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A model for tonal progressions of seventh chords

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In this paper we propose a model for idealized diatonic and chromatic voice leadings between seventh chords and study its computational aspects. The model provides mathematical formalization of the concept of tonal seventh-chord realizations including augmented sixth chords. Certain contrapuntal aspects, such as preparation and resolution of certain dissonant intervals, are taken into account. Possible applications of the model are discussed using a graph-theoretic approach. In particular, we present an algorithm for generating concrete voicings from sequences of seventh-chord symbols.

Keywords: seventh chord; harmony; voice leading; line of fifths; parsimony; chord progression; graph theory; shortest-path algorithm; pitch spelling

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1. Introduction

Mathematical approaches to music theory often involve voice-leading models and chord transformations in order to investigate and classify harmonic relations between chords and to provide insights into compound structures such as chord progressions. In particular, the concept of parsimonious voice leading attracted considerable attention in the past couple of decades, especially in Neo-Riemannian studies. Voice-leading parsimony for trichords is studied by Cohn (1997, 2012), while Gollin (1998); Childs (1998); Reenan and Bass (2016); Kerkez (2012); Cannas and Andreatta (2018) propose various generalizations of the concept for seventh chords based on the Relation definition by Douthett and Steinbach (1998). Transformations between seventh chords are also studied by Jacobus (2012). The problem of measuring the distance between two chords is addressed by Tymoczko (2006, 2009, 2011); Harasim, Schmidt, and Rohrmeier (2016); del Pozo and Gómez (2019), to name a few. Callender, Quinn, and Tymoczko (2008) and Wells Hall and Tymoczko (2011) investigate geometrical approach to voice leading and its applications. Kochavi (2008) generalizes the concept of voice-leading parsimony by introducing degrees of parsimony for voice leadings between two collections of tones, taking both diatonic and chromatic aspects into account.

We present a new mathematical model of bijective, stepwise voice leading between seventh chords which, unlike pitch-class-based models, makes a distinction between enharmonically equivalent realizations of seventh chords as well as between two types of semitonal voice leading: diatonic and chromatic. The goal is to refine the concept of

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parsimonious voice leading for seventh chords, formalize some rules from the theory of four-part writing and provide tools for generation and classification of relations between individual chords. Algorithms resulting from this approach form a computational framework implemented in a freely available software library which may be useful to music theorists, educators, and composers.

Throughout this paper, $\mathbb{Z}_N$ denotes the quotient group of the integers modulo N, which is a cyclic group of order N. Given $k \in \mathbb{Z}$, the unique element $m \in \mathbb{Z}_N$ such that $m \equiv k \pmod{N}$ is denoted by $k \bmod N$. Furthermore, let $\mathbb{N}_0$ be the totally ordered monoid $(\{0, 1, 2, \dots\}, +, 0)$ and $\Delta : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{N}_0$ the mapping defined by

$$\Delta(i, j) = \min\{w_D(i, j), w_D(j, i)\},$$

where $w_D(a, b)$ is the length of the path from a to b in the directed cyclic graph D on the elements of $\mathbb{Z}_N$ in which there is an arc from i to j if and only if $i + 1 = j$ in $\mathbb{Z}_N$. It can be shown that Δ is a (generalized) metric in $\mathbb{Z}_N$ and that the mapping $\|k\| := \Delta(k, 0)$, where $k \in \mathbb{Z}_N$, is a group norm in $\mathbb{Z}_N$ (Harasim, Schmidt, and Rohrmeier 2016). Since only the cyclic groups with $N = 7$ and $N = 12$ are discussed here, their elements are distinguished from integers by denoting $\mathbb{Z}_7 = \{\underline{0}, \underline{1}, \dots, \underline{6}\}$ and $\mathbb{Z}_{12} = \{\overline{0}, \overline{1}, \dots, \overline{11}\}$.

The rest of this paper is organized as follows. Section 2 presents the typical algebraic representation of seventh chords. Section 3 describes a model of musical tones and intervals using the line of fifths and elements of diatonic set theory. Section 4 defines tonal realizations of seventh chords. Section 5 introduces the novel concept of elementary transitions between tonal realizations which represent diatonic and chromatic relations between seventh chords in a form suitable for discussing contrapuntal aspects such as preparation and resolution of sevenths. Algorithms for generating such transitions are provided. In Section 6 it is shown how to construct chord graphs which represent relations between individual seventh chords as established in Section 5; walks in such graphs correspond to sequences (progressions) of seventh-chord symbols. Algorithms for finding and enumerating progressions are discussed and the theory of complex networks is applied to estimate the relative importance of different seventh-chord types. Finally, Section 7 addresses the problem of finding four-part realizations of seventh-chord progressions by using a computational, preference-rule-based approach.

2. Seventh chords

A stacked-thirds-based seventh chord in the circular pitch-class space is modeled as a pair $c = (p, \mathbf{s})$, where $p \in \mathbb{Z}_{12}$ is the *root pitch* and $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{Z}_{12}^3$, such that $s_k \in \{\overline{3}, \overline{4}\}$ for $k = 1, 2, 3$ and $s_1 + s_2 + s_3 \neq \overline{0}$, is the *type* of c. Each seventh chord has one of the seven possible types, which comprise the set

$$\mathcal{S} = \{[333], [433], [343], [334], [443], [344], [434]\},$$

where $[abc]$ is short for $(\overline{a}, \overline{b}, \overline{c})$ (Jedrzejewski 2019). In music theory, the elements of $\mathcal{S}$—in the order as denoted above—correspond to diminished, dominant, minor-seventh, half-diminished, augmented major, minor major, and major seventh chords, respectively (Piston 1987, p. 355). Each type specifies the bottom-up pitch structure of the respective chords. For example, the triple [433] indicates that a dominant seventh chord is built from a major third surmounted by two minor thirds.

The pitch-class set (pcset) $\{p, p + s_1, p + s_1 + s_2, p + s_1 + s_2 + s_3\} \subset \mathbb{Z}_{12}$ is called the

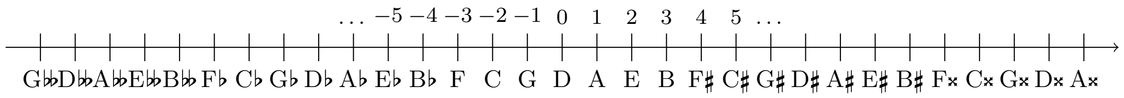


Figure 1. Line of fifths.

pitch-class representation of c and is denoted by P_c. It is easily verified that $|P_c| = 4$. Two seventh chords are considered equal if their pitch-class representations coincide. For instance, the diminished seventh chords $(\overline{0}, [333])$ and $(\overline{6}, [333])$ are equal according to this definition.

In this text, seventh chords are notated by using symbols common in jazz and popular music (Harte et al. 2005; Cannas and Andreatta 2018). For example, the chords $(\overline{0}, [433])$, $(\overline{2}, [333])$, $(\overline{3}, [343])$, $(\overline{6}, [334])$, $(\overline{8}, [434])$, and $(\overline{9}, [344])$ correspond to C^7, D^{o7}, $\mathrm{E}\flat\mathrm{m}^7/\mathrm{D}\sharp\mathrm{m}^7$, $\mathrm{F}\sharp^{ø}/\mathrm{G}\flat^{ø}$, $\mathrm{A}\flat^{\triangle}/\mathrm{G}\sharp^{\triangle}$, and $\mathrm{Am}^{\triangle}$, respectively.

Our study is limited to the five most commonly used types of seventh chords: diminished, dominant, minor, half-diminished, and major. The set containing all seventh chords of these types, which we denote by $\mathcal{C}$ in the following, has exactly 51 elements: there are three diminished seventh chords and 12 chords for each other type (Cannas and Andreatta 2018). Augmented major and minor major seventh chords stand apart due to the specific interval of augmented fifth and are usually resolved to triads (rarely they are followed by other seventh chords, as for example in mm. 28–29 of Bach's Scherzo from Partita in A minor, BWV 827). Furthermore, by excluding these types one achieves a better harmonic consistency of the remaining ones.

3. Musical tones and intervals

In this paper, a tone is identified with its position $x \in \mathbb{Z}$ on the line of fifths (LOF) which is shown in Figure 1 (Temperley 2000; Hook 2007). The tone D is mapped to 0 and consequently x and $-x$ always have the same number of accidental modifiers. Taking a step to the right in LOF corresponds to the movement by perfect fifth upwards, i.e. by four scale steps or seven semitones. Hence the scale degree and pitch-class of t, denoted by $\tau(x) \in \mathbb{Z}_7$ and $\pi(x) \in \mathbb{Z}_{12}$, respectively, are determined from the following relations:

$$\tau(x) \equiv 4x + 1 \pmod 7, \quad \pi(x) \equiv 7x + 2 \pmod{12}. \tag{1}$$

This approach to modeling tones relies on the octave equivalence but allows us to distinguish between enharmonically equivalent tones, such as C♯ and D♭. Although tones are defined as integers, the standard music notation will also be used to denote them when appropriate.

Let $X \subset \mathbb{Z}$ be a set of tones. The *diameter* of X is the largest LOF-distance between two tones in X, denoted by $\operatorname{diam}(X) := \max\{|x - y| : x, y \in X\}$.

Given two tones $x, y \in \mathbb{Z}$, the *interval from x to y* is defined as

$$\langle x, y\rangle := (g, h) \in \mathbb{Z}_7 \times \mathbb{Z}_{12}, \quad g = \tau(y) - \tau(x), \; h = \pi(y) - \pi(x). \tag{2}$$

The components g and h are called *generic size* and *specific size*, respectively (Clough and Myerson 1985; Harasim, Schmidt, and Rohrmeier 2016). Generic sizes of intervals are $\underline{0}$ (unison), $\underline{1}$ (second), $\underline{2}$ (third), $\underline{3}$ (fourth), $\underline{4}$ (fifth), $\underline{5}$ (sixth), and $\underline{6}$ (seventh).

