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Bijections for Weyl Chamber walks ending on an axis, using arc diagrams and Schnyder woods

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ABSTRACT

In the study of lattice walks there are several examples of enumerative equivalences which amount to a trade-off between domain and endpoint constraints. We present a family of such bijections for simple walks in Weyl chambers which use arc diagrams in a natural way. One consequence is a set of new bijections for standard Young tableaux of bounded height. A modification of the argument in two dimensions yields a bijection between Baxter permutations and walks ending on an axis, answering a recent question of Burrill et al. (2016). Some of our arguments (and related results) are proved using Schnyder woods. Our strategy for simple walks extends to any dimension and yields a new bijective connection between standard Young tableaux of height at most $2k$ and certain walks with prescribed endpoints in the k -dimensional Weyl chamber of type D.

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1. Introduction

In the context of directed 2D lattice paths with unit steps, there is a classic bijection between *meanders* and *bridges* of equal length. This maps lattice walks with steps $(1, 1)$ and $(1, -1)$ starting at the origin, staying above the x -axis (meanders) to those ending at height zero (bridges)—see Fig. 1. This example illustrates a common trade-off in lattice walks between domain constraints and endpoint constraints [15,6]. Note that the natural bijection shown in Fig. 1 proceeds via an intermediate class

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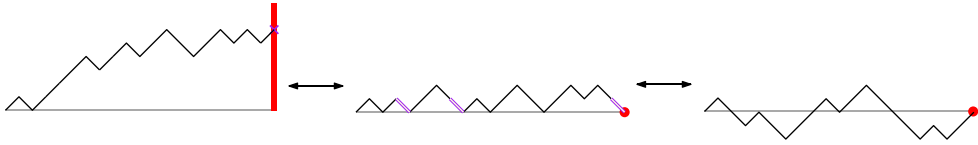


Fig. 1. An example of the classical bijection between meanders and bridges. From a Dyck walk D with some marked steps d_1, \dots, d_k reaching the x -axis, one gets (bijectively) a meander by turning every marked step into an up-step. To get (bijectively) a bridge from D , for $1 \leq i \leq k$ we let u_i be the up-step matched with d_i , and we switch every step between u_i and d_i (included).

of walks (Dyck walks) where both the stronger domain and endpoint restrictions are imposed, and the elements of this class carry additional “decorations” (here, marked down-steps reaching the x -axis).

This paper introduces a similar but new strategy which can be successfully applied to several models of Weyl chamber walks. In particular, for two classical step sets (simple walks and hesitating walks), we have found explicit bijections that exchange a domain constraint with an endpoint constraint. In the two-dimensional case, these bijections match walks in the quadrant $\{x \geq 0, y \geq 0\}$ ending at the origin (*excursions*¹), and walks in the octant $\{x \geq y \geq 0\}$ and ending on the x -axis (*axis-walks*). For both step sets, these bijections pass through decorated excursions restricted to the octant. Deciding exactly how to mark the steps in the decorated intermediary is less obvious than the Dyck walk example. We do this by using *open arc diagrams* (corresponding to partial matchings and set partitions) that are associated to the walks via the robust bijection of Chen et al. [11]—or more precisely, via its extension to open arc diagrams due to Burrill et al. [8]. In their full generality, these bijections map open arc diagrams with no $(k+1)$ -crossing² to walks in the k -dimensional Weyl chamber of type C $\{(x_1, \dots, x_k) : x_1 \geq \dots \geq x_k \geq 0\}$ that end on the x_1 -axis, where the number of open arcs gives the abscissa of the endpoint. It is at the level of arc diagrams that the marking of the object is easiest to describe: we map walks that end on the x -axis to open arc diagrams, mark the location of the open arcs, remove them, and then apply the inverse bijection to get marked excursions. The schematic outline of our core idea is illustrated in Fig. 2. The advantage of our approach is that it very easily generalizes to walks in arbitrary dimension.

Once these decorated excursions are obtained, it remains to process the marks. This processing is handled differently for simple walks and for hesitating walks (where a further step of transfer of decorations is needed), but in both cases the marks are used to produce an unmarked walk in a larger domain.

Part of the bijection for simple walks has the nice feature that it extends to higher dimension, unveiling a new bijective connection with standard Young tableaux of even-bounded height (which are known to be related to Weyl chamber axis-walks [22,8,26]).

1.1. Bijection for 2D simple walks

A lattice model is said to be *simple* if the step set consists of all of the elementary vectors and their negatives. In two dimensions, the steps correspond to the compass directions, that we denote N, E, S, W . Our first main result is the following Theorem, which is proved in Section 3.

Theorem 1. *There exists an explicit bijection (preserving the length) between simple axis-walks of even length staying in the first octant, and simple excursions staying in the first quadrant.*

As announced in the introduction, our strategy uses open arc diagrams to turn the simple axis-walk of length $2n$ into a decorated excursion. This is then transformed to a simple walk of length $2n$ in the tilted quadrant $\{(x, y) : x \geq 0, |y| \leq x\}$ starting and ending at $(1/2, 1/2)$, and finally mapped to a pair of Dyck paths of respective lengths $2n+2$ and $2n$. These are known [12,2] to be in bijection with simple excursions of length $2n$ in the quadrant. This chain of bijections is illustrated by Fig. 3.

¹ More generally, we use the term excursion to indicate the set of walks with a prescribed start and end point. When they are not specified, the prescribed start and end is assumed to be the origin.

² A k -crossing is a set of k mutually crossing arcs.

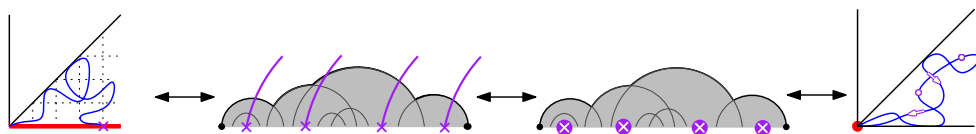


Fig. 2. In the first part of our bijections for 2D walks, we map axis-walks in the octant to open arc diagrams with no 3-crossing; we mark the positions of the open arcs, and then remove them; then we apply the inverse bijection on the resulting arc diagram with no open arcs, which yields an excursion with some markings.

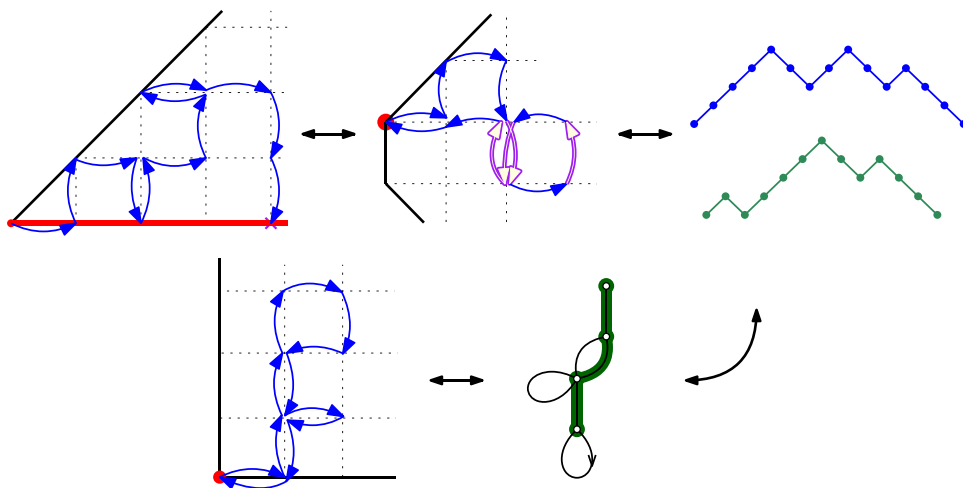


Fig. 3. Illustration of the bijective proof of [Theorem 1](#). The top part connects simple axis-walks in the octant and pairs of Dyck paths, as explained by [Theorem 7](#). The rest comes from the works of Bernardi [\[2\]](#), where he links pair of Dycks paths to simple excursions in the quadrant via the so-called *tree-rooted planar maps*. Note that this bijection seems to lose track of the statistic “*x*-coordinate of the endpoint”.

