The Even Subalgebras of Euclidean Geometric Spaces

A Geometric Proof of the Frobenius Classification of Finite-Dimensional Associative Division Algebras over the Reals

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A thesis presented for the degree of
Bachelor of Science

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May 12, 2022

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Abstract

Lorem ipsum dolor...

Acknowledgements

I want to thank...

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Historical Background

1.2 Aim and Content

We start with an overview of the field, aiming to address the following questions: what is a Geometric Algebra? How is it defined? What are its main components? How do different Geometric Algebras relate to one another?

I should preface this discussion by noting that Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms (the equivalence of which will be demonstrated), inspired by *Eric Chisolm*'s take on the topic [Chi12]. I have also taken care to discuss a couple of points which I think have been overlooked in other treatments of the topic: mainly, the **Universal Geometric Algebra** as a **template**[CATEGORY] and the relationship of **Geometric Algebra** to **Clifford Algebra**.

Geometric Algebra concerns itself with the construction of a family[CATEGORY] of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek view of mathematics in terms of geometric constructions: numbers as segments, ...

In his original paper, *Hestenes* proposes that Geometric Algebra (and Calculus) can and should provide a unified framework for mathematical physics. In his own words: *"Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."* (p. ix)[HS84].

The main

Our main focus in this paper, will be in the mathematical development of the theory and its usefulness in the field of Abstract Algebra: specifically, in the classification of Scalar Algebras.

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

2.1.1 Groups, Rings and Fields

Definition 2.1.1 (Group)**.** A group is a set closed under an associative, invertible product.

Definition 2.1.2 (Ring)**.** A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

Definition 2.1.3 (Field)**.** A field is a commutative ring in which every non-zero element has a multiplicative inverse.

2.1.2 Modules, Vector Spaces and Algebras

Definition 2.1.4 (Module)**.** A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

Definition 2.1.5 (Vector space)**.** A vector space is a module over a field.

Definition 2.1.6 (Algebra)**.** A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$\begin{aligned}
+ &: \text{associative, invertible, commutative} \\
\cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\
&\quad = ab * xz + a * xt + b * yz + yt \\
* &: \text{associative, distributive}
\end{aligned}$$

Definition 2.1.7 (Subalgebra)**.** A subalgebra is a subset of an algebra which is closed under the operations and the action of the algebra.

Definition 2.1.8 (Homomorphisms)**.** A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \to B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Definition 2.1.9 (Free Algebra)**.** A free algebra is...

Theorem 2.1.1 (Universal Property of Free Algebras)**.** *Given...*
DIAGRAM

Proof. Given...
DIAGRAM ☐

Definition 2.1.10 (Division Algebra)**.** A Division Algebra is an algebra in which all non-zero elements have a multiplicative inverse.

Definition 2.1.11 (Zero Divisor)**.** A zero divisor of an algebra A is a non-zero element $a \in A$ s. that there exists non-zero $b \in B$ for which $ab = 0_A$. (i.e. a divides zero)

Theorem 2.1.2 (Zero Divisor)**.** *If an algebra contains a zero divisor, then it is not a division algebra.*

Proof. Let a be a zero divisor in an algebra A... ☐

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained. CITATION?!

I have not been able to convince myself that the standard formulation is justified without addressing the peculiarities of infinite-dimensional inner product spaces; so we shall do otherwise, and define it as follows:

Definition 3.1.1 (Universal Geometric Algebra)**.** The **Universal Geometric Algebra** is the category of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product. We will later on observe, that this is in fact the Universal Property of Clifford Algebras.

A note on nomenclature is urgently in order: what we call **Universal Geometric Algebra** is in fact not an algebra; we will continue to refer to it as such, and in fact we will discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high-dimension such that it does not get in the way of the argument.

3.2 Axioms, Elements and Products

A Geometric Algebra $\mathcal{G}$ is a finite-dimensional unitary associative algebra over the reals obeying the following axioms:

Axiom 3.2.1. $\mathcal{G}$ contains $\mathbb{R}$ as a subalgebra and a real (finite) vector space $\mathcal{V}$ as a subspace; these generate the entire algebra.

Axiom 3.2.2. The formal product of a scalar and a vector corresponds with the multiplication by a scalar of the vector space.

Axiom 3.2.3. The square of any vector is a real number.

Axiom 3.2.4. The formal product on $\mathcal{V}$ is positive-definite, i.e.:

$$\forall v \neq 0 \in \mathcal{V} \quad vv > 0$$

Axiom 3.2.5. The antisymmetrized product of linearly independent vectors produces an element which does not belong to $\mathbb{R} \bigoplus \mathcal{V}$, we call this product the **exterior product** and denote it $u \wedge v = \frac{1}{2}(uv - vu)$.

