Asymptotics of Weighted Reflectable Walks in Weyl Chambers

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Problem statement. We aim to find asymptotics on the number of weighted lattice walks in Weyl chambers by applying the machinery from Pemantle and Wilson's *Analytic Combinatorics in Several Variables*. Previous results on lattice walk asymptotics include an analysis of the weighted B_2 model, the weighted ${A_1}^n$ model, and the unweighted A_d and B_d models. Our goal is to find a systematic analysis of walks in more weighted Weyl chambers and in higher dimensions.

First steps. We began the week by reviewing previously known results. Using *Asymptotic Lattice Path Enumeration Using Diagonals* by Melczer and Mishna for guidance, we discussed how to encode the number of walks with a given step set as the diagonal of a rational generating function. We then reviewed the steps to extract asymptotics from a multivariate generating function by using singularity analysis and identifying relevant critical points. To ensure we understood the machinery, we successfully rederived asymptotics for Example 6.5.2 in Melczer and Wilson's *Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior*. We did this by modifying Sage code from the website accompanying Melczer's *An Invitation to Analytic Combinatorics*. Next, we rederived asymptotic expressions for Gouyou-Beauchamps excursions, noting that the existence of two critical points lead to periodic behavior in the asymptotics. Finally, we reviewed a transformation between walks in the root lattice and walks in $\mathbb{Z}^d$.

Ongoing work. When looking at weighted walks in higher dimensions, we found that the nature of the critical points controlling the asymptotics depends on the weights placed on the walks, mirroring what Mishna and Simon found in *The asymptotics of reflectable weighted walks in arbitrary dimension*. We noted that focusing on excursions and walks with large weights would simplify our analyses. Thus, we will look at walks with large weights first, and gradually allow some weights to be small. Theorem 10.3.1 in Pemantle and Wilson's textbook will cover the case when a single weight is small, and Theorem 10.3.4 or a direct residue computation could handle additional small weights. Ideally, a provable pattern will emerge depending on the number of small weights like in Mishna and Simon's work.

We also noticed that in Melczer and Wilson's work, they used an algebraic trick similar to partial fractions to reduce a generating function into the sum of two simpler generating functions. From a combinatorial perspective, this algebraic trick likely describes a relationship between different sets of walks. We propose using the decomposition to find a bijection between walks in the positive quadrant in $\mathbb{Z}^2$ with a given step set ending anywhere, and slightly modified walks that end on an axis.