Weyl chambers for short step Quarter-plane Lattice Paths.

Richard Brak

February 19, 2020

School of Mathematics
The University of Melbourne
Parkville, VIC 3052,
Australia

rb1@unimelb.edu.au

Abstract

We consider four examples of short step lattice paths confined to the quarter plane. These are the Kreweras, Reverse Kreweras, Gessel, and Mishna-Rechnitzer lattice paths. The Reverse Kreweras are straightforward to solve and thus interesting as a contrast to the Kreweras paths and Gessel paths as the latter two have historically been *significantly* more difficult to solve. The Mishna-Rechnitzer paths are interesting as they are associated with and infinite order group. We will give some geometrical insight into all these properties by considering the Weyl chambers associated with their step sets.

For Reverse Kreweras paths the Weyl chamber walls coincide with the quarter plane boundary and hence the problem is readily solvable by Bethe Ansatz or by using the Gessel-Zeilberger Theorem. For Kreweras paths the quarter plane corresponds to the union of two adjacent Weyl Chambers and hence neither the Bethe Ansatz nor the Gessel-Zeilberger Theorem are directly applicable making the problem considerably more difficult to solve. Similarly, the quarter plane for Gessel paths is the union of three Weyl chambers.

For Mishna-Rechnitzer paths the step set has non-zero barycenter leading to an affine dihedral reflection group. The affine structure corresponds to the drift in the random walk. The quarter plane is the union of an infinite number of Weyl alcoves.

Keywords: Random walks, lattice paths, quarter plane, isometry group, affine group, Weyl chamber

1 Introduction and definitions

The lattice $\mathcal{L}$ is $\mathcal{L} = \mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers. The **quarter plane**, $\mathcal{Q}$, is the subset $\mathcal{Q} = \{(a, b) \in \mathcal{L} : a > 0 \text{ and } b > 0\}$. Define the y-**boundary** to be the subset $\mathcal{Q}_y = \{(0, b) \in \mathcal{L} : b \geq 0\}$ and the x-**boundary** to be the subset $\mathcal{Q}_x = \{(a, 0) \in \mathcal{L} : a \geq 0\}$. The **boundary** of the quarter plane, $\mathcal{Q}_\partial$ is defined as the subset of points $\mathcal{Q}_\partial = \mathcal{Q}_x \cup \mathcal{Q}_y$.

A **step set**, $\mathcal{S}$, is any finite subset of $\mathcal{L}$. A **step** is an element of $\mathcal{S}$. A **short step** is a step (a, b) for which $a^2 + b^2 \leq 2$ ie. a nearest-neighbour step. A **lattice path** or **random walk** of length t, or t-path, with step set $\mathcal{S}$ is a finite sequence $s_1 s_2 s_3 \dots s_t$ of steps, $s_i \in \mathcal{S}$. If a path starts at $\mathbf{a} \in \mathcal{L}$ and has steps $s_1 s_2 s_3 \dots s_t$ then it is said to end at $\mathbf{b} = \mathbf{a} + \sum_{i=1}^{t} s_i$. The **step set generating function** is the Laurent polynomial

$$\lambda(\mathcal{S}) = \sum_{(a,b)\in\mathcal{S}} x^a y^b \,. \tag{1.1}$$

We wish to compute the number of length t lattice paths that start at some point $\mathbf{a} = (a_1, a_2) \in \mathcal{Q}$ and end at some point $\mathbf{b} = (b_1, b_2) \in \mathcal{Q}$ with the condition that the path is not allowed to step on the quarter plane boundary ie. *all* of the partial sums $\mathbf{a} + \sum_{i=1}^{p} s_i$, $1 \leq p \leq t$ are points in $\mathcal{Q}$. We will refer to such paths as **quarter plane paths** [1]. Note, this definition differs slightly from that used in [2] where the quarter plane boundary is shifted off the x and y axes.

Let

$$\mathcal{Z}_t^{\mathrm{P}}(\mathbf{a} \to \mathbf{b}) = \text{Number of } t\text{-paths in } \mathcal{Q} \text{ from } \mathbf{a} \text{ to } \mathbf{b}, \text{ with step set } \mathcal{S}_P \tag{1.2}$$

We will also consider paths in $\mathcal{L}$ *without* the quarter plane boundary constraint. We will refer to these a **free lattice paths** with

$$\mathcal{F}_t^{\mathrm{P}}(\mathbf{a} \to \mathbf{b}) = \text{Number of free } t\text{-paths in } \mathcal{L} \text{ from } \mathbf{a} \text{ to } \mathbf{b}, \text{ with step set } \mathcal{S}_P \tag{1.3}$$

Note, for free paths $\mathbf{a}$ and $\mathbf{b}$ are no longer constrained to the quarter plane. We define the t-step **free generating function**

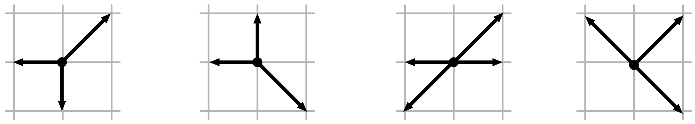
$$G_t^{\mathrm{P}}(\mathbf{a}; x, y) = \sum_{\mathbf{b}\in\mathcal{L}} \mathcal{F}_t^{\mathrm{P}}(\mathbf{a} \to \mathbf{b}) x^{b_1} y^{b_2} \tag{1.4}$$

where $\mathbf{b} = (b_1, b_2)$.

