LATTICE WALK CLASSIFICATION

ALGEBRAIC, ANALYTIC, AND GEOMETRIC PERSPECTIVES

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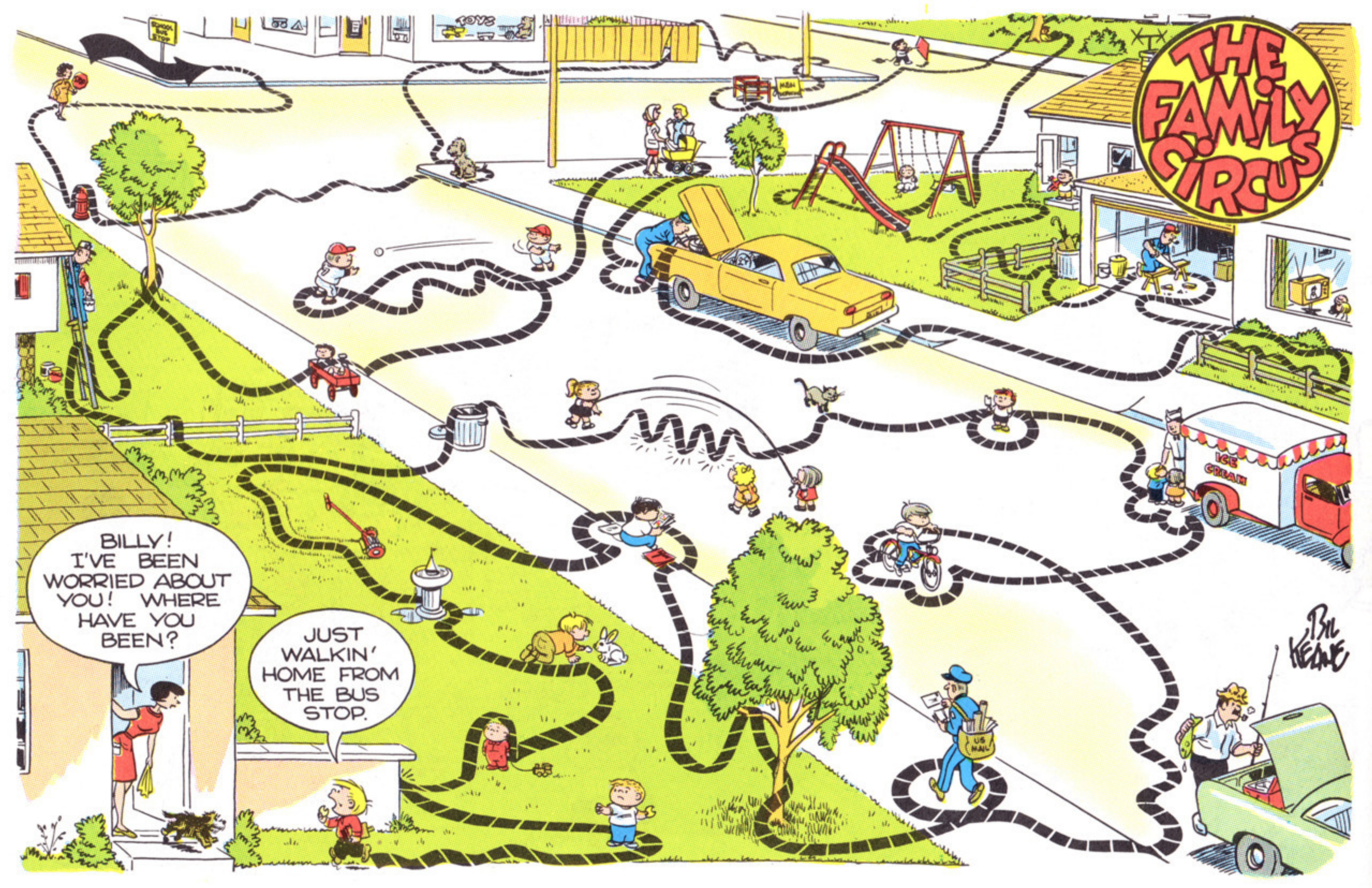
SFU

1. LATTICE PATHS

A WALK IS A SEQUENCE OF STEPS



Recall we consider fixed, finite sets of possible steps.

+ constraints.

+ non-uniform weighting/ probability for the steps.

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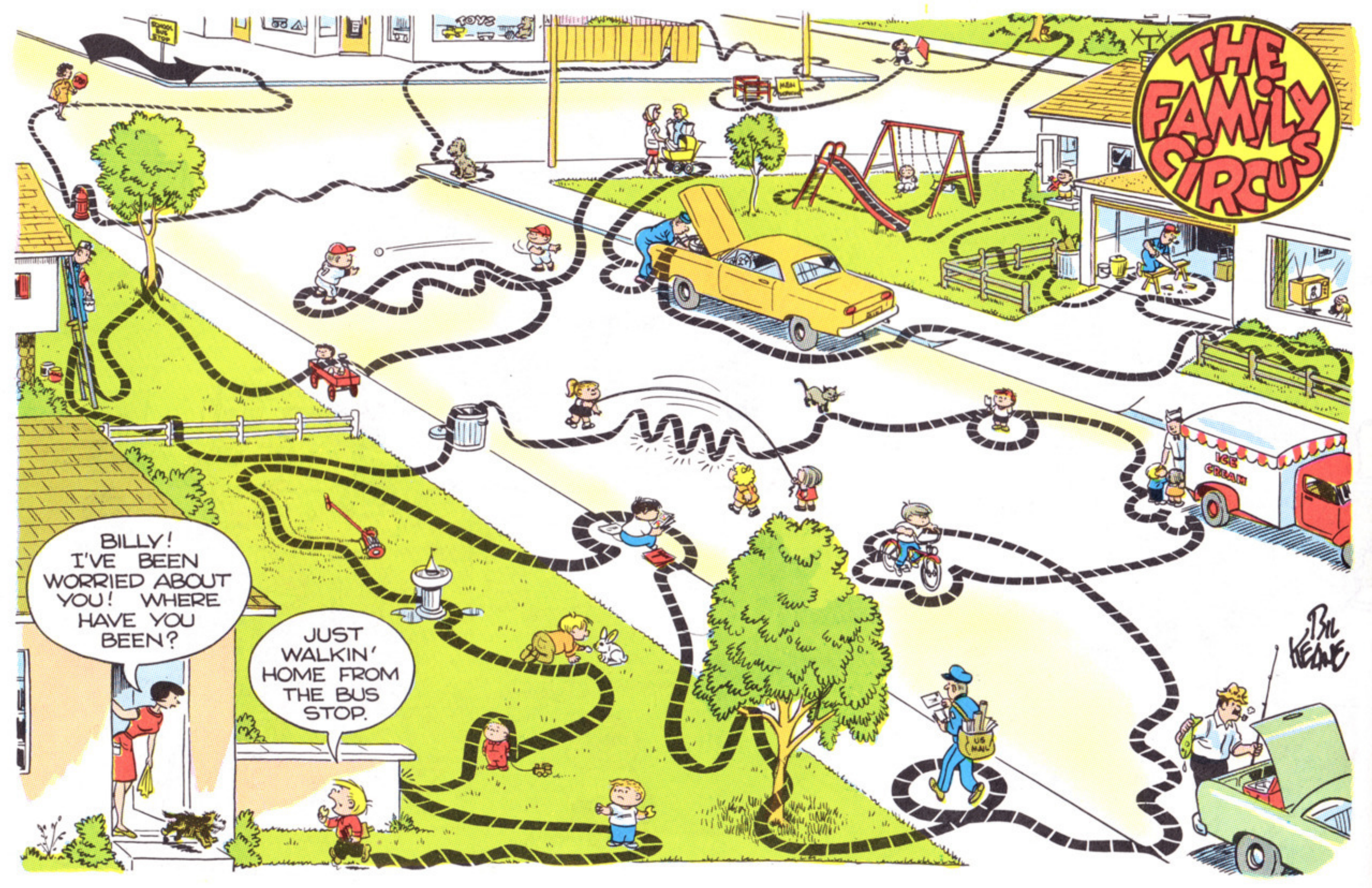

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STRATEGY:
ENCODE WALKS WITH MONOMIALS

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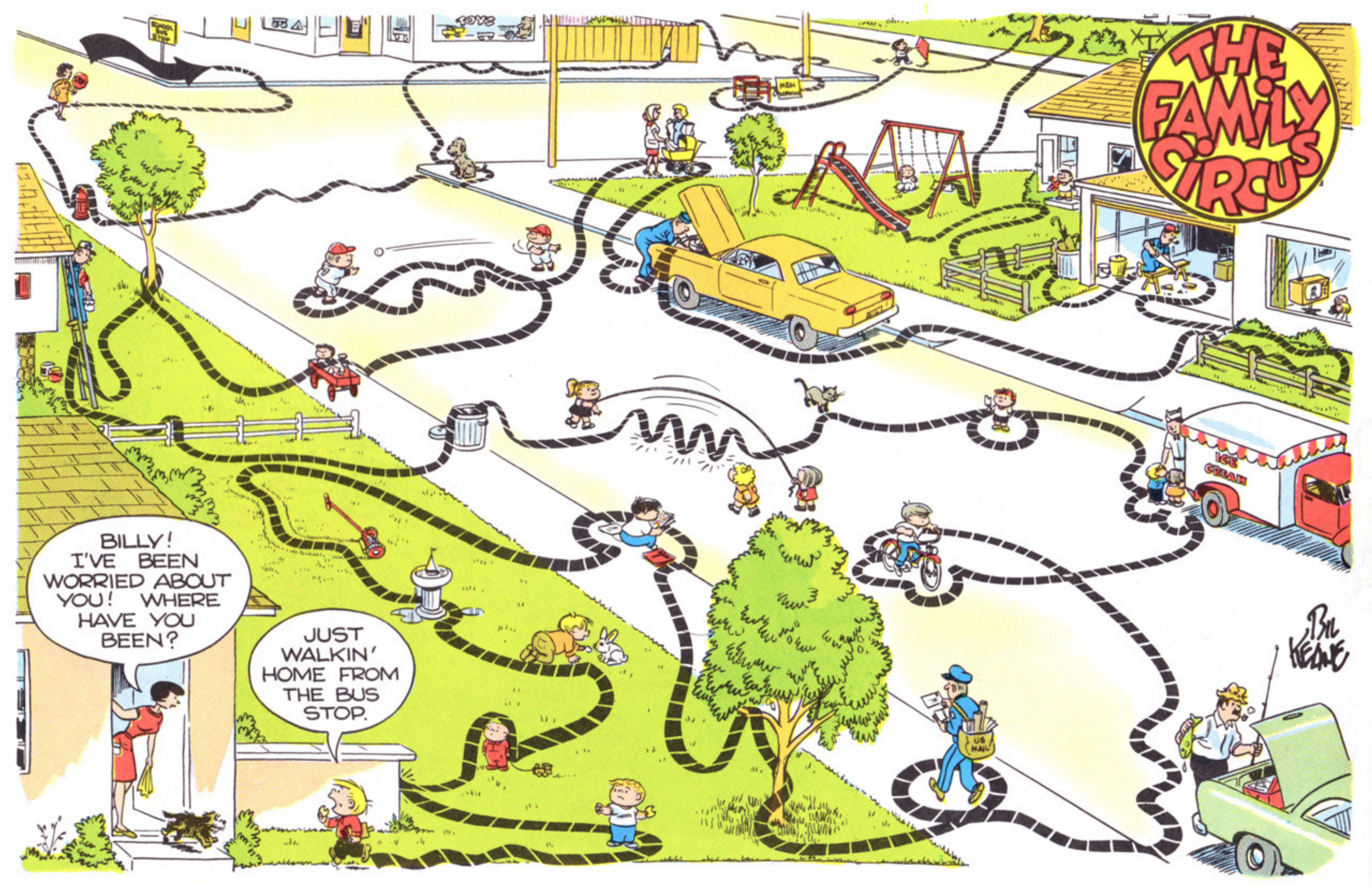

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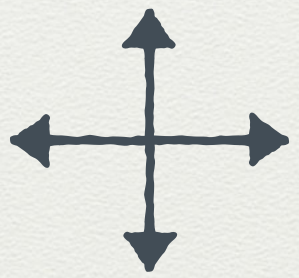

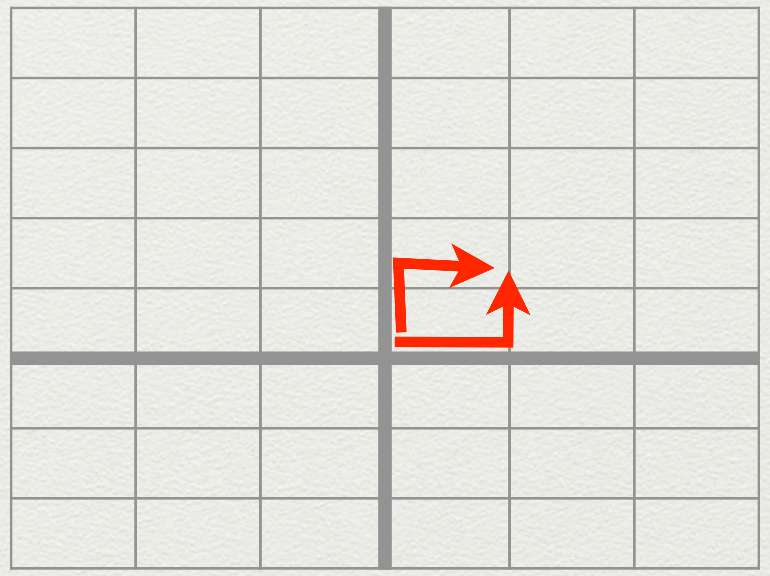

$$(x + 1/x + y + 1/y) \times (x + 1/x + y + 1/y)$$

$$= x^2 + 1/x^2 + y^2 + 1/y^2 + 4 + 2xy + 2x/y + 2y/x + 2/xy$$

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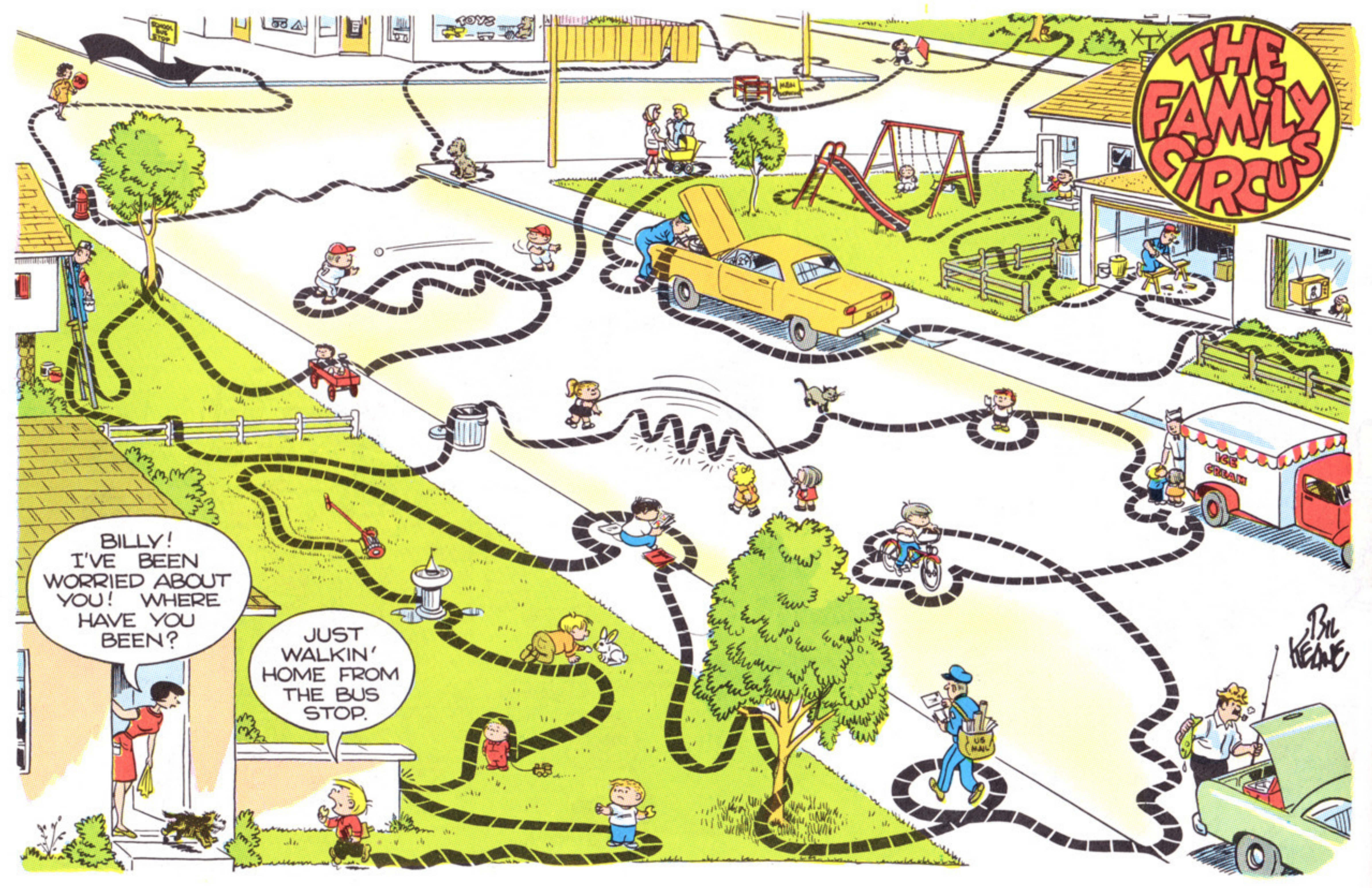

$$\frac{1}{1-z\left(x+1/x+y+1/y\right)}$$

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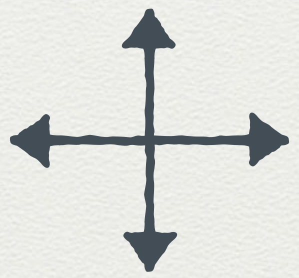