We refer to the formal product of such an algebra, as a **geometric product**.

An immediate result of axioms 3.2.3 and 3.2.4 is:

Lemma 3.2.1. All vectors are invertible with respect to the geometric product and their inverse is given by:

$$v^{-1} = \frac{v}{v^2}$$

We first prove the following lemma.

Lemma 3.2.2 (Inner Product)**.** The symmetrized product of two vectors,

$$u|v = \frac{1}{2}(uv + vu)$$

is an inner product.

Proof. Consider the following expression, and recall Axiom 3.2.3:

$$\begin{aligned} (u+v)^2 &= u^2 + v^2 + uv + vu \Leftrightarrow \\ 2u|v &= (u+v)^2 - u^2 - v^2 \in \mathbb{R} \end{aligned}$$

So the symmetrized product maps into $\mathbb{R}$ (i.e. $| : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$). By Definition 2.1.6, the geometric product is bilinear and so is the symmetrized product as it is a linear function of the bilinear products; moreover, by Axiom 3.2.4, it is also positive-definite. We conclude that $u|v = uv + vu$ is indeed an inner product on $\mathcal{V}$.

□

From the above, we obtain the principal property of the geometric product.

Theorem 3.2.3 (Geometric Product of Vectors)**.** *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv + uv) = \frac{1}{2}(uv + vu + uv - vu) = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

The result follows immediately from Lemma 3.2.2 and Axiom 3.2.5:

$$uv = u|v + u \wedge v$$

□

As a corollary, we obtain the following self-evident propositions.

Corollary 3.2.4. The exterior product of a linearly dependent set of vectors is zero.

Corollary 3.2.5. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

We will now go on to consider general expressions and properties of products between arbitrary multivectors; but first, we ought to familiarise ourselves with some specific terminology.

Definition 3.2.6 (Identities)**.** As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Definition 3.2.7 (Scalars)**.** We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.2.8 (1-vectors)**.** Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.2.9 (k-blades)**.** Products $\boldsymbol{e}_1\boldsymbol{e}_2 \dots \boldsymbol{e}_k$ of k anticommuting vectors; we say that a k-blade has grade k. These are also called simple k-vectors.

Definition 3.2.10 (k-versors)**.** Arbitrary products $\boldsymbol{v}_1\boldsymbol{v}_2 \dots \boldsymbol{v}_k$ of k vectors.

Definition 3.2.11 (Multivectors)**.** Finite sums of versors. By Axiom 3.2.1, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector. We say A is a **homogeneous** multivector iff it is a sum of k-blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade. We introduce the grade operator $\langle A \rangle_k$ which returns the k-grade component of A. The grade of a multivector $\operatorname{gr}(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A.

A brief note on conventions: here on out, whenever left unspecified, the precedence of products is inner→wedge→geometric.

First off, we generalize the definition of the inner and outer products to homogeneous multivectors.

Definition 3.2.12. The inner product between two homogeneous multivectors A_r and B_s is the lowest grade term in their product.

Definition 3.2.13. The outer product between two homogeneous multivectors A_r and B_s is the highest grade term in their product.

We now prove the following lemma regarding the product of a vector with a homogenous multivector.

Lemma 3.2.6. The inner and outer products of a vector with a homogeneous multivector have the following expressions:

$$\begin{aligned} a|A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ a \wedge A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

Proof. We shall assume that A_r is an r-blade: the case of a homogeneous multivector follows directly using distributivity of the geometric product (since an r-grade homogeneous multivector is a sum of r-blades).

Recall the definition of the inner product between two vectors (def. 3.2.12)

$$a|b = \frac{1}{2}(ab + ba)$$

We can reverse it to obtain:

$$ab = 2a|b - ba$$

Repeated application of the above allows us to permute indices in a product, as follows:

$$\begin{aligned} aA_r = aa_1a_2\dots a_r =& 2a|a_1a_2\dots a_r - a_1aa_2\dots a_r \\ =& 2a|a_1a_2\dots a_r - 2a|a_2a_1\dots a_r + a_1aa_2\dots a_r \\ =& \dots \\ =& 2\sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r + (-1)^r a_1a_2\dots a_r a \\ =& \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r + \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1\dots\check{a}_k\dots a_r \\ & + (-1)^r a_1a_2\dots a_r a \end{aligned}$$

Notice that the first term above has grade $r-1$: we will demonstrate that this is indeed the lowest grade term and denote it $a|A_r$.

