The Even Subalgebras of Euclidean Geometric Spaces

The Realisation of Real Finite-Dimensional Associative Division Algebras in the Context of Euclidean Geometric Algebras

Marcelo Guimarães Neto

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HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
Finland

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Introduction

1.1 Historical Background

Geometric Algebra has its roots in the long-running pursuit of a mathematical framework for the description of physical space. Though the concept of vectors is ancient, in this section we shall focus on the developments starting in the 19th century.

In the early 1800s, **Grassmann** was the first to formulate the notions of 'modern' linear algebra (vector spaces, bases, inner product and orthogonality); his development of exterior algebra through the introduction of the outer product, laid the key theoretical groundwork for Clifford's Geometric Algebra as it provided a formal algebraic system for vectors which is capable of describing higher-dimensional geometric primitives (p.28-29)[Hes02]:

$$\begin{aligned} \boldsymbol{e_i} | \boldsymbol{e_j} &= \delta_{ij} \\ \boldsymbol{e_i} \wedge \boldsymbol{e_j} &= -\boldsymbol{e_i} \wedge \boldsymbol{e_j} \end{aligned}$$

Around the same time, **Hamilton** extends the 2-dimensional algebra of the complex numbers into the 4-dimensional algebra of the quaternions, aiming at a formal vector algebra for $\mathbb{R}^3$.

$$\begin{aligned} &q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \\ &i^2 = j^2 = k^2 = ijk = -1 \\ &ij = k, ki = j, jk = i \end{aligned}$$

The quaternion algebra, though arithmetically complete and initially adopted by Maxwell in his formulation of electrodynamics, was not very popular due to its noncommutativity and the negative square of vectors, and eventually became relegated to very specific applications. Fundamentally, the shortcomings of the quaternions are grounded in the fact that quaternions provide a representation for the algebra of rotations in three-dimensions, and are not suitable to describing cartesian vectors in $\mathbb{R}^3$ [JA16].

Gibbs is responsible for the standard vector algebra that is most widely used today: his insight was to separate the product of quaternion 'vectors' into dot and cross products and formally replace the imaginary units with unit vectors that square to +1.

$$\begin{aligned} vu &= -\sum v_i u_i + \sum_{i \to j \to k} (v_i u_j - v_j u_i)\boldsymbol{k} \\ &\equiv -v|u + v \times u \end{aligned}$$

The development of standard vector calculus (e.g. $\nabla, \nabla\cdot, \nabla\times$), applied extensively in Electrodynamics by **Heaviside**, lead to the widespread adoption of the formalism.

Though it became the adopted formalism, **Gibbs**' vector algebra has its own deficiencies: it requires two different vectors product, lacks a division operation, the cross product is noncommutative and does not generalize to higher-dimensions.

Clifford's Geometric Algebra - introduced in the second half of the 19th century - aimed to supersed both formalisms by providing an arithmetically complete algebra that could be generalized to higher dimensions, and adequately describe both vectors and transformations in space; it extends Grassmann's work, incorporating Hamilton's quaternions into an abstractable and generalizable algebraic system for vectors, based on the geometric product:

$$\begin{aligned} &uv = u|v + u \wedge v \\ &V = \sum \langle V \rangle_i \\ &\langle V \rangle_r = \sum v_i \boldsymbol{e_{i_1}} \wedge ... \wedge \boldsymbol{e_{i_r}} \end{aligned}$$

Though **Clifford**'s work was neglected in his time, it has recently been repopularized by **David Hestenes** [HS84]. We will be discussing the modern formalism in this thesis.

1.2 Content and Aim

Geometric Algebra - as the mathematical project proposed by *David Hestenes* and *Garret Sobczyk* in [HS84] - is a relatively new subject: as such, there seems to be no consolidated consensus as to the precise manner in which it ought to be presented. This paper will follow most closely the original approach in [HS84]; nonetheless, I have taken the liberty to formulate and present some of these concepts in a slightly different manner (though only superficially so). In particular, this applies to the choice of axioms - inspired by *Eric Chisolm*'s take on the topic [Chi12] - and the formulation of the **Universal Geometric Algebra** as a family of abstract algebras.

