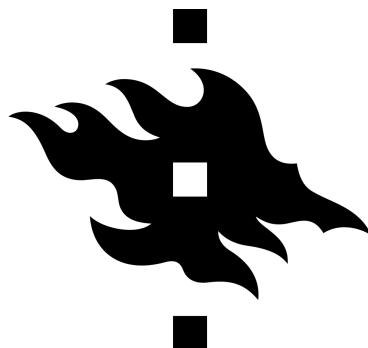


The Even Subalgebras of Euclidean Geometric Spaces

A Geometric Proof of the Frobenius Classification
of Finite-Dimensional Associative Division
Algebras over the Reals

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A thesis presented for the degree of
Bachelor of Science



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Abstract

Lorem ipsum dolor...

Acknowledgements

I want to thank...

Contents

1	Introduction	7
1.1	Historical Background	7
1.2	Aim and Content	7
2	Mathematical Background	9
2.1	Abstract Algebra: A Brief Primer	9
2.1.1	Groups, Rings and Fields	9
2.1.2	Modules, Vector Spaces and Algebras	9
3	Geometric Algebra	11
3.1	The Universal Geometric Algebra	11
3.1.1	The UGA as a Category	11
3.1.2	Axioms, Elements and Products	12
3.2	Constructing a Geometric Algebra	17
4	Classification of Scalar Algebras	18
4.1	Euclidean Even Geometric Subalgebras	18
4.1.1	$\mathcal{G}(\mathbb{E}_1)$	18
4.1.2	$\mathcal{G}(\mathbb{E}_2)$	18
4.1.3	$\mathcal{G}(\mathbb{E}_3)$	19
4.2	Non-divisibility of $\mathcal{G}_+(\mathbb{E}_n)$ for $n \leq 3$	20
4.3	The Geometric Isomorphism Theorem of Scalar Algebras	21

5	Conclusion	22
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List of Figures

List of Tables

4.1	Multiplication table of $\mathcal{G}(\mathbb{E}_2)$	18
4.2	Multiplication tables of $\mathcal{G}(\mathbb{E}_2)$ and \mathbb{C}	19

Chapter 1

Introduction

1.1 Historical Background

1.2 Aim and Content

We start with an overview of the field, aiming to address the following questions: what is a Geometric Algebra? How is it defined? What are its main components? How do different Geometric Algebras relate to one another?

I should preface this discussion by noting that Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms (the equivalence of which will be demonstrated), inspired by *Eric Chisolm*'s take on the topic [Chi12]. I have also taken care to discuss a couple of points which I think have been overlooked in other treatments of the topic: mainly, the **Universal Geometric Algebra** as a **template**[CATEGORY] and the relationship of **Geometric Algebra** to **Clifford Algebra**.

Geometric Algebra concerns itself with the construction of a family[CATEGORY] of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek view of mathematics in terms of geometric

constructions: numbers as segments, ...

In his original paper, *Hestenes* proposes that Geometric Algebra (and Calculus) can and should provide a unified framework for mathematical physics. In his own words: “*Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content.*” (p. ix)[HS84].

The main

Our main focus in this paper, will be in the mathematical development of the theory and its usefulness in the field of Abstract Algebra: specifically, in the classification of Scalar Algebras.

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

2.1.1 Groups, Rings and Fields

Definition 2.1.1 (Group). A group is a set closed under an associative, invertible product.

Definition 2.1.2 (Ring). A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

Definition 2.1.3 (Field). A field is a commutative ring in which every non-zero element has a multiplicative inverse.

2.1.2 Modules, Vector Spaces and Algebras

Definition 2.1.4 (Module). A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

Definition 2.1.5 (Vector space). A vector space is a module over a field.

Definition 2.1.6 (Algebra). A real algebra A is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations $(+, \cdot : A \times A \rightarrow A)$ and an action $(* : \mathbb{R} \times A \rightarrow A)$, with the properties:

$$\begin{aligned} + & : \text{associative, invertible, commutative} \\ \cdot & : \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\ & = ab * xz + a * xt + b * yz + yt \\ * & : \text{associative, distributive} \end{aligned}$$

Definition 2.1.7 (Subalgebra). A subalgebra is a subset of an algebra which is closed under the operations and the action of the algebra.

Definition 2.1.8 (Homomorphisms). A homomorphism is a structure-preserving mapping between two algebraic structures, i.e. $\phi : A \rightarrow B$ s. that:

$$\phi(a * xz + y) = a * \phi(x)\phi(z) + \phi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Definition 2.1.9 (Free Algebra). A free algebra is...

