Asymptotics of Weighted Reflectable Walks in Weyl Chambers

Consider d-dimensional weighted walks encoded in the generating function,

$$\frac{G(\mathbf{x})}{(1-x_1)\cdots(1-x_d)(1-x_1\cdots x_d tS(1/\mathbf{x}))},$$

where $S(\mathbf{x})$ is the weighted inventory of the model. Then, consider the case of large weights, where the critical point occurs at $\mathbf{z}_* := (\mathbf{1}, 1/S(\mathbf{1}))$, and where $G(\mathbf{1}) \neq 0$. Applying Theorem 10.3.1 from Pemantle and Wilson, we have:

$$\Gamma_\Psi := \begin{pmatrix} x_i \frac{\partial}{\partial x_i}(1-x_1) \\ x_i \frac{\partial}{\partial x_i}(1-x_2) \\ \vdots \\ x_i \frac{\partial}{\partial x_i}(1-x_d) \\ x_i \frac{\partial}{\partial x_i}(1-x_1\cdots x_d tS(1/\mathbf{x})) \end{pmatrix} = \begin{pmatrix} -x_1 & 0 & \dots & 0 & 0 \\ 0 & -x_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -x_d & 0 \\ * & * & * & * & -x_1x_2\dots x_d tS(1/\mathbf{x}) \end{pmatrix},$$

where the starred terms are complicated but irrelevant. Thus,

$$\det \Gamma_\Psi \Big|_{(\mathbf{1},1/S(\mathbf{1}))} = (-1)^{d+1}.$$

Plugging into the Theorem statement, the number of walks of length n is asymptotic to

$$(\mathbf{z}_*)^{-\mathbf{n}} \frac{G(\mathbf{z}_*)}{\det \Gamma_\Psi} = (-1)^{d+1} S(\mathbf{1})^n G(\mathbf{1}).$$

It looks like a sign is off here.