Let us rewrite the sum using invertibility of vectors (th. 3.2.1):

$$\begin{aligned}
a|A_r &= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_1 \dots \check{a}_k \dots a_r \\
&= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1} a_k a_1 \dots \check{a}_k \dots a_r \\
&= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1} A_r
\end{aligned}$$

Substracting the above from aA_r and factoring:

$$aA_r - a|A_r = (a - \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1})A_r \equiv bA_r$$

Since by construction, $b|a_k = 0 \forall k \in \{1, \dots r\}$, it follows by Corollary 3.2.5 that the above is a product of $r+1$ anticommuting vectors and thus has grade $r+1$ by Definition 3.2.9. We are justified in writing:

$$\begin{aligned}
aA_r &= a|A_r + a \wedge A_r \\
a|A_r &= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_1 \dots \check{a}_k \dots a_r \\
a \wedge A_r &= a|A_r + (-1)^r A_r a
\end{aligned}$$

From which the lemma follows straightforwardly by substituting the third expression into the first:

$$\begin{aligned}
aA_r = 2a|A_r + (-1)^r A_r a &\Rightarrow a|A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \\
aA_r = 2a \wedge A_r - (-1)^r A_r a &\Rightarrow a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a)
\end{aligned}$$

□

ATTRIBUTE APPROACH OF THE PROOF TO HESTENES

Using the above lemma, we can prove the following important property of the geometric product between homogeneous multivectors.

Theorem 3.2.7 (Product of Homogeneous Multivectors)**.** *The product of homogeneous multivectors A_r, B_s (for $s \leq r$) can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^{r}\langle A_r B_s\rangle_{s-r+2k}$$

Proof. We prove this by induction on $r \leq s$ when A_r and B_s are simple r- and s-vectors respectively.

The case $r = 1, \quad s = 1$ is true by Definition 3.2.3. The case $r = 1, \quad s > 1$ is true by Lemma 3.2.6. Assume the expression holds for $r = q, \quad s > r$, we show that it holds for $r + 1$:

$$\begin{aligned} A_{r+1}B_s =& a_{r+1}A_rB_s = a_{r+1}\sum_{k=0}^{r}\langle A_rB_s\rangle_{s-r+2k} = \\ =& \sum_{k=0}^{r} a_{r+1}\langle A_rB_s\rangle_{s-r+2k} = \\ =& \sum_{k=0}^{r}\left[a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1}\wedge\langle A_rB_s\rangle_{s-r+2k}\right] \\ =& a|\langle A_rB_s\rangle_{s-r} \\ &+ \sum_{k=1}^{r}\left[a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1}\wedge\langle A_rB_s\rangle_{s-r+2(k-1)}\right] \\ &+ a\wedge\langle A_rB_s\rangle_{s+r} \end{aligned}$$

where in the last step, we have grouped terms together by grade:

$$\begin{aligned} &\langle A_{r+1}B_s\rangle_{s-(r+1)} \equiv a_{r+1}|\langle A_rB_s\rangle_{s-r} \\ &\langle A_{r+1}B_s\rangle_{s-(r+1)+2k} \equiv a|\langle A_rB_s\rangle_{s-r+2k} + a\wedge\langle A_rB_s\rangle_{s-r+2(k-1)} \\ &\langle A_{r+1}B_s\rangle_{r+1+s} \equiv a\wedge\langle A_rB_s\rangle_{s+r} \end{aligned}$$

The case where $s \leq r$ follows by induction on s (the same argument as above). The general case for homogeneous multivectors follows by distributivity of the geometric product. $\square$

ATTRIBUTE APPROACH OF THE PROOF TO CHISOLM

As a corollary, we obtain our sought-after result.

Corollary 3.2.8 (Even Subalgebras)**.** The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

Moreover, we are now equipped to provide general, explicit definitions for the inner and outer products in the case of arbitrary multivectors.

