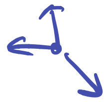
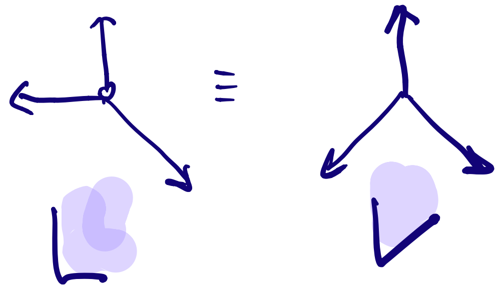
Marni Mishna.

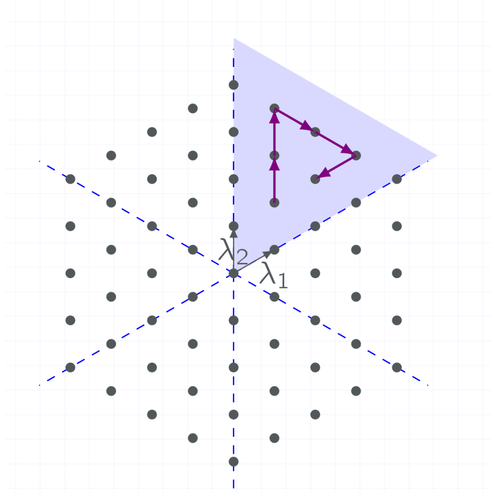
$T =$ set of walks that start @ (0,0) and stay in first quadrant.

Step set : 

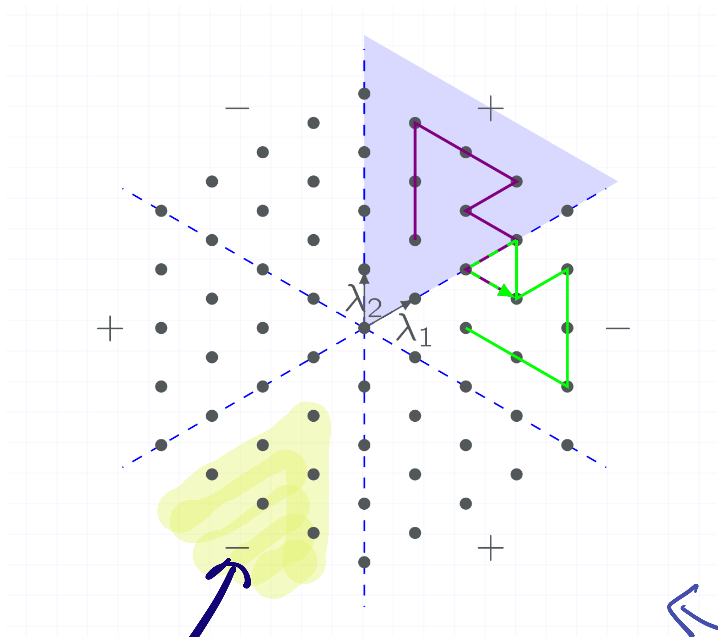
- In bijection w/ motzkin paths + standard young tableaux of bounded height. OEIS A001006.

- Equivalent to walks in triangular lattice restricted to a $\pi/3$ - wedge.





- The reflection principle argument works well here because the step set is invariant under reflections that generate the chamber.

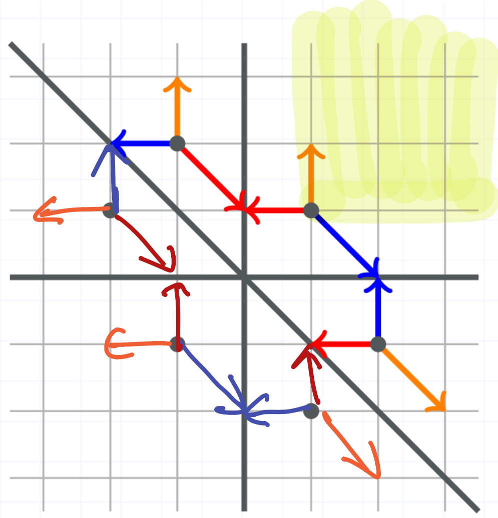


In this context, we pair a walk, and the walk that is formed by reflecting the part of the walk up to its first boundary touch across that boundary. The remainder is the same.

