarter plane models

S	Annihilating DE
	$t^2(4t-1)(4t+1)D_t^3+2t(4t+1)(16t-3)D_t^2$ $+\left(-6+28t+224t^2\right)D_t+(12+64t)$
	$t^2(4t+1)(4t-1)^2D_t^3+t(4t-1)(112t^2-5)D_t^2$ $+4(8t-1)(20t^2-3t-1)D_t+\left(-4-48t+128t^2\right)$
	$t^2(6t-1)(6t+1)(2t+1)(2t-1)(12t^2-1)D_t^3$ $+t(2t-1)(6048t^5+2736t^4-672t^3-336t^2+6t+5)D_t^2$ $+\left(-4+16t+516t^2+96t^3-5520t^4-2304t^5+17280t^6\right)D_t$ $+\left(8+132t+96t^2-1104t^3-1152t^4+3456t^5\right)$
	$-t^2(4t+1)(8t-1)(2t-1)(t+1)D_t^3$ $+t\left(-5+33t+252t^2-200t^3-576t^4\right)D_t^2$ $+\left(-4+48t+468t^2-88t^3-1152t^4\right)D_t$ $+\left(12+144t+72t^2-384t^3\right)$

developed and implemented in Mathematica by Koutschan [23]. Table 3 displays the annihilating differential equations for the simple model in dimensions 3 and 4.

Given an annihilating linear differential operator of the univariate generating function $F(\mathbf{1},t)$, one can easily compute a linear recurrence relation that the counting sequence (s_n) must satisfy. The Birkhoff-Trjitzinsky method (see [15,31]) can then be used to determine a basis of solutions to this recurrence. At a regular singular point, each element of the basis has dominant asymptotic growth of the form

$$s_n^{(k)} \sim C_k \rho^n n^{\beta_k} (\log n)^{l_k},$$

for computable constants C_k, ρ, β_k, l_k. This technique has already been used in the asymptotic analysis of lattice walks in restricted regions—for instance in the work of Bostan and Kauers [4] on two dimensional lattice walks confined to the positive quadrant—however it is not apparent how the number of walks in a model, s_n, is represented as a linear combination of the basis elements $s_n^{(k)}$. Determining this linear combination is known in the literature as the *connection problem*, as it describes how the generating function is connected to a local basis of singular solutions. This highlights a severe drawback to using the differential equation for asymptotics, when compared to the other methods described above: there is no known effective procedure to solve the connection problem in general, even when the coefficients of the differential equation are rational functions (see the discussion below Theorem VII.10 on page 521 of [15] for more details). In essence, this implies that the multiplicative growth con-

Table 3 Annihilating differential equations for the models $\{e_1, -e_1, \dots, e_d, -e_d\}$

Dimension d	Annihilating DE
3	$-t^3(2t-1)(2t+1)(6t-1)(6t+1)D_t^4$
	$-4t^2\left(576t^4+36t^3-140t^2-5t+3\right)D_t^3$
	$-4t\left(2592t^4+324t^3-531t^2-40t+9\right)D_t^2$
	$-8\left(1728t^4+324t^3-282t^2-34t+3\right)D_t$
	$-24\left(144t^3+36t^2-17t-3\right)$
4	$-t^4(4t-1)(4t+1)(8t-1)(8t+1)D_t^5$
	$-4t^3(4t+1)\left(1536t^3-320t^2-30t+5\right)D_t^4$
	$-4t^2\left(47104t^4+3968t^3-2976t^2-145t+30\right)D_t^3$
	$-12t\left(45056t^4+5760t^3-2368t^2-191t+20\right)D_t^2$
	$-24\left(21504t^4+3712t^3-848t^2-106t+5\right)D_t$
	$-96\left(1024t^3+224t^2-24t-5\right)$

stant of the dominant asymptotic term cannot, in general, be rigorously determined through this approach (Bostan and Kauers used numerical approximations to non-rigorously solve the connection problem for their work on two dimensional models).

