The Tandem Model

Working Notes

MRC1b Group[*]

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1 Introduction

Consider the model of lattice path walks defined by steps from the set $\mathcal{T} = \{[1,0],[-1,1],[0,-1]\}$ restricted to the first quadrant. In the literature, these have taken the name the "Tandem Model" owing to the fact that they can model two queues operating in tandem. The generating function of this model is algebraic, and there is an explicit form.

Our goal here is two-fold: First, we would are interested in the enumeration of walks in *weighted* tandem models in a form such that generic weights appear in the formula. This in turn permits a clear picture of the reliance of the final asymptotic formulas on the drift of the initial model. Secondly, we strive to understand the interaction and connections between several different techniques. Can the use of multiple techniques simultaneously facilitate the computations? This may imply easier computations, nicer forms for answers, or suggest means for random generation. In each case we derive expressions for the dominant singularity of the generating function, and we compare them. It also allows us ot interpolate between $Q(0,0)$ and $Q(1,0)$.

Once we have a conclusion, we can add to this. $\mathcal{M}$

2 The unweighted case

2.1 Background

Connection to Young tableau of bounded height $\mathcal{M}$

There is a simple size preserving bijection from the set of walks in the tandem model to standard Young tableaux of height bounded by 3. Recall, a standard Young tableau is a filling of a Ferrer's diagram that is strictly increasing along the rows and columns.

We denote the steps of a tandem walk by N, SE, W and remark that the condition of staying in the quarter plane implies that at any point in the walk the number of W is at most the number

[*]using computations of S Melczer and J Courtiel

X

Figure 1: A random tandem walk

of SE steps, which is in turn bounded by the number of N steps. The aforementioned bijection is as follows. The image of a walk $w = (w_1, w_2, \ldots, w_n) \in \{N, SE, W\}^n$ is a filling of the partition $(\lambda_1, \lambda_2, \lambda_3)$ of n, where λ_1 is the number of N steps, λ_2 is the number of SE steps and λ_3 is the number of W steps. The quarter plane condition ensures that this is a Ferrers diagram. To fill this diagram, parse the w. If $w_i = N$, (resp. SE, W) then the number i is added to the top row (resp. second, third). Remark that the endpoint of a walk is $(\#N - \#SE, \#SE - \#W)$.

For example the image of $w = (N, N, SE, N, W, SE, N)$, which is a tandem walk that ends at $(2, 1)$ is the tableau $(1, 2, 4, 7)(3, 6)(5)$. *Format this.* $\mathcal{M}$

In fact, there are many combinatorial classes in bijection. Many of them are listed in the OEIS entry A001006. For example, number of ways of drawing any number of nonintersecting chords joining n (labelled) points on a circle is the same as the number of tandem walks of length n.

2.2 Exact enumeration

Let $a_{i,j}(n)$ denote the number of walks with steps from $\mathcal{T}$, starting at the origin, ending at the point (i, j) and staying in the first quadrant. We can use the bijection to derive an explicit formula for $a_{i,j}(n)$.

The enumeration of standard Young tableaux is a well explored topic, and in this case, we use the well-known hook formula to enumerate tableaux of a given shape. The number of standard young tableaux of shape λ is given by

$$|SYT(\lambda)| = \frac{n!}{\prod_{x\in[\lambda]} h_x}$$

where the product is taken over all boxes x in the ferrer's diagram, and h_x is the sum of the boxes to the right and below the box x in the diagram.

If a walk ends at point (i, j) and is of length n, then $n = \#N + \#SE + \#W$ and $i = \#N - \#SE$ $j = \#SE - \#W$ so it is in bijection with a partition of shape $\lambda = (n + 2i + j, n - i + j, n - i - 2j)$.

I don't recall how to translate the hook length formula into the one below. Remark: $n-i-2j = 3\#W$, $n+2i+j = 3\#N$, $n-i+j = 3\#S$, hence it is of the shape $\lambda = (n+2i+j, n-i+j, n-i-2j)$ $\mathcal{M}$

$$a_{ij}(n) = \begin{cases} \frac{(i+1)(j+1)(i+j+2)n!}{(n/3-i/3-2/3\,j)!\,(n/3-i/3+j/3+1)!\,(n/3+2/3\,i+j/3+2)!} & \text{if } 3|n-i-2j, 3|n-i+j, \text{ and } 3|n+2i+j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

2.3 Generating function expression

Expand upon this $\mathcal{M}$

$$Q(1, 1; t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2}.$$

Exponential GF: $\exp(t)\,\mathrm{BesselI}(1, 2t)/t$.

