The Even Subalgebras of Euclidean Geometric Spaces

The Realisation of Real Finite-Dimensional Associative Division Algebras in the Context of Euclidean Geometric Algebras

Author: **Marcelo Guimarães Neto**

Supervisor: **Konstantin Izyurov**

A thesis presented for the degree of
Bachelor of Science

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
Finland

June 10, 2022

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Abstract This paper is an introduction to the field of Geometric Algebra as the proper vector algebra for Euclidean space. The field - though originating in the work of William K. Clifford in the 1800s (thus also known as Clifford Algebra) - has recently been rediscovered, especially with regards to its multitude of applications in physics. Our approach to the subject follows most closely that of David Hestenes', who has been its main proponent; we will diverge slightly in our choice of axioms and a few definitions, though the content is equivalent. Our main focus will be in the study of the even subalgebras of the Geometric Algebras of Euclidean Spaces: in particular, we will show that for the 1, 2 and 3 dimensional Euclidean Spaces, these are isomorphic to the Real, Complex and Quaternion algebras respectively. We also show that higher-dimensional algebras are not division algebras, as we would indeed expect since the geometric product is associative and the Euclidean Algebras are finite.

Acknowledgements

I want to thank...

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Chapter 1

Introduction

1.1 Historical Background

Geometric Algebra has its roots in the long-running pursuit of a mathematical framework for the description of physical space. Though the concept of vectors is ancient, in this section we shall focus on the developments starting in the 19th century.

In the early 1800s, **Grassmann** was the first to formulate the notions of 'modern' linear algebra (vector spaces, bases, inner product and orthogonality) [Gra44]; his development of exterior (or as he called it, extended) algebra laid the key theoretical groundwork for Clifford's Geometric Algebra. (p.28-29)[Hes02].

Around the same time, **Hamilton** extends the 2-dimensional algebra of the complex numbers into the 4-dimensional algebra of the quaternions, aiming at a formal vector algebra for $\mathbb{R}^3$. The quaternion algebra - though a division algebra and initially adopted by Maxwell in his formulation of electrodynamics - faced some pushback and was eventually mostly supplanted by Gibbs' vector algebra (especially in physical applications) and became relegated to specific applications. Some would argue that the shortcomings of the quaternions are grounded in the fact that quaternions provide a representation for the algebra of rotations in three-dimensions, and are not natural representations of cartesian vectors in $\mathbb{R}^3$ (a major shortcoming being that quaternions do not generalize to higher dimensions) [JA16].

Gibbs is responsible for the standard vector algebra that is most widely used today: his insight was to separate the product of quaternion 'vectors' into dot and cross products and formally replace the imaginary units with unit vectors that square to $+1$.

The development of standard vector calculus (e.g. $\nabla, \nabla\cdot, \nabla\times$), applied extensively in Electrodynamics by **Heaviside**, lead to the widespread adoption of the formalism.

Though it became the adopted formalism, **Gibbs**' vector algebra has its own deficiencies: it requires two separate vectors product, lacks a division operation and its cross product does not generalize to higher-dimensions.

Clifford's Geometric Algebra - introduced in the second half of the 19th century - aimed to supersede both formalisms by providing an algebra that could be generalized to higher dimensions, and adequately and comprehensively describe both vectors and transformations in space; it extends Grassmann's work, incorporating Hamilton's quaternions into an abstractable and generalizable algebraic system for vectors, based on the geometric product.

Though **Clifford**'s work was mostly neglected in his time, it finds modern applications in a few areas of mathematics including differential geometry and mathematical physics (especially with regards to spinors). A mathematical project which aims to establish Geometric Algebra as a comprehensive, unifying framework for a variety of mathematical theories and tools related to the description of space, has also been put forth recently by **David Hestenes** [HS84].

1.2 Content and Aim

Our main focus in this paper, will be in the axiomatic approach to Geometric Algebra: we will be proving basic properties and theorems from the axioms and use these towards constructing the Euclidean Geometric Algebras and their Even Subalgebras in 1, 2 and 3 dimensions, demonstrating that the latter are actually isomorphic to the Real, Complex and Quaternion algebras.

In section 3.2, we provide an axiomatic formulation of Geometric Algebra as well as present the main objects and their definitions. The most important results in this section are Theorem 3.2.3 which separates the geometric product of two vectors into an inner and an outer product and Lemma 3.2.6 which affirms that every multivector has a unique decomposition in terms of more basic objects called k-blades. We also make some remarks about the construction of a Geometric Algebra for a given Inner Product Space.

