Geometric Algebras

BSc. Seminar

Marcelo Guimarães Neto

University of Helsinki

May 2, 2022

Outline

Historical Background (early 1800s)

Historical Background (early 1800s)

Hamilton

Extends the complex numbers into quaternions, aiming at a formal vector algebra for $\mathbb{R}^3$, effectively representing (rotations in) 3D space:

$$q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = k, ki = j, jk = i$$

Product of two 'vectors':

$$-\sum v_i u_i + \sum_{i \to j \to k} (v_i u_j - v_j u_i)\boldsymbol{k}$$

Historical Background (early 1800s)

Hamilton

Extends the complex numbers into quaternions, aiming at a formal vector algebra for $\mathbb{R}^3$, effectively representing (rotations in) 3D space:

$$q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = k, ki = j, jk = i$$

Product of two 'vectors':

$$-\sum v_i u_i + \sum_{i \to j \to k} (v_i u_j - v_j u_i)\boldsymbol{k}$$

Grassmann

First formulation of 'modern' linear algebra (vector spaces, bases, inner product and orthogonality) as well as the outer product:

$$\boldsymbol{e_i} | \boldsymbol{e_j} = \delta_{ij}$$
$$\boldsymbol{e_i} \wedge \boldsymbol{e_j} = -\boldsymbol{e_i} \wedge \boldsymbol{e_j}$$
$$\boldsymbol{v} = \sum v_i \boldsymbol{e_i}, \boldsymbol{u} = \sum u_j \boldsymbol{e_j}$$

The outer products of linearly independent vectors represents their linear span.

Historical Background (late 1800s)

Gibbs-Heaviside

Separates the product of quaternion 'vectors' into dot and cross products; formally replaces the imaginary units with the unit vectors.

Development of standard vector calculus (e.g. $\nabla, \nabla\cdot, \nabla\times$), applied extensively in Electrodynamics, leading to the widespread adoption of the formalism.

Historical Background (late 1800s)

Gibbs-Heaviside

Separates the product of quaternion 'vectors' into dot and cross products; formally replaces the imaginary units with the unit vectors.

Development of standard vector calculus (e.g. $\nabla, \nabla\cdot, \nabla\times$), applied extensively in Electrodynamics, leading to the widespread adoption of the formalism.

Clifford

Extends Grassmann's work into his Geometric Algebra, incorporating Hamilton's quaternions into an abstractable and generalizable algebraic system for vectors, based on the geometric product:

$$uv = u \rfloor v + u \wedge v$$

$$V = \sum \langle V \rangle_i$$

$$\langle V \rangle_r = \sum v_i \boldsymbol{e}_{\boldsymbol{i_1}} \wedge \ldots \wedge \boldsymbol{e}_{\boldsymbol{i_r}}$$

Algebras, Substructures, Homomorphisms

Definition (Algebra)

A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$\begin{aligned} + &: \text{associative, invertible, commutative} \\ \cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\ &= ab * xz + a * xt + b * yz + yt \\ * &: \text{associative, distributive} \end{aligned}$$

Algebras, Substructures, Homomorphisms

Definition (Algebra)

A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$\begin{aligned} + &: \text{associative, invertible, commutative} \\ \cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\ &\quad = ab * xz + a * xt + b * yz + yt \\ * &: \text{associative, distributive} \end{aligned}$$

Examples

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $(\mathbb{R}^3, +, \times, *)$, $\mathbb{R}[X]$

Substructures and Homomorphisms

Definition (Subalgebra)

A subalgebra is a subset of an algebra which is closed under the operations and the action.

Definition (Homomorphisms)

A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \rightarrow B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Theorem (Basis defines the algebra)

A linear map $\varphi : A \rightarrow B$ is a homomorphism iff for every basis vector: $\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axiom 3

The square of any vector is a real number.

Axioms of Geometric Algebra

A *Real (Finite) Geometric Algebra* is a set $\mathcal{G}(\mathcal{V})$ satisfying:

Axiom 1

$\mathcal{G}(\mathcal{V})$ is a unitary associative algebra.

Axiom 2

$\mathcal{G}(\mathcal{V})$ contains $\mathbb{R}$ and the real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

Axiom 3

The square of any vector is a real number.