2.4 Functional equation

Although we consider several strategies, each relies on an analysis of the *complete generating function* defined as the formal series in $\mathbb{Q}[x, y][[t]]$:

$$Q(x, y; t) = \sum_{i,j,n} a_{ij}(n)\, x^i y^j t^n.$$

We remark that several evaluations are of interest. The series $Q(x, 0; t)$ marks the walks that return the x-axis and tracks the position. The series $Q(0, 0; t)$ is the generating function of *excursions*, that is walks that return to $(0, 0)$, and the series $Q(1, 1; t)$ is the generating function of walks that end anywhere.

The complete generating function satisfies the following functional equation

$$K(x, y)Q(x, y; t) = xy - K(x, 0)Q(x, 0; t) - K(0, y)Q(0, y; t) \quad \text{with} \quad K(x, y) = xy(1 - t(x + \frac{y}{x} + \frac{1}{y})). \tag{2}$$

2.5 Differential equation

Given a series development, it is possible to guess a candidate differential equation for which the series is a solution. This is implemented in Maple using the `gfun` package [], for example. The initial series development of $Q(x, y; t)$ is straightfoward to compute using a basic recurrence. This process is robust enough to handle additional variables, although practically speaking, we were only able to guess differential equations for functions with one additional variable. Using the differential equations guessed for $K(x, 0)Q(x, 0)$ and $K(0, y)Q(0, y)$, using equation Eqn. (**??**), and the closure properties of differential equations, we found a candidate differential equation satisfied by $K(x, y)Q(x, y; t)$. These are all located in the appendix. *Put these in the appendix. PRobably not the one for $Q(x, y; t)$ though.* $\mathcal{M}$

The dominant singularity of the generating function is a root of the leading term. Once the value is identified, a local solution expansion around this point gives the sub-exponential factor in the asymptotic growth.

The leading coefficient *mention the order* for the candidate DE for $K(x, 0)Q(x, 0)$ is: $\mathcal{M}$

$$\begin{aligned} p(x,t) &= t^4\,(3\,t-1)\left(9\,t^2+3\,t+1\right)\left(x^3t^2-2\,tx^2-4\,t^2+x\right) \\ &\qquad \left(63\,t^4x^3-5\,x^4t^2-99\,t^3x^2+288\,t^4+5\,x^3t-52\,xt^2+4\,x^2\right) \\ &= t^4\,(3\,t-1)\;p_1(x,t)\,p_2(x,t)\,p_3(x,t). \end{aligned}$$

We are interested in positive real valued solutions to $p(x, t)$ for fixed non-negative real values of x. Such a value exists by Pringsheim's theorem. These are the candidates for dominant singularity. When $x = 0$ the only real postive root is $t = 1/3$. Thus, the dominant singularity of $Q(0, 0; t)$ is $1/3$. There are two real roots when $x = 1$, $t = 1/3$ and

$$t = \frac{\left(25381+819\,\sqrt{518}\right)^{2/3} - 2\,\sqrt[3]{25381+819\,\sqrt{518}} + 667}{117\,\sqrt[3]{25381+819\,\sqrt{518}}} \sim 0.4461\ldots.$$

The latter is a root of $p_3(1, t)$.

Is it fair to say that from this process we cannot decide which one is the dominant singularity? $\mathcal{M}$

Other evaluations will be important for the weighted case, as we shall see in Section **??**. Most important, in fact, will be how the dominant singularity changes as a function of x. The point $x = 1$ is very important, and behaviour changes both in this set up, and in the solution given by an integral.

$\mathcal{M}$

(i) *Note how many terms would have been necessary for the $Q(x, y; t)$ DE. Can we do this compuation?*

(ii) *Does desingularization help?*

(iii) *We need to prove that the DEs are correct. Possibilities: (1) Using the explicit form of $a_{ij}(n)$ of Equation **??**, possibly with computer algebra, if the computations get too messy; (2) Try to get a diagonal form for $Q(x, 0)$ (3) Are there bounds that we can compute?.*

2.6 Analytic expression for $Q(x, 0)$ as an integral

Since the tandem model is a small step model, it fits into the framework of Fayolle/Kurkova/Raschel. As a first result, since the drift is zero, by Fayolle/Raschel [?] the dominant singularity is the number of steps, that is $\rho = \frac{1}{3}$.