According to its specific size, an interval can be either perfect (unison, fourth, or fifth), minor/major (third, sixth, second, or seventh), augmented or diminished (Piston 1987, pages 4–5). By definition, the inverse interval from y to x is equal to $\langle y,x\rangle = (-g,-h)$. Also, letting $(g',h') = \langle y,z\rangle$, it readily follows that $\langle x,z\rangle = (g+g',h+h')$. The tones x and y are enharmonically equivalent if $h=\overline{0}$, i.e. if $x \equiv y \pmod{12}$.

Although LOF is theoretically infinite, the vast majority of classical music works—including those from the 20[th] century—has been written using exclusively the segment from G𝄫 to A𝄪 shown in Figure 1 (for instance, J. S. Bach uses the tones from E𝄫 to A𝄪 in the Well Tempered Clavier[1]). Those are precisely the tones which have at most two sharps/flats. Triple accidentals are extremely rare (Byrd 2018a; Hook 2007). The indicated LOF-domain corresponds to the set $\{-15,-14,\dots,15\} \subset \mathbb{Z}$ which is denoted by $\mathcal{T}$ in the following. Its importance has been noted in previous works; for instance, pitch-spelling algorithms by Stoddard, Raphael, and Utgoff (2004) and Longuet-Higgins (1987) restrict their output to $\mathcal{T}$.

4. Tonal realizations of seventh chords

A seventh chord $c \in \mathcal{C}$ can be realized in the musical staff by choosing, for each pitch $p \in P_c$, a tone $x \in \mathbb{Z}$ such that $\pi(x) = p$. Any set $R \subset \mathbb{Z}$ such that $|R| = 4$ and $\pi(R) = P_c$ is therefore called a *realization* of c.

In this section we focus on realizations which are meaningful in the context of tonal harmony (Piston 1987, p. 31). For example, $\{\mathrm{C},\mathrm{E},\mathrm{G},\mathrm{B}\flat\}$ and $\{\mathrm{C},\mathrm{E},\mathrm{G},\mathrm{A}\sharp\}$, which are both realizations of C^7, have functional roles in keys of F and E, respectively. On the other hand, {C, D𝄪, F𝄪, A♯} is a realization C^7 which does not have a functional interpretation in any key and hence is not tonal.[2]. A formal definition is proposed as follows.

Definition 4.1 A realization R of $c \in \mathcal{C}$ is *tonal* if the following conditions are satisfied:

(1) R can be arranged into a column of stacked generic thirds, as in Figure 2(b), i.e. $\tau(R)$ is a maximally even four-element subset of $\mathbb{Z}_7$ (Clough and Douthett 1991),
(2) each of two interlocking generic fifths in the column, which are indicated by brackets, is either perfect or diminished.

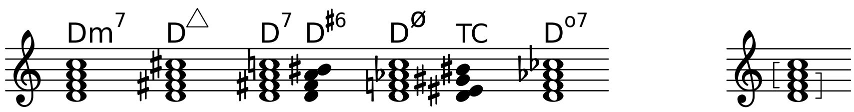
Tonal realizations of seventh chords above the tone D are shown in Figure 2(a). There are seven distinct types of tonal realizations in total; the corresponding LOF patterns are shown in Figure 3. Dominant and half-diminished seventh chords can both be spelled in two ways, as regular seventh chords or augmented sixth chords. The latter alternatives, which are displayed with black note-heads, are called German sixth chord (Piston 1987, p. 279) and Tristan chord[3] (TC) (Martin 2008).

To prove that the set of types shown in Figure 3 is exhaustive with respect to Definition 4.1, it is enough to list all integer partitions (s_1,s_2,s_3,s_4) of the number 12 such that $1 \le s_k \le 4$ for all k and $s_1+s_2, s_2+s_3 \in \{6,7\}$. There are exactly nine of them:

[1]The tone E𝄫 occurs in measure 53 of Fugue in B♭ minor, Book II, while A𝄪 occurs in measure 22 of Fugue in C♯ major, Book I.

[2]Composers sometimes choose non-tonal realizations of seventh chords, for instance in Balakirev's 10 Songs, No. 4, m. 1 (Reenan and Bass 2016, Fig. 5a), where $\mathrm{D}\flat^7$ is spelled as $\{\mathrm{F},\mathrm{B},\mathrm{A}\flat,\mathrm{C}\sharp\}$ and in Wagner's Prelude to *Tristan*, m. 10, where $\mathrm{D}^{ø}$ is spelled as $\{\mathrm{C},\mathrm{F},\mathrm{G}\sharp,\mathrm{D}\}$. However, such decisions are usually dependent on the surrounding voice-leading context rather than the chord itself.

[3]In this text, the term *Tristan chord* does not refer to the particular chord from Wagner's *Tristan*, but rather to the intervallic structure exhibited by that chord.



(a) (b)

Figure 2. Intervallic structure of tonal realizations of seventh chords.

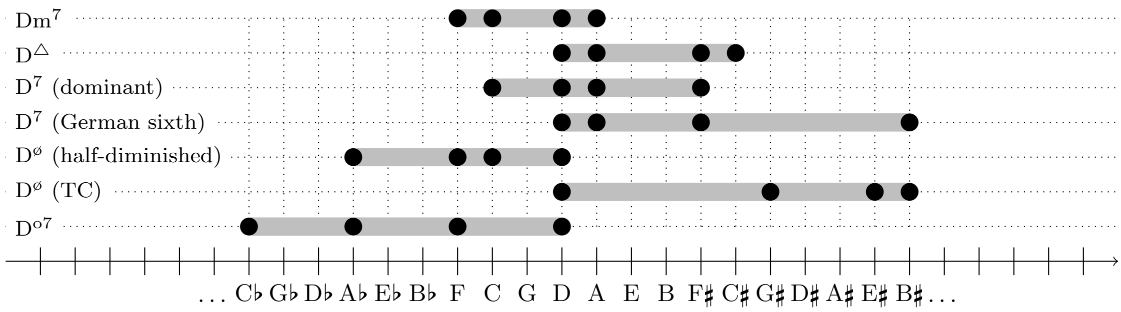


Figure 3. Tonal realization LOF-patterns.

$(3,4,3,2)$, $(4,3,4,1)$, $(4,3,3,2)$, $(2,4,3,3)$, $(3,3,4,2)$, $(3,4,2,3)$, $(3,3,3,3)$, $(4,2,4,2)$, and $(2,4,2,4)$. The first seven partitions correspond to seven realization types in Figure 2 (in the given order), while the last two represent the intervallic structure of the augmented sixth chord commonly called the French sixth (Piston 1987, p. 279), which is not contained in $\mathcal{C}$.

Given a chord $c \in \mathcal{C}$, one can efficiently find all tonal realizations of c in a finite domain $S \subset \mathbb{Z}$ by sliding the respective pattern(s) in Figure 3 along S and taking the sets $R \subset S$ for which $\pi(R) = P_c$. The set of all tonal realizations of c is denoted by $\mathcal{R}(c)$ in the following.

The *structural inverse* of $R = \{x, y, z, w\} \in \mathcal{R}(c)$, defined by $-R := \{-x, -y, -z, -w\}$, is a tonal realization of some $d \in \mathcal{C}$. This follows from the fact that the set of tonal-realization patterns in Figure 2 is closed with respect to reflection symmetry in LOF[4].

Since $R \in \mathcal{R}(c)$ can be arranged into a column of stacked thirds, there exists unique $x, y \in R$ such that $\langle x, y\rangle$ is a generic second. The tones x and y are called *generic seventh* and *generic root* of R, respectively. These correspond to the top and bottom tone in the arrangement shown in Figure 2(b).

If R does not spread too much in LOF, then the requirement of maximal evenness in Definition 4.1 can be omitted; this is a consequence of Theorem 4.3 stated below. Its proof relies on the following lemma.

Lemma 4.2 *Let $x, y \in \mathbb{Z}$ be two tones such that $|x-y| \le N \in \mathbb{N}_0$ and let $g \in \mathbb{Z}_7$, $h \in \mathbb{Z}_{12}$ be the generic and specific size of the interval $\langle x, y\rangle$, respectively. Then the statements*

(a) $h \in \{\overline{5}, \overline{6}, \overline{7}\} \implies g \neq \underline{0}$,
(b) $h \in \{\overline{7}, \overline{8}, \overline{9}\} \implies g \neq \underline{1}$,
(c) $h \in \{\overline{8}, \overline{9}, \overline{10}, \overline{11}\} \implies g \neq \underline{2}$,

are guaranteed to hold for $N \le 30$, but not for $N > 30$.

[4]Structural inverse of a minor, major, or diminished seventh chord is again a chord of the same type. The inverse of a dominant seventh chord (German sixth) is a half-diminished seventh chord (TC) and vice versa.

Proof. From (1) and (2) it follows that $g \equiv 4k \pmod 7$ and $h \equiv 7k \pmod{12}$, where $k = y - x$. Now we find k such that $|k|$ is minimal, $h \in \{\overline{5}, \overline{6}, \overline{7}\}$, and $g = \underline{0}$ (naively, by computing of g and h for $k = 0, 1, -1, 2, -2$ and so on). The solution indicates where (a) first fails. One obtains $k \in \{-35, 35\}$, which implies that (a) holds for $N \leq 34$. Analogously, one finds that (b) and (c) hold for $N \leq 32$ and $N \leq 30$ since they fail as soon as $k = -33$ and $k = -31$, respectively. Therefore the statements always hold for $N \leq 30$, but given $N > 30$ there exist x and y for which one of them fails. ∎

THEOREM 4.3 *Let $c \in \mathcal{C}$ and R be a realization of c such that* $\operatorname{diam}(R) \leq 30$ *and, given $x, y \in R$ and letting $(g, h) = \langle x, y \rangle$, the following holds:*

$$g = \underline{4} \implies h \in \{\overline{6}, \overline{7}\}, \tag{3}$$

i.e. that each generic fifth formed by a pair of tones in R is either perfect or diminished. Then R is tonal.