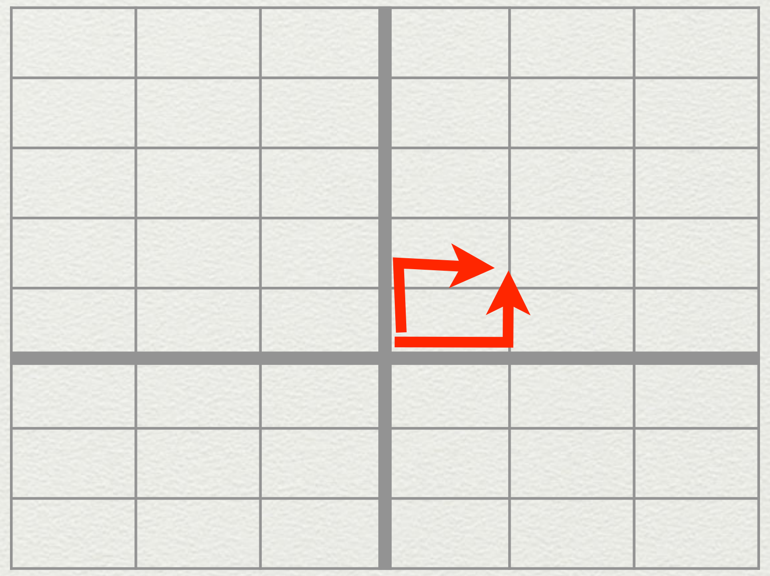

$\rightarrow \quad \leftarrow \quad \uparrow \quad \downarrow$

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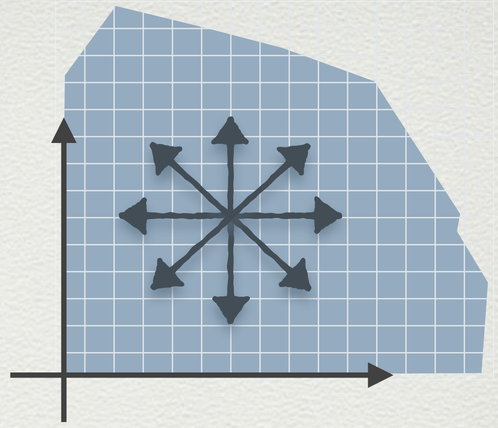
$$= x^2+1/x^2+y^2+1/y^2+4+2xy+2x/y+2y/x+2/xy$$

Case Study: Small Step Walks in the Quarter Plane



- Consider a set of vectors in $\mathcal{S} \subseteq \{(i,j) \mid i,j \in \{0,1,-1\}\} \backslash \{(0,0)\}$.

  A walk stays in the quarter plane if each cumulative sum of steps is in $\mathbb{N}^2$

- Of interest:

  $\text{walks}\,(0,0) \xrightarrow{n} (0,0)$:= #Walks of length n that start and end at (0,0)

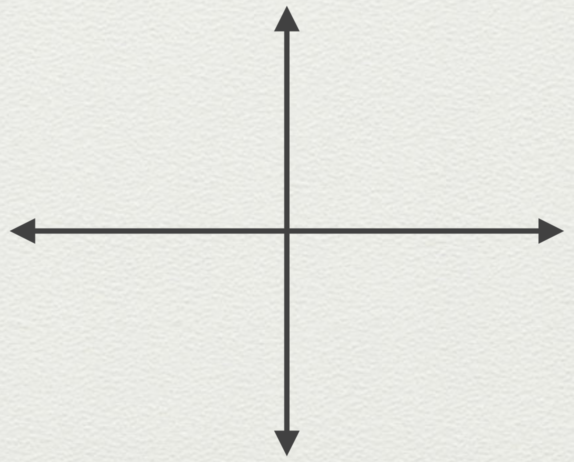
  $\text{walks}\,(0,0) \xrightarrow{n} (i,j)$:= #Walks of length n at start at (0,0) and end at (i,j)

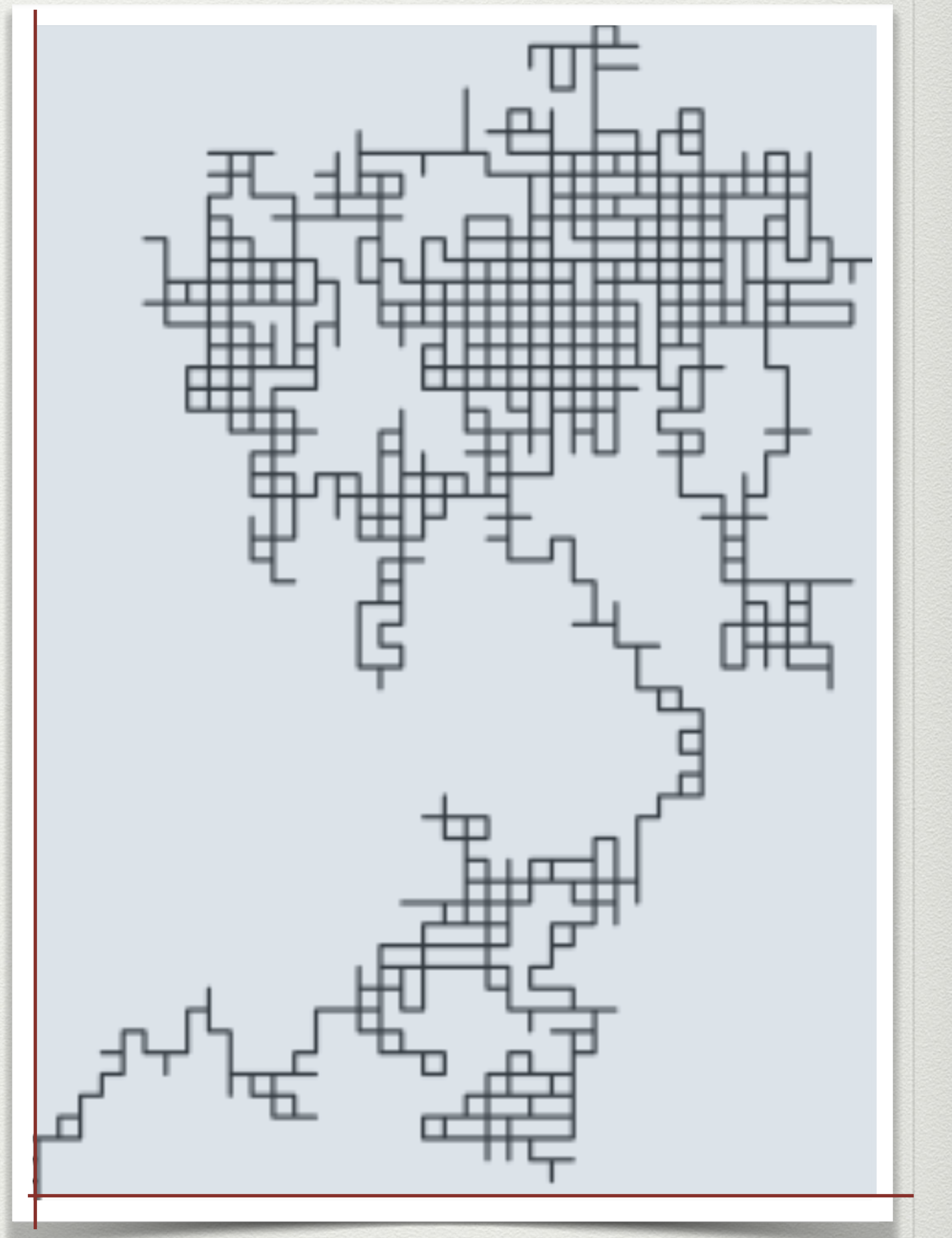
  Total number of walks of length n

  Is #walks to the point (i,j) >0?

  Example: The Simple Walks

  $\mathcal{S} = \{(1,0), (-1,0), (0,1), (0,-1)\}$ ={N, E, S, W}





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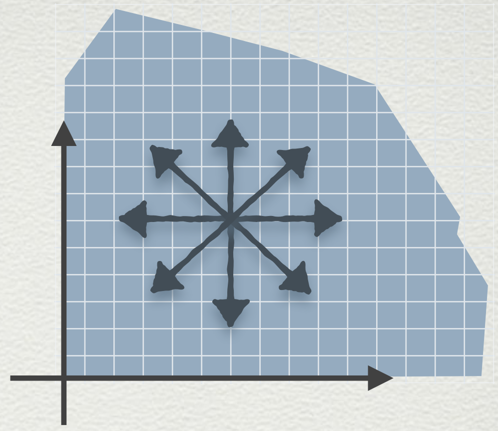
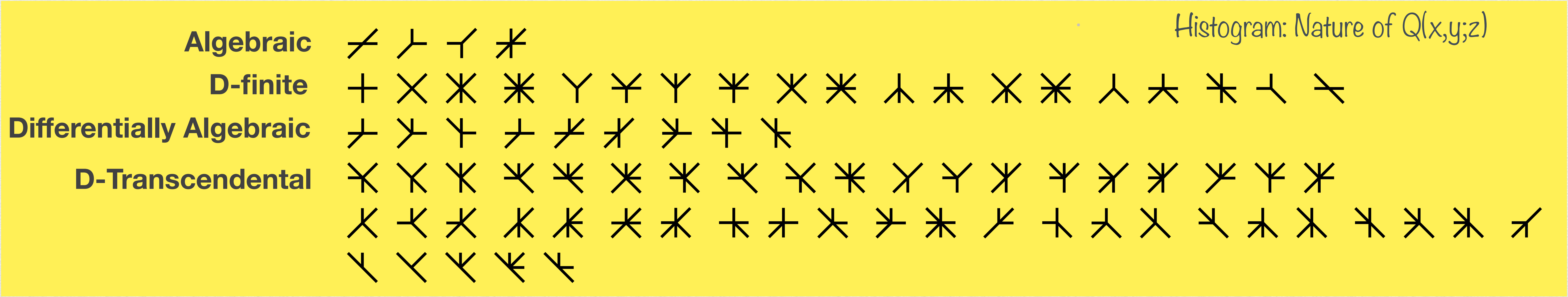


- $\mathcal{S} \subseteq \{(i,j) \mid i,j \in \{0,1,-1\}\} \backslash \{(0,0)\}$
- Generating function marking endpoint and length:

$$Q(x,y;z) := \sum_{n\geq 0} \sum_{(i,j)\in\mathbb{N}^2} \text{walks } (0,0) \xrightarrow{n} (i,j)\, x^i y^j z^n \qquad \in \mathbb{N}[x,y][[z]]$$

- Classify Q(x,y;z). Classify Q(1,1; z).

REMARKABLE MULTI-FACETED EFFORT SINCE 2004



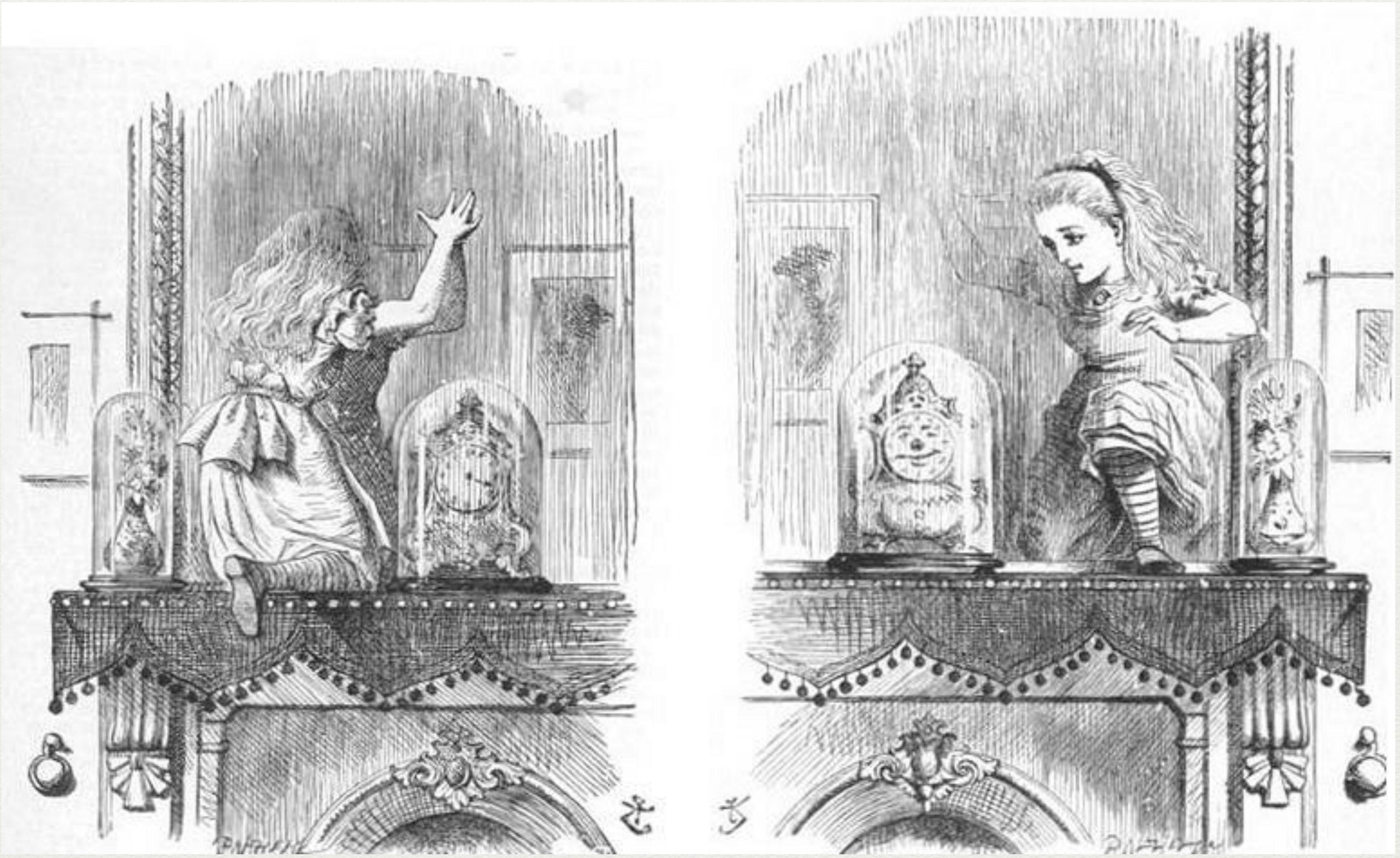
CLASSIFICATION VIA ENUMERATION

LATTICE WALKS

bounding region

symmetry of step sets

underlying lattice



FORMAL SERIES

ANALYTIC FUNCTIONS

Rational functions

Algebraic functions

Solutions of linear ODEs

Solutions of non-linear ODEs

THERE IS A TWO WAY FLOW OF INFORMATION

TODAY: WALKS WITH D-FINITE GENERATING FUNCTIONS

Transcendance established via asymptotics

Today: walks with D-finite generating functions

- D-finite functions: algebraic, $e^{(\text{polynomial}(z))}$, not e^{e^z}

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- D-finite functions: algebraic, $e^{(\text{polynomial}(z))}$, not e^{e^z}
- Diagonals of multivariate rational functions a common form in the realm of walks. Today we explore a core combinatorial source.

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- D-finite functions are a signal of structure.

  "Almost anything is non-holonomic unless it is holonomic by design" - Flajolet Gerhold Salvy

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  "Almost anything is non-holonomic unless it is holonomic by design" - Flajolet Gerhold Salvy
- Next goal: Combinatorial understanding of..

  - Models with algebraic GF that aren't defined by an algebraic grammar; Transcendance established via asymptotics
  - Models with non-D-finite generating function that are differentiably algebraic.

DIAGONAL EXTRACTIONS

- Taylor Series Expansion $\frac{1}{1-x-y-z} = \sum \binom{i+j+k}{i\, j\, k} x^i y^j z^k$

- Consider the central diagonal $F(z) = \text{Diag}\,\frac{1}{1-x-y-z} = \text{Diag} \sum \binom{i+j+k}{i, j, k} x^i y^j z^k = \sum \binom{3n}{n, n, n} z^n$

- In this case, we can compute asymptotics $\binom{3n}{n\, n\, n} \sim \kappa\, 27^n n^{-1}$ hence F(z) is transcendental

- The diagonal of a multivariate rational function is D-finite. We can explicitly construct algebraic functions as diagonals.

- Related operations:
  CT (constant term) $CT_{x,y}(x+3+y+y^2)t = 3t$ **Positive extraction** $[x^{>0}y^{>0}](x+2/x+3+4y^2)t\Big|_{x=y=1} = 5t$

- **Longstanding open question:** Do all combinatorial D-finite functions arise this way? (Modulo some suitable conditions...)

II. WALKS IN ALGEBRA

WALKS ARE NATURAL IN ALGEBRAIC CONTEXTS

Walks are sequences of steps

Many algebraic objects are products of generators

We can answer questions about the **number of ways** something can be expressed as a product of generators…

… with constraints on intermediary products

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WHEN IS A PRODUCT OF N GENERATORS = 1?