Bousquet-Mélou and Misha [2] considered 56 different short step quarter plane path problems. Integral to the many solutions obtained by Bousquet-Mélou and Misha was the use of a "substituion" group associated with the step set. This group is defined as follows: Let g_1 and g_2 be rational functions of x and y. The action of the map $g : (x, y) \mapsto (g_1(x, y), g_2(x, y))$, on $\lambda(\mathcal{S})$ is defined as $g \cdot \lambda(\mathcal{S}) := \sum_{(a,b)\in\mathcal{S}} g_1(x, y)^a g_2(x, y)^b$. The set of such maps that leave λ invariant i.e. $g \cdot \lambda = \lambda$, form a group (under composition).

In this paper we will revisit four of the 56 cases. In particular, the three cases are Kreweras(K) [3], Reverse Kreweras($\bar{\mathrm{K}}$), Gessel(G), and Mishna-Rechnitzer(MR) paths [2] which have respective step sets (illustrated below):

$$\begin{aligned} \text{Reverse Kreweras}: &\quad \mathcal{S}_{\bar{\mathrm{K}}} = \{(1,0),(0,1),(-1,-1)\}, && (1.5)\\ \text{Kreweras}: &\quad \mathcal{S}_{\mathrm{K}} = \{(-1,0),(0,-1),(1,1)\} && (1.6)\\ \text{Gessel}: &\quad \mathcal{S}_{\mathrm{G}} = \{(-1,0),(1,0),(-1,-1),(1,1)\} && (1.7)\\ \text{Mishna-Rechnitzer}: &\quad \mathcal{S}_{\mathrm{MR}} = \{(-1,1),(1,-1),(1,1)\} && (1.8) \end{aligned}$$



and respective step set generating functions:

$$\text{Reverse Kreweras}: \quad \lambda_{\bar{\mathrm{K}}}(x,y) = x + y + \bar{x}\bar{y} \tag{1.9}$$

$$\text{Kreweras}: \quad \lambda_K(x,y) = \bar{x} + \bar{y} + xy \tag{1.10}$$

$$\text{Gessel}: \quad \lambda_G(x,y) = x + \bar{x} + \bar{x}\bar{y} + xy \tag{1.11}$$

$$\text{Mishna-Rechnitzer}: \quad \lambda_{MR}(x,y) = \bar{x}y + x\bar{y} + xy \tag{1.12}$$

where we use the notation

$$\bar{x} := \frac{1}{x}, \qquad \bar{y} := \frac{1}{y}\,. \tag{1.13}$$

Kreweras paths and Gessel paths are interesting as they have historically been significantly more difficult to solve exactly [4]. We will give some geometrical insight into why this is the case.

The Mishna-Rechnitzer paths are of interest as [2] and [4] found an infinite substitution group associated with the step set but did not characterise it.

In this paper, rather than substitution groups, we consider reflection groups (acting on $\mathcal{L}$ rather than λ) associated with the various step sets. All the finite reflection groups turn out to be finite dihedral reflection groups (isomorphic to the the corresponding substitution group).

Of particular relevance are the Weyl chambers associated with each group. We shall see that the relationship between the Weyl chamber of the refection group and the quarter plane is integral to the method of solution. In particular if the Weyl chamber corresponds to the quarter plane then the Bethe Ansatz method [5] and Gessel-Zeilberger theorem [6] are readily applied. This is the case for the Reverse Kreweras paths but *not* for Kreweras or Gessel paths.

How the infinite reflection group associated with the Mishna-Rechnitzer step set can be used to solve the quarter plane problem is not clear. In this context the Bethe Ansatz manifests itself as the same group orbit sum (1.14) that appears in the Gessel-Zeilberger theorem [6].