Geometric Algebra concerns itself with the construction of a collection of abstract algebras which adequately extend the arithmetic of the real numbers to higher-dimensional, coordinate-free settings, under the guidance of geometric intuition. In a **Geometric Algebra** elements are fully characterised by three (geometric) properties: **magnitude**, **direction** and **orientation**. In practice, the theory identifies linear spaces with Euclidean geometric primitives, and in doing so equips any (finite-dimensional) linear space with a powerful algebra whose elements are the subspaces themselves.

This is reminiscent of the Greek pre-algebraic view of mathematics, whereby arithmetic was intrinsically tied to geometric constructions. Perhaps for this very reason, **Geometric Algebra** has had made significant progress in its express goal of providing a unified framework for mathematical physics and applied mathematics. In his book, *Hestenes* details this more specifically (p. ix)[HS84]:

> "Our long-range aim is to see Geometric Calculus established as a unified system for handling linear and multilinear algebra, multivariable calculus, complex variable theory, differential geometry and other subjects with geometric content."

Our main focus in this paper, will be in the mathematical exposition of the theory: we will be proving basic properties and theorems from the axioms and use these towards constructing the Euclidean Geometric Algebras and their Even Subalgebras in 1, 2 and 3 dimensions, demonstrating that the latter are actually isomorphic to the Real, Complex and Quaternion algebras.

Chapter 2

Mathematical Background

2.1 Abstract Algebra: A Brief Primer

In this section I will present background definitions and results which will be of use in the subsequent chapters of the thesis.

2.1.1 Groups, Rings and Fields

Definition 2.1.1 (Group)**.** A group is a set closed under an associative, invertible product.

Definition 2.1.2 (Ring)**.** A ring is an Abelian group (we refer to the group operation as addition) equipped with a second associative and distributive binary operation (we refer to it as multiplication).

Definition 2.1.3 (Field)**.** A field is a commutative ring in which every non-zero element has a multiplicative inverse.

2.1.2 Modules, Vector Spaces and Algebras

Definition 2.1.4 (Module)**.** A module is an Abelian group closed under a left-right multiplication by a ring that is associative and distributive.

Definition 2.1.5 (Vector space)**.** A vector space is a module over a field.

Definition 2.1.6 (Algebra)**.** A real algebra A is a real vector space equipped with a bilinear product; i.e. a set closed under two operations $(+, \cdot : A \times A \to A)$ and an action $(* : \mathbb{R} \times A \to A)$, with the properties:

- The addition operation is associative, invertible and commutative.

- The multiplication by a scalar is associative and distributive with regards to the addition operation.
- The product is bilinear:

$$(a * x + y) \cdot (b * z + t) = ab * xz + a * xt + b * yz + yt$$

Definition 2.1.7 (Subalgebra)**.** A subalgebra is a subset of an algebra which is closed under the operations and the action of the algebra.

Definition 2.1.8 (Homomorphisms)**.** A homomorphism is a structure-preserving mapping between two algebraic strucutures, i.e. $\phi : A \to B$ s. that:

$$\varphi(a * xz + y) = a * \varphi(x)\varphi(z) + \varphi(y)$$

If the mapping is 1-to-1, we call it an isomorphism.

Theorem 2.1.1 (Basis defines the algebra)**.** *Let A and B be two finite-dimensional algebras. A linear map $\varphi : A \to B$ is a homomorphism iff for every element in the basis of A:*

$$\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$$

Moreover if j is a one-to-one mapping between bases $\mathcal{B}_A$ and $\mathcal{B}_B$ of their respective algebras, and j commutes with multiplication - there exists one unique such linear map $\varphi : A \to B$ that is an isomorphism and for which $\varphi|_{\mathcal{B}_A} \equiv j$.