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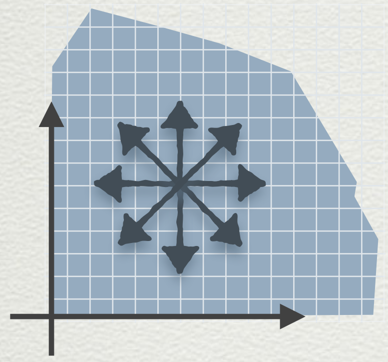
WHEN IS A PRODUCT OF N GENERATORS = 1?

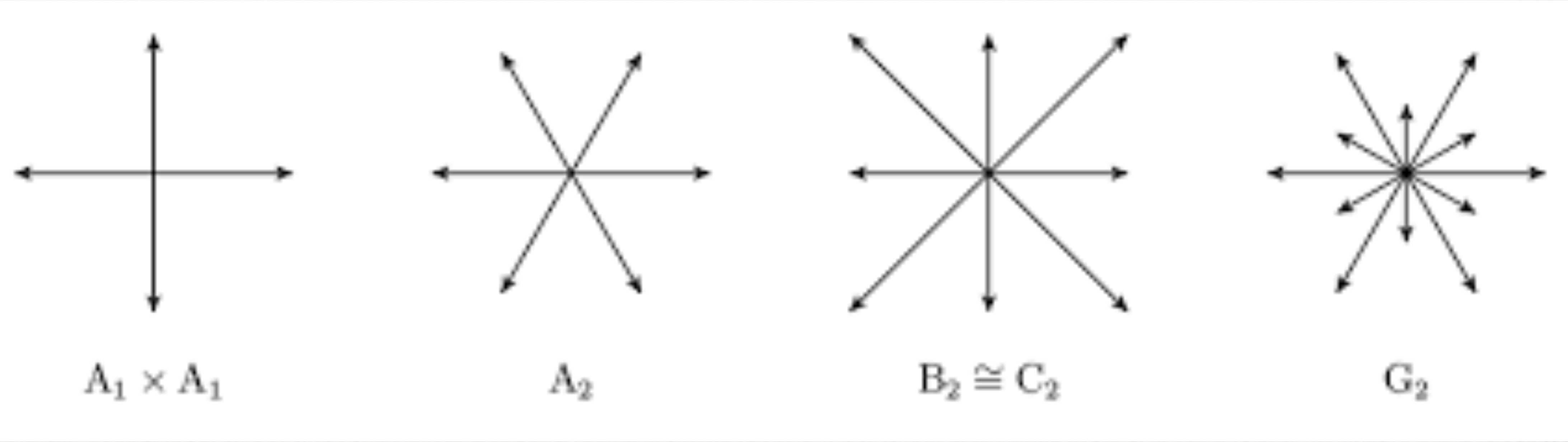
HOW MANY DISTINCT WAYS TO DECOMPOSE A PRODUCT INTO IRREDUCIBLES?

ROOT SYSTEMS

- A root system R is a configuration of vectors in a Euclidean space satisfying certain geometrical properties:
    - If you consider the line perpendicular to any root, say β, then reflection in that line sends any other root, say α, to another root. Moreover, the root to which it is sent equals $\alpha + n\beta$, where n is an integer.
- They are classified and are central to Lie Algebra.
- Use them to define a lattice, with a primary Weyl chamber. We can count walks on this lattice that are restricted to the Weyl chamber. The reflections form a group.



- A1xA1: models symmetric across both axes
  A2: "Tandem" and "Double Tandem" models
  B2: Gouyou-Beauchamps



Rank 2 systems

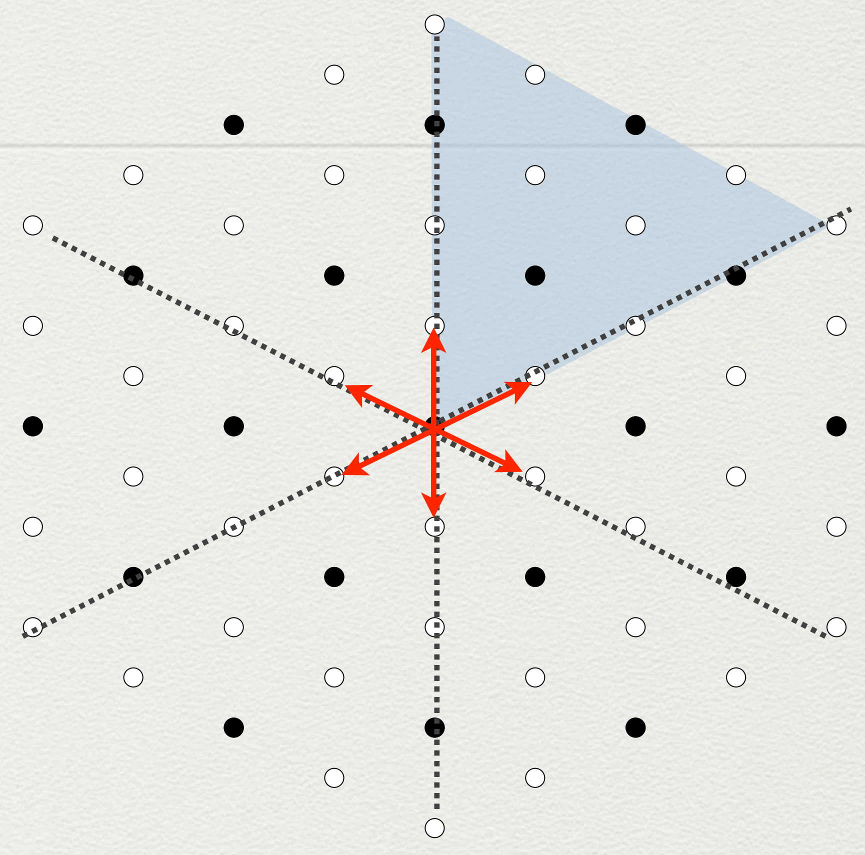
REFLECTABLE WALKS RESTRICTED TO A WEYL CHAMBER.

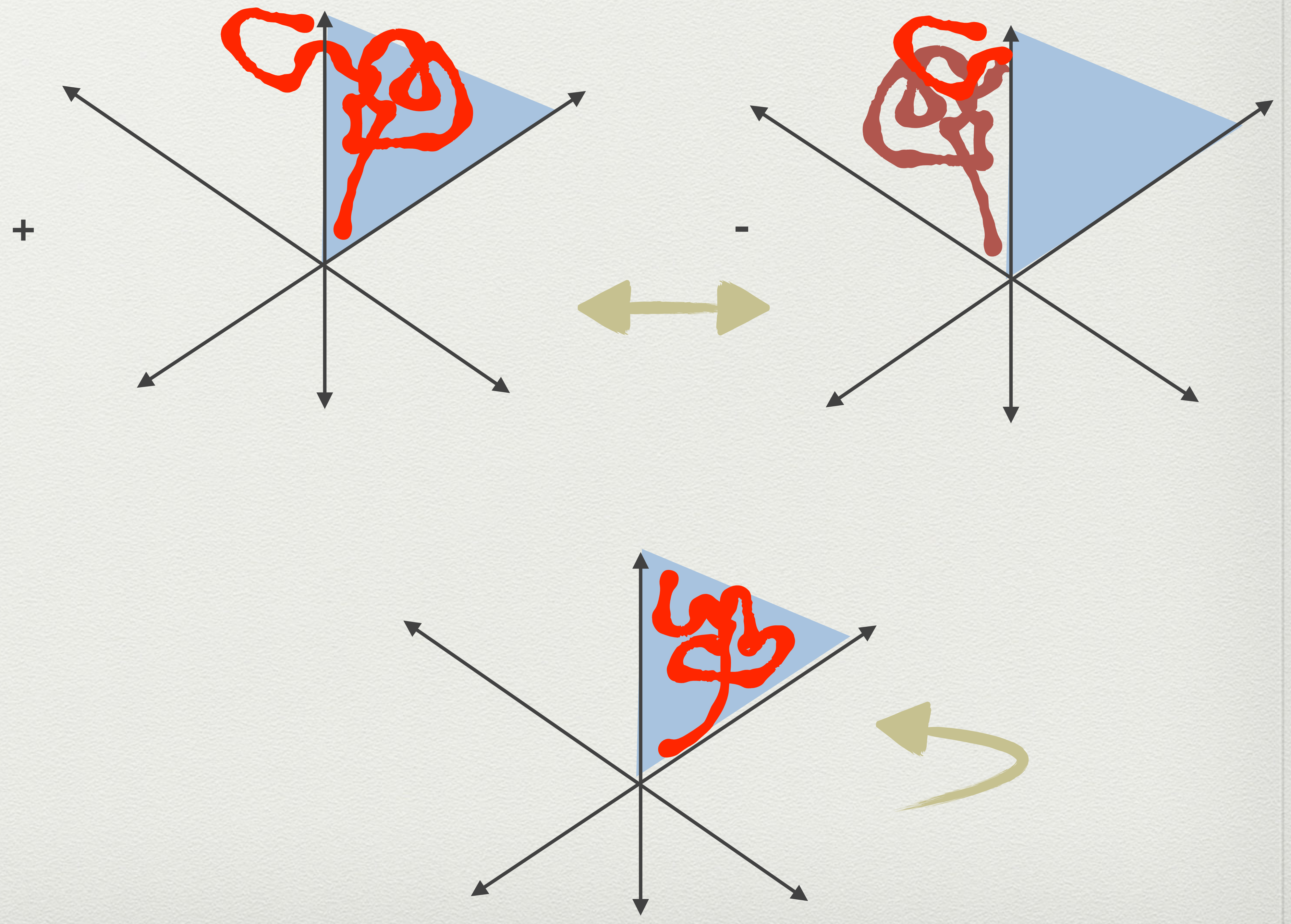
- Consider a step sets stable fixed by the reflection group, and steps that **do not** permit to **jump over boundaries**

- **Reflection principle** ~ A sign reversing involution.

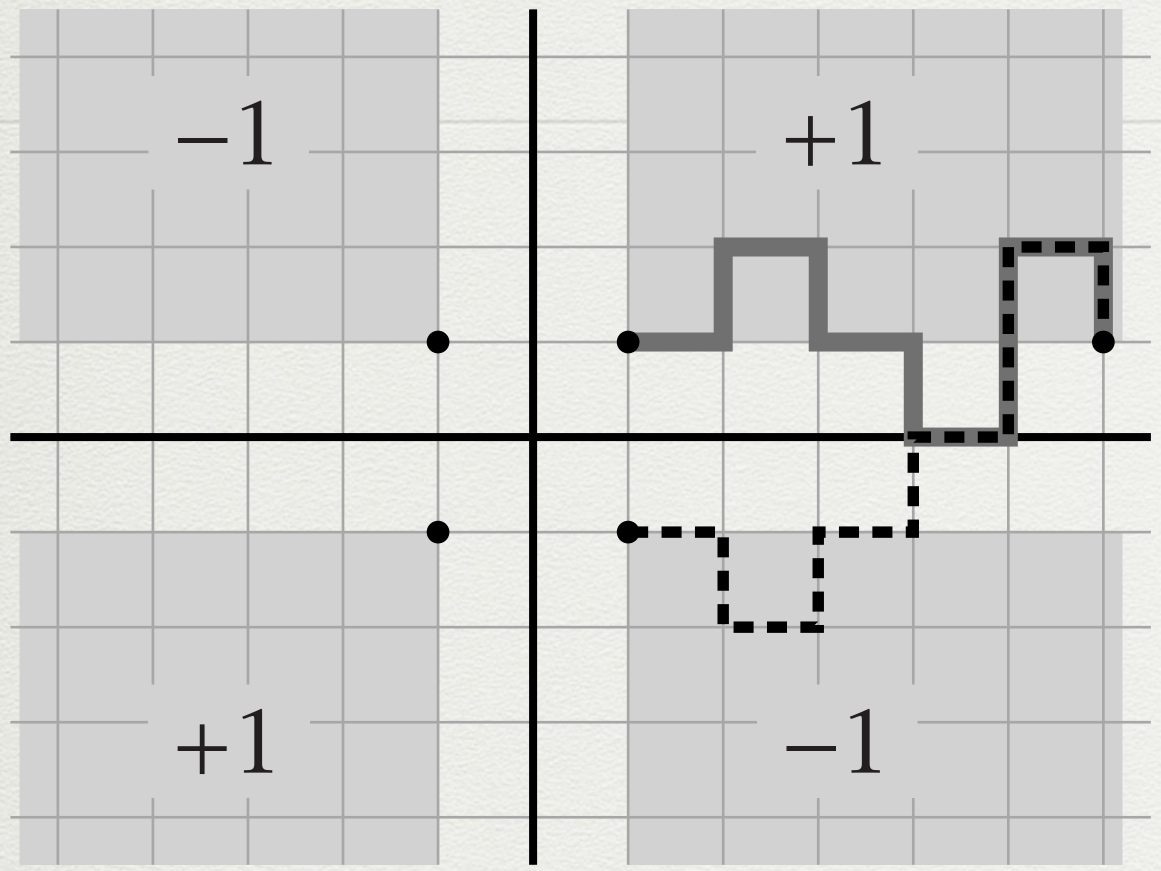
- Two walks are paired using the first time they hit a wall. Walks that never hit a wall are fixed by the map.

- Underlying: Weyl character formula





Simple walks in the first quadrant



Sum of walks that stay in their chamber = Signed sum of all walks starting at (1,1) and it image under the reflections.

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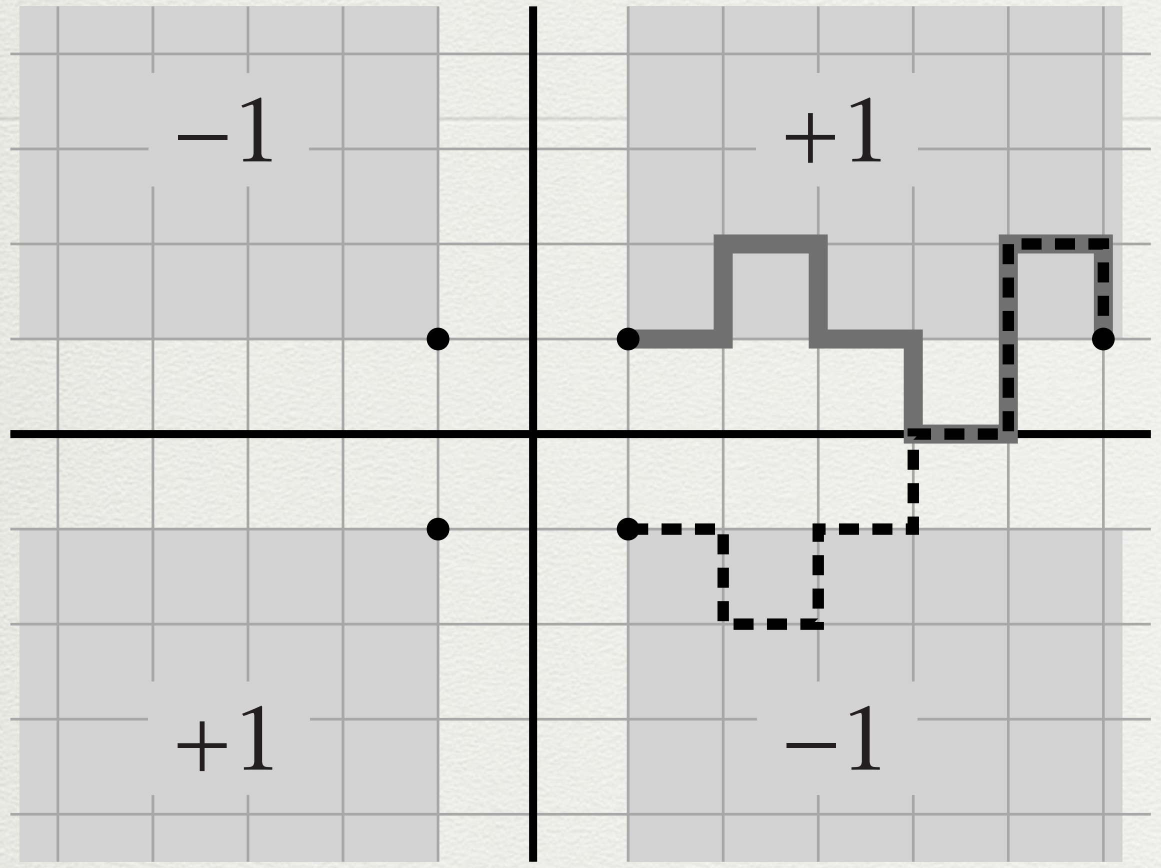