Section 3.3 contains a few definitions and a minor lemma leading up to Theorem 3.3.2 which characterises the geometric product between homogeneous multivectors from which we obtain the main result of Chapter 3 in the form of Corollary 3.3.3 which states that the set of even-graded elements constitutes a subalgebra.

In Chapter 4, we construct the Geometric Algebras of the 1, 2 and 3-dimensional Euclidean Spaces and prove that their Even Subalgebras are indeed isomorphic to the scalar algebras. We conclude by showing that the higher-dimensional even subalgebras are non-divisible, as required by the Frobenius Classification of the Real Associative Finite-Dimensional Division Algebras.

1.3 Geometric Algebra as a Project

Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This thesis will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner. In particular, this applies to the choice of axioms - inspired by *Eric Chisolm*'s take on the topic [Chi12] and our limitation to finite-dimensional inner product spaces.

Geometric Algebra concerns itself with the construction of a collection of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. Intuitively, the theory identifies linear spaces with Euclidean geometric primitives (think points, segments, parallelograms...), and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek pre-algebraic view of mathematics, whereby arithmetic was intrinsically tied to geometric constructions. Perhaps for this very reason, **Geometric Algebra** has had made significant progress in its express goal of providing a unified framework for mathematical physics and applied mathematics. In his book, *Hestenes* details this more specifically (p. ix)[HS84]:

> "Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

In this section we will go over background definitions and results which will be of use in the subsequent chapters of the thesis.

Definition 2.1.1 (Algebra)**.** A real algebra A is a real vector space equipped with a bilinear product; i.e. a set equipped with two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$ under which it is closed, having the following properties:

- The addition operation is associative, invertible and commutative.
- The action (or multiplication by scalar) is associative and distributive with regards to the addition operation.
- The product (or multiplication) is bilinear:

$$(a * x + y) \cdot (b * z + t) = ab * xz + a * xt + b * yz + yt$$

We say that an algebra is **associative** when its multiplication is associative: in this thesis, we will always be dealing with real, associative algebras (unless otherwise mentioned).

Definition 2.1.2 (Subalgebra)**.** A subalgebra is a subset of an algebra which is closed under the operations and the action of the algebra.

Definition 2.1.3 (Homomorphisms)**.** An algebra homomorphism is a map between two algebras which preserves their algebraic structure, i.e. $\phi : A \to B$ s. that $\forall a \in \mathbb{R}$ and $x, y, z \in A$:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Theorem 2.1.1 (Basis defines the algebra)**.** *Let A and B be two finite-dimensional algebras. A linear map $\varphi : A \to B$ is a homomorphism iff for every element in the basis of A:*

$$\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$$

Moreover if j is a one-to-one mapping between bases $\mathcal{B}_A$ and $\mathcal{B}_B$ of their respective algebras, and j preserves the multiplication tables of the bases (i.e. $j(e_i e_j) = j(e_i)j(e_j)$) - then, there exists one unique such linear map $\varphi : A \to B$ that is an isomorphism and for which $\varphi|_{\mathcal{B}_A} \equiv j$.

The latter part of the above theorem allows us to simply compare multiplication tables for the basis elements of two finite-dimensional algebras in order to determine whether they are isomorphic. We say that the mapping j induces the isomorphism φ.

Definition 2.1.4 (Division Algebra)**.** A Division Algebra is an algebra in which all non-zero elements have a multiplicative inverse.

Definition 2.1.5 (Zero Divisor)**.** A zero divisor of an algebra A is a non-zero element $a \in A$ s. that there exists non-zero $b \in B$ for which $ab = 0_A$. (i.e. a divides zero)

Theorem 2.1.2 (Zero Divisor)**.** *If an (associative) algebra contains a zero divisor, then it is not a division algebra.*

Proof. Let a be a zero divisor in an algebra A, then $\exists b \neq 0_A \in A$ such that:

$$ab = 0_A$$

Assume a has an inverse in A, then:

$$a^{-1}ab = a^{-1}0_A \Leftrightarrow b = 0_A$$

This contradicts our assumptions, and so we have proven that a has no inverse in A and thus A is not a division algebra. $\square$

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

In the standard formalism, the fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short): an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained (as subalgebras) [HS84].