Axiom 4

The symmetrized product on $\mathcal{V}$ is nondegenerate, i.e.:

$$vu + uv = 0 \; \forall u \in \mathcal{V} \iff v = 0$$

Elements of $\mathcal{G}(\mathcal{V})$

Identities

As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Elements of $\mathcal{G}(\mathcal{V})$

Identities

As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Scalars

We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Elements of $\mathcal{G}(\mathcal{V})$

Identities

As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Scalars

We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

1-vectors

Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Elements of $\mathcal{G}(\mathcal{V})$

k-blades

Products $\boldsymbol{e_1 e_2 ... e_k}$ of k anticommuting vectors; we say that a k-blade has grade k.

Elements of $\mathcal{G}(\mathcal{V})$

k-blades

Products $\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_k$ of k anticommuting vectors; we say that a k-blade has grade k.

k-versors

Arbitrary products $\boldsymbol{v}_1\boldsymbol{v}_2\ldots\boldsymbol{v}_k$ of k vectors.

Elements of $\mathcal{G}(\mathcal{V})$

k-blades

Products $\boldsymbol{e_1 e_2 \ldots e_k}$ of k anticommuting vectors; we say that a k-blade has grade k.

k-versors

Arbitrary products $\boldsymbol{v_1 v_2 \ldots v_k}$ of k vectors.

Multivectors

Finite sums of versors. By axiom (2), every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector. We say A is a **homogeneous** (or simple) multivector iff it is a sum of k-blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade.

Vector Products: Inner, Outer and Geometric

Definition (Inner Product)

The symmetrized product $u|v \equiv \frac{1}{2}(uv + vu)$ between two vectors is a (pseudo) inner product on $\mathcal{V}$.

Vector Products: Inner, Outer and Geometric

Definition (Inner Product)

The symmetrized product $u|v \equiv \frac{1}{2}(uv + vu)$ between two vectors is a (pseudo) inner product on $\mathcal{V}$.

Proof.

Axioms (1) and (4) imply it is bilinear, nondegenerate and by definition symmetric; we show it maps into $\mathbb{R}$:

$$(v+u)^2 = v^2 + u^2 + vu + uv \Leftrightarrow$$
$$vu + uv = (v+u)^2 - v^2 - u^2 \in \mathbb{R} \text{ (A3)}$$

$\square$

Vector Products: Inner, Outer and Geometric

Definition (Inner Product)

The symmetrized product $u|v \equiv \frac{1}{2}(uv + vu)$ between two vectors is a (pseudo) inner product on $\mathcal{V}$.

Definition (Outer Product)

We define the outer product of two vectors as the anti-symmetrized product $u \wedge v \equiv \frac{1}{2}(uv - vu)$.

Vector Products: Inner, Outer and Geometric

Definition (Inner Product)

The symmetrized product $u|v \equiv \frac{1}{2}(uv + vu)$ between two vectors is a (pseudo) inner product on $\mathcal{V}$.

Definition (Outer Product)

We define the outer product of two vectors as the anti-symmetrized product $u \wedge v \equiv \frac{1}{2}(uv - vu)$.

Remark (Geometric Product)

The geometric product between vectors can thus be written:

$$uv = u|v + u \wedge v$$

Homogeneous Products

Between homogeneous multivectors, the product can be shown to decompose as:

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots \langle A_r B_s \rangle_{r+s-2} + \langle A_r B_s \rangle_{r+s}$$

Homogeneous Products

Between homogeneous multivectors, the product can be shown to decompose as:

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots \langle A_r B_s \rangle_{r+s-2} + \langle A_r B_s \rangle_{r+s}$$

Definitions

Generalized Inner Product:

$$A_r | B_s \equiv \langle A_r B_s \rangle_{|s-r|}$$

Generalized Outer Product:

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s} = a_1 \wedge \ldots \wedge a_r \wedge b_1 \wedge \ldots \wedge b_s$$

Scalar Product:

$$A_r * B_s \equiv \langle A_r B_s \rangle$$

The Standard Geometric Algebra of $\mathbb{R}^3$

We define the geometric product of two vectors in $\mathcal{G}(\mathbb{R}^3)$

$$uv = u \cdot v + u \wedge v$$

For the cartesian basis:

$$e_i e_j = \begin{cases} 1 & \text{if } i = j \\ e_i \wedge e_j & \text{if } i \neq j \end{cases}$$

Elements $e_i \wedge e_j$ can be thought of as representing the rectangle spanned by the two vectors.