Example 6.1 As seen in Table 2, the generating function of the quarter-plane model  is annihilated by the differential operator

$$\begin{aligned}\mathcal{L} = &\left(-t^2+52t^4-624t^6+1728t^8\right)D_t^3\\ &+\left(-5t+4t^2+348t^3-4080t^5-576t^6+12096t^7\right)D_t^2\\ &+\left(-4+16t+516t^2+96t^3-5520t^4-2304t^5+17280t^6\right)D_t\\ &+\left(8+132t+96t^2-1104t^3-1152t^4+3456t^5\right),\end{aligned}$$

which implies that the sequence (s_n) satisfies the following linear recurrence relation with polynomial coefficients

$$\begin{aligned}0 = &\left(-n^3-20n^2-133n-294\right)s_{n+6}\\ &+\left(4n^2+52n+168\right)s_{n+5}+\left(52n^3+816n^2+4304n+7620\right)s_{n+4}\\ &+(96n+384)\,s_{n+3}+\left(-624n^3-5952n^2-19008n-20304\right)s_{n+2}\\ &+\left(-576n^2-2880n-3456\right)s_{n+1}+\left(1728n^3+6912n^2+8640n+3456\right)s_n.\end{aligned}$$

Using the Birkhoff–Trjitzinsky method one computes a basis of local solutions at infinity to this degree six linear recurrence relation (the basis given here was computed using the Sage package of [22]):

$$
\begin{aligned}
s_n^{(1)} &= \frac{6^n}{n}\left(1 - \frac{17}{16}n^{-1} + \frac{605}{512}n^{-2} + O\left(n^{-3}\right)\right)\\
s_n^{(2)} &= \frac{6^n}{n^2}\left(1 - \frac{33}{16}n^{-1} + \frac{1565}{512}n^{-2} + O\left(n^{-3}\right)\right)\\
s_n^{(3)} &= \frac{(2\sqrt{3})^n}{n^4}\left(1 - \frac{14+3\sqrt{3}}{2}n^{-1} + O\left(n^{-2}\right)\right)\\
s_n^{(4)} &= \frac{(-2\sqrt{3})^n}{n^4}\left(1 - \frac{14-3\sqrt{3}}{2}n^{-1} + O\left(n^{-2}\right)\right)\\
s_n^{(5)} &= \frac{2^n}{n^3}\left(1 - \frac{51}{16}n^{-1} + \frac{3341}{512}n^{-2} + O\left(n^{-3}\right)\right)\\
s_n^{(6)} &= \frac{(-2)^n}{n^2}\left(1 - \frac{35}{16}n^{-1} + \frac{1805}{512}n^{-2} + O\left(n^{-3}\right)\right),
\end{aligned}
$$

so that $s_n = O(6^n/n)$. Note that the results of Example 5.1 imply

$$
s_n = \frac{\sqrt{6}}{\pi}s_n^{(1)} + \frac{\sqrt{6}}{4\pi}s_n^{(2)} + O\left((2\sqrt{3})^n\right),
$$

and we can partially resolve the connection problem, however this is only possible because leading term asymptotics for s_n were already calculated through the techniques of Pemantle, Raichev, and Wilson. $\lhd$

Although differential operators are very useful data structures for the D-finite functions which they annihilate, the discussion above illustrates that the representation of $F(\mathbf{1}, t)$ as a rational diagonal can yield easier access to its asymptotic information when coupled with the results of analytic combinatorics in several variables (at least in the smooth case). Furthermore, the combinatorial properties of lattice path models often naturally give representations of their generating functions as rational diagonals, and determining annihilating differential operators for these diagonals can be difficult. Creative telescoping methods—although always improving (see, for example, [6])—do not scale well with degree and must be calculated on a model by model basis.

7 Walks in Weyl Chambers

The highly symmetric models fit into the extension of the reflection principle to Weyl chambers initiated by Gessel and Zeilberger [17]. Recasting these lattice path models into their context has three main benefits: it provides insight into the kernel method computations (specifically, how to find the group); it allows us to determine diagonal expressions for the generating function of excursions; and finally, it provides a potential course of action for approaching other types of models.