Theorem 2.1.1 (Universal Property of Free Algebras). *Given...
DIAGRAM*

Proof. Given...
DIAGRAM

□

Definition 2.1.10 (Division Algebra). A Division Algebra is an algebra in which all non-zero elements have a multiplicative inverse.

Definition 2.1.11 (Zero Divisor). A zero divisor of an algebra A is a non-zero element $a \in A$ s. that there exists non-zero $b \in B$ for which $ab = 0_A$. (i.e. a divides zero)

Theorem 2.1.2 (Zero Divisor). *If an algebra contains a zero divisor, then it is not a division algebra.*

Proof. Let a be a zero divisor in an algebra A ...

□

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

3.1.1 The UGA as a **Category**

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained. **CITATION?!**

I have not been able to convince myself that the standard formulation is justified without addressing the peculiarities of infinite-dimensional inner product spaces; so we shall do otherwise, and define it as follows:

Definition 3.1.1 (Universal Geometric Algebra). The **Universal Geometric Algebra** is the **category** of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product. We will later on observe, that this is in fact the Universal Property of Clifford Algebras.

A note on nomenclature is urgently in order: what we call **Universal Geometric Algebra** is in fact not an algebra; we will continue to refer to it as such, and in fact we will discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high-dimension such that it does not get in the way of the argument.

3.1.2 Axioms, Elements and Products

A Geometric Algebra \mathcal{G} is a finite-dimensional unitary associative algebra over the reals obeying the following axioms:

Axiom 3.1.2. \mathcal{G} contains \mathbb{R} as a subalgebra and a real (finite) vector space \mathcal{V} as a subspace; these generate the entire algebra.

Axiom 3.1.3. The formal product of a scalar and a vector corresponds with the multiplication by a scalar of the vector space.

Axiom 3.1.4. The square of any vector is a real number.

Axiom 3.1.5. The formal product on \mathcal{V} is positive-definite, i.e.:

$$\forall v \neq 0 \in \mathcal{V} \quad vv > 0$$

Axiom 3.1.6. The antisymmetrized product of linearly independent vectors produces an element which does not belong to $\mathbb{R} \oplus \mathcal{V}$, we call this product the **exterior product** and denote it $u \wedge v = \frac{1}{2}(uv - vu)$.

We refer to the formal product of such an algebra, as a **geometric product**.

An immediate result of axioms 3.1.4 and 3.1.5 is:

Lemma 3.1.1. All vectors are invertible with respect to the geometric product and their inverse is given by:

$$v^{-1} = \frac{v}{v^2}$$

We first prove the following lemma.

Lemma 3.1.2 (Inner Product). The symmetrized product of two vectors, $u|v = \frac{1}{2}(uv + vu)$, is an inner product.

Proof. Consider the following expression, and recall Axiom 3.1.4:

$$\begin{aligned} (u + v)^2 &= u^2 + v^2 + uv + vu \Leftrightarrow \\ 2(u|v) &= (u + v)^2 - u^2 - v^2 \in \mathbb{R} \end{aligned}$$

So the symmetrized product maps into \mathbb{R} (i.e. $| : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$). By Definition 2.1.6, the geometric product is bilinear; and so is the symmetrized product as it is a linear function of the bilinear products; moreover, by Axiom 3.1.5, it is also positive-definite. We conclude that $u|v = uv + vu$ is indeed an inner product on \mathcal{V} .

□

From the above, we obtain the principal property of the geometric product.

Theorem 3.1.3 (Geometric Product of Vectors). *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv + uv) = \frac{1}{2}(uv + vu + uv - vu) = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

The result follows immediately from Lemma 3.1.2 and Axiom 3.1.6:

$$uv = u|v + u \wedge v$$

□

As a corollary, we obtain the following self-evident propositions.

Corollary 3.1.4. The exterior product of a linearly dependent set of vectors is zero.

Corollary 3.1.5. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

We will now go on to consider general expressions and properties of products between arbitrary multivectors; but first, we ought to familiarise ourselves with some specific terminology.

Definition 3.1.7 (Identities). As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Definition 3.1.8 (Scalars). We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.1.9 (1-vectors). Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.1.10 (k-blades). Products $e_1 e_2 \dots e_k$ of k anticommuting vectors; we say that a k -blade has grade k .

Definition 3.1.11 (k-versors). Arbitrary products $v_1 v_2 \dots v_k$ of k vectors.