Theorem (Grade-Dimension)

The maximum grade of an element of $\mathcal{G}(\mathbb{R}^3)$ is 3; moreover $\mathcal{G}(\mathbb{R}^3)$ is itself a vector space of dimension 8 with basis $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, I \equiv e_1e_2e_3\}$.

Pseudoscalars, Duality and Invertibility

Definition (Pseudoscalar)

The 3-blade is unique up to scalar multiplication; we call it 'pseudoscalar' and denote it I as it has the following property:

$$I^2 = e_1 e_2 e_3 e_1 e_2 e_3 = -e_1 e_2 e_3 e_3 e_2 e_1 = -1$$

The unit bivectors are pseudoscalars of their respective subspaces: $(e_i e_j)^2 = -e_i e_j e_j e_i = -1$

Definition (Dual)

The dual of a multivector: $V^* \equiv IV$

Remark (Invertibility)

All vectors in $\mathcal{G}(\mathbb{R}^3)$ are invertible: $v^{-1} \equiv \frac{v}{v^2}$

Gibbs' Cross Product

Definition (Cross Product)

We define the cross product between two vectors:

$$u \times v \equiv -Iu \wedge v$$

Remark

This is indeed the cross product we are all familiar with, as:

$$e_1 \times e_2 = -Ie_1 e_2 = e_3$$
$$e_1 \times e_3 = -Ie_1 e_3 = -e_2$$
$$e_2 \times e_3 = -Ie_2 e_3 = e_1$$

Projections, Reflections and Rotations

Reflection by an axis

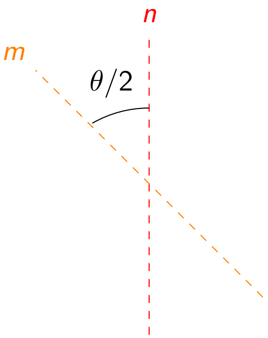
Let v be a vector which we wish to reflect through the axis represented by the vector n, then:

$$v = (v|n)n^{-1} + (v \wedge n)n^{-1} \equiv P_n(v) + R_n(v)$$
$$v' \equiv -P_n(v) + R_n(v) = -nvn^{-1}$$

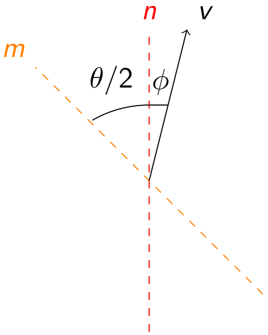
Rotation on a plane

Rotation of a vector v through the plane represented by B is given by: $v' = RvR^{-1}$, where $R = exp(-B\theta/2)$

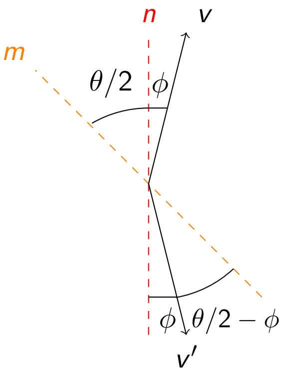
Hamilton's method for rotations



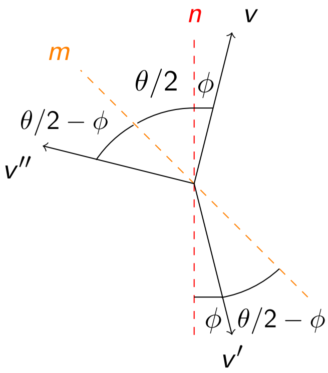
Hamilton's method for rotations



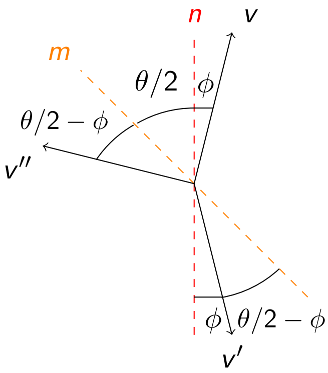
Hamilton's method for rotations



Hamilton's method for rotations



Hamilton's method for rotations



The rotated vector is thus:

$$
\begin{aligned}
v'' &= -m(-nvn^{-1})m^{-1} \\
&= (mn)v(mn)^{-1}
\end{aligned}
$$

We identify mn as the rotor R; moreover, if B is the unit bivector:

$$
\begin{aligned}
R &= |m||n|(\cos(\theta/2) - \sin(\theta/2)B) \\
&\equiv |R|\exp(-B\theta/2)
\end{aligned}
$$

We say B generates rotations in the plane and call it **spinor**.