← Purple + Green are matched.

Ultimately, we will consider a translation of the region. Walks that do not touch the boundary in purple region will give rise to walks that do not leave the region. (once been started.)

For us, it is easier to manipulate generating series in rectangular co-ordinates:



The 6 regions under linear transformation and the reflected steps in each case.

purple region

Now: walks leaving yellow region are cancelled, leaving only those that stay in yellow region.

$$\#\,\mathrm{freewalks}_{(1,1)\to(k,l)}(n) = [x^k y^\ell]\, xy(1/x + x/y + y)^n$$

$$\#\,\mathrm{restrictedwalks}_{(1,1)\to(k,l)}(n) = [x^k y^\ell] \underbrace{\left(xy - \frac{x^2}{y} + \frac{x}{y^2} - \frac{1}{xy} + \frac{y}{x^2} - \frac{y^2}{x}\right)}_{\text{signed starting points}} \underbrace{\left(x + \frac{y}{x} + \frac{1}{y}\right)^n}_{\text{unrestricted walks}}$$

With a little bit of algebra,

1) $x_i \mapsto \frac{1}{x_i}$
2) $t \mapsto xyzt$

$$T(z) = \Delta \frac{(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y)}{\underbrace{(1 - zxy(1/x + x/y + y))}_{H_1}(1-x)(1-y)}$$

Critical points?

There are 4 strata, and we determine critical points from each of them.

- Strata:

  $S_1 = \{(x,y) : H_1(x,y) = 0, x \neq 1, y \neq 1\} \to (1,1,1/3)(w, w^2, w^2/3), (w^2, w, w/3)$

  $S_2 = \{(1,y) : H_1(1,y) = 0, y \neq 1\} \to (1,1,1/3)$

  $S_3 = \{(x,1) : H_1(x,1) = 0, x \neq 1\} \to (1,1,1/3)$

  $S_4 = \{(1,1)\} \to (1,1,1/3)$

$$(x,y)^{(n,m)} = (x^n y^m)$$

Only $(1,1,1/3)$ turns out to affect dominant asymptotics: with a contribution of $3^n \cdot n^{-3/2} \cdot \frac{3\sqrt{3}}{4\sqrt{\pi}} + O(3^n \cdot n^{-5/2})$ for both summands.

Theorem 10.3.3 (complete intersection) *Let $F = G/\prod_{j=1}^d H_j^{m_j}$ in $\mathbf{R}_z$ with each H_j squarefree and all divisors intersecting transversely at z. Suppose that G is holomorphic in a neighborhood of z and $G(z) \neq 0$. Then*

$$\frac{1}{(2\pi i)^d} \int_T z^{-r-1} F(z)\, dz \sim \Phi_z(r)$$

with

$$\Phi_z(r) := \frac{1}{(m-1)!} \frac{z^{-r} G(z)}{\det \Gamma_\Psi(z)} (r \Gamma_\Psi^{-1})^{m-1}. \qquad (10.3.3)$$

use (n,n) for r

The remainder term is of a lower exponential order, $\exp[|r|(\hat{r} \cdot \log z - \varepsilon)]$, uniformly as $\hat{r}$ varies over compact subsets of the interior of $\mathbf{N}(z)$.

Γ_Ψ is the matrix whose rows are the logarithmic gradients $\nabla_{\log H_j(z)}$.

$$\Gamma_\Psi = [x_i \frac{\partial H_j}{\partial x_j}]_{i,j}$$

TORIN'S OBSERVATION

Recall: Generating function for walks that end on an axis:

$$\Delta \frac{(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y)}{(1 - zxy(1/x + x/y + y))(1-x)}$$