Definition 3.1.12 (Multivectors). Finite sums of versors. By Axiom 3.1.2, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector. We say A is a **homogeneous** (or simple) multivector iff it is a sum of k -blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade. We introduce the grade operator $\langle A \rangle_k$ which returns the k -grade component of A . The grade of a multivector $\text{gr}(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A .

A brief note on conventions: here on out, whenever left unspecified, the precedence of products is inner \rightarrow wedge \rightarrow geometric.

First off, we generalize the definition of the inner and outer products to homogeneous multivectors.

Definition 3.1.13. The inner product between two homogeneous multivectors A_r and B_s is the lowest grade term in their product.

Definition 3.1.14. The outer product between two homogeneous multivectors A_r and B_s is the highest grade term in their product.

We now prove the following steppingstone lemma regarding the product of a vector with a homogenous multivector.

Lemma 3.1.6. The inner and outer products of a vector with a homogeneous multivector have the following expressions:

$$\begin{aligned} a|A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ a \wedge A_r &= \langle aA_r \rangle_{r+1} = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

Proof. We shall assume that A_r is an r -blade: the case of a homogeneous multivector follows directly using distributivity of the geometric product (since an r -grade homogeneous multivector is a sum of r -blades).

Recall the definition of the inner product between two vectors (def. 3.1.13)

$$a|b = \frac{1}{2}(ab + ba)$$

We can reverse it to obtain:

$$ab = 2a|b - ba$$

Repeated application of the above allows us to permute indices in a product, as follows:

$$\begin{aligned} aA_r &= aa_1a_2 \dots a_r = 2a|a_1a_2 \dots a_r - a_1aa_2 \dots a_r \\ &= 2a|a_1a_2 \dots a_r - 2a|a_2a_1 \dots a_r + a_1aa_2 \dots a_r \\ &= \dots \\ &= 2 \sum_{k=1}^r (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r + (-1)^r a_1 a_2 \dots a_r a \\ &= \sum_{k=1}^r (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r + \sum_{k=1}^r (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r + (-1)^r a_1 a_2 \dots a_r a \end{aligned}$$

Notice that the first term above has grade $r - 1$: we will demonstrate that this is indeed the lowest grade term and denote it $a|A_r$.

Let us rewrite the sum using invertibility of vectors (th. 3.1.1):

$$\begin{aligned} a|A_r &= \sum_{k=1}^r (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^r (-1)^{k+1} a|a_k a_k^{-1} a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^r (-1)^{k+1} a|a_k a_k^{-1} A_r \end{aligned}$$

Subtracting the above from aA_r and factoring:

$$aA_r - a|A_r = \left(a - \sum_{k=1}^r (-1)^{k+1} a|a_k a_k^{-1}\right) A_r \equiv bA_r$$

Since by construction, $b|a_k = 0 \forall k \in \{1, \dots, r\}$, it follows that the above is a product of $r + 1$ anticommuting vectors and thus has grade $r + 1$ by axiom 3.1.6. We are justified in writing:

$$\begin{aligned} aA_r &= a|A_r + a \wedge A_r \\ a|A_r &= \sum_{k=1}^r (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ a \wedge A_r &= a|A_r + (-1)^r A_r a \end{aligned}$$

From which the lemma follows straightforwardly by substituting the third expression into the first:

$$\begin{aligned} aA_r &= 2a|A_r + (-1)^r A_r a \Rightarrow a|A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ aA_r &= 2a \wedge A_r - (-1)^r A_r a \Rightarrow a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

□

Using the above lemma, we can prove the following important property of the geometric product between homogeneous multivectors.

Theorem 3.1.7 (Product of Homogeneous Multivectors). *The product of homogeneous multivectors A_r, B_s (for $s \leq r$) can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^r \langle A_r B_s \rangle_{s-r+2k}$$

ATTRIBUTE APPROACH OF THE PROOF TO HESTENES

As a corollary, we obtain our sought-after result.

Corollary 3.1.8 (Even Subalgebras). The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

Moreover, we are now equipped to provide general, explicit definitions for the inner and outer products in the case of arbitrary multivectors.