Definition 3.2.14 (Generalized Inner Product)**.** On homogeneous multivectors $(s > r)$:

$$A_r|B_s \equiv \langle A_rB_s\rangle_{s-r}$$

On arbitrary multivectors:

$$A|B \equiv \sum_{r=0}^{\mathrm{gr}(A)}\sum_{s=r}^{\mathrm{gr}(B)}\langle A\rangle_r|\langle B\rangle_s$$

THE INNER PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

Definition 3.2.15 (Generalized Outer Product)**.** On homogeneous multivectors ($s > r$):

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

THE EXTERIOR PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

3.3 Bases and Frames

DISCUSS THE VALIDITY OF THE AXIOMS AND PROVE THAT GEOMETRIC ALGEBRAS ARE SIMPLY CLIFFORD ALGEBRAS (DECIDE IF I SHOULD RELAX POSITIVE-DEFINITENESS REQUIREMENT AND DRAW DISTINCTION THERE OR IF I DRAW DISTINCTION AT KEEPING THE INNER PRODUCT FORMAL OR BOTH); PROVE UNIVERSAL PROPERTY

OR

PROVE UNIVERSAL PROPERTY (FORMAL CONSTRUCTION) FROM AXIOM, DISCUSS BASES (PRACTICAL CONSTRUCTION; before universal property?), GRADE-DIMENSION AND PSEUDOSCALAR

A Geometric Algebra is only realized into a concrete algebra once an inner-product space is specified. It is then straightforward to construct the whole algebra given a basis for the inner-product space. But first, we need some specific terminology.

Definition 3.3.1 (Basis)**.** The **basis** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is simply the basis of the generating inner-product space $\mathcal{V}$.

Definition 3.3.2 (Frame)**.** The **frame** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is the basis of $\mathcal{G}(\mathcal{V})$ as a vector space. It is constructed by taking all product combinations of the unit vectors in an orthonormal basis for $\mathcal{V}$ where no vector is repeated. Frame elements for a given orthonormal basis are unique up to permutation of factors.

Definition 3.3.3 (Pseudoscalar)**.** The **pseudoscalar** of a geometric algebra $\mathcal{G}(\mathcal{V})$ is the highest grade element in its frame. It is unique up to scalar multiplication (including permutation of factors). The square of a pseudoscalar is also a scalar, by convention, it is normalized such that it squares to 1 or -1 (which defines the orientation of the frame).

The distinction between basis and frame is very important: a frame is to a basis, as a geometric algebra is to its generating vector space. For finite-dimensional spaces, we can always produce an orthonormal basis by the Gran-Schmidt process, so we will henceforth assume all bases to be orthonormal unless otherwise specified.

A frame can be identified one-to-one with the power set of a basis, as it is composed of all the possible combinations of the basis elements without repetition and up to permutation (we identify the unit scalar in the frame with the empty subset of the basis). It follows that the dimension of a geometric algebra $\mathcal{G}$ (as a vector space) is $2^{\mathrm{gr}(\mathcal{G})}$.

The recipe for constructing a Geometric Algebra is thus, as follows:

1. construct an orthonormal basis for the inner-product space: this will be the basis of the algebra
2. construct the frame of the algebra by taking products between non-repeating unit vectors

3. generate the rest of the algebra as a vector space with the frame as its basis

Chapter 4

Classification of Scalar Algebras

4.1 Even Subalgebras

We start off with a discussion on the construction of even subalgebras and introduce some notation and results that will be of use in the next section.

An even subalgebra has its own basis which is distinct from that of the original algebra; this is made explicit in the following self-evident lemma:

Lemma 4.1.1. Let $\mathcal{G}$ be a geometric algebra with basis $\{e_1, e_2, \dots e_n\}$. Its even subalgebra $\mathcal{G}_+$ has basis $\{e_1e_2, \dots e_1e_n, e_2e_3 \dots e_{n-1}e_n\}$; we will make use of the following shortened notation: $e_ie_j = e_{ij}$.

The basis 2-vectors have the following properties:

$$
\begin{aligned}
&e_{ij} = 0 \text{ if } i = j \\
&e_{ij} = -e_{ji} \\
&{e_{ij}}^2 = -1 \\
&e_{ij}e_{rs} = e_{rs}e_{ij} \text{ if the 2-vectors have no indices in common} \\
&e_{ij}e_{rs} = -e_{rs}e_{ij} \text{ if the 2-vectors have one index in common}
\end{aligned}
$$

We will make use of the above properties in computation, without explicit mention.

An even subalgebra is constructed from its basis precisely in the same manner as a geometric algebra.

4.2 Euclidean Geometric Algebras

We will follow the approach laid out in the previous section in order to construct the Geometric Algebra of the 1, 2 and 3 dimensional Euclidean spaces and consider their even subalgebras. The main result of this chapter will be our proof that these are isomorphic to the well-known scalar algerbras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

4.2.1 $\mathcal{G}(\mathbb{E}_1)$ and $\mathcal{G}_+(\mathbb{E}_1)$

$\mathcal{G}(\mathbb{E}_1)$ is trivial: the inner product is simply given by the magnitude of the standard product (i.e. $a|b = |ab|$); and its frame is just $\{1, e_1\}$. It follows that its even subalgebra has frame $\{1\}$ and thus corresponds to $\mathbb{R}$.