Proof. Since $\operatorname{diam}(R) \leq 30$, the statements in Lemma 4.2 hold for any pair of tones in R. To prove that R is tonal it suffices to show that $\tau(R)$ it is maximally even.

Let $c = (p, \mathbf{s}) \in \mathcal{C}$ where $\mathbf{s} = (s_1, s_2, s_3)$. Because $|R| = 4$ and $\pi(R) = P_c$, one may write $R = \{x_1, x_2, x_3, x_4\}$ such that $\pi(x_1) = p$ and $\langle x_{k+1}, x_k \rangle = (g_k, s_k)$ for $k = 1, 2, 3$ (thus, the tones in R are initially ordered with respect to to $\mathbf{s}$). From $s_k \in \{\overline{3}, \overline{4}\}$ and (3) it follows that $g_k \neq \underline{4}$. Since (3) is equivalent to $g = \underline{3} \implies h \in \{\overline{5}, \overline{6}\}$ (i.e. each generic fourth is either perfect or augmented), it also follows that $g_k \neq \underline{3}$. Additionally, since $-s_k \in \{\overline{8}, \overline{9}\}$, (b) and (c) imply $-g_k \notin \{\underline{1}, \underline{2}\}$ from which it follows that $g_k \notin \{\underline{5}, \underline{6}\}$. Hence $g_k \in \{\underline{0}, \underline{1}, \underline{2}\}$ for $k = 1, 2, 3$. Letting $(g_4, s_4) = \langle x_4, x_1 \rangle$, from $s_1 + s_2 + s_3 + s_4 = \overline{0}$ it follows that $s_4 \in \{\overline{1}, \overline{2}, \overline{3}\}$ and therefore $g_4 \in \{\underline{0}, \underline{1}, \underline{2}, \underline{6}\}$ follows by reasoning similar to above.

Let $g = -g_4 = g_1 + g_2 + g_3$ and $h = -s_4 = s_1 + s_2 + s_3$ denote the components of $\langle x_1, x_4 \rangle$. Since $s_1 + s_2, s_2 + s_3 \in \{\overline{6}, \overline{7}\}$, from (a) it follows that $g_1 + g_2, g_2 + g_3 \neq \underline{0}$ (these are components of intervals $\langle x_1, x_3 \rangle$ and $\langle x_2, x_4 \rangle$, respectively). Now assume that $g = \underline{1}$. It follows that $g_1 + g_2 = g_2 + g_3 = \underline{1}$. According to the contrapositive of (b), the specific sizes $s_1 + s_2$ and $s_2 + s_3$ must be both different from $\overline{7}$ and therefore equal to $\overline{6}$. Hence $\mathbf{s} = [333]$ and $h = \overline{9}$. But now (b) contradicts the assumption and hence $g \neq \underline{1}$, i.e. $g_4 \neq \underline{6}$. Therefore $g_k \in \{\underline{0}, \underline{1}, \underline{2}\}$ for $k = 1, 2, 3, 4$.

Since $g_1 + g_2 + g_3 + g_4 = \underline{0}$, there now exists a unique $m \in \{1, 2, 3, 4\}$ such that $g_m = \underline{1}$ and $g_k = \underline{2}$ for $k \neq m$, and therefore $\tau(R)$ is maximally even in $\mathbb{Z}_7$. Letting $\sigma = (2341) \in S_4$, the cyclic permutation $\rho = \sigma^m$ maps n to 4 and therefore represents the order of tones in which any two adjacent tones form a generic third. ∎

COROLLARY 4.4 *A realization $R \subset \mathcal{T}$ of $c \in \mathcal{C}$ is tonal if (3) holds for all $x, y \in R$.*

The condition $\operatorname{diam}(R) \leq 30$ is important because Theorem 4.3 fails if R spreads too much in LOF. For example, $\{-16, -14, -5, 16\} = \{\text{C}♭\!♭, \text{D}♭\!♭, \text{E}♭, \text{E}𝄪\}$ is a realization of $\text{C}^{ø}$ with diameter 32. The condition (3) is satisfied since there are no generic fifths, but the realization is not tonal because it does not satisfy the first condition in Definition 4.1.

We implemented a simple algorithm which finds all distinct (tonal) realizations of chords in $\mathcal{C}$ in a finite domain $S \subset \mathbb{Z}$. It returns the set of all $R \in \binom{S}{4}$ such that $\pi(R) = P_c$ for some $c \in \mathcal{C}$ and labels the realizations which satisfy the condition (3) as tonal. For $S = \mathcal{T}$, 2281 realizations with 167 of them being tonal, which is about 7.32%, are obtained. For $S = \mathcal{T} \cup \{16\} = \{-15, \ldots, 16\}$ one obtains exactly 174 realizations which satisfy (3), each of them fitting a pattern shown in Figure 3 which indicates that Theorem 4.3 holds even if $\mathrm{diam}(R) = 31$. The algorithm is implemented as `Realization::lof_patterns` in a C++ library SEPTIMA which is available under GPL license[5]. This library contains implementations of routines described further in this text.

5. Elementary transitions between seventh chords

The objective of this section is to provide a model of scalar-parsimonious transformations[6] between seventh chords in which voices move chromatically or stepwise diatonically (such voice leading is called *conjunct* in the following). The concept is similar to that of idealized voice leadings (Cohn 2012; Yust 2015) in which voices move by shortest routes. We begin by some preliminaries regarding the voice leading between seventh chords.

Definition 5.1 Let $c, d \in \mathcal{C}$, $X \in \mathcal{R}(c)$, and $Y \in \mathcal{R}(d)$. Then a bijection $v : X \to Y$ is called *voice leading* from X to Y. The pair $(x, v(x))$ constitutes the *voice* starting with $x \in X$. The *transition* T from X to Y with respect to v is denoted by $T = X \xrightarrow{v} Y$.

The set $\{X \xrightarrow{v} Y : X \in \mathcal{R}(c), Y \in \mathcal{R}(d), v \in \mathrm{Bij}(X, Y)\}$ of all transitions from c to d is denoted by $\mathrm{Tr}(c, d)$. Transitions $T^{-1} = Y \xrightarrow{v^{-1}} X$ and $-T = -X \xrightarrow{w} -Y$, where $w(-x) = -v(x)$ for $x \in X$, are *retrograde* and *structural* inversion of T, respectively.

Given a voice leading $v : X \to Y$, the ℓ^∞ norm of v is defined as follows:

$$\|v\|_\infty := \max\{|v(x) - x| : x \in X\}.$$

Clearly, the conditions $\|v\|_\infty \geq 0$, $\|v\|_\infty = 0 \iff X = Y \wedge v = \mathrm{id}$, and $\|v\|_\infty = \|v^{-1}\|_\infty$ hold. Furthermore, given a voice leading $w : Y \to Z$ and letting $y = v(x)$, from

$$|(w \circ v)(x) - x| = |v(x) - x + (w \circ v)(x) - v(x)| \leq |v(x) - x| + |w(y) - y|,$$

one obtains $\|w \circ v\|_\infty \leq \|v\|_\infty + \|w\|_\infty$, i.e. the triangle inequality. Therefore, $\|\cdot\|_\infty$ is indeed a norm. Similarly, ℓ^p norm of v, where $p > 0$, is defined by

$$\|v\|_p := \left(\sum_{x \in X} |x - v(x)|^p \right)^{1/p}.$$

5.1. *The model*

Let $c, d \in \mathcal{C}$, $X \in \mathcal{R}(c)$, and $Y \in \mathcal{R}(d)$. The total scalar displacement of a voice leading $v : X \to Y$ is defined as follows:

$$\mu(v) := \sum_{x \in X} \Delta(\tau(x), (\tau \circ v)(x)) = \sum_{x \in X} \|4(v(x) - x) \bmod 7\|.$$

From Theorem 3.8 by Harasim, Schmidt, and Rohrmeier (2016) it follows that v minimizes μ among all bijections from X to Y if tones with the same scale degree appear in the same voice and movement in the remaining voices is minimal. Since tonal realizations are built of stacked thirds, it is now easy to verify that there exist exactly seven generic

[5] https://github.com/marohnicluka/septima

[6] Tones with the same scale degree are retained while the other move stepwise, c.f. (Plotkin 2019).

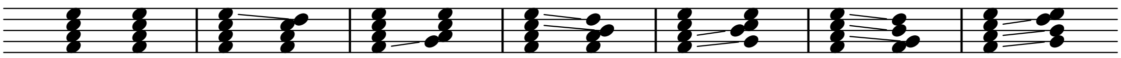


Figure 4. Generic types of scalar-parsimonious transitions between tonal realizations of seventh chords.

types of voice leadings that minimize μ, which are shown in Figure 4. The following theorem provides a simple characterization.

THEOREM 5.2 *Let x_0 and y_0 denote the generic roots of X and Y. A voice leading $v : X \to Y$ is minimal with respect to μ if and only if*

$$\mu(v) = \|2(x_0 - y_0) \bmod 7\|. \tag{4}$$

Proof. Generic types shown in Figure 4 correspond to seven possible generic sizes of $\langle x_0, y_0\rangle$: unison, sixth, third, fourth, fifth, second, and seventh, respectively. For each of these there exists $m = x_0 - y_0 \bmod 7$. One obtains $m = 0, 4, 3, 1, 6, 2, 5$. Now by computing $\|2(x_0 - y_0) \bmod 7\| = \|2m \bmod 7\|$ for each m one obtains $0, 1, 1, 2, 2, 3, 3$, i.e. the values of μ for the respective types. ■

Definition 5.3 A transition $X \xrightarrow{v} Y$ is *μ-efficient* (i.e. scalar-parsimonious) if v satisfies the condition (4).

