On The Euclidean Geometric Algebras and their Even Subalgebras

The Realisation of Real Finite-Dimensional Associative Division Algebras in the Context of Euclidean Geometric Algebras

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A thesis presented for the degree of
Bachelor of Science

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June 25, 2022

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Abstract This paper is an introduction to the field of Geometric Algebra. The field - originating in the work of William K. Clifford in the 1800s (thus also known as Clifford Algebra) - has recently been repopularised as a mathematical project, especially with regards to its multitude of applications in physics. Our approach to the subject follows along broad lines that of David Hestenes; we will diverge in our choice of axioms as well as a few definitions: particularly in regards to our strict requirement of an inner product and our limitation to finite-dimensional vector spaces. Our main focus will be in the study of the even subalgebras of the Geometric Algebras of Euclidean Spaces: in particular, we will show that for the 1, 2 and 3 dimensional Euclidean Spaces, these are isomorphic to the Real, Complex and Quaternion algebras respectively. We also show that higher-dimensional algebras are not division algebras, as we would indeed expect since the geometric product is associative and the Euclidean Algebras are finite.

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Chapter 1

Introduction

1.1 Historical Background

Geometric Algebra has its roots in the long-running pursuit of a mathematical framework for the description of physical space. Though the concept of vectors is ancient, in this section we shall focus on the developments starting in the 19th century.

In the early 1800s, **Grassmann** was the first to formulate the notions of 'modern' linear algebra (vector spaces, bases, inner product and orthogonality) [Gra44]; his development of exterior (or as he called it, extended) algebra laid the key theoretical groundwork for Clifford's Geometric Algebra. [Hes02, p.28-29].

Around the same time, **Hamilton** extends the 2-dimensional algebra of the complex numbers into the 4-dimensional algebra of the quaternions, aiming at a formal vector algebra for $\mathbb{R}^3$. The quaternion algebra - though a division algebra and initially adopted by Maxwell in his formulation of electrodynamics - faced some pushback and was eventually mostly supplanted by Gibbs' vector algebra (especially in physical applications) and became relegated to specific applications. Some would argue that the shortcomings of the quaternions are grounded in the fact that quaternions provide a representation for the algebra of rotations in three-dimensions, and are not natural representations of cartesian vectors in $\mathbb{R}^3$ (a major shortcoming being that quaternions do not generalize to higher dimensions) [JA16].

Gibbs is responsible for the standard vector algebra that is most widely used today: his insight was to separate the product of quaternion 'vectors' into dot and cross products and formally replace the imaginary units with unit vectors that square to $+1$.

The development of standard vector calculus (e.g. $\nabla, \nabla\cdot, \nabla\times$), applied extensively in Electrodynamics by **Heaviside**, lead to the widespread adoption of the formalism.

Though it became the adopted formalism, **Gibbs**' vector algebra has its own deficiencies: it requires two separate vectors product, lacks a division operation and its cross product does not generalize to higher-dimensions.

Clifford's Geometric Algebra - introduced in the second half of the 19th century - aimed to supersede both formalisms by providing an algebra that could be generalized to higher dimensions, and adequately and comprehensively describe both vectors and transformations in space; it extends Grassmann's work, incorporating Hamilton's quaternions into a generalizable algebraic system for vectors, based on the geometric product.

Though **Clifford**'s work was mostly neglected in his time, it finds modern applications in a few areas of mathematics including differential geometry and mathematical physics (especially with regards to spinors). A mathematical project which aims to establish Geometric Algebra as a comprehensive, unifying framework for a variety of mathematical theories and tools related to the description of space, has also been put forth recently by **David Hestenes** [HS84].

1.2 Content and Aim

Our main focus in this paper, will be in the axiomatic approach to Geometric Algebra: we will be proving basic properties and theorems from the axioms and use these towards constructing the Euclidean Geometric Algebras and their Even Subalgebras in 1, 2 and 3 dimensions, demonstrating that the latter are actually isomorphic to the Real, Complex and Quaternion algebras.

In section 3.1, we provide an axiomatic formulation of Geometric Algebra as well as present the main objects and their definitions. The most important results in this section are Theorem 3.1.4 which separates the geometric product of two vectors into an inner and an outer product and Theorem 3.1.7 which affirms that every multivector has a unique decomposition in terms of more basic objects called k-blades. We also make some remarks about the construction of a Geometric Algebra for a given Inner Product Space.