7.1 Weyl Chambers and Reflectable Walks

The following definitions are taken from Gessel and Zeilberger [17], Grabiner and Magyar [19], and Humphreys [20], and the reader is directed to these manuscripts for more details.

A (*reduced*) *root system* is a finite set of vectors $\Phi \subset \mathbb{R}^n$ such that

- for any $x, y \in \Phi$, the set Φ contains the reflection of y through the hyperplane with normal x

$$\sigma_x(y) = y - 2\frac{(x, y)}{(x, x)}x;$$

- for any $x, y \in \Phi$, $x - \sigma_y(x)$ is an integer multiple of y;
- the only scalar multiples of $x \in \Phi$ to be in Φ are x and $-x$.

The set of linear transformations generated by the reflections σ_x is always a finite Coxeter group, and is called the *Weyl group* W of the root system. The complement of the union of the hyperplanes whose normals are the root system is an open set, and a connected component of this open set is called a *Weyl chamber*. For the root system Φ, a set of *positive roots* Φ^+ is a subset of Φ such that

(1) for each $x \in \Phi$ exactly one of x and $-x$ is in Φ^+;
(2) for any two distinct $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$.

An element of Φ^+ is called a *simple root* if it cannot be written as the sum of two elements of Φ^+, and a maximal set Δ of simple roots is called a *basis* for the root system. It can be shown that for a basis Δ any $x \in \Phi$ is a linear combination of members of Δ with all non-negative or non-positive coefficients, and that the set $\{\sigma_x : x \in \Delta\}$ generates the Weyl group W.

Fix a root system Φ and a basis Δ, and let

- $\mathcal{S} \subset \mathbb{Z}^n$ be a set of steps such that $W \cdot \mathcal{S} = \mathcal{S}$—i.e., $\mathcal{S}$ is preserved under each element of the Weyl group;
- L be a lattice, restricted to the linear span of elements of $\mathcal{S}$, such that $W \cdot L = L$;
- C be the Weyl chamber

$$C = \{\mathbf{z} \in \mathbb{R}^n : (\alpha, \mathbf{z}) > 0 \quad \text{for all } \alpha \in \Delta\}.$$

The lattice path model in the Weyl chamber C using the steps $\mathcal{S}$ beginning at a point $\mathbf{a} \in C$ is the combinatorial class of all sequences of steps in $\mathcal{S}$ beginning at $\mathbf{a}$ and never leaving C (when viewed as a walk on L in the typical manner). If, in addition to the requirements above, the two conditions

(1) For all $\alpha \in \Delta$ and $s \in \mathcal{S}$, $(\alpha, s) = \pm k(\alpha)$ or 0, where $k(\alpha)$ is a constant depending only on α;
(2) For all $\alpha \in \Delta$ and $\lambda \in L$, (α, λ) is an integer multiple of $k(\alpha)$ depending only on α;

are met, we say that the lattice path model is *reflectable*, and any step $s \in S$ taken from any lattice point inside C will not leave C except possibly to land on its boundary (one of the hyperplanes whose normals are the elements of Φ).

We restate the main result of Gessel and Zeilberger [17], modified by converting a constant term extraction to a diagonal extraction.

Theorem 7.1 (Gessel and Zeilberger [17]) *Given a reflectable walk as defined above such that* $(\mathbf{a}, \alpha)$ *is an integer multiple of* $k(\alpha)$ *for each* $\alpha \in \Delta$*, and an element* $\mathbf{b} \in C$ *such that* $(\mathbf{b}, \alpha)$ *is also an integer multiple of* $k(\alpha)$ *for each* $\alpha \in \Delta$*, the generating function for the number of walks which begin at* $\mathbf{a}$*, end at* $\mathbf{b}$*, and stay in* C *is*

$$F_{\mathbf{a}\to\mathbf{b}}(t) = \Delta\left[\frac{1}{1 - t(z_1\cdots z_d)S(\mathbf{z})}\cdot \mathbf{z}^{-\mathbf{b}} \cdot \sum_{w\in W}(-1)^{l(w)}\mathbf{z}^{w(\mathbf{a})}\right], \tag{22}$$

where $l(w)$ *is the minimal length of* w *represented as a product of elements in* $\{\sigma_x : x \in \Delta\}$.