Compare [Theorem 1](#) to the following result recently proved by Elizalde:

Theorem 2 (Elizalde [\[15\]](#)). *There exists an explicit bijection (preserving the length) between simple walks staying in the first octant and ending on the diagonal, and simple excursions staying in the first quadrant.*

In [Section 5](#) we provide an alternative proof of [Theorem 2](#) using Schnyder woods. Note that [Theorems 1](#) and [2](#) together yield a bijection for simple walks of length $2n$ staying in the octant, mapping those ending on the x -axis to those ending on the diagonal. This answers an open question of Bousquet-Mélou and Mishna [\[6\]](#).

Moreover, in [Section 6](#) we give an extension for dimension $k \geq 1$ of the aforementioned bijection between simple axis-walks in the octant and simple walks from $(\frac{1}{2}, \frac{1}{2})$ to itself in the tilted quadrant. This yields a new bijective connection between standard Young tableaux of height at most $2k$ and simple excursions in the k -dimensional Weyl chamber of type D.

Grabiner and Magyar gave explicit enumeration formulas for excursions in Weyl Chambers, and hence this bijection permits a straightforward application of their results. In [Section 6.2](#) we use their results to illustrate a new derivation of Gessel's formulas for standard Young tableaux of even-bounded height.

1.2. Bijection for 2D hesitating walks

A (2-dimensional) *hesitating* walk is a sequence of steps s_1, \dots, s_{2n} such that every step of odd index is either in $\{N, E\}$ (a *positive* step) or is $\mathbf{0} = (0, 0)$, every step of even index is either in $\{W, S\}$

(a *negative step*) or is **0**, and for every $i \in \{1, \dots, n\}$, s_{2i-1} and s_{2i} cannot both be zero. It is convenient to not represent the null step in the drawings, but to group the steps by pairs of the form (s_{2i-1}, s_{2i}) instead. In Section 4 we show the analogous, although more difficult, result for hesitating walks, which answers a recent question of Burrill et al. [8]:

Theorem 3. *There exists an explicit bijection (preserving the length) between hesitating axis-walks in the first octant, and hesitating excursions in the first quadrant.*

It is then easy to derive a bijection between Baxter permutations of size $n + 1$ (known to be in bijection with hesitating excursions of half-length n in the quadrant) and open matching-diagrams with n points and no enhanced 3-nesting (known to be in bijection with hesitating axis-walks of half-length n in the octant). This answers a conjecture formulated by Burrill et al. [9].

In order to show Theorem 3, again the first step is to use the strategy of Fig. 2 to turn the axis-walks into decorated hesitating excursions, where the decoration consists in marking some W-steps on the x -axis. A further ingredient here is to turn the decoration into marked steps leaving the diagonal, after which the decorated excursions in the octant are known [8] to be equivalent to hesitating excursions in the quadrant.

Hesitating excursions of half-length $n - 1$ in the quadrant are known to be counted by the Baxter numbers $B_n = \frac{2}{n(n+1)^2} \sum_{k=1}^n \binom{n+1}{k+1} \binom{n+1}{k} \binom{n+1}{k-1}$. Indeed, as shown by Burrill et al. [8], they are in easy bijection with the classical Baxter family of non-intersecting triples of directed lattice walks. On the other hand it has been first shown in [31] (and more recently in [8]) that hesitating axis-walks of half-length n in the octant are also counted by B_{n+1} . Both of these proofs involve an equality of generating functions, and neither proof retains significant combinatorial intuition. Our result is the first bijective proof that these walks are counted by B_{n+1} . Such a result is not obvious to find since the family of hesitating axis-walks in the octant does not seem to be naturally endowed with the classical (bivariate) symmetric generating tree common to the Baxter families such as Baxter permutations, twin pairs of binary trees, 2-oriented plane quadrangulations, and plane bipolar orientations [14,1,17]. These families share the same generating tree, and hence there exists a “canonical” bijection relating them [5]. We cannot rely on such a systematic bijective strategy here.

Theorem 3 can be extended to a similar kind of plane walks, namely vacillating walks, leading to some new enumerative results on such walks.

In the case of simple walks, we saw that both the excursions in the quadrant and the axis-walks in the octant are in bijection with the simple walks in the octant ending on the diagonal. Does this hold for hesitating walks? In fact, a computational enumeration (up to half-length 50) suggests the following conjecture (where the conjectural part is the bijective link to the third family).

Conjecture 4. *The following families are in bijection:*

- hesitating excursions of length $2n$ in the quadrant,
- hesitating axis-walks of length $2n$ in the octant,
- hesitating walks of length $2n$ in the octant, ending on the thick diagonal : $\{(n, n), n \in \mathbb{N}\} \cup \{(n+1, n), n \in \mathbb{N}\}$.

If we denote by u_n (resp. v_n) the number of hesitating walks of length $2n$ in the octant ending on $\{x = y\}$ (resp. ending on $\{x = y + 1\}$), then Conjecture 4 would imply that $u_n + v_n$ equals the Baxter number B_{n+1} . We have not been able yet to find a computational proof that $u_n + v_n = B_{n+1}$, but have numerically checked that both u_n and v_n seem to be P-recursive (of order 2). We have not found any natural subfamily of hesitating axis-walks of length $2n$ (for instance with a parity constraint on the ending point) that are counted by u_n .

2. Open arc diagrams

Arc diagrams are a graphic representation of combinatorial structures such as partitions or matchings, which enables a convenient visualization of certain patterns, such as crossings. A *partition diagram* is defined for a set partition π of $\{1, \dots, n\}$: draw n points on a line, labeled from 1 to n ; for



Fig. 4. Left: an open matching-diagram with 4 open arcs. Right: an open partition-diagram with 2 open arcs.

each (ordered) block $\{a_1, \dots, a_k\}$ of π , we draw an arc from a_i to a_{i+1} for $1 \leq i \leq k-1$. A *matching diagram* is a partition diagram where the underlying set partition is a matching (i.e. every block has size 2).

A point of a partition diagram can be an *opening point*, if it is the first point of a block of size ≥ 2 ; a *closing point*, if it is the last point of a block of size ≥ 2 ; a *transition point*, if it is a non-extremal point in a block of size ≥ 3 ; or a *fixed point*, if it is a block of size 1. A matching diagram only has opening and closing points.