Section 3.2 contains a few definitions and a minor lemma leading up to Theorem 3.2.2 which characterises the geometric product between homogeneous multivectors from which we obtain the main result of Chapter 3 in the form of Corollary 3.2.3 which states that the set of even-graded elements constitutes a subalgebra.

In Chapter 4, we construct the Geometric Algebras of the 1, 2 and 3-dimensional Euclidean Spaces and prove that their Even Subalgebras are indeed isomorphic to the scalar algebras. We conclude by showing that the higher-dimensional even subalgebras are non-divisible, as required by the Frobenius Classification of the Real Associative Finite-Dimensional Division Algebras.

1.3 Geometric Algebra as a Project

Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This thesis will diverge from the approach in [HS84] in a few important ways; in particular, this applies to the choice of axioms - closer to Vaz and da Rocha's take on the topic [VR16] and our limitation to finite-dimensional inner product spaces.

Geometric Algebra concerns itself with the construction of a collection of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. Intuitively, the theory identifies linear spaces with Euclidean geometric primitives (think points, segments, parallelograms...), and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek pre-algebraic view of mathematics, whereby arithmetic was intrinsically tied to geometric constructions. Perhaps for this very reason, **Geometric Algebra** has had made significant progress in its express goal of providing a unified framework for mathematical physics and applied mathematics. In his book, *Hestenes* details this more specifically [HS84, p. ix]:

> "Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

In this section we will go over background definitions and results which will be of use in the subsequent chapters of the thesis.

Definition 2.1.1 (Algebra)**.** A real algebra A is a real vector space equipped with a bilinear multiplication operation ($\cdot : A \to A$).

That the product is bilinear simply means that $\forall x, y, z, t \in A,\ a, b \in \mathbb{R}$:

$$(a * x + y) \cdot (b * z + t) = ab * xz + a * xt + b * yz + yt$$

We say that an algebra is **associative** when its multiplication is associative; an algebra is unital if its multiplication has an identity element in the algebra (for an algebra A, we denote this element 1_A): in this thesis, we will be dealing with real associative unital algebras (unless otherwise mentioned).

Definition 2.1.2 (Subalgebra)**.** A subalgebra is a subset of an algebra which is closed under the operations and the action of the algebra.

Definition 2.1.3 (Homomorphisms)**.** An algebra homomorphism is a map between two algebras which preserves their algebraic structure, i.e. $\varphi : A \to B$ s. that $\forall a \in \mathbb{R}$ and $x, y, z \in A$:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Definition 2.1.4 (Basis)**.** A subset B of an algebra A is said to be a **basis** for the algebra A iff it is the basis of A as a vector space. If such a set exists, we say that the algebra is free. We say that the algebra A is finite-dimensional if it has a finite-dimensional basis.

Remark. Later on, we will introduce the distinct concept of the basis of a Geometric Algebra - to differentiate the two we will sometimes refer to the above definition of basis as the linear (or vector-space) basis of the algebra, and we will refer to the linear basis of a Geometric Algebra as a frame (this is standard terminology that will be introduced later). It should be clear in any case that whenever we refer to the basis of an algebra that is not a Geometric Algebra, we are referring to the above definition.

Theorem 2.1.1 (Basis defines the algebra). *Let A and B be two finite-dimensional algebras. A linear map $\varphi : A \to B$ is a homomorphism iff for every element in the basis of A:*

$$\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$$

Moreover if j is a one-to-one mapping between bases $\mathcal{B}_A$ and $\mathcal{B}_B$ of their respective algebras, and j preserves the multiplication on the bases (i.e. $j(e_i e_j) = j(e_i)j(e_j)$) - then, there exists one unique such linear map $\varphi : A \to B$ that is an isomorphism and for which $\varphi|_{\mathcal{B}_A} \equiv j$.

The latter part of the above theorem allows us to simply compare products of the basis elements of two finite-dimensional algebras in order to determine whether they are isomorphic. We say that the mapping j induces the isomorphism φ.