We can also describe the integral form.

The following expressions for $Q(x, 0)$ and $Q(0, y)$ can be found in [?, Theorem 1] (with $(i-1, j-1)$ denoting the starting point of the walk):

$$\begin{cases} K(x,0)Q(x,0) &= \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{M}} u^i Y_0(u)^j \frac{w'(u)}{w(x)-w(u)}\mathrm{d}u, \\ K(0,y)Q(0,y) &= \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{L}} X_0(u)^i u^j \frac{\widetilde{w}'(u)}{\widetilde{w}(y)-\widetilde{w}(u)}\mathrm{d}u, \end{cases} \tag{3}$$

provided that $w(0) = \widetilde{w}(0) = \infty$, which will hold with our choice (**??**).

$$Y_1(x) = 1/2\,\frac{-t+x+\sqrt{-4\,t^2x^3+t^2-2\,tx+x^2}}{tx} \quad Y_0(x) = -1/2\,\frac{t-x+\sqrt{-4\,t^2x^3+t^2-2\,tx+x^2}}{tx}$$

$$X_1(y) = 1/2\,\frac{-ty^2+y+\sqrt{t^2y^4-2\,ty^3-4\,t^2y+y^2}}{t} \quad X_0(y) = -1/2\,\frac{ty^2+\sqrt{t^2y^4-2\,ty^3-4\,t^2y+y^2}-y}{t}$$

$\mathcal{M}$ is the curve $X([y_1, y_2])$ and $\mathcal{L} = Y([x_1, x_2])$. Equation (**??**) provides contour integral expressions. In particular, with this formula the generating functions are defined only in the interior of the curves $\mathcal{M}$ and $\mathcal{L}$. The conformal gluing functions w and $\widetilde{w}$ are defined by (see [?, Theorem 3 (iv)])

$$\begin{cases} w(x) &= \dfrac{t}{x^2} - \dfrac{1}{x} - tx, \\ \widetilde{w}(y) &= ty^2 - y - \dfrac{t}{y}. \end{cases} \tag{4}$$

Functions w and $\widetilde{w}$ satisfy $w = \widetilde{w}(Y_0)$ and $\widetilde{w} = w(X_0)$, and in particular $w(X_0(Y_0)) = w$.

Update the caption for the figure $\mathcal{M}$

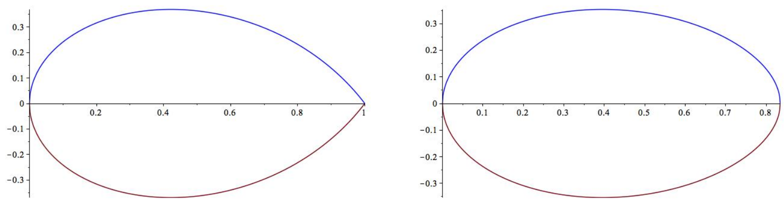

Figure 2: The curve $\mathcal{M}$ for $t = 1/3$ and $t = 1/3 - \epsilon$

2.6.1 Singularities of $Q(0,0)$

2.6.2 Analytic continuation

One has

$$K(x,0)Q(x,0) = K(X_0(Y_0(x)),0)Q(X_0(Y_0(x)),0) + Y_0(x)(x - X_0(Y_0(x))). \tag{5}$$

See [?, Equation (4.4)] for the original derivation of the above identity.

The computation of $Q(\alpha, 0; \gamma t)$ and $Q(0, \beta; \gamma t)$ depend on the location of α and β w.r.t. the curves $\mathcal{M}$ and $\mathcal{L}$. If there are inside we can use the integral expressions given in (**??**). If not, one has to use first (**??**).

2.7 Orbit Sum

This is a candidate for the orbit sum method, and it was computed by Bousquet-Mélou and Mishna [**?**].