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$$\sum_{n\geq 0} \text{walk}_{\mathbb{N}^2}((0,0) \xrightarrow{n} (0,0))\, z^n$$

$$= [x^1y^1]\frac{xy - x/y + (xy)^{-1} + y/x}{(1 - z(x + 1/x + y + 1/y))}$$

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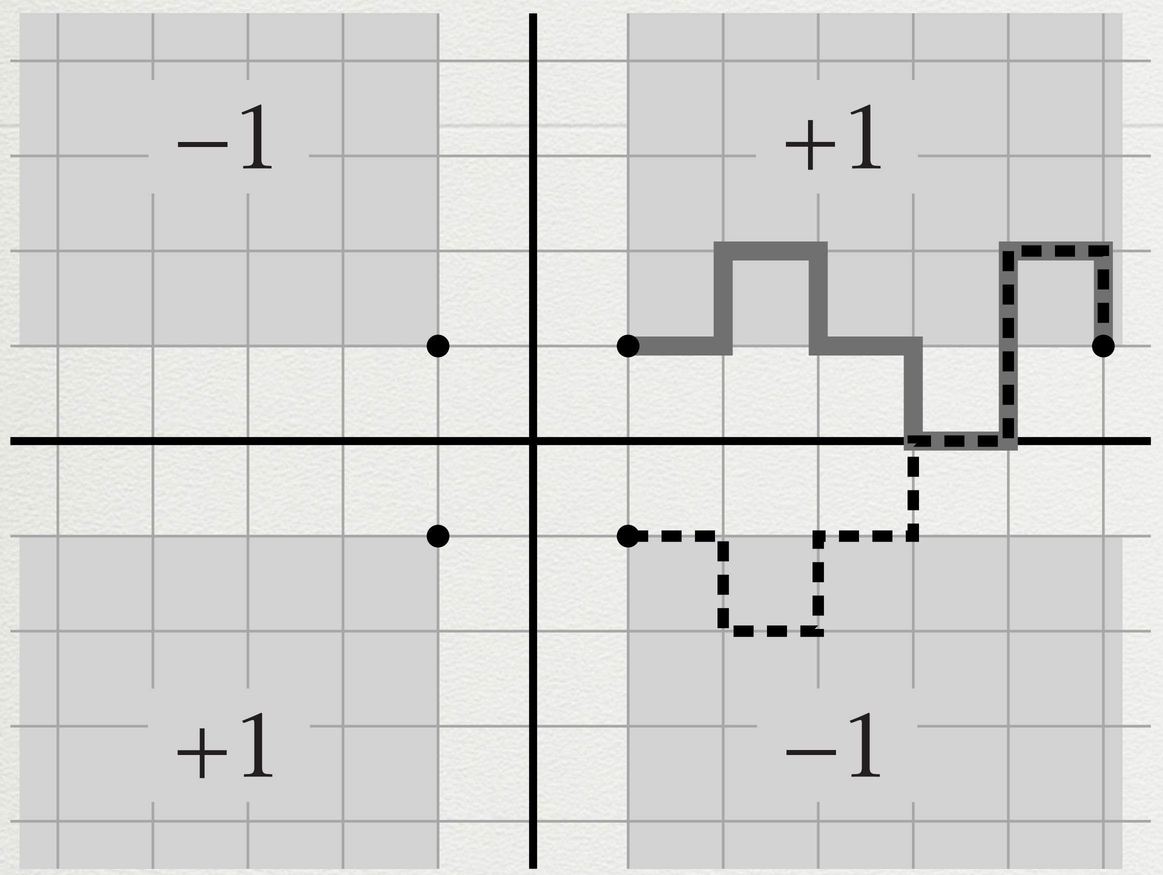

walks that end at a given point ...

$$\text{Diag}\ \frac{(x^2-1)(y^2-1)}{1-z(x^2y+y+xy^2+x)}$$

walks that end on an axis ...

$$\text{Diag}\ \frac{(x^2-1)(y^2-1)}{1-z(x^2y+y+xy^2+x)}(1-x)$$

and walks that end anywhere.

$$\text{Diag}\ \frac{(x^2-1)(y^2-1)}{1-z(x^2y+y+xy^2+x)}(1-x)(1-y)$$

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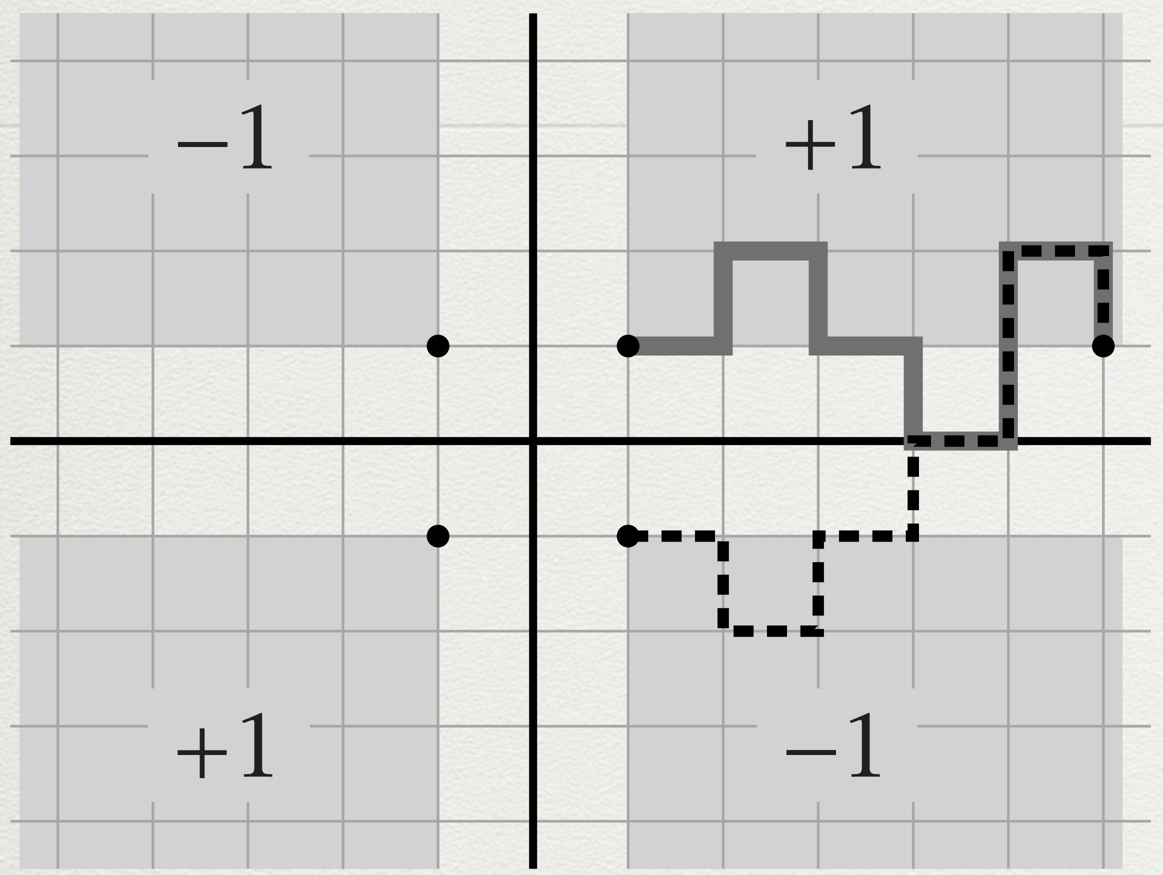
SIMPLE WALKS IN THE FIRST QUADRANT



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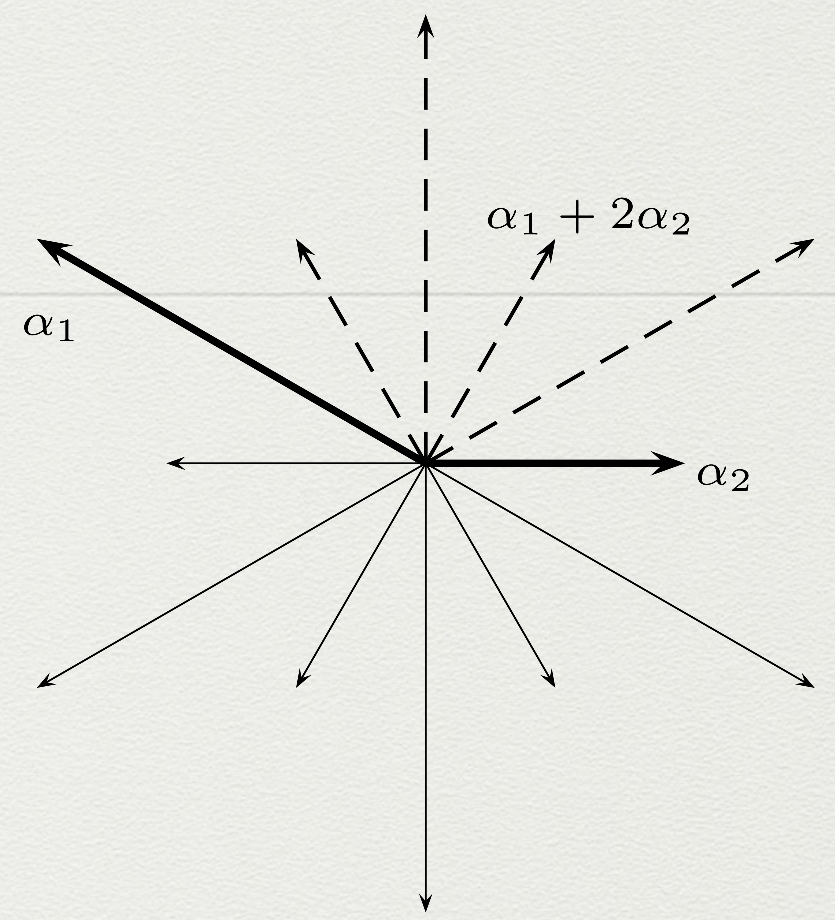
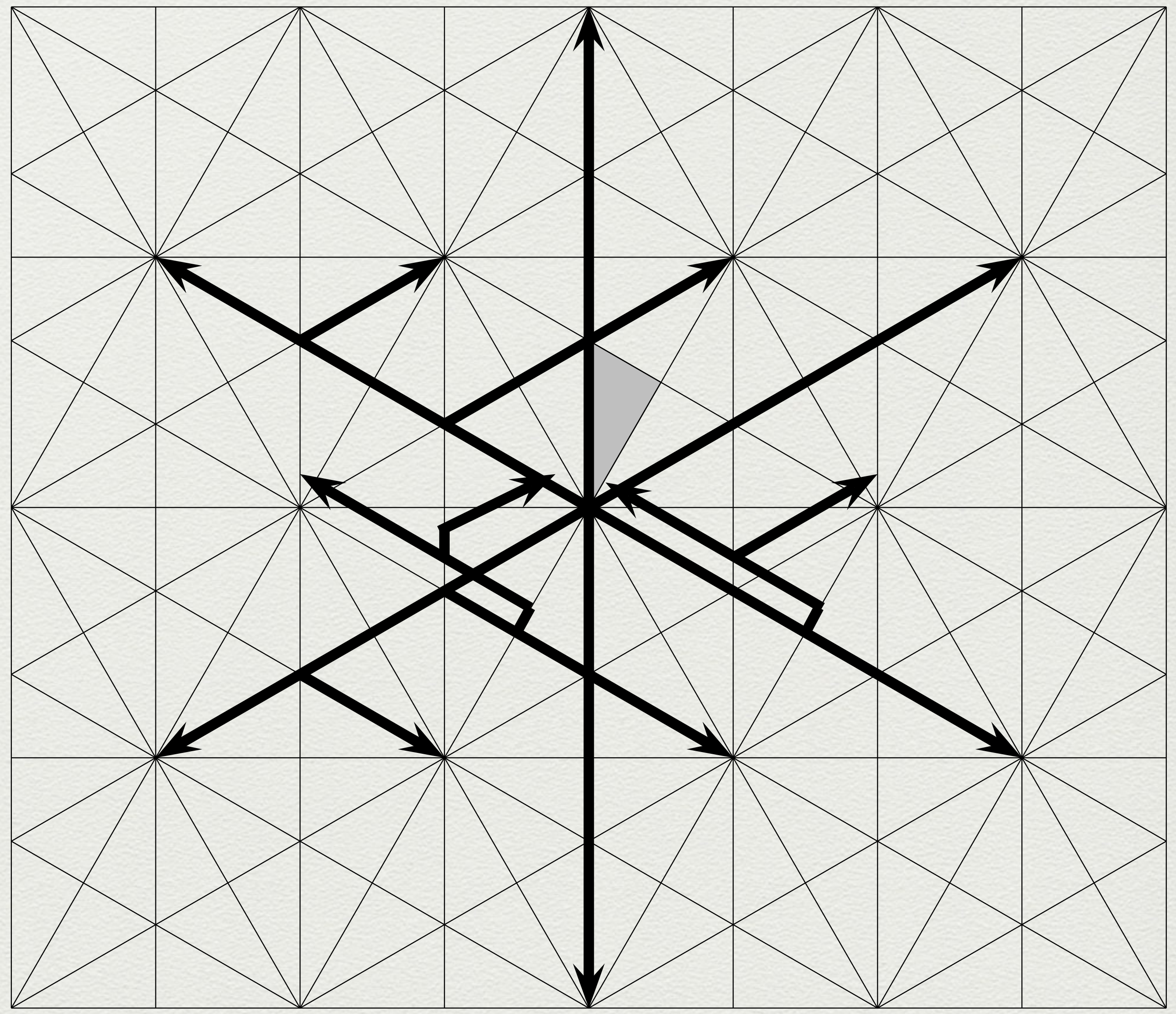
WHAT CAN WE DO WITH THESE EXPRESSIONS?

An unusual family: G2



- Root system of type G2 give rise to interesting walks.
- Plane is tiled into 12 chambers
- Each step is built of mini-steps. At no point can it leave the Weyl chamber.
- We can find a generating function expression to count excursions, walks that end on a boundary, and walks that end anywhere.
- By similar machinery, GFs are extractions of rational functions.
- What "big steps" that are manageable can look like.



Images from Admissible subsets and Littelmann paths in affine Kazhdan-Lusztig theory Jérémie Guilhot

Reflections and Diagonals

- Other reflection arguments are possible when we consider a winding plane.
- Reflectable/Algebraic walks: The GFs are expressible as diagonals of rational functions
- Orbit sum: S={E, SE, W, NW} in the quarter plane

$$Q(1,1,t) = [x^{\geq}y^{\geq}]\frac{(1-\bar{x})(1+\bar{x})(1-\bar{y})\left(1-\bar{x}^2y\right)(1-x\bar{y})(1+x\bar{y})}{1-t(x+\bar{x}+x\bar{y}+\bar{x}y)}\Big|_{x=y=1} \quad (\bar{x}=\frac{1}{x};\bar{y}=\frac{1}{y}).$$

- If we cannot determine an explicit form for Q(1,1;z), can we estimate

$$[z^n]Q(1,1;z) = \sum_{(i,j)\in\mathbb{N}^2} \text{walks}\,(0,0) \xrightarrow{n} (i,j)?$$