Definition 2.1.5 (Division Algebra). An associative algebra A is a **division algebra** iff all non-zero elements have a multiplicative inverse, i.e.

$$\forall a \in A\ \exists x \in A \text{ s.that } ax = xa = 1$$

Definition 2.1.6 (Zero Divisor). A zero divisor of an algebra A is a non-zero element $a \in A$ s. that there exists non-zero $b \in B$ for which $ab = 0_A$. (i.e. a divides zero)

Theorem 2.1.2 (Zero Divisor). *If an associative algebra contains a zero divisor, then it is not a division algebra.*

Proof. Let a be a zero divisor in an algebra A, then $\exists b \neq 0_A \in A$ such that:

$$ab = 0_A$$

Assume a has an inverse in A, then:

$$b = (a^{-1}a)b = a^{-1}(ab) = a^{-1}0_A = 0_A$$

This contradicts our assumptions, and so we have proven that a has no inverse in A and thus A is not a division algebra. □

Chapter 3

Geometric Algebra

3.1 Axioms and Definitions

Let $(\mathcal{V}, \cdot|\cdot)$ be a real finite-dimensional inner product space, $\mathcal{G}$ be a unital associative algebra over $\mathbb{R}$ and $l : \mathcal{V} \to A$ be a linear map.

Definition 3.1.1 (Geometric Algebra)**.** The pair $(\mathcal{G}, l)$ is the Geometric Algebra for the space $(\mathcal{V}, \cdot|\cdot)$ when $\mathcal{G}$ satisfies the following axioms:

Axiom 3.1.2. $\mathcal{G}$ is spanned by the products of elements in the set

$$\{l(v)|v \in \mathcal{V}\} \bigcup \{1_{\mathcal{G}}\}$$

We say that the algebra is generated by the above set.

Axiom 3.1.3. For all vectors $v \in \mathcal{V}$, the square of their mapping $l(v)$ corresponds to the square of their magnitued, i.e.:

$$\forall v \in \mathcal{V} : l(v)^2 = v|v = |v|^2$$

Axiom 3.1.4. Given any unital associative algebra A over $\mathbb{R}$ and any linear map $j : \mathcal{V} \to A$ such that

$$\forall v \in \mathcal{V} : j(v)^2 = (v|v)1_A$$

there is a unique algebra homomorphism $f : \mathcal{G} \to A$ such $f \odot i = j$.

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{G} \arrow[d, dashed, "f"] \\
 & A
\end{tikzcd}$$

Before anything else, we present a few theorems and remarks regarding existence and universality.

Theorem 3.1.1 (Uniqueness)**.** *The Geometric Algebra $\mathcal{G}$ of an inner product space $(\mathcal{V}, \cdot|\cdot)$, if it exists, is unique up to isomorphism.*

Proof. If (A, i) and (B, j) are two Geometric Algebras of $(\mathcal{V}, \cdot|\cdot)$, then by Axiom 3.1.4 there exist algebra homomorphisms:

$$\begin{aligned} f &: A \to B \text{ s. that } f \odot i = j \\ g &: B \to A \text{ s. that } g \odot j = i \end{aligned}$$

It follows that the compositions obey:

$$\begin{aligned} g \odot f &: A \to A \text{ s. that } i = (g \odot f) \odot i \\ &\Rightarrow g \odot f = id_A : A \to A \\ f \odot g &: B \to B \text{ s. that } j = (f \odot g) \odot j \\ &\Rightarrow f \odot g = id_B : B \to B \end{aligned}$$

We conclude that $g = f^{-1}$ and thus the homomorphisms are 1-to-1 so that the Geometric Algebras are isomorphic. □

Definition 3.1.5 (Existence and Dimensionality)**.** For every finite-dimensional real inner product space $(\mathcal{V}, \cdot|\cdot)$ there exists a Geometric Algebra $\mathcal{G}$ with dimension 2^n, where $n = \dim(\mathcal{V})$.

A generalized proof of existence and dimensionality is outside the scope of this thesis; later on, we will construct the Geometric Algebras of 1, 2 and 3-dimensional Euclidean Spaces explicitly and that will be sufficient for our discussion. For the interested reader, you may refer to a general proof by construction in [VR16, Section 3.2]

Remark. From now on, we will make the identifications

$$\begin{aligned} l(v) &\equiv v \ \forall v \in \mathcal{V} \\ \alpha 1_A &\equiv \alpha \ \forall \alpha \in \mathbb{R} \end{aligned}$$

This is justified due to the linearity of the mapping $l : \mathcal{V} \to A$, and the bilinearity of the product of an algebra, which implies that $\alpha 1_A \cdot \beta 1_A + \gamma 1_A = (\alpha\beta + \gamma) 1_A$.

We are now equipped to prove some basic properties about the Geometric Algebra.