For our purposes, whereby we will limit ourselves to geometric algebras over finite-dimensional inner product spaces, I have found this to be unnecessary and not the best suited approach. Instead we shall formulate as our starting point, a family of geometric algebras:

Definition 3.1.1 (Geometric Family of Algebras)**.** The **Geometric Family of Algebras** is a family of algebras obeying a specific set of axioms. Its elements (the Geometric Algebras) are specified by the choice of a finite-dimensional inner product space over the real numbers.

A **GFA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product.

We will, however, discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high (finite) dimension such that it does not get in the way of the argument.

3.2 Axioms and Definitions

The Geometric Algebra $\mathcal{G}$ of a finite-dimensional inner product space $\mathcal{V}$ is the 'freest' (i.e. the most general) unitary associative algebra over the reals obeying the following axioms:

Axiom 3.2.1. $\mathcal{G}$ contains $\mathbb{R}$ as a subalgebra and $\mathcal{V}$ as a subspace; these generate the entire algebra. We call elements of $\mathbb{R}$ scalars, and elements of $\mathcal{V}$ vectors.

Axiom 3.2.2. The formal product of a scalar and a vector corresponds with the multiplication by a scalar of the vector space.

Axiom 3.2.3. The square of a vector corresponds to its inner product with itself.

$$\forall v \neq 0 \in \mathcal{V} \quad v^2 \equiv vv = v|v = |v|^2$$

Remark. The above axiom distinguishes a Clifford Algebra from a Geometric Algebra in our treatment: a Clifford Algebra only requires a quadratic form on a vector space $\mathcal{V}$. Standard treatments of Geometric Algebra also do not make such a strict requirement and thus the terms are often used interchangeably: we will not be doing so in this thesis.

We refer to the product of such an algebra as a **geometric product**.

Remark. That the algebra is the 'freest' simply means that we specify the product just enough that it satisfies the axioms, and impose no further relations. One could for example define a geometric product on $\mathbb{R}^3$ as the sum of the inner and cross products: the generated algebra would satisfy the axioms, however it would not be the freest since the cross product imposes the additional equivalence relation:

$$e_i e_{[i+1]_3} \equiv e_{[i+2]_3} \quad \forall i \in \{1, 2, 3\}$$

where $[a]_3$ is a shorthand notation for a mod 3. A modern, abstract formulation of the concept has this property naturally since the Geometric Algebra is constructed from more abstract objects by imposing only the required relations: I have not taken this route given the scope and target audience of the thesis. FETCH A REFERENCE?!

Two results follow immediately from Axiom 3.2.3:

Lemma 3.2.1. All vectors are invertible with respect to the geometric product and the inverse is given by:

$$v^{-1} = \frac{v}{v^2}$$

Lemma 3.2.2 (Inner Product)**.** The symmetrized product of two vectors,

$$s(u, v) = \frac{1}{2}(uv + vu)$$

corresponds to the inner product on $\mathcal{V}$.

Proof. Consider the following expression, and recall Axiom 3.2.3:

$$\begin{aligned}
(u+v)^2 &= u^2+v^2+uv+vu \Leftrightarrow \\
2s(u,v) &= (u+v)^2-u^2-v^2 \\
&\overset{3.2.3}{=} (u+v)|(u+v)-u|u-v|v \\
&= u|v+v|u = 2u|v
\end{aligned}$$

We conclude

$$s(u,v) \equiv u|v$$

□

We have observed that the axioms place a restriction on the symmetric part of the product between two vectors. The antisymmetric part, on the other hand, bears no restriction: since the Geometric Algebra is the most general algebra obeying the axioms, we identify this part of the product with a formal bilinear and antisymmetric product, we call it the **exterior** product. We thus obtain the principal property of the geometric product.

Theorem 3.2.3 (Geometric Product of Vectors)**.** *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv+uv) = \frac{1}{2}(uv+vu+uv-vu) = \frac{1}{2}(uv+vu)+\frac{1}{2}(uv-vu)$$

The result follows immediately from Lemma 3.2.2 and the above remarks. □

As a corollary, we obtain the following self-evident propositions.

Corollary 3.2.4. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

Corollary 3.2.5. An orthonormal basis $\{e_i\}_{i=1}^n$ obeys the following relations:

$$\begin{aligned}
e_ie_j &= -e_je_i \\
e_ie_i &= 1
\end{aligned}$$

Before we make some final remarks about existence, uniqueness and construction, we ought to familiarise ourselves with some specific terminology.