If $(\mathbf{b}, \alpha)$ is an integer multiple of $k(\alpha)$ for each $\alpha \in \Delta$ and $\mathbf{b} \in C$, and the formal power series $\sum_{\mathbf{b}\in C}\mathbf{z}^{-\mathbf{b}}$ exists (see [2] for a discussion on the existence of multivariate Laurent series), then summing Eq. (22) over all possible endpoints implies that the generating function for the number of walks beginning at $\mathbf{a}$ and staying in C (which are allowed to end anywhere) is

$$F_a(t) = \Delta\left[\frac{1}{1 - t(z_1\cdots z_d)S(\mathbf{z})}\cdot \sum_{\mathbf{b}\in C}\mathbf{z}^{-\mathbf{b}} \cdot \sum_{w\in W}(-1)^{l(w)}\mathbf{z}^{w(\mathbf{a})}\right]. \tag{23}$$

7.2 Classification of Weyl Chambers and Reflectable Walks

Given two root systems $\Phi_1 \subset \mathbb{R}^n$ and $\Phi_2 \subset \mathbb{R}^m$, one can create a new root system $\Phi_1 \times \Phi_2$ by treating the two vector spaces spanned by the elements of Φ_1 and Φ_2 as mutually orthogonal subspaces of $\mathbb{R}^{n+m}$. To this end, a root system Φ is called *reducible* if it can be decomposed as $\Phi = \Phi_1 \cup \Phi_2$, where Φ_1 and Φ_2 are root systems whose elements are pairwise orthogonal, and *irreducible* otherwise.

One of the main results in the study of root systems—which arises in relation to Lie algebras and representation theory—is a complete classification of the irreducible root systems, consisting of four infinite families (A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, and D_n for $n \geq 4$) and five exceptional cases (E_6, E_7, E_8, F_4, and G_2). The interested reader is directed to Section 11.4 of Humphreys [20] for more details and a proof of the classification.

Example 7.2 There is, up to scaling by a constant, one root system in $\mathbb{R}$: the system $\Phi_1 = \{\pm 1\}$ with basis $\Delta_1 = \{1\}$, which is called A_1. From this, the root system $A_1 \times A_1 = A_1^2 \subset \mathbb{R}^2$ is defined as the direct sum of two copies of A_1, giving elements $\Phi_2 = \{\pm e_1, \pm e_2\}$ and basis $\Delta_2 = \{e_1, e_2\}$. In general, for any $d \in \mathbb{N}$ the root system A_1^d will be the system with elements $\Phi = \{\pm e_1, \ldots, \pm e_d\}$, which admits the basis $\Delta = \{e_1, \ldots, e_d\}$. ◁

7.3 Highly Symmetric Models are Walks in Weyl Chambers

The root system A_1^d, described in Example 7.2, has corresponding Weyl chamber

$$C = \{\mathbf{z} : z_1 > 0 \text{ and } z_2 > 0 \text{ and } \cdots \text{ and } z_d > 0\} = (\mathbb{Z}_{>0})^d,$$

and it follows directly from the definitions above that a step set $\mathcal{S} \subset \mathbb{Z}^d$ is a reflectable walk with respect to Δ if and only if it is highly symmetric. As C does not include the hyperplanes $\{z_1 = 0\}, \cdots, \{z_d = 0\}$, we shift the origin of the walks under consideration by starting them at the point $\mathbf{a} = \mathbf{1}$. The Weyl group W corresponding to this set of roots is isomorphic to $\mathbb{Z}_2^d$ (in fact, it is equal to the group $\mathcal{G}$ as defined in Sect. 2.2) and

$$\sum_{\mathbf{b}\in C} \mathbf{z}^{-\mathbf{b}} = \frac{1}{z_1 - 1} \cdots \frac{1}{z_d - 1}$$

$$\sum_{w\in W} (-1)^{l(w)} \mathbf{z}^{w(\mathbf{a})} = (z_1 - \overline{z}_1) \cdots (z_d - \overline{z}_d).$$

