# The Even Subalgebras of Euclidean Geometric Spaces

## A Geometric Proof of the Frobenius Classification of Finite-Dimensional Associative Division Algebras over the Reals

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A thesis presented for the degree of
Bachelor of Science

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May 9, 2022

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## Abstract

Lorem ipsum dolor...

# Acknowledgements

I want to thank...

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# Chapter 1

# Introduction

## 1.1 Historical Background

## 1.2 Aim and Content

We start with an overview of the field, aiming to address the following questions: what is a Geometric Algebra? How is it defined? What are its main components? How do different Geometric Algebras relate to one another?

I should preface this discussion by noting that Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms (the equivalence of which will be demonstrated), inspired by *Eric Chisolm*'s take on the topic [Chi12]. I have also taken care to discuss a couple of points which I think have been overlooked in other treatments of the topic: mainly, the **Universal Geometric Algebra** as a **template**[CATEGORY] and the relationship of **Geometric Algebra** to **Clifford Algebra**.

**Geometric Algebra** concerns itself with the construction of a family[CATEGORY] of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek view of mathematics in terms of geometric constructions: numbers as segments, ...

In his original paper, *Hestenes* proposes that Geometric Algebra (and Calculus) can and should provide a unified framework for mathematical physics. In his own words: *"Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."* (p. ix)[HS84].

The main

Our main focus in this paper, will be in the mathematical development of the theory and its usefulness in the field of Abstract Algebra: specifically, in the classification of Scalar Algebras.

# Chapter 2

# Mathematical Background

## 2.1 Abstract Algebra: A Brief Primer

### 2.1.1 Groups, Rings and Fields

**Definition 2.1.1** (Group)**.** A group is a set closed under an associative, invertible product.

**Definition 2.1.2** (Ring)**.** A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

**Definition 2.1.3** (Field)**.** A field is a commutative ring in which every non-zero element has a multiplicative inverse.

### 2.1.2 Modules, Vector Spaces and Algebras

**Definition 2.1.4** (Module)**.** A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

**Definition 2.1.5** (Vector space)**.** A vector space is a module over a field.

**Definition 2.1.6** (Algebra)**.** A real algebra $A$ is a real vector space equipped with a bilinear product; i.e.: a set closed under two operations ($+, \cdot : A \times A \to A$) and an action ($* : \mathbb{R} \times A \to A$), with the properties:

$$\begin{aligned}
+ &: \text{associative, invertible, commutative} \\
\cdot &: \text{bilinear} : (a * x + y) \cdot (b * z + t) = \\
&\quad = ab * xz + a * xt + b * yz + yt \\
* &: \text{associative, distributive}
\end{aligned}$$

**Definition 2.1.7** (Homomorphisms)**.** A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \to B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

DIVISIBILITY

# Chapter 3

# Geometric Algebra

## 3.1 The Universal Geometric Algebra

### 3.1.1 The UGA as a Category

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained. CITATION?!

I have not been able to convince myself that the standard formulation is justified without addressing the peculiarities of infinite-dimensional inner product spaces; so we shall do otherwise, and define it as follows:

**Definition 3.1.1** (Universal Geometric Algebra)**.** The **Universal Geometric Algebra** is the category of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product. We will later on observe, that this is in fact the Universal Property of Clifford Algebras.

A note on nomenclature is urgently in order: what we call **Universal Geometric Algebra** is in fact not an algebra; we will continue to refer to it as such, and in fact we will discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high-dimension such that it does not get in the way of the argument.

### 3.1.2 The Axioms

A Geometric Algebra $\mathcal{G}$ is a finite-dimensional unitary associative algebra over the reals obeying the following axioms:

**Axiom 3.1.2.** $\mathcal{G}$ contains $\mathbb{R}$ and a real (finite) vector space $\mathcal{V}$ as subspaces; these generate the entire algebra.

**Axiom 3.1.3.** The square of any vector is a real number.

**Axiom 3.1.4.** The symmetrized product on $\mathcal{V}$ is positive-definite, i.e.:

$$\forall v \neq 0 \in \mathcal{V} \quad vv > 0$$

### 3.1.3 The Elements

### 3.1.4 The Products

## 3.2 Clifford Algebras

## 3.3 The Euclidean Geometric Algebras: $\mathcal{G}(\mathbb{E}_1), \mathcal{G}(\mathbb{E}_2), \mathcal{G}(\mathbb{E}_3)$

### 3.3.1 $\mathcal{G}(\mathbb{E}_1)$

### 3.3.2 $\mathcal{G}(\mathbb{E}_2)$

### 3.3.3 $\mathcal{G}(\mathbb{E}_3)$

# Chapter 4

# Classification of the Scalar Algebras

## 4.1 The Even Subalgebras $\mathcal{G}_+(\mathbb{E}_1), \mathcal{G}_+(\mathbb{E}_2), \mathcal{G}_+(\mathbb{E}_3)$

## 4.2 Divisibility of the Even Subalgebras

## 4.3 Non-divisibility of $\mathcal{G}_+(\mathbb{E}_n)$ for $n \leq 3$

## 4.4 The Geometric Isomorphism Theorem of Scalar Algebras

# Chapter 5

# Conclusion

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