Group of the walk The group is generated by two involutions, and is of order 6.

$$[[[\iota, \psi, \iota], [y^{-1}, x^{-1}]], [[\psi, \iota], [\frac{y}{x}, x^{-1}]], [[\iota, \psi], [y^{-1}, \frac{x}{y}]], [[\psi], [x, \frac{x}{y}]], [[\iota], [\frac{y}{x}, y]], [[], [x, y]]]$$

Positive series extraction expression

$$Q(1,1;t) = [x^{>0}][y^{>0}]\frac{-y^3x^3 + x^4y + xy^4 - x^3 - y^3 + xy}{x^2y^2\,(tx^2y + ty^2 + tx - xy)} \tag{6}$$

Problem 1. *We can express the generating function for $Q(x, 0; t)$ as a diagonal. Use this to verify the differential equation. The program of mathematica.*

2.8 ACSV

Put in the details from the expression for the counting generating function [?] $\mathcal{M}$

2.9 Creative Telescoping

Has solution as an integral of hypergeometric functions (van Heoij)

$Q(1,0;t) = (1+t)^{1/2}(1-3t)^{3/2} Int((hypergeom([-2/3,-1/3],[1],27t)-1+4t(hypergeom([-1/3,1/3],[2],27t^3)-1)+$

Get this entry from the table of Chyzak. $\mathcal{M}$

3 The weighted case

The weighted version of a model uses the same directions and assigns non-negative real valued weights. When the weights are integers, it can be interpreted as multiple distinct copies of the same step. When they are positive and sum to one, it is a probability. Kauers and Yatchuk [?] determined that all possible non-negative weightings of this model will lead to D-finite generating functions.

Let the weights, respectively, be $w = (\alpha, \beta, \gamma)$ for $[1, 0]$ (E), $[-1, 1]$ (NW) and $[0, -1]$ (S). The weight of a walk is the product of the weight of each step. For example the weight of the walk E E NW S is $\alpha^2\beta\gamma$. We are interested in the series

$$Q^{(w)}(x, y; t) = \sum_{w \in \mathcal{W}} \text{weight}(w)\, x^{x(w)} y^{y(w)} t^{\text{length}(w)}. \tag{7}$$

3.1 From weights to parameters

The generating function for the weighted model can be expressed by an algebraic substitution evaluation of the unweighted model.

Proposition 1. *The complete generating function for $\mathcal{T}^{(w)}$ the weighted tandem model with weight vector (α, β, γ), denoted $Q^{(w)}(x, y; t)$ satisfies:*

$$Q^{(w)}(x, y; t) = Q\left(\frac{\alpha}{\gamma\beta}x, \frac{\beta}{\gamma}y; \frac{t}{\alpha\beta\gamma}\right). \tag{8}$$

Here $Q(x, y; t)$ is the generating function for the unweighted model, equivalently for $w = (1, 1, 1)$.

Proof. *Proof using the explicit formula.* □ $\mathcal{M}$

We recall Lemma 16 from [?]. If $F(a, u) = \sum f_n(a)u^n$ is a formal power series in u with coefficients in $\mathbb{N}[a]$ such that $f_n(a)$ has degree at most n. Assume further that F is not a polynomial.

For $a \geqslant 0$, let $\rho(a)$ be the radius of convergence of the series $F(a, \cdot)$. Then ρ is a non-increasing function on $[0, \infty)$ which is finite and and continuous on $(0, +\infty)$.

This means that the radius of convergence of $Q(x, 0; t)$ as a series in t is continuous as a function of x.

3.2 Analysis of differential equations

As we noted earlier, the radius of convergence of $Q(x, 0; t)$ as a function of t can be expressed as a function of x for non-negative real values of x.

I think a few well chosen figures would make this very clear. $\mathcal{M}$

3.3 Integral form

The integral form in Equation (**??**) holds only for $|x| < 1$. This means that we can only find an expression for certain weightings using this strategy.