Definition 3.2.4 (Identity Elements)**.** As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identity elements: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Definition 3.2.5 (Scalars)**.** We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.2.6 (1-vectors)**.** Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.2.7 (k-blades)**.** Products $e_1 e_2 \ldots e_k$ of orthogonal vectors; we say that a k-blade has grade k. These are also called simple k-vectors.

Definition 3.2.8 (Versors)**.** Arbitrary products $\boldsymbol{v}_1 \boldsymbol{v}_2 \ldots \boldsymbol{v}_k$ of vectors.

Definition 3.2.9 (Multivectors)**.** Finite sums of versors. By Axiom 3.2.1, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector.

A critical lemma to which we will often make implicit reference is the following:

Lemma 3.2.6. Every element $V \in \mathcal{G}(\mathcal{V})$ has a unique decomposition (with respect to a given basis) as a sum of k-blades.

Proof. Every multivector can be written as a finite sum of versors, so it suffices to prove the statement for an arbitrary versor. Consider an arbitrary set of vectors expressed in a given orthonormal basis $\{v_i \equiv \sum_{j=1}^{n} v_i^j e_j\}_{i=1}^{m}$; we expand their product using bilinearity:

$$\prod_{i=1}^{m} v_i = \prod_{i=1}^{m} \sum_{j=1}^{n} v_i^j e_j = \sum_{j_i:\{1..m\} \to \{1..n\}} \left(\prod_{i=1}^{m} v_i^{j_i} \right) \left(\prod_{i=1}^{m} e_{j_i} \right)$$

It follows from Corollary 3.2.5 that the product of the basis vectors in each term is a k-blade, where k is the number of unit vectors appearing an odd number of times in the product. Uniqueness follows from the freeness of the algebra: the product of orthogonal vectors is a formal product subject only to bilinearity and antisymmetry, so that distinct products of orthonormal basis vectors are linearly independent and form in fact a vector space-basis for the whole algebra. $\square$

We are now justified in presenting the following definitions:

Definition 3.2.10 (Grade)**.** We introduce the grade operator $\langle A \rangle_k$ which returns the k-grade component of A. The grade of a multivector $\mathrm{gr}\,(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A. We say A is a **homogeneous** multivector of grade k iff it is a sum of k-blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade.

Definition 3.2.11 (Basis)**.** The **basis** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is simply the basis of the generating inner-product space $\mathcal{V}$.

Definition 3.2.12 (Frame)**.** The **frame** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is the basis of $\mathcal{G}(\mathcal{V})$ as a vector space. It is constructed by taking all product combinations of the unit vectors in an orthonormal basis for $\mathcal{V}$ where no vector is repeated. Frame elements for a given orthonormal basis are unique up to permutation of factors.

Definition 3.2.13 (Pseudoscalar)**.** The **pseudoscalar** of a geometric algebra $\mathcal{G}(\mathcal{V})$ is the highest grade element in its frame. It is unique up to scalar multiplication (including permutation of factors). The square of a pseudoscalar is also a scalar, by convention, it is normalized such that it squares to 1 or -1 (which defines the orientation of the frame).

The distinction between basis and frame is very important: a frame is to a basis, as a geometric algebra is to its generating vector space. For finite-dimensional spaces, we can always produce an orthonormal basis by the Gram-Schmidt process, so we will henceforth assume all bases to be orthonormal unless otherwise specified.

A frame can be identified one-to-one with the power set of a basis, as it is composed of all the possible combinations of the basis elements without repetition and up to permutation (we identify the unit scalar in the frame with the empty subset of the basis). It follows that the dimension of a geometric algebra $\mathcal{G}$ (as a vector space) is $2^{\mathrm{gr}(\mathcal{G})}$.

It is then clear that for every inner product space there exists a unique Geometric Algebra, and the recipe for its construction is quite simple:

1. construct an orthonormal basis for the inner-product space: this will be the basis of the algebra
2. construct the frame of the algebra by taking products of basis vectors
3. generate the rest of the algebra as a vector space with the frame as its basis

A brief note on conventions: here on out, whenever left unspecified, the precedence of products is inner$\rightarrow$wedge$\rightarrow$geometric.

3.3 Products

We will now go on to consider general expressions and properties of products between arbitrary multivectors. First off, we generalize the definition of the inner and outer products to homogeneous multivectors.