The latter part of the above theorem is the Universal Property of Free Algebras, and it allows us to simply compare multiplication tables for the basis elements of two finite-dimensional algebras in order to determine whether they are isomorphic. We say that the mapping j induces the isomorphism φ.

Definition 2.1.9 (Division Algebra)**.** A Division Algebra is an algebra in which all non-zero elements have a multiplicative inverse.

Definition 2.1.10 (Zero Divisor)**.** A zero divisor of an algebra A is a non-zero element $a \in A$ s. that there exists non-zero $b \in B$ for which $ab = 0_A$. (i.e. a divides zero)

Theorem 2.1.2 (Zero Divisor)**.** *If an algebra contains a zero divisor, then it is not a division algebra.*

Proof. Let a be a zero divisor in an algebra A, then $\exists b \neq 0_A \in A$ such that:

$$ab = 0_A$$

Assume a has an inverse in A, then:

$$a^{-1}ab = a^{-1}0_A \Leftrightarrow b = 0_A$$

This contradicts our premises, and so we have proven that a has no inverse in A and thus A is not a division algebra. □

Chapter 3

Geometric Algebra

3.1 The Universal Geometric Algebra

The fundamental concept in Geometric Algebra is that of the **Universal Geometric Algebra** (UGA for short). It is usually formulated as an infinite-dimensional abstract algebra obeying a certain set of axioms, within which all the Geometric Algebras are contained [HS84].

I have not been able to convince myself that the standard formulation is justified without addressing the peculiarities of infinite-dimensional inner product spaces; so we shall do otherwise, and define it as follows:

Definition 3.1.1 (Universal Geometric Algebra)**.** The **Universal Geometric Algebra** is a family of algebras obeying a specific set of axioms. Its elements (the Geometric or Clifford Algebras) are specified by the choice of a finite-dimensional inner product space over the reals.

A **UGA** is then a template for Geometric Algebras: given a finite-dimensional linear space, and an inner product, there exists a unique axiom-abiding algebra which contains the linear space and whose symmetrized product corresponds to the prescribed inner product.

A note on nomenclature is urgently in order: what we call **Universal Geometric Algebra** is in fact not an algebra; we will continue to refer to it as such, and in fact we will discuss the generally applicable definitions and results in the context of an abstract algebra whose formal product obeys the axioms, and whose abstract linear space will always be assumed to be of a sufficiently high-dimension such that it does not get in the way of the argument.

3.2 Axioms, Elements and Products

A Geometric Algebra $\mathcal{G}$ is a finite-dimensional unitary associative algebra over the reals obeying the following axioms:

Axiom 3.2.1. $\mathcal{G}$ contains $\mathbb{R}$ as a subalgebra and a real (finite) vector space $\mathcal{V}$ as a subspace; these generate the entire algebra.

Axiom 3.2.2. The formal product of a scalar and a vector corresponds with the multiplication by a scalar of the vector space.

Axiom 3.2.3. The square of any vector is a real number.

Axiom 3.2.4. The formal product on $\mathcal{V}$ is positive-definite, i.e.:

$$\forall v \neq 0 \in \mathcal{V} \quad vv > 0$$

Axiom 3.2.5. The antisymmetrized product of linearly independent vectors produces an element which does not belong to $\mathbb{R} \bigoplus \mathcal{V}$, we call this product the **exterior product** and denote it $u \wedge v = \frac{1}{2}(uv - vu)$.

We refer to the formal product of such an algebra, as a **geometric product**.

An immediate result of axioms 3.2.3 and 3.2.4 is:

Lemma 3.2.1. All vectors are invertible with respect to the geometric product and their inverse is given by:

$$v^{-1} = \frac{v}{v^2}$$

We first prove the following lemma.

Lemma 3.2.2 (Inner Product)**.** The symmetrized product of two vectors,

$$u|v = \frac{1}{2}(uv + vu)$$

is an inner product.