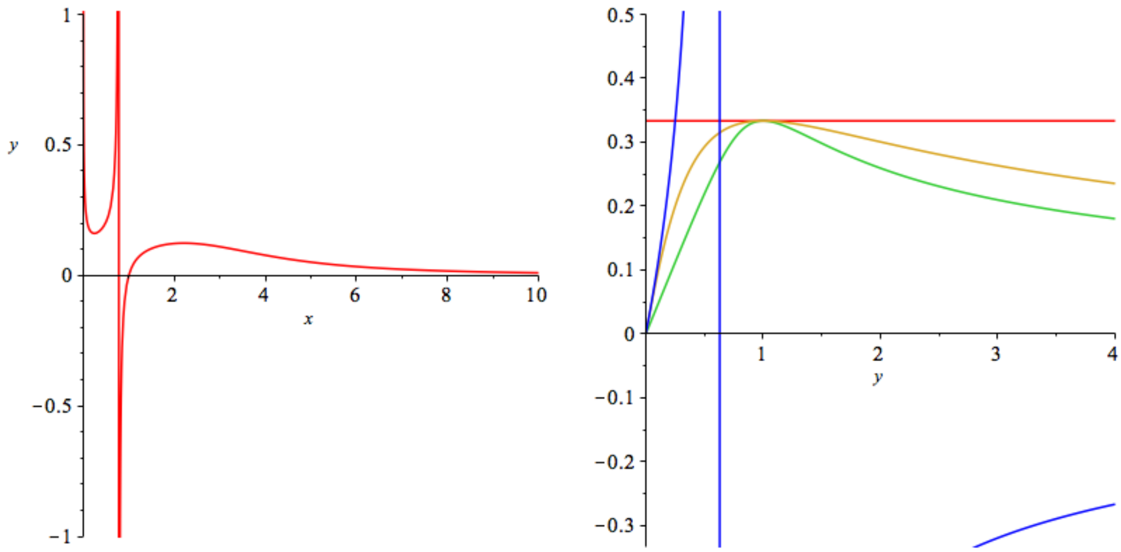

Figure 3: *Left.* Solutions of $p_3(x, t)$ for $0 \leqslant x$ and t. *Right.* Real solutions to factors of $p(x, t)$

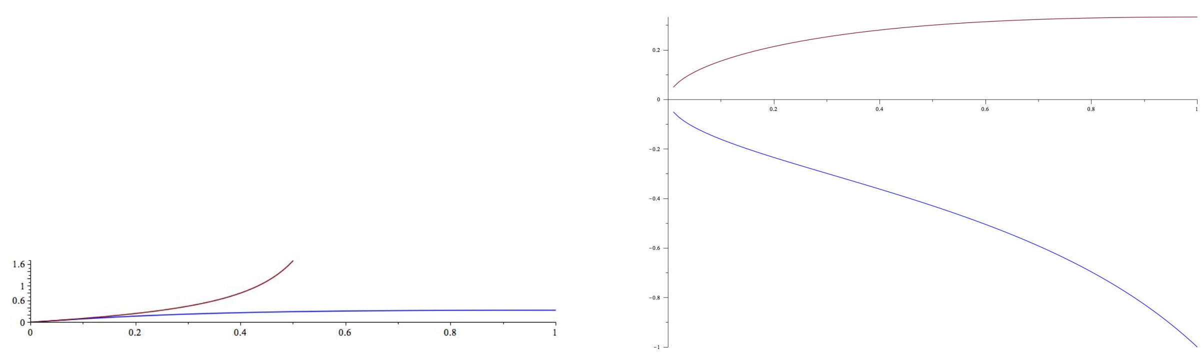

Figure 4: *Left.* Singularities coming from $X_0(x)$ for different values of x. t. *Right.* Singularities coming from $Y_0(y)$ for different values of y.

3.4 Orbit Sum

The orbit sum still holds under the weightings, and we can find an expression for the generating function as a diagonal using the parameterized group

The group

$$[[[\iota,\psi,\iota],[\frac{\gamma}{y\alpha},\frac{\gamma}{\alpha\,x}]],[[\psi,\iota],[\frac{\beta\,y}{\alpha\,x},\frac{\gamma}{\alpha\,x}]],[[\iota,\psi],[\frac{\gamma}{y\alpha},\frac{\gamma\,x}{\beta\,y}]],[[\psi],[x,\frac{\gamma\,x}{\beta\,y}]],[[\iota],[\frac{\beta\,y}{\alpha\,x},y]],[[],[x,y]]] \tag{9}$$

The orbit sum

$$Q^{(w)}(x,y;t)=[x^{\geqslant 0}][y^{\geqslant 0}]-\frac{1}{\alpha\,tx^2y+\beta\,ty^2+\gamma\,tx-xy}\left(-\frac{\gamma^2}{y\alpha^2x}+\frac{\beta\,y\gamma}{x^2\alpha^2}+\frac{\gamma^2x}{y^2\alpha\,\beta}-\frac{x^2\gamma}{\beta\,y}-\frac{\beta\,y^2}{\alpha\,x}+xy\right) \tag{10}$$

3.5 ACSV

Regular orbit sum.