Definition 3.1.15 (Generalized Inner Product). On homogeneous multivectors ($s > r$):

$$A_r | B_s \equiv \langle A_r B_s \rangle_{s-r}$$

On arbitrary multivectors:

$$A|B \equiv \sum_{r=0}^{\text{gr}(A)} \sum_{s=r}^{\text{gr}(B)} \langle A \rangle_r | \langle B \rangle_s$$

THE INNER PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

Definition 3.1.16 (Generalized Outer Product). On homogeneous multivectors ($s > r$):

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\text{gr}(A)} \sum_{s=r}^{\text{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

THE EXTERIOR PRODUCT IS A GRADE LOWERING OPERATION, REPRESENTING WHAT? represents the orthogonal complement of the smaller space in the larger space (left-right)

3.2 Constructing a Geometric Algebra

DISCUSS THE VALIDITY OF THE AXIOMS AND PROVE THAT GEOMETRIC ALGEBRAS ARE SIMPLY CLIFFORD ALGEBRAS (DECIDE IF I SHOULD RELAX POSITIVE-DEFINITENESS REQUIREMENT AND DRAW DISTINCTION THERE OR IF I DRAW DISTINCTION AT KEEPING THE INNER PRODUCT FORMAL OR BOTH); PROVE UNIVERSAL PROPERTY

OR

PROVE UNIVERSAL PROPERTY (FORMAL CONSTRUCTION) FROM AXIOM, DISCUSS BASES (PRACTICAL CONSTRUCTION; before universal property?), GRADE-DIMENSION AND PSEUDOSCALAR

Chapter 4

Classification of Scalar Algebras

4.1 Euclidean Even Geometric Subalgebras

We will follow the approach laid out in the previous section in order to construct the Geometric Algebra of the 1, 2 and 3 dimensional Euclidean spaces, and consider their even subalgebras. The main result of this section will be our proof that these are isomorphic to the well-known scalar algebras \mathbb{R} , \mathbb{C} and \mathbb{H} .

4.1.1 $\mathcal{G}(\mathbb{E}_1)$

$\mathcal{G}(\mathbb{E}_1)$ is trivial, it corresponds to the algebra \mathbb{R} over itself: the scalar and bilinear multiplications are one and the same; the inner product is simply given by the magnitude of the standard product (i.e. $a|b = |ab|$); and the basis is just $\{1\}$.

It follows that its even subalgebra is itself, and thus clearly isomorphic to \mathbb{R} .

4.1.2 $\mathcal{G}(\mathbb{E}_2)$

\mathbb{E}_2 is the standard euclidean plane. We denote its orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and work out its multiplication table:

\cdot	1	\mathbf{e}_1	\mathbf{e}_2	$\mathbf{e}_1\mathbf{e}_2$
1	1	\mathbf{e}_1	\mathbf{e}_2	$\mathbf{e}_1\mathbf{e}_2$
\mathbf{e}_1	\mathbf{e}_1	1	$\mathbf{e}_1\mathbf{e}_2$	\mathbf{e}_2
\mathbf{e}_2	\mathbf{e}_2	$-\mathbf{e}_1\mathbf{e}_2$	1	$-\mathbf{e}_1$
$\mathbf{e}_1\mathbf{e}_2$	$\mathbf{e}_1\mathbf{e}_2$	$-\mathbf{e}_2$	\mathbf{e}_1	-1

Table 4.1: Multiplication table of $\mathcal{G}(\mathbb{E}_2)$

It is clear then that the basis of the algebra $\mathcal{G}(\mathbb{E}_2)$ is simply $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$;

the even subalgebra $\mathcal{G}_+(\mathbb{E}_2)$ is thus generated by $\{1, \mathbf{e}_1\mathbf{e}_2\}$, and it becomes clear that the mapping $\mathbf{e}_1\mathbf{e}_2 \leftrightarrow i$ induces an isomorphism $j : \mathcal{G}(\mathbb{E}_2) \leftrightarrow \mathbb{C}$ by comparing multiplication tables:

\cdot	1	$\mathbf{e}_1\mathbf{e}_2$
1	1	$\mathbf{e}_1\mathbf{e}_2$
$\mathbf{e}_1\mathbf{e}_2$	$\mathbf{e}_1\mathbf{e}_2$	-1

\cdot	1	i
1	1	i
i	i	-1

Table 4.2: Multiplication tables of $\mathcal{G}(\mathbb{E}_2)$ and \mathbb{C}

4.1.3 $\mathcal{G}(\mathbb{E}_3)$

4.2 Non-divisibility of $\mathcal{G}_+(\mathbb{E}_n)$ for $n \leq 3$

4.3 The Geometric Isomorphism Theorem of Scalar Algebras

Chapter 5

Conclusion

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