A 3-crossing pattern in an arc diagram is a set of three mutually crossing arcs, i.e. three arcs (i_1, j_1) , (i_2, j_2) and (i_3, j_3) with $i_1 < i_2 < i_3 < j_1 < j_2 < j_3$. An *enhanced 3-crossing* is a 3-crossing where arcs sharing an endpoint are also considered to be crossing. More formally, three arcs (i_1, j_1) , (i_2, j_2) , (i_3, j_3) form an enhanced 3-crossing if $i_1 < i_2 < i_3 \leq j_1 < j_2 < j_3$. These definitions are naturally generalized to k -crossings for $k \geq 2$.

Arc diagrams can be extended to *open arc diagrams*, by allowing arcs with only a left endpoint, and no right endpoint (see Fig. 4 for examples). A 3-crossing and an enhanced 3-crossing can now include one open arc but only if this open arc starts as position i_3 (informally it is as if $j_3 = +\infty$). The two examples of Fig. 4 have multiple 2-crossings but no 3-crossing. For all types of arc diagrams the size is defined as the number of points in the diagram.

In [11], Chen et al. describe a bijection between arc diagrams with no $(k+1)$ -crossings, and excursions staying in the k -dimensional Weyl chamber of type C. It was subsequently extended in [7,8] by Burrill et al. to map open arc diagrams to axis-walks (see the left part of Fig. 6 for an example).

Theorem 5 (Burrill et al. [8], Restricted to 3-crossings). *There exists an explicit bijection between open matching (resp. open partition) diagrams of size n , with m open arcs and no 3-crossing (resp. no enhanced 3-crossing), and simple (resp. hesitating) walks of length n (resp. of half-length n) staying in the first octant $\{(x, y), 0 \leq y \leq x\}$, starting at the origin and ending at $(m, 0)$.*

We refer the reader to the original paper [8] for a description of the bijection in its complete form, which is based on the Robinson–Schensted insertion algorithm on tableaux, but can also conveniently be reformulated in terms of growth diagrams [25,26].

Here are important properties of this correspondence that we use in our bijections. The first part is extracted from Proposition 3 of [8], and the second part follows straightforwardly from the description of the bijection.

Property 6. *Let π be a closed matching (resp. partition) diagram of size n with no 3-crossing (resp. enhanced 3-crossing), and ω the simple (resp. hesitating) walk of length n corresponding to π via the bijection due to Burrill et al. [8].*

Open arcs can be inserted into intervals at positions i_1, i_2, \dots, i_k in π without forming a 3-crossing if and only if for every $j \in \{1, \dots, k\}$ the y -coordinate after i_j steps in ω is zero.

Given a partition diagram π , the fixed points of π correspond to the factors EW in ω , and the closing points of π correspond to the factors {OW, OS} in ω (with O denoting the zero step). In addition, an open arc can be added on a fixed point or a closing point without creating an enhanced 3-crossing if and only if an open arc could be added into the interval just to the left of that point without creating a 3-crossing.

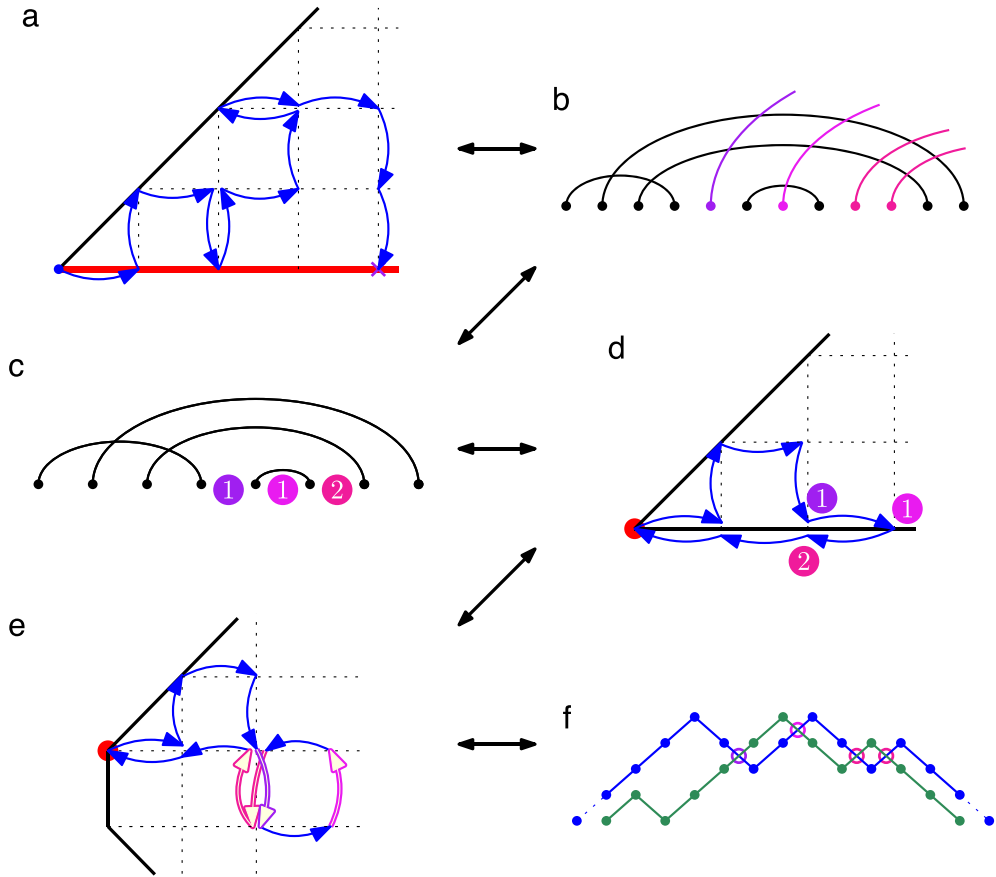


Fig. 5. An example of the bijection of [Theorem 7](#). Successive objects are: (a) a simple axis-walk in the octant; (b) an open matching diagram with no 3-crossing; (c) a matching diagram with no 3-crossing and with integer weights at some intervals to record the positions of the former open arcs; (d) a simple excursion in the octant where each visit to the x -axis carries a nonnegative integer weight; (e) a simple excursion in the tilted quadrant; (f) a pair of Dyck paths whose lengths differ by 2.

3. Proof of [Theorem 1](#): simple walks

The first main ingredient lies in the results from [\[12,2\]](#), where the respective authors describe a correspondence between simple excursions of length $2n$ in the quadrant, and pairs of Dyck paths of lengths $2n$ and $2n + 2$. To have a bijective proof of [Theorem 1](#), we then need to connect such pairs of Dyck paths to simple axis-walks of even length in the octant. This is given by the following theorem, with an extension to odd length.

Theorem 7. Let \mathcal{C}_n be the set of Dyck paths of length $2n$, and let \mathcal{U}_n be the set of simple axis-walks of length n in the first octant. There is an explicit bijection for each $n \geq 0$ between \mathcal{U}_{2n} and $\mathcal{C}_n \times \mathcal{C}_{n+1}$, and between \mathcal{U}_{2n+1} and $\mathcal{C}_{n+1} \times \mathcal{C}_{n+1}$.