Proof. Consider the following expression, and recall Axiom 3.2.3:

$$(u + v)^2 = u^2 + v^2 + uv + vu \Leftrightarrow$$
$$2u|v = (u + v)^2 - u^2 - v^2 \in \mathbb{R}$$

So the symmetrized product maps into $\mathbb{R}$ (i.e. $| : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$). By Definition 2.1.6, the geometric product is bilinear and so is the symmetrized product as it is a linear function of the bilinear products; moreover, by Axiom 3.2.4, it is also positive-definite. We conclude that $u|v = uv + vu$ is indeed an inner product on $\mathcal{V}$.

$\square$

From the above, we obtain the principal property of the geometric product.

Theorem 3.2.3 (Geometric Product of Vectors)**.** *The geometric product of two vectors can be broken down into an inner and an exterior product:*

$$uv = u|v + u \wedge v$$

Proof. We first show how the geometric product decomposes into a symmetric and antisymmetric part:

$$uv = \frac{1}{2}(uv + uv) = \frac{1}{2}(uv + vu + uv - vu) = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

The result follows immediately from Lemma 3.2.2 and Axiom 3.2.5:

$$uv = u|v + u \wedge v$$

□

As a corollary, we obtain the following self-evident propositions.

Corollary 3.2.4. The exterior product of a linearly dependent set of vectors is zero.

Corollary 3.2.5. A pair of vectors are orthogonal if and only if they anticommute with respect to the geometric product.

We will now go on to consider general expressions and properties of products between arbitrary multivectors; but first, we ought to familiarise ourselves with some specific terminology.

Definition 3.2.6 (Identities)**.** As a unitary algebra, $\mathcal{G}(\mathcal{V})$ has unique additive and multiplicative identities: $0, 1 \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$

Definition 3.2.7 (Scalars)**.** We refer to reals $a \in \mathbb{R} \subset \mathcal{G}(\mathcal{V})$ as scalars, or *grade-0* vectors.

Definition 3.2.8 (1-vectors)**.** Elements $v \in \mathcal{V} \subset \mathcal{G}(\mathcal{V})$ are called 1-vectors, or simply vectors.

Definition 3.2.9 (k-blades)**.** Products $\boldsymbol{e}_1\boldsymbol{e}_2 \ldots \boldsymbol{e}_k$ of k anticommuting vectors; we say that a k-blade has grade k. These are also called simple k-vectors.

Definition 3.2.10 (k-versors)**.** Arbitrary products $\boldsymbol{v}_1\boldsymbol{v}_2 \ldots \boldsymbol{v}_k$ of k vectors.

Definition 3.2.11 (Multivectors)**.** Finite sums of versors. By Axiom 3.2.1, every element $V \in \mathcal{G}(\mathcal{V})$ is a multivector. We say A is a **homogeneous** multivector iff it is a sum of k-blades for a given $k \in \mathbb{N}$; otherwise we say A is of mixed grade. We introduce the grade operator $\langle A \rangle_k$ which returns the k-grade component of A. The grade of a multivector $\mathrm{gr}\,(A) \in \mathbb{N}$ is the grade of the maximum-grade term in A.

A brief note on conventions: here on out, whenever left unspecified, the precedence of products is inner→wedge→geometric.

First off, we generalize the definition of the inner and outer products to homogeneous multivectors.

Definition 3.2.12. The inner product between two homogeneous multivectors A_r and B_s is the lowest grade term in their product.

Definition 3.2.13. The outer product between two homogeneous multivectors A_r and B_s is the highest grade term in their product.

We now prove the following lemma regarding the product of a vector with a homogenous multivector.

Lemma 3.2.6. The inner and outer products of a vector with a homogeneous multivector have the following expressions:

$$\begin{aligned} a|A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ a \wedge A_r &= \langle aA_r \rangle_{r-1} = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

Proof. We shall assume that A_r is an r-blade: the case of a homogeneous multivector follows directly using distributivity of the geometric product (since an r-grade homogeneous multivector is a sum of r-blades).