III. The Geometry of Reflectable Walks

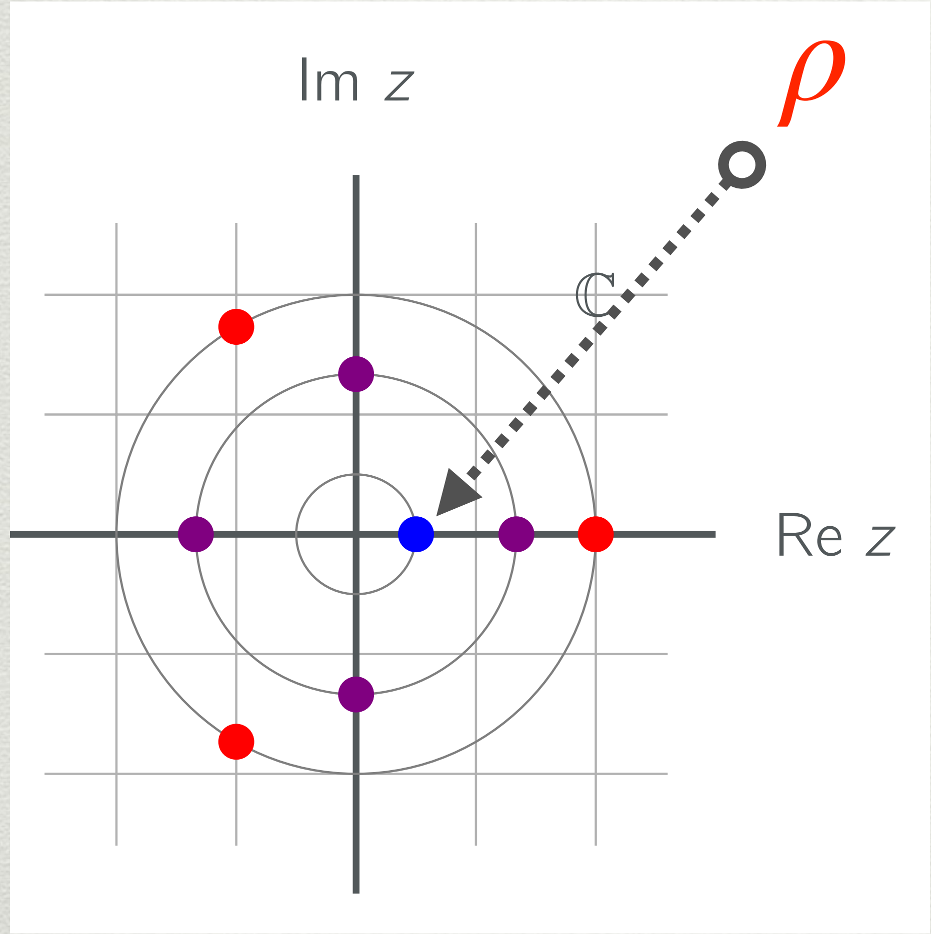
MANTRA OF UNIVARIATE ANALYTIC COMBINATORICS

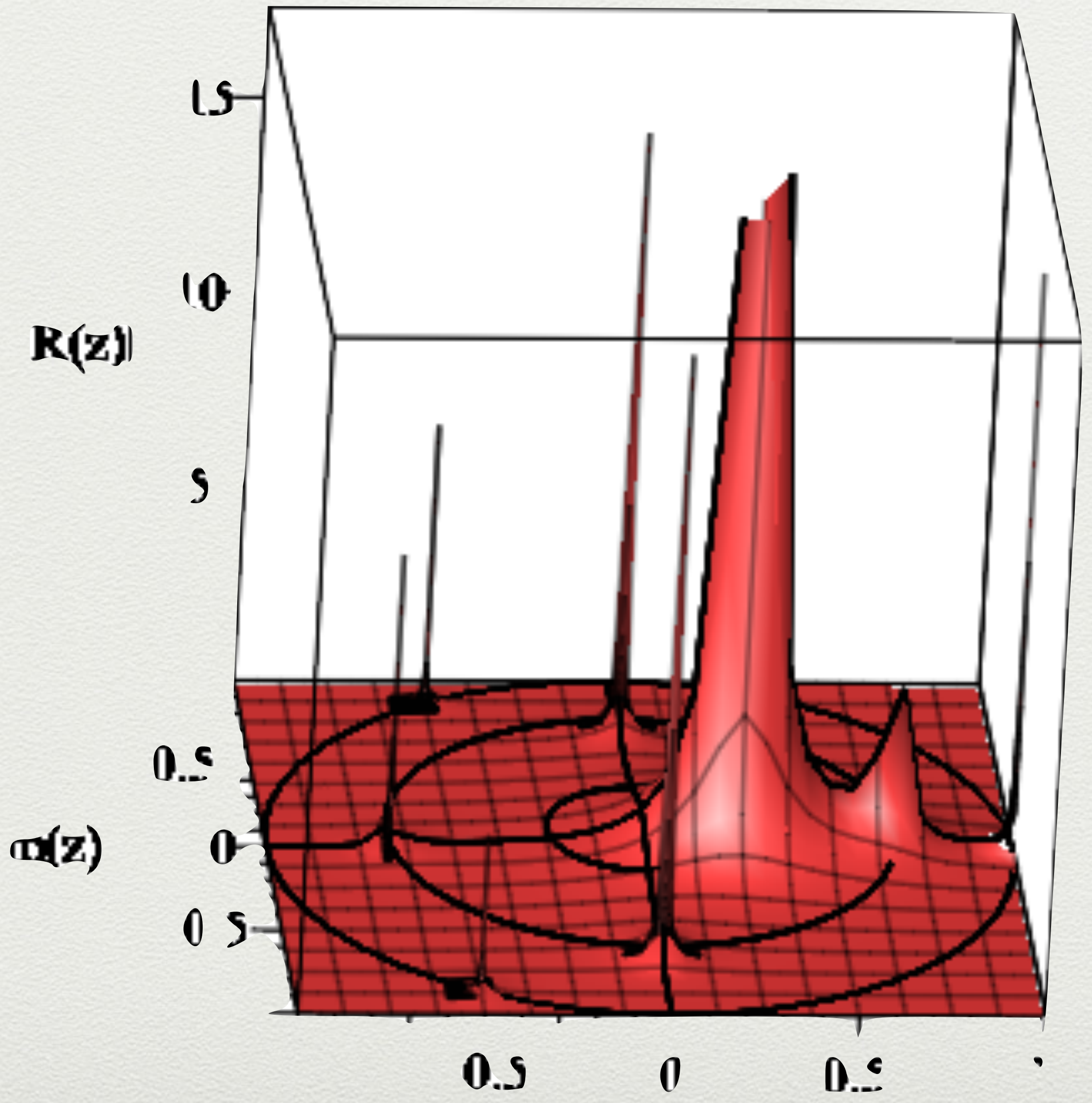
- The **location** of the dominant singularity ρ of C(z) tells you about the **exponential growth** of c(n). (Radius of convergence)

- The **nature** of the dominant singularity (eg. pole of order k, branch cut) of C(z) tells you about the **sub-exponential** growth

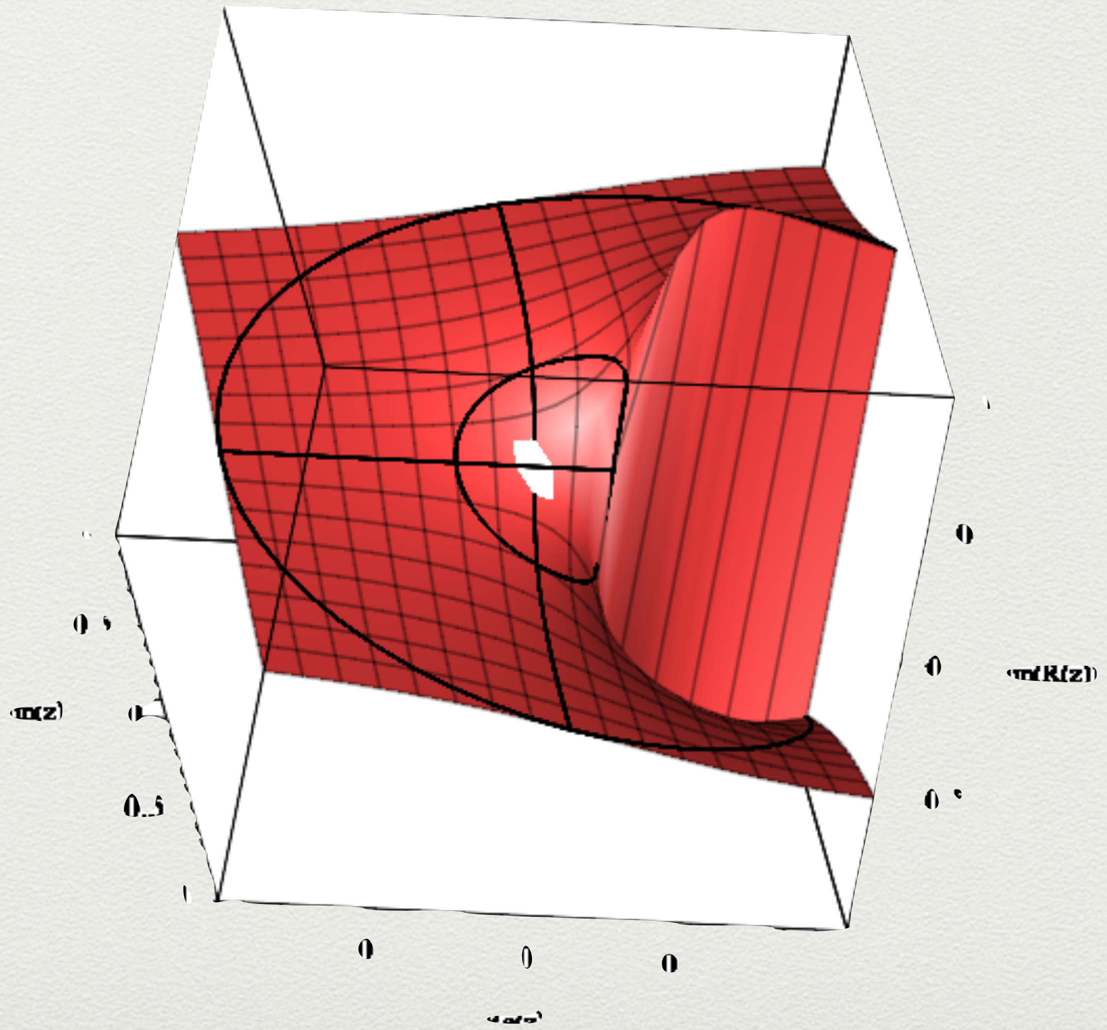
$$C(z) = \sum_{n \in \mathbb{N}} c(n)\, z^n$$

$$c(n) \sim \kappa \rho^{-n} n^{-r}$$





Poles
|F(z)|



Square root singularity
Im(F(z))

Visualizing Domains in Higher Dimension Complex Space

$$\frac{1}{1-x-y} = \sum \binom{k+\ell}{k,\ell} x^k y^\ell$$

The domain of convergence $\mathscr{D}$ is a subset of $\mathbb{C}^2$

Visualizing domains in higher dimension complex space

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The domain of convergence $\mathscr{D}$ is a subset of $\mathbb{C}^2$

- Power series are absolutely convergent!
- Plot points in $\mathbb{R}^2$ where the series is convergent.
- What is the multivariable equivalent of dominant singularity?

VISUALIZING DOMAINS IN HIGHER DIMENSION COMPLEX SPACE

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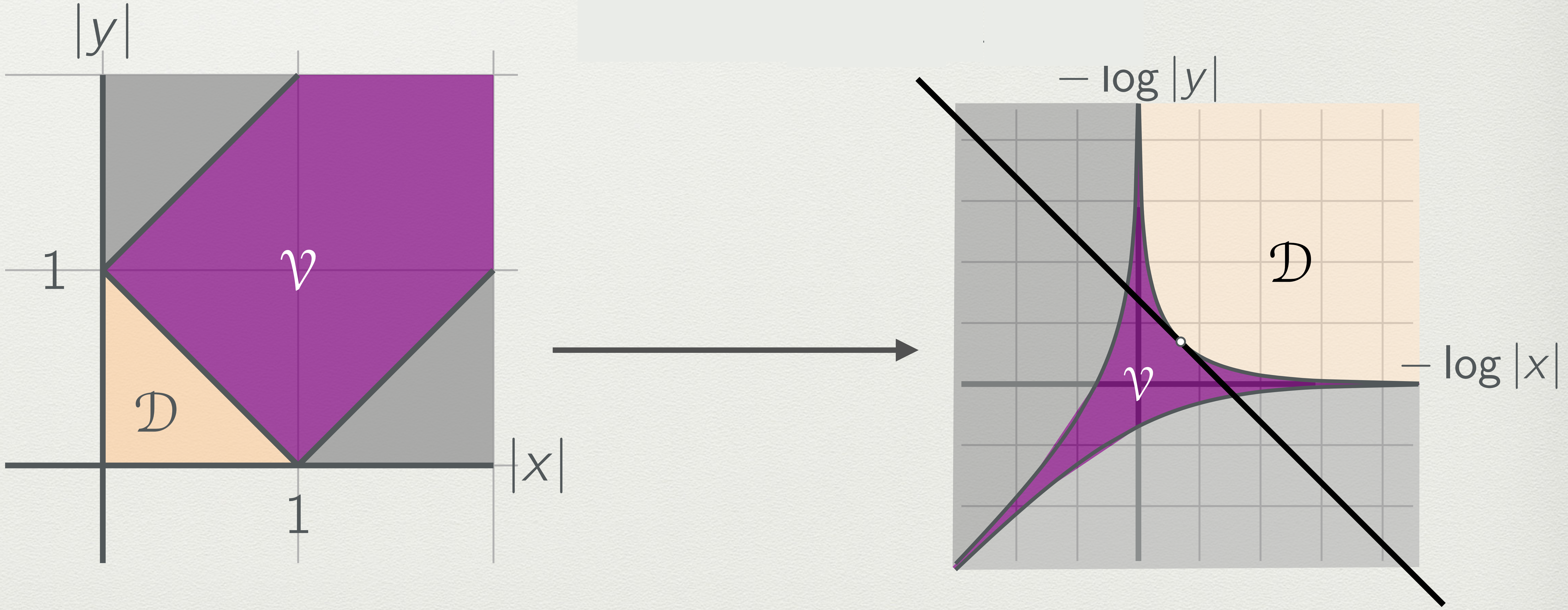


$(x, y) \mapsto (-\log|x|, -\log|y|)$

HIGHER DIMENSIONAL MANTRA

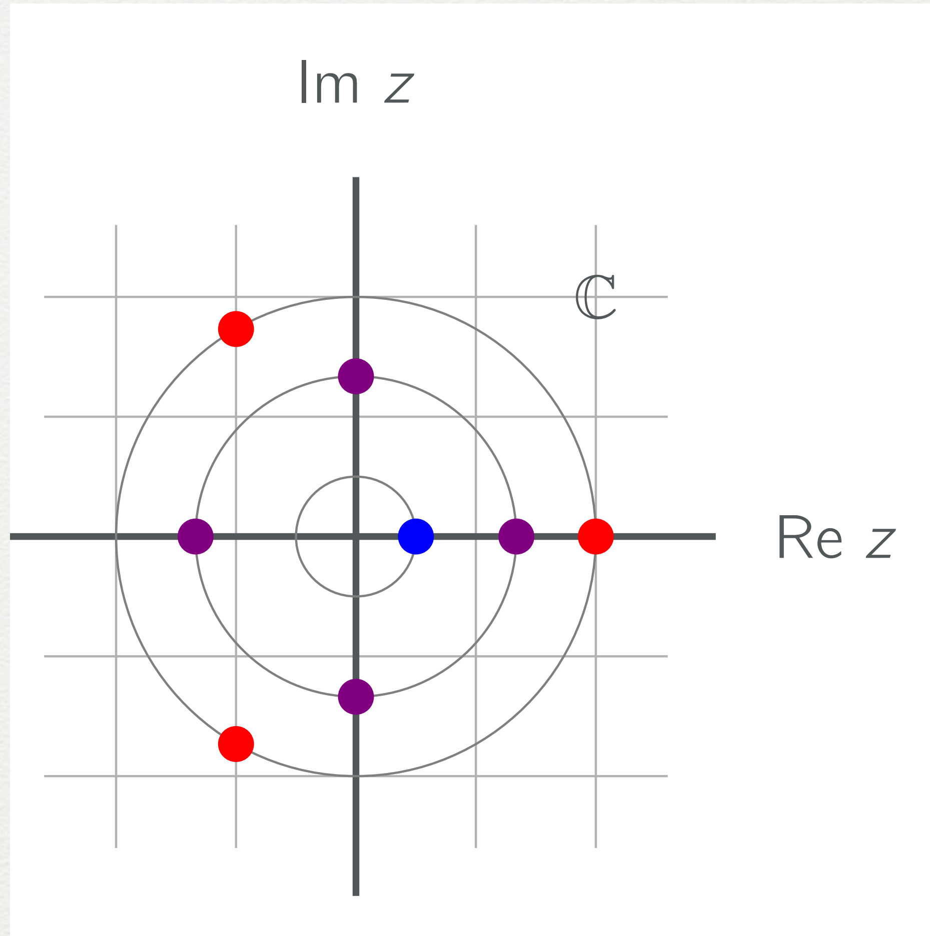
The boundary of convergence of the series (~variety defined by the denominator) identifies **critical points.** Roughly the analogue of **dominant singularity.**

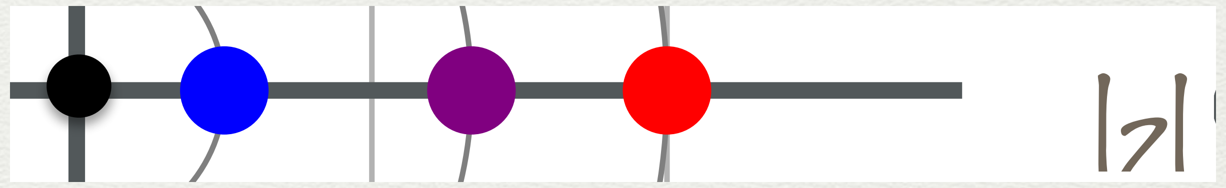
The location of the critical point determines the **exponential growth.**

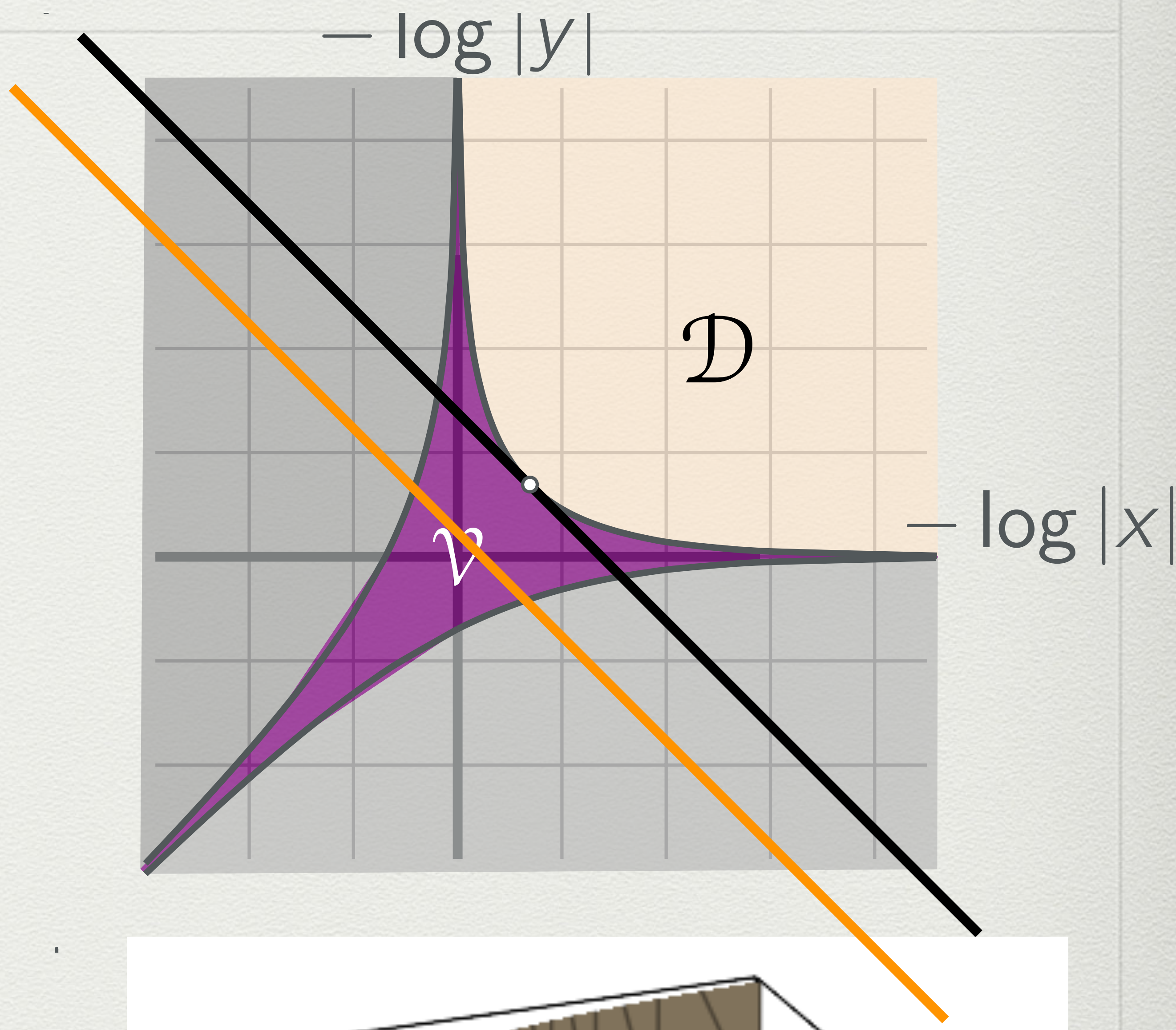
The geometry at that point determines **the sub-exponential growth**. The formulas incorporate the value of the numerator at the critical point.