Definition 3.3.1. The inner product between two homogeneous multivectors A_r and B_s is the lowest-possible grade term in their product. We will later see that this is the $|r-s|$-grade term; if the product does not have such a term, then we say that the inner product is 0.

Definition 3.3.2. The outer product between two homogeneous multivectors A_r and B_s is the highest-possible grade term in their product. We will later see that this is the $|r+s|$-grade term; if the product does not have such a term, then we say that the outer product is 0.

We now prove the following lemma regarding the product of a vector with a homogenous multivector.

Lemma 3.3.1. The inner and outer products of a vector with a homogeneous multivector have the following expressions:

$$\begin{aligned} a|A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ a \wedge A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

Proof. We shall assume that A_r is an r-blade: the case of a homogeneous multivector follows directly using distributivity of the geometric product (since an r-grade homogeneous multivector is a sum of r-blades).

Recall the definition of the inner product between two vectors (def. 3.3.1)

$$a|b = \frac{1}{2}(ab + ba)$$

We can reverse it to obtain:

$$ab = 2a|b - ba$$

Repeated application of the above allows us to permute indices in a product, as follows:

$$\begin{aligned} aA_r = aa_1a_2 \dots a_r =& 2a|a_1a_2 \dots a_r - a_1aa_2 \dots a_r \\ =& 2a|a_1a_2 \dots a_r - 2a|a_2a_1 \dots a_r + a_1aa_2 \dots a_r \\ =& \dots \\ =& 2\sum_{k=1}^{r}(-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r + (-1)^r a_1a_2 \dots a_r a \\ =& \sum_{k=1}^{r}(-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r + \sum_{k=1}^{r}(-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ & + (-1)^r a_1a_2 \dots a_r a \end{aligned}$$

Notice that the first term above has grade $r-1$: we will demonstrate that this is indeed the lowest grade term. For now, we preemptively denote it $a|A_r$.

Let us rewrite the sum using invertibility of vectors (th. 3.2.1):

$$\begin{aligned} a|A_r &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_k^{-1} a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r} a|a_k a_k^{-1} A_r \end{aligned}$$

Substracting the above from aA_r and factoring:

$$aA_r - a|A_r = (a - \sum_{k=1}^{r} a|a_k a_k^{-1}) A_r \equiv bA_r$$

Since by construction, $b|a_k = 0 \forall k \in \{1, \dots r\}$, it follows by Corollary 3.2.4 that the above is a product of $r+1$ anticommuting vectors and thus has grade $r+1$ by Definition 3.2.7. We are justified in writing:

$$\begin{aligned} aA_r &= a|A_r + a \wedge A_r \\ a|A_r &= \sum_{k=1}^{r} (-1)^{k+1} a|a_k a_1 \dots \check{a}_k \dots a_r \\ a \wedge A_r &= a|A_r + (-1)^r A_r a \end{aligned}$$

From which the lemma follows straightforwardly by substituting the third expression into the first:

$$\begin{aligned} aA_r = 2a|A_r + (-1)^r A_r a &\Rightarrow a|A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ aA_r = 2a \wedge A_r - (-1)^r A_r a &\Rightarrow a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

□

The above proof is due to *Hestenes* (p. 8-10)[HS84]

Using the above lemma, we can prove the following important property of the geometric product between homogeneous multivectors.

Theorem 3.3.2 (Product of Homogeneous Multivectors)**.** *The product of homogeneous multivectors A_r, B_s (for $s \leq r$) can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^{r} \langle A_r B_s \rangle_{s-r+2k}$$

Proof. We prove this by induction on $r \leq s$ when A_r and B_s are simple r- and s-vectors respectively.

The case $r = 1,\ s = 1$ is true by Definition 3.2.3. The case $r = 1,\ s > 1$ is true by Lemma 3.3.1. Assume the expression holds for $r = q,\ s > r$, we show that it holds for $r + 1$:

$$\begin{aligned}
A_{r+1}B_s = a_{r+1}A_rB_s &= a_{r+1}\sum_{k=0}^{r}\langle A_rB_s\rangle_{s-r+2k} \\
&= \sum_{k=0}^{r} a_{r+1}\langle A_rB_s\rangle_{s-r+2k} \\
&= \sum_{k=0}^{r} [a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1} \wedge \langle A_rB_s\rangle_{s-r+2k}] \\
&= a|\langle A_rB_s\rangle_{s-r} \\
&\quad + \sum_{k=1}^{r} \left[a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1} \wedge \langle A_rB_s\rangle_{s-r+2(k-1)}\right] \\
&\quad + a \wedge \langle A_rB_s\rangle_{s+r}
\end{aligned}$$