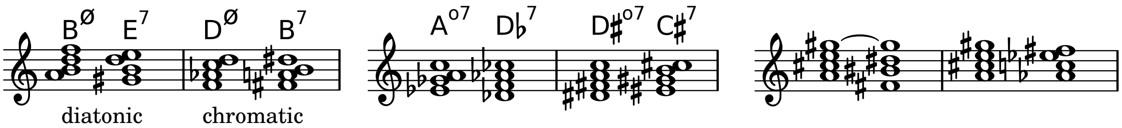
A transition $T = X \xrightarrow{v} Y \in \mathrm{Tr}(c, d)$ uniquely defines a bijection $\overline{v} : P_c \to P_d$ between pitch-class representations of c and d which satisfies the relation $(\overline{v} \circ \pi)(x) = (\pi \circ v)(x)$ for $x \in X$. The function $\overline{v}$ is a voice leading between pcsets P_c and P_d which is obtained from v by discarding the scale-related data. Letting $\mathfrak{D}$ denote the voice-leading efficiency metric on $\binom{\mathbb{Z}_{12}}{4}$ as defined by Harasim, Schmidt, and Rohrmeier (2016), T is *efficient* if its voice leading work (Cohn 2012) (i.e. the total pitch displacement), defined by

$$\overline{\mu}(v) := \sum_{p \in P_c} \Delta(p, \overline{v}(p)) = \sum_{x \in X} \Delta(\pi(x), (\pi \circ v)(x)) = \sum_{x \in X} \|7(v(x) - x) \bmod 12\|,$$

is equal to $\mathfrak{D}(P_c, P_d)$. In that case, $\overline{v}$ is minimal among all bijections from P_c to P_d.

Since stepwise voice leading in diatonic space does not imply any particular level of efficiency in chromatic space (Kochavi 2008, p. 180), specific sizes of intervals $\langle v, v(x)\rangle$ in a transition $X \xrightarrow{v} Y$ have to be restricted separately. This is achieved by setting an upper bound for ℓ^∞ norm of v, thereby limiting the voice-leading displacement in LOF. Indeed, assuming that $\|v\|_\infty \leq M \in \mathbb{N}_0$, the total number of allowed types of intervals $\langle x, v(x)\rangle$ monotonically increases with M. For $M = 5$ the movement of voices is restricted to diatonic steps (minor and major seconds), while for $M = 7$ both diatonic and chromatic movement are allowed. These two cases are the most important ones; setting M to larger values introduces interval types that are either seldom used or redundant. For instance, the augmented second, which is sometimes found in stepwise voice leading (Holmes 2017), is enabled by setting $M = 9$. The case $M = 12$ includes the diminished second, which acts as an enharmonic change[7]. Lastly, the case $M = 14$ accepts the doubly-augmented unison, an interval which occurs very rarely (Byrd 2018b).

[7]For example, in Chopin's Mazurka op. 67 No. 2 the tone F♭ at the end of m. 25 changes into E♮ at the beginning of m. 26.



(a) Examples of elementary transitions between seventh chords.

(b) Two elementary transitions from $\mathrm{C}^{\mathrm{o}7}$ to $\mathrm{C}\sharp^7$.

(c) Which transition from $\mathrm{A}^{\triangle}$ to $\mathrm{G}\sharp^7$ is more parsimonious?

Figure 5.

In this paper we focus on the case $M = 7$ which features conjunct voice leading. Hence we propose the following definition of basic transitions between seventh chords.

Definition 5.4 A transition $T = X \xrightarrow{v} Y$ is *elementary* if it is μ-efficient and $\|v\|_\infty \leq 7$. The *degree* of T is the minimal $M \in \mathbb{N}_0$ such that $\|v\|_\infty \leq M$.

Elementary transitions of degree 7 are called *chromatic* and the rest are called *diatonic*. It is easy to prove that if T is elementary then so are T^{-1} and $-T$. Transitions can be notated in musical staff by choosing a register for each voice. Figure 5(a) shows two examples of elementary transitions. The set of all elementary transitions in $\mathrm{Tr}(c, d)$, where $c, d \in \mathcal{C}$ and $c \neq d$, is denoted by $\mathcal{E}(c, d)$.

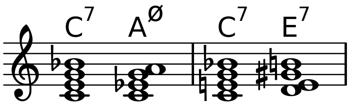
Elementary transitions are not necessarily efficient. For instance, consider two transitions in $\mathcal{E}(\mathrm{C}^{\mathrm{o}7}, \mathrm{C}\sharp^7)$ which are shown in Figure 5(b). Voice leading work in the corresponding voice leadings amounts to 5 and 7 semitones, respectively, which means that the second transition is not efficient—but nevertheless exhibits a high degree of parsimony.

Note that the concept of elementary transitions does not imply common-pitch retention in chromatic case, unlike in traditional definitions of parsimonious relations between pcsets where it is usually explicitly required, e.g. in Cohn (1997); Douthett and Steinbach (1998); Cook (2005). Although such requirement does not prevent one from finding an efficient voice leading, as shown by Harasim, Schmidt, and Rohrmeier (2016), there exist cases in which it is not desirable. For example, Figure 5(c) shows two transitions between $\mathrm{A}^{\triangle}$ and $\mathrm{G}\sharp^7$ which use efficient voice leadings (the total displacement of voices equals 5 semitones in both cases). In the first transition, the common pitch-class $\overline{8} = \pi(\mathrm{G}\sharp)$ is fixed, which forces one of the other voices to move by more than two semitones. Thus, although efficient, the resulting voice leading is not conjunct. It is also contrapuntally awkward; the seventh of $\mathrm{A}^{\triangle}$ is not resolved in any acceptable way. On the other hand, the second transition (in which each voice makes a movement) is elementary and intuitively seems more parsimonious than the first. However, the measure of tonal parsimony proposed by Kochavi (2008, Def. 8.1.8), which hierarchizes both chromatic and diatonic measures, assigns a higher degree of parsimony to the first transition because common-pitch retention is strongly favored. This study suggests that, for seventh chords, it is better to require common-note-name retention (i.e. μ-efficiency) instead.

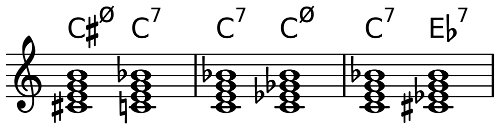
5.2. *Contrapuntal aspects*

In this section we discuss certain contrapuntal considerations which should be taken into account when studying tonal transitions between seventh chords.

First, certain dissonant vertical intervals have a tendency towards resolution, which affects the subsequent relative motion of the respective voices. The only such interval to be addressed in an elementary transition $T = X \xrightarrow{v} Y$ is the generic seventh of X, which tends to shrink into a sixth unless it is propagated to the next chord (see Figure 6

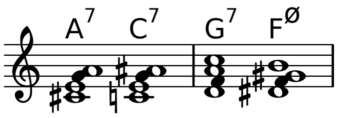


(a) Resolving the generic seventh by sliding into a sixth.

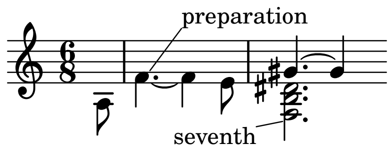


(b) Examples of propagation of generic sevenths.

Figure 6.



(a)



(b) the beginning of Prelude to *Tristan*

Figure 7. Examples of preparing generic sevenths in augmented sixth chords.

for examples). Since T is μ-efficient, that is indeed always the case (see Figure 4), so elementary transitions behave consistently in that respect.

Second, preparation of dissonant chords—in this case, seventh chords—is another important issue in classical Western music. Regarding the traditional practice of counterpoint, a vertical interval $\langle x, y\rangle$ of generic seventh occurring at a stressed beat is usually preceded by a consonant interval $\langle z, y\rangle$, thus preparing its upper tone y (Salzer and Schachter 1989, p. 78). Letting y denote the generic root and seventh of Y, respectively, it is therefore and often desirable that y is prepared in X within the same voice. Since T is μ-efficient, it is enough to require that $y \in X$. If the generic seventh of Y is either minor or major, this requirement—called *preparation rule* (PR) in the following—clearly incorporates the traditional practice in the present model of voice leading. The cases in which Y is an augmented sixth chord are illustrated in Figure 7(a). For example, the note F in the first measure of Wagner's prelude to *Tristan* can be seen as the preparation for TC in the second measure, as shown in Figure 7(b).

It should be noted that preparation of seventh chords is far from being applied consistently in practice. For instance, the seventh may come about as a passing tone or be prepared indirectly in some other voice (Aldwell, Schachter, and Cadwallader 2010, p. 455). Furthermore, dominant seventh chords have been regularly used without preparation ever since Monteverdi (Christensen 2019, p. 69)[8]. Although composers like Bach, Mozart, Beethoven, and Chopin usually prepare non-dominant sevenths when coming from other seventh chords, Reenan and Bass (2016) present a few excerpts from the late 19$^{\text{th}}$-century music in which such chords are introduced without preparation. Early 20$^{\text{th}}$-century composers are usually not concerned with preparation of sevenths at all (see e.g. Trois Gymnopédies by Erik Satie).

5.3. *Generating elementary transitions*

Elementary transitions between two distinct seventh chords $c, d \in \mathcal{C}$, which use only the tones from a given finite LOF-domain $S \subset \mathbb{Z}$, can be listed using a simple algorithm

[8]This is possibly related to the fact that the generic root in a dominant seventh chord forms a tritone interval with one of the other tones, which seems to affect the quality of the seventh.

