Weighted Tandem Models: Elements of Asymptotic enumeration
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Preliminaries

```
> #libname:= "/cecm/home/mmishna/maple/algolib", libname;
> restart;
  libname:= "/Users/mmishna/maple/algolib", libname:
> gfun:-version();
```

$$3.53 \tag{1.1}$$

```
> GenStep:= proc(i,j,n, Steps)option remember;
        if i<0 or j<0 then 0
        elif n=0 then if i=0 and j=0 then 1 else 0 fi
        else add(procname(i-s[1], j-s[2], n-1, Steps), s=Steps)
        fi;
  end proc:
> ## Our representative classes
  Weights:= [[1,1,1], [1,2,1], [2, 1, 1], [1, 1, 2]]:
> for wt in Weights do
    St[wt]:= [[1,0]$wt[1], [-1,1]$wt[2], [0,-1]$wt[3]]:
    ## Generate the counting sequence
    # List[wt]:= [seq(add(add(GenStep(i,j,n, St[wt]), i=0..n), j=0..
  n), n=0..120)];
  od:
```

Exact enumeration

```
> count:= proc (i,j,n)
  option remember;
      if  (n-i-2*j) <0 or (n-i+j+3)<0 then return( 0) fi;
      if  type((n-i-2*j)/3, integer) and type((n-i+j+3)/3, integer)
  and type((n+2*i+j+6)/3, integer) then
     ((i+1)*(j+1)*(i+j+2)*n!
         /((n-i-2*j)/3)!/((n-i+j+3)/3)!/((n+2*i+j+6)/3)!)
      else return(0);
      fi:
  end proc:
> LList:=
  [seq(add(add(count(i,j,n)*x^i*y^j, i=0..n), j=0..n), n=0..200)]:
> with(gfun):
> St[[1,1,1]];
```

$$[[1, 0], [-1, 1], [0, -1]] \tag{2.1}$$

```
> K:= unapply(expand(x*y*(1-t*add(x^s[1]*y^s[2], s= St[[1,1,1]]))),
  x, y, t):
> QXY_ser:= add(LList[i]*t^(i-1), i=1..10):
  QX0_ser:= subs(y=0, QXY_ser);
  Q0Y_ser:= subs(x=0, QXY_ser);
  Q00_ser:= subs([x=0, y=0], QXY_ser);
  map(simplify, series(K(x,y, t)*QXY_ser-
  (x*y-x*t*QX0_ser-y^2*t*Q0Y_ser), t, 7));
```

$$QX0\_ser := 1 + t\,x + x^2\,t^2 + \left(x^3 + 1\right)t^3 + \left(x^4 + 3\,x\right)t^4 + \left(x^5 + 6\,x^2\right)t^5 + \left(x^6 + 10\,x^3\right.$$

$$+5)\,t^6+\left(x^7+15\,x^4+21\,x\right)t^7+\left(x^8+21\,x^5+56\,x^2\right)t^8+\left(x^9+28\,x^6+120\,x^3+42\right)t^9$$

$$Q0Y\_ser:=1+y\,t^2+t^3+2\,y^2\,t^4+5\,y\,t^5+\left(5\,y^3+5\right)t^6+21\,y^2\,t^7+\left(14\,y^4+42\,y\right)t^8+\left(84\,y^3+42\right)t^9$$

$$Q00\_ser:=42\,t^9+5\,t^6+t^3+1$$

$$\mathrm{O}\left(t^7\right) \tag{2.2}$$

```
> ## QX0:= subs(y=0, LList):
  ## DE_QX0:=listtodiffeq(QX0, f(t))[1]:
  ## Q0Y:= subs(x=0, LList):
  ## DE_Q0Y:=listtodiffeq(Q0Y, f(t))[1]:
> KQXY_ser:= series(K(x,y,t)*add(LList[i]*t^(i-1), i=1..200), t, 200)
  :
  KQX0_ser:= subs(y=0, KQXY_ser):
  KQ0Y_ser:= subs(x=0, KQXY_ser):
  KQ00_ser:= subs([x=0, y=0], KQXY_ser):
> ## DEs for the walks that return to an axis
> DE_KQX0:=seriestodiffeq(KQX0_ser, f(t))[1]:
  DE_KQ0Y:=seriestodiffeq(KQ0Y_ser, f(t))[1]:
  # save([DE_KQX0, DE_KQ0Y], "DES1.mpl"):
> with(DEtools): with(PDEtools):
  _Envdiffopdomain := [Dt, t]:
> DE_KQXY:=`diffeq+diffeq`(DE_KQX0, DE_KQ0Y, f(t)):
  lprint("***", DE_KQXY):
  # save([DE_KQXY], "DES2.mpl"):

  for k from 1 to 10 do
     lprint(k, coeff(DE_KQXY[1], diff(f(t), t$k)));
  od;
  if assigned(DE_KQXY) then
    DOp:= Desingularize(de2diffop(gfun[diffeqtohomdiffeq](DE_KQXY, Q
  (t)), Q(t)), Dt, t):
    DEG:= degree(DOp, Dt):
  lprint(coeff(DOp, Dt, DEG));
  lprint(fsolve(coeff(DOp, Dt, DEG)));
  fi:

Warning,  computation interrupted
"***", DE_KQXY
1, 0
2, 0
3, 0
4, 0
5, 0
6, 0
7, 0
8, 0
9, 0
10, 0
> with(SumTools[Hypergeometric]):
> Gosper(((i+1)*(j+1)*(i+j+2)*n!
         /((n-i-2*j)/3)!/((n-i+j+3)/3)!/((n+2*i+j+6)/3)!), i,j,n);
Error, (in SumTools:-Hypergeometric:-Gosper) the input is expected
to be a hypergeometric term in i
```