Recall the definition of the inner product between two vectors (def. 3.2.12)

$$a|b = \frac{1}{2}(ab + ba)$$

We can reverse it to obtain:

$$ab = 2a|b - ba$$

Repeated application of the above allows us to permute indices in a product, as follows:

$$\begin{aligned} aA_r = aa_1a_2 \dots a_r =& 2a|a_1a_2 \dots a_r - a_1aa_2 \dots a_r \\ =& 2a|a_1a_2 \dots a_r - 2a|a_2a_1 \dots a_r + a_1aa_2 \dots a_r \\ =& \dots \\ =& 2\sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1 \dots \check{a}_k \dots a_r + (-1)^r a_1a_2 \dots a_r a \\ =& \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1 \dots \check{a}_k \dots a_r + \sum_{k=1}^{r}(-1)^{k+1}a|a_ka_1 \dots \check{a}_k \dots a_r \\ &+ (-1)^r a_1a_2 \dots a_r a \end{aligned}$$

Notice that the first term above has grade $r-1$: we will demonstrate that this is indeed the lowest grade term and denote it $a|A_r$.

Let us rewrite the sum using invertibility of vectors (th. 3.2.1):

$$\begin{aligned} a|A_r &= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1} a_k a_1 \dots \check{a}_k \dots a_r \\ &= \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1} A_r \end{aligned}$$

Substracting the above from aA_r and factoring:

$$aA_r - a|A_r = (a - \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_k^{-1})A_r \equiv bA_r$$

Since by construction, $b|a_k = 0 \forall k \in \{1, \dots r\}$, it follows by Corollary 3.2.5 that the above is a product of $r+1$ anticommuting vectors and thus has grade $r+1$ by Definition 3.2.9. We are justified in writing:

$$\begin{aligned} &aA_r = a|A_r + a \wedge A_r \\ &a|A_r = \sum_{k=1}^{r}(-1)^{k+1}a|a_k a_1 \dots \check{a}_k \dots a_r \\ &a \wedge A_r = a|A_r + (-1)^r A_r a \end{aligned}$$

From which the lemma follows straightforwardly by substituting the third expression into the first:

$$\begin{aligned} aA_r = 2a|A_r + (-1)^r A_r a \Rightarrow a|A_r = \frac{1}{2}(aA_r - (-1)^r A_r a) \\ aA_r = 2a \wedge A_r - (-1)^r A_r a \Rightarrow a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \end{aligned}$$

□

The above proof is due to *Hestenes* (p. 8-10)[HS84]

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Using the above lemma, we can prove the following important property of the geometric product between homogeneous multivectors.

Theorem 3.2.7 (Product of Homogeneous Multivectors)**.** *The product of homogeneous multivectors A_r, B_s (for $s \leq r$) can be decomposed as follows:*

$$A_r B_s = \sum_{k=0}^{r}\langle A_r B_s\rangle_{s-r+2k}$$

Proof. We prove this by induction on $r \leq s$ when A_r and B_s are simple r- and s-vectors respectively.

The case $r = 1, \quad s = 1$ is true by Definition 3.2.3. The case $r = 1, \quad s > 1$ is true by Lemma 3.2.6. Assume the expression holds for $r = q, \quad s > r$, we show that it holds for $r + 1$:

$$\begin{aligned}
A_{r+1}B_s =& a_{r+1}A_rB_s = a_{r+1}\sum_{k=0}^{r}\langle A_rB_s\rangle_{s-r+2k} = \\
=& \sum_{k=0}^{r} a_{r+1}\langle A_rB_s\rangle_{s-r+2k} = \\
=& \sum_{k=0}^{r} [a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1} \wedge \langle A_rB_s\rangle_{s-r+2k}] \\
=& a|\langle A_rB_s\rangle_{s-r} \\
&+ \sum_{k=1}^{r} \left[a_{r+1}|\langle A_rB_s\rangle_{s-r+2k} + a_{r+1} \wedge \langle A_rB_s\rangle_{s-r+2(k-1)}\right] \\
&+ a \wedge \langle A_rB_s\rangle_{s+r}
\end{aligned}$$