$$Q(1,1;t)=\Delta-\frac{\left(-y^2+x\right)(xy-1)\left(x^2-y\right)}{\left(txy^2+tx^2+ty-1\right)(-1+x)(-1+y)\,xy}$$

$$Q(u,v;t)=\Delta_{x,y,t}-\frac{\left(-y^2+x\right)(xy-1)\left(x^2-y\right)}{\left(txy^2+tx^2+ty-1\right)(-1+ux)(-1+vy)\,xy}$$

$$Q^{(w)}(1,1;t)=\Delta-\frac{\left(-y^2+x\right)(xy-1)\left(x^2-y\right)}{\left(ctxy^2+btx^2+aty-1\right)(-1+x)(-1+y)\,xy}$$

```
H1 := c*t*x*y^2+b*t*x^2+a*t*y-1;
H2 := 1-x;
H3 := 1-y;
```

$$H1 := ctxy^2+btx^2+aty-1$$

$$H2 := 1-x$$

$$H3 := 1-y$$

```
# On V_\{1,2,3\}
c123 := [1,1,solve(subs(x=1,y=1,H1),t)];
```

$$c123 := [1,1,(a+b+c)^{-1}]$$

```
with(Groebner):
GB1 := Basis([H1,diff(H1,x)*x-diff(H1,t)*t,diff(H1,y)*y-diff(H1,t)*t],plex(x,y,t));
```

$$GB1 := [27\,a^4bt^3-c^2,-9\,a^3bt^2+c^2y,-3\,a^2t+cx]$$

```
solve(%,\{x,y,t\})
```

$$\left\{ t = \frac{RootOf\left(27\,\_Z^3ab - c^2\right)}{a}, x = 3\,\frac{aRootOf\left(27\,\_Z^3ab - c^2\right)}{c}, y = 9\,\frac{ab\left(RootOf\left(27\,\_Z^3ab - c^2\right)\right)^2}{c^2} \right\}$$

```
# On V1:
c1 := [a^(2/3)/(b^(1/3)*c^(1/3)), a^(1/3)*b^(1/3)/c^(2/3), (1/3)*c^(2/3)/(a^(4/3)*b^(1/3))];
mul(k,k=c1);
subs(x=c1[1],y=c1[2],t=c1[3],GB1);
```

$$c1 := [\frac{a^{2/3}}{\sqrt[3]{b}\sqrt[3]{c}}, \frac{\sqrt[3]{a}\sqrt[3]{b}}{c^{2/3}}, 1/3\,\frac{c^{2/3}}{a^{4/3}\sqrt[3]{b}}]$$

$$1/3\,\frac{1}{\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c}}$$

$$[0, 0, 0]$$

```
# On V12
subs(x=1,H1);
```

$$cty^2 + aty + bt - 1$$

```
GB12 := Basis(subs(x=1,[H1,diff(H1,y)*y-diff(H1,t)*t]),plex(y,t));
```

$$GB12 := [\left(a^2b - 4\,b^2c\right)t^2 + 4\,bct - c, \left(-a^2b + 4\,b^2c\right)t + cya - 2\,bc]$$

```
c12 := [1,sqrt(b/c),(a*sqrt(b*c)-2*b*c)/(b*(a^2-4*b*c))];
simplify(mul(k,k=c12)) assuming a>0,b>0,c>0;
simplify(subs(x=c12[1],y=c12[2],t=c12[3],GB12)) assuming a>0,b>0,c>0;
```

$$c12 := [1, \sqrt{\frac{b}{c}}, \frac{a\sqrt{bc} - 2\,bc}{b\left(a^2 - 4\,bc\right)}]$$

$$\frac{a\sqrt{b}\sqrt{c} - 2\,bc}{\sqrt{c}\sqrt{b}\left(a^2 - 4\,bc\right)}$$

$$[0, 0]$$

```
# On V13
subs(y=1,H1);
```

$$btx^2 + ctx + at - 1$$

```
GB13 := Basis(subs(y=1,[H1,diff(H1,x)*x-diff(H1,t)*t]),plex(x,t));
```

$$GB13 := [\left(4\,a^2b - ac^2\right)t^2 - 4\,abt + b, t\left(4\,a^2b - ac^2\right) + bxc - 2\,ab]$$

```
c13 := [-sqrt(a)/sqrt(b),1,(c*sqrt(a*b)+2*a*b)/(a*(4*a*b-c^2))];
simplify(mul(k,k=c13)) assuming a>0,b>0,c>0;
simplify(subs(x=c13[1],y=c13[2],t=c13[3],GB13)) assuming a>0,b>0,c>0;
```

