The Even Subalgebras of Euclidean Geometric Spaces

A Geometric Proof of the Frobenius Classification of Finite-Dimensional Associative Division Algebras over the Reals

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A thesis presented for the degree of
Bachelor of Science

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May 10, 2022

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Abstract

Lorem ipsum dolor...

Acknowledgements

I want to thank...

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Chapter 1

Introduction

1.1 Historical Background

1.2 Aim and Content

We start with an overview of the field, aiming to address the following questions: what is a Geometric Algebra? How is it defined? What are its main components? How do different Geometric Algebras relate to one another?

I should preface this discussion by noting that Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms (the equivalence of which will be demonstrated), inspired by *Eric Chisolm*'s take on the topic [Chi12]. I have also taken care to discuss a couple of points which I think have been overlooked in other treatments of the topic: mainly, the **Universal Geometric Algebra** as a **template**[CATEGORY] and the relationship of **Geometric Algebra** to **Clifford Algebra**.

Geometric Algebra concerns itself with the construction of a family[CATEGORY] of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek view of mathematics in terms of geometric constructions: numbers as segments, ...

In his original paper, *Hestenes* proposes that Geometric Algebra (and Calculus) can and should provide a unified framework for mathematical physics. In his own words: *"Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."* (p. ix)[HS84].

The main

Our main focus in this paper, will be in the mathematical development of the theory and its usefulness in the field of Abstract Algebra: specifically, in the classification of Scalar Algebras.

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

2.1.1 Groups, Rings and Fields

Definition 2.1.1 (Group)**.** A group is a set closed under an associative, invertible product.

Definition 2.1.2 (Ring)**.** A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

Definition 2.1.3 (Field)**.** A field is a commutative ring in which every non-zero element has a multiplicative inverse.

2.1.2 Modules, Vector Spaces and Algebras

Definition 2.1.4 (Module)**.** A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

Definition 2.1.5 (Vector space)**.** A vector space is a module over a field.

Definition 2.1.6 (Algebra)**.** A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

$$\begin{aligned}
+ &: \text{associative, invertible, commutative} \\
\cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\
&\quad = ab * xz + a * xt + b * yz + yt \\
* &: \text{associative, distributive}
\end{aligned}$$

Definition 2.1.7 (Homomorphisms)**.** A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \to B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

DIVISIBILITY

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

3.1.1 The UGA as a Category

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained. CITATION?!

I have not been able to convince myself that the standard formulation is justified without addressing the peculiarities of infinite-dimensional inner product spaces; so we shall do otherwise, and define it as follows:

Definition 3.1.1 (Universal Geometric Algebra)**.** The **Universal Geometric Algebra** is the category of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product. We will later on observe, that this is in fact the Universal Property of Clifford Algebras.

A note on nomenclature is urgently in order: what we call **Universal Geometric Algebra** is in fact not an algebra; we will continue to refer to it as such, and in fact we will discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high-dimension such that it does not get in the way of the argument.

3.1.2 Axioms, Elements and Products

A Geometric Algebra $\mathcal{G}$ is a finite-dimensional unitary associative algebra over the reals obeying the following axioms:

Axiom 3.1.2. $\mathcal{G}$ contains $\mathbb{R}$ as a subalgebra and a real (finite) vector space $\mathcal{V}$ as a subspace; these generate the entire algebra.

Axiom 3.1.3. The formal product of a scalar and a vector corresponds with the multiplication by a scalar of the vector space.

Axiom 3.1.4. The square of any vector is a real number.

Axiom 3.1.5. The formal product on $\mathcal{V}$ is positive-definite, i.e.:

$$\forall v \neq 0 \in \mathcal{V} \quad vv > 0$$

Axiom 3.1.6. The antisymmetrized product of linearly independent vectors produces an element which does not belong to $\mathbb{R} \bigoplus \mathcal{V}$, we call this product the **exterior product** and denote it $u \wedge v = \frac{1}{2}(uv - vu)$.

We refer to the formal product of such an algebra, as a **geometric product**.

The axioms give rise to a large variety of different objects: it is thus useful to familiarise ourselves with some specific terminology before we continue with our discussion.

Definition 3.1.7 (Identities)**.** As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Definition 3.1.8 (Scalars)**.** We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.1.9 (1-vectors)**.** Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.1.10 (k-blades)**.** Products $\boldsymbol{e_1 e_2 ... e_k}$ of k anticommuting vectors; we say that a k-blade has grade k.

Definition 3.1.11 (k-versors)**.** Arbitrary products $\boldsymbol{v_1 v_2 ... v_k}$ of k vectors.

