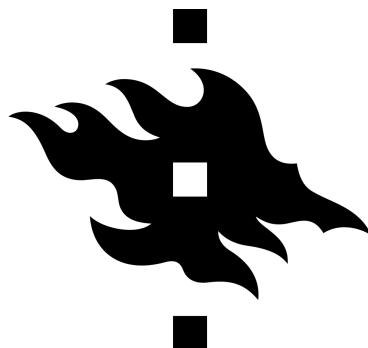


The Even Subalgebras of Euclidean Geometric Spaces

A Geometric Proof of the Frobenius Classification
of Finite-Dimensional Associative Division
Algebras over the Reals

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A thesis presented for the degree of
Bachelor of Science



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Abstract

Lorem ipsum dolor...

Acknowledgements

I want to thank...

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Chapter 1

Introduction

1.1 Historical Background

1.2 Aim and Content

We start with an overview of the field, aiming to address the following questions: what is a Geometric Algebra? How is it defined? What are its main components? How do different Geometric Algebras relate to one another?

I should preface this discussion by noting that Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms (the equivalence of which will be demonstrated), inspired by *Eric Chisolm*'s take on the topic [Chi12]. I have also taken care to discuss a couple of points which I think have been overlooked in other treatments of the topic: mainly, the **Universal Geometric Algebra** as a **template**[CATEGORY] and the relationship of **Geometric Algebra** to **Clifford Algebra**.

Geometric Algebra concerns itself with the construction of a family[CATEGORY] of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek view of mathematics in terms of geometric

constructions: numbers as segments, ...

In his original paper, *Hestenes* proposes that Geometric Algebra (and Calculus) can and should provide a unified framework for mathematical physics. In his own words: “*Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content.*” (p. ix)[HS84].

The main

Our main focus in this paper, will be in the mathematical development of the theory and its usefulness in the field of Abstract Algebra: specifically, in the classification of Scalar Algebras.

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

2.1.1 Groups, Rings and Fields

Definition 2.1.1 (Group). A group is a set closed under an associative, invertible product.

Definition 2.1.2 (Ring). A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

Definition 2.1.3 (Field). A field is a commutative ring in which every non-zero element has a multiplicative inverse.

2.1.2 Modules, Vector Spaces and Algebras

Definition 2.1.4 (Module). A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

Definition 2.1.5 (Vector space). A vector space is a module over a field.

Definition 2.1.6 (Algebra). A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \rightarrow A)$ and an action $(* : \mathbb{R} \times A \rightarrow A)$, with the properties:

- $+$: associative, invertible, commutative
- \cdot : bilinear : $(a * x + y) \cdot (b * z + t) =$
 $= ab * xz + a * xt + b * yz + yt$
- $*$: associative, distributive

Definition 2.1.7 (Homomorphisms). A homomorphism is a structure-preserving mapping between two algebraic structures, i.e. $\phi : A \rightarrow B$ s. that:

$$\phi(a * xz + y) = a * \phi(x)\phi(z) + \phi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

DIVISIBILITY

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

3.1.1 The UGA as an Abstract Algebra Template[CATEGORY]

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained. CITATION?!

We shall do otherwise, and define it as follows:

Definition 3.1.1 (Universal Geometric Algebra). The **Universal Geometric Algebra** is the category of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product.

3.1.2 The Axioms**3.1.3 The Universal Property****3.1.4 The Elements****3.1.5 The Products****3.2 Clifford Algebras****3.3 The Euclidean Geometric Algebras: $\mathcal{G}(\mathbb{E}_1), \mathcal{G}(\mathbb{E}_2), \mathcal{G}(\mathbb{E}_3)$** **3.3.1 $\mathcal{G}(\mathbb{E}_1)$** **3.3.2 $\mathcal{G}(\mathbb{E}_2)$** **3.3.3 $\mathcal{G}(\mathbb{E}_3)$**

Chapter 4

Classification of the Scalar Algebras

4.1 The Even Subalgebras $\mathcal{G}_+(\mathbb{E}_1), \mathcal{G}_+(\mathbb{E}_2), \mathcal{G}_+(\mathbb{E}_3)$

4.2 Divisibility of the Even Subalgebras

4.3 Non-divisibility of $\mathcal{G}_+(\mathbb{E}_n)$ for $n \leq 3$

4.4 The Geometric Isomorphism Theorem of Scalar Algebras

Chapter 5

Conclusion

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