Exponential Growth Factor

Find the exponential growth factor using JMY.

```
> # curve plotting
```

```
> lprint("******* Exponential growth factor **********"):
  for wt in Weights do
  printf("\n ----------------------------- \n"):
     drift:= add(s, s= St[wt]):
     numsteps:= nops(St[wt]):
     printf("\n Wt: %a \n Model: %a \n Drift: %a\n # Steps: %a\n",
  wt, St[wt], drift, numsteps ):

     ## Inventory
     P:= expand(add(x^s[1]*y^s[2], s=St[wt])):

     ## Compute K(theta) for theta = 0.. Pi/2
     func[wt]:= NULL:
     slp[wt]:= NULL:
     for theta in [seq(Pi/80*k, k=0..40)] do

       P2:= subs(x=u^sin(theta), y=u^cos(theta), P);

       slp[wt]:= slp[wt], [theta, subs(u=1, diff(P2, u))]:
       if evalf(subs(u=1, diff(P2, u)))>=0 then func[wt]:= func[wt],
  [theta, nops(St[wt])] :
       else
         func[wt]:= func[wt],  [theta, subs(u=fsolve(diff(P2,u)=0, u=
  0.1), P2)];

        ##func[wt]:= func[wt],  [theta, subs(u=2, P2)];
       fi;
     od;

     ## plot this curve
     PLT[wt]:= plot([func[wt]]);
   ##  SPLT[wt]:= plot([slp[wt]]);


     #lprint("from List", evalf(List[wt][80]/List[wt][79]));

      if drift[1]>= 0 and drift[2]>= 0 then EGF[wt]:= numsteps ;
      else
         # determine the phase transition in the range



         P:= expand(add(x^s[1]*y^s[2], s=St[wt]));
         sublist:= fsolve({diff(P, x)=0, diff(P, y)=0}, {x=0.1, y=
  0.1});
         alpha:= subs(sublist, x):
         beta:= subs(sublist, y):
   lprint("critical points", alpha, beta);
         if beta=1 then
            thetastar:= Pi/2;

         else
            thetastar:= arctan(ln(alpha)/ln(beta)):
         fi;

         if drift[2]*drift[1]<0 then
           theta_switch:= evalf(arctan(-drift[2]/drift[1])):
```

```
          if drift[1]<0  and evalf(thetastar)< theta_switch then
                  thetastar:= Pi/2:
          elif drift[2]<0 and  theta_switch<evalf(thetastar) then
              thetastar:= 0;
          fi;
        fi:

  printf("thetastar %a\n", thetastar);
        P2:= subs(x=u^sin(thetastar), y=u^cos(thetastar), P);
  lprint("tau", fsolve(diff(P2,u)=0, u=0.1));
        EGF[wt]:=subs(u=fsolve(diff(P2,u)=0, u=0.1), P2):


        ## covariance:= subs([x=1, y=1], diff(P, x, y))-drift[1]*
  drift[2];
        ## K_x:= subs(y=1, coeff(P, x, 0)+2*sqrt(coeff(P, x, 1)*
  coeff(P, x, -1)));
        ## K_y:= subs(x=1, coeff(P, y, 0)+2*sqrt(coeff(P, y, 1)*
  coeff(P, y, -1)));
        ## K_e:= subs(sublist, P);
        ## lprint("x,y,z",  evalf([K_x, K_y, K_e]));

      fi:
  printf("EGF %a\n", EGF[wt] );
  od:
```

```
"******* Exponential growth factor **********"

 -----------------------------

 Wt: [1, 1, 1]
 Model: [[1, 0], [-1, 1], [0, -1]]
 Drift: [0, 0]
 # Steps: 3
EGF 3

 -----------------------------

 Wt: [1, 2, 1]
 Model: [[1, 0], [-1, 1], [-1, 1], [0, -1]]
 Drift: [-1, 1]
 # Steps: 4
"critical points", 1.259921050, .7937005260
thetastar 1/2*Pi
"tau", 1.414213562
EGF 3.828427125

 -----------------------------

 Wt: [2, 1, 1]
 Model: [[1, 0], [1, 0], [-1, 1], [0, -1]]
 Drift: [1, 0]
 # Steps: 4
EGF 4

 -----------------------------

 Wt: [1, 1, 2]
 Model: [[1, 0], [-1, 1], [0, -1], [0, -1]]
 Drift: [0, -1]
 # Steps: 4
"critical points", 1.259921050, 1.587401052
```

```
thetastar .4636476092
"tau", 1.676387876
EGF 3.779763150
```