Two results follow immediately from Axiom 3.1.3:

Lemma 3.1.2. All vectors are invertible with respect to the geometric product and the inverse is given by:

$$v^{-1} = \frac{v}{v^2}$$

Lemma 3.1.3 (Inner Product)**.** The symmetrized product of two vectors,

$$s(u,v) = \frac{1}{2}(uv + vu)$$

corresponds to the inner product on $\mathcal{V}$, i.e.:

$$s(u,v) = u|v \ \forall u, v \in \mathcal{V}$$

Proof. Consider the following expression, and recall Axiom 3.1.3:

$$\begin{aligned}
(u+v)^2 &= u^2 + v^2 + uv + vu \Leftrightarrow \\
2s(u,v) &= (u+v)^2 - u^2 - v^2 \\
&\overset{3.1.3}{=} (u+v)|(u+v) - u|u - v|v \\
&= u|v + v|u = 2u|v
\end{aligned}$$

We conclude

$$s(u,v) \equiv u|v$$

□

Remark. We have seen that $\mathbb{R} \bigcup \mathcal{V} \subset \mathcal{G}$ is closed under the symmetric part of the product; the algebra must thus be generated by the antisymmetric part of the product: we denote this bilinear antisymmetric product $u \wedge v$ and call it the **exterior product**.

One of the main results of this section is the principal property of the geometric product.

Theorem 3.1.4 (Geometric Product of Vectors)**.** *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv + uv) = \frac{1}{2}(uv + vu + uv - vu) = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

The result follows immediately from Lemma 3.1.3 and the above remarks. □

As a corollary, we obtain the following self-evident propositions.

Corollary 3.1.5. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

Corollary 3.1.6. An orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathcal{V}$ obeys the following relations:

$$\begin{aligned}
e_i e_j &= -e_j e_i \\
e_i e_i &= 1
\end{aligned}$$

Before we move on to consider more general results, we ought to familiarise ourselves with some specific terminology.

Definition 3.1.6 (Scalars)**.** We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.1.7 (1-vectors)**.** Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.1.8 (k-blades)**.** Products $e_{i_1} e_{i_2} \ldots e_{i_k}$ of orthogonal vectors; these are also called simple k-vectors.

Definition 3.1.9 (Versors)**.** Arbitrary products $v_1 v_2 \ldots v_k$ of vectors.

Definition 3.1.10 (Multivectors)**.** Finite sums of versors. By Axiom 3.1.2, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector.

Definition 3.1.11 (Basis)**.** A **basis** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is any basis of the generating inner-product space $\mathcal{V}$.

Definition 3.1.12 (Frame)**.** A **frame** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is the set of distinct (up to permutation) k-blades formed from an orthogonal basis of $\mathcal{G}(\mathcal{V})$.

Remark. It is important to note that it is a non-trivial fact of combinatorics and group theory that the sign of a permutation is well-defined. We will make implicit use of this in arguments and definitions that refer to permutation of orthogonal vectors in a product: in practice this means that we are justified in assigning a unique sign to a permutation based on counting the number of pairwise transposition of factors, and saying things such as 'distinct up to permutation'.

Theorem 3.1.7 (A Frame is a Linear Basis)**.** *Any frame is a linear basis for the geometric algebra.*

Proof. By definition a frame is a linearly independent set, we need only show that a frame spans the algebra. Consider an arbitrary set of vectors expressed in a given orthogonal basis $\{v_i \equiv \sum_{j=1}^{n} v_i^j e_j\}_{i=1}^{m}$; we expand their product using bilinearity:

$$\prod_{i=1}^{m} v_i = \prod_{i=1}^{m} \sum_{j=1}^{n} v_i^j e_j = \sum_{j_i : \{1..m\} \to \{1..n\}} \left(\prod_{i=1}^{m} v_i^{j_i} \right) \left(\prod_{i=1}^{m} e_{j_i} \right)$$

It follows from Corollary 3.1.6 that the product of the basis vectors in each term is a k-blade, where k is the number of unit vectors appearing an odd number of times in the product. The above shows that the product of vectors is indeed a linear combination of elements in the frame generated by the orthogonal basis above.

A frame can be mapped into the power set of a basis as it constitutes a subset of all the possible combinations of the basis elements without repetition and up to permutation (we identify the unit scalar in the frame with the empty subset of the basis). As such, the maximal number of distinct elements of a frame is 2^n.

Since the algebra has dimension 2^n, the frame must have 2^n linearly independent elements and indeed constitute a linear basis for the algebra. □

We are now justified in presenting the following definitions:

Definition 3.1.13 (Grade)**.** We introduce the grade operator $\langle A \rangle_k$ which returns the k-grade component of A. The grade of a multivector $\mathrm{gr}(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A. We say A is a **homogeneous** multivector of grade k iff it is a sum of k-blades for a given $k \in \mathbb{N}$ (when we wish to make it explicit, we denote it A_k); otherwise we say A is of mixed grade.