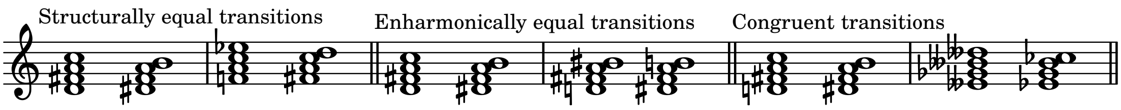


Figure 8. Three types of transition equivalence.

that, for all $X \in \mathcal{R}(c), Y \in \mathcal{R}(d)$ such that $X, Y \subset S$, outputs each $v \in \mathrm{Bij}(X, Y)$ which satisfies the conditions in Definition 5.4. Optionally, one can output only those transitions in which the generic seventh of Y is prepared (see Section 5.2). This algorithm is implemented in SEPTIMA as `Transition::elementary_transitions`.

To obtain a better overview of different voice-leading types, it is desirable to find only classes of elementary transitions which are identical up to transposition and/or pitch spelling. The following relations are useful in that respect.

Definition 5.5 Transitions $T_1 = X_1 \xrightarrow{v} Y_1$ and $T_2 = X_2 \xrightarrow{w} Y_2$ are

(1) *structurally equal*, which is denoted by $T_1 \sim T_2$, if there exists $k \in \mathbb{Z}$ such that $X_2 = \{x + k : x \in X_1\}$ and $w(x + k) = v(x) + k$ for all $x \in X_1$ (i.e. T_2 is obtained from T_1 by sliding the latter in LOF),
(2) *enharmonically equal*, which is denoted by $T_1 \equiv T_2$, if $\overline{v} = \overline{w}$,
(3) *congruent*, which is denoted by $T_1 \cong T_2$, if they are structurally and enharmonically equal.

Examples in Figure 8 are provided for clarification. All three relations defined above are obviously reflexive, symmetric and transitive, thus splitting a set of transitions into equivalence classes. In particular, the quotient set $\mathcal{E}(c, d)/\cong$ is finite because $\|v\|_\infty$ is bounded and the set of tonal-realization patterns (see Figure 3) is finite. The corresponding equivalence classes can be represented by transitions near the given point $z \in \mathbb{Z}$ in LOF. We propose the following measure of closeness in that respect.

Definition 5.6 Let $z \in \mathbb{Z}$ and, given $T = X \xrightarrow{v} Y$,

$$f(T) := \frac{1}{|X \cup Y|} \sum_{x \in X \cup Y} |x - z|, \quad g(T) := \sum_{x \in X} |x + v(x) - 2z|.$$

Then T_1 is *closer to* z than T_2 if either

- $f(T_1) < f(T_2)$ or
- $f(T_1) = f(T_2)$ and $g(T_1) < g(T_2)$.

In Definition 5.6, two measures f and g are hierarchized. The first corresponds to the mean absolute deviation of the elements of $X \cup Y$ from z, while the second depends only on the voice-leading function v and is proportional to the average distance of $\frac{x+v(x)}{2}$ from z on the real line. Normally one would choose $z = 0$, in which case the obtained transitions are easily notated since they contain the minimal number of accidentals.

THEOREM 5.7 *Let $c, d \in \mathcal{C}$, $e \in \mathcal{E}(c, d)/\cong$, and $S \subset \mathbb{Z}$ be a LOF segment such that* $\mathrm{diam}(S) \geq 28$. *Then there exists $X \xrightarrow{v} Y \in e$ such that $X, Y \in S$.*

Proof. Let $T = X \xrightarrow{v} Y \in \mathcal{E}(c, d)$. From Figure 2 it is clear that $\mathrm{diam}(X), \mathrm{diam}(Y) \leq 10$. From $\|v\|_\infty \leq 7$ it follows that $\mathrm{diam}(X \cup Y) \leq 17$. Since each transition which is congruent

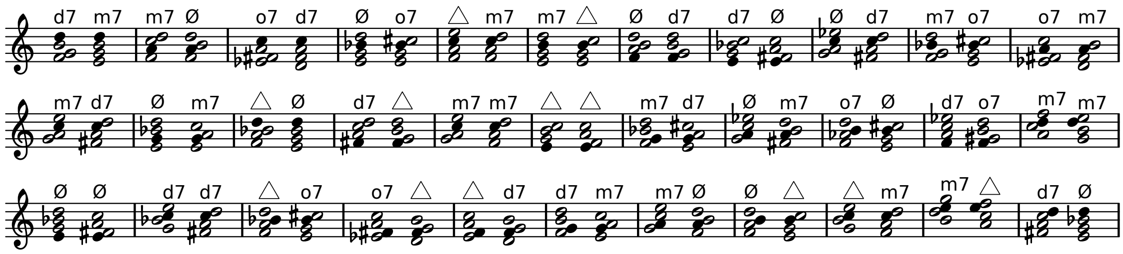


Figure 9. Diatonic elementary transitions.

to T is obtained by shifting X and Y by $12k$ steps in LOF for some $k \in \mathbb{Z}$ and adjusting the voice leading accordingly, for a given $m \in \mathbb{Z}$ there exists a unique $T' = X' \xrightarrow{w} Y'$ such that $T' \cong T$ and $\min\{X' \cup Y'\} \in S_m = \{m, m+1, \ldots, m+11\}$. Therefore, any segment $S \subset \mathbb{Z}$ such that $\operatorname{diam}(S) \geq \operatorname{diam}(S_m) + 17 = 28$ contains a transition which is congruent to T. ∎

Corollary 5.8 *Each elementary transition is realizable in $\mathcal{T}$.*

The algorithm `Transition::elementary_classes`, implemented in septima, uses Theorem 5.7 for computing transitions representing the classes $\left(\bigcup_{c,d\in\mathcal{C}} \mathcal{E}(c,d)\right) / \cong$. Given $z \in \mathbb{Z}$, representatives are sought in $\{z-14, z-13, \ldots, z+14\}$ and are required to be as close to z as possible. This algorithm is used by `Transition::elementary_types` which chooses representatives of the classes in $\left(\bigcup_{c,d\in\mathcal{C}} \mathcal{E}(c,d)\right) / \sim$ in a similar way. The result, which is a maximal set of elementary transitions which are structurally different from each other, is further simplified by performing the following additional steps in the given order: (1) augmented sixth chords are respelled whenever possible, (2) any transition for which there exists an enharmonic equivalent with a smaller voice-leading ℓ^1 norm is discarded[9], (3) (only if PR is ignored) transitions which are retrograde inverses of each other are considered as duplicates.

It is convenient to sort the output of the above algorithms in ascending order with respect to the parsimony metric proposed by Kochavi (2008, Def. 8.1.8). If that metric yields equal parsimony for a pair of transitions, then the one with smaller LOF-spread is considered more parsimonious than the other. The spread of $T = X \xrightarrow{v} Y$ is defined as the standard deviation of elements in $X \cup Y$.

Transitions obtained by `Transition::elementary_types` for $z = 0$ are shown in Figure 9 and 10. Each of these illustrate a particular type of voice leading between two seventh chords. In total, 33 diatonic and 83 chromatic transition types are obtained[10]. Each diatonic transition is either consistent with PR or its retrograde inversion is (but not both). The voice in which preparation occurs is written using black note-heads. German sixths and Tristan chords, which are present only in some chromatic transitions (when it is not possible to respell them), are indicated by Ger and TC, respectively.

By computing $P_{m,n}$-relations between c and d, the following can be shown to hold.

(1) If $c, d \in \mathcal{C}$ are $P_{m,n}$-related, then $\mathcal{E}(c,d)$ contains an efficient transition.
(2) If c and d are not $P_{m,n}$-related, then $\mathcal{E}(c,d) = \emptyset$ unless one chord is a major

[9] Diatonic motion of voices is hence favored over chromatic one.
[10] If the first two steps in the above output-simplification process are skipped, 219 structural classes are obtained in total (45 diatonic and 174 chromatic).

Figure 10. Chromatic elementary transitions.

seventh and the other is either the dominant seventh lying one semitone below it[11] or the half-diminished seventh lying two semitones above it. These two exceptions, which are structural inverses of each other, are enclosed by bracket in Figure 10.

The above results show that our model involves a refined concept of parsimony, generalizing the theories of $P_{1,0}$ (Cannas and Andreatta 2018), $P_{2,0}$ (Childs 1998), and $P_{3,0}$ (Reenan and Bass 2016) transformations. Furthermore, it extends beyond the Relation definition, providing two additional transformations described above, which seem to be unaccounted for in harmony textbooks. The model includes six types of inefficient transitions and five types of transitions with voice leadings not acting identically on intersections of the respective pcsets. The former group is marked by bracket and the latter by asterisks in Figure 10. However, generality of the statement (1) is not affected by removing these transitions from consideration; the last seven transition types shown in Figure 10 have efficient, common-tone-fixing alternatives.

5.4. *An example: finding resolutions of Tristan chord*

To demonstrate a practical application of elementary transitions, we consider the problem of finding all possible successions in which TC moves to a dominant/half-diminished seventh chord. For example, Tymoczko (2011, p. 302) collects various resolutions of TC ap-

[11]This succession is found in Chopin's Prelude No. 4 in E minor, Op. 28 (see the excerpt in Figure 14 on page 16), but is typically overlooked by analysts. Indeed, $C^{\triangle}$ occurring at the end of m. 1 slides into B^7 at the beginning of m. 2. The tone E marked with an asterisk is a suspension resolving to E♭, which is equivalent to D♯. The succession $C^{\triangle} \to B^7$ is also found in popular music, e.g. in *The Black Swan* (1968) by Jake Thackray.