Substitution into Eq. 23 recovers Eq. (9), shifted by a factor of $(z_1 \cdots z_d)$ to account for the shifted walk origin $\mathbf{a}$:

$$F_{\mathbf{1}}(t) = \Delta \left[\frac{(1 + z_1) \cdots (1 + z_d)}{1 - t(z_1 \cdots z_d)S(\mathbf{z})} \cdot (z_1 \cdots z_d) \right].$$

We note that the argument presented in Sect. 2 mirrors the proof of Theorem 7.1 given by Gessel and Zeilberger.

7.4 Excursions

One advantage of this setup is that it is straightforward to modify several combinatorial parameters. For example, there is straightforward access to the generating function for walks that return to the origin. Specifically, if we set $\mathbf{a} = \mathbf{b} = \mathbf{1}$ in Eq. (22), we deduce that the number of excursions e_n is given by

$$\begin{aligned} e_n &= [t^n]\Delta \left(\frac{(z_1 - \overline{z}_1) \cdots (z_d - \overline{z}_d)}{1 - t(z_1 \cdots z_d)S(\mathbf{z})} \cdot (z_1 \cdots z_d)^{-1} \right) \\ &= [t^n]\Delta \left(\frac{t^2(z_1^2 - 1) \cdots (z_d^2 - 1)}{1 - t(z_1 \cdots z_d)S(\mathbf{z})} \cdot (t z_1 \cdots z_d)^{-2} \right) \\ &= [t^{n+2}]\Delta \left(\frac{t^2(z_1^2 - 1) \cdots (z_d^2 - 1)}{1 - t(z_1 \cdots z_d)S(\mathbf{z})} \right). \end{aligned}$$

Note that the form of the final rational function on the right hand side implies that the same minimal points will appear in the analysis of excursion asympotics—however, due to the factors of $(z_1 - 1) \cdots (z_d - 1)$ now present in the numerator the finitely minimal point $\rho = (1, \ldots, 1, 1/|S|)$ will vanish, bringing down the polynomial growth factor of excursions compared to the asymptotics of walks ending anywhere. Furthermore, as more than one minimal point can now determine the dominant asymptotics, closed form results are not easily obtainable. Despite that, as the minimal points are still classified by Proposition 3.2, one can use the machinery available to calculate higher term asymptotic expansions (as in Sect. 5) to determine the dominant asymptotics of specific models.

Example 7.3 Consider the highly symmetric 2D step set $\{N, S, NE, SE, NW, SW\}$ = . Here we have

$$e_n = [t^{n+2}]\Delta\left(\frac{t^2(x^2-1)(y^2-1)}{1-(tx^2y^2+ty^2+tx^2+t+txy^2+tx)}\right),$$

and as discussed in Example 5.1 this rational function has the expected minimal point $\boldsymbol{\rho} = (1, 1, 1/6)$ along with the point $\boldsymbol{\sigma} = (1, -1, 1/6)$. Computing the terms in expansion (19) at these two minimal points—again aided by the Sage implementation of [28]—gives the asymptotic contributions (after properly shifting index):

$$e_n^{(\boldsymbol{\rho})} = 6^n\left(\frac{3\sqrt{6}}{2\pi n^3} + O(1/n^4)\right) \qquad e_n^{(\boldsymbol{\sigma})} = (-6)^n\left(\frac{3\sqrt{6}}{2\pi n^3} + O(1/n^4)\right).$$

Thus, the counting sequence for the number of excursions of length n has the asymptotic expansion

$$e_n = 6^n\left(\frac{3\sqrt{6}}{2\pi n^3}(1+(-1)^n) + O(1/n^4)\right),$$

where we note that there are no excursions of odd length. ◁

As the denominator of the rational function under consideration is smooth, and the numerator $t^2(z_1^2 - 1)\cdots(z_d^2 - 1)$ vanishes at any minimal point to order d, the asymptotic expansion given in Eq. (19) implies the following.