Spinor subalgebra: $\mathcal{G}_2(\mathbb{R}^3) \simeq \mathbb{C}$

Theorem

For any choice of $(i, j) : i \neq j$, the set $\{1, e_i e_j\}$ generates a subalgebra $\mathcal{G}_2(\mathbb{R}^3)$ isomorphic to $\mathbb{C}$.

Proof.

Consider the multiplication table of the set:

$\cdot$	1	$e_i e_j$
1	1	$e_i e_j$
$e_i e_j$	$e_i e_j$	-1

It is clear that the isomorphism is given by $e_i e_j \longleftrightarrow i$.

$\square$

Spinor subalgebra: $\mathcal{G}_+(\mathbb{R}^3) \simeq \mathbb{H}$

Theorem

The set of even grade elements $\{1, e_i e_j : i \neq j\}$ generates a subalgebra $\mathcal{G}_+(\mathbb{R}^3)$ isomorphic to $\mathbb{H}$.

Proof.

Compare multiplication tables:

$\cdot$	1	$e_1 e_2$	$e_2 e_3$	$e_1 e_3$
1	1	$e_1 e_2$	$e_2 e_3$	$e_1 e_3$
$e_1 e_2$	$e_1 e_2$	-1	$e_1 e_3$	$-e_2 e_3$
$e_2 e_3$	$e_2 e_3$	$-e_1 e_3$	-1	$e_1 e_2$
$e_1 e_3$	$e_1 e_3$	$e_2 e_3$	$-e_1 e_2$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

□

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition
5. Simplifies algebraic manipulations of vectors

Disadvantages

1. Requires working with a large number of different objects

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition
5. Simplifies algebraic manipulations of vectors

Disadvantages

1. Requires working with a large number of different objects
2. Depends on a quadratic form, so it is not applicable in more abstract topological settings

Conclusion

Advantages

1. Generalizes Gibbs' algebra for $\mathbb{R}^3$ to any dimension
2. Contains $\mathbb{C}$ and $\mathbb{H}$ as subalgebras, so it handles rotations and orientations elegantly and efficiently
3. Unifies objects and transformations into a single algebra
4. Geometric intuition
5. Simplifies algebraic manipulations of vectors

Disadvantages

1. Requires working with a large number of different objects
2. Depends on a quadratic form, so it is not applicable in more abstract topological settings
3. Requires embedding manifolds into a higher dimensional Euclidean space

References and Additional material

Historical background and Geometric intuition

1. Chappell, J. & Iqbal, A. & Hartnett, J. & Abbott, D. 2016. *The Vector Algebra War: A Historical Perspective*. IEEE Access. 4. 1997-2004. 10.1109/ACCESS.2016.2538262.
2. sudgylacmoe (2020) *A Swift Introduction to Geometric Algebra*. 17 August. Available at: `https://www.youtube.com/watch?v=60z_hpEAtD8&t=2076s` (Accessed: 20 February 2022).
3. Marc ten Bosch (2020) *Let's remove Quaternions from every 3D Engine: Intro to Rotors from Geometric Algebra*. 30 January. Available at: `https://www.youtube.com/watch?v=Idlv83CxP-8` (Accessed: 20 February 2022).

References and Additional material

Literature

1. Chisolm, E., 2012. *Geometric Algebra*. arXiv:1205.5935
2. Hestenes, D. and Sobczyk, G., 1984. *Clifford Algebra to Geometric Calculus*. Dordrecht: Springer Netherlands.
3. Doran, C. and Lasenby, A., 2003. *Geometric algebra for physicists*. Cambridge: Cambridge University Press.
4. Hestenes, D., 2002. *New Foundations for Classical Mechanics*. 2nd ed. New York, Boston, Dordrecht, London, Moscow: Kluwer Academic Publishers.

Questions?