Lemma 3.1.8. The grade of a multivector is a well-defined property, that is to say every blade has a unique grade that is independent of the choice of orthogonal vectors.

Proof. First, we prove that for a given orthogonal basis, the grade of a blade is unique, then we show that the grade of a blade is independent of the choice of basis. The result extends directly to general multivectors.

By Theorem 3.1.7, a blade is simply an element of some frame of $\mathcal{G}$: since a frame is a linear basis for the Geometric Algebra, it follows that frame elements of different grade are linearly independent. The grade is thus unique w.r.t. to a specific choice of frame.

Let $\{e_j\}_{j=1}^n$ and

$$w_i = \sum_{j=1}^n A_{ij} e_j$$

be two orthonormal bases (orthogonality suffices, we choose this purely for notational convenience).

The orthogonality of $\{w_i\}_{i=1}^k$ implies that for $a \neq b$:

$$w_a | w_b = (\sum_{j=1}^n A_{aj} e_j) | (\sum_{r=1}^n A_{br} e_r) = \sum_{j=1}^n A_{aj} A_{bj} = 0$$

Expanding the full product, we have:

$$w_1 ... w_k = (\sum_{j_1=1}^n A_{1j_1} e_{j_1}) ... (\sum_{j_k=1}^n A_{kj_k} e_{j_k})$$

We can index the terms in the above product as follows: at each of the k factors choose one of the terms in the sum (i.e. $A_{it} e_t$), construct a sequence $t(i)$ out of subsequent choices.

The terms with grades lesser than k (w.r.t. the orthonormal basis) are those terms for which at least one choice is repeated, we can use this to index all possible such terms: given a choice of $(q, m) \in \{1..k\}^2$ where the repetition ought to occur, we are free to choose any $c \in \{1..n\}$ for which $t(q) = t(m) = c$, moreover we are free to vary all other terms independently so we must sum over all subsequences $F(q, m) \equiv \{f(i) : \{1..k\} \setminus \{q, m\} \to \{1..n\}\}$.

Thus, we may collect all lower grade terms in the following sum:

$$\sum_{(q,m)} \sum_{f \in F} \sum_{c=1}^{n} (-1)^{|m-q-1|} (A_{1f(1)} ... A_{qc} A_{mc} ... A_{kf(k)}) e_{f(1)} ... e_{f(k)}$$

$$= \sum_{(q,m)} \sum_{f \in F} (-1)^{|m-q-1|} (A_{1f(1)} ... A_{kf(k)}) e_{f(1)} ... e_{f(k)} \sum_{c=1}^{n} (A_{qc} A_{mc}) = 0$$

So all lower-grade terms vanish. That the total product is non-zero simply follows from the fact that the algebra would otherwise permit a frame of dimension smaller than 2^n. □

Definition 3.1.14 (Pseudoscalar)**.** The **pseudoscalar** of a geometric algebra $\mathcal{G}(\mathcal{V})$ is the highest grade element in its frame. It is unique up to scalar multiplication (including permutation of factors). The square of a pseudoscalar is also a scalar, by convention, it is normalized such that it squares to 1 or -1 (which defines the orientation of the frame).

The distinction between basis and frame is very important: a frame is to a basis, as a geometric algebra is to its generating vector space. For finite-dimensional spaces, we can always produce an orthonormal basis by the Gram-Schmidt process, so we will henceforth assume all bases to be orthonormal unless otherwise specified.

It is then clear that for every inner product space, we may construct its Geometric Algebra by the following recipe:

1. construct an orthonormal basis for the inner-product space: this will be the basis of the algebra
2. construct the frame of the algebra by taking products of basis vectors
3. generate the rest of the algebra as a vector space with the frame as its basis

A brief note on conventions: here on out, whenever left unspecified, the precedence of products is inner→wedge→geometric.

3.2 Products

We will now go on to consider general expressions and properties of products between arbitrary multivectors. First off, we generalize the definition of the inner and outer products to homogeneous multivectors.

Definition 3.2.1. The inner product $A_r|B_s$ between two homogeneous multivectors A_r and B_s is the lowest-possible grade term in their product. We will later see that this is the $|r-s|$-grade term.