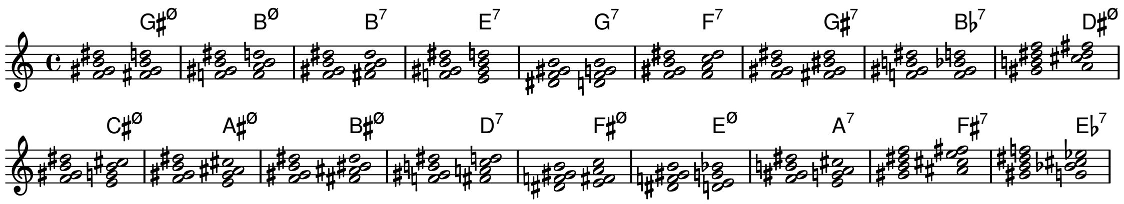


Figure 11. Transitions from Tristan chord to a dominant/half-diminished seventh chord.

pearing in Wagner's *Tristan*, while Martin (2008, p. 15) provides a partial solution to the problem by applying the DOUTH2 relation, obtaining nine distinct successions. In this study the following approach is taken. All elementary transitions from $\mathrm{F}^{\o}$, spelled as TC, to a dominant/half-diminished seventh chord are found using SEPTIMA; augmented-sixth realizations of destination chords are respelled whenever possible. In total, 18 transitions are obtained, as shown in Figure 11. There are 8 transitions to half-diminished seventh chords; their roots form the set $\{\mathrm{C}, \mathrm{C}\sharp, \mathrm{D}\sharp, \mathrm{E}, \mathrm{F}\sharp, \mathrm{G}\sharp, \mathrm{A}\sharp, \mathrm{B}\}$. That leaves 10 transitions to dominant seventh chords (only C^7 and $\mathrm{C}\sharp^7$ are not reached).

The transition $T = \{\mathrm{F}, \mathrm{B}, \mathrm{D}\sharp, \mathrm{G}\sharp\} \xrightarrow{v} \{\mathrm{F}, \mathrm{C}, \mathrm{D}, \mathrm{A}\flat\} \in \mathrm{Tr}(\mathrm{F}^{\o}, \mathrm{D}^{\o})$ is the only one among those provided by Martin (2008) which does not appear in Figure 11; namely, the set $\mathcal{E}(\mathrm{F}^{\o}, \mathrm{D}^{\o})$ does not contain a transition in which $\mathrm{F}^{\o}$ is spelled as TC. Indeed, T is not a chromatic resolution of TC since $v(\mathrm{G}\sharp) = \mathrm{A}\flat$, implying that $\|v\|_\infty \geq 12$.

6. Chord graphs

Definition 6.1 Let $V \subset \mathcal{C}$ and $S \subset \mathbb{Z}$. A simple directed graph $G(V, E)$, where $(c, d) \in V \times V$ belongs to the edge set E if and only if there exists a transition $X \xrightarrow{v} Y \in \mathcal{E}(c, d)$ such that $X, Y \subset S$, is called the *chord graph on V with support S*. Optionally, it may be required that Y is prepared. Augmented sixths may be allowed or not, which affects the number of elementary transitions from/to dominant and half-diminished seventh chords.

Edges in G represent relations between seventh chords in V. Chords c and d are related if an elementary transition from c to d exists in the given LOF-domain S. If PR is ignored, then these relations are symmetric and hence edge directions in G may be discarded, reducing G to an undirected graph. In SEPTIMA, chord graphs can be created using the `ChordGraph` class.

Figure 12 shows a chord graph on the set of 12 dominant seventh chords with support $\mathcal{T}$ with vertex labels corresponding to pitch-classes of the respective chords' roots. It is a regular, strongly connected, non-Hamiltonian graph with 24 edges and contains an Eulerian circuit (each vertex has even in- and out-degree). PR is applied and transitions with augmented-sixth realizations are not considered (if PR was ignored and augmented sixths taken into account, a complete graph would have been obtained, which means that there is an elementary transition between any pair of dominant seventh chords[12]). Similar graphs on sets of 12 half-diminished and 12 minor seventh chords both have 36 edges (they are not isomorphic), which means that these chord types are better interconnected than dominant sevenths with respect to PR. The graph on major seventh chords has only 12 edges which form a directed cycle, while for three diminished seventh chords one obtains an empty graph (with no edges) since they have no tones in common.

[12]Chromatic transformations between dominant seventh chords are listed by Schönberg (1983, Ex. 300, p. 363).

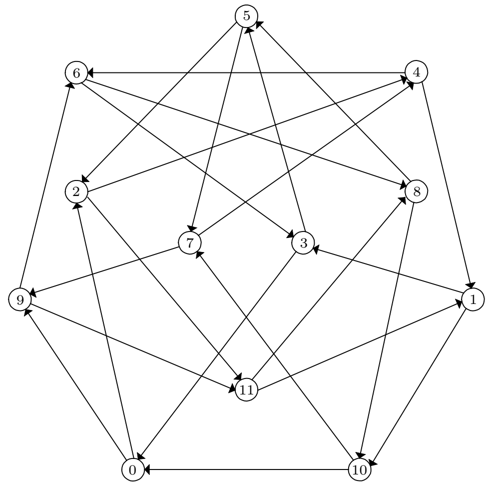


Figure 12. Diatonic/chromatic relations (with preparation) between dominant seventh chords.

6.1. *Progressions of seventh chords*

A walk $(c_1, \ldots, c_n)$ in a chord graph corresponds to a progression which is denoted by $c_1 \to c_2 \to \cdots \to c_n$. Edges of the walk represent elementary transitions between successive chords. Using graph theory, one can easily count the number of progressions of the given length from one chord to another. Generally, the total number of walks with $n-1$ edges starting with the i-th and ending with the j-th vertex in a (di)graph is equal to the entry in the i-th row and the j-th column of the matrix A^{n-1}, where A is the adjacency matrix of the (di)graph (Biggs 1993, p. 9). For example, in the graph shown in Figure 12 there exist exactly 120 mutually different progressions of 6 chords starting and ending with the same chord; this number is obtained by computing the trace of A^5.

Fairly long progressions of seventh chords can be found in musical works. For example, two excerpts from Chopin's opus are shown in Figures 13 and 14. Additional examples from his Mazurkas include Op. 6 No. 1 (mm. 5–9), Op. 7 No. 2 (mm. 19–24), Op. 30 No. 4 (mm. 129–132), Op. 67 No. 2 (mm. 21–26), and Op. 68 No. 4 (mm. 1–6). These are mostly sequential progressions exhibiting semitonal voice leading and moving steadily downwards. Analysis of the progressions in Op. 6 No. 1, Op. 7 No. 2, and Op. 28, no. 4 can be found in Tymoczko (2011, pp. 287, 290). We provide a more detailed analysis of the latter prelude (see Figure 14) which includes $\mathrm{C}^{\triangle}$ chord in m. 1[13].

Further examples by other composers include Mozart's Concerto for piano and orchestra in C minor KV 491, 1st movement, mm. 234–238 (a descending sequence of diminished seventh chords) and Rachmaninov's Vocalise Op. 34, mm. 1–3 (the progression starts with Dm^7 at the third beat of m. 1). Additional examples by Mozart, Beethoven, and V. Herbert are found in Tymoczko (2011, p. 292), and also by R. Strauss, Liszt, and Rachmaninov in Reenan and Bass (2016). Progressions of seventh chords are common in jazz music; for example, Jedrzejewski (2019, p. 167) refers to a sequence from *Giant Steps* by John Coltrane.

Each of the above mentioned progressions corresponds to a walk in chord graph on $\mathcal{C}$ with support $\mathcal{T}$, which is denoted by $\mathcal{G}$ in the following. That graph, in its undirected form (i.e. when PR is ignored), contains 1215 edges and represents the "universe" of diatonic and chromatic relations between seventh chords with edge between c and d

[13] c.f. Jacobus (2012, p. 87)

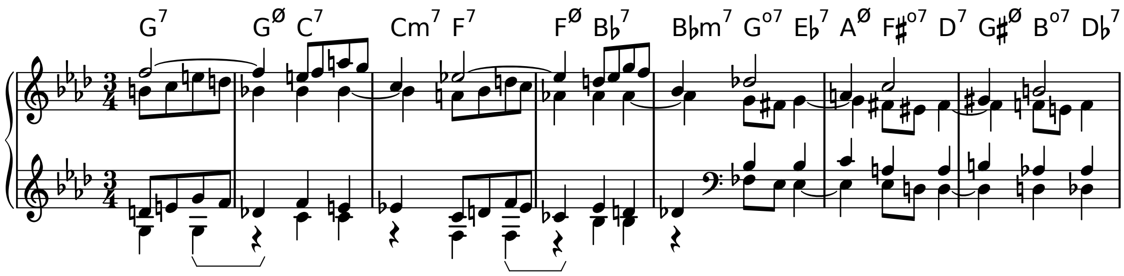


Figure 13. F. Chopin: Mazurka Op. 68 No. 4, mm. 33–39.

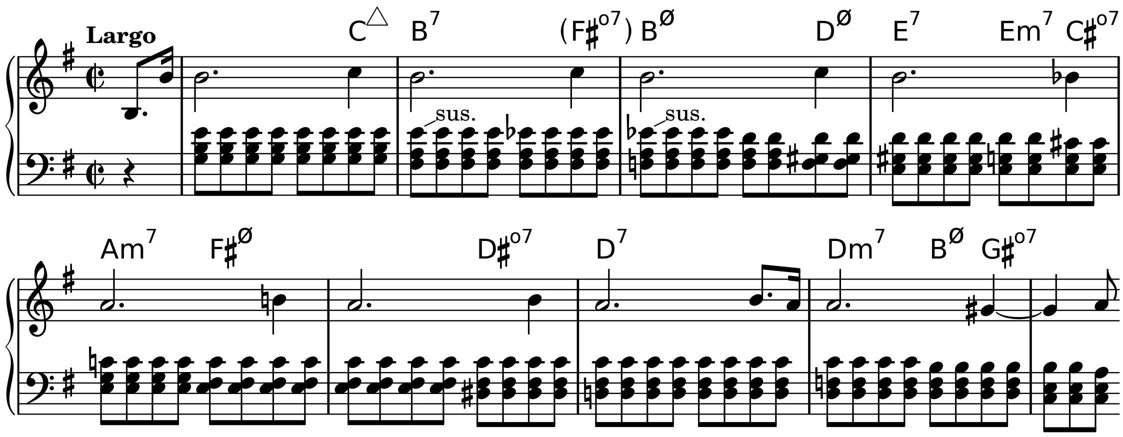


Figure 14. F. Chopin: Prelude No. 4 in E minor, Op. 28, mm. 1–8.

representing the set $\mathcal{E}(c, d)$ for all $c, d \in \mathcal{C}$, $c \neq d$. When PR is used, $\mathcal{G}$ is a digraph with 984 arcs. (In both cases augmented sixths are allowed). By removing augmented-sixth realizations from consideration, the number of edges/arcs in $\mathcal{G}$ drops to 1119 without preparation and to 768 with preparation.