Definition 3.1.12 (Multivectors)**.** Finite sums of versors. By Axiom 3.1.2, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector. We say A is a **homogeneous** (or simple) multivector iff it is a sum of k-blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade. We introduce the grade operator $\langle A \rangle_k$ which returns the k-grade component of A. The grade of a multivector $\mathrm{gr}\,(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A.

We first prove the following lemma.

Lemma 3.1.1 (Inner Product)**.** The symmetrized product of two vectors, $u|v = \frac{1}{2}(uv + vu)$, is an inner product.

Proof. Consider the following expression, and recall Axiom 3.1.4:

$$(u+v)^2 = u^2 + v^2 + uv + vu \Leftrightarrow$$
$$2(u|v) = (u+v)^2 - u^2 - v^2 \in \mathbb{R}$$

So the symmetrized product maps into $\mathbb{R}$ (i.e. $| : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$). By Definition 2.1.6, the geometric product is bilinear; and so is the symmetrized product as it is a linear function of the bilinear products; moreover, by Axiom 3.1.5, it is also positive-definite. We conclude that $u|v = uv + vu$ is indeed an inner product on $\mathcal{V}$.

□

From the above, we obtain the principal property of the geometric product.

Theorem 3.1.2 (Geometric Product of Vectors)**.** *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv + uv) = \frac{1}{2}(uv + vu + uv - vu) = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

The result follows immediately from Lemma 3.1.1 and Axiom 3.1.6:

$$uv = u|v + u \wedge v$$

□

As a corollary, we obtain the following self-evident propositions.

Corollary 3.1.3. The exterior product of a linearly dependent set of vectors is zero.

Corollary 3.1.4. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

We will now go on to consider general expressions and properties of products between arbitrary multivectors; but first, a brief note on conventions: here on out, whenever left unspecified, the precedence of product is inner→wedge→geometric.

PROVE GENERAL FORMULA IN ORDER TO PROVE THAT EVEN MULTIVECTORS FORM A SUBALGEBRA

The geometric product between homogeneous multivectors has the following important property.

Theorem 3.1.5 (Product of Homogeneous Multivectors)**.** *The product of homogeneous multivectors A_r, B_s (for $s \leq r$) can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^{r} \langle A_r B_s \rangle_{s-r+2k}$$

Corollary 3.1.6 (Even Subalgebras)**.** The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

We are now equipped to provide the general definitions for the inner and outer products in the case of arbitrary multivectors.

Definition 3.1.13 (Generalized Inner Product)**.** On homogeneous multivectors ($s > r$):

$$A_r | B_s \equiv \langle A_r B_s \rangle_{s-r}$$

On arbitrary multivectors:

$$A|B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r | \langle B \rangle_s$$

THE INNER PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

Definition 3.1.14 (Generalized Outer Product)**.** On homogeneous multivectors ($s > r$):

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

THE EXTERIOR PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

3.2 Clifford Algebras

DISCUSS THE VALIDITY OF THE AXIOMS AND PROVE THAT GEOMETRIC ALGEBRAS ARE SIMPLY CLIFFORD ALGEBRAS (DECIDE IF I SHOULD RELAX POSITIVE-DEFINITENESS REQUIREMENT AND DRAW DISTINCTION THERE OR IF I DRAW DISTINCTION AT KEEPING THE INNER PRODUCT FORMAL OR BOTH); PROVE UNIVERSAL PROPERTY

3.3 The Euclidean Geometric Algebras: $\mathcal{G}(\mathbb{E}_1), \mathcal{G}(\mathbb{E}_2), \mathcal{G}(\mathbb{E}_3)$

DISCUSS APPROACH: CONSTRUCT BASIS OF THE ALGEBRA FROM MULTIPLICATION TABLE OF VECTOR SPACE BASIS, DISCUSS THE EVEN SUBALGEBRAS?

3.3.1 $\mathcal{G}(\mathbb{E}_1)$

3.3.2 $\mathcal{G}(\mathbb{E}_2)$

3.3.3 $\mathcal{G}(\mathbb{E}_3)$

Chapter 4

Classification of the Scalar Algebras

4.1 The Even Subalgebras $\mathcal{G}_+(\mathbb{E}_1), \mathcal{G}_+(\mathbb{E}_2), \mathcal{G}_+(\mathbb{E}_3)$

4.2 Divisibility of the Even Subalgebras

4.3 Non-divisibility of $\mathcal{G}_+(\mathbb{E}_n)$ for $n \leq 3$

4.4 The Geometric Isomorphism Theorem of Scalar Algebras

Chapter 5

Conclusion

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