Definition 3.2.2. The outer product $A_r \wedge B_s$ between two homogeneous multivectors A_r and B_s is the highest-possible grade term in their product. We will later see that this is the $|r+s|$-grade term.

We now prove the following lemma regarding the product of a vector with a homogenous multivector.

Lemma 3.2.1. The inner and outer products of a vector with a homogeneous multivector have the following expressions:

$$a|A_r = \langle aA_r\rangle_{r-1} = \frac{1}{2}(aA_r - (-1)^r A_r a)$$
$$a \wedge A_r = \langle aA_r\rangle_{r+1} = \frac{1}{2}(aA_r + (-1)^r A_r a)$$

Proof. We shall assume that A_r is an r-blade: the case of a homogeneous multivector follows directly using distributivity of the geometric product (since an r-grade homogeneous multivector is a sum of r-blades).

Recall the definition of the inner product between two vectors (def. 3.2.1)

$$a|b = \frac{1}{2}(ab + ba)$$

We can reverse it to obtain:

$$ab = 2a|b - ba$$

Repeated application of the above allows us to permute indices in a product, as follows:

$$\begin{aligned}
aA_r = aa_1a_2\dots a_r =& 2a|a_1a_2\dots a_r - a_1aa_2\dots a_r \\
=& 2a|a_1a_2\dots a_r - 2a|a_2a_1\dots a_r + a_1a_2aa_3\dots a_r \\
=& \dots \\
=& 2\sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r + (-1)^r a_1a_2\dots a_r a \\
=& \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r + \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r \\
&+ (-1)^r a_1a_2\dots a_r a
\end{aligned}$$

Notice that the first term above (denote it T_1 for convenience) has grade $r-1$: we will prove that this is indeed the lowest grade term by showing that $aA_r - T_1$ has grade $r+1$.

Let us rewrite the sum using invertibility of vectors (th. 3.1.2):

$$\begin{aligned} T_1 &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_k^{-1} a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r} a|a_k a_k^{-1} A_r \end{aligned}$$

Substracting the above from aA_r and factoring:

$$aA_r - a|A_r = (a - \sum_{k=1}^{r} a|a_k a_k^{-1})A_r \equiv bA_r$$

Since $a_k^{-1} = |a_k|^{-2} a_k$ (Lemma 3.1.2), we have by construction that

$$b|a_k = (a - \sum_{k=1}^{r} (a|a_k) |a_k|^{-2} a_k)|a_k = a|a_k - a|a_k = 0$$

It follows by Corollary 3.1.5 that bA_r is a product of $r+1$ orthogonal vectors and thus has grade $r+1$ by Definition 3.1.13. We are justified in writing:

$$\begin{aligned} aA_r &= a|A_r + a \wedge A_r \\ a|A_r &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ a \wedge A_r &= a|A_r + (-1)^r A_r a \end{aligned}$$

From which the lemma follows straightforwardly by substituting the third expression into the first:

$$\begin{aligned} aA_r = 2a|A_r + (-1)^r A_r a &\Rightarrow a|A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ aA_r = 2a \wedge A_r - (-1)^r A_r a &\Rightarrow a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

□

The above proof is due to *Hestenes* [HS84, p. 8-10]

Using the above lemma, we can prove the following important property of the geometric product between homogeneous multivectors.

Theorem 3.2.2 (Product of Homogeneous Multivectors). *The product of homogeneous multivectors A_r, B_s can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^{\min r,s} \langle A_r B_s \rangle_{|s-r|+2k}$$

Proof. We prove this by induction on $r \leq s$ when A_r and B_s are simple r- and s-vectors respectively.

The case $r = 1,\ s = 1$ is true by Definition 3.1.4. The case $r = 1,\ s > 1$ is true by Lemma 3.2.1. Assume the expression holds for $r = q,\ s > r$, we show that it holds for $r + 1$:

$$\begin{aligned}
A_{r+1} B_s &= a_{r+1} A_r B_s = a_{r+1} \sum_{k=0}^{r} \langle A_r B_s \rangle_{s-r+2k} \\
&= \sum_{k=0}^{r} a_{r+1} \langle A_r B_s \rangle_{s-r+2k} \\
&= \sum_{k=0}^{r} [a_{r+1} | \langle A_r B_s \rangle_{s-r+2k} + a_{r+1} \wedge \langle A_r B_s \rangle_{s-r+2k}] \\
&= a_{r+1} | \langle A_r B_s \rangle_{s-r} \\
&\quad + \sum_{k=1}^{r} \left[a_{r+1} | \langle A_r B_s \rangle_{s-r+2k} + a_{r+1} \wedge \langle A_r B_s \rangle_{s-r+2(k-1)} \right] \\
&\quad + a_{r+1} \wedge \langle A_r B_s \rangle_{s+r}
\end{aligned}$$