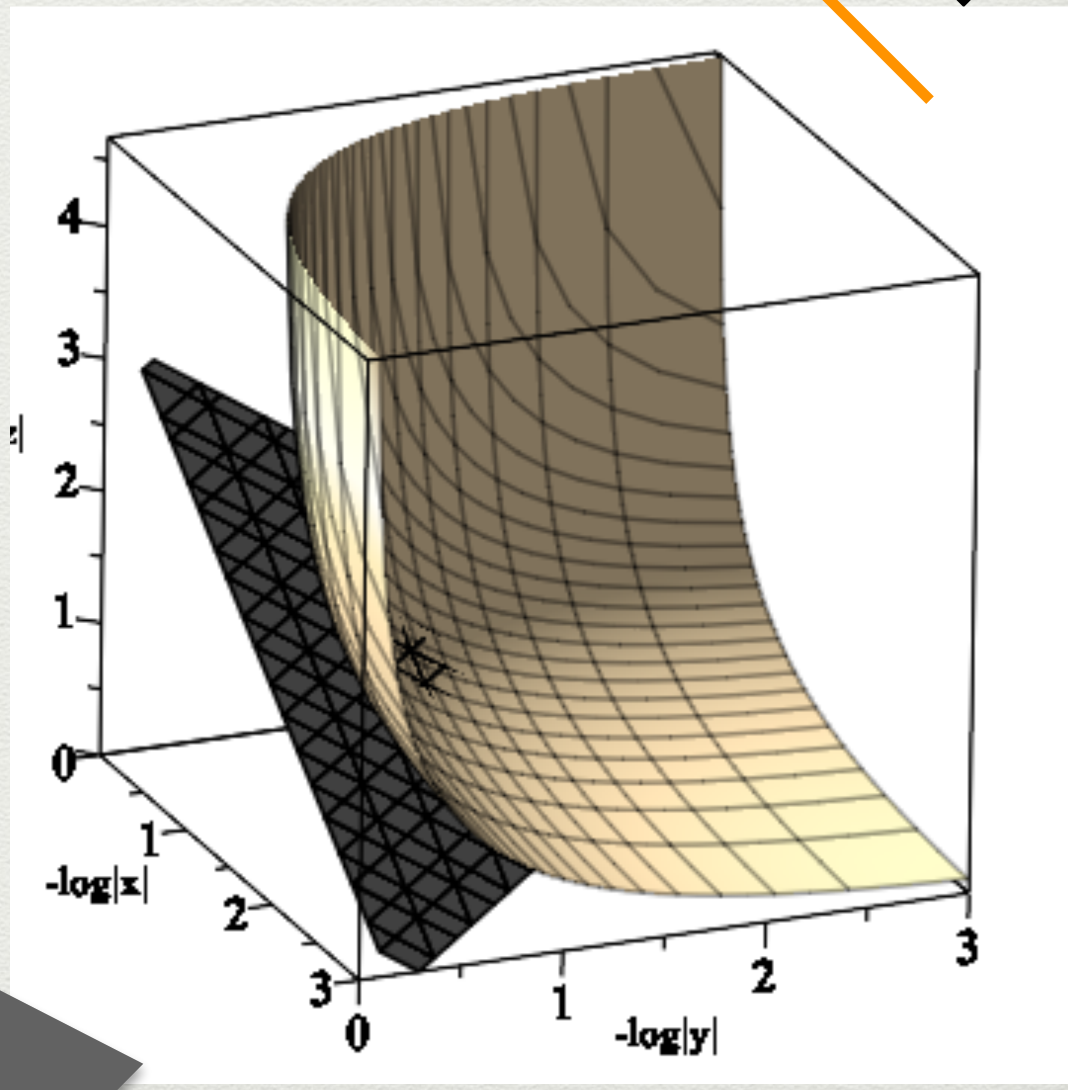
The formulas relevant for walks give sub-exponential growth exponent in $\frac{1}{2}\mathbb{N}$

(links transcendence to the order of the zero in the denominator)

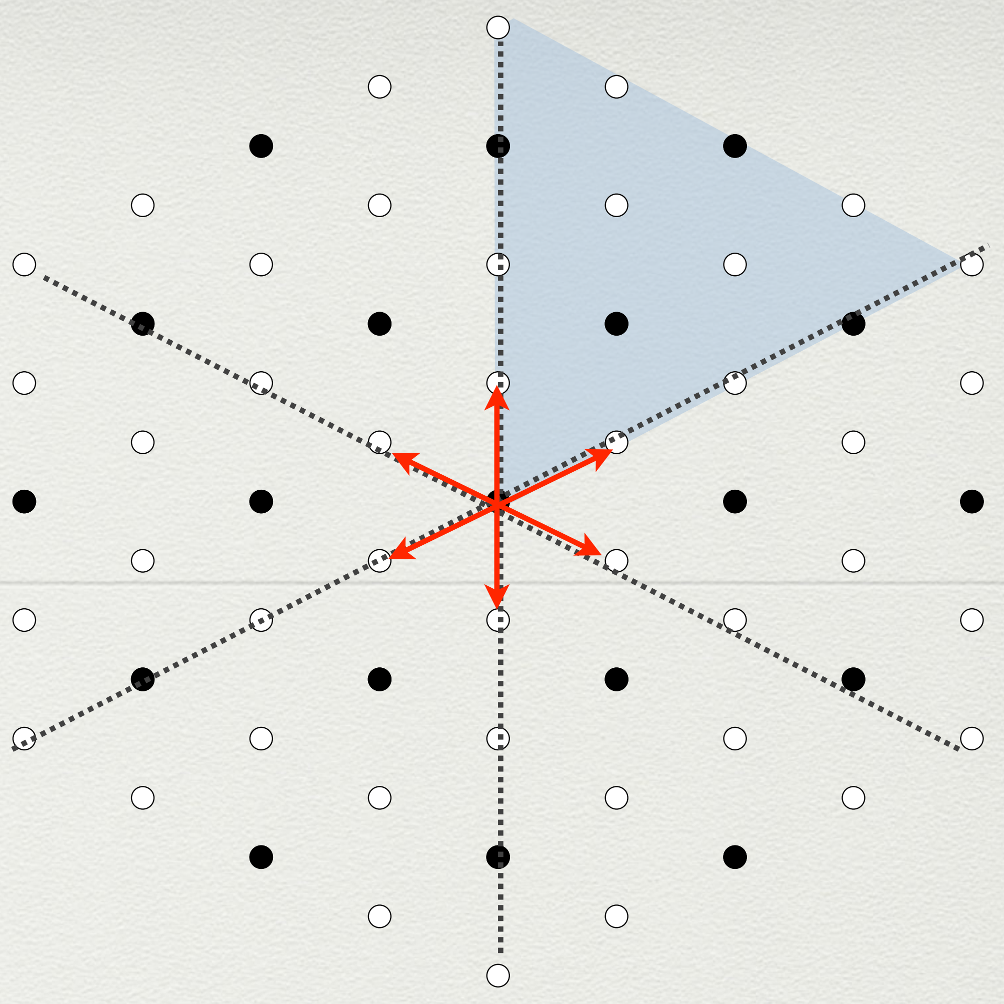






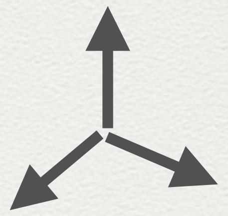


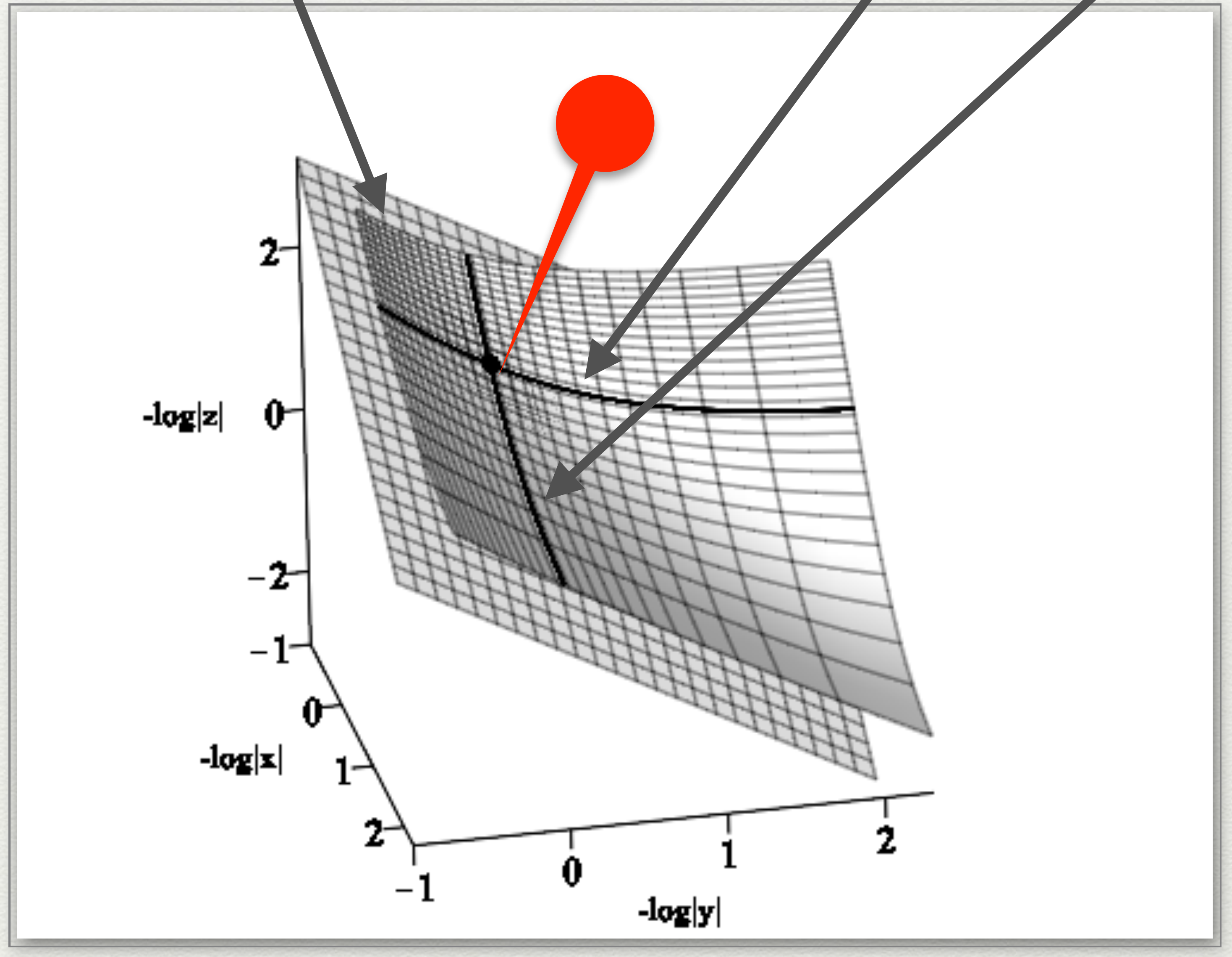
Tandem model / Walk in A_2

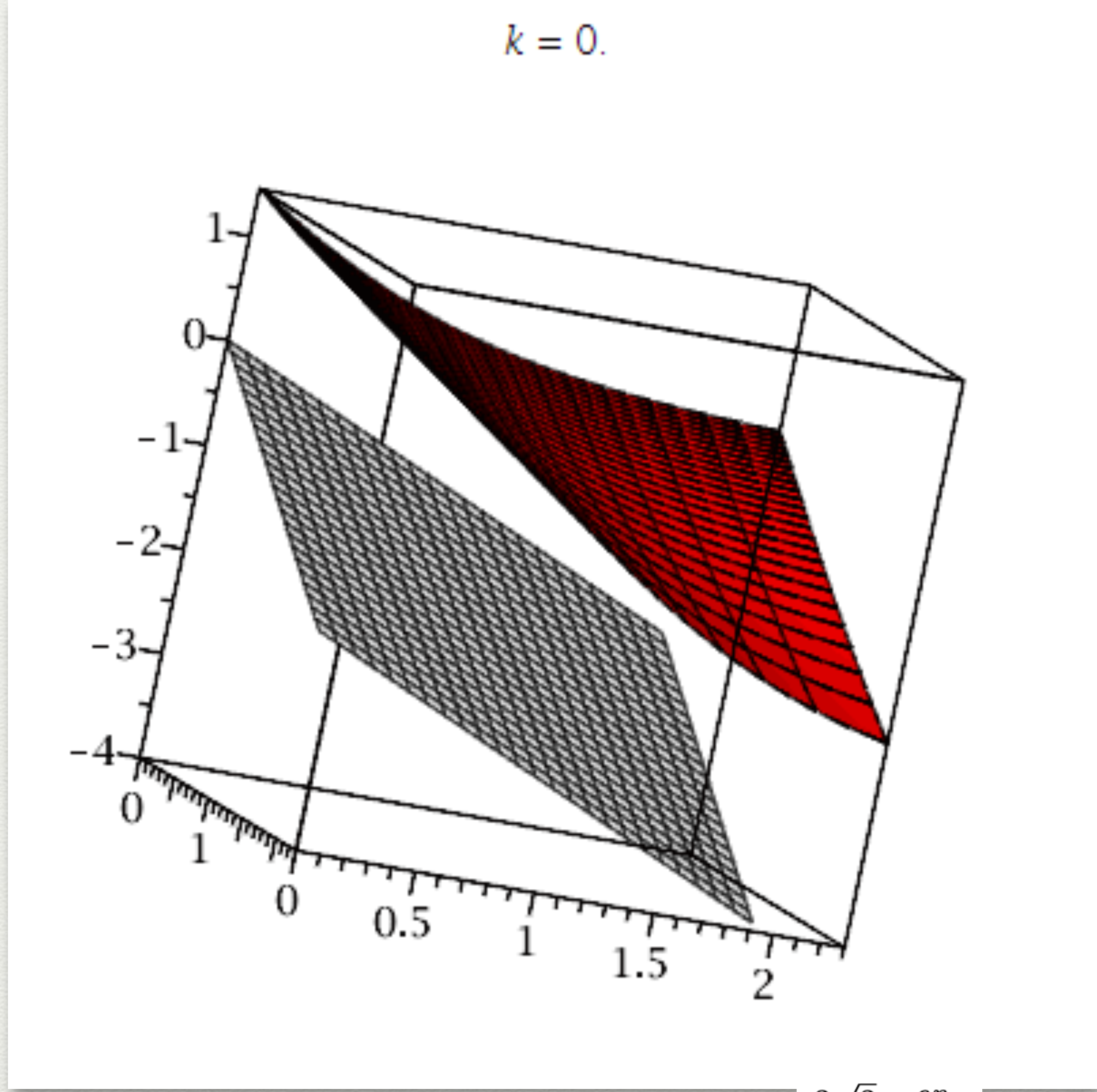


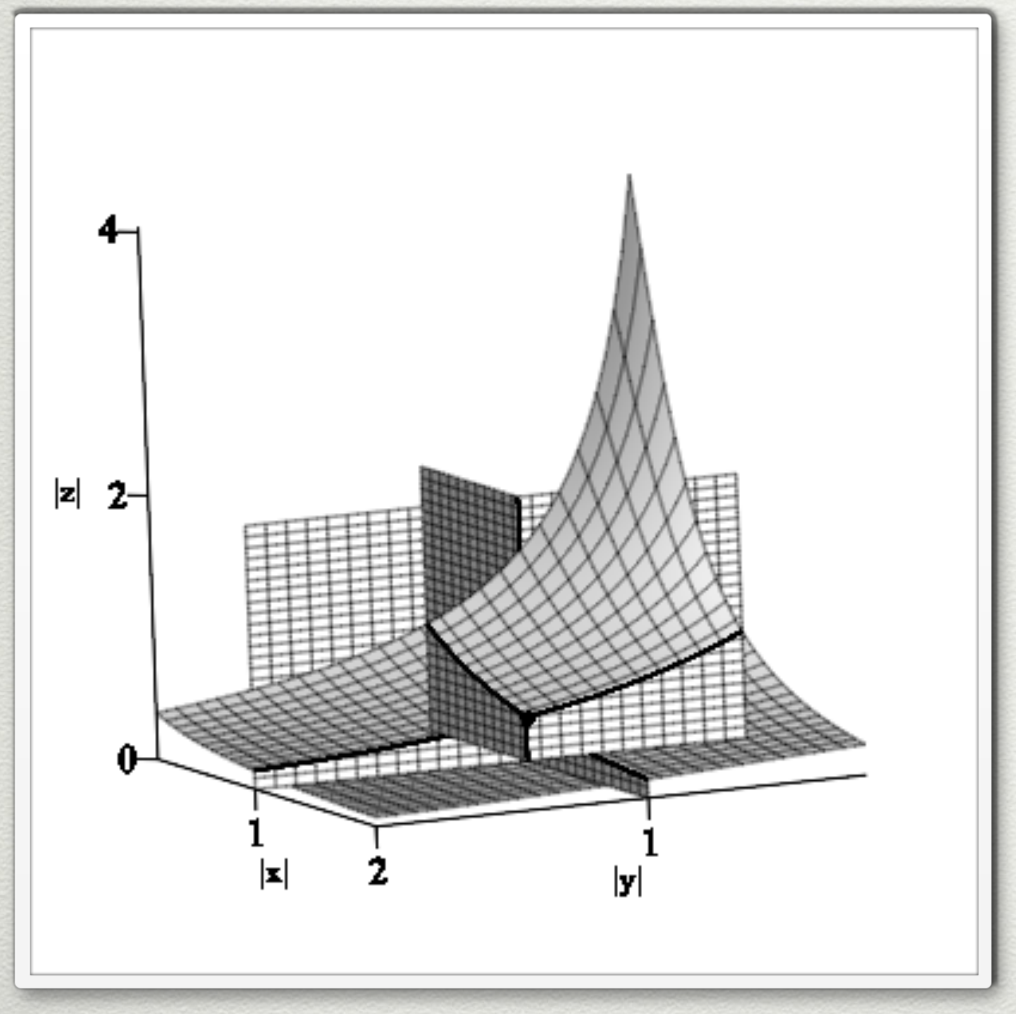
- Allows us to see the connection between walks that end at the origin, an axis, or anywhere.
- This model has algebraic GF for walks that end anywhere and transcendental GF for excursions. $\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$ $\sqrt{3}\frac{3^{3m}}{\pi m^4}$