This section provides a bijective proof of this result, as illustrated by [Fig. 5](#). Gouyou-Beauchamps [\[22\]](#) showed that the cardinality of simple axis-walks in the octant is indeed $\text{Cat}_n \text{Cat}_{n+1}$ or Cat_{n+1}^2 , depending on the parity, where Cat_n is the n th Catalan number. However, his proof uses a reflection principle argument of Gessel Viennot which involves subtractions and cancellations of terms.

As described in the introduction, our strategy relies on the bijection of [Theorem 5](#), which allows us to turn simple axis-walk into simple excursion with decorations consisting of weights assigned to each visit to the x -axis.

Lemma 8. *Simple axis-walks of length n in the octant ending at $(m, 0)$ are in bijection with simple excursions of length $n - m$ in the octant, where each visit to the x -axis carries a non-negative integer weight, such that the sum of the weights is m .*

Proof. Using [Theorem 5](#), such an axis-walk is mapped to an open matching diagram of size n with m open arcs and without 3-crossing. We then remove the open arcs along with their nodes to obtain a (closed) matching diagram π of size $n - m$, and we record their former positions as follows: for each interval of π that contained at least one open arc, we assign to the interval a positive weight equal to the number of open arcs it formerly contained (see [Fig. 5\(b\)](#) to (c)). The sum of these weights is thus m . Only specific intervals can carry weights since adding arcs in some intervals might create a 3-crossing.

By [Theorem 5](#) (again), the diagram π is mapped to an excursion in the octant. By [Property 6](#), we know that the intervals of π where insertion of open arcs is possible exactly correspond to the visits of the excursion to the x -axis. We then transfer the weights to the corresponding positions (see [Fig. 5\(c\)](#) to (d)). \square

As a final step, we transform the weighted excursions in the octant into pairs of Dyck paths. To do so, we define an intermediary class of walks in the tilted quadrant

$$\tilde{Q} = \{(x, y) : x \geq 0, |y| \leq x\}.$$

This domain is constituted of two copies of the octant, a positive one $\{(x, y) : x \geq y \geq 0\}$, and an upside-down negative one $\{(x, y) : x \geq -y \geq 0\}$.

Lemma 9. *For n, m both even (resp. both odd), the set of simple decorated excursions of length $n - m$ in the octant with a total weight m on the visits to the x -axis is in bijection with the set of simple walks of length n in the tilted quadrant \tilde{Q} from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$ (resp. to $(\frac{1}{2}, -\frac{1}{2})$) where exactly m steps change the sign of y in the walk. This set is in bijection with $C_{\lfloor (n+1)/2 \rfloor} \times C_{\lceil (n+1)/2 \rceil}$, in such a way that if the two Dyck paths are drawn with respective starting points $((0, 0), (-1, 0))$, they cross exactly m times.*

Proof. The idea is illustrated by [Fig. 5\(d\)–\(f\)](#).

First we translate the whole excursion by $(\frac{1}{2}, \frac{1}{2})$.

A weight on a position at height $\frac{1}{2}$ indicates the number of switches that are to be inserted at this position. The switches partition the path into a certain number of factors. We apply the transformation $(x, y) \rightarrow (x, -y)$ to factors of even rank (where the first factor has rank 1); and for each switch, we insert a vertical step crossing the x -axis in order to join the end of the preceding factor to the beginning of the next one.

Concerning the second bijection, we map any walk $(x_i, y_i)_{i \in \{0, \dots, n\}}$ of \tilde{Q} to the pair P_1, P_2 of paths

$$((x_i + y_i)_{i \in \{0, \dots, n\}}, (x_i - y_i)_{i \in \{0, \dots, n\}}),$$

i.e., the successive heights of P_1 (resp. of P_2) are the successive values of $x_i + y_i$ (resp. of $x_i - y_i$). We easily see that the constraint of staying in \tilde{Q} is mapped to the constraint that both P_1 and P_2 remain nonnegative. In addition, for even length $2n$ the endpoint conditions ensures that P_1 starts and ends at 1, while P_2 starts and ends at 0; so Q is a Dyck path of length $2n$ and P_1 which can be identified with a Dyck path of length $2n + 2$, upon prepending an up-step and appending a down-step. For odd length $2n + 1$, the endpoint conditions ensure that P starts at 1 and ends at 0, while P_2 starts at 0 and ends at 1; hence both P_1 and P_2 are identified with a Dyck path of length $2n + 2$ (upon prepending a down-step to P_1 and appending a down-step to P_2). \square

We obtain the bijection for [Theorem 7](#) by composing [Lemma 8](#) with [Lemma 9](#).

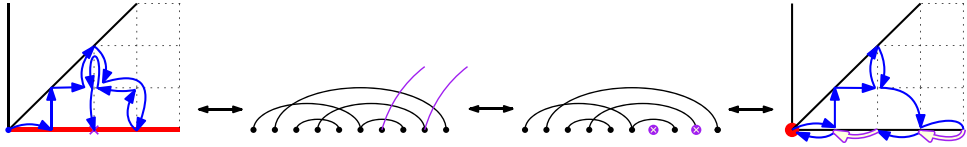


Fig. 6. An example of the first part of the bijection. From left to right: a hesitating axis-walk in the octant; an open partition diagram; a decorated partition diagram; a decorated hesitating excursion in the octant.

4. Proof of Theorem 3: hesitating walks

We next give the details of the bijection between hesitating axis-walks in the octant, and hesitating excursions to prove Theorem 3. The initial part is similar to before, and we provide two variants for the second.

4.1. Transformation into decorated hesitating excursions in the octant

The general strategy is the same as for simple walks: turn an axis-walk in the octant into an open partition diagram, remove the open arcs while marking their locations, and transform the decorated diagram back to a decorated excursion. In contrast, the second part is different from the simple walk case, since the marking does not easily induce an excursion in a larger domain. That is why we need an additional step of decoration transfer.

Lemma 10. *Hesitating walks of length $2n$ staying in the octant and ending at $(m, 0)$ are in bijection with hesitating excursions of length $2n$ staying in the octant in which m W -steps on the x -axis have been marked.*

Proof. Using Property 6, it is easy to check that for π a partition-diagram of size n and ω the corresponding hesitating excursion of length $2n$ in the octant, the (closing or fixed) points of π where an open arc can be added exactly correspond to the W -steps of ω on the x -axis. If we mark m such steps we obtain an open partition diagram of size n with m open arcs and no enhanced 3-crossing, which itself corresponds (by Theorem 5) to an hesitating walk of length $2n$ in the octant that ends at $(m, 0)$. \square

It turns out that hesitating excursions in the quadrant (the second family of objects appearing in Theorem 3) are also in bijection with some decorated hesitating excursions in the octant, as stated by the following lemma.

Lemma 11. *Hesitating (resp. simple) excursions of length $2n$ in the first quadrant are in bijection with hesitating (resp. simple) excursions of length $2n$ in the first octant with marked steps leaving the diagonal $y = x$.*

As proved in [8], this lemma is a consequence of the reflection principle with respect to the diagonal.

In terms of hesitating excursions of the quadrant, the number of marked steps corresponds to a parameter called the *switch-multiplicity* of the walk, which is roughly speaking the number of times that the walk crosses the diagonal, or similarly to the number of times the walk goes over the diagonal.

4.2. Moving the marks around

In view of Lemmas 10 and 11, Theorem 3 holds if there is an equidistribution for the hesitating excursions of the octant between the parameter counting the steps leaving the diagonal, and the parameter counting W -steps on the x -axis. This is true, and furthermore, they are symmetrically distributed.