where in the last step, we have grouped terms together by grade:

$$\begin{aligned}
&\langle A_{r+1}B_s\rangle_{s-(r+1)} \equiv a_{r+1}|\langle A_rB_s\rangle_{s-r} \\
&\langle A_{r+1}B_s\rangle_{s-(r+1)+2k} \equiv a|\langle A_rB_s\rangle_{s-r+2k} + a \wedge \langle A_rB_s\rangle_{s-r+2(k-1)} \\
&\langle A_{r+1}B_s\rangle_{r+1+s} \equiv a \wedge \langle A_rB_s\rangle_{s+r}
\end{aligned}$$

The case where $s \leq r$ follows by induction on s (the same argument as above). The general case for homogeneous multivectors follows by distributivity of the geometric product. □

The above proof is due to *Chisolm* (p. 20-21)[Chi12]

As a corollary, we obtain our sought-after result.

Corollary 3.2.8 (Even Subalgebras)**.** The set of even-grade elements of a Geometric Algebra constitutes a subalgebra.

Moreover, we are now equipped to provide general, explicit definitions for the inner and outer products in the case of arbitrary multivectors.

Definition 3.2.14. (Generalized Inner Product)

On homogeneous multivectors:

$$A_r|B_s \equiv \langle A_rB_s\rangle_{|s-r|}$$

On arbitrary multivectors:

$$A|B \equiv \sum_{r=0}^{\mathrm{gr}(A)}\sum_{s=r}^{\mathrm{gr}(B)}\langle A\rangle_r|\langle B\rangle_s$$

Definition 3.2.15. (Generalized Outer Product)

On homogeneous multivectors:

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{s+r}$$

On arbitrary multivectors:

$$A \wedge B \equiv \sum_{r=0}^{\mathrm{gr}(A)} \sum_{s=r}^{\mathrm{gr}(B)} \langle A \rangle_r \wedge \langle B \rangle_s$$

3.3 Bases and Frames

A Geometric Algebra is only realized into a concrete algebra once an inner-product space is specified. It is then straightforward to construct the whole algebra given a basis for the inner-product space. But first, we need some specific terminology.

Definition 3.3.1 (Basis)**.** The **basis** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is simply the basis of the generating inner-product space $\mathcal{V}$.

Definition 3.3.2 (Frame)**.** The **frame** of a Geometric Algebra $\mathcal{G}(\mathcal{V})$ is the basis of $\mathcal{G}(\mathcal{V})$ as a vector space. It is constructed by taking all product combinations of the unit vectors in an orthonormal basis for $\mathcal{V}$ where no vector is repeated. Frame elements for a given orthonormal basis are unique up to permutation of factors.

Definition 3.3.3 (Pseudoscalar)**.** The **pseudoscalar** of a geometric algebra $\mathcal{G}(\mathcal{V})$ is the highest grade element in its frame. It is unique up to scalar multiplication (including permutation of factors). The square of a pseudoscalar is also a scalar, by convention, it is normalized such that it squares to 1 or -1 (which defines the orientation of the frame).

The distinction between basis and frame is very important: a frame is to a basis, as a geometric algebra is to its generating vector space. For finite-dimensional spaces, we can always produce an orthonormal basis by the Gran-Schmidt process, so we will henceforth assume all bases to be orthonormal unless otherwise specified.

A frame can be identified one-to-one with the power set of a basis, as it is composed of all the possible combinations of the basis elements without repetition and up to permutation (we identify the unit scalar in the frame with the empty subset of the basis). It follows that the dimension of a geometric algebra $\mathcal{G}$ (as a vector space) is $2^{\mathrm{gr}(\mathcal{G})}$.