We briefly summarise the Gessel-Zeilbger theorem. Let R be a finite or affine root system, let $\mathcal{G}$ be a Weyl group and $\mathit{\Delta}$ any of basis for R. The length of an element $g \in \mathcal{G}$, denoted $\ell(g)$, is the least number of terms possible to express g as a product of fundamental reflections σ_α, $\alpha \in R$ (equivalently, the length of the reduced Coxeter word representing g). Let $\langle \cdot, \cdot \rangle$ be an inner product of the Euclidean space in which the root system is embedded. If $\mathcal{L}$ is a lattice of points invariant under a Weyl group $\mathcal{G}$ then the Weyl chamber (or fundamental region) of the group is the set of points

$$\mathcal{C}(\mathcal{G}) = \{x \in \mathcal{L} \,:\, \forall \alpha \in \mathit{\Delta},\ \langle x, \alpha \rangle > 0\}\,.$$

The walls of the chamber $\mathcal{C}(\mathcal{G})$ is the set of points

$$\bar{\mathcal{C}}(\mathcal{G}) = \{x \in \mathcal{L} \,:\, \forall \alpha \in \mathit{\Delta},\ \langle x, \alpha \rangle = 0 \text{ and } \forall \beta \in \mathit{\Delta} \setminus \alpha,\ \langle x, \beta \rangle \geq 0\}.$$

In the affine case the chamber becomes an alcove.

The Gessel-Zeilberger theorem expresses the number of lattice paths (using steps from some step set) that are confined to the Weyl chamber can be expressed in terms of lattice paths using steps from the same step set that are *not* confined to the chamber.

Thus, let $\mathrm{Walk}_m^{\mathcal{G}}(\mathbf{a} \to \mathbf{b})$ be the number of length m paths on $\mathcal{L}$ with step set $\mathcal{S}$ which start at $\mathbf{a}$, end at $\mathbf{b}$ and all points of are points the Weyl chamber $\mathcal{C}(\mathcal{G})$. Let $\mathrm{Free}_m(\mathbf{a} \to \mathbf{b})$ be the number of length m paths on $\mathcal{L}$ with step set $\mathcal{S}$ which start at $\mathbf{a}$, end at $\mathbf{b}$ (with no chamber constraint).

Theorem 1 (Gessel-Zeilberger [6]). *Let $\mathcal{L}$ be a lattice of points, $\mathcal{S}$ the path step set and $\mathcal{G}$ a Weyl group satisfying*

i) $\mathcal{L}$ is invariant under $\mathcal{G}$, and

ii) $\mathcal{S}$ is invariant under $\mathcal{G}$, and

iii) For all $\alpha \in \Delta$ and for all $s \in \mathcal{S}$, $\langle \alpha, s \rangle = \pm k(\alpha)$ where $k(\alpha)$ is a fixed number that only depends on α.

is given by

$$\mathit{Walk}_m^{\mathcal{G}}(\mathbf{a} \to \mathbf{b}) = \sum_{g \in \mathcal{G}} (-1)^{\ell(g)} \, \mathit{Free}_m(g(\mathbf{a}) \to \mathbf{b}) \tag{1.14}$$

where $g(a)$ is the action g on the lattice point a.

Note, the theorem *assumes* the Weyl chamber associated with a lattice path problem is known. This paper gives a method for determining the Weyl chamber and hence whether or not the theorem can be used. Since the step set is a subset of $\mathcal{L}$ if $\mathcal{G}$ leaves $\mathcal{S}$ invariant is also leaves $\mathcal{L}$ invariant. Condition iii) enforces the constraint the every path contributing to $\mathrm{Free}_m(g(a) \to b)$ that cross a wall of the chamber can only do so by stepping onto the wall first ie. it cannot "step over" a chamber wall.

Thus Theorem 1 can only give a solution to a short step quarter plane path problem if the quarter plane boundary corresponds to the walls of the Weyl chamber of the Weyl group of the step set (and lattice).

We will determine the Weyl group of the above four short step problems. Only the Reverse Kreweras paths have a Weyl chamber corresponding to the quarter plane. For the Kreweras paths the quarter plane us the union of two chambers (and the dividing wall). For the Gessel paths the quarter plane is the union of three adjacent chambers (and the chamber walls). For the Mishna-Rechnitzer paths the group is affine and hence the quarter plane is the union of an infinite number of chambers.

2 The step set and Weyl groups

The following follows from elementary combinatorics. The free Reverse Kreweras, Kreweras Gessel and Mishna-Rechnitzer t-path fixed length generating functions are

$$G_t^{\bar{\mathrm{K}}}(\mathbf{a}; x, y) = x^{a_1} y^{a_2} (x + y + \bar{x}\bar{y})^t \tag{2.1}$$
$$G_t^{\mathrm{K}}(\mathbf{a}; x, y) = x^{a_1} y^{a_2} (\bar{x} + \bar{y} + xy)^t \tag{2.2}$$
$$G_t^{\mathrm{G}}(\mathbf{a}; x, y) = x^{a_1} y^{a_2} (x + \bar{x} + \bar{x}\bar{y} + xy)^t \tag{2.3}$$
$$G_t^{\mathrm{MR}}(\mathbf{a}; x, y) = x^{a_1} y^{a_2} (\bar{x}y + x\bar{y} + xy)^t \tag{2.4}$$

where $\mathbf{a} = (a_1, a_2)$.

