Secrets from my step set

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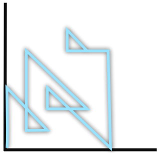
The story so far

Given $\mathcal{S}$={N, SE, W} $\rightarrow \#$  $=?$

We have seen

- Methods for finding the generating function
  - *MBM:* Define kernel; define associated group; generate and manipulate equations;
    Solve using positive series extraction of a rational function
  - *KR:* Define associated boundary value problem; solve using template solution
  - *AB:* Use power of computer algebra for intelligent guessing
- Nature of OGF (Rational? Algebraic? Holonomic?) correlated to finiteness of group

What remains to be told

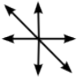
- Nice combinatorial conjectures
- The case of the infinite group
  - "How to" do the enumeration
  - Why it is plausible to connect infinite group with non-holonomic
- Explore the underlying combinatorics
  What is the fundamental difference between two boundaries and one?
- How does this help us understand non-holonomic series, from a combinatorial point of view?

Some nice problems for the bijective specialist

Explain this nice relation

For a step set $\mathcal{S}$, let $W_{\mathcal{S}}(t)$ be the ogf for walks counted by length, ending anywhere in the plane.

Theorem 1. Step set: $\mathcal{A}$ ={N, SE, W}
$a(n) = [t^n]W_{\mathcal{A}}(t) = M_n \leftarrow$ MOTZKIN numbers

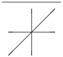


Theorem 2. Step set: $\mathcal{B}$ ={N, E, SE, S, W, NE}
$b(n) = [t^n]W_{\mathcal{B}}(t) = 2^n M_n$

Open: Find a *combinatorial explanation* for

$$a(n) = 2^n b(n)$$

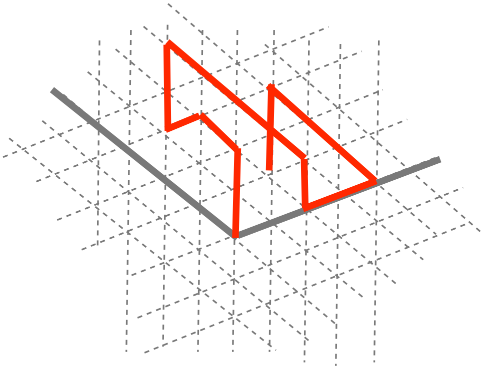
Explain the algebraic formula



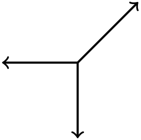
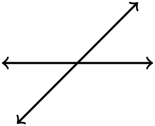
Step set: {N, NE, E, S, SW, W}

$$W(t) + 3tW(t)^2 + 4t^2W(t)^3 + 2t^3W(t)^4 = \frac{1}{1-6t}$$

This is equivalent to walks in the $\frac{2}{3}\pi$ wedge of the triangular lattice with steps along any direction:



Much harder: Why are these algebraic?

Kreweras:  Gessel: 

- In each case the pumping lemma shows that you will *not* find a direct CFG, so start with a different strategy.
- *Idea:* Characterization in terms of k-regular sequences

The case of the infinite group

Recall the group of a walk

Consider $\mathcal{S}$ ={NW, NE, SW}. The kernel is

$$\begin{aligned} K(x,y) &= 1 - t(y/x + xy + x/y) \\ &= t\left(y + \frac{1}{y}\right)x + 1 + ty\frac{1}{x} = t\left(x + \frac{1}{x}\right)y + 1 + tx\frac{1}{y} \end{aligned}$$

$G(\mathcal{S})$ is generated by $\Phi = \left(\frac{y}{x(1+\frac{1}{y})}, y\right)$ and $\Psi = \left(x, \frac{x}{y(1+\frac{1}{x})}\right)$