Analyze the DEs

```
> with(DEtools): with(PDEtools):
  _Envdiffopdomain := [Dt, t]:
```

The basic model: steps returning to an axis

```
> # DE_KQX0:=seriestodiffeq(KQX0_ser, f(t))[1]:
  # DE_KQ0Y:=seriestodiffeq(KQ0Y_ser, f(t))[1]:
  # save([DE_KQX0, DE_KQ0Y], "DES1.mpl"):
> with(DEtools): with(PDEtools):
  _Envdiffopdomain := [Dt, t]:
> #DE_KQXY:=`diffeq+diffeq`(DE_KQX0, DE_KQ0Y, f(t)):
  #save([DE_KQXY], "DES2.mpl")

  if assigned(DE_KQX0) then
    DOp:= Desingularize(de2diffop(gfun[diffeqtohomdiffeq](DE_KQX0, Q
  (t)), Q(t)), Dt, t):
    DEG:= degree(DOp, Dt):
  lprint(coeff(DOp, Dt, DEG));
  lprint(fsolve(coeff(DOp, Dt, DEG)));
  fi:
Error, (in gfun:-formatdiffeq) invalid differential equation, {
(4/81)*x^2*(2*x-t*x^2-54*t^4*x^2-30*t^2+18*t^3*x+27*t^5*x^3+378*
t^5)*f(t)/(7*x^6+4*x^3-128)+(4/81)*t*(-2*x^3-30*t^3*x^6-124*t^3*
x^3+3402*t^9*x^6+1944*t^9*x^3-62208*t^9+21*t*x^4-4*t^2*x^5+30*x^2*
t^2+15*t^4*x^7-339*t^4*x^4+57*t^5*x^5-2136*t^5*x^2+900*t^6*
x^6+2340*t^6*x^3+15264*t^7*x-450*t^7*x^7+2520*t^7*x^4-9234*t^8*
x^5+15066*t^8*x^2)*(diff(f(t), t))/(7*x^6+4*x^3-128)+(2/81)*t^2*
(28*t^2*x^5+286*x^2*t^2-30618*t^8*x^5+20412*t^8*x^2-18*x^3+43*t*
x^4-95*t^3*x^6+224*t^3*x^3-1632*t^4*x+40*t^4*x^7-1510*t^4*x^4-21*
t^5*x^5-9006*t^5*x^2+2406*t^6*x^6+4422*t^6*x^3+5376*t^6+55008*t^7*
x-1125*t^7*...
> coeff(DE_KQX0[1], diff(f(t), t$4));
  solve(%, {t, x});
  coeff(DE_KQX0[1], diff(f(t), t$5));
```

$$\frac{1}{243}\,\frac{1}{7x^6+4x^3-128}\Big(t^4\big(-4x^3+3tx^4+11t^2x^5+68x^2t^2-15t^3x^6+123t^3x^3-496t^4x+5t^4x^7-310t^4x^4-72t^5x^5-1656t^5x^2+342t^6x^6-441t^6x^3+1152t^6+13392t^7x-135t^7x^7+6183t^7x^4-6075t^8x^5-4860t^8x^2+1701t^9x^6+972t^9x^3-31104t^9\big)\Big)$$

$$\{t=0,x=x\},\left\{t=\frac{1}{3},x=x\right\},\left\{t=-\frac{1}{6}+\frac{1}{6}\cdot\mathrm{I}\sqrt{3},x=x\right\},\left\{t=-\frac{1}{6}-\frac{1}{6}\cdot\mathrm{I}\sqrt{3},x=x\right\},\left\{t=\frac{x^2+2\sqrt{x}}{x^3-4},x=x\right\},\left\{t=-\frac{-x^2+2\sqrt{x}}{x^3-4},x=x\right\},\left\{t=\mathit{RootOf}\big((288+63x^3)\_Z^4-99\_Z^3x^2+(-52x-5x^4)\_Z^2+5\_Zx^3+4x^2\big),x=x\right\}$$