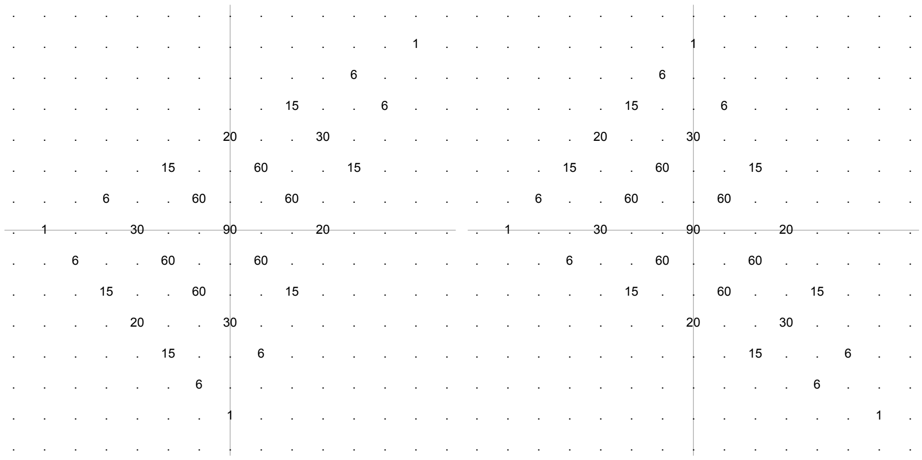


Figure 1: Pascal slices for $t = 6$ for (left) free Kreweras and (right) free Reverse Kreweras paths.

The primary objects of interest are $\mathcal{F}_t^{\mathrm{P}}(\mathbf{0} \to \mathbf{b})$, where $\mathbf{0} = (0,0)$, which we will consider as defining integer valued counting functions on $\mathcal{L}$, that is

$$F_t^{\mathrm{P}} : \mathcal{L} \to \mathbb{Z}\,;\,(b_1, b_2) \mapsto [x^{b_1} y^{b_2}] G_t^{\mathrm{P}}(\mathbf{0}; x, y) \tag{2.5}$$

The coefficients of the free generating functions are trinomial coefficients giving the following result.

Proposition 1. *The free Reverse Kreweras, Kreweras Gessel and Mishna-Rechnitzer t-path counting functions are:*

$$F_t^{\bar{K}}(a,b) = \binom{t}{\frac{1}{3}(t-2a-b), \frac{1}{3}(t+a+2b), \frac{1}{3}(t+a-b)} \tag{2.6a}$$

$$F_t^{K}(a,b) = \binom{t}{\frac{1}{3}(t-2a+b), \frac{1}{3}(t+a-2b), \frac{1}{3}(t+a+b)} \tag{2.6b}$$

$$F_t^{G}(a,b) = \sum_{k\geq 0} \binom{t}{k, k-a+b, -k+\frac{1}{2}(a-2b+t), -k+\frac{1}{2}(a+t)} \tag{2.6c}$$

$$F_t^{MR}(a,b) = \binom{t}{\frac{1}{2}(t-a), \frac{1}{2}(t-b), \frac{1}{2}(a+b),} \tag{2.6d}$$

where the multinomials are assumed to vanish if the value is not a positive integer.

Examples of theses functions are shown in figure 1. These plots are the analogues of rows in Pascal triangle (but now correspond to slices of a "Pascal" cone). We will thus call them **Pascal slices**.

Definition 1. *Let F_t^P be the map (2.5) and $g\colon \mathcal{L} \to \mathcal{L}$ a distance preserving bijection that leaves F_t^P (or the Pascal plot) invariant. The set of such maps will be called the* ***reflection group****, or isometry group, of the step set $\mathcal{S}_P$.*

For the Kreweras, Reverse Kreweras and Gessel paths we shall see that the same finite group leaves F_t^P invariant *independent* of t, whilst for Mishna-Rechnitzer paths the group is different for every path length t (but they are all subgroups of a single infinite group - an affine dihedral group).

If G is a group of isometries of $\mathcal{L}$ then the action of $g \in G$ on F_t^{P}, denoted $g \cdot F_t^{\mathrm{P}}$ is defined in the usual way as

$$(g \cdot F_t^{\mathrm{P}})(\mathbf{b}) = F_t^{\mathrm{P}}(g^{-1} \cdot \mathbf{b}). \tag{2.7}$$

Since $F_t^{\mathrm{P}}(\mathbf{b})$ is given by trinomial coefficients the isometry group is related to the symmetry of the trinomial coefficients. Clearly the trinomial

$$\binom{t}{n_1, n_2, n_3} = \frac{t!}{n_1! n_2! n_3!} \tag{2.8}$$

is invariant under any permutation of n_1, n_2 and n_3. Thus to find the isometry groups of each of the types of free path we look for isometries of $\mathcal{L}$ that permute the lower three parameters of the trinomials (2.8). For example, for the free Kreweras paths from (2.6b):

$$n_1(a,b) = \frac{1}{3}(t - 2a + b), \tag{2.9a}$$

$$n_2(a,b) = \frac{1}{3}(t + a - 2b), \tag{2.9b}$$

$$n_3(a,b) = \frac{1}{3}(t + a + b) \tag{2.9c}$$

and using the reflection

$$g\colon (a,b) \mapsto (a,b) \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

give immediately the permutation

$$n_1 \mapsto n_1, \qquad n_2 \mapsto n_3, \qquad n_3 \mapsto n_2.$$

and hence g leaves (2.6b) unchanged.