Proposition 12. *There is an explicit involution over the set of hesitating excursions of length $2n$ in the octant that exchanges the number of W -steps on the x -axis and the number of steps leaving the diagonal.*

Proof. The proof passes through four main intermediaries. We first map a hesitating walk to a simple walk, tracking enough information to recover the hesitating walk. We use a classic mapping of steps to convert a simple excursion to a pair of Dyck paths. Then, we apply an involution on pairs of Dyck paths which swaps a key parameter. The final deduction comes from tracing parameters through these bijections, and back. This gives the stated result.

We now give the details of the individual steps.

From hesitating walks to simple walks. We call a *sailing point* a positive step, E or N , followed by a negative step, W or S . We transform every hesitating walk into a simple walk in which some sailing points are marked. To do so, we gather the steps of the hesitating walk in pairs, discarding the zero-steps, and marking every sailing point induced by the gathering of two non-zero steps. Thus, every hesitating excursion in the octant is identified with a simple excursion in the octant where some sailing points are marked.

From simple walks to pairs of Dyck paths. Simple excursions in the octant are mapped to non-crossing pairs of Dyck paths by the transformation $(x_i, y_i) \rightarrow ((x_i + y_i), (x_i - y_i))$. The sailing points of the excursion become the *upper peaks*, i.e. the peaks of the upper path; the W -steps on the x -axis become the *upper contacts*, i.e. down steps occurring at the same time and the same height for both paths; and the steps leaving the diagonal become the *lower contacts*, i.e. up steps leaving the x -axis in the lower path. The proof is now reduced to the following claim.

Involution for non-crossing pairs of Dyck paths.

Claim 13. *There is an explicit involution on pairs of non-crossing Dyck paths of length $2n$, which preserves the number of upper peaks and maps the number of upper contacts to the number of lower contacts.*

Such an involution is given by an article of Elizalde and Rubey [16]. More specifically, their involution operates on non-crossing pairs of Dyck paths, preserves the upper path (hence the upper peaks), and exchanges the number of upper and lower contacts.

In the next section, we also give an alternative proof of the previous claim, using Schnyder woods. \square

4.3. Bonus: vacillating walks

In the literature (see [11,10] for example), there are commonly three families of walks related to tableaux: the simple and hesitating walks, which we have already dealt with, and vacillating walks. A (2-dimensional) *vacillating* walk is an even sequence of steps such that every step of odd index is W , S or $\mathbf{0}$ (a negative or zero step), and every step of even index is N , E or $\mathbf{0}$ (a positive or zero step). Unlike hesitating walks, vacillating walks can have successive zero steps.

It turns out that an analogue of Theorems 1 and 3 for vacillating walks can be seen as a corollary of Theorem 3.

Corollary 14. *For every integer $n \geq 1$, the number of vacillating axis-walks of half-length n in the first octant is twice the number of vacillating excursions of half-length n in the first quadrant. The latter number is equal to $\sum_{k=0}^{n-1} \binom{n-1}{k} B_{k+1}$, where B_k is the k th Baxter number.*

Proof. First, we claim that the vacillating axis-walks of half-length n in the quadrant (resp. in the octant) are in bijection with the triples $(hw, P, \text{laststep})$ where:

- hw is a hesitating axis-walk of half-length $k \leq n - 1$ in the quadrant (resp. in the octant);
- P is a multiset of size $n - 1 - k$ and with elements in $\{0, \dots, k\}$;
- laststep is either E or $\mathbf{0}$.

Indeed, removing the first step and the last step from a vacillating walk w of half-length n induces a hesitating walk of half-length $n - 1$ where we allow some consecutive steps s_{2i-1} and s_{2i} to be zero. Thus, to obtain a valid hesitating walk hw , we have to remove the consecutive double-zero steps, and store them in a multiset P , which describes the positions where we have to insert back the double-zero steps to recover the original vacillating walk. To recover the original vacillating walk we also have to put back the first step, which has to be $\mathbf{0}$, because a walk starting at the origin cannot begin by a negative step, and the last step, which can be E or $\mathbf{0}$, but not N because the walk ends on the axis.

If we restrict the axis-walk to be an excursion, then under the previous bijection, hw is also an excursion, and *laststep* is necessarily $\mathbf{0}$ (hence the factor 2).

We now use the bijection of [Theorem 3](#), which matches the hesitating axis-walks in the first octant and the hesitating excursions in the first quadrant. Translated in terms of vacillating walks via the above bijection, it means that the vacillating axis-walks of the first octant are in bijection with the pairs formed by a vacillating excursion in the first quadrant, and a step in $\{E, \mathbf{0}\}$.

As for the counting formula for vacillating excursions of half-length n in the first quadrant, it directly follows from the fact that hesitating excursions of half-length k in the first quadrant are counted by B_{k+1} . \square

5. Alternative proofs using Schnyder woods

In this section, we propose some alternative proofs of [Theorem 2](#) and [Claim 13](#) by using as an intermediate step a famous combinatorial family: the *Schnyder woods*.

5.1. Schnyder woods and pairs of non-crossing Dyck paths

A *rooted triangulation* is the embedding of a simple planar graph in the plane such that all faces are triangles, with a distinguished *external face* and a distinguished *root vertex* on the external face. We are interested in a particular kind of edge-coloring and orientation of triangulations, known as *Schnyder woods*. One of them is depicted by [Fig. 7\(d\)](#). Schnyder woods were introduced by Schnyder in [\[28\]](#) and [\[29\]](#) for triangulations and were later extended to different families of maps. There are several applications of these colorings, ranging from graph drawing to map encoding.

We describe a Schnyder wood using some conventions of the triangulation. The *size* of a triangulation is the number of vertices minus 3 (we do not count the vertices of the external face). The root vertex is labeled v_0 , and the two other vertices in counterclockwise order around the external face are labeled v_1 and v_2 . The vertices that are not incident to the external face and edges that do not bound the external face are called *internal*.

A Schnyder wood of a simple triangulation is an edge-coloring into 3 colors, along with an orientation of all internal edges, such that the following properties are satisfied:

- for each $i \in \{0, 1, 2\}$, the set of i -colored edges forms a spanning tree T_i , rooted and oriented towards v_i ;
- if, for $i \in \{0, 1, 2\}$, an i -tail (resp. i -head) with respect to a vertex denotes an edge colored by i oriented away from (resp. toward) this vertex, then in clockwise order around any internal vertex, there are: one 0-tail, some 1-heads, one 2-tail, some 0-heads, one 1-tail, some 2-heads.

Schnyder woods are in bijective correspondence with several combinatorial families. A pair P, Q of Dyck paths of length $2n$ (both starting at $(0, 0)$ and ending at $(2n, 0)$) is called *non-crossing* if for each $0 \leq i \leq 2n$, the height of Q after i steps is at most the height of P after i steps.