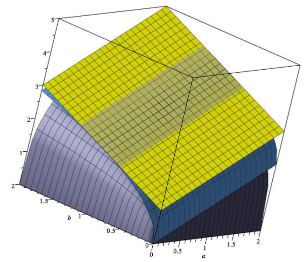

Figure 5: $a+b+1$, $3\sqrt[3]{ab}$, $a+2\sqrt{b}$ for various values of a, b

$$c13 := [-\frac{\sqrt{a}}{\sqrt{b}}, 1, \frac{c\sqrt{ab}+2\,ab}{a\,(4\,ab-c^2)}]$$

$$-\frac{c\sqrt{b}\sqrt{a}+2\,ab}{\sqrt{a}\sqrt{b}\,(4\,ab-c^2)}$$

$$[0,0]$$

The exponential growth is one of:

$$(a+b+1)^n$$

$$\left(3\sqrt[3]{ab}\right)^n$$

$$\left(a+2\sqrt{b}\right)^n$$

4 Generalizations in higher dimensions

The combinatorial interpretation of the model (without weights) as a young tableau of height bounded by 3 is naturally generalized by considering other heights, say height bounded by $d+1$. This is in bijective correspondence with a d-dimensional lattice model restricted to a positive cone, with a step set made of $d+1$ steps:

$$\mathcal{T}_d = \{e_{i+1} - e_i : i = 1 \ldots d-1\} \cup \{e_1, -e_d\}.$$

These all have D-finite generating functions, from combinatorial results [**?**], and expressions as diagonals of rational functions [**?**].

Does the same hold in the weighted case?

The interpretation as Young tableau in the weighted case is clear, but not necessarily motivated.

Conjecture 2. *The weighted, d-dimensional Tandem model has finite group, and a D-finite generating function.*

This is likely easily proved given the appropriate change of variables as before.

Determine the appropriate change of variables in the higher dimension case.

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5 Conclusion

Different starting points

$\mathcal{M}$

$$R := \frac{1}{x^2y^2\left(atx^2y+bty^2+ctx-xy\right)}\left(\left(y^{-1}\right)^i\left(x^{-1}\right)^j xy-\left(\frac{y}{x}\right)^i y^3\left(x^{-1}\right)^j-\left(y^{-1}\right)^i\left(\frac{x}{y}\right)^j x^3+x^{i+4}\left(\frac{x}{y}\right)^j y+\left(\frac{y}{x}\right)^i y^{4+j}x-x^{i+3}y^{j+3}\right)$$

6 Appendix

$$\begin{aligned}Q(1,1;t) = {} & 1+t+2\,t^2+4\,t^3+9\,t^4+21\,t^5+51\,t^6+127\,t^7+323\,t^8+835\,t^9+2188\,t^{10}+5798\,t^{11}+15511\,t^{12}\\ & +41835\,t^{13}+113634\,t^{14}+310572\,t^{15}+853467\,t^{16}+2356779\,t^{17}+6536382\,t^{18}+18199284\,t^{19}\\ & +50852019\,t^{20}+142547559\,t^{21}+400763223\,t^{22}+1129760415\,t^{23}+3192727797\,t^{24}\\ & +9043402501\,t^{25}+25669818476\,t^{26}+73007772802\,t^{27}+208023278209\,t^{28}+O\left(t^{30}\right) \quad (OEIS A001006)\end{aligned}$$

$$\begin{aligned}Q(x,0;t) = {} & 1+xt+x^2t^2+\left(x^3+1\right)t^3+\left(x^4+3\,x\right)t^4+\left(x^5+6\,x^2\right)t^5+\left(x^6+10\,x^3+5\right)t^6+\left(x^7+15\,x^4+21\,x\right)t^7\\ & +\left(x^8+21\,x^5+56\,x^2\right)t^8+\left(x^9+28\,x^6+120\,x^3+42\right)t^9+\left(x^{10}+36\,x^7+225\,x^4+210\,x\right)t^{10}\\ & +\left(x^{11}+45\,x^8+385\,x^5+660\,x^2\right)t^{11}+\left(x^{12}+55\,x^9+616\,x^6+1650\,x^3+462\right)t^{12}+O\left(t^{13}\right)\end{aligned}$$

$$\begin{aligned}Q(0,0;t) = {} & 1+t^3+5\,t^6+42\,t^9+462\,t^{12}+6006\,t^{15}+87516\,t^{18}+1385670\,t^{21}+23371634\,t^{24}+414315330\,t^{27}\\ & +7646001090\,t^{30}+O\left(t^{33}\right)\end{aligned}$$