The recipe for constructing a Geometric Algebra is thus, as follows:

1. construct an orthonormal basis for the inner-product space: this will be the basis of the algebra
2. construct the frame of the algebra by taking products between non-repeating unit vectors
3. generate the rest of the algebra as a vector space with the frame as its basis

Chapter 4

Classification of Scalar Algebras

4.1 Even Subalgebras

We start off with a discussion on the construction of even subalgebras and introduce some notation and results that will be of use in the next section.

An even subalgebra has its own basis which is distinct from that of the original algebra; this is made explicit in the following self-evident lemma:

Lemma 4.1.1. Let $\mathcal{G}$ be a geometric algebra with basis $\{e_1, e_2, \dots e_n\}$. Its even subalgebra $\mathcal{G}_+$ has basis $\{e_1e_2, \dots e_1e_n, e_2e_3 \dots e_{n-1}e_n\}$; we will make use of the following shortened notation: $e_ie_j = e_{ij}$.

The basis 2-vectors have the following properties:

$$
\begin{aligned}
&e_{ij} = 0 \text{ if } i = j\\
&e_{ij} = -e_{ji}\\
&{e_{ij}}^2 = -1\\
&e_{ij}e_{rs} = e_{rs}e_{ij} \text{ if the 2-vectors have no indices in common}\\
&e_{ij}e_{rs} = -e_{rs}e_{ij} \text{ if the 2-vectors have one index in common}
\end{aligned}
$$

We will make use of the above properties in computation, without explicit mention.

An even subalgebra is constructed from its basis precisely in the same manner as a geometric algebra.

4.2 Euclidean Geometric Algebras

We will follow the approach laid out in the previous section in order to construct the Geometric Algebra of the 1, 2 and 3 dimensional Euclidean spaces and consider their even subalgebras. The main result of this chapter will be our proof that these are isomorphic to the well-known scalar algerbras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

4.2.1 $\mathcal{G}(\mathbb{E}_1)$ and $\mathcal{G}_+(\mathbb{E}_1)$

$\mathcal{G}(\mathbb{E}_1)$ is trivial: the inner product is simply given by the magnitude of the standard product (i.e. $a|b = |ab|$); and its frame is just $\{1, e_1\}$. It follows that its even subalgebra has frame $\{1\}$ and thus corresponds to $\mathbb{R}$.

4.2.2 $\mathcal{G}(\mathbb{E}_2)$ and $\mathcal{G}_+(\mathbb{E}_2)$

$\mathbb{E}_2$ is the standard euclidean plane. We denote its orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and work out its multiplication table with respect to the geometric product:

$\cdot$	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$
1	1	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	1	$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1\boldsymbol{e}_2$	1	$-\boldsymbol{e}_1$
$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	-1

Table 4.1: Multiplication table of $\mathcal{G}(\mathbb{E}_2)$

The frame of $\mathcal{G}(\mathbb{E}_2)$ is simply $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2\}$, and the even subalgebra $\mathcal{G}_+(\mathbb{E}_2)$ is thus generated by $\{1, \boldsymbol{e}_1\boldsymbol{e}_2\}$. It is apparent that the mapping $\boldsymbol{e}_1\boldsymbol{e}_2 \leftrightarrow i$ induces an isomorphism $j : \mathcal{G}(\mathbb{E}_2) \leftrightarrow \mathbb{C}$ by comparing multiplication tables:

$\cdot$	1	$\boldsymbol{e}_1\boldsymbol{e}_2$
1	1	$\boldsymbol{e}_1\boldsymbol{e}_2$
$\boldsymbol{e}_1\boldsymbol{e}_2$	$\boldsymbol{e}_1\boldsymbol{e}_2$	-1

$\cdot$	1	i
1	1	i
i	i	-1

Table 4.2: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_2)$ and $\mathbb{C}$

4.2.3 $\mathcal{G}(\mathbb{E}_3)$ and $\mathcal{G}_+(\mathbb{E}_3)$

$\mathbb{E}_3$ is the standard euclidean space. We denote its orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\, \boldsymbol{e}_3\}$.