Theorem 7.4 *Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ define a d-dimensional, nontrivial highly symmetric lattice path model. The number of excursions e_n of length n taking steps in $\mathcal{S}$, beginning and ending at the origin, and never leaving the positive orthant satisfies*

$$e_n = O\left(\frac{|\mathcal{S}|^n}{n^{3d/2}}\right).$$

8 Conclusion

The purpose of this article, aside from the specific combinatorial results it contains, is to reinforce the notion that there are many possibilities for studying lattice walks in restricted regions through the use of diagonals and analytic combinatorics in several variables: in this context the diagonal data structure often permits a very general analysis. Furthermore, walks with symmetry across each axis all have a smooth singular variety, making them the perfect entry point to this confluence of the kernel method, the reflection principle and analytic combinatorics in several variables.

8.1 Generalizations: Other Weyl Chambers

A major goal moving forward is to deal with more general step set models. As a first attempt, we have considered reflectable walks in the Weyl chamber A_2, obtaining compact diagonal expressions for the generating functions of models with group order 6 in the classification of [10]. The expressions are more difficult to analyze with these asymptotic techniques, as the rational functions in the diagonal representations no longer have smooth singular varieties. This work is ongoing.

More generally, Grabiner and Magyar [19] have classified, for every irreducible root system Φ, the step sets which give rise to reflectable lattice path models in each corresponding Weyl chamber. This combinatorial classification gives a large collection of future objects to study through the means of analytic combinatorics in several variables. Assuming one can get the generating function for the number of walks in a more general setting as a rational diagonal, many asymptotic results can be reduced to an analysis of this rational function. Both [27,30] give results for singular varieties which are non-smooth, but whose critical points are *multiple points*. Due to the constraints on the rational functions arising from the combinatorial nature of lattice paths in restricted regions, there is hope for a completely systematic treatment.

One natural starting point is the following question: can the infamous Gessel model in the first quadrant be expressed as a walk in a Weyl Chamber? Understanding why (or why not) such a representation exists may give insight into why this D-finite model has resisted a more straightforward orbit sum analysis [5], although there has been some important recent progress [9].

Furthermore, the asymptotic enumeration of excursions has received much attention lately, due to the recent work of Denisov and Wachtel [13]. It could be interesting to link their probabilistic work to the enumerative expressions using diagonals available for many D-finite models. The results of [7] suggest very compelling evidence that differences between D-finite and non-D-finite models are detectable through asymptotic enumeration.

8.2 Are All D-Finite Models Diagonals of Rational Functions?

Across the study of lattice path models to date, it has been true that every model with a D-finite generating function is accompanied by an expression for the generating function as the diagonal of a rational function (or equivalent). A conjecture of Christol [12] posits that any globally bounded D-finite function (which includes power series convergent at the origin with integer coefficients) can be written as the diagonal of a multivariate rational function. Could one prove a lattice path version of this conjecture?

More practically, could such a result be made effective with an automatic method of writing known D-finite functions as diagonals?

Finally, it would be interesting to understand if there is a direct combinatorial interpretation for the diagonal operator acting on rational functions. Recent work of Garrabrant and Pak [16] gives a tiling interpretation for diagonals of $\mathbb{N}$-rational functions. The rational functions encountered in our analysis are very combinatorial, although they have some negative coefficients off the diagonal. Very possibly a signed version of their construction might capture the diagonals that we build; i.e., one might hope to find a combinatorial interpretation (possibly in terms of tilings) for the multi-dimensional sequence obtained by taking the absolute value of each coefficient in our power series expansions, where the sign of each power series coefficient corresponds to some combinatorial property of the objects in this interpretation.

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