Geometric Algebras

BSc. Seminar

Marcelo Guimarães Neto

University of Helsinki

May 2, 2022

Outline

Algebras, Substructures, Homomorphisms

Definition (Algebra)

A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$
\begin{aligned}
+ &: \text{associative, invertible, commutative} \\
\cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\
&= ab * xz + a * xt + b * yz + yt \\
* &: \text{associative, distributive}
\end{aligned}
$$

Algebras, Substructures, Homomorphisms

Definition (Algebra)

A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$
\begin{aligned}
+ &: \text{associative, invertible, commutative} \\
\cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\
&\quad = ab * xz + a * xt + b * yz + yt \\
* &: \text{associative, distributive}
\end{aligned}
$$

Examples

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $(\mathbb{R}^3, +, \times, *)$, $\mathbb{R}[X]$

Substructures and Homomorphisms

Definition (Subalgebra)

A subalgebra is a subset of an algebra which is closed under the operations and the action.

Definition (Homomorphisms)

A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \to B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Theorem (Basis defines the algebra)

A linear map $\varphi : A \to B$ is a homomorphism iff for every basis vector: $\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axiom 3

The square of any vector is a real number.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axiom 3

The square of any vector is a real number.

Axiom 4

The symmetrized product on $\mathcal{V}$ is nondegenerate, i.e.:

$$vu + uv = 0 \; \forall u \in \mathcal{V} \iff v = 0$$

The Standard Geometric Algebra of $\mathbb{R}^3$

We define the geometric product of two vectors in $\mathcal{G}(\mathbb{R}^3)$

$$uv = u \cdot v + u \wedge v$$

For the cartesian basis:

$$e_i e_j = \begin{cases} 1 & \text{if } i = j \\ e_i \wedge e_j & \text{if } i \neq j \end{cases}$$

Elements $e_i \wedge e_j$ can be thought of as representing the rectangle spanned by the two vectors.

Theorem (Grade-Dimension)

The maximum grade of an element of $\mathcal{G}(\mathbb{R}^3)$ is 3; moreover $\mathcal{G}(\mathbb{R}^3)$ is itself a vector space of dimension 8 with basis $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, I \equiv e_1e_2e_3\}$.

Gibbs' Cross Product

Definition (Cross Product)

We define the cross product between two vectors:

$$u \times v \equiv -Iu \wedge v$$

Remark

This is indeed the cross product we are all familiar with, as:

$$\begin{aligned} e_1 \times e_2 &= -Ie_1 e_2 = e_3 \\ e_1 \times e_3 &= -Ie_1 e_3 = -e_2 \\ e_2 \times e_3 &= -Ie_2 e_3 = e_1 \end{aligned}$$

Projections, Reflections and Rotations

Reflection by an axis

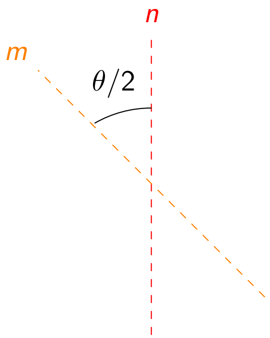
Let v be a vector which we wish to reflect through the axis represented by the vector n, then:

$$v = (v|n)n^{-1} + (v \wedge n)n^{-1} \equiv P_n(v) + R_n(v)$$
$$v' \equiv -P_n(v) + R_n(v) = -nvn^{-1}$$

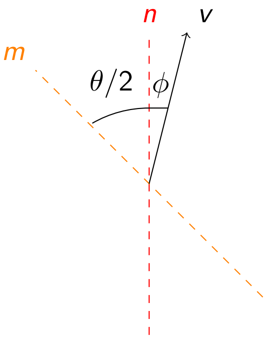
Rotation on a plane

Rotation of a vector v through the plane represented by B is given by: $v' = RvR^{-1}$, where $R = exp(-B\theta/2)$