$$\frac{(1-y^2/x+y^3-x^2y^2+x^3-x^2/y)}{(1-zxy(1/x+x/y+y))\,(1-x)(1-y)}$$









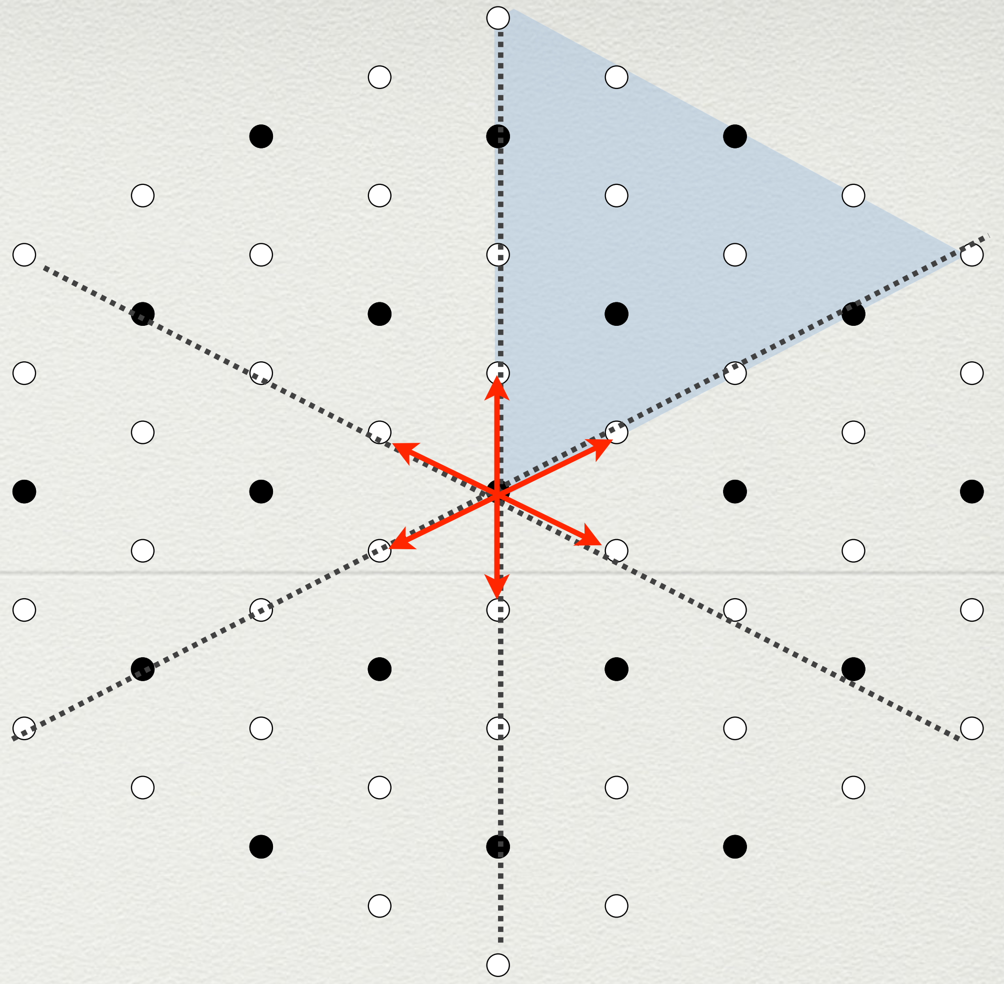
We conclude: Three critical points:

$$(w, w^2, w^2/3), (w^2, w, w/3), (1, 1, 1/3)$$

(Potential for periodicity..)

$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{6^n}{n^{3/2}}$

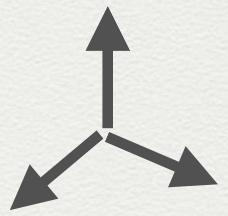
Tandem model / Walk in A_2

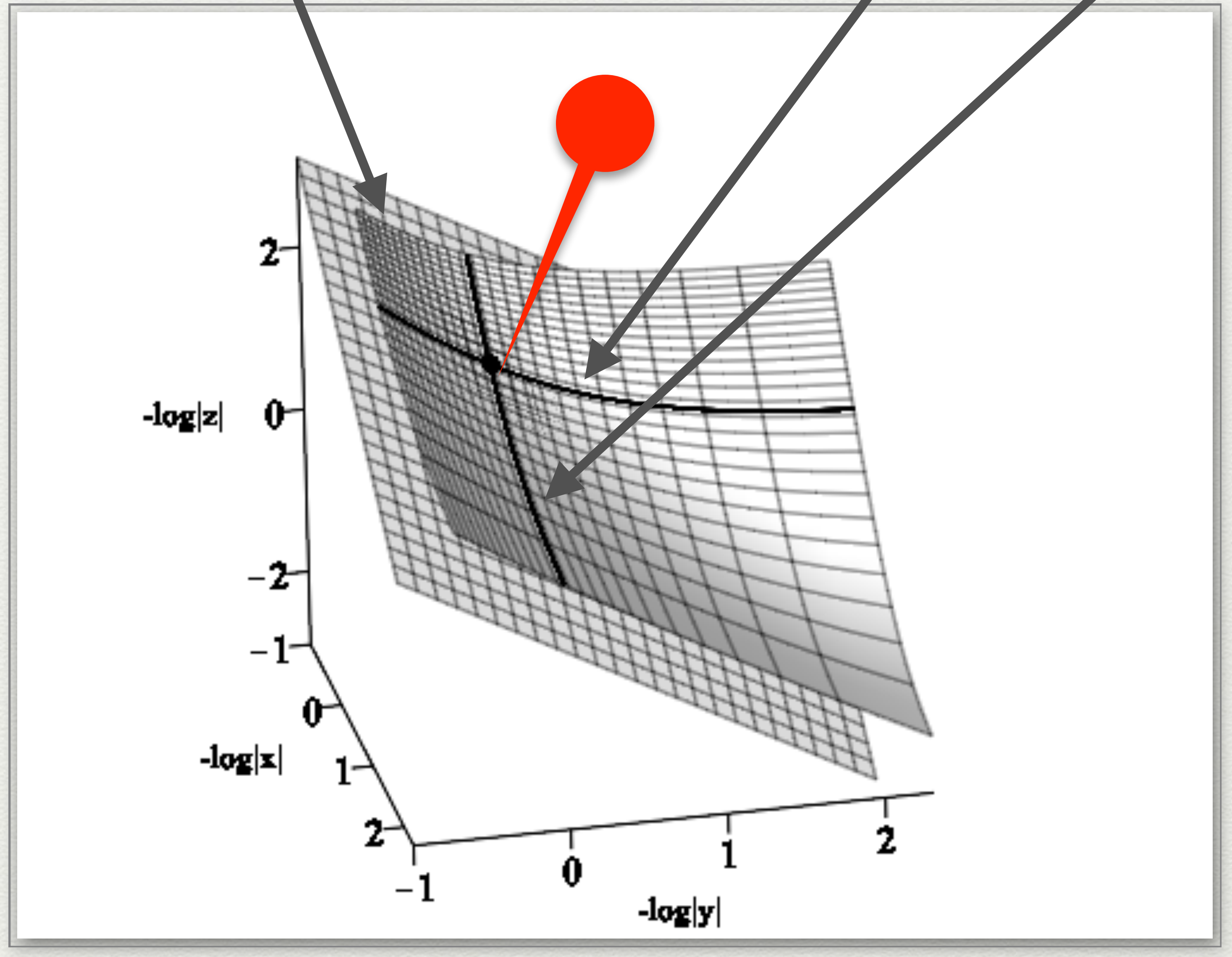


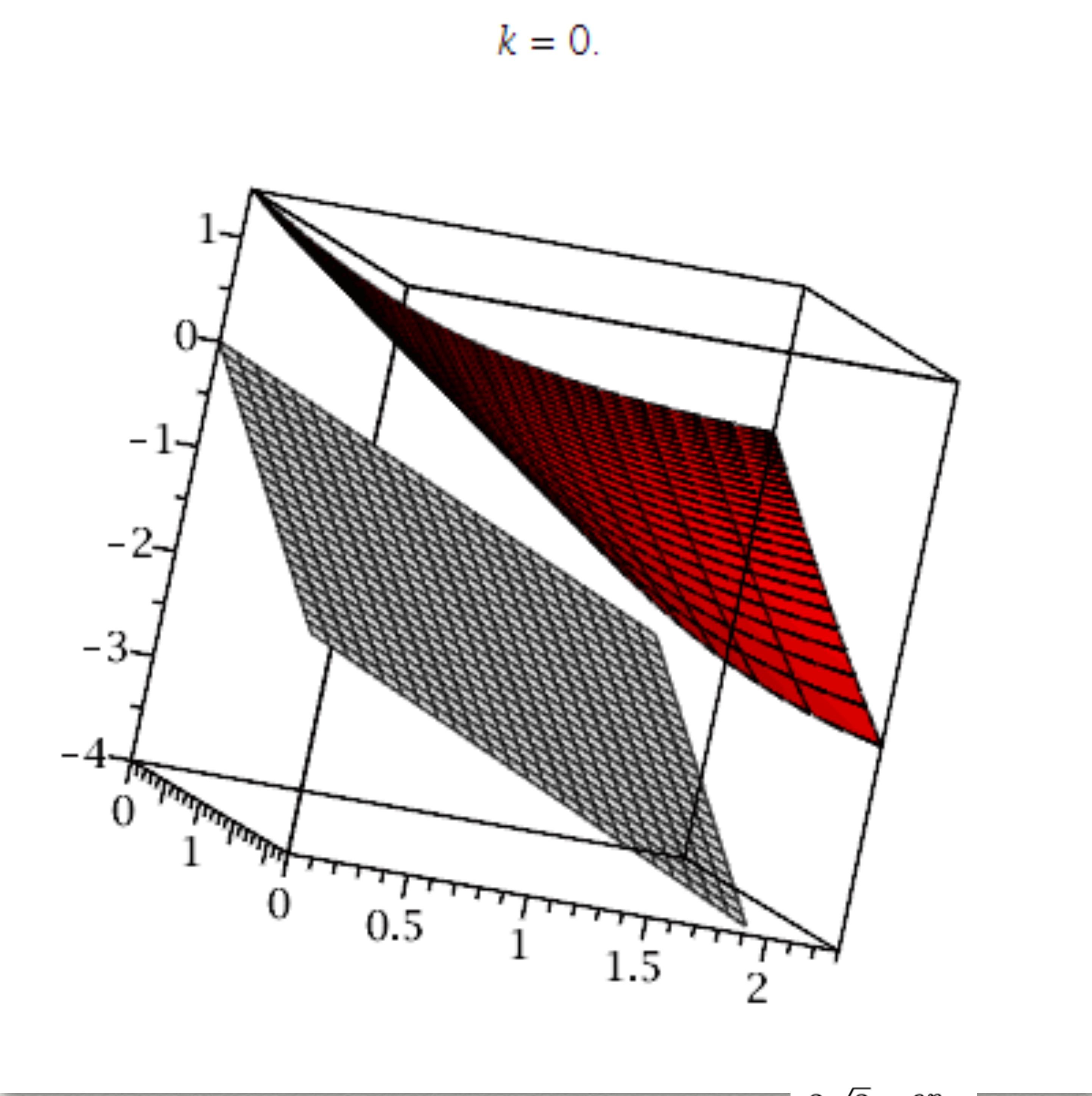
- Allows us to see the connection between walks that end at the origin, an axis, or anywhere.

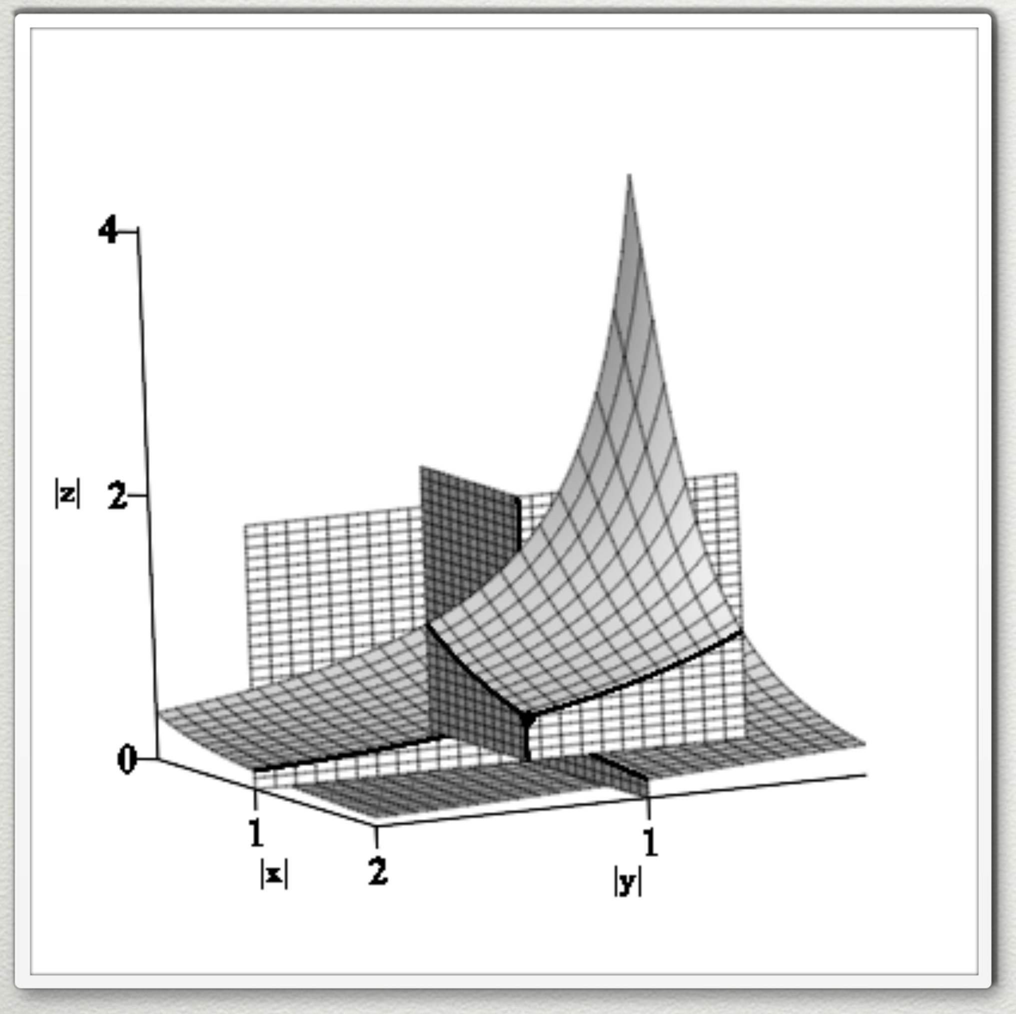
- This model has algebraic GF for walks that end anywhere and transcendental GF for excursions. $\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$ $\sqrt{3}\frac{3^{3m}}{\pi m^4}$

$$\frac{(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y)}{(1 - zxy(1/x + x/y + y))\,(1-x)(1-y)}$$









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IV. Next Steps

DICHOTOMIES: EITHER ELEMENTARY OR COMPLICATED

- Undergrad: Complex analytic functions are **either** unbounded or constant
- Fatou: If $F(z)$ is a series with positive integer coefficients, and it converges in the unit disc, the function is **either** rational, or has a natural boundary. (=> not D-finite)
- There are numerous conditions that imply that function is **either rational**, **or hypertranscendental.**
  - Satisfying both a difference equation AND a differential equation
  - Satisfying multiple p-Mahler equations

What is the combinatorial explanation? How can we systematically exploit these results?

MOVING FORWARD

- What is the META THEOREM for D-finite functions in combinatorics.
- How can the geometric understanding inform our understanding of transcendency?
- How can do we translate the geometric conditions of [Hardouin & Singer] to combinatorial conditions on the model?
- **Open Question:** What are the questions we should be considering?