The frame of $\mathcal{G}(\mathbb{E}_3)$ is simply $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\}$; the even subalgebra $\mathcal{G}_+(\mathbb{E}_3)$ is thus generated by the set $\{1, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_1\boldsymbol{e}_3\}$; by observing its multiplication table (Table 4.3), it becomes clear that the mapping

$$\begin{aligned}\boldsymbol{e}_1\boldsymbol{e}_2 &\leftrightarrow i\\ \boldsymbol{e}_2\boldsymbol{e}_3 &\leftrightarrow j\\ \boldsymbol{e}_1\boldsymbol{e}_3 &\leftrightarrow k\end{aligned}$$

induces an isomorphism $j : \mathcal{G}(\mathbb{E}_3) \leftrightarrow \mathbb{H}$.

$\cdot$	1	e_1e_2	e_2e_3	e_1e_3
1	1	e_1e_2	e_2e_3	e_1e_3
e_1e_2	e_1e_2	-1	e_1e_3	$-e_2e_3$
e_2e_3	e_2e_3	$-e_1e_3$	-1	e_1e_2
e_1e_3	e_1e_3	e_2e_3	$-e_1e_2$	-1

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 4.3: Multiplication tables of $\mathcal{G}_+(\mathbb{E}_3)$ and $\mathbb{H}$

We conclude with the following lemma which shows that for $n > 3$, the even subalgebras are no longer division algebras.

Lemma 4.2.1 (Non-divisibility of higher-dimensional even euclidean subalgebras)**.** Let $\mathcal{G}_+(\mathbb{E}_n)$ denote the even geometric subalgebra of the n-dimensional Euclidean space. $\mathcal{G}_+(\mathbb{E}_n)$ is a division algebra if and only if $n \leq 3$.

Proof. Let $n = 4$ and consider the following element

$$u = e_{12}e_{13}e_{34}$$

We show that it squares to 1:

$$\begin{aligned}u^2 =& e_{12}e_{13}e_{34}e_{12}e_{13}e_{34}\\ =& e_{12}e_{12}e_{13}e_{13}e_{34}e_{34}\\ =& 1\end{aligned}$$

It thus follows that $u^2 - 1 = 0$, by factoring this expression we have found two zero-divisors:

$$\begin{aligned}&u^2 - 1 = (u-1)(u+1) = 0\\ &u \pm 1 \neq 0\end{aligned}$$

By Theorem 2.1.2, $\mathcal{G}_+(\mathbb{E}_4)$ is not a division algebra. Moreover since all higher-dimensional even subalgebras contain $\mathcal{G}(\mathbb{E}_4)$, they all contain the above zero divisors and as such are also not division algebras. □

Chapter 5

Conclusion

In this paper, we have developed the fundamentals of Geometric Algebra.

Starting from a formal set of axioms, we have thoroughly derived the general properties of the algebra, its elements and its products.

We have observed that the algebra does indeed encompass the standard Gibbs' algebra, the algebra of the Complex Numbers and that of the Quaternions. All while having the marked advantage of invertibility with regards to vectors, a natural generalization to higher dimensions, as well as the capability to encode Euclidean geometric primitives through the exterior product.

Geometric Algebra - and its extension to the domain of analysis, Geometric Calculus - have many promising applications that unfortunately were outside the scope of this thesis: linear and multilinear algebra can be expressed in the language of frames and outermorphisms; differential geometry can be formulated in the framework of vector manifolds and the geometric derivative with the promise of coordinate-free computations; the even subalgebras of Euclidean geometric spaces can be shown to account for the theory of Spinors; advances have even been made in the theory of Lie Algebras and Groups.

Geometric Algebra postulates that geometry is a fundamental guide towards meaningful mathematical pursuits even in abstract settings. It should be no surprise that such a theory would be so adept at describing a wide range of geometric phenomena. As *Hestenes* puts it (p. xii)[HS84]:

> "Geometry without algebra is dumb! Algebra without geometry is blind!"

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