Theorem 15 (Bonichon [\[4\]](#)). *Non-crossing pairs of Dyck paths of length $2n$ are in bijection with Schnyder woods of size n .*

[Theorem 15](#) was first proved by Bonichon in [\[4\]](#). Bernardi and Bonichon improved the description of this bijection a few years later [\[3\]](#). It is the latter form that we use here. We give a quick description of the map Ψ from Schnyder woods to pairs of non-crossing Dyck paths—see (c) and (d) from [Fig. 7](#)

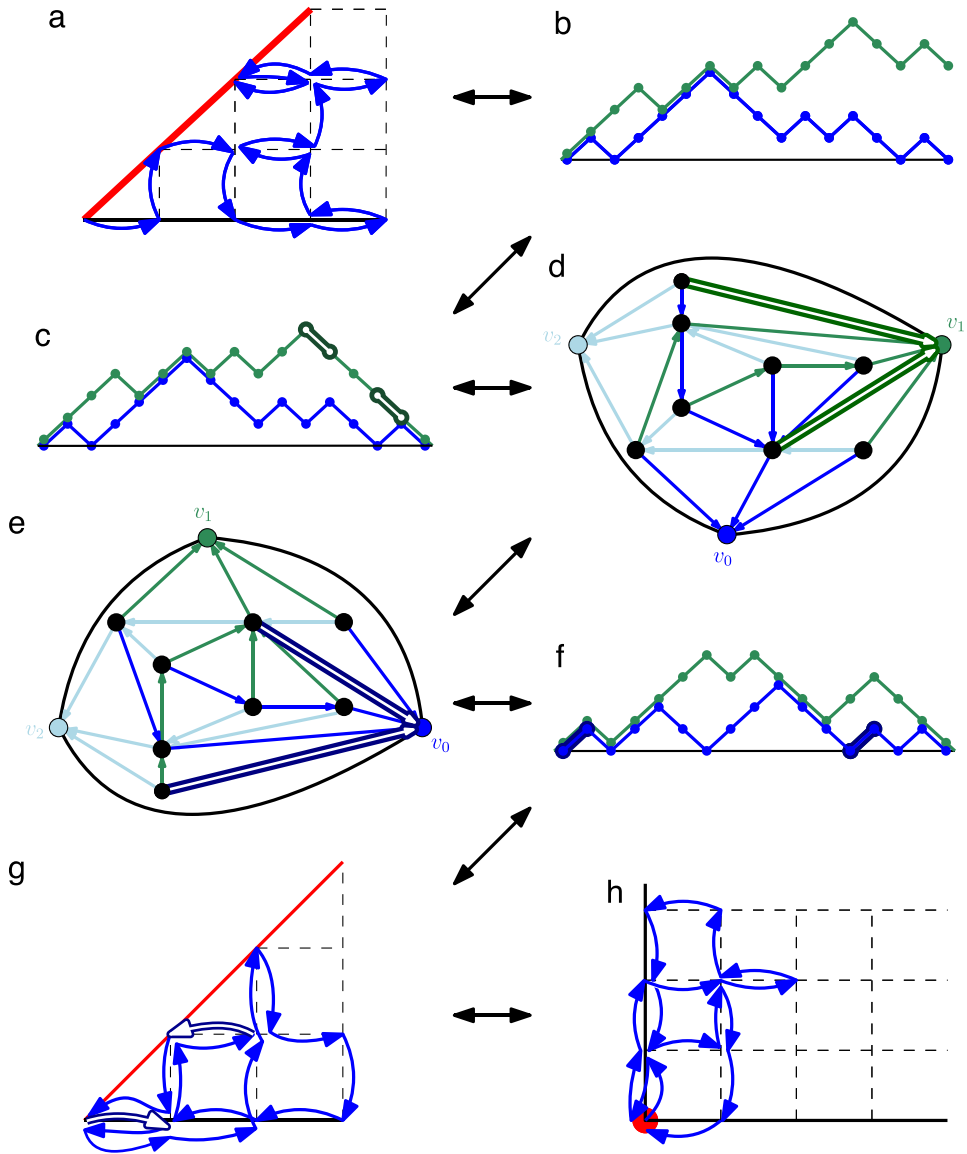


Fig. 7. Illustration of the bijective proof of [Theorem 2](#) using Schnyder woods. Successive objects are: (a) a simple walk ending on the diagonal and staying in the octant; a non-crossing pair formed by a Dyck path and a non-negative walk starting at $(0, 0)$; (c) a non-crossing pair of Dyck paths where some steps on the last descent of the upper path are marked; (d) and (e) two Schnyder woods (the roles of T_0 and T_1 have been swapped thanks to a vertical reflection); (f) a non-crossing pair of Dyck paths where some up-steps of the lower path that leave the x -axis are marked; (g) a simple excursion staying in the quadrant where some steps leaving the diagonal are marked; (h) a simple excursion staying in the quadrant.

for an example of the bijection. First, remember the classical bijection Ω between plane trees and Dyck paths: take a plane tree, turn around it clockwise, starting and ending at the root; the first time an edge is visited, write an up-step, the second time, write a down step. From a Schnyder wood, we now generate a pair (P, Q) of non-crossing Dyck paths. The bottom path $Q = UD^{\alpha_1} \dots UD^{\alpha_n}$, where

Table 1The correspondence of parameters through the bijections Ψ and Φ .

	Schnyder wood	Pair of non-crossing Dyck paths
1	Size	Half-length
2	Number of leaves of T_0	Number of lower peaks
3	Number of internal nodes of T_1	Number of upper peaks
4	Degree of v_0 in T_0	Number of lower contacts
5	Degree of v_1 in T_1	Length of the last upper descent
6	Degree of v_2 in T_2	Number of reversed upper contacts

U stands for up-step and D stands for down-step, is the path representing the tree T_0 of color 0: $Q = \Omega(T_0)$. The tour around T_0 induces an order on the internal vertices, that we subsequently call u_1, \dots, u_n (the first vertex visited by the tour is v_0). Let β_i be the number of 1-heads incident to u_i and let β_{n+1} be the number of 1-heads incident to v_1 . Note that β_1 has to be 0. The upper path is now defined as $P = UD^{\beta_2} \dots UD^{\beta_{n+1}}$. We refer the reader to Bernardi and Bonichon's paper [3] for a proof that P is positive and does not cross Q , a description of the reverse map Φ , and a proof that Ψ and Φ are reciprocal bijections.

For our own purpose, we need to review how certain parameters are transformed by Ψ , as shown in Table 1 where the first column corresponds to Schnyder woods, and the second column corresponds to pairs of Dyck paths. The bijections Ψ and Φ map each parameter to its counterpart on the same row. In the proof of Proposition 12, we defined the *upper contacts* as common down-steps. Similarly, *reversed upper contacts* are common up-steps. *Upper* (resp. *lower*) *peaks* are peaks of the upper (resp. lower) path.

Lemma 16. *The bijection Φ satisfies the correspondence of parameters given by Table 1.*

Proof. This proof uses references and notation from Bernardi and Bonichon's paper [3]. Row 1 is trivial, and Rows 2 and 4 are well-known properties on the bijection between Dyck paths and trees. Row 5 is a direct consequence of the construction of the bijection, and Row 3 is a direct consequence of the fact that an upper peak is a descent i of positive length $\beta_i > 0$, which corresponds to an internal node of T_1 .