CURRENT / FUTURE PROJECTS

- Asymptotic enumeration problems from root systems
  (Cedric Lecouvey & Students)

- Classification theorems for walks on Cayley graphs
  (Jason Bell, Haggai Liu)

- Crack the nut of what it means to be differentiably algebraic
  (Lucia Di Vizio, Charlotte Hardouin, Alin Bostan, Kilian Raschel,...)

THANK YOU!

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- Gessel and Zeilberger (1992). Random walk in a Weyl chamber. Proceedings of the American Mathematical Society, 115(1), 27-31.
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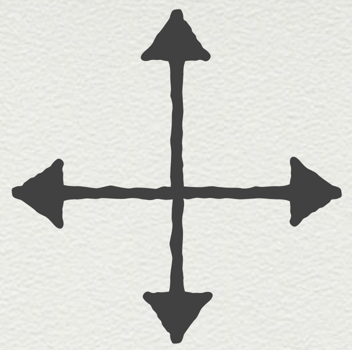
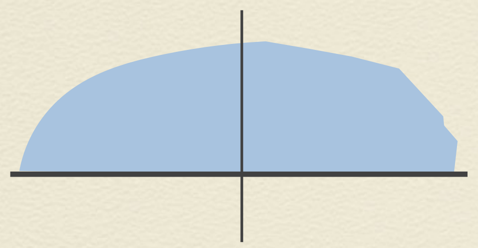
- Elder and Rechnitzer (2014). The cogrowth series for BS (N, N) is D-finite International J. Algebra and Computation 24.02 171-187.

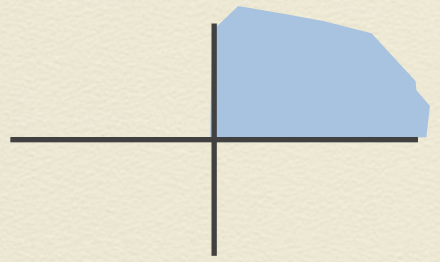
BONUS!

Simple Steps in Various Cones

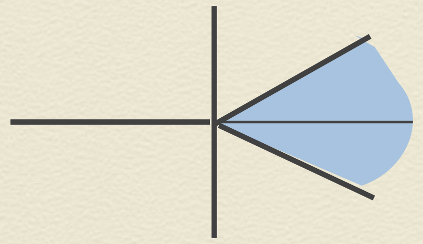




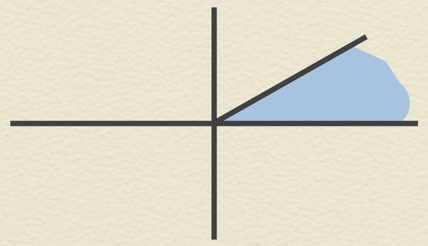
Half plane
Algebraic



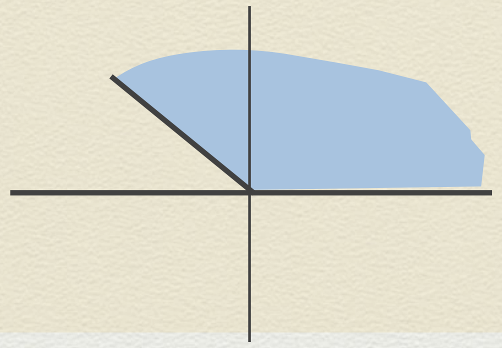
D-finite



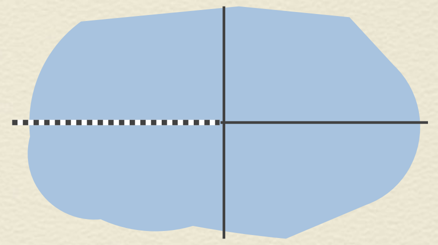
D-finite



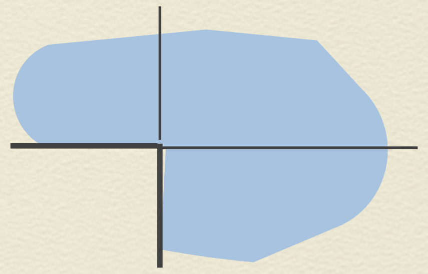
D-finite
(Gouyou-Beauchamps)



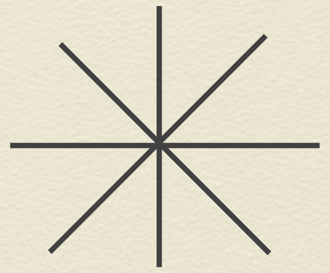
Nasty algebraic
(Bostan & Kauers…)



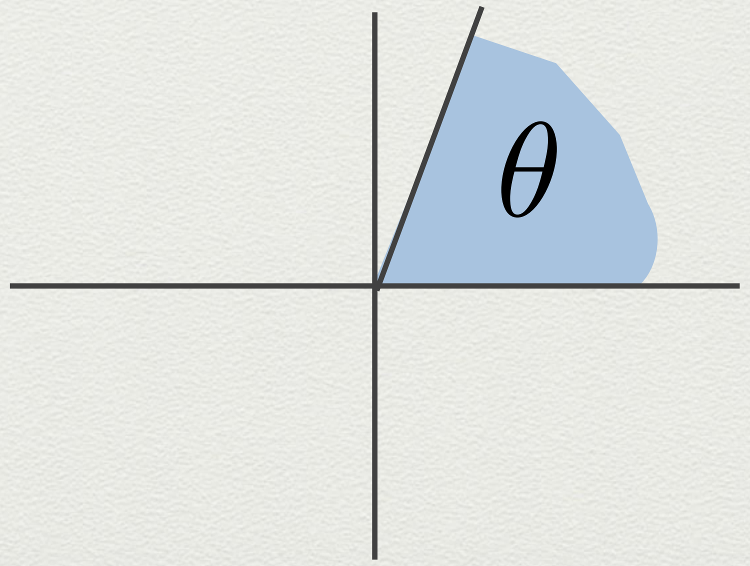
Slit plane model
Algebraic
(Bousquet-Mélou/Schaeffer)



3/4 plane model
D-finite (Bousquet-Mélou…)



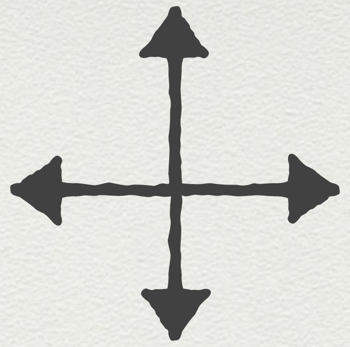
Excursions (Budd)
Conditions for Algebraicity
Winding walks Elvey-Price

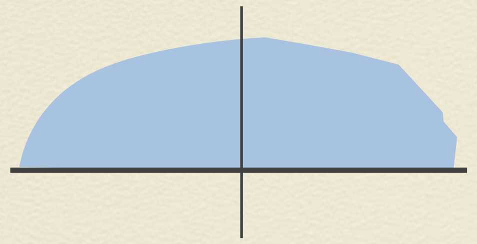


Under some conditions, *Q(x,y;z)* is provably not D-finite (Denisov & Wachtel 2016)

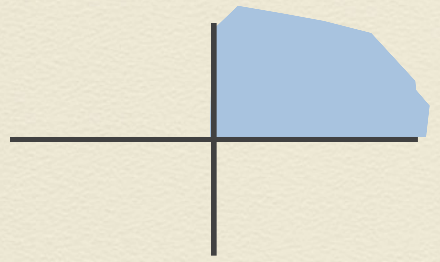
$$e(n) \sim \alpha \rho^{-n} n^{f(\theta)}$$

SIMPLE STEPS IN VARIOUS CONES

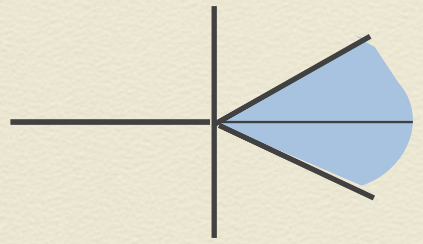




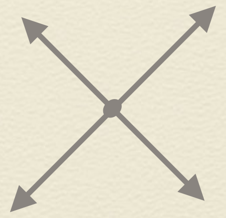
Half plane
Algebraic

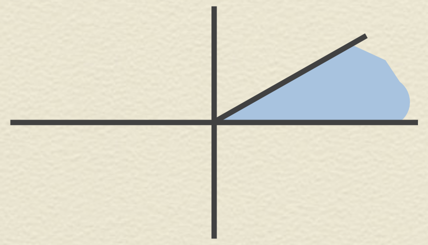


D-finite

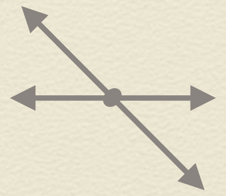


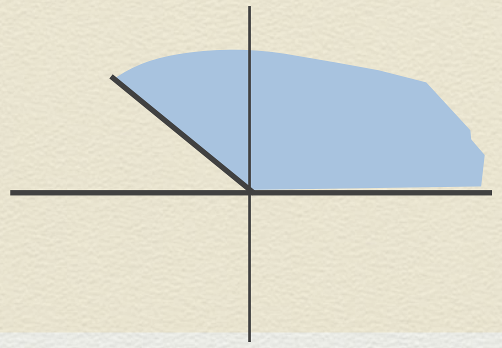
D-finite





D-finite
(Gouyou-Beauchamps)





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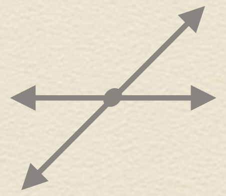
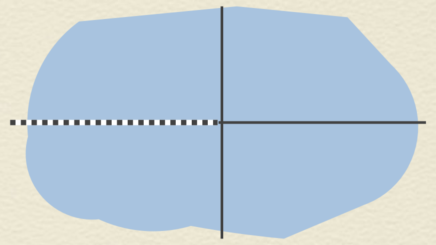
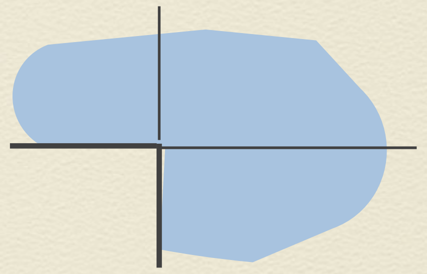


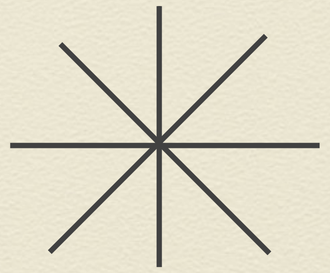
Equivalent quarter
plane models



Slit plane model
Algebraic
(Bousquet-Mélou/Schaeffer)



3/4 plane model
D-finite (Bousquet-Mélou…)



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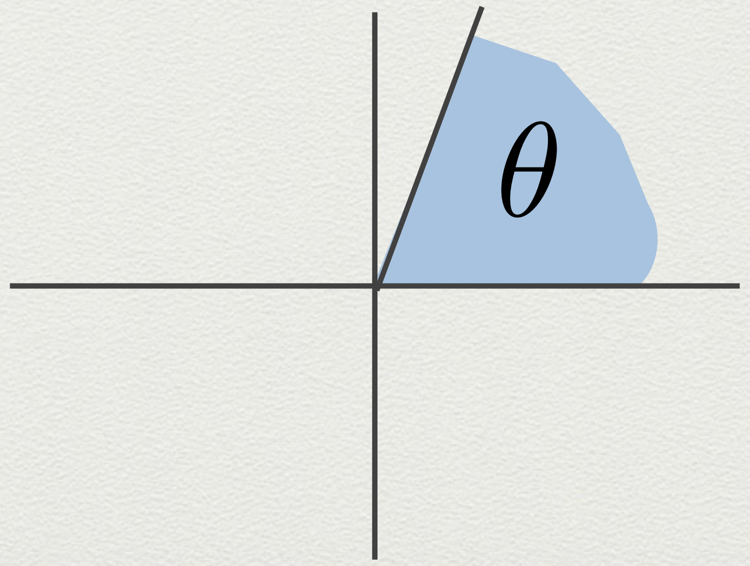

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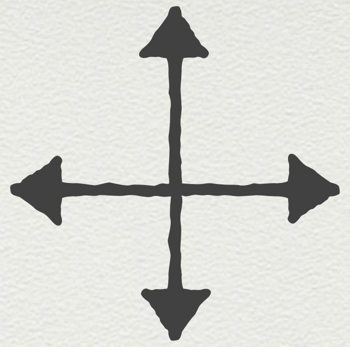

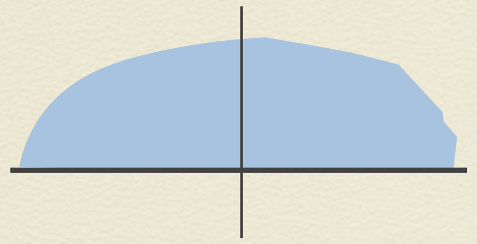

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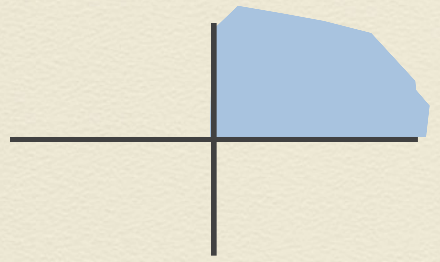
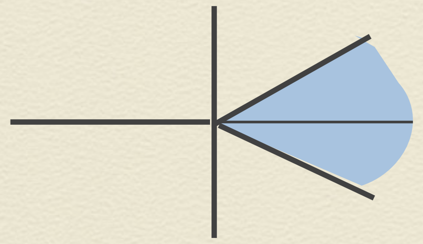


D-finite



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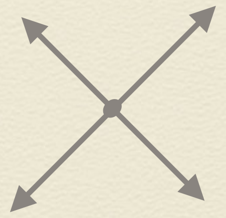

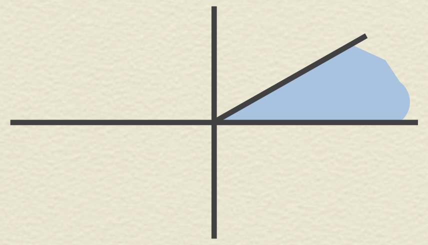

D-finite
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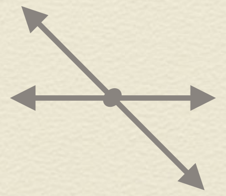

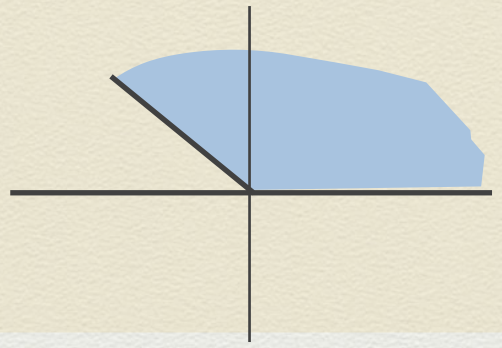

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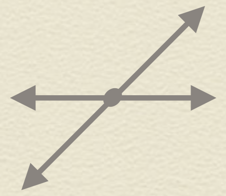
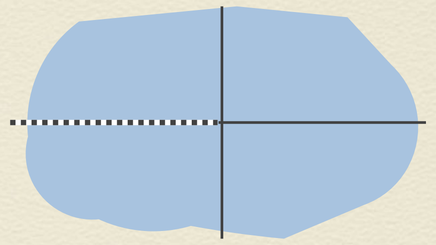


Equivalent quarter plane models



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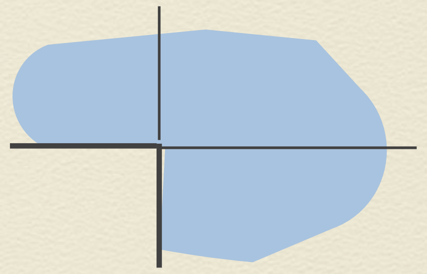

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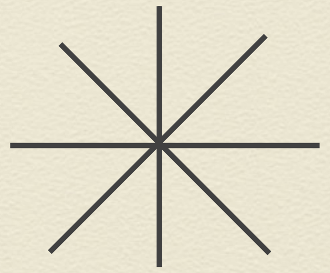
D-finite (Bousquet-Mélou...)



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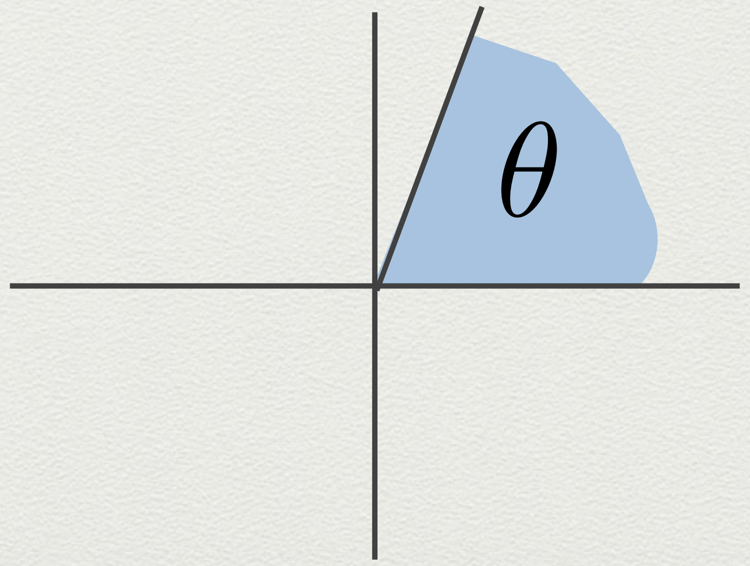

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USEFUL, BUT DOES NOT CAPTURE ALL MODELS.