where in the last step, we have grouped terms together by grade:

$$\begin{aligned}
\langle A_{r+1} B_s \rangle_{s-(r+1)} &\equiv a_{r+1} | \langle A_r B_s \rangle_{s-r} \\
\langle A_{r+1} B_s \rangle_{s-(r+1)+2k} &\equiv a | \langle A_r B_s \rangle_{s-r+2k} + a \wedge \langle A_r B_s \rangle_{s-r+2(k-1)} \\
\langle A_{r+1} B_s \rangle_{r+1+s} &\equiv a \wedge \langle A_r B_s \rangle_{s+r}
\end{aligned}$$

The case where $s \leq r$ follows by induction on s (the same argument as above). The general case for homogeneous multivectors follows by distributivity of the geometric product. □

The above proof is due to *Chisolm* [Chi12, p. 20-21]

As a corollary, we obtain our sought-after result.

Corollary 3.2.3 (Even Subalgebras). The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

It also follows that the inner and outer products can be naturally generalized to arbitrary multivectors as well-defined bilinear operations on $\mathcal{G}(\mathcal{V})$.

Corollary 3.2.4. (Generalized Inner Product)

On homogeneous multivectors:

$$A_r | B_s \equiv \langle A_r B_s \rangle_{|s-r|}$$

On arbitrary multivectors:

$$A | B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r | \langle B \rangle_s$$

Corollary 3.2.5. (Generalized Outer Product)

On homogeneous multivectors:

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

Chapter 4

Euclidean Even Subalgebras

4.1 Even Subalgebras

We start off with a discussion on the construction of even subalgebras and introduce some notation and results that will be of use in the next section.

An even subalgebra is constructed from its generating elements precisely in the same manner as a geometric algebra; its linear basis is the subset of even grade elements of the linear basis of the original algebra.

This is made explicit in the following self-evident lemma:

Lemma 4.1.1. Let $\mathcal{G}$ be a geometric algebra with basis $\{e_1, e_2, \ldots e_n\}$. Its even subalgebra $\mathcal{G}_+$ is generated by the set of unit bivectors $\{e_1e_2, \ldots e_1e_n, e_2e_3 \ldots e_{n-1}e_n\}$; we will make use of the following shortened notation: $e_ie_j = e_{ij}$.

The unit bivectors have the following basic properties:

$$\begin{aligned} e_{ij} &= -e_{ji} \\ {e_{ij}}^2 &= -1 \end{aligned}$$

Moreover, it follows from straightforward algebraic manipulations that the product of two unit bivectors anti-commutes if they share a common index and commutes otherwise.

We will make use of the above properties in computation, without explicit mention.

4.2 Euclidean Geometric Algebras

We will follow the approach laid out in the previous section in order to construct the Geometric Algebra of the 1, 2 and 3 dimensional Euclidean spaces and consider their even subalgebras. The main result of this chapter will be our proof that these are isomorphic to the well-known scalar algerbras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

4.2.1 $\mathcal{G}(\mathbb{E}_1)$ and $\mathcal{G}_+(\mathbb{E}_1)$

We define the 1-dimensional Euclidean space as follows in order to distinguish it from the real numbers:

$$\mathbb{E}_1 \equiv \{\alpha e_1 | \alpha \in \mathbb{R}\} \sim \mathbb{R}$$

$\mathcal{G}(\mathbb{E}_1)$ is trivial: its frame is just $\{1, e_1\}$. It follows that its even subalgebra has basis $\{1\}$ and thus corresponds to $\mathbb{R}$.