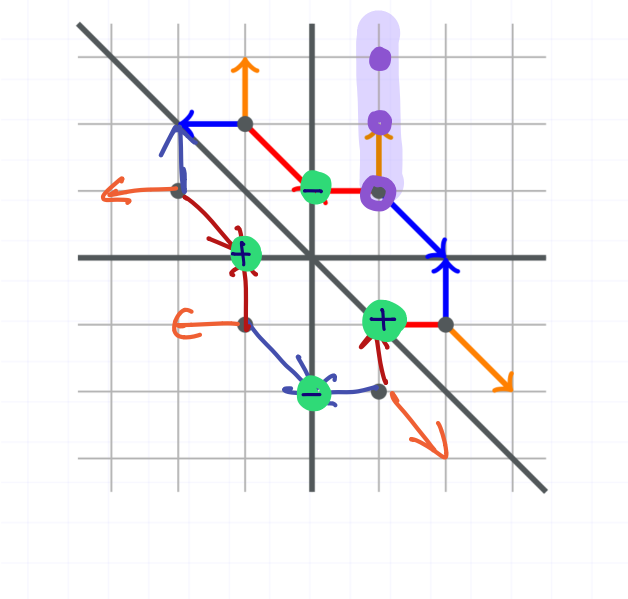
Remark, as $x^2 - y = (x-1)(x+1) - (y-1)$

look like GFs for walks that end on axis

$$\Delta \frac{(x^2 - y)(1 - \overline{x}\overline{y})(x - y^2)}{(1-x)(1-y)(1 - xyt(\overline{y} + y\overline{x} + x))} = -\frac{(1 - \overline{x}\overline{y})(x - y^2)(x+1)}{(1-y)(1 - xyt(\overline{y} + y\overline{x} + x))} + \frac{(1 - \overline{x}\overline{y})(x - y^2)}{(1-x)(1 - xyt(\overline{y} + y\overline{x} + x))}$$

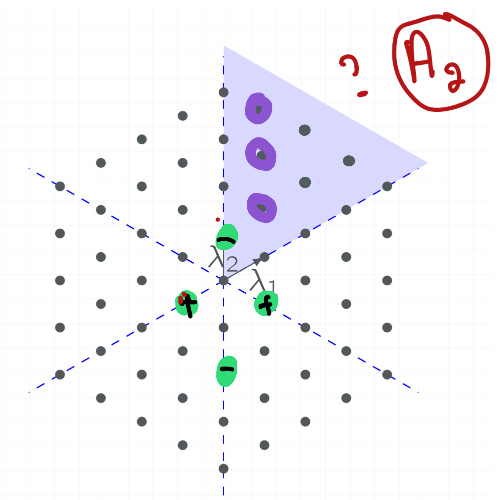
$= \Delta$

$$\Delta (\ldots) = [x^{=1} y] \frac{(1 - xy)(\overline{x} - y^2)}{1 - t(y + \overline{y}x + \overline{x})} \to \overline{x} - \overline{y}^2 - y + \frac{x}{y}$$



① expand terms of

- $\dfrac{\bar{x}}{1-t(y+\bar{y}x+\bar{x})}$
- $-\dfrac{y^2}{1-t(y+\bar{y}x+\bar{x})}$
- ...



If we expand these series, can we observe cancellation? Can we interpret this combinatorially (e.g. via a sign reversing involution?)

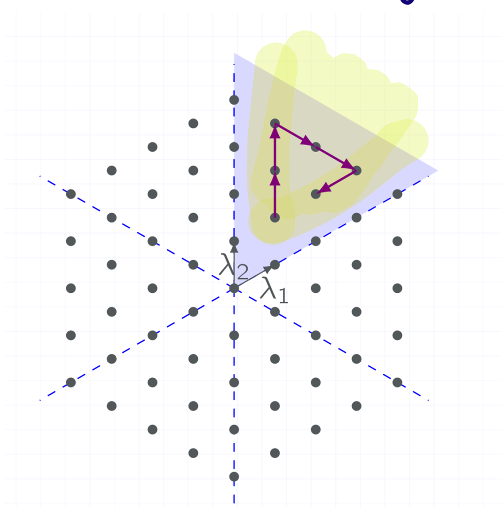
When we do this with the other generating function

$$-\frac{(1-\overline{xy})(x-y^2)(x+1)}{(1-y)(1-xyt(\bar{y}+y\bar{x}+x))}$$

what happens?

(Where are the "green points" etc.)

Finally put it together to count walks that stay in yellow zone.



Marni's guess: this is an example of a known phenomenon with walks. You reduce the possible end points by a dimension, and you double the enclosed region and there is a bijective relation to the original problem.

eg $\{\nearrow, \searrow\}$-walks: # above or on axis ~~ending~~ at any height

= # above or below axis ending at height 0.