Row 6 is less straight-forward. It corresponds to the tight case in [3] when proving that the pair of walks is non-crossing; we sketch the main arguments. Let u_i be an internal vertex of T (for $1 \leq i \leq n$), and let h_i be the tail of color 2 at u_i . Then one can check that u_i is a neighbor of v_2 in T_2 if and only if h_i is not below a 1-arc (an arc of color 1), i.e., there is no 1-arc e such that h_i is inside the (unique) cycle formed by e and T_0 . In such a situation, h_i comes after the tail of e and before the head of e during a clockwise tour around T_0 . The number of 1-tails before h_i in such a tour is $\sum_{j < i} \alpha_j$, while the number of 1-heads before h_i is $\sum_{j \leq i} \beta_j$. Consequently, given that a 1-tail always comes in the tour before its corresponding 1-head, u_i is a neighbor of v_2 in T_2 if and only if the previous numbers coincide, i.e., $\sum_{j < i} \alpha_j = \sum_{j \leq i} \beta_j$. Finally, we remark that $-i + \sum_{j < i} \alpha_j$ (resp. $-i + \sum_{j \leq i} \beta_j$) gives the height of the i th up-step of the lower (resp. upper) Dyck path; hence the two numbers match if and only if the i th up-steps of the lower and upper Dyck paths form a reversed upper contact. \square

5.2. Consequences

From Lemma 16, we can prove several non-trivial properties on non-crossing pairs of Dyck paths, and subsequently on excursions in the octant. Schnyder woods have more evident symmetries than pairs of Dyck paths, and the very expression of these symmetries gives involutions that are not easily phrased in terms of pairs of Dyck paths.

We give an alternative proof of Theorem 2 (see Fig. 7 for an outline of the bijection).

Proof of Theorem 2. We are going to describe the bijection from excursions to walks ending at the diagonal.

First, we use [Lemma 11](#) to map an excursion in the quadrant to an excursion in the octant with k marked steps leaving the diagonal. This excursion is then mapped to a pair of non-crossing Dyck paths with k marked lower contacts.

We apply Φ , move the root from v_0 to v_1 and change the orientation (meaning that “clockwise” becomes “counterclockwise”). Moving the root and changing the orientation amounts to exchanging the roles of T_0 and T_1 . Hence, according to [Lemma 16](#) (rows 4 and 5), when we then apply Ψ to get back to a pair of non-crossing Dyck paths, we get k marked steps on the descent of the upper path.

We reverse these k steps, and the upper path now ends at height $2k$. We map the pair of paths back to a walk in octant that ends at coordinates (k, k) . \square

A similar strategy can be applied to prove the next theorem, which is a stronger version of [Claim 13](#) (that was used to complete the proof of [Theorem 3](#)), but still weaker than the result from [16], which preserves the upper path.

Theorem 17. *There is an explicit involution on pairs of non-crossing Dyck paths of length $2n$, which preserves the number of upper peaks and exchanges the number of upper contacts and the number of lower contacts.*

Proof. We take a pair (P, Q) of non-crossing Dyck paths, reverse them to transform upper contacts into reversed upper contacts, while keeping the same number of peaks and lower contacts, and apply Φ to get a Schnyder wood.

Then we move the root of the map from v_0 to v_2 , and flip the orientation of the plane. This has the effect of exchanging the roles played by T_0 and T_2 .

Finally we apply Ψ to get back to a pair of non-crossing Dyck paths, and reverse again the two paths. [Lemma 16](#) (rows 3, 4 and 6) is enough to conclude the proof. \square

5.3. An extension of the Narayana symmetry

The Narayana number $N(n, p)$ is defined as the number of Dyck paths of length $2n$ with p peaks. These numbers refine the Catalan numbers Cat_n , in the sense that $\sum_{p=1}^n N(n, p) = \text{Cat}_n$. The following statement is well-known.

Property 18. *The Narayana numbers satisfy the following symmetry property*

$$N(n, p) = N(n, n - p + 1).$$

The symmetry can be obtained from a classical bijection between Dyck paths of length $2n$ and rooted binary trees with $n + 1$ leaves: the number of peaks of the Dyck path is mapped to the number of left leaves, and the symmetry follows by applying a reflexion to the tree.

The Narayana numbers can be extended to any k -tuple of non-crossing Dyck paths in the following way: $N(n, p_1, \dots, p_k)$ is the number of non-crossing k -tuples D_1, \dots, D_k of Dyck paths (ordered from bottom to top) such that D_i has p_i peaks.

Theorem 19. *For $k = 2$, the extended Narayana numbers satisfy the symmetry property*

$$N(n, p, q) = N(n, n - q + 1, n - p + 1).$$

Proof. The method we use is similar both to the case of classical Narayana numbers and to the previous subsection. Starting from a non-intersecting pair of Dyck paths, we apply Φ , move the root from v_0 to v_1 , change the orientation (similarly as before, this amounts to exchanging the roles of T_0 and T_1), and apply Ψ back to a pair of paths. This yields an involution on non-crossing pairs of Dyck paths that has the desired peak-parameter correspondence, according to [Lemma 16](#) (rows 2 and 3). \square

However, a similar symmetry does not seem to hold for higher values of k . For example, one could expect that $N(n, p_1, \dots, p_k) = N(n, n - p_k + 1, \dots, n - p_1 + 1)$, but we present a minimal counterexample to that in [Fig. 8](#).

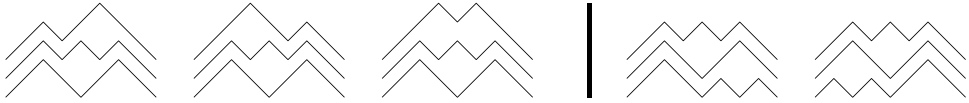


Fig. 8. A minimal counterexample to the conjectural identity $N(n, p_1, \dots, p_k) = N(n, n - p_k + 1, \dots, n - p_1 + 1)$ (which holds for $k \in \{1, 2\}$). We have $3 = N(4, 2, 3, 2) \neq N(4, 3, 2, 3) = 2$.

6. A new bijection for Young tableaux of even-bounded height

As we have seen in Section 3, the main step in the proof of Theorem 1 is an explicit bijection between simple axis-walks of length n staying in the octant $\{x \geq y \geq 0\}$, and simple walks of length n from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{(-1)^n}{2})$ staying in the tilted quadrant. As it turns out, this bijection can be easily generalized to any dimension, and infers new connections with standard Young tableaux with even-bounded height.

6.1. Walks in higher dimensional Weyl chambers

For $k \geq 1$, we define the k -dimensional Weyl chamber³ of type C as

$$W_C(k) := \{(x_1, x_2, \dots, x_k) \mid x_1 \geq x_2 \geq \dots \geq x_k \geq 0\},$$

and the k -dimensional Weyl chamber of type D as

$$W_D(k) := \{(x_1, x_2, \dots, x_k) \mid x_1 \geq x_2 \geq \dots \geq x_{k-1} \geq |x_k|\}.$$

In this context, an *axis-walk* is any walk starting at the origin and ending on the x_1 -axis. With these definitions, the generalization of Theorem 1 reads as follows.

Theorem 20. For $k \geq 1$ and $n \geq 0$, there is an explicit bijection between simple axis-walks of length n staying in $W_C(k)$ and simple excursions of length n staying in $W_D(k)$, starting from $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, and ending at $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(-1)^n}{2})$. The ending x_1 -coordinate of a walk from $W_C(k)$ corresponds to the number of steps that change the sign of x_k in its bijective image.