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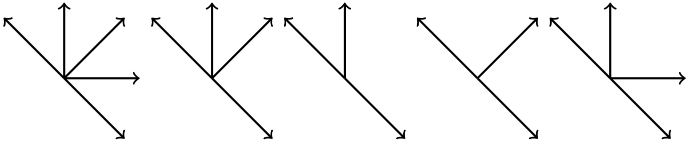
Let us see what happens to the point (t^2, t) under $\Phi \circ \Psi$:

$$
\left(t^2, t\right) \to \left(t^4 + O(t^6), t^3 + O(t^5)\right) \to \left(t^6 + O(t^8), t^5 + O(t^7)\right) \to \dots
$$

Not all infinite groups are created equal

The “best” walks with infinite groups are:



- Iterated Kernel approach works for all of these examples
- Distinguished from other infinite group classes in the Boundary Value Methodology: “singular”

Iterated kernel method [JaPrRe08;MiRe09]

Kernel equation:

$$K(x,y)Q(x,y) = xy - tx^2Q(x,0) - ty^2Q(y,0)$$

Consider one root of the kernel $Y(x) = Y(x;t) = \frac{1-\sqrt{1-4t^2(1+x^2)}}{2t(1+x^2)}$

$$\implies \qquad K(x, Y(x)) = 0$$

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Rewrite equation:

$$0 = xY(x) - tx^2Q(x,0) - tY(x)^2Q(Y(x),0)$$

Iterate the root: $(Y_0(x) := x)$

$$Y_n(x) := Y(Y_{n-1}(x)) = xt^n + O(xt^2) \implies K(Y_{n-1}(x), Y_n(x)) = 0$$

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Rewrite equation in general form:

$$0 = Y_{n-1}(x)Y_n(x) - tY_{n-1}^2(x)Q(Y_{n-1}(x),0) - tY_n(x)^2Q(Y_n(x),0)$$

Take an alternating sum

$$
\begin{array}{rllll}
0 & = xY_1(x) & -tx^2Q(x,0) & & -tY_1(x)^2Q(Y_1(x),0) \\
0 & = Y_1(x)Y_2(x) & -tY_1^2(x)Q(Y_1(x),0) & & -tY_2(x)^2Q(Y_2(x),0) \\
0 & = Y_2(x)Y_3(x) & -tY_2^2(x)Q(Y_2(x),0) & & -tY_3(x)^2Q(Y_3(x),0) \\
0 & = Y_3(x)Y_4(x) & -tY_3^2(x)Q(Y_3(x),0) & & -tY_4(x)^2Q(Y_4(x),0) \\
 & & \vdots & & \\
\hline
\end{array}
$$

$$0 = \sum (-1)^n Y_n(x) Y_{n+1}(x) - tx^2 Q(x,0)$$

This works because $Y_n(x) = xt^n + O(xt^2)$, hence the sum converges as a formal power series.

Singularities spring eternal

Theorem

$$W(t) = (1-3t)^{-1}(1 - 2\sum_{n\geq 0}(-1)^n Y_n(1)Y_{n+1}(1)).$$

The set $\bigcup_n \mathrm{poles}(Y_n(1))$ *is infinite, and is a subset of poles($W(t)$). Consequently, $W(t)$ is not holonomic.*

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Proof

- $Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \dots;$

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- $Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \ldots;$
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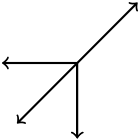
- $Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \ldots;$
- Valid power series in q;
- Show $\sum_{n\geq 0}(-1)^n Y_n(1)Y_{n+1}(1)$ convergent, except: when denominator is zero and along the branch cut of Y_1.
- Show singularities don't cancel.

IKM: Selectively applicable

The next models to analyze are non-singular. A good one to check is {N, NE, E, SW}

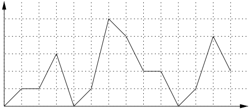


nice symmetry, but $Y(Y(x)) = x$

The combinatorics of restricted lattice paths

One dimensional case

Meanders: Walks with steps of the form $(1, k), k \in \mathbb{Z}$ that start at (0,0), end above or on the axis and never go below the axis.