In the case of the Gessel paths it is only the sum of multinomials, (2.6c), that must be invariant. Thus a permutation of the trinomial indexes or a permutation of the trinomials in the sum would suffice. We only consider the former.

2.1 Weyl groups for Reverse Kreweras and Kreweras paths

Reverse Kreweras. The six possible permutations of the trinomial parameters give the following theorem.

Theorem 2. *The isometry group of the free Reverse Kreweras paths is an order six dihedral group D_3 and have the representation*

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{2.10}$$

$$g_3 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad g_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.11}$$

The elements g_1 and g_2 represent reflections with reflection lines having the Cartesian equations $x = 0$ and $y = 0$ respectively. As a Coxeter system D_3 is generated by the two reflections g_1 and g_2 and has presentation

$$g_1^2 = g_0 \qquad g_1^2 = g_0 \qquad (g_1 g_2)^3 = g_0 \tag{2.12}$$

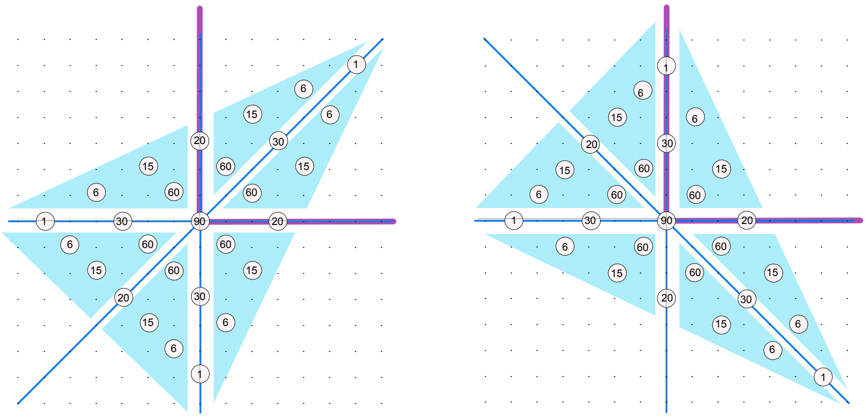


Figure 2: Weyl Chambers: (left) Kreweras and (right) Reverse Kreweras

Thus $g_4 = g_1 g_2$ and $g_5 = g_2 g_1$ are order three rotations.

The two generating reflections are *not* perpendicular to their reflection lines but reflect at an oblique angle. The directions of the reflections are given by the left eigenvectors corresponding to eigenvalue -1 which are $(-2, 1)$ and $(-1, 2)$ for g_1 and g_2 respectively.

The oblique reflections can be seen as non-diagonal bilinear functional as follows. Conventionally reflections $s_{\boldsymbol{\alpha}}$ across a hyperplane (a line in this case) defined by a vector $\boldsymbol{\alpha}$ corresponds to the map

$$s_{\boldsymbol{\alpha}} : \mathbf{a} \mapsto \mathbf{a} - \frac{2}{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle} \langle \mathbf{a}, \boldsymbol{\alpha} \rangle \boldsymbol{\alpha}$$

where the bilinear functional $\langle \cdot, \cdot \rangle$ is symmetric and positive definite and represented by the matrix $\mathbf{B}$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{B} \cdot \mathbf{b}^T$$

where $\mathbf{b}^T$ is the transpose. For the Reverse Kreweras the matrix is

$$\mathbf{B}^{\bar{\mathrm{K}}} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

A basis for $\mathcal{L}$ can be chosen that would diagonalise $\mathbf{B}$ and hence change the reflections to perpendicular but this would then mean the quarter plane boundaries would not be perpendicular.

The dihedral symmetry is readily seen in the Pascal slices - figure 2 (right) shows the case for $t = 6$ and shows the three reflection lines.

Kreweras. For free Kreweras paths the result is similar, differing by a rotation of π from the Kreweras counting function. The trinomial parameters are:

$$n_1(a, b) = \frac{1}{3}(t - 2a + b), \tag{2.13a}$$

$$n_2(a, b) = \frac{1}{3}(t - 2a + b), \tag{2.13b}$$

$$n_3(a, b) = \frac{1}{3}(t + a + b) \tag{2.13c}$$

Isometries of $\mathcal{L}$ which permute n_1, n_2 and n_3 are given by the following theorem.