Note that the case $k = 1$ is precisely our introductory example, and the case $k = 2$ is the first part of Lemma 9. The arguments to show Theorem 20 are very similar to those in the proofs of Lemma 8 and the first part of Lemma 9. They use the general formulation of Theorem 5 (specifically, the bijection between open matching diagrams without $(k + 1)$ -crossing and simple axis-walks in $W_C(k)$), and the property that the intervals where an open arc can be added (without creating a $(k + 1)$ -crossing) correspond to the visits of the walk to the hyperplane defined by $x_k = 0$.

Moreover, it has been recently shown [8,26] that the set of standard Young tableaux of size n with height at most $2k$ is in bijection with simple axis-walks of length n in $W_C(k)$, with the ending x_1 -coordinate mapped to the number of columns of odd length. Composing this bijection with Theorem 20 infers the following result.

Corollary 21. For $n, k \geq 1$, there is an explicit bijection between the standard Young tableaux of size n with height at most $2k$, and the simple walks of length n staying in $W_D(k)$, starting from $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, and ending at $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(-1)^n}{2})$. The number of odd columns corresponds to the number of steps that change the sign of x_k .

³ For convenience we define the chambers using non-strict inequalities, our bijective statements can equivalently be given under strict inequalities, upon applying the coordinate shift $\tilde{x}_i = x_i + k + 1 - i$.

6.2. Recovering Gessel's formula

Thanks to the lattice path enumeration techniques of Grabiner and Magyar [23], the previous corollary has an interesting consequence: a combinatorial interpretation of the determinant expression of Gessel [19] for the generating function of standard Young tableaux of even-bounded height.

Proposition 22 (Gessel [19]). *Let $Y_d[n]$ be the number of Young tableaux of size n with at most d rows, and $Y_d(x) = \sum_{n \geq 0} \frac{1}{n!} Y_d[n] x^n$ the associated generating function. Then for each $k \geq 1$,*

$$Y_{2k}(x) = \det(I_{i-j}(2x) + I_{i+j-1}(2x))_{1 \leq i, j \leq k}, \quad (1)$$

where (for $m \in \mathbb{Z}$) $I_m(2x) = \sum_{i \geq 0} \frac{1}{(m+i)! i!} x^{m+2i}$.

Let us now explain how we can recover this result from Corollary 21. First, it proves to be convenient to take here the Weyl chamber of type D under the form

$$\widehat{W}_D(k) := \{ |x_1| < x_2 < \dots < x_k \}.$$

For each point $\lambda = (\lambda_1, \dots, \lambda_k)$ (in \mathbb{R}^k) we denote by λ' the point $(-\lambda_1, \lambda_2, \dots, \lambda_k)$. Let $\rho = (1/2, 3/2, \dots, k - 1/2)$. Then Corollary 21 states that for n even (resp. odd), $Y_{2k}[n]$ is the number of walks of length n in $\widehat{W}_D(k)$ from ρ to ρ (resp. to ρ').

For every points $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ both in $\widehat{W}_D(k)$, let $N_{\lambda, \mu}[n]$ be the number of simple walks of length n from λ to μ staying in $\widehat{W}_D(k)$, and let $N_{\lambda, \mu}(x) := \sum_{n \geq 0} \frac{1}{n!} N_{\lambda, \mu}[n] x^n$ be the associated generating function.

Lemma 23 (Grabiner and Magyar [23]). *For every points $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ in $\widehat{W}_D(k)$ such that λ_i and μ_i belong to $1/2 + \mathbb{Z}$ for $i \in \{1, \dots, k\}$, we have*

$$N_{\lambda, \mu}(x) + N_{\lambda, \mu'}(x) = \det(I_{\lambda_i - \mu_j}(2x) + I_{\lambda_i + \mu_j}(2x))_{1 \leq i, j \leq k},$$

where, for $m \in \mathbb{Z}$, $I_m(2x)$ is defined as $I_m(2x) = \sum_{i \geq 0} \frac{1}{(m+i)! i!} x^{m+2i}$.

The proof techniques in [23] rely on a general reflexion principle (see [20, Theorem 1]) for Weyl chamber walks, which results in determinant expressions for the relevant generating functions (the determinant is naturally expressed in terms of $I_m(2x)$, which is the exponential generating function of simple 1d walks that start at 0 and end at m).

By Corollary 21 we have

$$Y_{2k}(x) = N_{\rho, \rho}(x) + N_{\rho, \rho'}(x),$$

where $N_{\rho, \rho}(x)$ gathers the coefficients of even power and $N_{\rho, \rho'}(x)$ gathers the coefficients of odd power. Hence, applying Lemma 23 to $\lambda = \mu = \rho$, we recover Proposition 22.

6.3. Related formulas

For odd d , the expression for the generating function $Y_d(x)$ found in Gessel's paper [19] is

$$Y_{2k+1}(x) = e^x \det(I_{i-j}(2x) - I_{i+j}(2x))_{1 \leq i, j \leq k}.$$

In that case the combinatorial derivation is easier. Indeed, a Young tableau with at most $2k + 1$ rows is identified (via the Robinson–Schensted correspondence) to an involutive permutation without $(2k + 2)$ -decreasing subsequence, which itself maps to a partial matching diagram without $(k + 1)$ -nesting (*partial* here means that there can be isolated points along the line). This implies that

$$\sum_{n \geq 0} \frac{1}{n!} Y_{2k+1}(n) x^n = e^x M_k(x),$$

where $M_k(x)$ is the exponential generating function for matching diagrams without $(k + 1)$ -nesting. By the Chen et al. bijection [11], the series $M_k(x)$ is the exponential generating function of simple walks

in the Weyl chamber of type C starting and ending at the origin; and it is shown by Grabiner and Magyar [23, Section 6.2] that this generating function is $\det(l_{i-j}(2x) - l_{i+j}(2x))_{1 \leq i, j \leq k}$.

Similarly, a determinant formula is known for the enumeration of pairs of Young tableaux of bounded height. More precisely, let $u_d[n]$ be the number of pairs of Young tableaux of the same shape with at most d rows (also by the Robinson–Schensted correspondence, the number of permutations in \mathfrak{S}_n with no $(d+1)$ -increasing subsequence). Then for every $d \geq 1$, we have as shown by Gessel [19] (note that this time the expression is uniform in d , with no dependence on the parity)

$$\sum_{n \geq 0} \frac{u_d[n]}{n!} x^{2n} = \det(l_{i-j}(2x))_{1 \leq i, j \leq k}.$$

A combinatorial proof of this expression has been given in [21] via simple walks ending at so-called Toeplitz points, see also [30] for a combinatorial derivation based on arc diagrams.

Regarding asymptotic enumeration, it should in principle be possible to use recent results by Denisov and Wachtel for the asymptotic enumeration of walks in cones [13, Theo. 6] in order to recover from Corollary 21 the expression found by Regev [27] for the asymptotic number of Young tableaux of size n with at most $2k$ rows, which is, for each fixed $k \geq 1$,

$$Y_{2k}[n] \sim_{n \rightarrow \infty} (2/\pi)^{k/2} (2k)^n (k/n)^{k(k-1/2)} \prod_{i=0}^{k-1} (2i)!.$$

(The relevant constants for the Weyl chamber of type D can be computed from [24] and from Selberg integrals [18, Eq. 1.20].)

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