Combinatorics well understood + asymptotic formulas [BaFl01]

Theorem 4. *Consider a simple aperiodic walk. The number of paths of length n, $[z^n]W(z,1)$, is $P(1)^n$ exactly. Set*

$$\overline{Y}_1(z) := \prod_{j=2}^{c}(1 - u_j(z)).$$

The asymptotic number of meanders depends on the sign of the drift $\delta = P'(1)$ as follows:

$$\begin{aligned}
\delta = 0: \quad [z^n]F(z,1) \quad &\sim \quad \nu_0 \frac{P(1)^n}{\sqrt{\pi n}}\left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right) \\
&\quad\ \ \nu_0 := \sqrt{2\frac{P(1)}{P''(1)}}\overline{Y}_1(\rho), \quad \rho = P(\tau)^{-1} = P(1)^{-1}; \\
\delta < 0: \quad [z^n]F(z,1) \quad &\sim \quad \nu_0^{\pm} \frac{P(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \frac{c_1'}{n} + \frac{c_2'}{n^2} + \cdots\right) \\
&\quad\ \ \nu_0^{\pm} := -\sqrt{2\frac{P(\tau)^3}{P''(\tau)}}\frac{\overline{Y}_1(\rho)}{P(\tau) - P(1)}, \quad \rho = P(\tau)^{-1}; \\
\delta > 0: \quad [z^n]F(z,1) \quad &\sim \quad \xi_0 P(1)^n + \nu_0^{\pm} \frac{P(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \frac{c_1''}{n} + \frac{c_2''}{n^2} + \cdots\right) \\
&\quad\ \ \xi_0 := (1 - u_1(\rho_1))\overline{Y}_1(\rho_1), \quad \rho_1 := P(1)^{-1}.
\end{aligned}$$

Asymptotics: Finite group cases

$$w_n = [z^n]W_S(t) \quad w_n \sim \kappa n^\alpha \rho^{-n}$$

drift = vector sum of elements in the step set

	Description	Example	ρ	α
1	no drift; symmetric	{N, E, S, W}	# steps	-1 (-$\frac{1}{2}$)
2	up drift; symmetric	{NW, N, NE, S}	# steps	-$\frac{1}{2}$ (-1)
3	down drift; symmetric	{N, SW, S, WE }	$P(\tau)$	-2 (-$\frac{3}{2}$)
4	no drift; tandem/d. tandem	{N, SE, W }	#steps	-$\frac{3}{2}$
5	no drift; Krewerases	{NE, S, W }	#steps	-$\frac{3}{4}$
6	no drift; Gessel	{NE, E, SW, W}	#steps	-$\frac{2}{3}$
7	no drift; G-B	{NW, W, SE, E}	#steps	-2

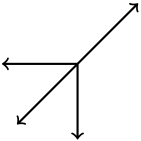
These results (guessed numerically [BoKa09]) are predicted by the meander arising as a horizontal projection:

e.g. $\{N, SW, S, SE\} \to \{\nearrow, \searrow, \searrow, \searrow\}$

$P(u) = \#\text{up steps } u + \#\text{side steps} + \#\text{down steps } u^{-1};\ \tau : P'(\tau) = 0.$

Asymptotics: Infinite group case

There are no walks with an infinite group and drift=0. Infinite group case is similar for positive drift, and negative drift along an axis. Numerical studies are inconclusive in the case of negative drift in two directions



The big picture

A combinatorial understanding of holonomy

GOAL
A theory of holonomic functions akin to the Chomsky-Schützenberger understanding of algebraic functions.

Holonomic functions in the combinatorial context only pop out when there is substantial structure. What is it?

Example: Lattice paths

- symmetry across y-axis is sufficient.
- symmetry across line $x = y$ is insufficient
- zero drift/rotational/reversal symmetry sufficient in 2D but maybe not in 3D
- *Which symmetries affect the Galois group of the kernel?*

Merci beaucoup.