Theorem 3. *The elements of the isometry group of the free Kreweras is an order six dihedral group D_3 and have have representation*

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \tag{2.14}$$

$$g_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.15}$$

$$\tag{2.16}$$

The proof of the theorem is a simple verification of the permutations of (2.13) using the matrices.

The elements g_1 and g_2 represent oblique reflections with reflection lines having the Cartesian equations $x = 0$ and $y = 0$ respectively. The reflections are *not* perpendicular to the lines but are at an oblique angle. The direction of the reflection is given by the eigenvector corresponding to eigenvalue -1 which are $(2, 1)$ and $(1, 2)$ for g_1 and g_2 respectively. The dihedral symmetry is readily seen from a Pascal slice - Fig. 2 (left) shows the case for $t = 6$ and shows the three reflection lines.

Weyl Chambers Rather than giving a formal definition of Weyl chambers (which can be found in any textbook on reflection/Coxeter groups) we give a brief informal description. For $\mathcal{L}$ the orbits of points in $\mathcal{L}$ under the action of a dihedral group partitions $\mathcal{L}$ is subsets defined by the orbits. For $\mathcal{L}$ the orbits of size less than n correspond to points on any reflection line. The set of points in orbits of size n partition into disjoint "cones" which are separated by the reflection lines. The action of the group on any cone will generate all the points of the other cones. Conventionally, a cone whose points have positive coordinates is chosen - this is the Weyl chamber. For Reverse Kreweras and Kreweras the cones and Weyl chamber can be seen in Fig. 2 and correspond to the set of points between the lines $y = x$ and $y = 0$ and $x = 0$ and $y = 0$ respectively.

The essential difference between Reverse Kreweras and Kreweras is apparent from figure Fig. 2. For the Reverse Kreweras paths the quarter plane corresponds exactly to the Weyl chamber for the free paths, whilst for the Kreweras paths the quarter plane is bisected by a reflection line. This means a theorem such as that due to Gessel and Zielberger [6] cannot be used for the latter path problem (which requires the chamber walls correspond to the boundary of the domain).

It also means is that the standard Bethe Ansatz orbit Ansatz,

$$G_t^{\bar{\mathrm{K}}}(\mathbf{a} \to \mathbf{b}) = F_t^{\bar{\mathrm{K}}}(\mathbf{b}) \sum_{g \in D_3} (-1)^{|g|} x^{g^{-1}(a_1)} y^{g^{-1}(a_2)} \tag{2.17}$$

where $\mathbf{a} = (a_1, a_2)$, D_3 is given by the representation in Theorem 2 and $|g|$ is the number of reflections g, solves the Reverse Kreweras enumeration problem (ie. gives the correct boundary conditions).

In the case of Kreweras

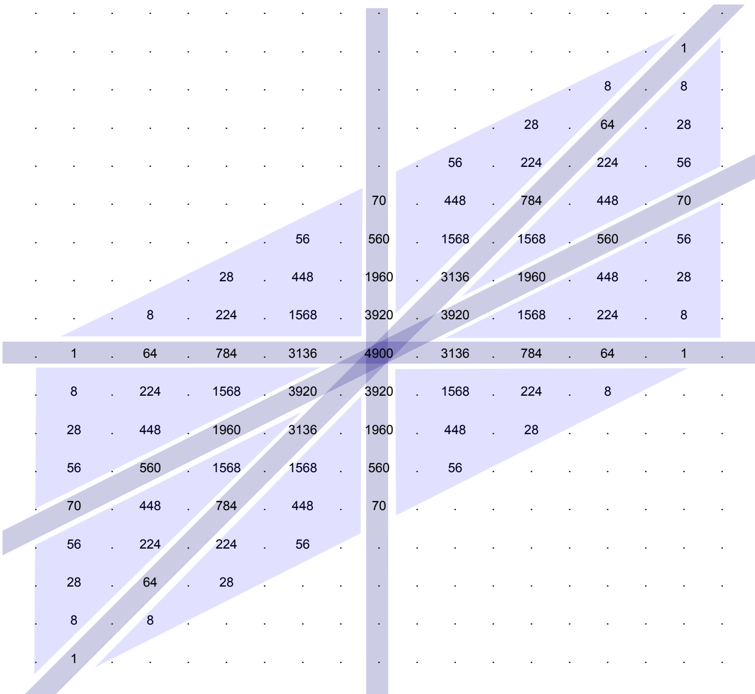
$$G_t^{\bar{\mathrm{K}}}(\mathbf{a} \to \mathbf{b}) \neq F_t^{\bar{\mathrm{K}}}(\mathbf{b}) \sum_{g \in D_3} (-1)^{|g|} x^{g^{-1}(a_1)} y^{g^{-1}(a_2)} \tag{2.18}$$

where D_3 is given by the representation in Theorem 3, the right hand side enforces a line of zeros along *all* reflection lines. Since the Reverse Kreweras has no reflection line through the quarter plane (2.17) is the solution, however for the Kreweras there is reflection line bisecting the quarter plane and thus the orbit Ansatz does not solve Kreweras.