4.2.2 $\mathcal{G}(\mathbb{E}_2)$ and $\mathcal{G}_+(\mathbb{E}_2)$

$\mathbb{E}_2$ is the standard euclidean plane. We denote its orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and work out its multiplication table with respect to the geometric product:

$\cdot$	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$
1	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	1	$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1\boldsymbol{e}_2$	1	$-\boldsymbol{e}_1$
$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	-1

Table 4.1: Multiplication table of $\mathcal{G}(\mathbb{E}_2)$

The frame of $\mathcal{G}(\mathbb{E}_2)$ is simply $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2\}$, and the even subalgebra $\mathcal{G}_+(\mathbb{E}_2)$ is thus generated by $\{1, \boldsymbol{e}_1\boldsymbol{e}_2\}$. It is apparent that the mapping $\boldsymbol{e}_1\boldsymbol{e}_2 \leftrightarrow i$ induces an isomorphism $j : \mathcal{G}(\mathbb{E}_2) \leftrightarrow \mathbb{C}$ by comparing multiplication tables:

$\cdot$	1	$\boldsymbol{e}_1\boldsymbol{e}_2$
1	1	$\boldsymbol{e}_1\boldsymbol{e}_2$
$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$	-1

$\cdot$	1	i
1	1	i
i	i	-1

Table 4.2: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_2)$ and $\mathbb{C}$

4.2.3 $\mathcal{G}(\mathbb{E}_3)$ and $\mathcal{G}_+(\mathbb{E}_3)$

$\mathbb{E}_3$ is the standard euclidean space. We denote its orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\, \boldsymbol{e}_3\}$.

The frame of $\mathcal{G}(\mathbb{E}_3)$ is simply $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\}$; the even subalgebra $\mathcal{G}_+(\mathbb{E}_3)$ is thus generated by the set $\{1, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_3\}$; by observing its multiplication table (Table 4.3), it becomes clear that the mapping

$$\begin{aligned}\boldsymbol{e}_1\boldsymbol{e}_2 &\leftrightarrow i\\ \boldsymbol{e}_2\boldsymbol{e}_3 &\leftrightarrow j\\ \boldsymbol{e}_1\boldsymbol{e}_3 &\leftrightarrow k\end{aligned}$$

induces an isomorphism $j : \mathcal{G}(\mathbb{E}_3) \leftrightarrow \mathbb{H}$.

$\cdot$	1	e_1e_2	e_2e_3	e_1e_3
1	1	e_1e_2	e_2e_3	e_1e_3
e_1e_2	e_1e_2	-1	e_1e_3	$-e_2e_3$
e_2e_3	e_2e_3	$-e_1e_3$	-1	e_1e_2
e_1e_3	e_1e_3	e_2e_3	$-e_1e_2$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 4.3: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_3)$ and $\mathbb{H}$

We conclude with the following lemma which shows that for $n > 3$, the even subalgebras are no longer division algebras.

Lemma 4.2.1 (Non-divisibility of higher-dimensional even euclidean subalgebras)**.** Let $\mathcal{G}_+(\mathbb{E}_n)$ denote the even geometric subalgebra of the n-dimensional Euclidean space. $\mathcal{G}_+(\mathbb{E}_n)$ is a division algebra if and only if $n \leq 3$.

Proof. Let $n = 4$ and onsider the following element

$$u = e_{12}e_{13}e_{34}$$

We show that it squares to 1:

$$\begin{aligned}u^2 =& e_{12}e_{13}e_{34}e_{12}e_{13}e_{34}\\ =& e_{12}e_{12}e_{13}e_{13}e_{34}e_{34}\\ =& 1\end{aligned}$$

It thus follows that $u^2 - 1 = 0$, by factoring this expression we have found two zero-divisors:

$$\begin{aligned}&u^2 - 1 = (u-1)(u+1) = 0\\ &u \pm 1 \neq 0\end{aligned}$$

By Theorem 2.1.2, $\mathcal{G}_+(\mathbb{E}_4)$ is not a division algebra. Moreover since all higher-dimensional even subalgebras contain $\mathcal{G}(\mathbb{E}_4)$, they all contain the above zero divisors and as such are also not division algebras. □

Chapter 5

Conclusion

Bibliography

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