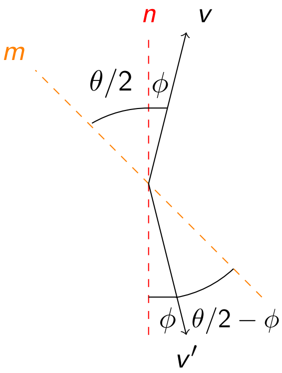
Hamilton's method for rotations



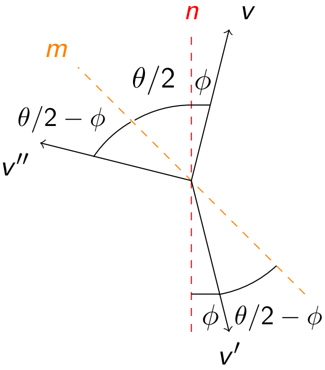
Hamilton's method for rotations



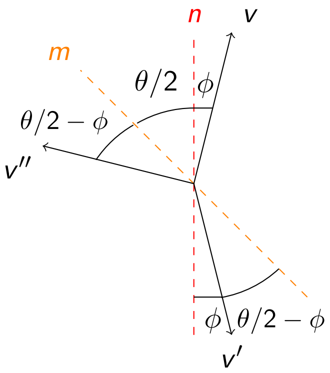
Hamilton's method for rotations



Hamilton's method for rotations



Hamilton's method for rotations



The rotated vector is thus:

$$\begin{aligned} v'' &= -m(-nvn^{-1})m^{-1} \\ &= (mn)v(mn)^{-1} \end{aligned}$$

We identify mn as the rotor R; moreover, if B is the unit bivector:

$$\begin{aligned} R &= |m||n|(\cos(\theta/2) - \sin(\theta/2)B) \\ &\equiv |R| \exp(-B\theta/2) \end{aligned}$$

We say B generates rotations in the plane and call it **spinor**.

Spinor subalgebra: $\mathcal{G}_+(\mathbb{R}^3) \simeq \mathbb{H}$

Theorem

The set of even grade elements $\{1, e_i e_j : i \neq j\}$ generates a subalgebra $\mathcal{G}_+(\mathbb{R}^3)$ isomorphic to $\mathbb{H}$.

Proof.

Compare multiplication tables:

$\cdot$	1	e_1e_2	e_2e_3	e_1e_3
1	1	e_1e_2	e_2e_3	e_1e_3
e_1e_2	e_1e_2	-1	e_1e_3	$-e_2e_3$
e_2e_3	e_2e_3	$-e_1e_3$	-1	e_1e_2
e_1e_3	e_1e_3	e_2e_3	$-e_1e_2$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

□

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition
5. Simplifies algebraic manipulations of vectors

References and Additional material

Historical background and Geometric intuition

1. Chappell, J. & Iqbal, A. & Hartnett, J. & Abbott, D. 2016. *The Vector Algebra War: A Historical Perspective*. IEEE Access. 4. 1997-2004. 10.1109/ACCESS.2016.2538262.
2. sudgylacmoe (2020) *A Swift Introduction to Geometric Algebra*. 17 August. Available at: `https://www.youtube.com/watch?v=60z_hpEAtD8&t=2076s` (Accessed: 20 February 2022).
3. Marc ten Bosch (2020) *Let's remove Quaternions from every 3D Engine: Intro to Rotors from Geometric Algebra*. 30 January. Available at: `https://www.youtube.com/watch?v=Idlv83CxP-8` (Accessed: 20 February 2022).

References and Additional material

Literature

1. Chisolm, E., 2012. *Geometric Algebra*. arXiv:1205.5935
2. Hestenes, D. and Sobczyk, G., 1984. *Clifford Algebra to Geometric Calculus*. Dordrecht: Springer Netherlands.
3. Doran, C. and Lasenby, A., 2003. *Geometric algebra for physicists*. Cambridge: Cambridge University Press.
4. Hestenes, D., 2002. *New Foundations for Classical Mechanics*. 2nd ed. New York, Boston, Dordrecht, London, Moscow: Kluwer Academic Publishers.

Questions?