This gives some insight as to why Kreweras is so much more difficult to solve. The method used by Bousquet-Mélou uses a "half-orbit" sum which almost solves the problem except for a "line of defects". This line can be removed by using Gessel factorisation [4].

2.2 Weyl group for Gessel paths

A Pascal slice for $n = 8$ is shown below. From this example it is clear there are four lines of reflection with oblique angles of reflection across all reflection line. In this form not all the angles between the lines of reflection are rational multiples of π, however the rotations are rational multiples of π (see Fig. 3) as required of a finite dihedral group.



The isometry group of the free Gessel paths is an order eight dihedral group D_4 which have matrix representations given in Fig. 3 and Fig. 4.

$g_5 = g_1 g_4$	$g_6 = g_1 g_3$	$g_7 = g_1 g_2$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{5\pi}{4}$

Figure 3: The three rotation elements of the order eight dihedral group D_4 of the free Gessel paths showing the representations and the angle of rotation.

g_1	g_2	g_3	g_4
$\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$
$(1,1)\,,(1,0)$	$(2,1)\,,(0,1)$	$(0,1)\,,(2,1)$	$(1,0)\,,(1,1)$
$-1 \qquad 1$	$-1 \qquad 1$	$-1 \qquad 1$	$-1 \qquad 1$
$y=0$	$x=0$	$y=x/2$	$y=x$
$\frac{\pi}{4}$	$\frac{\pi}{2}-\gamma$	γ	$\frac{3\pi}{4}$

Figure 4: The four reflection elements of the order eight dihedral group D_4 of the free Gessel paths showing the representations, the left eigenvectors, the eigenvalues, the line of reflection and the oblique angle of reflection where $\gamma = \arctan(2) = 1.10715\ldots$

The Gessel path Weyl chamber is the set of points between the line $y = 0$ and $y = x/2$. Similar to the Kreweras paths, the quarter plane does not coincide with the Weyl chamber so neither the Gessel-Zeilberger theorem nor the Bethe Ansatz will solve the problem.

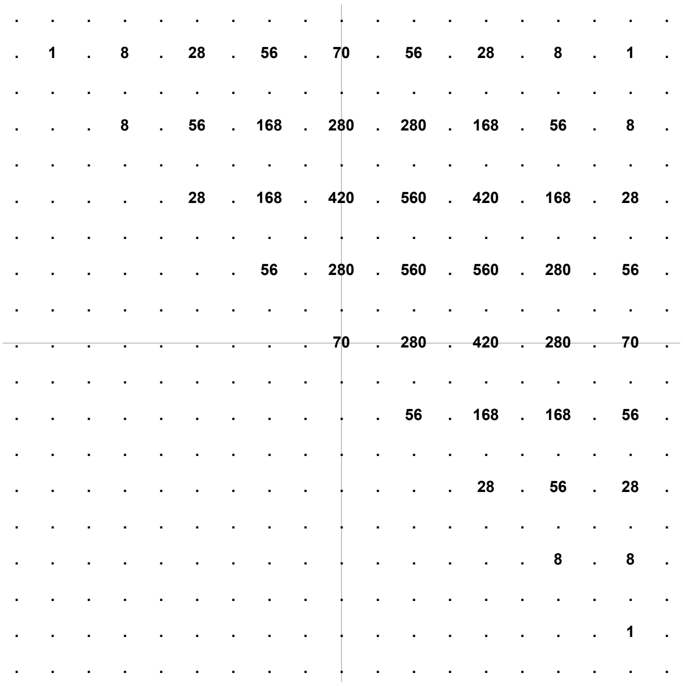


Figure 5: A Pascal slice for Mishna-Rechnitzer paths corresponding to $t = 8$.

2.3 Affine dihedral group for Mishna-Rechnitzer paths

The group structure of Mishna-Rechnitzer paths is more complex. From (1)

$$F_t^{\mathrm{MR}}(a,b) = \left(\begin{matrix} t \\ \frac{1}{2}(t-a), \frac{1}{2}(t-b), \frac{1}{2}(a+b), \end{matrix}\right)$$

The step set barycenter is at $(\frac{1}{3}, \frac{1}{3})$ giving a path drift of $(\frac{t}{3}, \frac{t}{3})$.