6.2. *Complex networks*

Wixey and Sturman (2016) use various centrality measures in undirected voice-leading graphs to explain the importance and role of individual vertices (chords) in triad progressions. Their approach is applicable to chord graphs obtained by using Definition 6.1. In this section, we discuss yet another tool for quantifying the relative importance of particular chords in progressions, namely the communicability betweenness centrality (CBC) (Estrada, Higham, and Hatano 2009). The respective value $C_{\mathrm{CB}}(k)$ for the k-th vertex is interpreted as a weighted proportion of walks in G which pass through it; longer walks receive larger penalties. In the context of chord progressions, this is a more appropriate measure than the classic betweenness centrality (Freeman 1977), which takes only shortest walks into account.

The original definition of CBC is suitable only for strongly connected graphs. Since chord graphs do not necessarily have this property, the definition is generalized as follows. Let $\delta(i, j) \in \{0, 1\}$ be equal to 1 if there is a path from i to j and 0 otherwise. Now we

Table 1. Communicability betweenness centrality for types of chords in $\mathcal{G}$.

Chord type	without preparation	with preparation
dominant seventh	0.626896	0.369888
half-diminished seventh	0.626896	0.352248
minor seventh	0.612719	0.270448
major seventh	0.548063	0.188849
diminished seventh	0.640837	0.454371

define

$$C_{\mathrm{CB}}(k) := \frac{1}{N_k} \sum_{i \neq k} \sum_{\substack{j \notin \{i,k\} \\ \delta(i,j) \neq 0}} \left(1 - \frac{\mathrm{e}^{A_k}_{ij}}{\mathrm{e}^{A}_{ij}} \right), \qquad N_k = \sum_{i \neq k} \sum_{j \notin \{i,k\}} \delta(i,j). \tag{5}$$

Here e^A and e^{A_k} stand for matrix exponentials of the adjacency matrix A and the matrix A_k in which the k-th column replaced by zeros. An algorithm for computing (5) (using GSL[14] for computation of matrix exponential) is implemented in SEPTIMA as `ChordGraph::communicability_betweenness_centrality`[15].

Table 1 shows CBC-measures of seventh chords in chord graph $\mathcal{G}$, with and without using PR (both variants are strongly connected). Chords of the same type have the same CBC-measure, which is due to transpositional symmetry. When PR is ignored, the seventh chords are approximately equally represented in progressions (indeed, $\mathcal{G}$ is only 60 edges short from being complete), although major seventh chords appear little less frequently. However, the relative differences get substantially bigger if preparation is mandatory. Dominant and half-diminished seventh chords appear more frequently in progressions than minor and major seventh chords. Diminished seventh chords are even more common; although there are only three such vertices in $\mathcal{G}$, they are most frequently found in progressions. Major seventh chords are by far the least frequent ones; this may explain why they are comparatively rare in the Classical period, since during that time non-dominant seventh chords were usually introduced with preparation.

7. Automatic voicings from sequences of seventh chords

In this section we describe how a shortest-path algorithm can be used to automatically generate concrete four-part realization of a sequence of chord symbols using conjunct voice leading. This approach is similar to finding shortest paths in chord spaces of Callender, Quinn, and Tymoczko (2008) and also to the algorithm of del Pozo and Gómez (2019), which uses the concept of nabla distance. The notable difference is that our algorithm performs pitch-spelling inference.

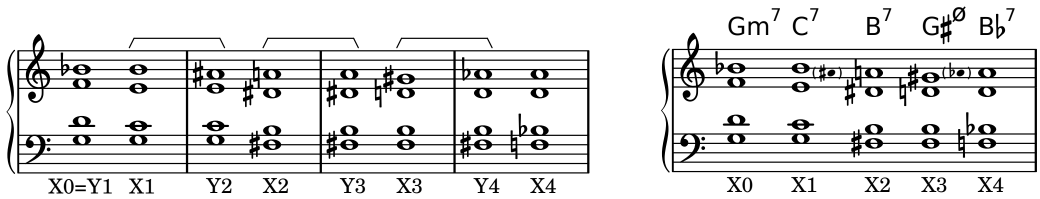
7.1. *Transition chains*

Definition 7.1 A *pitch spelling* for $c \in \mathcal{C}$ in a domain $S \subset \mathbb{Z}$ is a mapping $f : P_c \to S$ such that $f(P_c) \in \mathcal{R}(c)$.

Let $n \in \mathbb{N}$ and $\mathbf{c} = (c_0, c_1, \ldots, c_n) \in \mathcal{C}^{n+1}$ be a walk in a chord graph G with finite

[14] GNU Scientific Library

[15] SEPTIMA documentation contains an example graph with CBC centrality measures assigned to vertex colors.



(a) individual transitions

(b) transitions joined together

Figure 15. A staff-notation (right) for the progression Gm^7 $\to$ C^7 $\to$ B^7 $\to$ G♯^{ø} $\to$ B♭^{7} obtained by joining consecutive elementary transitions (left) along enharmonically equivalent realizations, paired by brackets.

support $S \subset \mathbb{Z}$. Now choose $X_0 \in \mathcal{R}(c_0)$ such that $X_0 \subset S$ and, for each $k = 1, \ldots, n$, a transition

$$T_k = Y_k \xrightarrow{v_k} X_k \in \mathcal{E}(c_{k-1}, c_k), \quad X_k, Y_k \subset S. \tag{6}$$

Since for each $k = 0, 1, \ldots, n$ there exists a unique bijection $f_k : X_k \to P_{c_k}$ such that $f_k(x) = \pi(x)$ for each $x \in X_k$, the inverse $f_k^{-1} : P_{c_k} \to X_k \subset S$ satisfies the conditions of Definition 7.1 and therefore is a pitch spelling for c_k in S. Furthermore, since realizations X_{k-1} and Y_k are both members of $\mathcal{R}(c_{k-1})$, it follows $\pi(X_{k-1}) = \pi(Y_k)$ and hence for each $k = 1, 2, \ldots, n$ there exists a unique bijection $\varphi_k : X_{k-1} \to Y_k$ such that $(\pi \circ \varphi_k)(x) = \pi(x)$ for each $x \in X_{k-1}$. The tuple

$$(X_0, \varphi_1, T_1, \varphi_2, T_2, \ldots, \varphi_n, T_n) \tag{7}$$

is called a *transition chain*. Mappings $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the "glue" which joins the inner realizations of consecutive transitions together using pitch-class equivalence. An example is shown in Figure 15. Firstly, an elementary transition is chosen for each pair of consecutive chords; these are shown in Figure 15(a). Then, consecutive transitions are joined together, yielding the result shown in Figure 15(b).

Realizations $X_0, X_1, \ldots, X_n$ are pitch spellings for individual chords in $\mathbf{c}$ in S. Since $\mathbf{c}$ is a progression, these realizations slide into one another. For $k = 1, 2, \ldots, n$, the link between c_{k-1} and c_k is represented by the transition $X_{k-1} \xrightarrow{v_k \circ \varphi_k} X_k$. If $\|v_k \circ \varphi_k\|_\infty > 7$, then the enharmonic change of X_{k-1} to Y_k must be indicated by cue notes. Otherwise, that is not mandatory. For example, the cue A♯ in Figure 15 does not have to be indicated while the cue A♭ is mandatory.

Given a transition chain (7), the corresponding voices can be extracted using a simple routine, available in SEPTIMA as `TransitionNetwork::realize_path`, which outputs the voices as sequences of tones and inserts mandatory cue notes in each voice. Such output is called a *voicing*. Two transition chains are considered to be equivalent if they yield the same voicing.

7.2. *Algorithm*

In this section we provide an algorithm for obtaining an optimal transition chain for the progression $\mathbf{c}$. Such chain is obtained by minimizing the penalty function which is defined as follows. Let, for $z \in S$ and $k = 1, 2, \ldots, n$,

$$\alpha_k = \frac{1}{14}\sqrt{\sum_{x \in X_k} (x - z)^2}, \qquad \beta_k = \frac{\|v_k \circ \varphi_k\|_2}{7} = \frac{1}{14}\sqrt{\sum_{x \in X_{k-1}} (x - (v_k \circ \varphi_k)(x))^2},$$

and let, for $k = 0, 1, \ldots, n$, $\gamma_k \in \{0, 1\}$ be equal to 1 if X_k is an augmented sixth and 0 otherwise. Quantities α_k and β_k measure the distance of X_k from z (in units equal to seven steps in LOF) and the complexity of voice-leading notation in the transition from X_{k-1} to X_k, respectively. Smaller voice-leading LOF-shift indicates simpler intervals in the notation, e.g. minor second is simpler than augmented unison (chromatic step). Also, there are fewer mandatory enharmonic cues. The parameter z represents the "center of gravity" in LOF. It is preferred that means $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\gamma}$ of the above quantities are as small as possible, thereby corresponding to the following rules, respectively:

(1) Realizations closer to z are preferred.
(2) Melodic intervals in voice leading should be as simple as possible.
(3) It is preferred not to use augmented sixths.