$$0 \qquad \textbf{(4.1.1)}$$

```
> coeff(DE_KQ0Y[1], diff(f(t), t$4));
  solve(%, {t, y});
  coeff(DE_KQ0Y[1], diff(f(t), t$5));
```

$$\frac{1}{243} \frac{1}{8 y^6 + 2 y^3 - 1} \left(t^4 \left(-4 y^6 + 13 t y^5 + 48 t^2 y^7 + 5 t^2 y^4 - 48 t^3 y^6 - 60 t^3 y^3 - 128 t^4 y^8 - 233 t^4 y^5 + 82 y^2 t^4 - 45 y t^5 - 1008 t^5 y^7 - 135 t^5 y^4 + 4140 t^6 y^6 + 1602 t^6 y^3 + 9 t^6 + 3456 t^7 y^8 - 3186 t^7 y^5 - 2214 y^2 t^7 + 1215 t^8 y - 7776 t^8 y^7 + 1944 t^9 y^6 + 486 t^9 y^3 - 243 t^9\right)\right)$$

$$\{t=0, y=y\}, \left\{t=\frac{1}{3}, y=y\right\}, \left\{t=-\frac{1}{6}+\frac{1}{6} \cdot \mathrm{I}\sqrt{3}, y=y\right\}, \left\{t=-\frac{1}{6}-\frac{1}{6} \cdot \mathrm{I}\sqrt{3}, y=y\right\}, \left\{t = \frac{\left(-1+2 y^{3/2}\right) y}{4 y^3 - 1}, y=y\right\}, \left\{t=-\frac{\left(1+2 y^{3/2}\right) y}{4 y^3-1}, y=y\right\}, \{t = \mathit{RootOf}\left(\left(9+18 y^3\right) \_Z^4 - 4 + 5\_Z + \left(32 y^3+19\right)\_Z^2 + \left(-72 y^3 - 27\right)\_Z^3\right) y, y=y\}$$

$$0 \tag{4.1.2}$$

Differential equations

```
>
  with(gfun):
> for wt in Weights do
  print("-----", wt);
  lprint(evalf(1/EGF[wt]));
  if assigned(DE[wt]) then
    DOp[wt]:= Desingularize(de2diffop(gfun[diffeqtohomdiffeq](DE[wt],
  Q(t)), Q(t)), Dt, t):
    DEG:= degree(DOp[wt], Dt):
  lprint(fsolve(coeff(DOp[wt], Dt, DEG)));
    ebgt := map(simplify,(dchange({t=1/EGF[wt]-s},DE[wt]))):
    ct := eval(subs(Q(s)=0,ebgt));
    FORSOL[wt]:=formal_sol(ebgt-ct,Q(s),s=0);
  else print("To do");
  fi;
  od:
```

"-----", [6, 2, 1, 3]

```
0.83333333333333333333e-1
-.10206207261596575409, -0.88388347648318440550e-1, 0., 0., 0., 0.
, 0.83333333333333333333e-1, 0.84040820577345752144e-1,
0.88388347648318440550e-1, .10206207261596575409,
.10502425616176292537, .47595917942265424786
```

"-----", [4, 2, 1, 2]

```
.11111111111111111111
```

"To do"

"-----", [2, 1, 1, 2]

```
.16666666666666666667
```

```
-.1767766952966368811O, -.16666666666666666667, 0., 0., 0., 0.,
.16666666666666666667, .17157287525380990240,
.1767766952966368811O, 5.8284271247461900976
```

"-----", [1, 1, 2, 2]

```
.1767766952966368811O
```

"To do"

"-----", [2, 2, 1, 1]

```
.17157287525380990240
```

"To do"

"-----", [1, 2, 4, 2]

```
.12500000000000000000
-.12500000000000000000, -.11785113019775792073, 0., 0., 0., 0.,
.11111111111111111111, .11785113019775792073,
.12500000000000000000, 1.
```

"-----", [1, 2, 2, 1]

```
.1767766952966368811O
-.1767766952966368811O, -.16666666666666666667, 0., 0., 0., 0.,
.16666666666666666667, .17157287525380990240,
.1767766952966368811O, 5.8284271247461900976
```

"-----", [1, 3, 6, 2]

```
.10206207261596575409
-.10206207261596575409, -0.88388347648318440550e-1, 0., 0., 0., 0.
, 0.83333333333333333333e-1, 0.84040820577345752144e-1,
0.88388347648318440550e-1, .10206207261596575409,
.10502425616176292537, .47595917942265424786
```

"-----", [1, 1, 1, 1]

```
.25000000000000000000
-.25000000000000000000, 0., 0., .25000000000000000000
```