From the trinomial, $F_t^{\mathrm{MR}}(a,b)$, is invariant if the isometry permutes the three functions

$$n_1(a,b) = t - a, \tag{2.19a}$$
$$n_2(a,b) = t - b, \tag{2.19b}$$
$$n_3(a,b) = a + b\,. \tag{2.19c}$$

A Pascal slice is shown in Fig. 5. These three equations differ fundamentally from the previous path problems in that one equation, n_3, does not depend on t. The absence of t from n_3 forces us to extend the isometries from reflections and rotations to include translations, that is, to use affine maps. For example, if we try

$$g_{321} \colon (a,b) \mapsto (a,b) \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} + t(\tau_1, \tau_2)$$

where (τ_1, τ_2) is some row vector. Substituting into (2.19) we see that the three equations are permuted

$$n_1 \mapsto n_3\,, \quad n_2 \mapsto n_2 \quad \text{and} \quad n_3 \mapsto n_1$$

only if $\tau_1 = 0$ and $\tau_2 = 1$. Using affine isometries gives the following theorem.

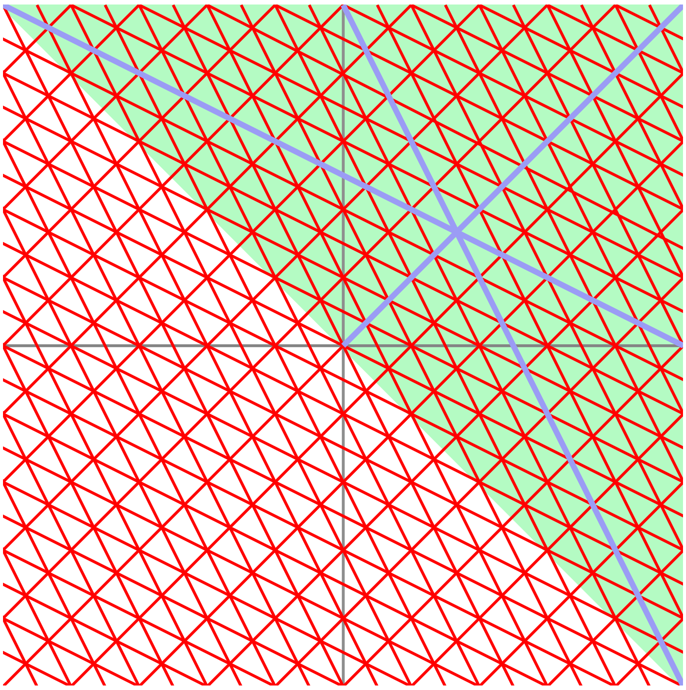


Figure 6: The alcoves of the affine $\check{D}_3$ group of the Mishna-Rechnitzer paths. The blue lines are the reflection planes which leave the $t = 10$ (hence they intersect at $(\frac{10}{3}, \frac{10}{3})$) Pascal slice invariant.

Theorem 4. *The elements of the isometry group of the free Mishna-Rechnitzer paths is an affine dihedral group $\check{D}_3$ where every element of the group is of the form*

$$T_\sigma(k_1, k_2)\colon (a, b) \mapsto (a, b)g_\sigma + (k_1, k_2) \tag{2.20}$$

where the linear maps g_σ have representations:

$$g_{123} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_{132} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad g_{321} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \tag{2.21}$$

$$g_{312} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad g_{231} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad g_{213} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.22}$$

The Mishna-Rechnitzer trinomial parameters (2.19) *are permuted by the subgroup of elements*

$$\begin{array}{ccc} T_{123}(0,0) & T_{213}(0,0) & T_{132}(0,t) \\ T_{321}(t,0) & T_{312}(t,0) & T_{231}(0,t) \end{array} \tag{2.23}$$

Again, the proof of the theorem is a simple substitution of the affine maps into (2.19) and showing each permutes n_1, n_2 and n_3. Since the group is affine we now have Weyl alcoves rather than chambers. Since the alcoves are a finite set of elements they cannot correspond to the quarter plane.

It is not clear how this group structure can be used in conjunction with some sort of orbit Ansatz to solve the enumeration problem.

3 Acknowledgement

We would like to thank the Australian Research Council (ARC) and the Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS) for financial support.

References

[1] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random Walks in the Quarter-Plane*. Springer- Verlag Berlin Heidelberg, 1999.

[2] Marni Mishna and Andrew Rechnitzer. Two non-holonomic lattice walks in the quarter plane. *Theoret. Comput. Sci.*, 410:3616–3630, 2009.

[3] G. Kreweras. Sur une classe de problèmes liés au treillis des partitions d'entiers. *Cahiers du B.U.R.O.*, 6:5–105, 1965.

[4] Mireille Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *The Annals of Applied Probability*, 15:1451–1491, 2005.

[5] R. Brak, J. Essam, and A. L. Owczarek. From the bethe ansatz to the gessel- viennot theorem. *Annals of Comb.*, 3:251–263, 1998.

[6] I. M. Gessel and D Zeilberger. Random walk in a weyl chamber. *Proc. Amer. Math. Soc.*, 115(1):27–31, 1992. readme.