Now the total penalty θ for the transition chain (7) is defined as a weighted mean of α_k, β_k, and γ_k with nonnegative weights $w_\alpha, w_\beta, w_\gamma$ representing the relative importance of the corresponding preferential rules:

$$\theta = \frac{w_\alpha \sum_{k=1}^{n} \alpha_k + w_\beta \sum_{k=1}^{n} \beta_k + w_\gamma \sum_{k=0}^{n} \gamma_k}{n(w_\alpha + w_\beta + w_\gamma)}. \tag{8}$$

A transition chain is optimal if it minimizes (8). Such chain can be obtained by finding a shortest path in the *transition network* for $\mathbf{c}$, which is defined as a digraph $N_{\mathbf{c}}(V, E)$ where, letting $V_k = \{(k, T) : T = X \xrightarrow{v} Y \in \mathcal{E}(c_{k-1}, c_k) \wedge X, Y \subset S\}$ for $k = 1, 2, \ldots, n$, the set of vertices V and the set of edges E are given by

$$V = \bigcup_{k=1}^{n} V_k, \qquad E = \bigcup_{k=1}^{n-1} \{(a, b) : a \in V_k, b \in V_{k+1}\}.$$

The vertices of $N_{\mathbf{c}}$ are divided into n levels $V_1, V_2, \ldots, V_n$. There is an arc between each two vertices in consecutive levels V_k and V_{k+1} for $k = 1, 2, \ldots, n-1$, starting in V_k and ending in V_{k+1}. Each vertex in V_k is an ordered pair consisting of k and a transition from c_{k-1} to c_k. Therefore, vertices in different levels are necessarily different, regardless of the fact that $\mathbf{c}$ is a walk and therefore can contain more than one instance of a single chord. Vertices in V_1 and V_n are the sources and sinks of $N_{\mathbf{c}}$, respectively. An example transition network is shown in Figure 16.

The arcs of $N_{\mathbf{c}}$ are weighted using a nonnegative function $W : E \to \mathbb{R}$, which is defined as follows. Let $e_k = (T_k, T_{k+1}) \in E$ be an arc from k-th to $k+1$-th level, where $k \in \{1, 2, \ldots, n-1\}$. Then, with respect to the notation introduced above,

$$W(e_k) = \begin{cases} w_\alpha(\alpha_1 + \alpha_2) + w_\beta(\beta_1 + \beta_2) + w_\gamma(\gamma_0 + \gamma_1 + \gamma_2), & k = 1, \\ w_\alpha \alpha_{k+1} + w_\beta \beta_{k+1} + w_\gamma \gamma_{k+1}, & k > 1. \end{cases} \tag{9}$$

Since $\alpha_k > 0$ for all $k = 1, 2, \ldots, n$, it follows that W is a positive function.

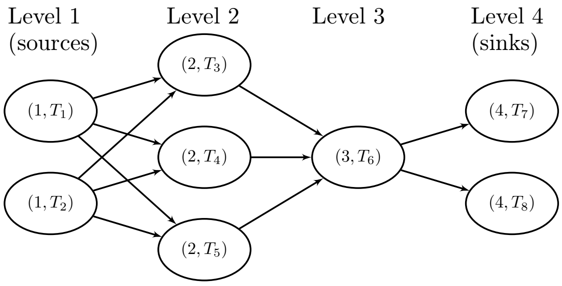


Figure 16. A transition network with four levels.

Every path in $N_{\mathbf{c}}$ from a source to a sink has the form

$$((1, T_1), (2, T_2), \ldots, (n, T_n)), \tag{10}$$

where $T_k \in V_k$ is defined in (6) for $k = 1, 2, \ldots, n$. Therefore, given $X_0 \in \mathcal{R}(c_0)$, the path (10) determines a unique transition chain given by (7). From (9) and (8) it follows that the weight of path (10) is equal to $\sum_{k=1}^{n-1} W(e_k) = n(w_\alpha + w_\beta + w_\gamma)\theta$, i.e. it is proportional to the penalty θ for (7).

An optimal voicing of $\mathbf{c}$ is now obtained as follows.

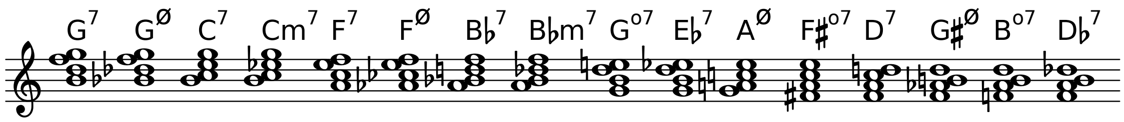
- For each pair $(X_0, z) \in \{X \in \mathcal{R}(c_0) : X \subset S\} \times \{\min S, \min S + 1, \ldots, \max S\}$, recompute arc weights in $N_{\mathbf{c}}$ and apply Dijkstra's shortest-path algorithm (West 2002, p. 97) to find cheapest paths between all source-sink pair of vertices in $V_1 \times V_n$. Among the $|V_1| \cdot |V_n|$ paths thus obtained for each particular value of z, choose the cheapest one and append it to a list L, which is initially empty.
- Compute the voicing for an element of L which minimizes the penalty θ.

This procedure is implemented in SEPTIMA as `ChordGraph::best_voicing`. It can be easily adapted to the task of finding all optimal voicings: after finding a cheapest source-sink path using Dijkstra's algorithm, we proceed by finding all source-sink paths of the same weight using Yen's algorithm (Yen 1971). Note that some of the obtained paths may represent identical voicings due to transition chain equivalence; the duplicates are not kept. Also, if two pitch spellings are transpositions of each other in LOF, then the one with larger value of $|z|$ is discarded. This variant of the algorithm is implemented in SEPTIMA as `ChordGraph::best_voicings`.

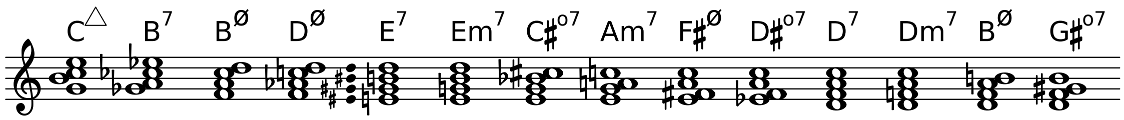
7.3. *Examples and applications*

We tested the algorithm described in Section 7.2 with $G = \mathcal{G}$, $w_\alpha = 1$, $w_\beta = 1.75$, and $w_\gamma = 0.2$, for progression shown in Figures 13 and 14, two sequences from Wagner's *Tristan*—the first from the opening of the Prelude and the second from the Immolation Scene—provided by Douthett and Steinbach (1998), and a sequence from Coltrane's *Giant Steps*, provided by Jedrzejewski (2019). In each case an unique optimal voicing was obtained; the results are shown in Figure 17. Voicings of Chopin's progressions move steadily downwards in a sequential manner while those of Wagner's progressions fluctuate about a fixed staff-position. In Coltrane's progression, the chord $\mathrm{B}^{\triangle}$ is spelled in two different ways, the first time as $\{\mathrm{B}, \mathrm{D}\sharp, \mathrm{F}\sharp, \mathrm{A}\sharp\}$ and the second time as $\{\mathrm{C}\flat, \mathrm{E}\flat, \mathrm{G}\flat, \mathrm{B}\flat\}$.

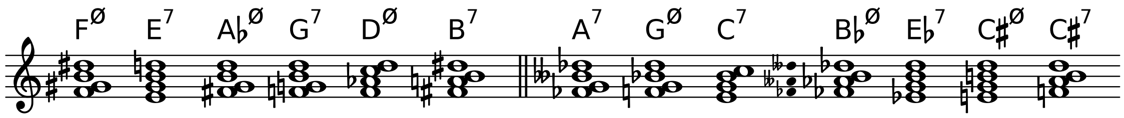
Although our algorithm obviously performs pitch-spelling inference, it does not aim to compete with existing pitch-spelling algorithms, which are more general and already



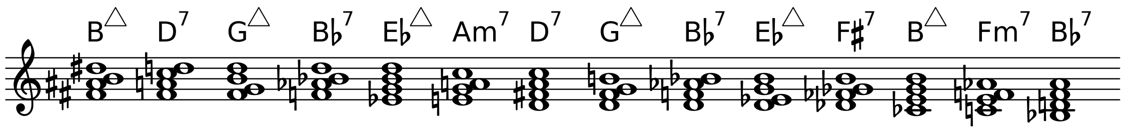
(a) Chopin's Mazurka Op. 68 No. 4 (the excerpt in Figure 13)



(b) Chopin's Prelude Op. 28, No. 4 (the excerpt in Figure 14)



(c) two progressions from Wagner's *Tristan*



(d) a progression from *Giant Steps* by John Coltrane

Figure 17. Optimal voicings returned by `ChordGraph::best_voicing` for $w_\alpha = 1$, $w_\beta = 1.75$, and $w_\gamma = 0.2$.

perform very well in practice (Meredith and Wiggins 2005). However, it may help such algorithms to be more accurate in certain cases. For instance, Stoddard, Raphael, and Utgoff (2004) report that a significant fraction of misspellings is due to notational ambiguities of augmented-sixth and diminished seventh chords. The present algorithm may be useful for choosing the right spelling in cases when these chords are followed or approached by other seventh chords.

8. Conclusion

The objective of this paper was to provide a model for bijective voice leading between seventh chords which refines the concept of parsimony. Since the model relies on the line of fifths for tone representation, it is capable of distinguishing between tonal and non-tonal realizations of seventh chords as well as between diatonic, chromatic, and other voice leadings. We have shown that the model provides efficient transitions between all $P_{m,n}$-related seventh chords and also for certain $P_{m,n}$-unrelated pairs. Finally, we presented an algorithm for generating concrete voicings from sequences of seventh-chord symbols.

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No potential conflict of interest was reported by the authors.

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