4.2.2 $\mathcal{G}(\mathbb{E}_2)$ and $\mathcal{G}_+(\mathbb{E}_2)$

$\mathbb{E}_2$ is the standard euclidean plane. We denote its orthonormal basis $\{e_1, e_2\}$ and work out its multiplication table with respect to the geometric product:

$\cdot$	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

Table 4.1: Multiplication table of $\mathcal{G}(\mathbb{E}_2)$

The frame of $\mathcal{G}(\mathbb{E}_2)$ is simply $\{1, e_1, e_2, e_1e_2\}$, and the even subalgebra $\mathcal{G}_+(\mathbb{E}_2)$ is thus generated by $\{1, e_1e_2\}$. It is apparent that the mapping $e_1e_2 \leftrightarrow i$ induces an isomorphism $j : \mathcal{G}(\mathbb{E}_2) \leftrightarrow \mathbb{C}$ by comparing multiplication tables:

$\cdot$	1	e_{12}
1	1	e_{12}
e_{12}	e_{12}	-1

$\cdot$	1	i
1	1	i
i	i	-1

Table 4.2: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_2)$ and $\mathbb{C}$

4.2.3 $\mathcal{G}(\mathbb{E}_3)$ and $\mathcal{G}_+(\mathbb{E}_3)$

$\mathbb{E}_3$ is the standard euclidean space. We denote its orthonormal basis $\{e_1, e_2\, e_3\}$.

The frame of $\mathcal{G}(\mathbb{E}_3)$ is simply $\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{12}e_3\}$; the even subalgebra $\mathcal{G}_+(\mathbb{E}_3)$ is thus generated by the set $\{1, e_{12}, e_{23}, e_{13}\}$; by observing its multiplication table (Table 4.3), it becomes clear that the mapping

$$\begin{aligned} e_{12} &\leftrightarrow i \\ e_{23} &\leftrightarrow j \\ e_{13} &\leftrightarrow k \end{aligned}$$

induces an isomorphism $j : \mathcal{G}(\mathbb{E}_3) \leftrightarrow \mathbb{H}$.

$\cdot$	1	e_{12}	e_{23}	e_{13}
1	1	e_{12}	e_{23}	e_{13}
e_{12}	e_{12}	-1	e_{13}	$-e_{23}$
e_{23}	e_{23}	$-e_{13}$	-1	e_{12}
e_{13}	e_{13}	e_{23}	$-e_{12}$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 4.3: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_3)$ and $\mathbb{H}$

We conclude with the following lemma which shows that for $n > 3$, the even subalgebras are no longer division algebras.

Lemma 4.2.1 (Non-divisibility of higher-dimensional even euclidean subalgebras)**.** Let $\mathcal{G}_+(\mathbb{E}_n)$ denote the even geometric subalgebra of the n-dimensional Euclidean space. $\mathcal{G}_+(\mathbb{E}_n)$ is a division algebra if and only if $n \leq 3$.

Proof. Let $n = 4$ and consider the following element

$$u = e_{12}e_{13}e_{34}$$

We show that it squares to 1:

$$\begin{aligned} u^2 &= e_{12}e_{13}e_{34}e_{12}e_{13}e_{34} \\ &= e_{12}e_{12}e_{13}e_{13}e_{34}e_{34} \\ &= 1 \end{aligned}$$

It thus follows that $u^2 - 1 = 0$, by factoring this expression we have found two zero-divisors:

$$\begin{aligned} &u^2 - 1 = (u-1)(u+1) = 0 \\ &u \pm 1 \ \neq 0 \end{aligned}$$

By Theorem 2.1.2, $\mathcal{G}_+(\mathbb{E}_4)$ is not a division algebra. Moreover since all higher-dimensional even subalgebras contain $\mathcal{G}(\mathbb{E}_4)$, they all contain the above zero divisors and as such are also not division algebras. $\square$

Chapter 5

Conclusion

In this paper, we have developed the fundamentals of Geometric Algebra.

Starting from a formal set of axioms, we have thoroughly derived the general properties of the algebra, its elements and its products.

We have observed that the algebra does indeed encompass the standard Gibbs' algebra, the algebra of the Complex Numbers and that of the Quaternions. All while having the marked advantage of invertibility of vectors and a natural generalization to higher dimensions.

Geometric Algebra - and its extension to the domain of analysis, Geometric Calculus - have many promising applications that unfortunately were outside the scope of this thesis: linear and multilinear algebra can be expressed in the language of frames and outermorphisms; differential geometry can be formulated in the framework of vector manifolds and the geometric derivative with the promise of coordinate-free computations; the even subalgebras of Euclidean geometric spaces can be shown to account for the theory of Spinors; advances have even been made in the theory of Lie Algebras and Groups.

Geometric Algebra postulates that geometry is a fundamental guide towards meaningful mathematical pursuits, even in abstract settings. It should be no surprise that such a theory would be so adept at describing a wide range of geometric phenomena. As *Hestenes* puts it [HS84, p. xii]:

> "Geometry without algebra is dumb! Algebra without geometry is blind!"

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