where in the last step, we have grouped terms together by grade:

$$\begin{aligned}
\langle A_{r+1}B_s\rangle_{s-(r+1)} &\equiv a_{r+1}|\langle A_rB_s\rangle_{s-r} \\
\langle A_{r+1}B_s\rangle_{s-(r+1)+2k} &\equiv a|\langle A_rB_s\rangle_{s-r+2k} + a \wedge \langle A_rB_s\rangle_{s-r+2(k-1)} \\
\langle A_{r+1}B_s\rangle_{r+1+s} &\equiv a \wedge \langle A_rB_s\rangle_{s+r}
\end{aligned}$$

The case where $s \leq r$ follows by induction on s (the same argument as above). The general case for homogeneous multivectors follows by distributivity of the geometric product. □

The above proof is due to *Chisolm* (p. 20-21)[Chi12]

As a corollary, we obtain our sought-after result.

Corollary 3.3.3 (Even Subalgebras)**.** The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

Moreover, we are now equipped to provide general, explicit definitions for the inner and outer products in the case of arbitrary multivectors.

Definition 3.3.3. (Generalized Inner Product)

On homogeneous multivectors:

$$A_r|B_s \equiv \langle A_rB_s\rangle_{|s-r|}$$

On arbitrary multivectors:

$$A|B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A\rangle_r|\langle B\rangle_s$$

Definition 3.3.4. (Generalized Outer Product)

On homogeneous multivectors:

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

Chapter 4

Euclidean Even Subalgebras

4.1 Even Subalgebras

We start off with a discussion on the construction of even subalgebras and introduce some notation and results that will be of use in the next section.

An even subalgebra is constructed from its basis precisely in the same manner as a geometric algebra, however its basis is distinct from that of the original algebra.

This is made explicit in the following self-evident lemma:

Lemma 4.1.1. Let $\mathcal{G}$ be a geometric algebra with basis $\{e_1, e_2, \dots e_n\}$. Its even subalgebra $\mathcal{G}_+$ has basis $\{e_1e_2, \dots e_1e_n, e_2e_3 \dots e_{n-1}e_n\}$; we will make use of the following shortened notation: $e_ie_j = e_{ij}$.

The basis bivectors have the following basic properties:

$$\begin{aligned} e_{ii} &= 0 \\ e_{ij} &= -e_{ji} \\ {e_{ij}}^2 &= -1 \end{aligned}$$

Moreover, it follows from straightforward algebraic manipulations that the product of two unit bivectors anti-commutes if they share a common index and commutes otherwise.

We will make use of the above properties in computation, without explicit mention.

4.2 Euclidean Geometric Algebras

We will follow the approach laid out in the previous section in order to construct the Geometric Algebra of the 1, 2 and 3 dimensional Euclidean spaces and consider their even subalgebras. The main result of this chapter will be our proof that these are isomorphic to the well-known scalar algerbras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

4.2.1 $\mathcal{G}(\mathbb{E}_1)$ and $\mathcal{G}_+(\mathbb{E}_1)$

We define the 1-dimensional Euclidean space as follows in order to distinguish it from the real numbers:

$$\mathbb{E}_1 \equiv \{\alpha e_1 | \alpha \in \mathbb{R}\} \sim \mathbb{R}$$

$\mathcal{G}(\mathbb{E}_1)$ is trivial: its frame is just $\{1, e_1\}$. It follows that its even subalgebra has frame $\{1\}$ and thus corresponds to $\mathbb{R}$.

4.2.2 $\mathcal{G}(\mathbb{E}_2)$ and $\mathcal{G}_+(\mathbb{E}_2)$

$\mathbb{E}_2$ is the standard euclidean plane. We denote its orthonormal basis $\{e_1, e_2\}$ and work out its multiplication table with respect to the geometric product:

$\cdot$	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

Table 4.1: Multiplication table of $\mathcal{G}(\mathbb{E}_2)$

The frame of $\mathcal{G}(\mathbb{E}_2)$ is simply $\{1, e_1, e_2, e_1e_2\}$, and the even subalgebra $\mathcal{G}_+(\mathbb{E}_2)$ is thus generated by $\{1, e_1e_2\}$. It is apparent that the mapping $e_1e_2 \leftrightarrow i$ induces an isomorphism $j : \mathcal{G}(\mathbb{E}_2) \leftrightarrow \mathbb{C}$ by comparing multiplication tables:

$\cdot$	1	e_{12}
1	1	e_{12}
e_{12}	e_{12}	-1

$\cdot$	1	i
1	1	i
i	i	-1

Table 4.2: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_2)$ and $\mathbb{C}$

4.2.3 $\mathcal{G}(\mathbb{E}_3)$ and $\mathcal{G}_+(\mathbb{E}_3)$

$\mathbb{E}_3$ is the standard euclidean space. We denote its orthonormal basis $\{e_1, e_2\, e_3\}$.

The frame of $\mathcal{G}(\mathbb{E}_3)$ is simply $\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{12}e_3\}$; the even subalgebra $\mathcal{G}_+(\mathbb{E}_3)$ is thus generated by the set $\{1, e_{12}, e_{23}, e_{13}\}$; by observing its multiplication table (Table 4.3), it becomes clear that the mapping

$$\begin{aligned} e_{12} &\leftrightarrow i \\ e_{23} &\leftrightarrow j \\ e_{13} &\leftrightarrow k \end{aligned}$$

induces an isomorphism $j : \mathcal{G}(\mathbb{E}_3) \leftrightarrow \mathbb{H}$.

$\cdot$	1	e_{12}	e_{23}	e_{13}
1	1	e_{12}	e_{23}	e_{13}
e_{12}	e_{12}	-1	e_{13}	$-e_{23}$
e_{23}	e_{23}	$-e_{13}$	-1	e_{12}
e_{13}	e_{13}	e_{23}	$-e_{12}$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 4.3: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_3)$ and $\mathbb{H}$

We conclude with the following lemma which shows that for $n > 3$, the even subalgebras are no longer division algebras.

Lemma 4.2.1 (Non-divisibility of higher-dimensional even euclidean subalgebras)**.** Let $\mathcal{G}_+(\mathbb{E}_n)$ denote the even geometric subalgebra of the n-dimensional Euclidean space. $\mathcal{G}_+(\mathbb{E}_n)$ is a division algebra if and only if $n \leq 3$.

Proof. Let $n = 4$ and consider the following element

$$u = e_{12}e_{13}e_{34}$$

We show that it squares to 1:

$$\begin{aligned} u^2 &= e_{12}e_{13}e_{34}e_{12}e_{13}e_{34} \\ &= e_{12}e_{12}e_{13}e_{13}e_{34}e_{34} \\ &= 1 \end{aligned}$$

It thus follows that $u^2 - 1 = 0$, by factoring this expression we have found two zero-divisors:

$$\begin{aligned} u^2 - 1 &= (u-1)(u+1) = 0 \\ u \pm 1 &\neq 0 \end{aligned}$$

By Theorem 2.1.2, $\mathcal{G}_+(\mathbb{E}_4)$ is not a division algebra. Moreover since all higher-dimensional even subalgebras contain $\mathcal{G}(\mathbb{E}_4)$, they all contain the above zero divisors and as such are also not division algebras. $\square$

Chapter 5

Conclusion

In this paper, we have developed the fundamentals of Geometric Algebra.

Starting from a formal set of axioms, we have thoroughly derived the general properties of the algebra, its elements and its products.

We have observed that the algebra does indeed encompass the standard Gibbs' algebra, the algebra of the Complex Numbers and that of the Quaternions. All while having the marked advantage of invertibility with regards to vectors, a natural generalization to higher dimensions, as well as the capability to encode Euclidean geometric primitives through the exterior product.

Geometric Algebra - and its extension to the domain of analysis, Geometric Calculus - have many promising applications that unfortunately were outside the scope of this thesis: linear and multilinear algebra can be expressed in the language of frames and outermorphisms; differential geometry can be formulated in the framework of vector manifolds and the geometric derivative with the promise of coordinate-free computations; the even subalgebras of Euclidean geometric spaces can be shown to account for the theory of Spinors; advances have even been made in the theory of Lie Algebras and Groups.

Geometric Algebra postulates that geometry is a fundamental guide towards meaningful mathematical pursuits even in abstract settings. It should be no surprise that such a theory would be so adept at describing a wide range of geometric phenomena. As *Hestenes* puts it (p. xii)[HS84]:

> "Geometry without algebra is dumb! Algebra without geometry is blind!"

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